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论文中，红色荧光表示系统模型的假设；蓝色荧光表示定义；黄色荧光表示文章的创新点；绿色荧光表示文章中某些重要定理。红色下划线表示文章中写的比较好的英文；绿色下划线表示文章中仿真结果和结论。

Double-spend Attack Models with Time Advantange for Bitcoin

Carlos Pinzón[1]

Mathematics Department
Escuela Colombiana de Ingeniería
Bogotá, Colombia

Camilo Rocha[2]

Department of Electronics and Computer Science
Pontificia Universidad Javeriana
Cali, Colombia

Abstract

Bitcoin is a digital currency in which the need for a trusted third party is avoided. Instead, this digital currency is based on the concept of 'proof of work' allowing users to execute payments by digitally signing their transactions. Since electronic files can be duplicated, fraudulent transactions in the form of double-spend attacks – where users spend the same money at least twice – can happen. This paper is about attack models that can assign possible time advantage to attacker agents in the Bitcoin network. In particular, this paper presents: (i) two attack models in which partial advancement towards block production can be influenced by time and not only by the hashpower used to produce blocks of hashes, and (ii) algorithmic experimentation comparing these models against existing well-known hashrate-based attack models that do not consider time advantage. As a conclusion, this paper presents evidence on the fact that advantages are not negligible for cases in which an attacker has had enough time for secretly mining fraudulent blocks or significant control over the network. Also, the models presented in this paper help in supporting previous claims in the literature about how to correctly model and detect double-spend attacks in the Bitcoin network.

Keywords: Bitcoin, double-spend attack, mathematical model, time-based model, attacker advantage

1 Introduction

Bitcoin is a digital currency. As of May 2016, it is worth US $7200+ million in the form of 15.8 million bitcoins, approximately; this digital currency is expected to reach 5+ million users by 2019. Bitcoin uses a scheme in which the need for a trusted third party is avoided: instead, this digital currency is based on the concept of 'proof of work' allowing users to execute payments by digitally signing their transactions

[1] Email: carlos.pinzon@mail.escuelaing.edu.co
[2] Email: camilo.rocha@javerianacali.edu.co

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using hashes through a distributed time-stamping service. Since electronic files can be duplicated and Bitcoin does not have a trusted third-party that can verify whether a digital coin has been spent, fraudulent transactions with users spending the same money at least twice can happen. These fraudulent schemes are known as *double-spend attacks* and have taken place already in the Bitcoin network [6].

Some mathematical models have been proposed to analyze how feasible really are double-spend attacks in Bitcoin. In particular, S. Nakamoto [7] and M. Rosenfeld [9] have proposed *hashrate-based* attack models in which, overall, an attack is successful whenever the number of confirmations of a honest transaction is surpassed by the number of confirmations of a fraudulent one. In this setting, hashrate means that the model considers how fast a block header can be hashed, while the number of confirmations refers to the number of blocks of transactions claiming a transaction to be valid. In particular, these two models are used to answer the following question:

(i) What is the probability of a double-spend attack with K confirmations given that an attacker has mined n blocks?

The models of S. Nakamoto and M. Rosenfeld produce very similar answers for Question (i), despite certain differences between them at the technical level.

One important aspect of the models of S. Nakamoto and M. Rosenfeld is the fact that blocks of transactions are considered either non-existent or complete, but never partially complete. That is, these models do not consider blocks to be partially built, neither by honest nor attacker nodes. More importantly, these models lack the expressiveness needed to analyze situations in which an attacker has probably already built fraudulent blocks because it had some time advantage. These features seem important in an attack model given the fact that, at some point, a honest block might just start being built from scratch while competing against an attacker that may have been already mining for a certain period of time. In this case, for instance, such a situation could result in an undetected attack in the attack models of S. Nakamoto and M. Rosenfeld.

This paper addresses the following more general research question:

(ii) What is the probability of a double-spend attack with K confirmations given that an attacker has mined n blocks and, in addition, has been mining for t time units?

For answering Question (ii), this paper presents two double-spend attack models for Bitcoin in which an attacker can be assigned some advantage in terms of time units. The first model presented in this paper results from generalizing the model of M. Rosenfeld by adding an extra parameter denoting some time advantage assigned to the attacker, assumed to be used in producing and mining fraudulent blocks. The second model, following a completely different approach, is a purely time-based model that takes into account the times at which both honest and attacker nodes last mined a block, in addition to their relative progress in terms of the length of chains already mined. In this paper, the former model is called the *generalized model* and the latter is called the *time-based model*.

Both the generalized and the time-based models are hashrate-based attack mod-

els in which partial advancement towards block production is influenced by time, and not only by the hashpower used to produce blocks of hashes. The equations governing these models use an Erlang probability distribution, which is specially suited for modeling waiting times in queuing systems [12]. This contrasts with the models of S. Nakamoto that uses Poisson probability distribution and of M. Rosenfeld that uses a negative binomial probability distribution. The choice of this distribution can be justified because waiting times between occurrences of the event of producing a block can be naturally considered to be Erlang distributed, even if these events occur independently with some average rate that can be modeled with a Poisson process. Furthermore, since partial block production is directly tied to the amount of time spent on mining a block, it is more of a continuous process than a discrete one. Because of these reasons, the Erlang probability distribution seems well-suited for both models.

This paper also presents some results of extensive experimentation performed on the generalized and time-based attack models. As a consequence of this experimentation, there is strong evidence indicating that these two attack models coincide and behave very much like the attack model by M. Rosenfeld, but produce more information because of the time advantage inclusion. That is, on the one hand, this paper presents evidence that partial block production is not negligible for analyzing and detecting double-spend attacks in Bitcoin. On the other hand, experimental data supports the claim that the time-based model presented in this paper is correct with respect to the existing attack model proposed by M. Rosenfeld, which is surprising given the intrinsic differences between these attack models. Furthermore, the models contributed by the authors in this paper witness the observation already made by M. Rosenfeld stating that the double-spend attack model proposed by S. Nakamoto was not entirely accurate [9].

In summary, this paper contributes two double-spend attack models for Bitcoin, more general than existing models commonly used in the analysis of double-spend attacks, and reports on extensive experimentation performed on them. To the best of the authors' knowledge, double-spend attack models for Bitcoin that incorporate time have never been reported before in the literature. Furthermore, the experimentation performed on both the generalized and the time-based models presented in this paper reinforce the believe that double-spend attacks are very unlikely in practice, even when an attacker has some reasonable time advantage.

Paper outline. The paper is organized as follows. Section 2 presents a technical overview of the Bitcoin digital currency network. Section 3 briefly explains what a double-spend attack is. Section 4 presents an overview of the double-spend attack models of S. Nakamoto and M. Rosenfeld. Section 5 presents the main contribution of this paper, namely, the generalized and time-based attack models proposed by the authors. Section 6 gathers some information obtained from experimentation on the double-spend attack models and establishes some properties about these models. Finally, Section 7 gathers some concluding remarks and future work.

2 Bitcoin Overview

This section presents a summary of Bitcoin and its dynamics. The user is referred to [7] and [3] for a more comprehensive explanation.

2.1 Bitcoin: the Network and the Currency

The term *Bitcoin* is used for two different notions [3]: (i) a payment service based on a decentralized network and (ii) the name of the currency used in the Bitcoin payment service (often abbreviated as BTC). Bitcoin's dynamics rely on the work of the members of a P2P network called the *Bitcoin network* whose purpose is to update and maintain the stability of a public database called the *blockchain*. The blockchain plays the role of bookkeeping by registering *all* the transactions that have ever taken place in the Bitcoin network. This network does not rely on a trusted third-party that can verify whether a digital coin has been spent. Instead, it relies on a 'proof of work' principle in which users can execute payments by digitally signing their transactions and registering them in the blockchain.

The nodes of the Bitcoin network are called *miners* because their main job is to group new transactions together in packets called *blocks* and to *mine* them. That is, miners try to solve the difficult problem of finding an appropriate nonce according to the block's information, which requires a significant amount of computational power. In this sense, miners compete to mine blocks. Once a miner fully mines a block, its solution is broadcast to the network and it receives 25 BTC as reward for the successful mining task. This is how the Bitcoin network issues currency and promotes miners to work. In the Bitcoin network it is common that once a block is mined, each one of the miners will stop mining and will start mining a fresh new block (even if the transactions in this new block are exactly the ones in a previously mined block).

2.2 Accounts and Transactions

Anyone can create a Bitcoin account, even offline or without the purpose of becoming a miner. Because accounts can be created offline, it is not possible to know how many user accounts are there. BTCs can be sent to any account, even to those created offline with the network confirming, in this case, that such a Joe Doe is entitled to spend that money. With an account, an user can create a transaction. A transaction can have more than one input account and more than one output account; it can also include a small tip for the miner that fully mines a block containing it.

Technically, a Bitcoin account is a public-private key pair based on asymmetric cryptography [8]. Each one of the keys plays the following roles:

- A public key acts as an account number or account identifier. BTCs can be sent to an account by just putting its corresponding public key as an output account in a given transaction.
- A private key is the digital representation of ownership of an account, that is, it is the mechanism used for entitling someone to be an account's owner.

Transaction 987				
Inputs			Outputs	
Transaction	Account	Value	Account	Value
198	A_0	1.30	B	1.70
432	A_1	0.35	A_0	0.24
258	A_1	0.20		
Fee for the miner who mines this first: 0.01				
Digitally signed by A_0 and A_1				

Transaction 1005				
Inputs			Outputs	
Transaction	Account	Value	Account	Value
987	A_0	0.24	C	0.4
987	B	1.70	A_0	0.04
			B	1.5
Fee for the miner who mines this first: 0.0				
Digitally signed by A_0 and B				

Figure 1. Example of transactions, with transaction 1005 depending on transaction 987.

Consequently, the holder of an account's private key has the ability to spend the BTCs in that account by signing transactions with the private key.

The balance of BTCs in a Bitcoin account is not represented by a single number. Instead, any balance is represented by pointing to the previous transactions in the blockchain that justify the ownership of the money. That is, a balance is a trace to the time in which each of its BTCs was first created.

For example, consider Transaction 987 depicted in in Figure 1, in which Alice wants to pay 1.7 BTC into Bob's account B. Alice uses two accounts, namely, A_0 and A_1. She has received 1.3 BTC in account A_0 (Transaction 198), 0.35 BTC in account A_1 (Transaction 432), and 0.2 BTC in A_1 (Transaction 258). The Bitcoin network requires that transactions consume all the inputs. Therefore, Alice must add one of her accounts to the output list of this transaction in order to have back the remaining change. In Transaction 1005, which depends on Transaction 987, Alice and Bob want to pay Charlie 0.2 BTC each.

In the Bitcoin network, the standard reward for a miner that first mines a block of transactions consists of 25 BTC. Also, fees are a mechanism used to tip those

Nonce
Pointer to a block already in the blockchain
Hash of block's body
Transaction 1465
Transaction 1504
⋮
Transaction 1433
Transaction 1505
Transaction 1492
Special transaction to the miner

Figure 2. Main parts of a block.

miners that include a transaction in a block of transactions to be mined. For example, Transaction 987 tips 0.01 BTC to those miners who successfully mine a block containing that transaction. However, as show by Transaction 1005, fees are optional. In this latter case, the miner who first mines the block will receive exactly the standard 25 BTC for mining a block but will not be tipped. The standard reward of 25 BTC for mining a block is likely to change in the future, when bitcoins will not to be issued by the system for mining a block.

Finally, note that the system somehow needs to identify if the output of a transaction has been already spent. Otherwise, an user could create multiple transactions from the same source of income. In the Bitcoin network, miners will check from the blockchain that there are enough funds for the transaction, before mining a block with such a transaction.

2.3 Mining a Block of Transactions

A block consists of three parts, namely, a *nonce*, a *header*, and a *body*:

- The nonce is a hash of the block's header having many leading zeros (around 17 these days).
- The header comprises a "hash-pointer" to a previous block and the hash of the block's body.
- The body comprises a set of transactions, including one that pays 25 BTC to an account of the miner.

Mining a block corresponds to finding a suitable value for the block's nonce which, as mentioned earlier, is a highly-demanding computational task.

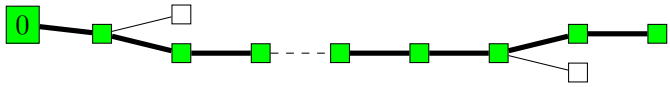


Figure 3. Example of a blockchain with its original block (0), its valid chain (filled), and two branches of length one each (shallow).

2.4 The Blockchain

The blockchain contains all transactions that have ever taken place and it plays the role of bookkeeper of the Bitcoin network. Technically, the blockchain is a tree of blocks with one very long branch, which makes it look like a single chain. In order to avoid confusion, the main branch in this tree will be called the *valid chain*, while other branches will be called *branches* (from the valid chain). Blocks in the valid chain are called *valid blocks*. Branches may appear because of delays in the network, users trying different scripts, or because of attacks. Nevertheless, the length of these branches is very small when compared to the length of the valid chain. Figure 3 depicts an example of a blockchain.

One important observation about the blockchain in Figure 3 is that the last filled block, from left to right, is considered valid. This is because valid nodes form a larger branch having 2 nodes when compared to invalid nodes which have branch length 1. In general, when there is a tie in the length of branches derived from the valid branch, the next valid block is chosen to be that in which the last transaction was first announced. Also, due to time lag, it can happen that some nodes consider the wrong branch to be the valid one, but this is very unlikely in practice for a significant amount of nodes.

As mentioned earlier in this section, mined blocks contain a hash-pointer to their previous block. Therefore, the blockchain can be seen as a collection of linked lists with hash-pointers acting as back-pointers. Note that this helps in preventing any updates from easily happening to the internal tree structure of the blockchain.

3 What is a Double-spend Attack?

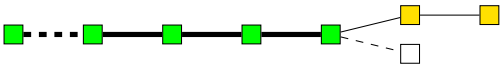
This section presents an overview of how double-spend attacks can happen in the Bitcoin network. For more details about this type of attack, the reader is referred to [4]; for information about other types of attack to the Bitcoin network, the reader is referred to [5].

It is known that falsifying a digital signature is a hard computational problem [8]. Therefore, it is practically useless to try to modify a valid transaction that has been signed. However, there's still a technique that can be used to deceive someone about the state of a transaction. This kind of attack to the Bitcoin network is called a *double-spend attack*. Even though it requires a tremendous computational power and it is very likely to fail, it may be profitable. As a matter of fact, it has happened already [6].

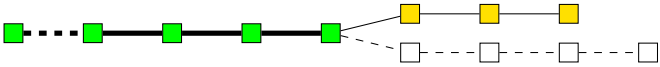
A double-spend attack can be performed as follows:

(i) The attacker A wants a service or a product from B.

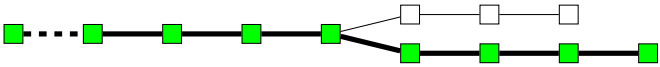
(a) Initial state of the blockchain in which all transactions are considered as valid.



(b) Honest nodes continue extending the valid chain by putting yellow blocks, while the attacker secretly starts mining a fraudulent branch.



(c) The attacker succeeds in making the fraudulent branch longer than the honest one.



(d) The attacker's branch is published and is now considered the valid one.

Figure 4. A successful double-spend attack.

(ii) A creates two transactions: one paying to B and another paying to himself, using the the same input for the transactions.

(iii) A publishes the "A to B" payment and secretly starts mining a block containing the "A to A" payment. Once the latter mining task succeeds, it continues to add blocks after it.

(iv) B gives the product or service to A, since the payment was confirmed or B did not wait long enough.

(v) A is lucky and the fraudulent branch becomes longer than the valid one. The attacker nodes publish all the blocks in the new branch and all the nodes agree on considering them as the valid ones since the branch is longer than the current valid branch.

(vi) B gave out the product or service to A without receiving any payment. At this point, B can not find A because it is anonymous or has left.

Figure 4 depicts some stages of a successful double-spend attack. Stage (a) depicts the initial state of the blockchain. In Stage (b), honest nodes are extending the valid chain by putting valid blocks, while the attacker secretly starts mining a fraudulent branch. In Stage (c), the attacker succeeds in making the fraudulent branch longer than the honest one. Finally, in Stage (d), the attacker's branch is published and is now considered the valid one.

4 Two Existing Attack Models

This section presents an overview of the double-spend attack models by S. Nakamoto [7] and M. Rosenfeld [9], together with some notation and observations. Some of these

notation and observations are also used in Section 5 for presenting the generalized and time-based double-spend attack models proposed by the authors.

The attack models by S. Nakamoto and M. Rosenfeld use similar notions and definitions:

- Quantity $q \in [0,1]$ is the probability of the attacker nodes mining a block first than the honest nodes when they all start mining at the same time. This is equivalent to saying that q is the proportion of attacker's computation power with respect to the total computational power in the network (see Proposition 4.1).
- Quantity $K \in \mathbb{N}$ is the minimum number of confirmations required to accept a block and its transactions as valid. This quantity is set by each seller and not by the network itself.
- Quantity $\tau \in \mathbb{R}_{>0}$ is the average time in seconds required for the whole network (i.e., honest and attacker nodes) to mine a block.

Additionally, functions exclusively used in the model of S. Nakamoto will have N as subscript, while those exclusively used in the model by M. Rosenfeld will have R as subscript. Functions shared between these two models can be written without any subscripts.

Each one of the attack models in this paper, including the models proposed in Section 5, is introduced in terms of three measures. Namely, each model is described in terms of the probability of an attacker successfully performing a double-spend attack under some assumptions (DS), a potential (P) progress function, and a catch-up function (C). Function DS depends on functions P and C, and it intuitively measures the double-spend attack vulnerability of the network as a probability. The potential progress function P describes the expected attacker's branch length once the valid branch is long enough, while the catch-up function C describes the probability of performing a double-spend attack given the expected attacker's branch length.

The ultimate goal of a double-spend attack model is to observe the behavior of function DS in terms of its parameters, as follows:

(i) $DS_N(q,K)$ and $DS_R(q,K)$ represent in the models of S. Nakamoto and M. Rosenfeld, respectively, the probability of an attacker successfully performing a double-spend attack given that the attacker node controls q percent of the network and the honest nodes just mined the K'th block.

(ii) $P_N(q,m,n)$ and $P_R(q,m,n)$ are the the progress functions in the models of S. Nakamoto and M. Rosenfeld, respectively; they represent the probability of an attacker mining exactly n blocks once the honest nodes mine the m'th block.

(iii) $C(q,z)$ represents the probability of an attacker's branch ever becoming longer than the honest one given an initial disadvantage of z blocks.

Since function C is the same for S. Nakamoto's and M. Rosenfeld's models, no subscript is used for it.

4.1 The Model of S. Nakamoto [7]

The double-spend attack model of S. Nakamoto computes the probability of a double-spend attack by combining the probability of the attacker having mined exactly n blocks once the honest nodes mine the K'th confirmation block, with the probability of catching-up from a $K - n$ block difference. The double-spend attack model of S. Nakamoto in [7] considers that the attacker has exactly mined 1 fraudulent block before starting the attack, but it can be easily modified to handle an advantage of n fraudulent blocks.

4.1.1 Attacker's potential progress function

In this model, the potential progress function corresponds to a Poisson distribution given by

$$P_N(q, m, n) = e^{-\lambda}\lambda^n/n!,$$

where $\lambda = mq/(1 - q)$.

4.1.2 The catch-up function

The catch-up function is based on a random walk in which success and failure are given to attacker or honest nodes mining a block, respectively.

Both S. Nakamoto and M. Rosenfeld agree on this function to be given by

$$C(q, z) = \begin{cases} \left(\frac{q}{p}\right)^{z+1} & \text{, if } q < 0.5 \wedge z > 0 \\ 1 & \text{, otherwise,} \end{cases}$$

where q represents the computational power of the attacker, $p = 1 - q$, and z is the initial disadvantage of the attacker.

4.1.3 Double-spend attack probability

The probability of a double-spend attack in S. Nakamoto's model is the probability of the attacker progressing from 1 precomputed block to n blocks and then catching up from a difference of $K - n$ blocks:

$$\begin{aligned} DS_N(q, K) &= \sum_{n=0}^{+\infty} P_N(q, K, n) C_N(q, K - n - 1) \\ &= 1 - \sum_{n=0}^{K} P_N(q, K, n)(1 - C_N(q, K - n - 1)). \end{aligned}$$

4.2 The Model of M. Rosenfeld [9]

The double-spend attack model of M. Rosenfeld uses the same approach used by S. Nakamoto. It also considers an initial advantage of 1 fraudulent block.

4.2.1 Attacker's potential progress function

M. Rosenfeld's potential progress function corresponds to a negative binomial distribution given by

$$P_R(q,m,n) = \begin{cases} 1 & \text{, if } m = n = 0 \\ \binom{m+n-1}{n} q^n (1-q)^m & \text{, otherwise,} \end{cases}$$

where $P_R(q,m,n)$ is the probability of the attacker having mined n blocks once the honest nodes mine the m'th block.

4.2.2 Catch-up function

M. Rosenfeld uses the same catch-up function C originally proposed by S. Nakamoto.

4.2.3 Double-spend attack probability

The double-spend attack probability in this model is denoted DS_R and it is given by:

$$\begin{aligned} DS_R(q,K) &= \sum_{n=0}^{+\infty} P_R(q,K,n) C_R(q,K-n-1) \\ &= 1 - \sum_{n=0}^{K} P_R(q,K,n)(1 - C_R(q,K-n-1)). \end{aligned}$$

Finally, it is shown that in these two models (as it is also the case for the attack models proposed in Section 5) there is a direct relation between computational power and the probability of first mining a block.

Proposition 4.1 *The probability of mining a block first than the others given a proportion of computational power of q with respect to the total computational power in the network is exactly q.*

Proof Since the probability of mining a block in a single trial is constant and independent of previous trials, the time needed for mining a block follows an exponential distribution. Let T denote the random variable describing the time needed to mine a block when using all the power available in the network. The density function for T is given for $x \geq 0$ by

$$f(x) = \frac{1}{\tau} e^{-\frac{1}{\tau}x},$$

where τ is the expected time, so that the probability of mining a block per time unit is $\frac{1}{\tau}$. Since the power in the network is proportional to the amount of hashes computed per time unit, the probability of mining a block per time unit is also proportional to such a power. Therefore, the probabilities of mining a block per time unit for the attacker and for the honest nodes will be $\frac{q}{\tau}$ and $\frac{p}{\tau}$, respectively. Let T_p and T_q be random variables denoting the time needed by the honest nodes

and the attacker nodes to mine a block, respectively. Then, the probability that the attacker nodes mine a block faster than the honest nodes is given by:

$$\begin{aligned} P(T_q < T_p) &= \int_0^\infty P(T_q = x)P(T_p > x)dx \\ &= \int_0^\infty \frac{q}{\tau} e^{-\frac{q}{\tau}x} e^{-\frac{p}{\tau}x} dx \\ &= q \int_0^\infty \frac{1}{\tau} e^{-\frac{1}{\tau}x} dx \\ &= q. \end{aligned}$$

□

5 Two New Attack Models

This section presents the generalized model and the time-based model for double-spend attacks proposed by the authors. Some of the notation, conventions, and functions used in this section are borrowed from Section 4. In addition, functions defined exclusively for the time-model have T as a subscript.

5.1 The Generalized Model

This is the first double-spend attack model proposed by the authors. This model results from generalizing the model of M. Rosenfeld by adding an extra parameter denoting some time advantage assigned to the attacker, i.e., time in which the attacker has been secretly trying to mine blocks. Henceforth, this extension is called the *generalized model*.

5.1.1 Attacker's potential progress function

It is represented by function

$$P(q, m, n, t)$$

which generalizes the potential progress functions in Section 4. Function P denotes the probability of an attacker mining exactly n blocks once the honest nodes mine the m'th block, assuming that the attacker has been mining secretly during $t\tau$ seconds. The extra parameter t represents the time advantage to be assigned to the attacker nodes towards fraudulent block production.

In order to define the progress function P for this model, it is necessary to consider the possible number of blocks that could have been mined during $t\tau$ seconds. Let $a(q, t, n)$ denote the probability of mining exactly n blocks if constantly mining during $t\tau$ seconds with a proportion of q hashpower respect to the total power in the network. Proposition 5.1 proposes a closed formula for computing a.

Proposition 5.1 *If $t > 0$ or $n > 0$, then*

$$a(q,t,n) = \frac{(qt)^n}{n!}e^{-qt}.$$

Proof Note that $a(q,t,0)$, which is the probability of mining 0 blocks in $t\tau$ seconds, represents the probability of mining the first block in a time greater than $t\tau$ seconds. That is

$$a(q,t,0) = \int_t^{+\infty} qe^{-qs}ds = e^{-qt}.$$

Consider $a(q,t,n) = \frac{(qt)^n}{n!}e^{-qt}$. The goal is to show that

$$a(q,t,n+1) = \frac{(qt)^{n+1}}{(n+1)!}e^{-qt}.$$

If s is the moment at which the next block is mined (in case there is a next block), then:

$$\begin{aligned}
a(q,t,n+1) &= \int_0^t qe^{-qs}a(q,t-s,n)ds \\
&= \int_0^t qe^{-qs}\frac{(q(t-s))^n}{n!}e^{-q(t-s)}ds \\
&= \int_0^t qe^{-qt}\frac{(q(t-s))^n}{n!}ds \\
&= qe^{-qt}\frac{q^n}{(n+1)!}\left((t-0)^{n+1} - (t-t))^{n+1}\right) \\
&= \frac{(qt)^{n+1}}{(n+1)!}e^{-qt}.
\end{aligned}$$

□

Thanks to Proposition 5.1, function P can be defined as

$$P(q,m,n,t) = \sum_{z=0}^{n} a(q,t,z)P_R(q,m,n-z),$$

where

$$a(q,t,n) = \begin{cases} 1 & \text{, if } t = n = 0 \\ 0 & \text{, if } t <= 0 \\ \frac{(qt)^n}{n!}e^{-qt} & \text{, otherwise.} \end{cases}$$

Note that functions P_R and P_N (introduced in Section 4) differ from P only in that they assume $t = 0$. Thus, it can be expected that the following assertion holds:

$$P_N(q,m,n) \approx P_R(q,m,n) = P(q,m,n,0).$$

5.1.2 Catch-up function

It uses the catch-up function C originally proposed by S. Nakamoto (see Section 4).

5.1.3 Double-spend attack probability

It is modeled by function

$$DS(q, K, n, t),$$

denoting the probability of an attacker successfully performing a double-spend attack given an initial advantage of n blocks and $t\tau$ seconds over the honest nodes.

Function DS is defined by:

$$\begin{aligned} DS(q, K, n, t) &= \sum_{z=0}^{+\infty} P(q, K, z, t) C_R(q, K - n - z). \\ &= 1 - \\ &\quad \sum_{z=0}^{K-n} P(q, K, z, t)(1 - C_R(q, K - n - z)). \end{aligned}$$

Note that functions DS_N and DS_R (introduced in Section 4) differ from DS only in that they assume $t = 0$ and an initial advantage of $n = 1$ blocks. In Section 6, it is shown that the following assertion holds:

$$DS_N(q, K) \approx DS_R(q, K) = DS(q, K, 1, 0).$$

5.2 The Time-based Model

This section presents the time-based model, the second double-spend attack model proposed by the authors. On the one hand, this model is as flexible as the generalized model in Section 5.1, and thus more general than the models of S. Nakamoto and M. Rosenfeld in Section 4. On the other hand, it uses a completely different approach to compute the probability of an attacker successfully performing a double-spend attack.

In the time-based model, states are identified by the length of the valid and fraudulent branches, which are assumed to have the same length, and the time difference at which both the honest and attacker nodes mined their last block. Namely, a state in the time-based model consists of two values t and n denoting the time difference t at which both the honest and attacker nodes mined their n'th block. It is key to note that the states in the time-based model differ from the previous attack-models in that the latter represents the states in terms of the length of the branches where the attacker and the honest nodes are mining, and they can be different. In the time-based model, sometimes it is enough to focus on the time parameter t, leaving the length parameter n to represent the maximum such length so that both the attacker and the honest nodes have mined their n'th block.

5.2.1 Attacker's potential progress function in time

It is represented by function

$$P_T(q, m, n, t)$$

denoting the probability of the time at which the n'th block mined by an attacker node is exactly (in the sense of a probability density function) $t\tau$ seconds after the time at which the m'th block is mined by a honest node.

As it is expressed in Proposition 4.1, and since the probability of mining a block in a single trial is constant and independent of previous trials, the time needed for mining a block follows an exponential distribution with expected value $\lambda = \frac{1}{q\tau}$. Consequently, the time $T_{q,n}$ needed to mine n blocks with hashpower q is given by the sum of the outcome of n independent exponential distributions, which follows an Erlang distribution [12] (i.e., a Gamma distribution with integer parameter), with shape n and scale $\mu = \frac{1}{q}$ having probability density function:

$$h_{q,n}(t) = \frac{t^{n-1}q^n e^{-qt}}{(n-1)!}.$$

In this way, an attacker's potential progress function in time P_T is the probability density function of the random variable $X = T_{q,n} - T_{p,m}$, with independent $T_{q,n}$ and $T_{p,m}$. Adopting this observations, function P_T is defined as (see [2] for more technical details):

$$P_T(q, m, n, t) = \int_{-\infty}^{+\infty} h_{q,n}(x)h_{p,m}(t-x)dx.$$

On a side note and given its complexity, for the experiments reported on Section 6, the authors have opted for integrating the expression using numerical packages instead of using directly its closed mathematical form.

5.2.2 Catch-up function

It is represented by function

$$C_T(q, t)$$

denoting the probability of an attacker's branch ever becoming longer than the valid one given that the honest nodes mined their last block $t\tau$ seconds before the attacker mined its last block.

For this model, it is required that function C_T satisfies the following two conditions:

(i) $C_T(q, t) = 1$ whenever $t < 0$, covering the case in which an attacker has mined the last block earlier than the honest nodes, meaning that the attacker's branch was already longer.

(ii) Since the probability of catching up from a time difference is equal to the probability of catching up with the next block plus the probability of catching up from the time difference once the next block is mined,

$$C_T(q, t) = \int_{-\infty}^{+\infty} P_T(q, 1, 1, x)C_T(q, x + t)dx.$$

A function that satisfies both requirements (i) and (ii) is the following function C_T, with $p = 1 - q$:

$$C_T(q,t) = \begin{cases} \frac{q}{p}e^{-(p-q)t} & \text{, if } t > 0 \\ 1 & \text{, otherwise,} \end{cases}$$

The correctness of this definition can be verified from the definition of function P_T and from the following observation:

$$P_T(q,1,1,t) = \begin{cases} pqe^{-qt} & \text{, if } t > 0 \\ pqe^{pt} & \text{, otherwise.} \end{cases}$$

5.2.3 Double-spend attack probability

It is represented by function

$$DS_T(q,K,n,t)$$

denoting, similar to DS in the generalized model, the probability of an attacker successfully performing a double-spend attack given an initial advantage of n blocks and $t\tau$ seconds over the honest nodes.

The double-spend attack probability in the time-based model DS_T can be expressed as the probability of having a time disadvantage of $t\tau$ seconds once the $K+1$'th block is mined times the probability of catching up from that disadvantage. Note that an attack will conclude with $K+1$ blocks and not with K blocks.

Function DS_T is defined by:

$$DS_T(q,K,n_0,t_0) = \int_{-\infty}^{+\infty} P(q,K+1,K-n_0+1,t)C_T(q,t-t_0)dt.$$

6 Model Comparison

This section summarizes experimental results obtained on the two attack models proposed by the authors in Section 5, namely on the generalized model and the time-based model for double-spend attacks in the Bitcoin network. In particular, the experimental results presented in this section are compared against the models of S. Nakamoto and M. Rosenfeld in Section 4. Additionally, this section includes snippets with the code, in the Python 3 programming language, used to run all the experiments.

One of the main conclusions of this section, supported mostly by the experimental data and regarding the correctness of the proposed models, is twofold:

(i) The generalized model proposed in this paper and the hashrate-based model of M. Rosenfeld [9] have the same results when predicting the probability of double-spend attacks in the Bitcoin network.

(ii) The generalized and the time-based models coincide.

In addition, the experimental data suggests that:

- The probability of a double-spend attack increases as the number of confirmations decreases or the attacker's power increases, which is a well-known fact about Bitcoin.
- Double-spend attacks are very unlikely in practice, when an attacker has limited time advantage.
- If an attacker gains control of 40% of the network's computational power, then double-spend attacks are almost impossible to contain. Since this situation is very unlikely, the best containment method for these sort of attacks is to set a high number of confirmations, as the double-spend attack models studied in this paper suggest.
- The model of S. Nakamoto behaves slightly different to the other three models.

6.1 $P_N(q,m,n) \approx P_R(q,m,n) = P(q,m,n,0)$

This is actually a fact which requires no experimental data to be shown. The 'equation' $P_N(q,m,n) \approx P_R(q,m,n)$ is a consequence of the similarities between the Poisson distribution and the Negative Binomial distribution. Moreover, the equation $P_R(q,m,n) = P(q,m,n,0)$ follows from

$$\begin{aligned} P(q,m,n,0) &= \sum_{z=0}^{n} a(q,0,z) P_R(q,m,n-z) \\ &= a(q,0,0) P_R(q,m,n) \\ &\quad + \sum_{z=1}^{n} a(q,0,z) P_R(q,m,n-z) \\ &= P_R(q,m,n), \end{aligned}$$

given that

$$a(q,0,n) = \begin{cases} 1 & \text{, if } n = 0 \\ 0 & \text{, otherwise.} \end{cases}$$

6.2 $DS_N(q,K) \approx DS_R(q,K) = DS(q,K,1,0)$

This is one of the main conclusions of this paper and it is also a fact that can be proven without experimental data. This follows from the definition of functions denoting the probability of an attacker successfully performing a double-spend attack, in the corresponding model, and the equation $P_R(q,m,n) = P(q,m,n,0)$ above-stated.

However, in order to give a more precise idea of how these functions behave, some experimental data in figures 5 and 6, and in Table 1 is included. In Figure 5, the plots of DS vs K are shown under two different cases, namely $q = 10\%$ and $q = 20\%$. Note that the curves representing DS_R and DS overlap, while the curve representing DS_N does not. For more detail, Figure 6 includes a zoomed version of

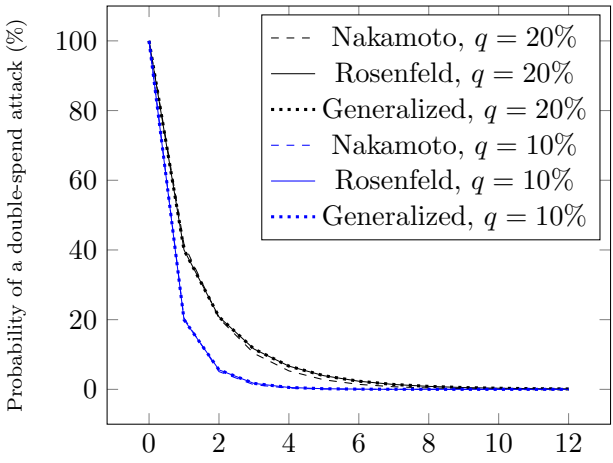


Figure 5. Comparison of the number of confirmations K for accepting a transaction as valid vs. the probability of a double-spend attack in three models (DS_N, DS_R, and DS). The curves appear to be the same but Nakamoto's are not.

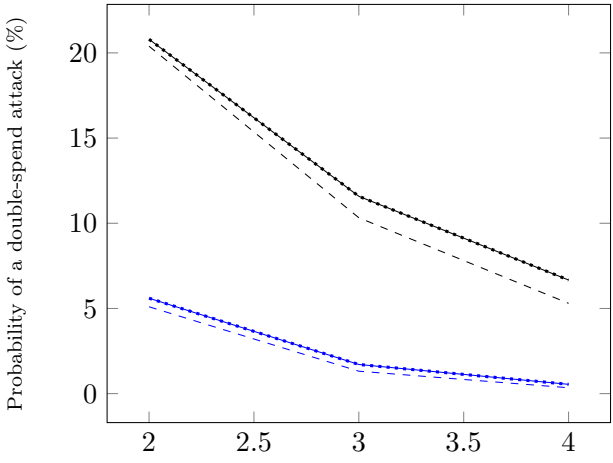


Figure 6. Figure 5 partially zoomed. It becomes clearer that Nakamoto's curves differ from the other two.

Figure 5 in which the gap between the curves is more noticeable. This experiment supports the claim about the model of S. Nakamoto not being entirely accurate for modeling realistic double-spend attacks in the Bitcoin network, as previously noted by M. Rosenfeld. Table 1 includes the values of DS_N, DS_R, and DS for some values for K and q, without any time advantage for the attacker nodes.

6.3 $DS(q, K, n_0, t_0) = DS_T(q, K, n_0, t_0)$

This claim is also a main result in this paper. Although it is not yet mathematically proven, the experimental data strongly supports this claim, as shown below.

Table 2 includes the values of DS and DS_T as a function of K and q assuming $n_0 = 0$ and $t_0 = 2.5$. This table summarizes a scenario in which the last block

Table 1
Probability of a double-spend attack according to three models (DS_N, DS_R, and DS) vs the hashpower q and the number of confirmations K for accepting a transaction as valid.

		Number of confirmations							
Model	q	0	1	2	3	4	5	6	7
Nakamoto	0%	100.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Rosenfeld	0%	100.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Generalized	0%	100.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Nakamoto	1%	100.00%	2.00%	0.05%	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0
Rosenfeld	1%	100.00%	2.00%	0.06%	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0
Generalized	1%	100.00%	2.00%	0.06%	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0
Nakamoto	5%	100.00%	10.12%	1.26%	0.16%	0.02%	≈ 0	≈ 0	≈ 0
Rosenfeld	5%	100.00%	10.00%	1.45%	0.23%	0.04%	0.01%	≈ 0	≈ 0
Generalized	5%	100.00%	10.00%	1.45%	0.23%	0.04%	0.01%	≈ 0	≈ 0
Nakamoto	10%	100.00%	20.46%	5.10%	1.32%	0.35%	0.09%	0.02%	0.01%
Rosenfeld	10%	100.00%	20.00%	5.60%	1.71%	0.55%	0.18%	0.06%	0.02%
Generalized	10%	100.00%	20.00%	5.60%	1.71%	0.55%	0.18%	0.06%	0.02%
Nakamoto	25%	100.00%	52.23%	31.54%	19.61%	12.35%	7.84%	4.99%	3.19%
Rosenfeld	25%	100.00%	50.00%	31.25%	20.70%	14.11%	9.79%	6.87%	4.86%
Generalized	25%	100.00%	50.00%	31.25%	20.70%	14.11%	9.79%	6.87%	4.86%
Nakamoto	40%	100.00%	82.89%	73.64%	66.42%	60.34%	55.06%	50.40%	46.23%
Rosenfeld	40%	100.00%	80.00%	70.40%	63.49%	57.96%	53.31%	49.30%	45.77%
Generalized	40%	100.00%	80.00%	70.40%	63.49%	57.96%	53.31%	49.30%	45.77%
Nakamoto	49%	100.00%	98.50%	97.77%	97.21%	96.74%	96.32%	95.94%	95.59%
Rosenfeld	49%	100.00%	98.00%	97.00%	96.25%	95.63%	95.08%	94.59%	94.14%
Generalized	49%	100.00%	98.00%	97.00%	96.25%	95.63%	95.08%	94.59%	94.14%

was published 2.5τ seconds ago, where τ is the average time between blocks, time in which the attacker has been mining but has not published any mined block (in case there is any). The more dense the time axis is in function P_T (default is $30K$, see Section 6.6.5 for more detail), the closer the results are. For small values of q, the curve P_T extends rapidly towards the right, increasing the window length, thus diminishing the absolute density of the time axis and the precision of function DS.

Table 3 includes the values of DS and DS_T as a function of K and q assuming $n_0 = 1$ and $t_0 = 0.5$. Notice that the results are similar, specially when $q \not\approx 0$.

6.4 Other results

The way in which the model of M. Rosenfeld was generalized in Section 5.1 allows for new properties to be studied. In fact, the generalized model as well as the time-based model are still consistent if $n_0 < 0$, which corresponds to the scenario in which the attacker nodes are not mining at the top of the blockchain, but in a branch $|n_0|$ blocks behind of the valid branch.

Figure 7 depicts the probability of a double-spend attack as the time advantage increases, which is the main feature of the models proposed in this paper. Note that if enough time has elapsed, it is highly possible that the attacker nodes have

Table 2
Comparison of DS vs DS_T assuming $n_0 = 0$ and $t_0 = 2.5$. $30K$ points used for time axis.

		Number of confirmations							
Model	q	0	1	2	3	4	5	6	7
Generalized	0%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Time-based	0%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Generalized	1%	3.45%	0.11%	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0
Time-based	1%	4.24%	0.17%	0.01%	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0
Generalized	5%	16.40%	2.50%	0.40%	0.07%	0.01%	≈ 0	≈ 0	≈ 0
Time-based	5%	16.34%	2.52%	0.41%	0.06%	0.01%	≈ 0	≈ 0	≈ 0
Generalized	10%	30.77%	8.97%	2.73%	0.86%	0.28%	0.09%	0.03%	0.01%
Time-based	10%	30.71%	8.92%	2.70%	0.85%	0.28%	0.09%	0.03%	0.01%
Generalized	25%	64.32%	40.90%	26.94%	18.24%	12.59%	8.79%	6.20%	4.41%
Time-based	25%	64.31%	40.88%	26.95%	18.25%	12.57%	8.79%	6.20%	4.40%
Generalized	40%	87.74%	77.44%	69.58%	63.31%	58.09%	53.61%	49.70%	46.22%
Time-based	40%	87.74%	77.44%	69.57%	63.30%	58.09%	53.61%	49.69%	46.21%
Generalized	50%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%
Time-based	50%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%

Table 3
Comparison of DS vs DS_T assuming $n_0 = 1$ and $t_0 = 0.5$.

		Number of confirmations							
Model	q	0	1	2	3	4	5	6	7
Generalized	0%	100.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Time-based	0%	100.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Generalized	1%	100.00%	2.49%	0.08%	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0
Time-based	1%	100.00%	4.54%	0.18%	0.01%	≈ 0	≈ 0	≈ 0	≈ 0
Generalized	5%	100.00%	12.22%	1.80%	0.29%	0.05%	0.01%	≈ 0	≈ 0
Time-based	5%	100.00%	12.53%	1.85%	0.30%	0.05%	0.01%	≈ 0	≈ 0
Generalized	10%	100.00%	23.90%	6.78%	2.08%	0.66%	0.22%	0.07%	0.02%
Time-based	10%	100.00%	23.94%	6.74%	2.06%	0.66%	0.21%	0.07%	0.02%
Generalized	25%	100.00%	55.88%	35.19%	23.36%	15.94%	11.06%	7.76%	5.49%
Time-based	25%	100.00%	55.87%	35.19%	23.36%	15.93%	11.06%	7.77%	5.49%
Generalized	40%	100.00%	83.63%	73.80%	66.61%	60.83%	55.97%	51.77%	48.06%
Time-based	40%	100.00%	83.63%	73.79%	66.60%	60.83%	55.97%	51.78%	48.07%
Generalized	50%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%
Time-based	50%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%

already built a fraudulent branch long enough to be considered valid. Also note that the two proposed models experimentally coincide.

Figure 8 shows an extension to negative values of the plots of DS and DS_T assuming $K = 6$, which is a commonly accepted value, and $t_0 = 0$ in order to be comparable with the previous models.

Table 4 considers a scenario in which the attacker has been mining during 2.7τ seconds and from the time it started mining, one valid block has been published.

6.5 Code Snippets

This section includes snippets with the code, in the Python 3 programming language, used to run the experiments presented earlier in this section.

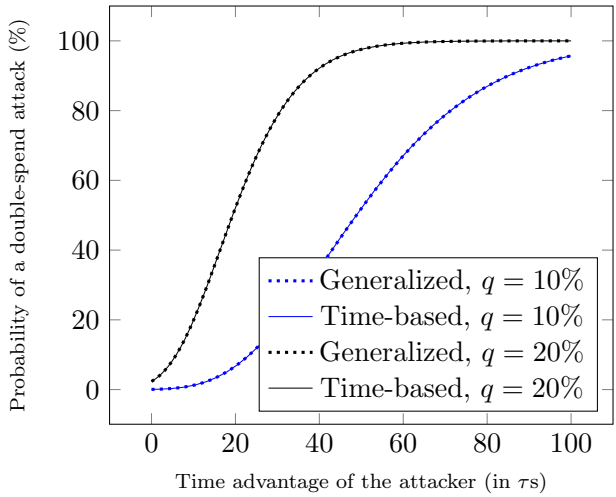


Figure 7. Probability of a double spend as a function of the time advantage, assuming $K = 6$ and $n_0 = 1$. The outputs of the proposed models appear to be the same.

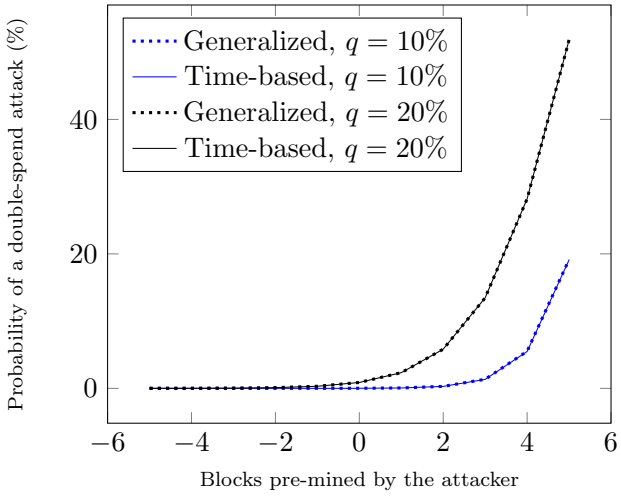


Figure 8. Probability of a double spend as a function of the block advantage, assuming $K = 6$ and $t_0 = 0$. The outputs of the proposed models appear to be the same. Negative values for n_0 represent an attacker which is not mining in the top of the blockchain, thus it is already losing the race against the honest nodes.

6.5.1 Model of S. Nakamoto

First, the code used to model the attack model by S. Nakamoto is presented. The catch-up function is implemented with function `C`, as follows:

```
def C(q,z):
  if z<0 or q>=0.5: prob = 1
  else: prob = (q/(1-q)) ** (z+1)
  return prob
```

The probability of success of a double-spend attack in the model of S. Nakamoto is implemented with functions `P_N` and `DS_N`, as follows:

```
from scipy.stats import poisson
def P_N(q,m,n):
  return poisson.pmf(n, m*q/(1-q))
```

Table 4
Comparison of DS vs DS_T assuming $n_0 = -1$ and $t_0 = 2.7$.

		Number of confirmations							
Model	q	0	1	2	3	4	5	6	7
Generalized	1%	0.07%	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0
Time-based	1%	≈ 0	0.01%	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0
Generalized	5%	1.70%	0.25%	0.04%	0.01%	≈ 0	≈ 0	≈ 0	≈ 0
Time-based	5%	1.62%	0.27%	0.04%	0.01%	≈ 0	≈ 0	≈ 0	≈ 0
Generalized	10%	6.28%	1.80%	0.55%	0.18%	0.06%	0.02%	0.01%	≈ 0
Time-based	10%	6.27%	1.81%	0.55%	0.17%	0.06%	0.02%	0.01%	≈ 0
Generalized	25%	31.83%	20.14%	13.38%	9.13%	6.34%	4.45%	3.15%	2.25%
Time-based	25%	31.78%	20.13%	13.39%	9.13%	6.33%	4.45%	3.16%	2.25%
Generalized	40%	68.91%	60.93%	55.01%	50.26%	46.26%	42.80%	39.75%	37.03%
Time-based	40%	68.89%	60.93%	55.01%	50.26%	46.26%	42.80%	39.74%	37.03%
Generalized	49%	96.57%	95.63%	94.91%	94.30%	93.77%	93.29%	92.85%	92.44%
Time-based	49%	96.57%	95.63%	94.91%	94.30%	93.77%	93.29%	92.85%	92.44%

```
def DS_N(q,K):
  return 1-sum(P_N(q,K,n)*(1-C(q,K-n-1)) for n in range(0,K+1))
```

6.5.2 Model of M. Rosenfeld

Next, functions `P_R`, `DS_R`, and an alternative way of computing DS_R using dynamic programming techniques, namely `DS_R_dp`, which corroborates the correctness of the function `DS_R`.

```
from scipy.stats import nbinom

def P_R(q,m,n):
  if q>=0.5 or ((q==0 or m==0) and n==0): prob = 1
  elif q==0 or m==0: prob = 0
  else: prob = ((1-q)/q) * nbinom.pmf(m-1,n+1,q)
  return prob

def DS_R(q,K):
  return 1-sum(P_R(q,K,n)*(1-C(q,K-n-1)) for n in range(0,K+1))

def DS_R_dp(q,K):
  dp1 = [C(q,n-1) for n in range(K+1)]
  for m in range(K):
    dp2 = [1]
    for n in range(K):
      dp2.append( q*dp2[n]+ (1-q)*dp1[n+1] )
    dp1 = list(dp2)
  return dp1[-1]
```

The function `DS_R_dp` is based on a recursive way of computing DS_R:

$$DS_R(q, K) = dp(q, K, K),$$

where

$$dp(q, m, n) = \begin{cases} q \cdot dp(q, m, n-1) + \\ (1-q)dp(q, m-1, n) & \text{, if } n > 0 \wedge m > 0 \\ C(q, m-n-1) & \text{, otherwise,} \end{cases}$$

which can be implemented either iteratively as shown above, or recursively as shown

below:

```
from functools import lru_cache
@lru_cache(maxsize=None) # Enables memorization
def dp(q,m,n):
  if n>0 and m>0:
    ans = (1-q)*dp(q,m-1,n)+ q*dp(q,m,n-1)
  else: ans = C(q,n-m-1)
  return ans
```

6.5.3 Generalized model

The generalized functions `a`, `P`, and `DS` are implemented as follows:

```
def a(q,t,n):
  if t==0 and n==0: prob = 1
  else:
    prob = np.exp(-q*t)
    for i in range(n):
      prob *= (q*t)/(i+1)
  return prob

def P(q,m,n,t):
  return sum(a(q,t,z)*P_R(q,m,n-z) for z in range(0,n+1))

def DS(q,K,n0,t0):
  return 1-sum(P(q,K,n,t0)*(1-C(q,K-n0-n)) for n in range(0,K-n0+1))
```

6.5.4 Time-based model

The time-based model consists of three functions, namely, `C_T`, `P_T`, and `DS_T`, and are presented below. The function `P_T` returns two arrays X and Y (with $30K$ samples), representing $P_T(q, m, n, t)$ as a function of t.

```
import numpy as np
from scipy.signal import fftconvolve
from scipy.stats import gamma

def C_T(q,t):
  if t<0 or q>=0.5: prob = 1
  else: prob = q/(1-q)*np.exp((2*q-1)*t)
  return prob

def P_T(q,m,n):
  hi = abs(n/q-m/(1-q)) + 3*(n/q**2+m/(1-q)**2)
  X = np.linspace(-hi,hi,30001)
  Ynq = gamma.pdf( X, n, scale=1/q )
  Ymp = gamma.pdf( X, m, scale=1/(1-q) )
  Y = fftconvolve(Ynq,Ymp[::-1],'same')
  Y = Y/np.trapz(Y,X)
  return X,Y

def DS_T(q,K,n0,t0):
  if q>=0.5 or n0>K: prob=1
  elif q==0: prob=0
  else:
    T,Y = P_T(q,K+1,K-n0+1)
    Y = [y*C_T(q,t-t0) for t,y in zip(T,Y)]
    prob = np.trapz(Y,T)
  return prob
```

7 Concluding Remarks

This paper has presented two models for double-spend attacks in the Bitcoin network. Overall, both models consider opartial advancement towards block produc-

tion and how fast a block header can be hashed. Previously existing attack models only considered how fast a block header can be hashed. In this regard, the extension with partial advancement towards block creation is a novelty, which to the best of the authors' knowledge, has not been previously reported in the literature.

The first model, called the *generalized model*, extends an attack-model developed by M. Rosenfeld [9] by adding a time parameter, enabling the model to assign time advantage to an attacker trying to achieve a double-spend attack. The second model, called *the time-based model*, follows a completely different approach by taking into account the times at which the honest and attacker nodes last mined a block, in addition to their relative progress in terms of the length of chains already mined. The two models proposed by the authors are more general than existing double-spend attack models.

The generalized and the time-based have been compared to two well-established hashrate-based models where partial production of a block is not considered. Namely, the attack models proposed by the authors in this paper have been extensively compared to the attack models previously developed by S. Nakamoto [7] and M. Rosenfeld [9]. With the help of experimental data, it is strongly suggested that the probability of catching up from a time difference is very similar to the probability of catching up from the new time difference obtained after the next block is found. That is, the new models presented in this paper, and the models of S. Nakamoto and M. Rosenfeld, surprisingly behave in a similar way. This reinforces the believe that double-spend attacks are very unlikely in practice given the current parameters in the Bitcoin network in terms of the number of confirmations required for a transaction to be valid and the number of leading zeroes required to sign a block of transactions. However, the double-spend attacks become a real risk when an attacker has enough time advantage towards fraudulent block production with enough computational power.

As usual, some work remains to be done. Proving that the generalized model and the time-based model coincide is an important and challenging task. Furthermore, it seems promising to specify the attack models in a language with a formal mathematical semantics and exercise them against temporal formulas, perhaps following some of the ideas in [11]. More precisely, it will be interesting to formally specify the models proposed by the authors in the rewrite-based specification language PMaude [1] and then use a statistical model checker such as MultiVeStA [10] to formally verify probabilistic temporal properties related to possible scenarios for double-spend attacks with time advantage in the Bitcoin network. Finally, the two models proposed by the authors in this paper can be used for comparing the effectiveness of double-spend attacks with time advantage against other types of attacks in the Bitcoin network.

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