Biometrika (1988), **75**, 4, *pp.* 803–4
Printed in Great Britain

On the classical choice of variance stabilizing transformations and an application for a Poisson variate

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SUMMARY

A simple method for obtaining a class of variance stabilizing transformations from a given normalizing transformation is presented. The method is based on Curtiss's (1943) classical results on such transformations. Illustrations are given for a Poisson variate.

Some key words: Poisson variate; Variance stabilizing transformation.

Consider a variate X_t with distribution depending on a parameter $t \in (0, \infty)$ such that for some nonnegative function $\psi_0(t)$, we have as $t \to \infty$ convergence in distribution in that

$$\psi_0(t)(X_t - t) \to Z \sim N(0, 1). \tag{1}$$

A transformation f, such that: (i) $f(X_t) - f(t)$ converges in distribution to cZ, for a nonzero constant c; and (ii) as $t \to \infty$, $\lim \operatorname{var}\{f(X_t)\} = c^2$, is termed variance stabilizing. For a survey of such transformations and their applications see Hoyle (1973).

In early and often overlooked work on this topic, Curtiss (1943) gave a rigorous general approach. His Theorems 3.1 and 3.2 provide sufficient conditions for the existence of a variance stabilizing transformation as the indefinite integral of $\psi_0(X_t)$. Curtiss's conditions, however, are cumbersome to verify for specific examples.

Our goal is to provide easily verifiable alternative conditions for Curtiss's conditions, and hence to delineate many variance stabilizing transformations. The following proposition, whose proof is in a technical report, achieves this.

PROPOSITION. *Let X_t be a variate satisfying* (1). *Let $\mathscr{A}$ be the class of nonnegative and continuous real-valued functions $\psi(t)$ such that $\psi(t) \sim c\psi_0(t)$ as $t \to \infty$ for some $c \neq 0$. Assume that $\psi_0 \in \mathscr{A}$ is such that*: (i) *ψ_0 is monotonic and differentiable*; (ii) *$d\{1/\psi_0(t)\}/dt \to 0$ as $t \to \infty$; and* (iii) *$\int \psi_0(t)\, dt$, where the integral is over (a, ∞), diverges. Then $\mathscr{A}$ generates a class of variance stabilizing transformations given by*

$$\tilde{\mathscr{A}} = \left\{ f(X_t) = \int_a^{X_t} \psi(x)\, dx,\ \psi \in \mathscr{A} \right\}.$$

This proposition is easily applied to a wide class of distributions and each member of $\mathscr{A}$ can be used to generate a parametric subclass of $\tilde{\mathscr{A}}$ from which an optimal stabilizer can be determined. Thus for a Poisson variate with mean t, $\psi_0(t) = t^{-\frac{1}{2}}$ satisfies the premises of the proposition. Fix $\alpha > 0$, $\beta \geq 0$ and $\gamma \geq 0$; then the functions

$$\psi_{\alpha,\beta}(t) = \tfrac{1}{2}(t+\beta)(t+\alpha)^{-3/2}, \quad \psi_{\alpha,\beta,\gamma}(t) = \psi_{\alpha,\beta}(t) - \tfrac{1}{2}(t+\gamma)^{-3/2},$$

are both asymptotically equivalent to $\frac{1}{2}t^{-\frac{1}{2}}$ and thus belong to $\mathscr{A}$. These generate the parametric subclasses of variance stabilizing transformations:

$$\tilde{\mathscr{A}}_{\alpha,\beta} = \{f_{\alpha,\beta}(X_t) = (X_t + 2\alpha - \beta)(X_t + \alpha)^{-\frac{1}{2}},\ \alpha > 0, \beta \geq 0\}, \tag{2}$$

$$\tilde{\mathscr{A}}_{\alpha,\beta,\gamma} = \{f_{\alpha,\beta,\gamma}(X_t) = f_{\alpha,\beta}(X_t) + (X_t + \gamma)^{-\frac{1}{2}},\ \gamma \geq 0\}. \tag{3}$$

Series expansions yield

$$\operatorname{var}\{f_{\alpha,\beta}(X_t)\} = \tfrac{1}{4} + 2^{-5}(3 - 24\alpha + 16\beta)t^{-1} + 2^{-7}(192\alpha^2 + 32\beta^2 - 192\alpha\beta - 156\alpha + 104\beta + 17)t^{-2}$$
$$-2^{-11}(5{,}120\alpha^3 - 6{,}144\alpha^2\beta + 1{,}536\alpha\beta^2 - 1{,}728\beta^2 - 11{,}328\alpha^2 + 11{,}136\alpha\beta$$
$$+ 6{,}384\alpha - 4{,}256\beta - 645)t^{-3} + O(t^{-4}). \quad (4)$$

The coefficients of t^{-1} and t^{-2} in (4) vanish if $\alpha = \alpha_0 = \frac{3}{8} + 3^{-\frac{1}{2}}/2$, $\beta = \beta_0 = \frac{3}{8} + 3^{\frac{1}{2}}/4$, in which case $\operatorname{var}\{f_{\alpha_0,\beta_0}(X_t)\} = \frac{1}{4} + O(t^{-3})$ and $f_{\alpha_0,\beta_0}(X_t)$ is the optimal stabilizer in $\mathscr{A}_{\alpha,\beta}$. For $\alpha = \beta$, (2) reduces to $\mathscr{A}_{\alpha,\alpha}$ (Anscombe, 1948). The optimal element in $\mathscr{A}_{\alpha,\alpha}$ is $(X_t + \frac{3}{8})^{\frac{1}{2}}$ with variance equal to $\frac{1}{4} + O(t^{-2})$. For $\alpha = \beta = 1$ and $\gamma = 0$ in (3), we obtain $(X_t + 1)^{\frac{1}{2}} + X_t^{1/2}$ (Freeman & Tukey, 1950). Numerical comparisons of the variances of $f_{\alpha_0,\beta_0}(X_t)$, $f_{3/8,3/8}(X_t)$ and $f_{1,1,0}(X_t)$, up to terms of order $O(t^{-4})$, show the superiority of $f_{\alpha_0,\beta_0}(X_t)$ as a stabilizer for small and large values of t. Similar results can be obtained for a chi-squared variate.

REFERENCES

ANSCOMBE, F. J. (1948). The transformation of Poisson, binomial, and negative binomial data. *Biometrika* **35**, 246–54.

CURTISS, J. H. (1943). On transformations used in the analysis of variance. *Ann. Math. Statist.* **14**, 107–22.

FREEMAN, M. F. & TUKEY, J. W. (1950). Transformations related to the angular and the square root. *Ann. Math. Statist.* **21**, 607–11.

HOYLE, M. H. (1973). Transformations—an introduction and a bibliography. *Int. Statist. Rev.* **41**, 203–23.

[*Received July* 1987. *Revised June* 1988]