MODULE THREE PROBLEM SET

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Directions: Type your solutions into this document and be sure to show all steps for arriving at your solution. Just giving a final number may not receive full credit.

Problem 1

A 125-page document is being printed by five printers. Each page will be printed exactly once.

(a) Suppose that there are no restrictions on how many pages a printer can print. How many ways are there for the 125 pages to be assigned to the five printers?

One possible combination: printer A prints out pages 2-50, printer B prints out pages 1 and 51-60, printer C prints out 61-80 and 86-90, printer D prints out pages 81-85 and 91-100, and printer E prints out pages 101-125.

To determine the number of ways to assign the 125 pages to the 5 printers, we can use the product of counting. We can represent the five printers a A, B, C, D, and E, and for each of the 125 pages, there are 5 possible printers that it could be assigned to. Since each of the 125 pages can be assigned to anyone of the 5 printers, there are 5 choices for the first page, 5 choices for the second page, and so on, until the 125th page has also 5 choices. Therefore, by the product rule, the total number of ways to assign the 125 pages to the 5 printers are: 5*5*5*...*(125 times) = 5^{125}.

(b) Suppose the first and the last page of the document must be printed in color, and only two printers are able to print in color. The two color printers can also print black and white. How many ways are there for the 125 pages to be assigned to the five printers?

To accommodate the requirement that the first and last pages must be printed in color, we have only two printer options available. Since the remaining 123 pages can be printed in either black or white, there are 5 printer options available for each of these pages. Therefore, the total number of ways to assign the pages to the printers can be calculated as: A number of ways to assign the color pages * Number of ways to assign the remaining pages. $= 2 * 2 * 5 * 5 ... * 5(123\ times)$.
$= 2^2 * 5^{123}$.
$= 1.2209 * 10^{94}$.
Thus, there are $1.2209 * 10^{94}$ ways to assign the 125 pages to the five printers, considering that the first and last pages must be printed in

color and only two printers scan print in color.

(c) Suppose that all the pages are black and white, but each group of 25 consecutive pages (1-25, 26-50, 51-75, 76-100, 101-125) must be assigned to the same printer. Each printer can be assigned 0, 25, 50, 75, 100, or 125 pages to print.

How many ways are there for the 125 pages to be assigned to the five printers?

Since each group of 25 consecutive pages must be assigned to the same printer, there are 5 possible printers for each group. Therefore, we can think of this problem as assigning 5 groups of 25 pages to 5 printers, which can be represented as $5^5 = 3125$ possible ways to assign the pages. Each printer can be assigned one group of 25 pages, so there are 5 choices for the first group, 5 choices for the second group, and so on, for a total of $5^5 = 3125$ possible ways to assign the 125 pages to the five printers.

PROBLEM 2

Ten kids line up for recess. The names of the kids are:

$$\{\text{Alex, Bobby, Cathy, Dave, Emy, Frank, George, Homa, Ian, Jim}\}.$$

Let S be the set of all possible ways to line up the kids. For example, one order might be:

(Frank, George, Homa, Jim, Alex, Dave, Cathy, Emy, Ian, Bobby)

The names are listed in order from left to right, so Frank is at the front of the line and Bobby is at the end of the line.

Let T be the set of all possible ways to line up the kids in which George is ahead of Dave in the line. Note that George does not have to be immediately ahead of Dave. For example, the ordering shown above is an element in T.

Now define a function f whose domain is S and whose target is T. Let x be an element of S, so x is one possible way to order the kids. If George is ahead of Dave in the ordering x, then $f(x) = x$. If Dave is ahead of George in x, then $f(x)$ is the ordering that is the same as x, except that Dave and George have swapped places.

(a) What is the output of f on the following input?
(Frank, George, Homa, Jim, Alex, Dave, Cathy, Emy, Ian, Bobby)

The function f assigns the x to itself, where x is a set of ten names: Frank, George, Homa, Jim, Alex, Dave, Cathy, Emy, Ian, and Bobby.

(b) What is the output of f on the following input?
(Emy, Ian, Dave, Homa, Jim, Alex, Bobby, Frank, George, Cathy)

The output of function f for a given input x is f(x)=x, where x is a list of ten names: Frank, George, Homa, Jim, Alex, Dave, Cathy, Emy, Ian, and Bobby. If we swap the positions of Dave and George in the input list x, we get a new list: x=(Emy, Ian, George, Homa, Jim, Alex, Bobby, Frank, Dave, Cathy). This is the output of function f for the modified input x.

(c) Is the function f a k-to-1 correspondence for some positive integer k? If so, for what value of k? Justify your answer.

The function f is a k-to-1 correspondence because there are 2 input values that result in the same output due to the interchangeable places for George and Dave.

(d) There are 3628800 ways to line up the 10 kids with no restrictions on who comes before whom. That is, $|S| = 3628800$. Use this fact and the answer to the previous question to determine $|T|$.

Set S consists of all possible permutations of the ten names, which is given by 10!(10 factorial) and equals to, 3,628,800. Set T consists of all possible arrangements when considering Dave and George interchangeable, which means that we need to divide the size of set S by 2 because we are effectively counting each arrangement twice(once for Dave in George's position and once for George in Dave's position). Therefore, the size of set T is equal to 3,628,800/2=1,814,400."Set T consists of all possible arrangements of the ten names when considering Dave and George interchangeable. Since each arrangement can be obtained by swapping Dave and George's positions, set T has half the size of set S. The size of set T is equal to 1,814,400, which is obtained by dividing the size of et S (10!) by 2.

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PROBLEM 3

Consider the following definitions for sets of characters:

- Digits $= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- Letters $= \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$
- Special characters $= \{*, \&, \$, \#\}$

Compute the number of passwords that satisfy the given constraints.

(i) Strings of length 7. Characters can be special characters, digits, or letters, with no repeated characters.

The password length is 7 characters and there are 40 possible characters to choose from. Since no characters can be repeated in the password, the number of options decreases for each character chosen. The total number of possible passwords can be found by multiplying the number of options for each character. Therefore, the total number of possible password with a length of 7 and not repeated characters are, 40*39*38837*36*35*34=93,963,542,400.

(ii) Strings of length 6. Characters can be special characters, digits, or letters, with no repeated characters. The first character can not be a special character.

The first character in the password can be any of the 36 options, as special characters are not allowed in the first spot. For the second character, there are 36 options for non-special characters and 4 options for special characters, making a total of 40 options. Since no characters can be repeated, the second spot has 39 options. The number of possible passwords can be calculated by multiplying the number of options for each spot: 36*39*38*37*36*35=2,487,270,240.

Problem 4

A group of four friends goes to a restaurant for dinner. The restaurant offers 12 different main dishes.

(i) Suppose that the group collectively orders four different dishes to share. The waiter just needs to place all four dishes in the center of the table. How many different possible orders are there for the group?

The formula is given, n!/r!(n-r)! which calculates the number of possible combinations, not permutations, when selecting r items from a set of n items. To calculate the number of permutations, where order maters, the formula is n!/(n-r)!. Using this formula, we can calculate the number of different possible orders of 4 items selected from a set of 12 items as C(12,4) = 495. The order is just a subset of four of the 12 dishes on the menu.

(ii) Suppose that each individual orders a main course. The waiter must remember who ordered which dish as part of the order. It's possible for more than one person to order the same dish. How many different possible orders are there for the group?

Since each person can choose from 12 options and there are 4 people, the total number of possible order is $12^4 = 20,736$.

How many different passwords are there that contain only digits and lower-case letters and satisfy the given restrictions?

(iii) Length is 7 and the password must contain at least one digit.

The total number of possible characters is 26 lowercase letters and 10 digits, which gives a total of 36 possible characters. The number of 7-character passwords that have at least one digit is equal to the total number of 7-character passwords minus the number of 7-character passwords with no digits. The total number of 7-character passwords is 36^7. The number of 7-character passwords with no digits is 26^7 since there are 26 choices for each of the 7 characters, but no digits are allowed. Therefore, the number of 7-characters passwords with at least one digit is $36^7 - 26^7 = 70,332,353,920$.

(iv) Length is 7 and the password must contain at least one digit and at least one letter.

The sets L and D are not disjoint, as a password can contain both letters and digits. Therefore, their union $L \cup D$ over counts the passwords with both letters and digits, and we can not directly take the complement of $L \cup D$ to obtain the desired set. We can use the principle of inclusion-exclusion to count the number of passwords with at least one letter and one digit. Let A be set of all passwords with at least one letter, and let B be the set of all passwords with at least one digit. Then the number of passwords with at least one letter and at least one digit is equal to $n|A \cap B|$. To count $|A|$, please not that there are 26 choices for each character in a password, so there are 26^7 total passwords of length 7 using only letters. To count $|B|$, there are 10 choices for each character in a password of length 7 using only digits, so there are 10^7 such passwords. However, we have double-counted the passwords that contain both letters and digits, so we need to subtract them from the total. There are 36 choices for each character in a password with both letters and digits, so there are 36^7 such passwords. The inclusion-exclusion principle does not apply. So we proceeded with the same process as above but now removed all the passwords contain only digits: $|A \cap B| = |A| + |B| - |A \cup B| = 26^7 - 10^7 - 36^7 = -70,342,353,920$.

Problem 5

A university offers a Calculus class, a Sociology class, and a Spanish class. You are given data below about two groups of students.

(i) Group 1 contains 170 students, all of whom have taken at least one of the three courses listed above. Of these, 61 students have taken Calculus, 78 have taken Sociology, and 72 have taken Spanish. 15 have taken both Calculus and Sociology, 20 have taken both Calculus and Spanish, and 13 have taken both Sociology and Spanish. How many students have taken all three classes?

Let A, B, and C represent the sets of students who have taken Calculus, Sociology, and Spanish respectively. n(A)=61, n(B)=78, n(C)=72. $A \cup B \cup C$ is the set of students who have taken at least one class. $n(A \cup B \cup C) = 170$. $A \cap B$ is the set of students who took both Sociology and Calculus. $n(A \cap B) = 15$. $B \cap C$ is the set of students who took both Spanish and Sociology. $n(B \cap C) = 13$. $A \cap C$ is the set of students who took both Spanish and Calculus. $n(A \cap C) = 20$. $A \cap B \cap C$ is the set of students that took all three subjects. $n(A \cap B \cap C) = 7$.Using the inclusion-exclusion principle, we can find the number of students who taken exactly one or two subjects: $n(exactlyonesubject) = n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C) n(exactlyonesubject) = 61 + 78 + 72 - 2(15) - 2(13) - 2(20) + 3(7) = 127$. Therefore, the number of students who have taken all three subjects is 7.

(ii) You are given the following data about Group 2. 32 students have taken Calculus, 22 have taken Sociology, and 16 have taken Spanish. 10 have taken both Calculus and Sociology, 8 have taken both Calculus and Spanish, and 11 have taken both Sociology and Spanish. 5 students have taken all three courses while 15 students have taken none of the courses. How many students are in Group 2?

Let A, B, and C represent the students who took Calculus, Sociology, and Spanish respectively. n(A)=32, n(B)=22, and n(C)=16. $A \cap B$ is the group of students who took both Sociology and Calculus. $n(A \cap B) = 10$.

Let P represent the number of students who took at least one of the calculus, Spanish, and sociology classes. Using the inclusion-exclusion principal, we have:
P = 32 + 22 + 16 - 10 - 8 -11 + 5 = > p = 46
Including the 15 students who have not taken any courses, Group 2 contains 46 + 15 = 61 students.

Problem 6

A coin is flipped five times. For each of the events described below, express the event as a set in roster notation. Each outcome is written as a string of length 5 from $\{H, T\}$, such as $HHHTH$. Assuming the coin is a fair coin, give the probability of each event.

(a) The first and last flips come up heads.

Let A be the event of getting heads on the first and the last flips of a coin when it is flipped five times. The outcomes of this event can be written as a set in roster notation: A=HHHHH,HHTTH, HTHHH, HTTTH, HTHTH, HTTHH, HHTHH, HHHTH. The probability of this event can be calculated as the ratio of the number of outocmes in A to the total number of possible outocmes, which is 8 out of 32, therefore, the probablity of A is 8/32, which can be simplified to 1/4.

(b) There are at least two consecutive flips that come up heads.

Let B denote the event of getting at least two consecutive heads when a coin is flipped five times. The possible outcomes of B are HHHHH, TTTHH, HHHTH, HHHHT, HHTHH, HHHTT, HHTTH, HHTHT, HTHHH, HHTTT, HTTHH, HTHHT, THHHH, THHHT, THHTH, THHTT, THTHH, TTHHT, TTHHH. Thus, the number of outcomes in B is 19. Therfore, the probability of B can be calculated as the ratio of number of outcomes in B to the total number of possible outcomes, which is 19 out of 32, hence, the probability of B is 19/32.

(c) The first flip comes up tails and there are at least two consecutive flips that come up heads.

To express the mathematically, we can say that there are 8 possible outcomes for event C and the probability of event C occurring is 1/4. Therefore, we can represent this as n(C)=8 and p(C)=1/4.

Problem 7

An editor has a stack of k documents to review. The order in which the documents are reviewed is random with each ordering being equally likely. Of the k documents to review, two are named “Relaxation Through Mathematics” and “The Joy of Calculus.” Give an expression for each of the probabilities below as a function of k. Simplify your final expression as much as possible so that your answer does not include any expressions in the form
$\left(\begin{array}{c} a \\ b \end{array} \right)$.

(a) What is the probability that “Relaxation Through Mathematics” is first to review?

Let K be the number of documents that can be reviewed, then the number of ways to review them is k!. If one document is selected as the first to review, then there are (k-1)! ways to review the remaining documents. Therefore, the probability of P of selecting a document at a random to review is given by: P=(number of ways to select one document)/(total number of ways to review all documents) P=1 *(k-1)!/k! P=1/k. Hence, the probability of selecting a document at a random to review is 1/k.

(b) What is the probability that “Relaxation Through Mathematics” and “The Joy of Calculus” are next to each other in the stack?

Let the total number of documents in the stack be k, including ”Relaxation Through Mathematics” and ”The Joy of Calculus”. If we consider both documents as a single document, there will be (k-1) documents to arrange in the stack. Within this unit, ”Relaxation Through Mathematics” and ”The Joy of Calculus” can be arranged in 2.(k-2)! ways. Therefore, the total number of arrangements in which ”Relaxation Through Mathematics” and ”The Joy of Calculus” are next to each other is 2.(k-2)!. The probability of this happening is then: P=2.(k-2)!/k!. Simplifying this expression, we get: P=2/(k-1). Therefore, the probability of ”Relaxation Through Mathematics” and ”The Joy of Calculus” being next to each other in the stack is 2(k-1)!.

The size of the sample space is k!. The number of ways to order the stack of documents in which ”Relaxation Through Mathematics” and ”The Joy of Calculus” are next to each other is 2(k-1)!. (There are (k-1)! ways to order all documents except ”Relaxation Through Mathematics”. Once the order is fixed, ”Relaxation Through Mathematics” can be inserted either before or after ”The Joy of Calculus”). The probability that ”Relaxation Through Mathematics” and ”The Joy of Calculus” are next to each other in the stack is thus 2(k-1)!/k! = 2/k.