A Law of Iterated Logarithm for Multi-Agent Reinforcement Learning

Gugan Thoppe
Computer Science and Automation
Indian Institute of Science
Bengaluru, Karnataka 560012, India
gthoppe@iisc.ac.in

Bhumesh Kumar
Electrical and Computer Engineering
University of Wisconsin at Madison
Madison, WI 53706, USA
bkumar@wisc.edu

Abstract

In Multi-Agent Reinforcement Learning (MARL), multiple agents interact with a common environment, as also with each other, for solving a shared problem in sequential decision-making. It has wide-ranging applications in gaming, robotics, finance, etc. In this work, we derive a novel law of iterated logarithm for a family of distributed nonlinear stochastic approximation schemes that is useful in MARL. In particular, our result describes the convergence rate on almost every sample path where the algorithm converges. This result is the first of its kind in the distributed setup and provides deeper insights than the existing ones, which only discuss convergence rates in the expected or the CLT sense. Importantly, our result holds under significantly weaker assumptions: neither the gossip matrix needs to be doubly stochastic nor the stepsizes square summable. As an application, we show that, for the stepsize $n^{-\gamma}$ with $\gamma \in (0, 1)$, the distributed TD(0) algorithm with linear function approximation has a convergence rate of $\mathcal{O}(\sqrt{n^{-\gamma} \ln n})$ a.s.; for the $1/n$ type stepsize, the same is $\mathcal{O}(\sqrt{n^{-1} \ln \ln n})$ a.s. These decay rates do not depend on the graph depicting the interactions among the different agents.

1 Introduction

Can a machine train itself in the same way an infant learns to sit up, crawl, and walk? That is, can a device interact with the environment and figure out the action sequence required to complete a given task? The study of *algorithms* that enable such decision-making is what the field of Reinforcement Learning (RL) is all about Sutton and Barto [2018]. In contrast, the *mathematics* needed to analyze such schemes is what forms the focus in Stochastic Approximation (SA) theory Benaïm [1999], Borkar [2009]. More generally, SA refers to an iterative scheme that helps find zeroes or optimal points of a function, for which only noisy evaluations are possible. In this work, we analyze a family of Distributed Stochastic Approximation (DSA) algorithms Mathkar and Borkar [2016] that is useful in Multi-Agent Reinforcement Learning (MARL) Littman [1994], Zhang et al. [2021a].

In the MARL framework, we have multiple agents or learners that continually engage with a shared environment: the agents pick local actions, and the environment responds by transitioning to a new state and giving each agent a different local reward. Additionally, the agents also *gossip* about local computations with each other. The goal of the agents is to cooperatively find action policies that maximize the collective rewards obtained over time. Algorithms useful in this endeavor have found empirical success in domains as diverse as gaming OpenAI: Openai five, robotics Ota [2006], autonomous driving Shalev-Shwartz et al. [2016], communication networks Littman and Boyan [2013], power grids Riedmiller et al. [2000], and economics Lee and Zhang [2002]. However, theoretical analyses of such schemes are still very minimal, and this is what this paper aims to address.

Preprint. Under review.

For the purpose of analysis, MARL methods are often viewed as special cases of DSA algorithms. The archetypical form of a DSA scheme with m distributed nodes can be described as follows. Let $\mathcal{G}$ be a directed graph representing the connections between these nodes, and $W \equiv (W_{ij}) \in [0,1]^{m \times m}$ a matrix whose ij-th entry denotes the strength of the edge $j \to i$. It is assumed that W is compatible with $\mathcal{G}$, i.e., $W_{ij} > 0$ only if $j \to i \in \mathcal{G}$. Then, at agent i, the above scheme has the update rule

$$x_{n+1}(i) = \sum_{j \in \mathcal{N}_i} W_{ij} x_n(j) + \alpha_n [h_i(x_n) + M_{n+1}(i)], \qquad n \geq 0, \tag{1}$$

where $x_n \in \mathbb{R}^{m \times d}$ is the joint estimate of the solution at time n, its j-th row, i.e., $x_n(j)$ denotes[1] the estimate obtained at agent j, $\mathcal{N}_i$ represents the set of in-neighbors of node i in $\mathcal{G}$, α_n is the stepsize, $h_i : \mathbb{R}^{m \times n} \to \mathbb{R}^d$ is the driving function at agent i, and $M_{n+1}(i) \in \mathbb{R}^d$ is the noise in its evaluation at time n. This update rule has two parts: a weighted average of the estimates obtained by gossip and a refinement based on local computations. Clearly, the joint update rule of all the agents is

$$x_{n+1} = W x_n + \alpha_n [h(x_n) + M_{n+1}], \tag{2}$$

where M_{n+1} is the $m \times d$ matrix whose i-th row is $M_{n+1}(i)$, and h is the function that maps $x \in \mathbb{R}^{m \times d}$ to the $m \times d$ matrix whose i-th row is $h_i(x)$.

Two important points about the above framework are as follows: i.) we allow h_i to be a function of all of x_n and not just of $x_n(i)$, as is commonly assumed, and ii.) the computations at different nodes in the above setup run synchronously on a common clock.

Related Work: We now give a summary of relevant theoretical results from the DSA and MARL literature. For ease of discussion, we categorize them into i.) asymptotic and ii.) finite-time results.

The asymptotic ones mainly concern almost sure (a.s.) convergence Tsitsiklis et al. [1986], Bianchi et al. [2013], Morral et al. [2014], Mathkar and Borkar [2016], Kar et al. [2013], Zhang et al. [2018b,a], Suttle et al. [2020], Lee et al. [2018]. The first four papers here provide convergence guarantees for a broad family of nonlinear DSA algorithms. The other articles also do the same, but in context of specific MARL schemes such as distributed Q-learning, distributed actor-critic methods, distributed TD methods, and their off-policy variations. Two other kinds of asymptotic results also exist in the literature. The first is the CLT shown in Morral et al. [2017] for the average of estimates obtained at different nodes in a generic DSA scheme. The other is the convergence in mean result obtained in Zeng et al. [2020a] for a distributed policy gradient method.

Finite-time literature, in contrast, majorly talks about expectation bounds. Assuming there exists a unique x_* that solves $\sum_{i=1}^m h_i(x) = 0$, these results describe the rate at which $\mathbb{E}\|x_n - x_*\|$ decays with n. Notable contributions for DSA here are Zeng et al. [2020b], Wai [2020]. Compared to ours, these look at a slightly different setup: the measurement noise at each node has a Markov component in place of a martingale difference term. In this new setup, Zeng et al. [2020b] shows that, for any sufficiently small but constant stepsize ϵ, the expected error decreases linearly to a ball of radius $\mathcal{O}(\sqrt{\epsilon \ln(1/\epsilon)})$. On the other hand, Wai [2020] deals with the case where h is additionally non-convex and shows that $\mathbb{E}\|x_n - x_*\| = \mathcal{O}(n^{-1/4}\sqrt{\ln n})$, which is comparable to the best known bound in the centralized setting.

Expectation bounds in the MARL framework primarily concern policy evaluation methods Doan et al. [2019, 2021], Sun et al. [2020], Chen et al. [2021]. The first three papers here deal with the distributed TD(0) method. These show that a result similar to the one in Zeng et al. [2020b] holds for this method under constant stepsizes. In contrast, when α_n is of the $1/n$ type, it is proven that $\mathbb{E}\|x_n - x_*\| = \mathcal{O}(1/\sqrt{n})$. Similar bounds have also been derived in Chen et al. [2021] for two distributed variants of the TDC method. There are also some other works that derive finite-time bounds Wai et al. [2018], Ding et al. [2019], Xu et al. [2020], Zhao et al. [2020], Heredia and Mou [2020], Stanković et al. [2020], Ren et al. [2021], Zhang et al. [2021b], but we do not discuss them in this paper since the algorithms proposed there do not fit the update rule given in (2).

The different finite-time results, as also the asymptotic CLT, do provide insights into the rate at which an iterative method converges. However, there are some significant issues with these studies. First, except Morral et al. [2017], all others require the gossip matrix to be doubly stochastic, at least in the mean. While this assumption simplifies the analysis, it also severely restricts the communication

[1] By default, all our vectors are row vectors. We use $'$ for transpose.

protocol choices. In fact, as pointed out in Morral et al. [2017], this condition even limits the use of a natural broadcast node, one that transmits its local estimate to all the neighbors without expecting all of them to respond. Second, these works only talk about convergence rates in the expected or the CLT sense. By their very nature, these results do not reveal much about the decay rates along different sample paths. Finally, all current results, including the ones on convergence, only apply to constant or square-summable stepsizes. Nothing is known about the slowly-decaying non-square-summable ones, which are generally preferable since they give similar benefits as constant stepsizes and, often, also guarantee convergence. Note that such issues also plague much of the distributed stochastic optimization literature Yuan et al. [2016], Sun et al. [2019], Lian et al. [2017], Koloskova et al. [2020], Pu et al. [2020], Pu and Nedić [2020].

Key Contributions: The highlights of this work that help us address the above issues are as follows.

1. **Law of Iterated Logarithm (LIL)**: We derive a novel law of iterated logarithm for the DSA scheme given in (2). That is, for a suitably defined x_*, we show that $\limsup[\alpha_n \ln t_{n+1}]^{-1/2}\|x_n - x_*\| \leq C$ a.s. on every sample path in

$$\mathcal{E}(x_*) := \{x_n \to x_*\}. \tag{3}$$

   Here, $C \geq 0$ is some constant[2] and $t_n = \sum_{k=0}^{n-1} \alpha_k$. Also, the norm that we work is the Euclidean norm. In particular, for any $B \in \mathbb{R}^{m \times d}$,

$$\|B\| = \sup_{u \in \mathbb{R}^m, v \in \mathbb{R}^d} \{|uBv'| : \|u\| = \|v\| = 1\}. \tag{4}$$

   This result is the first of its kind in the distributed setup. Further, as discussed in Remark 2.3 later, it provides deeper insights about the asymptotic behavior of $(x_n)_{n \geq 0}$ than other existing results, which only discuss convergence rates in the expected or the CLT sense.

2. **Analysis and Gossip Matrix**: The above result is obtained via a new approach we develop here for analyzing DSA schemes. Let $\pi \in \mathbb{R}^m$ be such that $\pi W = \pi$ and let

$$Q := \mathbb{I} - \mathbf{1}'\pi, \tag{5}$$

   where $\mathbf{1} \in \mathbb{R}^m$ denotes the vector of all ones. Then an outline of our approach is that we express $x_n - x_*$ as a sum of $\mathbf{1}'\pi(x_n - x_*)$ and Qx_n and, thereafter, analyze each summand by treating its update rule as a separate SA scheme. This contrasts the usual approach (e.g., Morral et al. [2017], Doan et al. [2019, 2021]) where the error is split into $(\mathbf{1}'\mathbf{1}/m)(x_n - x_*)$ and $(\mathbb{I} - \mathbf{1}'\mathbf{1}/m)x_n$. In fact, this is the main reason why, unlike other existing results, ours does not require that the gossip matrix be doubly stochastic.

3. **Concentration Inequality and Stepsizes**: We also improve upon an existing concentration result ([Duflo, 2013, Corollary 6.4.25]) for a sum of martingale differences; see Lemma 4.6. Specifically, by modifying the original proof from Duflo [2013], we show that the result stated there actually holds under a broader set of conditions. The key benefit of this is that, unlike other related results, our LIL result does not require that the stepsize sequence be square-summable.

4. **MARL Application**: We use our theory to prove a law of iterated logarithm for the distributed TD(0) algorithm with linear function approximation. This is the first such result in MARL.

Contents: The rest of the paper is structured as follows. In Section 2, we formally state our main result along with all the assumptions needed. We also pinpoint the new insights that our result provides. In Section 3, we give a demonstration of how our result can be applied in the MARL setup. In particular, there we talk about the distributed TD(0) algorithm with linear function approximation and prove that it indeed satisfies all the assumptions of our main result. Section 4 has two parts. In the first part, we state some key intermediate lemmas and then use the same to derive our main result. The latter part, in contrast, focuses on proofs of these intermediate results; note that we only sketch their proofs here and leave the details to the appendix. Finally, in Section 5, we conclude with a summary of our findings and discuss some interesting future directions.

[2]Throughout, C denotes a generic constant. Its value could be different each time it is used; in fact, it could be different even in the same line.

2 Assumptions and Main Result

Throughout this work, we assume that the following four technical assumptions, i.e., $\mathcal{A}_1, \ldots, \mathcal{A}_4$, hold for the DSA scheme in (2).

$\mathcal{A}_1$. **Property of the Gossip Matrix**: *W is an irreducible aperiodic row stochastic matrix.*

This condition implies there exists a unique vector $\pi \in \mathbb{R}^m$ such that

$$\pi W = \pi. \tag{6}$$

Subsequently, from [Mathkar and Borkar, 2016, Theorem 1], one would expect (2) to eventually converge to an invariant set of the m-fold product of the d-dimensional ODE

$$\dot{y}(t) = \sum_{i=1}^{m} \pi_i h_i(\mathbf{1}' y(t)) = \pi h(\mathbf{1}' y(t)). \tag{7}$$

By an m-fold product, we refer to the dynamics in $\mathbb{R}^{m \times d}$ where each row individually satisfies (7). A natural invariant set of this dynamics is $\mathcal{S} := \{\mathbf{1}' y : y \in \mathbb{R}^d\} \subset \mathbb{R}^{m \times d}$. With this in mind, let

$$x_* = \mathbf{1}' y_* \in \mathcal{S}, \tag{8}$$

where $y_* \in \mathbb{R}^d$ is an asymptotically stable equilibrium of (7). Notice that we don't assume y_* to be the only attractor of this ODE.

We remark that our main result concerns the behavior of the DSA scheme on the event $\mathcal{E}(x_*)$, where x_* is as defined above and $\mathcal{E}(x_*)$ is as defined in (3).

$\mathcal{A}_2$. **Nature of h near x_***: *There exists a neighbourhood $\mathcal{U}$ of x_* such that, for $x \in \mathcal{U}$,*

$$h(x) = -\mathbf{1}'\pi(x - x_*)A + \mathbf{1}'\pi f_1(x) + (\mathbb{I} - \mathbf{1}'\pi)(B + f_2(x)), \tag{9}$$

where $A \in \mathbb{R}^{d \times d}$ is such that $yAy' > 0$ for all $y \neq 0$, $B \in \mathbb{R}^{m \times d}$ is some constant matrix, $f_2 : \mathcal{U} \to \mathbb{R}^{m \times d}$ is some arbitrary continuous function, while $f_1 : \mathcal{U} \to \mathbb{R}^{m \times d}$ is another continuous function that additionally satisfies

$$\|\mathbf{1}'\pi f_1(x)\| = \mathcal{O}(\|\mathbf{1}'\pi(x - x_*)\|^a), \qquad \text{as } x \to x_*, \tag{10}$$

for some $a > 1$.

Note that $\mathcal{A}_2$ is a generalization of **Assumption (A1)** in Pelletier [1998]. As in Pelletier [1998], this also is local in nature, in that, it only prescribes a specific behavior for h close to x_*. We now construct a family of examples to show that this assumption broadly holds. The simplest member in this family is $h(x) = B - xA$, where B and A are as defined above[3]. Clearly, if $b(i)$ and $x(i)$ are the i-th rows of B and x, respectively, then the i-component function here is $h_i(x) = b(i) - x(i)A$. The fact that the scaling matrix A is the same for each i is crucial for $\mathcal{A}_2$ to hold. Also, observe that this function does not depend on π. The other members of the family are obtained by adding various π-dependent nonlinear perturbations to this simple setup, i.e., by making different choices[4] for f_1 and f_2.

$\mathcal{A}_3$. **Stepsize Behavior**: *There exists some decreasing positive function α defined on $[0, \infty)$ such that the stepsize $\alpha_n = \alpha(n)$. Further, α is either of Type 1 or Type γ;*

(a) *Type 1: $\alpha(n) = \alpha_0/n$ for some $\alpha_0 > 1/(2\lambda_{\min})$, where*

$$\lambda_{\min} := \min\{\mathscr{R}(\lambda) : \lambda \in \text{spectrum}(A)\} \tag{11}$$

with $\mathscr{R}(\lambda)$ denoting the real part of λ;

(b) *Type γ : The function α is differentiable and its derivative varies regularly with exponent $-1-\gamma$, where $0 < \gamma < 1$.*

[3]A verification of all the conditions mentioned in $\mathcal{A}_2$ for $h(x) = B - xA$ has been done in Section 3; there we also discuss usefulness of this function in the context of policy evaluation in MARL.

[4]For example, we can let $f_1(x) = g_1(\pi(x - x_*))$ and $f_2(x) = -xA + g_2(x)$, where $g_1 : \mathbb{R}^d \to \mathbb{R}^{m \times d}$ is such that, as $z \to 0$, $\|g_1(z)\| = \mathcal{O}(\|z\|^a)$ for some $a > 1$, and g_2 is an arbitrary continuous function.

The regularly varying condition above implies that $\left|\frac{\mathrm{d}\alpha(x)}{\mathrm{d}x}\right| = x^{-\gamma-1}L(x)$ for some slowly varying function L, e.g., $L(x) = C$ for some $C > 0$, or $L(x) = (\ln x)^\eta$ for some $\eta \in \mathbb{R}$. Thus, examples of α_n here include $Cn^{-\gamma}$ and $n^{-\gamma}(\ln n)^\eta$, which are non-square-summable for $\gamma \in (0, 1/2]$.

$\mathcal{A}_4$. **Noise Attributes**: *With $\mathcal{F}_n = \sigma(x_0, M_1, \ldots, M_n)$, and $\mathcal{E}(x_*)$, as in* (3)*, the following hold.*

(a) $\mathbb{E}(M_{n+1}|\mathcal{F}_n) = 0$ *a.s.*

(b) *There exists $C \geq 0$ such that $\|QM_{n+1}\| \leq C\,(1 + \|Q(x_n - x_*)\|)$ a.s. on $\mathcal{E}(x_*)$.*

(c) *There is a non-random symmetric positive semi-definite matrix $M \in \mathbb{R}^{d\times d}$ such that*

$$\lim_{n\to\infty} \mathbb{E}(M'_{n+1}\pi'\pi M_{n+1} \,|\, \mathcal{F}_n) = M \quad \textit{a.s. on } \mathcal{E}(x_*). \tag{12}$$

(d) *There exists $b > 2$ such that $\sup_{n\geq 0} \mathbb{E}(\|\pi M_{n+1}\|^b|\mathcal{F}_n) < \infty$ a.s. on $\mathcal{E}(x_*)$.*

These noise conditions are extensions of the standard assumptions in the SA literature Pelletier [1998], Mokkadem and Pelletier [2006], Borkar [2009].

Our main result can now be stated as follows. This generalizes Theorem 1 from Pelletier [1998].

Theorem 2.1 (**Main Result: Law of Iterated Logarithm**). *Suppose $\mathcal{A}_1, \ldots, \mathcal{A}_4$ hold and $\gamma > 2/b$ if α is of Type γ. Then, there exists some deterministic constant $C \geq 0$ such that*

$$\limsup [\alpha_n \ln t_{n+1}]^{-1/2} \|x_n - x_*\| \leq C \quad \textit{a.s. on } \mathcal{E}(x_*).$$

This result is called a law of iterated logarithm since its proof crucially relies on Lemma 4.6, which indeed is a law of iterated logarithm for a sum of scaled martingale differences. We end this section with some important comments about our main result.

Remark 2.2. *Our result shows that, a.s. on $\mathcal{E}(x_*)$, $\|x_n - x_*\|$ is $\mathcal{O}(\sqrt{n^{-1}\ln\ln n})$ in the Type 1 case, and $\mathcal{O}(\sqrt{n^{-\gamma}\ln n})$ in the Type γ case. Note that, since we require $\gamma > 2/b$, our result applies to smaller values of γ, only if $\mathcal{A}_4$.*(d) *holds for a sufficiently large b.*

Remark 2.3. *Our result provides deeper insights than the convergence rates that exist in the DSA/MARL literature. For this discussion, we suppose $\mathbb{P}\{\mathcal{E}(x_*)\} = 1$. As mentioned in Section 1, the existing results are of two kinds: finite-time expectation bounds and the CLT. Indeed a finite-time bound has several benefits and is not directly comparable to an asymptotic result. Nevertheless, an expectation bound only describes the average behavior, while ours characterizes the decay rate on almost every sample path. In fact, if we compare just the decay rate obtained in our result in the Type 1 case with that obtained in Doan et al. [2019, 2021], which show $\mathbb{E}\|x_n - x_*\| = \mathcal{O}(\sqrt{\ln n}/\sqrt{n})$, then ours is tighter (it has $\ln\ln n$ in place of $\ln n$). Furthermore, while a CLT can at the best show that $\limsup \alpha_n^{-1/2}\|x_n - x_*\| = \infty$ a.s., our result is more precise in stating that the expression becomes bounded if it is divided by an additional $\sqrt{\ln t_{n+1}}$ term.*

3 Application to Reinforcement Learning

We apply our result here to a variant of the distributed TD(0) algorithm Doan et al. [2019, 2021] with linear function approximation. This method is useful for policy evaluation in MARL. The discussion here is divided into the following three parts: i.) setup, ii.) objective and algorithm, and iii.) analysis.

Setup: We consider a distributed system of m agents modeled by a Markov Decision Process. This can be characterized by the tuple $(\mathcal{S}, \{\mathcal{U}_i\}, \mathcal{P}, \{\mathcal{R}_i\}, \gamma, \mathcal{G})$. Here, $\mathcal{S} = \{1, \ldots, L\}$ is the global state space, $\mathcal{U}_i$ and $\mathcal{R}_i$ are the set of actions and the reward function at agent i, respectively, $\mathcal{P}$ describes the transition probabilities, γ is the discount factor, and $\mathcal{G} \equiv (\mathcal{V}, \mathcal{E})$ is a directed graph that represents the connectivity structure among the m agents.

Let $\mathcal{N}_i$ and W be as in (1). We assume that this W satisfies the conditions in $\mathcal{A}_1$. Then, for this matrix, there is a unique vector $\pi \equiv (\pi_i)$ satisfying (6).

Let μ_i be the stationary policy of agent i and let $\mu \equiv (\mu_i)$. Also, let $\mu(a|s) = \prod_i \mu_i(a_i|s)$ be the probability for choosing the joint action $a \equiv (a_i) \in \prod_i \mathcal{U}_i$. This policy μ then induces a Markov chain on $\mathcal{S}$, which we assume is aperiodic and irreducible. Therefore, it also has a unique stationary distribution and we denote the same by $\varphi \in \mathbb{R}^L$.

At each step, the above system evolves as follows. First, each agent i sees the current state s and applies an action $a_i \in \mathcal{U}_i$ sampled from $\mu_i(\cdot|s)$. Based on the joint action a, the system then moves to a new state $\tilde{s}$. Equivalently, the joint action a and the state $\tilde{s}$ can be seen as samples of $\mu(\cdot|s)$ and $\mathcal{P}(\cdot|s, a)$, respectively. Finally, each agent i receives an instantaneous reward $\mathcal{R}_i(s, a, \tilde{s})$.

Objective and Algorithm: The goal of the multi-agent system is to cooperatively estimate the value function $J^\mu \in \mathbb{R}^L$ corresponding to μ. This is defined as the solution to the Bellman equation

$$J^\mu(s) = \mathbb{E}\bigg[\sum_i \pi_i \mathcal{R}_i(s, a, \tilde{s}) + \gamma J^\mu(\tilde{s})\bigg], \quad s \in \mathcal{S},$$

where the expectation is over $a \sim \mu(\cdot|s)$ and $\tilde{s} \sim \mathcal{P}(\cdot|s, a)$. This expression differs from the ones in Doan et al. [2019, 2021], in that, the coefficients π_i here is replaced by $1/m$ there. When L is large, estimating J^μ directly is intractable. An alternative then is to make use of linear function approximation. That is, for some d, we have a feature matrix $\Phi \in \mathbb{R}^{L\times d}$ with full column rank. And, with $\phi(s)$ denoting the s-th row of Φ, the goal is to find a $\theta \in \mathbb{R}^d$ such that $J^\mu(s) \approx \phi(s)\theta'$ for all $s \in \mathcal{S}$.

The distributed TD(0) algorithm is helpful in this latter context. Let $(s_n, a_n, \tilde{s}_n)$, $n \geq 0$, be IID[5] samples of $(s, a, \tilde{s})$, where $s \sim \varphi(\cdot)$, $a \sim \mu(\cdot|s)$, and $\tilde{s} \sim \mathcal{P}(\cdot|s, a)$. Then, at agent i, this distributed algorithm has the update rule:

$$\theta_{n+1}(i) = \sum_{j\in\mathcal{N}_i} W_{ij}\theta_n(j) + \alpha_n(b_n(i) - \theta_n(i)A_n), \tag{13}$$

where $b_n(i) = \mathcal{R}_i(s_n, a_n, \tilde{s}_n)\phi(s_n) \in \mathbb{R}^d$ and $A_n = \phi'(s_n)\phi(s_n) - \gamma\phi'(\tilde{s}_n)\phi(s_n) \in \mathbb{R}^{d\times d}$.

Analysis: We first express the update rule given in (13) for different i in the standard DSA form. Let[6] $A = \mathbb{E}[A_n]$ and $B = \mathbb{E}[B_n]$, where $B_n \in \mathbb{R}^{m\times d}$ is the matrix whose i-th row is $b_n(i)$. Both A and B do not depend on n since $(s_n, a_n, \tilde{s}_n)$, $n \geq 0$, is IID. Next, for $n \geq 0$, let $x_n \in \mathbb{R}^{m\times d}$ be the matrix whose i-th row is $\theta_n(i)$. Then, (13) for different i can be jointly written as shown in (2) for

$$h(x) = B - xA \qquad \text{and} \qquad M_{n+1} = (B_n - B) - x_n(A_n - A). \tag{14}$$

Next, we look at the limiting ODE given in (7). In our case, this has the form $\dot{y}(t) = \pi B - y(t)A$. Now, A is known to be positive definite Sutton and Barto [2018], i.e., $\theta A\theta' > 0$ for all $\theta \neq 0$. Hence, it is invertible and the real parts of all its eigenvalues are positive, i.e., $-A$ is Hurwitz stable. This then shows that $\theta_* = \pi BA^{-1}$ is the unique globally asymptotically stable equilibrium for the above ODE.

We now verify the assumptions stated in Section 2. $\mathcal{A}_1$ trivially holds due to assumptions on W. For $\mathcal{A}_2$, let $f_1(x) = 0$ and $f_2(x) = -xA$ for $x \in \mathbb{R}^{m\times d}$. Further, let $x_* = \mathbf{1}'\theta_*$. Then, $h(x) = -\mathbf{1}\pi(x - x_*)A + (\mathbb{I} - \mathbf{1}\pi)(B + f_2(x))$, as desired. In order to satisfy $\mathcal{A}_3$, we simply choose a stepsize sequence that fulfills one of the criteria mentioned there.

It now only remains to establish $\mathcal{A}_4$. Let $\mathcal{F}_n$ be as defined there. Then, part (a) follows from the definitions of A and B and the fact that $(s_n, a_n, \tilde{s}_n)$ is independent of the past. On the other hand, part (b) can be shown by building upon the arguments used in the proof of [Dalal et al., 2018a, Lemma 5.1]. Next observe that, since $(s_n, a_n, \tilde{s}_n)$ is independent of the past, the only quantity that is random in $\mathbb{E}[M_{n+1}'\pi'\pi M_{n+1}|\mathcal{F}_n]$ is x_n. Also, trivially, $\mathbb{E}[M_{n+1}'\pi'\pi M_{n+1}|\mathcal{F}_n]$ is a symmetric positive semi-definite matrix. Therefore, on the event $\mathcal{E}(x_*)$, it is easy to see that part (c) holds as well. Finally, notice that $\|\pi M_{n+1}\| \leq C(1 + \|x_n - x_*\|)$ for some $C \geq 0$; this follows as in part (b) above. Hence, on $\mathcal{E}(x_*)$, $\sup_{n\geq 0} \mathbb{E}[\|\pi M_{n+1}\|^b|\mathcal{F}_n] < \infty$ a.s. for any $b \geq 0$. This verifies part (d).

Thus, Theorem 2.1 holds for the distributed TD(0) algorithm with linear function approximation.

[5]The IID assumption is standard in literature Dalal et al. [2018a], Liu et al. [2015], Sutton et al. [2008], Tsitsiklis and Van Roy [1997] and is needed to ensure that the update rule only has martingale noise. Otherwise, the update rule will additionally have Markovian noise, the analysis of which is beyond the scope of this paper. The good news though is that, as shown in previous works Kaledin et al. [2020], Doan et al. [2019, 2021], the asymptotic behaviours with and without the Markovian noise are often similar.

[6]The matrix A is usually defined to be the transpose of the expression we use.

4 Theoretical Analysis: Proof of the Main Result

We now turn to the technical details of our analysis. With Q as in (5) and x_* as in (8), observe that $\mathbf{1}'\pi x_* = x_*$ and, hence, $x_n - x_* = \mathbf{1}'\pi(x_n - x_*) + Qx_n$. We refer to the first term in this decomposition as the agreement component of the error and the second as the disagreement component. This decomposition differs from the standard approaches Doan et al. [2019, 2021], Morral et al. [2017], wherein $x_n - x_*$ is split into $(\mathbf{1}'\mathbf{1}/m)(x_n - x_*)$ and $(\mathbb{I} - (\mathbf{1}'\mathbf{1}/m))x_n$. In fact, the success of our approach strongly hinges on this novel error decomposition.

The rest of the section is organized as follows. We first state our bounds for the two terms in our decomposition. Using these bounds, we then provide a formal proof for Theorem 2.1. Thereafter, we sketch the proofs of these intermediate bounds, leaving the details to the appendix.

Lemma 4.1. *(Agreement Error) Almost surely on $\mathcal{E}(x_*)$,*

$$\limsup_{n\to\infty} \frac{\|\boldsymbol{1}'\pi(x_n - x_*)\|}{\sqrt{\alpha_n \ln t_{n+1}}} \le C, \tag{15}$$

where $C \geq 0$ is some deterministic constant.

Lemma 4.2. *(Disagreement Error) Let $\delta > 0$. Then,*

$$\|Qx_n\| = \mathcal{O}\left(\alpha_n (\ln n)^{1+\delta}\right) \quad \textit{a.s. on } \mathcal{E}(x_*). \tag{16}$$

Remark 4.3. *Up to logarithmic factors, the rate at which the disagreement error decreases is the square of the rate at which the agreement error decreases. Thus, the overall convergence rate is essentially dictated by the agreement component of the error.*

With these two ingredients at hand, our main result is arrived at via the following short calculation.

Proof of Theorem 2.1. Observe that

$$\|x_n - x_*\| \le \|\mathbf{1}'\pi(x_n - x_*)\| + \|Qx_n\|.$$

Also, $\ln t_{n+1}$ is $O(\ln n)$ and $O(\ln\ln n)$ in the Type γ and Type 1 cases, respectively. The desired result is now easy to see from Lemmas 4.1 and 4.2. □

4.1 Bound on agreement error $\|\mathbf{1}'\pi(x_n - x_*)\|$

We first focus on the details of our analysis for the first ingredient, i.e., the agreement error. Let

$$\psi_{n+1} := \sum_{k=0}^{n} \alpha_k \mathbf{1}'\pi M_{k+1} \mathrm{e}^{-(t_{n+1} - t_{k+1})A}, \quad n \ge 0, \tag{17}$$

and

$$\Delta_n := \mathbf{1}'\pi(x_n - x_*) - \psi_n, \quad n \ge 0. \tag{18}$$

Clearly, to prove Lemma 4.1, it suffices to obtain bounds on the rate at which $\|\psi_n\|$ and $\|\Delta_n\|$ decay. These bounds are stated below. Note that these results are generalizations of Lemmas 1 and 3 from Pelletier [1998]. Specifically, the results there focused on one-timescale stochastic approximation, ours on the other hand handles the distributed case. Furthermore, the quantities of interest here, e.g, ψ_n, Δ_n, are matrix-valued, unlike the ones in Pelletier [1998] which were vector-valued.

Lemma 4.4. *Let b be as in $\mathcal{A}_4$. Suppose that either α is of Type 1 or that α is of Type γ with $\gamma > 2/b$. Then, there exists some deterministic constant $C \ge 0$ such that*

$$\limsup_{n\to\infty} (\alpha_n \ln t_{n+1})^{-1/2} \|\psi_{n+1}\| \le C \quad \textit{a.s. on } \mathcal{E}(x_*) \tag{19}$$

Lemma 4.5. *Suppose $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$ hold. Then, for any $\lambda \in (0, \lambda_{\min})$,*

$$\|\Delta_n\| = \mathcal{O}\left(\max\left\{e^{-\lambda\sum_{k=0}^{n}\alpha_k}, \sum_{j=0}^{n} \alpha_j e^{-\lambda\sum_{k=j+1}^{n}\alpha_k}[\alpha_j\|\psi_j\| + \|\psi_j\|^a]\right\}\right) \tag{20}$$

a.s. on $\mathcal{E}(x_*)$, where a is as in (10). *Furthermore,*

1. *If α is of Type 1, then*

$$\|\Delta_n\| = \mathcal{O}\left(\max\{n^{-\lambda\alpha_0}; n^{-\frac{a}{2}}; n^{-1.5}\}(\ln n)^{\frac{a}{2}+1}\right) \qquad \text{a.s. on } \mathcal{E}(x_*).$$

2. *If α is of Type γ with $2/b < \gamma < 1$, then*

$$\|\Delta_n\| = \mathcal{O}\left(\alpha_n(\alpha_n \ln t_{n+1})^{1/2} + (\alpha_n \ln t_{n+1})^{a/2}\right) \qquad \text{a.s. on } \mathcal{E}(x_*).$$

We refer the reader to the Appendix for the proofs of Lemmas 4.4 and 4.5. However, there is one point which we would like to emphasize here. That is, ψ_n is a sum of scaled (matrix-valued) martingale differences. And, to derive its decay rate, we use the following law of iterated logarithm.

Let $\mathrm{LL}(x) = \ln\ln(x)$.

Lemma 4.6. *For $n \geq 0$, let $U_{n+1} = \sum_{k=0}^{n} \phi_k \epsilon_{k+1}$, where $\{\epsilon_n\}$ is a real-valued martingale difference sequence adapted to a filtration $\{\mathcal{F}_n\}$, and $\{\phi_n\}$ is a sequence of real-valued scalars, again adapted to $\{\mathcal{F}_n\}$.*

Let $\{T_n\}$, also adapted to $\{\mathcal{F}_n\}$, be such that, for $n \geq 0$, $|\phi_n| \leq T_n$ a.s. and $\tau_n := \sum_{k=0}^{n} T_k^2$ satisfies $\lim_{n\to\infty} \tau_n = \infty$ a.s. Further, assume $\sup_{n\geq 0} \mathbb{E}[\epsilon_{n+1}^2|\mathcal{F}_n] \leq \sigma^2$ a.s. for some constant σ^2. Also, let $\beta > 0$ be such that $\sum_n T_n^{2+2\beta}\tau_n^{-1-\beta}[\mathrm{LL}(\tau_n)]^{\beta} < \infty$ and $\sup_{n\geq 0} \mathbb{E}\left[|\epsilon_{n+1}|^{2+2\beta}\middle|\mathcal{F}_n\right] < \infty$ a.s. Then,

$$\limsup [2\tau_n \mathrm{LL}\tau_n]^{-1/2}|U_{n+1}| \leq \sigma \quad \text{a.s.} \tag{21}$$

Remark 4.7. *The condition $\sum_n T_n^{2+2\beta}\tau_n^{-1-\beta}[\mathrm{LL}(\tau_n)]^{\beta} < \infty$ differs from the one in [Duflo, 2013, Corollary 6.4.25]; in that, it includes the additional term $[\mathrm{LL}(\tau_n)]^{\beta}$. The impact of this is that we no longer require β to be in $(0, 1)$ as was the case in [Duflo, 2013, Corollary 6.4.25]. Instead, β can now take any positive value. This is precisely what allows Theorem 2.1 to be applicable even when the stepsizes are non-square summable.*

Remark 4.8. *The above result goes through even if we have $\limsup_{n\to\infty} \mathbb{E}[\epsilon_{n+1}^2|\mathcal{F}_n] \leq \sigma^2$ instead of $\sup_{n\geq 0} \mathbb{E}[\epsilon_{n+1}^2|\mathcal{F}_n] \leq \sigma^2$; cf. [Pelletier, 1998, Result 1].*

We now present the proof of 4.1 which is a direct consequence of Lemmas 4.4 and 4.5.

Proof of Lemma 4.1. From (18), observe that

$$\|\mathbf{1}'\pi(x_n - x_*)\| \leq \|\psi_n\| + \|\Delta_n\|.$$

First consider the case where α is of Type γ. From Lemmas 4.4 and 4.5, we have

$$\limsup_{n\to\infty} \frac{\|\mathbf{1}'\pi(x_n - x_*)\|}{(\alpha_n \ln t_{n+1})^{1/2}} \leq C + \limsup_{n\to\infty} \mathcal{O}\left(\alpha_n + (\alpha_n \ln t_{n+1})^{(a-1)/2}\right) = C,$$

where the last display holds because $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \alpha_n \ln t_{n+1} = 0$, and $a > 1$.

Next consider the case where α is of Type 1. Again, from Lemmas 4.4 and 4.5, we get

$$\limsup_{n\to\infty} \frac{\|\mathbf{1}'\pi(x_n - x_*)\|}{(\alpha_n \ln t_{n+1})^{1/2}} \leq C + \limsup_{n\to\infty} \mathcal{O}\left(\frac{\max(n^{-\lambda\alpha_0}; n^{-\frac{a}{2}}; n^{-1.5})(\ln n)^{1+a/2}}{(n^{-1}\ln n)^{1/2}}\right) = C,$$

where the last display holds because $\lambda\alpha_0 > \frac{1}{2}$ and $a > 1$.

The desired result now follows. □

4.2 Bound on disagreement error $\|Qx_n\|$

We now turn to the detailed analysis of the disagreement component of the error. Let

$$\chi_{n+1} := \sum_{j=0}^{n} \alpha_j W^{n-j} Q M_{j+1} e^{-(t_{n+1}-t_{j+1})A}, \quad n \geq -1, \tag{22}$$

and

$$\Gamma_n := Qx_n - \chi_n, \quad n \geq 0. \tag{23}$$

Note that χ_n represents the cumulative noise in Qx_n. It is also easy to see that

$$\chi_{n+1} = W\chi_n e^{-\alpha_n A} + \alpha_n Q M_{n+1}, \quad n \geq 0. \tag{24}$$

We now state our bounds for $\|\chi_n\|$ and $\|\Gamma_n\|$. Note that χ_n and Γ_n are peculiar to the DSA setup and do not have analogues in the one-timescale analysis.

Lemma 4.9. *Let $\delta > 0$. Then,*

$$\|\chi_{n+1}\| = \mathcal{O}\left(\alpha_n (\ln n)^{1+\delta}\right) \qquad \textit{a.s. on } \mathcal{E}(x_*) \tag{25}$$

Lemma 4.10. *Almost surely on $\mathcal{E}(x_*)$,*

$$\|\Gamma_{n+1}\| = \mathcal{O}\left(\alpha_n\right). \tag{26}$$

We refer readers to the Appendix for proofs of Lemma 4.9 and 4.10. The overall disagreement error can now be bounded as shown below.

Proof of Lemma 4.2. Observe that

$$\|Qx_n\| \leq \|\Gamma_n\| + \|\chi_n\|.$$

Using lemmas 4.9 and 4.10, we then have

$$\|Qx_n\| = \mathcal{O}\left(\alpha_n\right) + \mathcal{O}\left(\alpha_n (\ln n)^{1+\delta}\right) = \mathcal{O}\left(\alpha_n (\ln n)^{1+\delta}\right),$$

as desired. □

5 Discussion

We derive a novel law of iterated logarithm for a family of nonlinear DSA algorithms that is useful in MARL. This law can also be seen as an asymptotic a.s. convergence rate result. It is the first of its kind in the distributed setup and holds under significantly weaker assumptions. Our proof uses a novel error decomposition and a novel law of iterated logarithm for a sum of martingale differences.

While our DSA framework is fairly general, a key limitation is that the scaling matrix (i.e., A) in each component function h_i needs to be the same. It would be interesting to see if our approach can be extended to cover the general case Zeng et al. [2020a] where the scaling matrices also depend on i. Another intriguing future direction is the setting with dynamic communication protocols, wherein the gossip matrix also evolves with time Doan et al. [2019, 2021]. A third direction is that of two-timescale DSA schemes Dalal et al. [2018b, 2020]. On the MARL side, important algorithms like distributed Q-learning Lauer and Riedmiller [2000] and its variants need more careful analysis and we believe our techniques would be instrumental for this as well. Finally, we would like to study the effect of momentum in MARL algorithms Avrachenkov et al. [2020].

Acknowledgments and Disclosure of Funding

We would like to thank Prof. Vivek Borkar for suggesting this exciting problem. We would also like to thank the anonymous reviewers for providing helpful and constructive feedback on the paper. Research of Gugan Thoppe is supported by IISc's start up grants SG/MHRD-19-0054 and SR/MHRD-19-0040.

References

Konstantin Avrachenkov, Kishor Patil, and Gugan Thoppe. Online algorithms for estimating change rates of web pages. *arXiv preprint arXiv:2009.08142*, 2020.

Michel Benaïm. Dynamics of stochastic approximation algorithms. In *Seminaire de probabilites XXXIII*, pages 1–68. Springer, 1999.

Pascal Bianchi, Gersende Fort, and Walid Hachem. Performance of a distributed stochastic approximation algorithm. *IEEE Transactions on Information Theory*, 59(11):7405–7418, 2013.

Vivek S Borkar. *Stochastic approximation: a dynamical systems viewpoint*, volume 48. Springer, 2009.

Ziyi Chen, Yi Zhou, and Rongrong Chen. Multi-agent off-policy td learning: Finite-time analysis with near-optimal sample complexity and communication complexity. *arXiv preprint arXiv:2103.13147*, 2021.

Gal Dalal, Balázs Szörényi, Gugan Thoppe, and Shie Mannor. Finite sample analyses for td (0) with function approximation. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 32, 2018a.

Gal Dalal, Gugan Thoppe, Balázs Szörényi, and Shie Mannor. Finite sample analysis of two-timescale stochastic approximation with applications to reinforcement learning. In *Conference On Learning Theory*, pages 1199–1233. PMLR, 2018b.

Gal Dalal, Balazs Szorenyi, and Gugan Thoppe. A tale of two-timescale reinforcement learning with the tightest finite-time bound. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 34(04), pages 3701–3708, 2020.

Dongsheng Ding, Xiaohan Wei, Zhuoran Yang, Zhaoran Wang, and Mihailo R Jovanovic. Fast multi-agent temporal-difference learning via homotopy stochastic primal-dual method. In *Optimization Foundations for Reinforcement Learning Workshop, 33rd Conference on Neural Information Processing Systems*, 2019.

Thinh Doan, Siva Maguluri, and Justin Romberg. Finite-time analysis of distributed td (0) with linear function approximation on multi-agent reinforcement learning. In *International Conference on Machine Learning*, pages 1626–1635, 2019.

Thinh T Doan, Siva Theja Maguluri, and Justin Romberg. Finite-time performance of distributed temporal-difference learning with linear function approximation. *SIAM Journal on Mathematics of Data Science*, 3(1):298–320, 2021.

Marie Duflo. *Random iterative models*, volume 34. Springer Science & Business Media, 2013.

Paulo Heredia and Shaoshuai Mou. Finite-sample analysis of multi-agent policy evaluation with kernelized gradient temporal difference. In *2020 59th IEEE Conference on Decision and Control (CDC)*, pages 5647–5652. IEEE, 2020.

Maxim Kaledin, Eric Moulines, Alexey Naumov, Vladislav Tadic, and Hoi-To Wai. Finite time analysis of linear two-timescale stochastic approximation with markovian noise. In *Conference on Learning Theory*, pages 2144–2203. PMLR, 2020.

Soummya Kar, José MF Moura, and H Vincent Poor. QD-learning: A collaborative distributed strategy for multi-agent reinforcement learning through consensus + innovations. *IEEE Transactions on Signal Processing*, 61(7):1848–1862, 2013.

Anastasia Koloskova, Nicolas Loizou, Sadra Boreiri, Martin Jaggi, and Sebastian Stich. A unified theory of decentralized sgd with changing topology and local updates. In *International Conference on Machine Learning*, pages 5381–5393. PMLR, 2020.

Martin Lauer and Martin Riedmiller. An algorithm for distributed reinforcement learning in cooperative multi-agent systems. In *In Proceedings of the Seventeenth International Conference on Machine Learning*. Citeseer, 2000.

Donghwan Lee, Hyungjin Yoon, and Naira Hovakimyan. Primal-dual algorithm for distributed reinforcement learning: distributed gtd. In *2018 IEEE Conference on Decision and Control (CDC)*, pages 1967–1972. IEEE, 2018.

Jae Won Lee and Byoung-Tak Zhang. Stock trading system using reinforcement learning with cooperative agents. In *Proceedings of the Nineteenth International Conference on Machine Learning*, pages 451–458, 2002.

Xiangru Lian, Ce Zhang, Huan Zhang, Cho-Jui Hsieh, Wei Zhang, and Ji Liu. Can decentralized algorithms outperform centralized algorithms? a case study for decentralized parallel stochastic gradient descent. *arXiv preprint arXiv:1705.09056*, 2017.

Michael Littman and Justin Boyan. A distributed reinforcement learning scheme for network routing. In *Proceedings of the international workshop on applications of neural networks to telecommunications*, pages 55–61. Psychology Press, 2013.

Michael L Littman. Markov games as a framework for multi-agent reinforcement learning. In *Machine learning proceedings 1994*, pages 157–163. Elsevier, 1994.

Bo Liu, Ji Liu, Mohammad Ghavamzadeh, Sridhar Mahadevan, and Marek Petrik. Finite-sample analysis of proximal gradient td algorithms. In *UAI*, pages 504–513. Citeseer, 2015.

Adwaitvedant S Mathkar and Vivek S Borkar. Nonlinear gossip. *SIAM Journal on Control and Optimization*, 54(3):1535–1557, 2016.

Abdelkader Mokkadem and Mariane Pelletier. Convergence rate and averaging of nonlinear two-time-scale stochastic approximation algorithms. *The Annals of Applied Probability*, 16(3):1671–1702, 2006.

Gemma Morral, Pascal Bianchi, and Gersende Fort. Success and failure of adaptation-diffusion algorithms for consensus in multi-agent networks. In *53rd IEEE Conference on Decision and Control*, pages 1476–1481. IEEE, 2014.

Gemma Morral, Pascal Bianchi, and Gersende Fort. Success and failure of adaptation-diffusion algorithms with decaying step size in multiagent networks. *IEEE Transactions on Signal Processing*, 65(11):2798–2813, 2017.

OpenAI: Openai five. Openai: Openai five. https://openai.com/blog/openai-five/, 2021. Accessed: 2021-05-24.

Jun Ota. Multi-agent robot systems as distributed autonomous systems. *Advanced engineering informatics*, 20(1):59–70, 2006.

Mariane Pelletier. On the almost sure asymptotic behaviour of stochastic algorithms. *Stochastic processes and their applications*, 78(2):217–244, 1998.

Shi Pu and Angelia Nedić. Distributed stochastic gradient tracking methods. *Mathematical Programming*, pages 1–49, 2020.

Shi Pu, Alex Olshevsky, and Ioannis Ch Paschalidis. Asymptotic network independence in distributed stochastic optimization for machine learning: Examining distributed and centralized stochastic gradient descent. *IEEE Signal Processing Magazine*, 37(3):114–122, 2020.

Jineng Ren, Jarvis Haupt, and Zehua Guo. Communication-efficient hierarchical distributed optimization for multi-agent policy evaluation. *Journal of Computational Science*, 49:101280, 2021.

Martin Riedmiller, Andrew Moore, and Jeff Schneider. Reinforcement learning for cooperating and communicating reactive agents in electrical power grids. In *Workshop on Balancing Reactivity and Social Deliberation in Multi-Agent Systems*, pages 137–149. Springer, 2000.

Shai Shalev-Shwartz, Shaked Shammah, and Amnon Shashua. Safe, multi-agent, reinforcement learning for autonomous driving. *arXiv preprint arXiv:1610.03295*, 2016.

Miloš S Stanković, Marko Beko, and Srdjan S Stanković. Distributed gradient temporal difference off-policy learning with eligibility traces: Weak convergence. *IFAC-PapersOnLine*, 53(2):1563–1568, 2020.

Jun Sun, Gang Wang, Georgios B Giannakis, Qinmin Yang, and Zaiyue Yang. Finite-time analysis of decentralized temporal-difference learning with linear function approximation. In *International Conference on Artificial Intelligence and Statistics*, pages 4485–4495. PMLR, 2020.

Tao Sun, Tianyi Chen, Yuejiao Sun, Qing Liao, and Dongsheng Li. Decentralized markov chain gradient descent. *arXiv preprint arXiv:1909.10238*, 2019.

Wesley Suttle, Zhuoran Yang, Kaiqing Zhang, Zhaoran Wang, Tamer Başar, and Ji Liu. A multi-agent off-policy actor-critic algorithm for distributed reinforcement learning. *IFAC-PapersOnLine*, 53(2):1549–1554, 2020.

Richard S Sutton and Andrew G Barto. *Reinforcement learning: An introduction*. MIT press, 2018.

Richard S Sutton, Csaba Szepesvári, and Hamid Reza Maei. A convergent o (n) temporal-difference algorithm for off-policy learning with linear function approximation. In *NIPS*, 2008.

John Tsitsiklis, Dimitri Bertsekas, and Michael Athans. Distributed asynchronous deterministic and stochastic gradient optimization algorithms. *IEEE transactions on automatic control*, 31(9): 803–812, 1986.

John N Tsitsiklis and Benjamin Van Roy. An analysis of temporal-difference learning with function approximation. *IEEE transactions on automatic control*, 42(5):674–690, 1997.

Hoi-To Wai. On the convergence of consensus algorithms with markovian noise and gradient bias. In *2020 59th IEEE Conference on Decision and Control (CDC)*, pages 4897–4902. IEEE, 2020.

Hoi To Wai, Zhuoran Yang, Mingyi Hong, and Zhaoran Wang. Multi-agent reinforcement learning via double averaging primal-dual optimization. *Advances in Neural Information Processing Systems*, 2018:9649–9660, 2018.

Yue Xu, Zengde Deng, Mengdi Wang, Wenjun Xu, Anthony Man-Cho So, and Shuguang Cui. Voting-based multi-agent reinforcement learning for intelligent iot. *IEEE Internet of Things Journal*, 2020.

Kun Yuan, Qing Ling, and Wotao Yin. On the convergence of decentralized gradient descent. *SIAM Journal on Optimization*, 26(3):1835–1854, 2016.

Sihan Zeng, Aqeel Anwar, Thinh Doan, Justin Romberg, and Arijit Raychowdhury. A decentralized policy gradient approach to multi-task reinforcement learning. *arXiv preprint arXiv:2006.04338*, 2020a.

Sihan Zeng, Thinh T Doan, and Justin Romberg. Finite-time analysis of decentralized stochastic approximation with applications in multi-agent and multi-task learning. *arXiv preprint arXiv:2010.15088*, 2020b.

Kaiqing Zhang, Zhuoran Yang, and Tamer Basar. Networked multi-agent reinforcement learning in continuous spaces. In *2018 IEEE Conference on Decision and Control (CDC)*, pages 2771–2776. IEEE, 2018a.

Kaiqing Zhang, Zhuoran Yang, Han Liu, Tong Zhang, and Tamer Basar. Fully decentralized multi-agent reinforcement learning with networked agents. In *International Conference on Machine Learning*, pages 5872–5881. PMLR, 2018b.

Kaiqing Zhang, Zhuoran Yang, and Tamer Başar. Multi-agent reinforcement learning: A selective overview of theories and algorithms. *Handbook of Reinforcement Learning and Control*, pages 321–384, 2021a.

Kaiqing Zhang, Zhuoran Yang, Han Liu, Tong Zhang, and Tamer Basar. Finite-sample analysis for decentralized batch multi-agent reinforcement learning with networked agents. *IEEE Transactions on Automatic Control*, 2021b.

Xiaoxiao Zhao, Peng Yi, and Li Li. Distributed policy evaluation via inexact admm in multi-agent reinforcement learning. *Control Theory and Technology*, 18(4):362–378, 2020.

6 Appendix - Proofs

Throughout the appendix, we will presume that $\mathbb{P}\{\mathcal{E}(x_*)\} = 1$. This is mainly to avoid writing "a.s. on $\mathcal{E}(x_*)$" in every statement.

6.1 Agreement Error Results: Proof of Lemma 4.4

The discussion here builds upon the ideas used in the proof of [Pelletier, 1998, Lemma 1]. In order to not repeat everything, our focus below will only be on those arguments that differ from the original ones. Our first goal is to derive a relation that is similar to [Pelletier, 1998, (22)].

Let μ be an eigenvalue of A with multiplicity ν and let w be a column vector in the null space of $(A - \mu I)^\nu$. That is, let w be a generalized right eigenvector of A corresponding to the eigenvalue μ. It is possible that both μ and w are complex valued. Then, for any $t > 0$,

$$\mathrm{e}^{-tA} w = \mathrm{e}^{-t(A-\mu\mathbb{I})}\mathrm{e}^{-t\mu\mathbb{I}} w = \mathrm{e}^{-t\mu}\mathrm{e}^{-t(A-\mu\mathbb{I})} w = \mathrm{e}^{-t\mu}\sum_{p=0}^{v-1}(-1)^p t^p w_p,$$

where $w_p = (A - \mu\mathbb{I})^p w/p!$. Consequently,

$$\begin{aligned}\psi_{n+1} w = {} & \sum_{k=0}^{n} \alpha_k \mathbf{1}' \pi M_{k+1} \mathrm{e}^{-\mu(t_{n+1}-t_{k+1})} \sum_{p=0}^{\nu-1}(-1)^p (t_{n+1}-t_{k+1})^p w_p \\ = {} & \mathbf{1}'\mathrm{e}^{-\mu t_{n+1}} \sum_{p=0}^{\nu-1}(-1)^p \sum_{k=0}^{n} \mathrm{e}^{\mu t_{k+1}} \alpha_k (t_{n+1}-t_{k+1})^p \pi M_{k+1} w_p. \end{aligned} \tag{27}$$

Let u be an arbitrary column vector and, for $0 \leq p \leq \nu - 1$, set

$$G_{n+1}^{(p)} := \sum_{k=0}^{n} \mathrm{e}^{\mu t_{k+1}} \alpha_k (t_{n+1}-t_{k+1})^p \pi M_{k+1} u.$$

Then, observe that

$$G_{k+1}^{(0)} - G_k^{(0)} = \mathrm{e}^{\mu t_{k+1}} \alpha_k \pi M_{k+1} u.$$

Hence, for $p \geq 1$,

$$\begin{aligned} G_{n+1}^{(p)} = {} & \sum_{k=0}^{n} (t_{n+1}-t_{k+1})^p (G_{k+1}^{(0)} - G_k^{(0)}) \\ = {} & \sum_{k=1}^{n} \left[(t_{n+1}-t_k)^p - (t_{n+1}-t_{k+1})^p\right] G_k^{(0)}, \end{aligned}$$

where the last relation follows since $G_0^{(0)} = 0$. Therefore, by the mean value theorem,

$$|G_{n+1}^{(p)}| \leq p \sum_{k=1}^{n} \alpha_k (t_{n+1}-t_k)^{p-1} |G_k^{(0)}|.$$

We now use Lemma 4.6 to derive an almost sure upper bound of $|G_{n+1}^{(0)}|$. We will use this later to derive an almost sure bound on $|G_{n+1}^{(p)}|$ for $1 \leq p \leq \nu - 1$.

By applying individually to the real and imaginary parts, it is not difficult to see that Lemma 4.6 is true even if the terms ϵ_k and ϕ_k in its statement are complex numbers. Keeping this in mind, let $\phi_k = e^{\mu t_{k+1}} \alpha_k$ and $\epsilon_{k+1} = \pi M_{k+1} u$.

We now verify the assumptions of Lemma 4.6. Clearly, ϕ_k and ϵ_{k+1} are complex scalars. Also, $\{\epsilon_{k+1}\}$ is a martingale difference sequence. Pick a $\beta > 0$ such that $1/\gamma < 1 + \beta < b/2$; this is possible since b in $\mathcal{A}_4$ is bigger than 2 and $\gamma > 2/b$. Then, because of $\mathcal{A}_4$ and the fact that

$$\mathbb{E}[|\epsilon_{n+1}|^{2+2\beta} | \mathcal{F}_n] \leq \|u\|^{2+2\beta}\, \mathbb{E}[\|\pi M_{n+1}\|^{2+2\beta} | \mathcal{F}_n],$$

we have $\sup_{n\geq 0} \mathbb{E}[|\epsilon_{n+1}|^{2+2\beta}|\mathcal{F}_n] < \infty$ a.s. Separately, observe that $\mathbb{E}[|\epsilon_{n+1}|^2|\mathcal{F}_n] = u^{\mathrm{H}}\mathbb{E}[M_{n+1}'\pi'\pi M_{n+1}|\mathcal{F}_n]u$, where H denotes the conjugate transpose operation. Combining this with $\mathcal{A}_4$, it then follows that $\limsup_{n\to\infty} \mathbb{E}[|\epsilon_{n+1}|^2|\mathcal{F}_n] = u^{\mathrm{H}}Mu$ a.s.

Let $\lambda = \mathscr{R}(\mu)$. Since $-A$ is Hurwitz, we have $\lambda \geq \lambda_{\min} > 0$. Hence, if $T_n = \alpha_n \mathrm{e}^{\lambda t_{n+1}}$, then $|\phi_n| \leq T_n$. Further, if α is of Type 1, then clearly $\tau_n \sim [\alpha_0/(2\lambda\alpha_0 - 1)]\alpha_n e^{2\lambda t_{n+1}}$; combining this with the fact that $\alpha_0 \geq 1/(2\lambda_{\min})$, then shows $\sum_{n\geq 0} T_n^{2+2\beta}\tau_n^{-1-\beta}[\mathrm{LL}(\tau_n)]^\beta \sim C\sum_{n\geq 0}[\mathrm{LL}(n)]^\beta/n^{1+\beta} < \infty$. On the other hand, if α is of Type γ, then [Pelletier, 1998, Lemma 4] shows that $\tau_n \sim 1/(2\lambda)\alpha_n e^{2\lambda t_{n+1}}$; this along with the fact that $\gamma(1+\beta) > 1$ then shows that $\sum_{n\geq 0} T_n^{2+2\beta}\tau_n^{-1-\beta}[\mathrm{LL}(\tau_n)]^\beta \sim (2\lambda)^{1+\beta}\sum_{n\geq 0}[\ln(n)]^\beta/n^{\gamma(1+\beta)} < \infty$. This verifies all the conditions needed in Lemma 4.6.

It now follows from Lemma 4.6 that, almost surely,

$$\limsup_{n\to\infty} \frac{e^{-\lambda t_{n+1}}|G_{n+1}^{(0)}|}{[\alpha_n \ln t_{n+1}]^{1/2}} \leq \sqrt{\frac{u^{\mathrm{H}}Mu}{\lambda - \zeta}},$$

where

$$\zeta = \begin{cases} 1/(2\alpha_0), & \text{if } \alpha \text{ is Type } 1,\\ 0, & \text{if } \alpha \text{ is Type } \gamma.\end{cases}$$

This expression is exactly of the form given in (22) in Pelletier [1998]; therefore, by repeating the arguments that follow (22) there, we get

$$\limsup_{n\to\infty} \frac{e^{-\lambda t_{n+1}}|G_{n+1}^{(p)}|}{[\alpha_n \ln t_{n+1}]^{1/2}} \leq \Lambda_p$$

for some deterministic constant Λ_p. Combining this with (27) then gives

$$\limsup_{n\to\infty} \frac{\|\psi_{n+1}w\|}{[\alpha_n \ln t_{n+1}]^{1/2}} \leq \sqrt{m}\Lambda_w$$

for some deterministic constant Λ_w; $\sqrt{m}$ comes in the expression due to the vector $\mathbf{1}'$.

Because w was arbitrary, the desired result now follows.

6.2 Agreement Error Results: Proof of Lemma 4.5

Using (17), observe that

$$\psi_{n+1} = \psi_n e^{-(t_{n+1}-t_n)A} + \alpha_n \mathbf{1}'\pi M_{n+1},$$

which, using a version of Taylor's theorem for matrix valued functions, can be written as

$$\psi_{n+1} = \psi_n[\mathbb{I} - \alpha_n A + O(\alpha_n^2)\mathbb{I}] + \alpha_n \mathbf{1}'\pi M_{n+1}.$$

Separately, recall from (18) that

$$\Delta_{n+1} = \mathbf{1}'\pi(x_{n+1} - x_*) - \psi_{n+1}.$$

Using (2), (6), (8), and $\mathcal{A}_2$, it then follows that

$$\Delta_{n+1} = \Delta_n(\mathbb{I} - \alpha_n A) + \alpha_n \mathbf{1}'\pi f_1(x_n) + \psi_n O(\alpha_n^2).$$

For $r_{n+1} := \mathbf{1}'\pi f_1(x_n)$, we finally have

$$\Delta_{n+1} = \Delta_n[\mathbb{I} - \alpha_n A] + \alpha_n[r_{n+1} + O(\alpha_n)\psi_n].$$

Next, let $0 < \lambda < \hat{\lambda} < \lambda_{\min}$. Then, for all sufficiently large n, $\|\mathbb{I} - \alpha_n A\| \leq (1 - \hat{\lambda}\alpha_n)$; therefore, for some $C \geq 0$,

$$\|\Delta_{n+1}\| \leq (1 - \alpha_n\hat{\lambda})\|\Delta_n\| + C\alpha_n(\|r_{n+1}\| + \alpha_n\|\psi_n\|),$$

$$\leq (1-\alpha_n\hat{\lambda})\|\Delta_n\| + C\alpha_n(\alpha_n\|\psi_n\| + \|\psi_n\|^a + \|\Delta_n\|^a),$$

where the last relation follows by using (10) and (18).

Equivalently, for all large enough n,

$$\|\Delta_{n+1}\| \leq (1-\lambda\alpha_n)\|\Delta_n\| - \alpha_n[(\hat{\lambda}-\lambda) - C\|\Delta_n\|^{a-1}]\|\Delta_n\| + C\alpha_n(\alpha_n\|\psi_n\| + \|\psi_n\|^a).$$

Since $\|\psi_n\| \to 0$ and $x_n \to x_*$ a.s., we have $\|\Delta_n\| \to 0$ a.s. This, along with the fact that $a > 1$, then shows that $(\hat{\lambda}-\lambda) - \|\Delta_n\|^{a-1} \geq 0$ for all large enough n. Hence, for all large enough n,

$$\|\Delta_{n+1}\| \leq (1-\lambda\alpha_n)\|\Delta_n\| + C\alpha_n[\alpha_n\|\psi_n\| + \|\psi_n\|^a].$$

Thus,

$$\|\Delta_n\| = \mathcal{O}\left(\max\left\{e^{-\lambda\sum_{k=0}^n \alpha_k}, \sum_{j=0}^n \alpha_j e^{-\lambda\sum_{k=j+1}^n \alpha_k}[\alpha_j\|\psi_j\| + \|\psi_j\|^a]\right\}\right),$$

as desired in (20).

To proceed with further calculations, let

$$\rho_n := \sum_{j=0}^n \alpha_j e^{-\lambda(t_{n+1}-t_{j+1})}(\alpha_j\|\psi_j\| + \|\psi_j\|^a).$$

When α is of Type 1, $e^{-\lambda t_{n+1}} = \Theta(n^{-\lambda\alpha_0})$. This, combined with Lemma 4.4, then shows that

$$\begin{aligned}
\rho_n &= \mathcal{O}\left(n^{-\lambda\alpha_0}\sum_{j=0}^n \frac{j^{\lambda\alpha_0}}{j}\left[\frac{1}{j}\left(\frac{\ln j}{j}\right)^{\frac{1}{2}} + \left(\frac{\ln j}{j}\right)^{\frac{a}{2}}\right]\right)\\
&= \mathcal{O}\left(n^{-\lambda\alpha_0}\sum_{j=0}^n \left[j^{\lambda\alpha_0-2.5}(\ln j)^{1/2} + j^{\lambda\alpha_0-1-\frac{a}{2}}(\ln j)^{\frac{a}{2}}\right]\right)\\
&= \mathcal{O}\left(n^{-\lambda\alpha_0}\left[n^{\lambda\alpha_0-1.5}(\ln n)^{1/2} + n^{\lambda\alpha_0-\frac{a}{2}}(\ln n)^{\frac{a}{2}} + (\ln n)^{\frac{a}{2}+1}\right]\right)\\
&= \mathcal{O}\left(n^{-1.5}(\ln n)^{1/2} + n^{-\frac{a}{2}}(\ln n)^{\frac{a}{2}} + n^{-\lambda\alpha_0}(\ln n)^{\frac{a}{2}+1}\right) \text{ a.s.;}
\end{aligned}$$

the $\log n$ factor in the last term of the third relation accounts for the possibility of $\lambda\alpha_0$ being either 2.5 or $1 + a/2$.

On the other hand, when α is of Type γ with $2/b < \gamma < 1$, Lemma 4.4 shows that

$$\rho_n = \mathcal{O}\left(e^{-\lambda t_{n+1}}\sum_{j=1}^n \alpha_j e^{\lambda t_{j+1}}\left[\alpha_j\left(\alpha_j \ln t_{j+1}\right)^{\frac{1}{2}} + \left(\alpha_j \ln t_{j+1}\right)^{\frac{a}{2}}\right]\right).$$

It then follows from [Pelletier, 1998, Lemma 4] that

$$\rho_n = \mathcal{O}\left(\alpha_n\left(\alpha_n \ln t_{n+1}\right)^{\frac{1}{2}} + \left(\alpha_n \ln t_{n+1}\right)^{\frac{a}{2}}\right) \text{ a.s.}$$

The desired result is now easy to see.

6.3 Disagreement Error Results: Proof of Lemma 4.9

Because of $\mathcal{A}_1$, note that all eigenvalues of W have magnitude less than or equal to 1. Furthermore, there is one and only one eigenvalue with magnitude 1 and that eigenvalue is 1 itself. Recall from (6) that the left eigenvector of W corresponding to the eigenvalue 1 is π.

For ease of exposition, we will presume that every other eigenvalue of W has multiplicity 1. Similarly, we will presume that every eigenvalue of A has multiplicity 1. The general case where some of the eigenvalues may have multiplicities larger than 1 can be handled using the ideas from this proof along with those in the proof of [Pelletier, 1998, Lemma 1] and Lemma 6.1.

Recall from (22) that

$$\chi_{n+1} = \sum_{j=0}^{n} \alpha_j W^{n-j} Q M_{j+1} e^{-(t_{n+1}-t_{j+1})A}.$$

To prove the desired result, it suffices to show there exists some universal constant $C \geq 0$ such that

$$\limsup_{n\to\infty} \frac{|u\chi_{n+1}v|}{\alpha_n(\ln n)^{1+\delta}} \leq C \tag{28}$$

for any arbitrary row vector $u \in \mathbb{R}^m$ and arbitrary column vector $v \in \mathbb{R}^d$, both with unit norm. Now, due to the above assumption on multiplicities, the eigenvalues of W span $\mathbb{R}^m$, while the eigenvalues of A span $\mathbb{R}^d$. Hence, it suffices to show (28) when u is a left eigenvector of W and v is a right eigenvector of A.

When $u = \pi$, we have $u\chi_{n+1}v = 0$. This follows from (6) and the fact that $\pi Q = 0$. The desired result thus trivially holds in this case. Keeping this in mind, suppose that u is some left eigenvector of W that is not equal to π. We will presume that the eigenvalue corresponding to u is $re^{\iota\theta}$, where $\iota := \sqrt{-1}$. Due to $\mathcal{A}_1$, $r < 1$; on the other hand, $\theta \in [0, 2pi)$ can be arbitrary. Similarly, let $\lambda + \iota\sigma$ denote the eigenvalue of A corresponding to v.

Now, observe that

$$\begin{aligned} u\chi_{n+1}v &= \sum_{j=0}^{n} \alpha_j u W^{n-j} Q M_{j+1} e^{-(t_{n+1}-t_{j+1})A} v, \\ &= \sum_{j=0}^{n} \alpha_j r^{n-j} e^{\iota\theta(n-j)} e^{-(t_{n+1}-t_{j+1})(\lambda+\iota\sigma)} u' Q M_{j+1} v \\ &= r^n e^{\iota\theta n} e^{-(\lambda+\iota\sigma)t_{n+1}} Z_{n+1}, \end{aligned}$$

where

$$Z_{n+1} := \sum_{j=0}^{n} \alpha_j r^{-j} e^{-j\iota\theta} e^{(\lambda+\iota\sigma)t_{j+1}} u' Q M_{j+1} v.$$

Hence,

$$|u\chi_{n+1}v| = r^n e^{-\lambda t_{n+1}} |Z_{n+1}|.$$

Note that $\{Z_{n+1}\}$ is one-dimensional martingale sequence, possibly complex valued. Hence, to derive a bound on $|Z_{n+1}|$, we now make use of [Duflo, 2013, Theorem 6.4.24]. The result in Duflo [2013] is stated for real-valued martingale sequences. To account for this discrepancy, we separately deal with the real and imaginary part of Z_{n+1}.

Let $R_{n+1} := \mathscr{R}(Z_{n+1})$ and $I_{n+1} := \Im(Z_{n+1})$. Also, let L be a positive finite random variable such that

$$\sup_{n\geq 0} \mathbb{E}[|uQM_{n+1}v|^2 | \mathcal{F}_n] \leq L.$$

Such a random variable exists on account of $\mathcal{A}_4$.b and the fact that $x_n \to x_*$ a.s. Then, it is not difficult to see that the quadratic variation of R_{n+1}, i.e.,

$$\langle R \rangle_{n+1} \leq C \sum_{k=0}^{n} \alpha_k^2 r^{-2k} e^{2\lambda t_{k+1}} \mathbb{E}[|uQM_{k+1}v|^2 | \mathcal{F}_k]$$

$$
\begin{aligned}
&\leq CL\sum_{k=0}^{n}\alpha_k^2 r^{-2k}e^{2\lambda t_{j+1}}\\
&\leq CL\alpha_n^2 r^{-2n}e^{2\lambda t_{n+1}}.
\end{aligned}
$$

Let $K_1 = CL$, where C is the constant present in the last relation above.

Next, define $s_n = \sqrt{K_1}\alpha_n r^{-n}e^{\lambda t_{n+1}}(\ln n)^{1/2+\delta}$ for some $\delta > 0$. Then, it is easy to see that $s_n \to \infty$ and $\langle R\rangle_{n+1} \leq s_n^2$. Further,

$$
\begin{aligned}
|R_{n+1} - R_n| &\leq \alpha_n r^{-n}e^{\lambda t_{n+1}}|uQM_{n+1}v|\\
&\leq C\alpha_n r^{-n}e^{\lambda t_{n+1}}[1 + ||Q(x_n - x^*)||],
\end{aligned}
$$

where the last relation follows due to $\mathcal{A}_4$.b and the fact that both $\|u\|$ and $\|v\|$ are bounded from above by 1. Let K_2 be the constant in the last relation above.

Then, for $h(x) = \sqrt{2x\ln\ln x}$, we have

$$
\begin{aligned}
|R_{n+1} - R_n| &\leq \frac{\sqrt{2}K_2}{\sqrt{K_1}}\frac{s_n^2}{h(s_n^2)}\frac{[1 + ||Q(x_n - x^*)||]}{(\ln n)^{1/2+\delta}}(\ln\ln s_n^2)^{1/2}\\
&= \frac{C_n s_n^2}{h(s_n^2)},
\end{aligned}
$$

where

$$
C_n = \frac{\sqrt{2}K_2}{\sqrt{K_1}}\frac{[1 + ||Q(x_n - x^*)||]}{(\ln n)^{1/2+\delta}}(\ln\ln(s_n^2))^{1/2}.
$$

Since $x_n \to x^*$, $||Q(x_n - x^*)||$ is bounded from above and, hence, $C_n \to 0$. Applying [Duflo, 2013, Theorem 6.4.24], it now follows that

$$
\limsup_{n\to\infty}\frac{|R_{n+1}|}{h(s_n^2)} \leq C.
$$

Similarly, it can be shown that

$$
\limsup_{n\to\infty}\frac{|I_{n+1}|}{h(s_n^2)} \leq C.
$$

Combining the two relations above then shows that

$$
\limsup_{n\to\infty}\frac{|Z_{n+1}|}{h(s_n^2)} \leq C.
$$

Consequently,

$$
\limsup_{n\to\infty}\frac{|u\chi_{n+1}v|}{\alpha_n(\log n)^{1+\delta}} \leq C.
$$

This verifies (28), as desired.

6.4 Disagreement Error Results: Proof of Lemma 4.10

The operator $Q = \mathbb{I} - \mathbf{1}'\pi$ satisfies the simple properties $WQ = QW$ and $Q\mathbf{1}'\pi = 0$. These properties and $\mathcal{A}_2$ lend (2) into

$$
\begin{aligned}
Qx_{n+1} &= WQx_n + \alpha_n(Qh(x_n) + QM_{n+1}),\\
&= WQx_n + \alpha_n(Q(B + f_2(x_n)) + QM_{n+1}).
\end{aligned}
$$

Using this and the definition of Γ_n from (23) then shows that

$$
\begin{aligned}
\Gamma_{n+1} &= Qx_{n+1} - \chi_{n+1}\\
&= W\Gamma_n + W\chi_n\kappa_n + \alpha_n Q(B + f_2(x_n)),
\end{aligned}
$$

where $\kappa_n = \mathbb{I} - e^{-\alpha_n A}$. Since $\Gamma_0 = Qx_0$, by unrolling the previous relation, we get

$$
\Gamma_{n+1} = W^{n+1}Qx_0 + \sum_{j=0}^{n}\alpha_j W^{n-j}Q(B + f_2(x_j)) + \sum_{j=0}^{n}W^{n+1-j}\chi_j\kappa_j.
$$

For ease of discussion, we will presume that W has unique eigenvalues. The general case where some of the eigenvalues may have multiplicities larger than 1 can be handled by building upon the ideas discussed in the proof of [Pelletier, 1998, Lemma 1] and Lemma 6.1.

Using (22), (6), and the fact that $\pi Q = 0$, it is easy to see that $\pi \Gamma_{n+1} = 0$. Hence, the desired result trivially holds then. Now, let the row vector $u \neq \pi$, of unit norm, be an arbitrary left eigenvector of W and suppose that its eigenvalue is $\rho e^{\iota\theta}$. Because of $\mathcal{A}_1$, it must be the case that $\rho < 1$.

It is then easy to see that

$$u\Gamma_{n+1} = \rho^{n+1} e^{\iota(n+1)\theta} uQx_0 + \sum_{j=0}^{n} \alpha_j \rho^{n-j} e^{\iota(n-j)\theta} uQ(B + f_2(x_n)) + \sum_{j=0}^{n} \rho^{n+1-j} e^{\iota(n+1-j)\theta} u\chi_j \kappa_j.$$

Now, since $x_n \to x_*$ a.s., it follows that x_n is bounded a.s. Hence,

$$\|u\Gamma_{n+1}\| \leq C\rho^{n+1} + C\sum_{j=0}^{n} \alpha_j \rho^{n-j} + C\sum_{j=0}^{n} \rho^{n+1-j} \|\chi_j\| \|\kappa_j\| \quad \text{a.s.}$$

From Lemma 4.9, $\|\chi_j\| = O(\alpha_j (\ln j)^{1+\delta})$ a.s. Separately, $\|\kappa_j\| = O(\alpha_j)$. Hence,

$$\|u\Gamma_{n+1}\| \leq C\rho^{n+1} + C\sum_{j=0}^{n} \alpha_j \rho^{n-j} + C\sum_{j=0}^{n} \alpha_j^2 \rho^{n-j} (\ln j)^{1+\delta} \quad \text{a.s.}$$

Next, note that $\alpha_j^{0.5} \ln j = o(1)$ for either type of the step sizes. Additionally, between $\sum_{j=0}^{n} \alpha_j \rho^{n-j}$ and $\sum_{j=0}^{n} \alpha_j^{1.5} \rho^{n-j}$, the dominant term is the former; this is because each term in its summation dominates the corresponding term in the latter. Separately,

$$\sum_{j=0}^{n} \alpha_j \rho^{n-j} \leq \left(\max_{0 \leq j \leq n} \rho^{(n-j)/2} \alpha_j\right) \sum_{j=0}^{n} \rho^{(n-j)/2} = O(\alpha_n).$$

The desired result is now easy to see.

6.5 Proof of Auxiliary Lemma 4.6

We only give a sketch of the proof since the arguments are similar to the ones used in the derivation[7] of [Duflo, 2013, Corollary 6.4.25]. In the latter's proof, a sequence $\{C_n\}$ adapted to $\{\mathcal{F}_n\}$ needs to be chosen such that

$$\lim_{n \to \infty} C_n = 0 \quad \text{and} \quad \sum_{n \geq 0} \frac{|\phi_n|^2 T_n^{2\beta}\, \mathrm{LL}(s_n^2)^{\beta - 1}}{C_n^{2\beta} s_n^{2+2\beta}} < \infty,$$

where $s_n^2 = \sigma^2 \tau_n$.

In Duflo [2013], C_n^2 was chosen to be $[\mathrm{LL}(s_n^2)]^{1-1/\beta}$. Because we need $\lim_n C_n$ to be 0, it necessarily follows that β should be in $(0, 1)$.

In contrast, we set $C_n^2 = [\mathrm{LL}(s_n^2)]^{-1/\beta}$. Since $\tau_n \to \infty$, we have that $s_n \to \infty$ as well. Combining this with the fact that $\beta > 0$, it then follows that $C_n \to 0$, as desired. Separately, due to the given conditions,

$$\sum_{n \geq 0} \frac{|\phi_n|^2 T_n^{2\beta}\, \mathrm{LL}(s_n^2)^{\beta-1}}{C_n^{2\beta} s_n^{2+2\beta}} \leq \sum_{n \geq 0} \frac{T_n^{2+2\beta} [\mathrm{LL}(s_n^2)]^{\beta - 1}}{C_n^{2\beta} s_n^{2+2\beta}} = \frac{1}{\sigma^{2+2\beta}} \sum_{n \geq 0} T_n^{2+2\beta} \tau_n^{-(1+\beta)} [\mathrm{LL}(s_n^2)]^{\beta} < \infty.$$

The desired result now follows.

[7] The definition of ξ_{k+1} and the expression for N_{n+1}, as given in [Duflo, 2013, p212], has typos. The correct versions are $\xi_{k+1} := \frac{C_k^\beta s_k^{2\beta}}{T_k^\beta [h(s_k^2)]^\beta} \eta_{k+1} \mathbb{1}[T_k \neq 0]$ and $N_{n+1} = \sum_{k=0}^{n} \frac{\phi_k T_k^\beta [h(s_k^2)]^\beta}{C_k^\beta s_k^{2\beta} h(s_k^2)} \xi_{k+1}$

6.6 Auxiliary Linear Algebraic Lemma

We state a linear algebra result here that is useful in the proof of our results when some of the eigenvalues of W have multiplicities bigger than 1.

Lemma 6.1. *Suppose that the Jordan normal form of W has L Jordan blocks with sizes $\ell_1, \ldots, \ell_L$, respectively. Also, let $1 = \theta_0, \theta_1, \ldots, \theta_{L-1}$ denote the corresponding distinct eigenvalues. Then, there exist some matrices $J_{i\ell}$, $1 \leq i \leq L-1$ and $0 \leq \ell \leq \ell_i - 1$, such that, for $k \geq 0$,*

$$W^k Q = \sum_{i=1}^{L-1} \sum_{\ell=0}^{\ell_i - 1} \theta_i^{k-\ell} \binom{k}{\ell} J_{i\ell}. \tag{29}$$

Proof. Due to $\mathcal{A}_1$, recall that precisely one eigenvalue of W equals 1. And, the left and right eigenvectors of W corresponding to this eigenvalue are π and $\mathbf{1}$, respectively. Therefore, a simple Jordan decomposition shows that

$$W^k = \sum_{i=1}^{L-1} \sum_{\ell=0}^{\ell_i - 1} \theta_i^{k-\ell} \binom{k}{\ell} J_{i\ell} + \mathbf{1}\pi.$$

for some suitably defined matrices $\{J_{i\ell}\}$. Separately, since $W\mathbf{1} = \mathbf{1}$, we have $W^k Q = W^k(\mathbb{I} - \mathbf{1}\pi) = W^k - \mathbf{1}\pi$. The desired result is now straightforward to see. $\square$