Recent Progress in Subdivision: a Survey

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Summary. After briefly establishing the traditional concepts in subdivision surfaces, we survey the way in which the literature on this topic has burgeoned in the last five or six years, picking out new trends, ideas and issues which are becoming important.

Subdivision surfaces were first described by Catmull and Clark [6] in 1978, soon, in fact, after the now-ubiquitous NURBS were identified as being a sensible standard for parametric surface descriptions. For twenty years they were an interesting generalisation of (a subset of) NURBS, with a paper on one aspect or another appearing in some relevant journal every year or so.

In the last five or six years the situation has totally changed, as indicated by the numbers of papers published which relate to subdivision reasonably directly. These are plotted against date in Fig. 1. Subdivision surfaces are now one of the methods of choice in Computer Graphics, and some consider that they might succeed NURBS as the standard in engineering CAD.

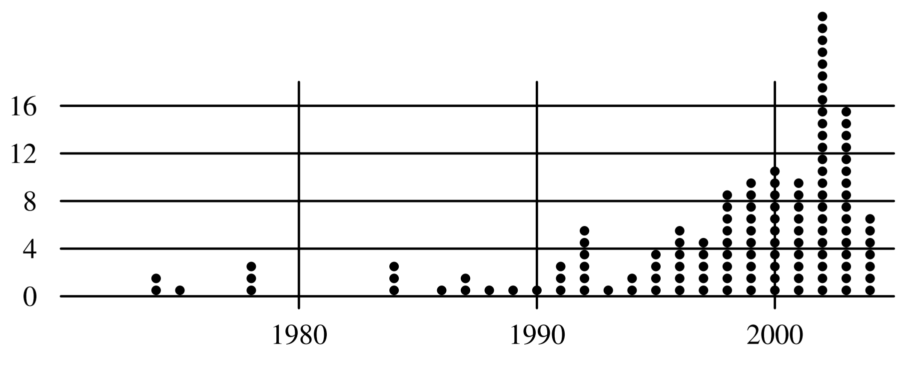

Fig. 1. Number of subdivision papers, plotted by year.

This paper looks at some of the technical changes that have happened in the last few years which are helping to drive that change in status. The paper is divided into the following aspects:

- Background – identifying the "classical" knowledge.
- New schemes and a classification.
- New domains and new ranges – a subdivision surface may be thought of as a map from a bivariate manifold into $\mathbb{R}^3$. We can map from other domains, and we can map into other ranges.
- New issues – the traditional focus on smoothness has now been joined by other criteria for judging the quality of a scheme.
- New ideas – it has become clear that linear stationary schemes may not provide the solution to all our problems, and newer ideas broaden our horizons.

1 Introduction and Background

The beginnings of the subdivision story can be dated back to the papers of de Rahm [1], over fifty years ago, but the relevance to the modelling of shape started with the proposal of Chaikin [2], who devised a method of generating smooth curves for plotting. This was soon analysed by Forrest [3] and by Riesenfeld [4] and linked with the burgeoning theory of B-spline curves. It became clear that uniform B-spline curves of any degree would have such a subdivision construction.

The extension to surfaces took just a few years, until 1978, when Catmull and Clark [6] published their descriptions of both quadratic and cubic subdivision surfaces, the exciting new point being that a surface could be described which was not forced to have a regular rectangular grid in the way that the tensor product B-spline surfaces were. The definition of a specific surface in terms of a control mesh could follow the needs of the boundaries and the curvature of the surface. This was made possible by the extension of the subdivision rules to allow for 'extraordinary points', being either 'extraordinary vertices' where either other than four faces came together at a vertex, or else 'extraordinary faces' where a face had other than four sides.

At about the same time Doo [5] and Sabin, who had also been working on quadratic subdivision, showed a way of analysing the behaviour of these schemes at the extraordinary points, treating the refinement process in terms of matrix multiplication, and using eigenanalysis of the spectrum of this matrix [7] . This aspect was followed up by Ball and Storry [9, 12] who made this analysis process more formal and succeeded in making some improvements to the coefficients used around the extraordinary points in the Catmull-Clark scheme, so that radial curves through an extraordinary point matched curvatures there at equal distances on opposite sides. In his PhD dissertation [11], Storry identified that in the limit, the configuration around an extraordinary point was always an affine transform (dependent on the original polyhedron) of a point distribution which was completely defined by the eigenvectors of the subdivision matrix. He called this the *natural configuration*.

The next two big ideas emerged in 1987. Loop, in his Masters' thesis [13], described a subdivision scheme defined over a grid of triangles. This not only gave a new domain over which subdivisions could be defined, but also showed that the eigenanalysis could be used explicitly in the original design of a scheme, in the choice of coefficients which should be used around extraordinary points.

The other significant publication that year was the description by Dyn, Levin and Gregory [14] of their four-point curve scheme. This was new in two ways: it was an interpolating scheme, rather than smoothing, and the limit curve did not consist of parametric polynomial pieces. The analysis of its continuity and differentiability therefore required new tools.

The first tool was provided in [14] and tools of a greater generality were provided in [19] and [20]. The method in the later paper together with the idea of the *symbol* of a subdivision scheme, presented in [19], was later expressed in terms of z-transforms [26], which turn convolution of sequences of numbers into multiplication of Laurent polynomials. Algebraic manipulation of these polynomials allows such processes as the taking of differences to be expressed very simply, and it has turned out that many of the arguments we need to deploy can be expressed very elegantly in this notation. It also provides sufficient conditions for a scheme to have a certain level of derivative continuity, whereas the eigenanalysis approach provides only necessary conditions.

The generalisation of the four-point ideas to an interpolating surface scheme came in 1990, with the description by Dyn, Levin and Gregory [17] of the butterfly scheme, an interpolating surface scheme defined over a triangular grid.

In 1995 Reif [34] showed that there was rather more to continuity than had been dreamt of. He identified that the natural configuration implies a parametrization of the rings of regular pieces which surround each extraordinary point and that it is essential, in order to obtain a scheme which generates well-behaved surfaces at the extraordinary points, to ensure that this parametrization is injective. (Later, Peters and Reif went further, and in [54] they constructed a scheme (a variant of the quadratic) for which the injectivity test fails, resulting in severe folding of the limit surface in every ring.)

The following year Reif [35] showed that the attempts to make a C2 variant of Catmull-Clark were not going to succeed, because a surface C2 at the extraordinary points would need to have regular pieces at least bi-sextic.

Thus as we passed the mid-1990s, subdivision theory stood like this:

- A surface subdivision scheme takes a manifold mesh of vertices joined by faces, usually called the polyhedron, and creates a new, finer, polyhedron by constructing new vertices as linear combinations of the old ones, in

groups defined by the connectivity of the polyhedron, and joining them up by new faces in a way related to the old connectivity.
The coefficients can be documented in diagrammatic form either in the *mask*, a diagram where the coefficients by which a given old vertex influences the surrounding new ones are laid out in the same pattern as those new vertices, or in the *stencils*, a set of diagrams where the coefficients by which nearby old vertices influence a given new one are laid out in the same pattern as those old vertices. These are totally equivalent. In the univariate case, they are the columns and rows, respectively, of the matrix by which the sequence of old vertices is multiplied to give the sequence of new ones.

- This refinement can be repeated as often as desired, and there are conditions on the scheme guaranteeing the existence of a well-defined limit surface to which the sequence of finer and finer polyhedra converges. During the refinement process the number of extraordinary points remains constant, and they become separated by regular mesh of a kind which is dependent on the topological rules of the scheme.
- The regular mesh is often well-described by box-spline theory (the butterfly scheme was almost alone in not being describable in those terms) but the z-transform analysis can always be applied to determine the smoothness of the limit surface in the regular regions. The extraordinary points are surrounded by rings of regular mesh, and close to the extraordinary point these are just affine transforms of the natural configuration, and can be parametrized by the characteristic map.
- Because every box-spline has a generating subdivision scheme [10], we had a way in principle of creating as many different subdivision schemes as we might want. Each such scheme would have to have its extraordinary point rules invented, of course, but nobody had bothered to go through the exercise. We also had a sequence of interpolating curve schemes, generated by letting an increasing number ($2n$) of points influence the new vertex in the middle of each span [16], but this had not led to a sequence of interpolating triangular surface schemes. In fact Catmull-Clark, Loop and butterfly were regarded as the significant surface schemes, and the cubic B-spline subdivision and the 4-point scheme as the significant curve schemes, any others being only of academic interest.
- The question of the behaviour of the limit surface in the immediate vicinity of the extraordinary points was still of interest. Indeed, the papers [7, 15, 18] before Reif's key result [35] on the lower bound of the polynomial order of patches surrounding an extraordinary point have been more than balanced by those after [46, 47, 58, 69, 70, 134].

This amount of classical subdivision theory is elaborated at greater length in the material prepared for the Primus Workshop earlier in the MINGLE project. Of particular relevance are the four chapters [100, 101, 102, 103].

2 New Schemes and a Classification

2.1 New Schemes

Kobbelt's 1996 scheme for interpolatory subdivision over quad grids [37] can be regarded as the last of the classical schemes, being essentially a tensor product of the four-point scheme with the awkward details sorted out.

The 'simplest' scheme, published by Peters and Reif in 1997 [41], however, had a new flavour to it. It was in fact a box-spline, and we should have been able to predict it by doing a systematic scan through all the box-splines with small numbers of shifts. This may be how they found it, but it still felt new, because the mesh changed orientation at each refinement stage.

This surprising aspect of that scheme is echoed in Kobbelt's $\sqrt{3}$ scheme [66] where a triangular grid becomes denser by the insertion of a new point in the middle of each triangle: the old edges disappear, being replaced by new ones joining each new point to the corners of its triangle and to the neighbouring new points. This scheme was not a box spline – its basis function is a fractal – and its publication led to an upsurge of interest in the analysis of smoothness[1] and in the determination of the support[2] of a scheme. In fact there is a box-spline with the same topology but a slightly larger mask. This was observed by Ron during the Dagstuhl meeting at which Kobbelt described his $\sqrt{3}$ scheme. As determined by Ron [127], over the regular grid it consists of quartic pieces and has the high continuity (C^3 in this case) expected from a box-spline scheme with many distinct generator directions. However, the translates of the basis function are not linearly independent, and so it is not possible to compute the control points from a set of points, similarly connected, which are to be interpolated. This topology of refinement of a triangular mesh was also explored by Guskov [53], who identified a bivariate family of schemes, one of which is an interpolating scheme.

The same theme led to Velho's 4-8 scheme [77, 78], a box-spline over the 4-direction grid (the quad grid with diagonals), which appears to have many nice properties, particularly in terms of being well behaved when there are diagonal features in the surface being designed. Yet another box-spline over this grid forms the quad part of Peters' 4-3 scheme [139].

Alexa [87] showed that the rotation can be even more general, the arity being described by a pair of integers[3]. This insight led to the exploration of skew schemes by Ivrissimtzis [129] and by Dodgson [111]. Direct exploration of

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[1] the number of continuous derivatives.

[2] The support of a scheme is the non zero domain of the basis function of a scheme. This was analysed fully by Ivrissimtzis [137] who showed exactly when the boundary of the support would follow lines of the grid, and when it would be fractal.

[3] Another view [113, 124] is that it is a dilation matrix, a 2×2 matrix with integer entries and eigenvalues greater than or equal to 1 and its spectral radius is greater than 1, and with certain symmetries [124] corresponding to those of the topology of refinement.

ternary univariate schemes by Hassan and Ivrissimtzis [88, 114] was also happening at about the same time. This thread was later followed by Loop [115], who explored ternary subdivision of the 3-direction box-spline and found an appropriate set of coefficients for the extraordinary point case. A ternary triangulation interpolating scheme was explored, so far unsuccessfully, by Dodgson et al. in [105].

2.2 Classification

We have also been able to see more structure in the set of known surface subdivision schemes. Whereas before we had half a dozen or so known patterns, with the tensor product box-splines providing one coherent family, we can now see a framework into which all the schemes likely to be of interest will probably fit. In fact there are a few variants on this but they do not differ greatly, and [128] is a good starting point. Another is [113].

Such a classification has a number of levels, the top level being the plane tessellation which appears in the regular case. Then whether the scheme is primal, dual, both or neither, and then the arity.

Once these attributes are defined, the topology of the refinement of the polyhedron from step to step is fully determined. We then identify the *footprint*, the set of non-zero coefficients in the mask. From this and the arity the region of support can be determined, and, most important, the question of whether the scheme gives a fractal limit surface or possibly a piecewise polynomial. The support boundary [137] gives a first clue to this.

The shapes of the stencils are also determined from the footprint.

The next step is to specify the actual coefficients in the regular grid case, and finally, schemes can differ by the special rules applied at extraordinary points, although two schemes, with differences in only this respect, should be regarded as variants rather than as distinct schemes.

This classification leads to a notation [128] for the description of almost any uniform linear stationary scheme, in which two letters code the regular grid type and the primality or duality of the scheme. These are followed by two integers defining the arity and then a list of lists of independent coefficients in the mask. This is a formal notation, (although ad hoc rather than following a more general structure such as L-systems [110]). In principle it should be possible to compile an implementation of a new scheme from this notation. The extraordinary point rules are covered by letting the coefficients be functions of the valency of the extraordinary point.

Curve schemes do not have, or need, quite such a rich classification. A sufficient analogous categorisation can use the primal/dual aspect as the first level, then the arity and the mask width, followed by the approximating/interpolating issue, and finally by the detail of the values in the mask. The formal notation has the primal/dual letter, the arity, and then a list of the independent coefficients in the mask. Everything in a linear stationary scheme is derivable from this information.

3 New Domains and New Ranges

3.1 Trivariate Data

Recently first attempts were made to extend subdivision schemes to volumetric data. The simplest case is that of refining trivariate function values given at the vertices of a cube-type lattice (hexahedral lattice), which is topologically equivalent to the integer lattice in $\mathbb{R}^3$, namely to $\mathbb{Z}^3$.

This was explored by Peters and Wittman [44], using a 7-direction box-spline and by Barthe et al. [89], using the trivariate analogue of Chaikin, as a convenient way of managing smooth data without excessive grid density for the definition of implicit free-form surfaces. Weimer and Warren [59] use a regular trivariate domain for the solution of flow fields. Chang et al. [107] derived a C^5 subdivision scheme based on box-splines, that must be applied over hybrid tetrahedral/octahedral meshes.

MacCracken and Joy [39] used 3D subdivision as a way of defining displacement fields for free-form deformations. Their scheme is the tensor product tri-cubic B-spline scheme over the cube-type lattice, and has an extension to similar lattices with arbitrary connectivities.

Bajaj et al. [95] proposed an alternative treatment of extraordinary edges and points to that presented in [39]. Their scheme also generates tri-cubic B-spline volumes at the regular parts of the lattice. In this tensor-product setting, an extraordinary edge is essentially just the tensor product of a bivariate extraordinary point with the curve associated with the sequence of extraordinary edges. This brings little new insight except that, because the subdominant eigenvalue associated with the curve is always 1/2, in order to keep the aspect ratios reasonable during refinement, that of the extraordinary point also needs to be 1/2.

The analysis of points where extraordinarities meet is yet to be developed. Due to the many possible combinations of the valency of the point and the valencies of the edges meeting at the point, such analysis is expected to be highly complicated. It appears to be enumerable only if we limit the extraordinary edges to have a valency at most one more or one less than the normal, when the number of possible interesting configurations (for a hexahedral grid) is only eight.

In contrast to triangulations in the 2D case, the tessellation of the integer lattice in 3D by a tetrahedralisation is more complicated, because there are no regular tetrahedralisations, only semi-regular[4], and there is not one obviously preferable choice. It is common in the Finite Element literature to use two different types of tetrahedra to cover the lattice.

[4] In fact there is only one regular grid in the trivariate case, the 'hexahedral' grid (a trivariate grid topologically equivalent to the regular cubical grid). All 'regular' tetrahedralisations are in fact only semiregular, with the underlying rotation group of the hexahedral grid.

There is one lattice of similar tetrahedra which tessellates $\mathbb{R}^3$. This is not a tessellation of $\mathbb{Z}^3$, but of the lattice generated by it together with the midpoints of the cubes of $\mathbb{Z}^3$ [22], but even this is semiregular in that some edges are 6-valent while others are 8-valent.

In work in progress, Dyn, Greiner and Duchamp study a subdivision scheme, based on local averaging steps, which generates C^1 functions over this lattice. This scheme is well defined over an extension of the above lattice to arbitrary connectivities, but at this stage the analysis of continuity and smoothness near the extraordinary edges and vertices is not satisfactory. For example, the situation round an extraordinary edge is not straightforwardly described in terms of matrices for grids of tetrahedra.

3.2 Higher Dimensions

It is quite clear that direct analogues exist for manifolds of higher dimensions. However, what problems will emerge as the dimension rises still further is not yet evident.

3.3 Associated Fields

There is no reason why the subdivision process need be carried out only on the coordinates of polyhedron points. DeRose [49, 50] uses additional data to carry coordinates for applying texture to subdivision surfaces, an important issue in Computer Graphics. Each vertex has two additional values, as well as the three coordinates, and all five values are refined as if the surface lay in a space of five dimensions. Such additional values can be used for any other properties for which we require a field of values to vary smoothly over the limit surface. All that is necessary is to think of the subdivision as being carried out in a space of many dimensions.

3.4 Field Analysis

In particular, a subdivision surface or volume can support an analogue of the finite element method by using the subdivision basis functions instead of the Lagrange functions conventionally used in stress-analysis elements. Subdivision is now logically carried out in the space of 3 position coordinates and 3 components of displacement, although the latter are treated as algebraic variables solved for by an energy minimisation process. This gives an extremely powerful method, because the subdivision process gives adaptive basis functions, strongly analogous to standard FE practice, but performing much better, in that they span a space in which the first derivative is continuous everywhere (like the actual field in an elliptic problem such as elasticity), so that freedoms are not wasted on modelling possible discontinuities of derivative.

This has been described by Cirak, Ortiz and Schröder [57, 91] in the surface case and by Weimer and Warren [59] in the solid case. A key result for surface analysis was obtained by Reif and Schröder in [74], which showed that it was not necessary to have bounded curvature for the bending energy of a subdivision surface to be well-defined. This meant that easily accessible untuned versions of Loop and Catmull-Clark could be used for this purpose.

The use of subdivision gives both good basis functions, capable of giving at least one order of magnitude improvement over conventional finite elements in speed for a given accuracy, and a very convenient h-refinement mechanism. This means that we can expect to see enormous benefits in systems which give almost real time stiffness and strength analysis during the interactive modification of not-too-complicated CAD models.

3.5 Compact Sets

In a series of papers, Dyn and Farkhi investigated spline subdivision schemes, with compact sets taking the rôle of the control points [71, 82, 86]. They found that for data consisting of convex, compact sets and with the Minkowski sum of sets replacing the usual addition of points, the limits generated by spline subdivision schemes approximate set-valued functions with convex images in a "shape preserving" way [71]. For set-valued functions with non-convex images these schemes converge to set-valued functions with convex images [140]. So no approximation can be expected.

In [82] the authors suggested performing spline subdivision by repeated averaging and replacing the averaging operation between two control points by the binary operation between two compact sets, called the "metric average". They proved that the resulting subdivision schemes are convergent. They also showed that any such scheme operating on data sampled from a Lipschitz continuous set-valued function, generates a limit set-valued function approximating the sampled set-valued function. The smoothness properties of the limit set-valued functions have yet to be explored.

3.6 Face-valued Subdivision

It is also possible to associate values with faces rather than with vertices and to have these values propagated through the refinement process. A natural quantity to attach to a face is the average of some function f over that face. In tensor-product schemes averages over the faces in the parametric domain can be regarded as point values of the primitive function of f, obtained by integrating f in the two parameters. This was observed by Donoho [27] for averages of univariate functions. In triangular grids refinement of averages over the triangles gives rise to new families of schemes.

Cohen et al. [72] present and analyse an "average interpolating" scheme. In such a scheme each triangle is refined, in the parametric domain, into four similar ones such that the averages assigned to the refined triangles generated from triangle T, sum to four times the average assigned to T. Cohen et al's scheme generates C^0 limit functions and is exact for linear bivariate polynomials.

A family of "approximating" face-value schemes, which are obtained by the elementary operation of repeated averaging of the face-values (in analogy to Box-splines schemes which can be described by simple repeated averaging of point values) are studied by Dyn et al. [112], where their smoothness is derived from convolution relations. Such schemes, also called "dual schemes" on triangulations, are investigated by Oswald and Schröder [122], as a continuation of the work of Zorin and Schröder [83] on alternating quad grids.

4 New Issues

4.1 Compatibility with Existing Parametric Surface Software

A requirement which has come from the move to incorporate subdivision surfaces into existing major commercial software systems already dealing with surface geometry held in other forms, is that a method is needed for the interrogation of these surfaces in a way which is compatible at a low level with the existing parametric surfaces. This is needed so that the breadth of functionality of those systems can be maintained without a complete recoding.

A large step in this direction can be taken by allowing a subdivision surface to be perceived as a collection of a small number of rectangular parametric surfaces. However, just using the regular regions of the surface and the rings around extraordinary points is not sufficient, because there are too many rings.

The alternative, developed by Stam [52], is to split the surface so that extraordinary points appear at corners of rectangular carpets, and have a special evaluator.

Whenever an evaluation request is received relating to a point in the region influenced by the extraordinary point, this evaluator does the subdivision to a sufficient level on the fly and then peeks at the right ring.

Although effective, this is inelegant, because the computation required depends on the closeness of the requested point to the extraordinary point. It also has a problem that very close to the extraordinary point, the mapping of a corner of the parameter square on to one sector of the subdivision surface has extremely large second derivatives. This means that the calling algorithms must be extremely robust to work effectively.

Zorin and Kristjansson [98] suggested that instead, in the region influenced by the extraordinary point, the surface should be examined in terms of the finite number of eigencomponents and evaluated direct from there.

It could be argued that divide and conquer algorithms, based on enclosures [93], would be a much more robust approach to interrogation, but the suppliers of commercial software have limited budgets for such complete reimplementations, and do have the requirement to be able to support any new surface type with the full range of functionality of the system. Being able to plug the new surfaces in at the evaluation level is a very practical way of achieving this in the short term.

4.2 Artifacts

Until about 2001 the only property of subdivision surfaces that anybody bothered with was the level of smoothness at extraordinary points, and at all points within schemes which did not have the box-spline underpinning which allows the level of continuity of regular regions to be determined *a priori*. Once subdivision surfaces started to be used for real tasks, it became apparent that there were other aspects equally important or more so.

One problem which was brought to light is that if an extraordinary point has a very high valency, the polyhedron after a few steps has what appears to be a cone centred at that point. This appears to be due to the fact that the subdominant eigenvalues tend to increase with valency, so that the rate of shrinkage towards such points is much slower than the average over the surface.

In particular, if the subdominant eigenvalue of an extraordinary point is significantly greater than $1/2$ in a binary scheme, the rings shrink more slowly towards that point than to normal points. After a few iterations this results in a 'polar cap' of long thin faces.

Because the distance of a control point from its limit position varies quadratically with the size of the local faces, the extraordinary control point remains far from the limit surfaces after all the other points have converged closely, and so if an image is drawn without projection of the vertices to their limit positions, it appears as if there is a cone-like failure of tangent continuity.

An even more serious problem is that in extremely extraordinary situations, where a high-valency extraordinary vertex is surrounded by low-valency ones, the shape of the limit surface can be significantly qualitatively different from that of the original polyhedron, showing features whose spatial frequencies are as high as the valency of the central extraordinarity. So far we have identified three possible causes for these:

1. the natural configuration of the high-valency point
2. inequality of the cup- and saddle-eigenvalues
3. plain interaction at the first level of refinement.

This is a high priority target for more understanding.

What is common to these effects is that features appear in the limit surface which have a spatial frequency too high, relative to the density of the original mesh, to be corrected by the careful placing of the control points. Once this is observed, we can identify other effects which also have this aspect in common.

In particular, all of our stationary curve schemes do in fact contain, in the limit curve, spatial frequencies of twice the Shannon limit. To avoid this would require that the basis function should be $\sin(x)/x$ which would in turn require a subdivision mask of infinite width. Indeed, the higher the degree, the smaller this artifact, and the benefit from looking on subdivision as being made out of sampling and smoothing occurs precisely because the smoothing reduces high frequencies more than lower.

A more interesting effect is that aliasing effects can also appear in directions where the original polyhedron has no features. A simple extruded ridge in the polyhedron can result in a limit surface with a 'dinosaur back', with one hump per original control point.

This is now fully understood. Every scheme misbehaves in this way if the original polyhedron has its vertices sampled from an extruded surface, except when the direction of extrusion is one of a small number of special ones – the directions in which the symbol of the scheme has at least one factor of $(1-z^a)/(1-z)$. Some schemes, such as Kobbelt's $\sqrt{3}$ [66] have no such directions. Catmull-Clark [6] has two, the directions of the edges of the mesh; Loop [13] has three; the 4-8 [77, 78] and 4-3 [139] schemes, despite being quad-based, have four, and the Dagstuhl scheme [127] has six. It turns out that there are also advantages of having more than one $(1-z^a)/(1-z)$ factor in a direction, as then a polyhedron formed by linearly varying extrusion will result in a limit surface with the same property.

The final effect, not strictly an aliasing one, is that the way that edges are handled in the raw versions of the standard subdivision schemes has a serious deficiency, in that the local curvature across the boundary is determined not by the shape of the polyhedron measured across the boundary but by the curvature along the local boundary. Indeed, in the Catmull-Clark scheme, the isoparametric lines crossing a boundary have zero curvature where they meet it.

These effects are described more fully, with their explanations, where they are known, by Sabin and Barthe in [116].

4.3 Street Wisdom

In the NURBS generation of surface descriptions, there seemed to be rather little know-how about how to choose control points to define the surface that you really wanted. Perhaps this was because there was little freedom to choose the topology and so recourse to just making a dense control polyhedron was frequently required.

In the subdivision surface case it is possible to use the extraordinary points to achieve a lot, and the need for 'street wisdom' is larger, because the opportunity to use it to reduce the density of the original mesh and thus give smoother surfaces easily is much higher. There seem to be two important rules.

1. Extraordinary points have two purposes:
   a) to allow the mesh to follow features even when those features do not form a regular grid. 'Features' in this context includes the edges of faces, and so if this principle is followed, the need for trimmed surfaces is significantly reduced.
   b) to match the local total curvatures, and thereby allow the mesh to remain more or less orthogonal everywhere. An extraordinary point of valency 3 in a quad grid provides enough positive discrete total curvature to satisfy an octant of a sphere. One of valency 5 provides the same amount of negative discrete total curvature.
   So the first rule is to *choose the topology of the initial polyhedron so that the only extraordinary vertices are those needed for these purposes*.
   In both cases the need for extremes of extraordinarity is very unusual. In almost all cases the variation of valency from the regular case need only be plus or minus one. Better results will almost always be obtained in a quad grid by using two separate 5-valent points rather than one 6-valent.
2. The second rule is to *keep the mesh as sparse as you can*. The filling in between the extraordinary points is exactly what the subdivision process does, and initial design can be carried out by providing a mesh with very few points except the necessary extraordinary ones (each such point being required either by 1a or by 1b above), carrying out one step at a time of subdivision and correcting the mesh to suit shorter wavelength features of the required shape at each step.

These rules are primitive. We hope that significant work will go into refining them into practical tools for surface designers to apply when using subdivision surfaces. The structures used by Skaria et al. [80], although derived in a different way, appear to be consistent with these guidelines.

4.4 Compression

Khodakovsky et al. [68] showed that this last idea can be exploited to give an effective compression of smooth surfaces for transmission over the web. The coarse mesh is transmitted (possibly coded, although it should be so sparse that coding does not save many bytes) together with the corrections by which the control points computed at each refinement level need to be moved so that the limit surface eventually matches the desired surface.

If the original surface is smooth and the subdivision scheme is a good one, the corrections drop dramatically in amplitude as refinement proceeds, so that the total information sent can be a very small fraction of the total mesh. Indeed it is likely to be independent of the original density of the mesh.

Guskov's idea of normal subdivision [65], where the corrections are applied only in a direction perpendicular to the surface, saves a further factor of 3 on the corrections, although it means that a given tessellation is not necessarily exactly matched. It was pointed out by a referee that entropy coding would normally exploit this redundancy almost as effectively as normal subdivision.

4.5 Tuning

The response to the artifacts which are associated with extraordinary points has to be to try to tune the behaviour round the extraordinary points by altering the local coefficients. There are two approaches to this: we can either think of the stencils as the primary representation of the scheme, or else the masks.

In the stencil-oriented approach we modify the values in those stencils which contain a reference to an extraordinary point. There are clear stages in which we can do this. The first stage is to modify only the stencil of the new extraordinary point itself. This was done in a half-hearted way in the original Catmull-Clark paper [6] as a response to the Doo-Sabin analysis [7] of the behaviour at the extraordinary point. It was done properly for the Catmull-Clark scheme by Ball and Storry [9]. Loop's scheme was explicitly designed with exactly this tuning as a design principle [13].

However, this can only control the continuity of cup-like curvatures. In order to control the saddle-like curvatures it is necessary also to modify the stencils of the 1-ring of new vertices. This was done for the Catmull-Clark scheme by Sabin [18], and for the Loop scheme by Holt [40], in a way which did not alter the footprints of the stencils. No new influences were introduced; only the already non-zero coefficients were altered.

More recent tunings, by Zorin of the butterfly scheme [38], by Prautzsch and Umlauf of the Loop and Butterfly schemes [46], by Loop of the Loop scheme [81, 99], and by Barthe of the Catmull-Clark scheme [133] have taken the attitude that it is OK to increase the sizes of the stencils if by doing so other properties, such as positivity, can be maintained as well as the required eigenratios. The tuning of the 4-3 scheme [139] however, maintains the footprints of the stencils.

The mask-oriented approach, on the other hand, says that as a first stage of tuning we will alter only the mask associated with the extraordinary point itself. The entries in the mask become expressions which depend on the extraordinary valency, and we choose them so that certain desired properties of the scheme (for example: good subdominant eigenvalue, bounded curvature, good characteristic maps) are achieved.

When this approach is applied, the stencil of each of the new vertices covered by the mask of the extraordinary vertex has to be renormalised so that the entries sum to unity, but this is straightforward, and covers the situation automatically where there are vertices which lie within the masks of more than one extraordinary point.

We now see the stages of stencil-oriented tuning appearing again in terms of which coefficients in the mask vary with valency. The central value controls the cup-curvature eigencomponent; the 1-ring value controls the ratio of the saddle-curvature and tangent plane eigencomponents; the rings further out control the eigenvectors, so that we can choose a neat natural configuration leading to a well-behaved characteristic map.

This approach is more limited, in that the influences of regular vertices are always regular, but it does ensure that the footprints of vertices near the extraordinary vertex remain standard, so that the support analysis remains valid in that region. It is argued here that this should be the approach of first resort, and that the larger number of freedoms available through stencil-based tuning should not be invoked until it is certain that the objectives cannot be met by mask-based tuning.

Both views bring their own insights and both need to be explored a lot more in the near future.

5 New Ideas

The theory described above has almost entirely focused on the uniform stationary case, in which the refinement rules are the same throughout the polyhedron and are also the same at every step, but we do not have to be limited in this way. We can consider non-uniform schemes, in which the rules vary from place to place, and non-stationary ones in which the rules vary with the refinement level.

5.1 Non-uniform Schemes

In a non-uniform scheme, the rules do not need to be the same at all parts of the grid.

Boundaries

In fact we have always had some measure of non-uniformity, in the sense that any practical scheme needed to have appropriate rules at the boundaries.

As was pointed out above, the boundary rules actually used turn out to be a little unfortunate, and Levin [55, 56] suggested the use of what he called "combined schemes" which defined a bounded piece of surface by both a polyhedron and a set of surface normals to be matched at the boundary. This effectively gives an analogue of the Bézier edge conditions normally used for NURBS surfaces.

Nasri and Sabin [109] surveyed and classified the ways in which constraints on the original polyhedron could be used to achieve interpolation of both boundaries and internal feature curves, with and without also controlling the variation of surface normal along those curves. Sabin and Bejancu [126] looked for an analogue of the "not-a-knot" condition found so valuable in the old days of interpolating cubic splines, and found that this could be expressed in terms of fourth differences over the polyhedron for both Catmull-Clark and Loop schemes.

Unequal-intervals

One of the potential barriers to adoption of subdivision surfaces as a successor to NURBS in the representation of engineering objects is the problem of compatibility. The possibility of having unequal spacing of knots in NURBS is frequently used, while all current subdivision schemes are implicitly the equivalent of equal interval splines.

Dyn, Gregory and Levin [31] made first steps in this by considering the quasi-uniform case, where all the intervals on one side of a central point have one size, all those on the other a different size. After one refinement all intervals were halved. Within this restricted variability the tools of eigenanalysis can still be applied, and they showed that all the important properties could be maintained. This work was continued by Dyn and Levin in [62] to the general unequal interval case.

The univariate case of B-spline unequal interval subdivision was explored by Qu and Gregory [24, 36] and by Warren [32], who showed that knot insertion in an unequal interval context could give a corresponding subdivision scheme, with all the expected properties.

Sederberg et al. [51] explored the possibility of labelling every edge in a polyhedron with a “knot interval”. A variant of Catmull-Clark was then used which produced a similarly-labelled refined polyhedron. This scheme contained all the unequal interval bicubic B-spline surfaces as a particular case when the grid was completely regular and the labels were equal on opposite edges of every quad. It also contained all Catmull-Clark surfaces as the particular case when all labels were equal.

It had two disadvantages. The first was the need to specify all the labels: some preprocess determining labels from some terser specification would be necessary in practice. The second was that the spectrum at extraordinary points had the subdominant eigenvalues split, so that proving smoothness was extremely hard. In fact slightly different rules for the new labellings could have solved this.

It seems that new work in this direction would be valuable, but it is now clear that the ambition of having all edges separately labellable is not in fact necessary to cover the regular non-uniform case. It is also clear that the rules for defining the new labellings should be derived from some logical argument about where the new “knots” should be. There is no reason why new knots should be either rigidly at the middles of old intervals or at wherever some convenient formula happens to put them.

Mixed Grid Schemes

Where a desired surface has a clear warp and weft, indicated by its very distinct principal curvatures, a quad grid is the natural choice, avoiding lateral artifacts by running the flow of the grid along the flow of the surface. However, for surfaces where there is no such bias, containing bumps, hollows and saddles (think of mild terrain), the natural way to define a polyhedron is to put a vertex at each maximum and each minimum and maybe at each saddle as well. In such a case it makes sense to use either a triangulation, or else a mixed grid containing both triangles and quads.

The special case of such a mixed grid, consisting of a triangular mesh on one side of an "extraordinary line" and a quad grid on the other was first raised by Loop at a Dagstuhl meeting, and he and Stam [118] produced an initial scheme. This case has been fully analysed by Levin and Levin [123] and an alternative scheme proposed.

The fuller case where quads and triangles can be mixed as appropriate for the surface being defined is the subject of a paper by Peters and Shiue [139]. In this case the quad part of the scheme, applied wherever a sufficiently large piece of the mesh consists entirely of quads is a four-direction box-spline, rather than Catmull-Clark, while the triangle part is a bounded-curvature variant of Loop.

The extraordinary line analysis is still an important part of the tool-box for such schemes, because as refinement proceeds, the interiors of original triangles become regions of triangle mesh and the interiors of original quads, quad mesh, with extraordinary lines between them. However, the corners, which can be of several different kinds, each need their own analysis, which is addressed by Levin in [121].

5.2 Non-stationary Schemes

In a non-stationary scheme the rules can change from level to level. In the simplest, uniform, form of non-stationary scheme the rules at each level of subdivision are the same everywhere.

There is a very interesting family of univariate non-stationary schemes generating Exponential B-splines (see [8] about this notion), and in particular trigonometric splines [30]. Any such scheme is similar to a corresponding B-spline scheme, has the same support of its mask, and its refinement rules converge to those of the corresponding B-spline scheme as the refinement level increases. In particular it is possible to construct a univariate scheme generating circles. (In several ways, the one of smallest support has refinement rules converging to those of the Chaikin's scheme). In [23] Dyn and Levin constructed the scheme of smallest support, which, when applied to the vertices of a regular n-gon, generates a circle.

Recently Morin, Warren and Weimer [79] constructed a circle-generating scheme, with refinement rules converging to the refinement rules of a cubic B-spline scheme, and then constructed its tensor-product with a stationary scheme, to obtain a subdivision scheme generating surfaces of revolution.

The analysis of convergence and smoothness of non-stationary scheme cannot use the analysis tools for stationary schemes. One method for studying the convergence is by comparison to a convergent stationary scheme [30, 94]. Recently Dyn, Levin and Luzzatto analysed with this method a family of non-stationary interpolatory schemes reproducing exponential functions [125]. These schemes are applicable to the processing of highly oscillatory signals.

Sabin [141] shows an analogue of the four-point scheme, in which the computation of the new vertices is not expressed in terms of linear combinations, but as a non-linear geometric construction which guarantees the preservation of circles. The non-stationarity is present in the non-linearity, but is hidden under an apparently constant construction.

5.3 Geometry Driven Schemes

This last scheme is non-uniform as well as non-stationary, and so is better described as a geometry-driven scheme. The rules both ensure that any set of points which lie in sequence on a circle, whatever their spacing, will give a circular limit curve, but the spacing also converges towards local uniformity, so that the results of Dyn and Levin [30] can then be invoked to determine the smoothness of the scheme.

In that sense it may well indicate a useful direction for other explorations. We define here the term "geometry-driven" as the name for a class of schemes in which the new polygon or polyhedron has its local shape determined geometrically from the locality of the old one. It is likely that such schemes will lose affine-invariance, and the actual value of affine invariance as against invariance under solid body, mirror and scaling transforms should be a matter for debate in the community.

Another example of a family of geometry driven schemes is described by Marinov [132] elsewhere in this book. It is derived from the "classical" linear 4-point scheme, by allowing a variable tension parameter instead of the fixed tension parameter. The tension parameter is adapted locally in various ways, according to the geometry of the control polygon within the 4-point stencil. With this change the new schemes retain the locality of the 4-point scheme, and at the same time achieve important shape-preservation properties, such as artifact elimination and co-convexity preservation. The proposed schemes are robust and have the special features of "double-knot" edges corresponding to continuity without smoothness in the limit curves (for artifact elimination), and inflection edges for co-convexity preservation.

5.4 Hermite Subdivision

In a Hermite scheme, not only point positions, but also derivatives or normals are carried forward from each stage to the next. This is not a totally new idea, but there have been significant developments recently.

Hermite schemes were first designed mainly for functional data. Merrien [25, 29] introduced interpolatory Hermite schemes of extremely local support. His univariate schemes are two-point schemes, namely the values of the function and a fixed number of its derivatives in the mid-point of an interval is determined by the values of the function and the same number of its derivatives at the two boundary points of the interval.

Similarly in his Hermite schemes on triangulations, the values of the function and its derivatives attached to a mid-point of an edge of a triangle in the refinement step, depend only on the values of the function and its derivatives attached to the boundary points of that edge. Merrien analysed the convergence and smoothness of his schemes. Dyn and Levin provided in [63] analysis tools for univariate interpolatory Hermite subdivision schemes, for any number of derivatives attached to the grid points and any support size of the scheme. Dyn and Lyche designed a specific bivariate Hermite subdivision scheme generating the Powell-Sabin twelve-split quadratic spline [64].

Recently new Hermite-type schemes were developed. Jüttler and Schwanecke in [97] represented Hermite schemes for curves corresponding to data of position and tangent vectors, in terms of spline curves with control points. They provided an analysis method based on their representation, and designed new schemes, interpolatory and non-interpolatory. van Damme [42] proposed several schemes. For bivariate functions he gave proofs for the convergence and smoothness of the corresponding schemes, while for surfaces he gave only experimental results. Han et al. [138] defined the notion of non-interpolatory Hermite subdivision schemes, and designed a family of such schemes for surface generation, whose refinement is done with a dilation matrix. They analyse the smoothness of their schemes which have certain symmetries of relevance to geometric modelling.

5.5 Non-linear Schemes for the Functional Case

In analogy to the geometry-driven extensions of the linear 4-point scheme, new non-linear 4-point schemes for interpolating given equidistant data, sampled from a piecewise smooth function, were designed by Cohen, Dyn and Matei [136]. This schemes are based on the ENO (essentially non-oscillatory) interpolation technique, and use a 4-point stencil for the insertion of a new point, which is either centred relative to the inserted point or moved by one point to one of the two sides. The choice of the stencil is data dependent and aims at using data from the smooth pieces of the approximated function. These schemes converge to Hölder continuous limit functions, and reproduce cubic polynomials.

5.6 Reverse Subdivision

Reverse subdivision is the process of taking a dense grid and determining a coarse one which, when refined by subdivision, gives a good approximation to the original. It is closely related to wavelets and to compression.

Although the idea of looking at subdivision through the insights of wavelet mathematics can be traced back to [28, 21] or even earlier, Warren [43] provides a very readable introduction to the idea of applying wavelet techniques to subdivision schemes, using the cubic B-spline univariate subdivision as a concrete example throughout.

In [60] and [84], Bartels and Samavati provide a much fuller description of the links between wavelets, least squares fitting and the reversal of specific subdivision schemes. Although the tensor product case is covered, the ideas are essentially univariate, but in [85] and [108] they apply these ideas to surfaces, reversing Loop and Butterfly, and the original Doo subdivision [5] respectively.

Hassan and Dodgson [130] show a simpler approach based on Chaikin subdivision with a purely local determination of the new (coarser) vertices and the error terms.

Suzuki, Takeuchi and Kanai [61] fit approximate subdivision surfaces by using an interactively defined initial polyhedron to indicate which parts of the given dense surface should be converged to from each original vertex, and then solve the linear system implied by the limit positions of each vertex to solve for a good set of coarse control points. Ma and Zhou [75] gain their initial polyhedron from a consideration of the boundary of the given dense data and carry out a similar operation. Ma et al. [92] derive their coarse initial polyhedron for Loop subdivision by using strong decimation and re-triangulation on the grid, and use least squares fitting rather than collocation. This fitting is done in a single swoop direct from a really coarse mesh, whereas that of the wavelet-based approaches is done one step at a time. It also has the effect that subsequent resubdivision will seldom give the same triangulation connectivity as the original. Taubin [96] observed that loss of the original connectivity can often be avoided if the original mesh had few extraordinary vertices, and those vertices are joined by chains of edges passing through normal vertices without turning. He identified a parallelisable algorithm for detecting such situations in both triangular and quad meshes.

Surazhsky and Gotsman [119] observe that in general dense triangular meshes can benefit from a process of remeshing, wherein the number of extraordinary vertices is reduced significantly (and the number of extremely extraordinary vertices reduced very significantly). Their resulting meshes are dramatically smoother then the originals, and are likely to provide a better start for recapturing of a coarse subdivision surface. Within their technology is a technique for migrating extraordinary vertices within a mesh, so that extraordinarities of opposite sign can be brought together to annihilate each other.

Alkalai and Dyn [131] also carry out an optimisation of the connectivity of a given triangulation, but at the coarse level. This improves the convexity of the initial polyhedron, minimising the discrete curvature merely by applying 2:2 swaps. The vertex coordinates are not changed.

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