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## Pion mass and decay constant

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## Abstract

Independent of assumptions about the form of the quark-antiquark scattering kernel we derive the explicit relation between the pion Bethe-Salpeter amplitude,  $\Gamma_{\pi}$ , and the quark propagator in the chiral limit;  $\Gamma_{\pi}$  necessarily involves a non-negligible  $\gamma_5 \gamma \cdot P$  term (P is the pion four-momentum). We also obtain exact expressions for the pion decay constant,  $f_{\pi}$ , and mass, both of which depend on  $\Gamma_{\pi}$ ; and demonstrate the equivalence between  $f_{\pi}$  and the pion Bethe-Salpeter normalisation constant in the chiral limit. We stress the importance of preserving the axial-vector Ward-Takahashi identity in any study of the pion itself, and in any study whose goal is a unified understanding of the properties of the pion and other hadronic bound states.

 ${\it Key\ words:}\ \ {\it Coldstone\ Bosons;}\ {\it Dynamical\ Chiral\ Symmetry\ Breaking;}$ 

Dyson-Schwinger equations; Nonperturbative QCD

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In the strong interaction spectrum the pion is identified as both a Goldstone mode, associated with dynamical chiral symmetry breaking (D $\chi$ SB), and a bound state composed of u- and d-quarks. This dichotomy is interesting because, while  $m_{\rho}/2 \simeq m_N/3 \equiv M_q$ : the constituent-quark mass,  $m_{\pi}/2$  is only  $\approx 0.2 M_q$ ; i.e., the pion is very much less-massive than other comparable strong interaction bound states. The constituent-quark mass,  $M_q \simeq 350 \,\mathrm{MeV}$ , provides an estimate of the "effective mass" of quarks bound in a hadron; and the ratio of this to the renormalisation-group-invariant current-quark mass of u- and d- quarks ( $\hat{m} \sim 10 \,\mathrm{MeV}$ ) indicates the magnitude of nonperturbative dressing effects on light-quark propagation characteristics. The particular nature of the pion can be represented by the question: "How does one form an almost-massless bound state from very massive constituents without fine-tuning?" The answer to this question is at the core of an understanding of

 $D\chi SB$  in terms of the elementary degrees of freedom in QCD.

In addressing this question we employ the Dyson-Schwinger equations [1] (DSEs), which provide a nonperturbative, renormalisable, continuum framework for analysing quantum field theories, and adapt the discussion of Ref. [2]. We begin with the renormalised, homogeneous, pseudoscalar Bethe-Salpeter Equation (BSE) <sup>1</sup>

$$\left[\Gamma_{\pi}^{j}(k;P)\right]_{tu} = \int_{q}^{\Lambda} \left[\chi_{\pi}^{j}(q;P)\right]_{sr} K_{tu}^{rs}(q,k;P),$$
(1)

where k is the relative and P the total momentum of the quark-antiquark pair,  $\chi^j_{\pi}(q;P) \equiv S(q_+)\Gamma^j_{\pi}(q;P)S(q_-)$ ,  $r,\ldots,u$  represent colour, flavour and Dirac indices,  $q_{\pm}=q\pm P/2$ , and  $\int_q^{\Lambda}\equiv \int^{\Lambda}d^4q/(2\pi)^4$  represents mnemonically a translationally-invariant regularisation of the integral, with  $\Lambda$  the regularisation mass-scale. The final stage of any calculation is to remove the regularisation by taking the limit  $\Lambda\to\infty$ . In (1), S is the dressed-quark propagator and K is the fully-amputated quark-antiquark scattering kernel; the important features of both are discussed below.

The homogeneous BSE is an eigenvalue problem. Solutions exist only for particular, separated values of  $P^2$ ; and the eigenvector associated with each eigenvalue, the Bethe-Salpeter amplitude (BSA):  $\Gamma(k; P)$ , is the one-particle-irreducible, fully-amputated quark-meson vertex. In the isovector, pseudoscalar channel the solution associated with the lowest eigenvalue is the pion. This solution of (1) has the general form [3]

$$\Gamma_{\pi}^{j}(k;P) = \tau^{j} \gamma_{5} \left[ i E_{\pi}(k;P) + \gamma \cdot P F_{\pi}(k;P) + \gamma \cdot k k \cdot P G_{\pi}(k;P) + \sigma_{\mu\nu} k_{\mu} P_{\nu} H_{\pi}(k;P) \right].$$

$$(2)$$

In Ref. [4] a "Feynman-like" gauge is employed:  $D_{\mu\nu}(k) \propto \delta_{\mu\nu}$ , in which case the Dirac algebra entails  $H_{\pi} \equiv 0$ . The general defects of such an Ansatz are discussed in Ref. [5]. Herein we note only that the Slavnov-Taylor identity:  $k_{\mu}D_{\mu\nu} = k_{\mu}D_{\mu\nu}^{\text{free}}(k)$ , states that the longitudinal part of the dressed-gluon propagator is *independent* of interactions and hence, even in Feynman gauge,  $D_{\mu\nu}(k) \not\propto \delta_{\mu\nu}$  and  $H_{\pi} \not\equiv 0$ . The interplay between  $F_{\pi}$ ,  $G_{\pi}$ ,  $H_{\pi}$  and  $F_{R}$ ,  $G_{R}$ ,  $H_{R}$ , discussed below in connection with (10)-(13), is also overlooked in Ref. [4].

We employ a Euclidean space formulation with  $\{\gamma_{\mu}, \gamma_{\nu}\} = 2\delta_{\mu\nu}, \ \gamma_{\mu}^{\dagger} = \gamma_{\mu}$  and  $a \cdot b = \sum_{i=1}^{4} a_{i}b_{i}$ . A spacelike vector,  $k_{\mu}$ , has  $k^{2} > 0$ .

Important in (1) is the renormalised dressed-quark propagator, which is obtained from the quark DSE

$$S(p)^{-1} = Z_2(i\gamma \cdot p + m_{\rm bm}) + Z_1 \int_q^{\Lambda} g^2 D_{\mu\nu}(p-q) \gamma_{\mu} S(q) \Gamma_{\nu}(q,p), \qquad (3)$$

where  $D_{\mu\nu}(k)$  is the renormalised dressed-gluon propagator,  $\Gamma_{\mu}(q;p)$  is the renormalised dressed-quark-gluon vertex and  $m_{\rm bm}$  is the current-quark bare mass that appears in the Lagrangian. The solution of (3) has the general form

$$S(p)^{-1} = i\gamma \cdot pA(p^2) + B(p^2).$$
 (4)

In (3),  $Z_1$  and  $Z_2$  are, respectively, the renormalisation constants for the quark-gluon vertex and quark wave function. <sup>2</sup> Each is fixed by the requirement that the associated Schwinger function (vertex, propagator) take a prescribed value at the renormalisation point,  $p^2 = \mu^2$ , large and spacelike. For example,  $Z_2$  is fixed by requiring that  $A(\mu^2)$  take a prescribed value. As with each renormalisation constant herein, they depend on the renormalisation point and the regularisation mass-scale. For example, at one-loop in QCD,  $Z_2 = [\ln(\mu^2/\Lambda_{\rm QCD}^2)/\ln(\Lambda^2/\Lambda_{\rm QCD}^2)]^{\gamma_S}$ , where  $\mu$  is the renormalisation point and the anomalous dimension is  $\gamma_S = -2\xi/(33 - 2N_f)$ , with  $\xi$  the gauge parameter and  $N_f$  the number of quark flavours. Also the renormalised current-quark mass is  $m_{(\mu)} \equiv Z_2 Z_4^{-1} m_{\rm bm} \equiv Z_m^{-1} m_{\rm bm}$ , which yields  $m_{(\mu)} = \hat{m}/[\frac{1}{2}\ln(\mu^2/\Lambda_{\rm QCD}^2)]^{\gamma_m}$ , where  $\hat{m}$  is the renormalisation-group-invariant current-quark mass and  $\gamma_m = 12/(33-2N_f)$ . We note that: at one-loop  $Z_2 \equiv 1$  in Landau gauge,  $\xi = 0$ ; the anomalous dimension,  $\gamma_m$ , of the mass renormalisation constant,  $Z_m = [\ln(\mu^2/\Lambda_{\rm QCD}^2)/\ln(\Lambda^2/\Lambda_{\rm QCD}^2)]^{\gamma_m}$ , is independent of  $\xi$  to all orders in perturbation theory; and the chiral limit is defined by  $\hat{m} = 0$ .

The quark condensate:  $\langle \bar{q}q \rangle \propto \int_q^{\Lambda} \mathrm{tr}[S(q)]$ , is an order parameter for D $\chi$ SB and (3) has been used extensively to study this phenomenon. As summarised in Ref. [1], using any form of the gluon propagator that is strongly enhanced in the neighbourhood of  $k^2 = 0$ ; i.e., in the infrared, consistent with the results of Refs. [9,10], and with any vertex that is free of kinematic singularities [9,11], the quark DSE admits a nonzero solution for  $B(p^2)$  in the chiral limit; i.e., one has D $\chi$ SB without fine-tuning. In addition, any model study that is able to provide a quantitatively good description of observables; such as: hadron masses, decay constants, scattering lengths, etc., involves the generation of

<sup>&</sup>lt;sup>2</sup> In discussing renormalisation we follow the conventions of Ref. [6]

<sup>&</sup>lt;sup>3</sup> The arguments presented herein cannot be applied in a straightforward fashion to models whose ultraviolet behaviour is that of quenched QED<sub>4</sub>, such as Ref. [7], where the chiral limit can't be defined in this way. The difficulties encountered in such cases are illustrated in Ref. [8].

a Euclidean constituent-quark mass:  $M_{u,d}^E \approx 400 \,\mathrm{MeV}$ ; i.e., the generation of a large "effective mass" for the u- and d-quarks <sup>4</sup>. This is illustrated, for example, in Refs. [7,12]. A large quark "effective mass" is therefore a direct and unavoidable consequence of the infrared enhancement of  $D_{\mu\nu}(k)$ .

As a final introductory remark, we note that the renormalised quark-antiquark scattering kernel in (1),  $K_{tu}^{rs}(q, k; P)$ , which also appears implicitly in (3) because it is the kernel in the inhomogeneous integral equation satisfied by  $\Gamma_{\mu}(q;p)$ , is the sum of a countable infinity of skeleton diagrams.<sup>5</sup> This is why we speak of model studies above: any quantitative study of (1) and (3) necessarily involves a truncation of this kernel whose reliability cannot be gauged a priori. It is, however, incumbent upon practitioners to avoid drawing conclusions that are artefacts of a given truncation. No truncation of the kernel is employed in deriving the general results presented herein.

In considering the pion, understanding chiral symmetry, and its explicit and dynamical breaking, is crucial. These features are expressed in the axial-vector Ward-Takahashi identity (AV-WTI), which involves the isovector axial-vector vertex:

$$\left[\Gamma_{5\mu}^{j}(k;P)\right]_{tu} = Z_{A} \left[\gamma_{5}\gamma_{\mu}\frac{\tau^{j}}{2}\right]_{tu} + \int_{q}^{\Lambda} [\chi_{5\mu}^{j}(q;P)]_{sr} K_{tu}^{rs}(q,k;P),$$
 (5)

 $\chi_{5\mu}^{j}(q;P) \equiv S(q_{+})\Gamma_{5\mu}^{j}(q;P)S(q_{-})$ , that has the general form

$$\Gamma^{j}_{5\mu}(k;P) = \frac{\tau^{j}}{2} \gamma_{5} \left[ \gamma_{\mu} F_{R}(k;P) + \gamma \cdot k k_{\mu} G_{R}(k;P) - \sigma_{\mu\nu} k_{\nu} H_{R}(k;P) \right]$$

$$+ \tilde{\Gamma}^{j}_{5\mu}(k;P) + \frac{P_{\mu}}{P^{2} + m_{\phi}^{2}} \phi^{j}(k;P) ,$$
(6)

where  $F_R$ ,  $G_R$ ,  $H_R$  and  $\tilde{\Gamma}^i_{5\mu}$  are regular as  $P^2 \to -m_\phi^2$ ,  $P_\mu \tilde{\Gamma}^i_{5\mu}(k;P) \sim \mathrm{O}(P^2)$  and  $\phi^j(k;P)$  has the structure depicted in (2). By convention, the renormalisation constant,  $Z_A$ , is chosen so as to fix the value of  $F_R(k;P=0)|_{k^2=\mu^2}$ . This form admits the possibility of at least one pole term in the axial-vector vertex but does not require it.

 $<sup>\</sup>overline{{}^4M_f^E}$  is defined [5] as the solution of:  $p^2A_f(p^2)^2=B_f(p^2)^2$ , where f labels the quark flavour. This is not the quark pole mass, which need not exist in a theory with confinement. The ratio  $M_f^E/\hat{m}_f$  is a single, indicative and quantitative measure of the nonperturbative effects of gluon-dressing on the quark propagator.

<sup>&</sup>lt;sup>5</sup> By definition, K does not contain quark-antiquark to single gauge-boson annihilation diagrams, such as would describe the leptonic decay of the pion, nor diagrams that become disconnected by cutting one quark and one antiquark line.

Substituting (6) into (5) and equating putative pole terms, it is clear that, if present,  $\phi^j(k;P)$  satisfies (1). Since (1) is an eigenvalue problem that only admits a  $\Gamma^j_{\pi} \neq 0$  solution for  $P^2 = -m_{\pi}^2$ , it follows that  $\phi^j(k;P)$  is nonzero only for  $P^2 = -m_{\pi}^2$  and the pole mass is  $m_{\phi}^2 = m_{\pi}^2$ . Hence, if K supports such a bound state, the axial-vector vertex contains a pion-pole contribution whose residue,  $r_A$ , is not fixed by these arguments; i.e.,

$$\Gamma_{5\mu}^{j}(k;P) = \frac{\tau^{j}}{2} \gamma_{5} \left[ \gamma_{\mu} F_{R}(k;P) + \gamma \cdot k k_{\mu} G_{R}(k;P) - \sigma_{\mu\nu} k_{\nu} H_{R}(k;P) \right]$$

$$+ \tilde{\Gamma}_{5\mu}^{i}(k;P) + \frac{r_{A} P_{\mu}}{P^{2} + m_{\pi}^{2}} \Gamma_{\pi}^{j}(k;P) .$$

$$(7)$$

In the chiral limit,  $\hat{m} = 0$ , the AV-WTI

$$-iP_{\mu}\Gamma_{5\mu}^{j}(k;P)\frac{Z_{2}}{Z_{A}} = S^{-1}(k_{+})\gamma_{5}\frac{\tau^{j}}{2} + \gamma_{5}\frac{\tau^{j}}{2}S^{-1}(k_{-}), \qquad (8)$$

where the ratio  $Z_2/Z_A$  is a finite, renormalisation-group-invariant quantity [13], and hence we are free to choose

$$Z_A(\mu, \Lambda) = Z_2(\mu, \Lambda). \tag{9}$$

If one assumes  $m_{\pi}^2 = 0$  in (7), substitutes it into the left-hand-side (l.h.s.) of (8) along with (4) on the right, and equates terms of order  $(P_{\nu})^0$  and  $P_{\nu}$ , one obtains the chiral-limit relations

$$r_A E_\pi(k;0) = B(k^2),$$
 (10)

$$F_R(k;0) + 2r_A F_\pi(k;0) = A(k^2), \qquad (11)$$

$$G_R(k;0) + 2r_A G_{\pi}(k;0) = 2A'(k^2),$$
 (12)

$$H_R(k;0) + 2 r_A H_\pi(k;0) = 0.$$
 (13)

In perturbation theory,  $B(k^2) \equiv 0$  in the chiral limit. The appearance of a  $B(k^2)$ -nonzero solution of (3) in the chiral limit signals  $D\chi SB$ : one has dynamically generated a quark mass term in the absence of a seed-mass. Equations (10)-(13) show that when chiral symmetry is dynamically broken: 1) the homogeneous, isovector, pseudoscalar BSE has a massless,  $P^2 = 0$ , solution; 2) the BSA for the massless bound state has a term proportional to  $\gamma_5$  alone, with  $E_{\pi}(k;0)$  completely determined by the scalar part of the quark self energy, in addition to other pseudoscalar Dirac structures,  $F_{\pi}$ ,  $G_{\pi}$  and  $H_{\pi}$ , that are nonzero in general; 3) the axial-vector vertex is dominated by the pion pole for  $P^2 \simeq 0$ . The converse is also true. Dynamical chiral symmetry breaking

is therefore a sufficient and necessary condition for the appearance of a massless pseudoscalar bound state of very-massive constituents that dominates the axial-vector vertex.

For  $\hat{m} \neq 0$ , the AV-WTI identity is

$$-iP_{\mu}\Gamma_{5\mu}^{j}(k;P) = S^{-1}(k_{+})\gamma_{5}\frac{\tau^{j}}{2} + \gamma_{5}\frac{\tau^{j}}{2}S^{-1}(k_{-}) - 2m_{(\mu)}\Gamma_{5}^{j}(k;P)\frac{Z_{4}}{Z_{P}}, (14)$$

where the isovector, pseudoscalar vertex is given by

$$\left[\Gamma_{5}^{j}(k;P)\right]_{tu} = Z_{P} \left[\gamma_{5} \frac{\tau^{j}}{2}\right]_{tu} + \int_{q}^{\Lambda} \left[\chi_{5}^{j}(q;P)\right]_{sr} K_{tu}^{rs}(q,k;P),$$
 (15)

with  $\chi_5^j(q;P) \equiv S(q_+)\Gamma_5^j(q;P)S(q_-)$ . In (15), the ratio  $Z_4/Z_P$  is a renormalisation-group-invariant quantity and hence we are free to choose

$$Z_P(\mu, \Lambda) = Z_4(\mu, \Lambda). \tag{16}$$

As argued in connection with (5), the solution of (15) has the general form

$$i\Gamma_{5}^{j}(k;P) = \frac{\tau^{j}}{2} \gamma_{5} \left[ iE_{R}^{P}(k;P) + \gamma \cdot P F_{R}^{P} + \gamma \cdot k \, k \cdot P G_{R}^{P}(k;P) + \sigma_{\mu\nu} \, k_{\mu} P_{\nu} \, H_{R}^{P}(k;P) \right] + \frac{r_{P}}{P^{2} + m_{\pi}^{2}} \Gamma_{\pi}^{j}(k;P) ,$$

$$(17)$$

where  $E_R^P$ ,  $F_R^P$ ,  $G_R^P$  and  $H_R^P$  are regular as  $P^2 \to -m_\pi^2$ ; i.e., the isovector, pseudoscalar vertex also receives a contribution from the pion pole. In this case equating pole terms in the AV-WTI entails

$$r_A m_\pi^2 = 2 \, m_{(\mu)} \, r_P \,. \tag{18}$$

The question arises: "What are the residues  $r_A$  and  $r_P$ ?"

To address this we note that (5) can be rewritten as

$$\frac{1}{Z_2} \left[ \Gamma_{5\mu}^j(k;P) \right]_{tu} =$$

$$\left[ \gamma_5 \gamma_\mu \frac{\tau^j}{2} \right]_{tu} + \int_q^{\Lambda} \left[ S(q_+) \frac{\tau^j}{2} \gamma_5 \gamma_\mu S(q_-) \right]_{sr} M_{tu}^{rs}(q,k;P) ,$$
(19)

where M is the renormalised, fully-amputated quark-antiquark scattering amplitude:  $M = K + K(SS)K + \dots$  [1], which can be decomposed as:

$$M_{tu}^{rs}(q,k;P) = \left[\bar{\Gamma}_{\pi}^{\ell}(q;-P)\right]_{rs} \frac{1}{P^2 + m_{\pi}^2} \left[\Gamma_{\pi}^{\ell}(k;P)\right]_{tu} + R_{tu}^{rs}(q,k;P), \quad (20)$$

where  $\bar{\Gamma}_{\pi}(k,-P)^{\mathrm{T}} = C^{-1}\Gamma_{\pi}(-k,-P)C$ , with  $C = \gamma_2\gamma_4$ , the charge conjugation matrix, and  $R_{tu}^{rs}(q,k;P)$  is regular as  $P^2 \to -m_{\pi}^2$ . The fact that here the pion pole has unit residue follows from the canonical normalisation of the BSA [3]:

$$2\delta^{ij}P_{\mu} = \int_{q}^{\Lambda} \left\{ \operatorname{tr} \left[ \bar{\Gamma}_{\pi}^{i}(q; -P) \frac{\partial S(q_{+})}{\partial P_{\mu}} \Gamma_{\pi}^{j}(q; P) S(q_{-}) \right] + \right.$$

$$\left. \operatorname{tr} \left[ \bar{\Gamma}_{\pi}^{i}(q; -P) S(q_{+}) \Gamma_{\pi}^{j}(q; P) \frac{\partial S(q_{-})}{\partial P_{\mu}} \right] \right\} +$$

$$\left. \int_{q}^{\Lambda} \int_{k}^{\Lambda} \left[ \bar{\chi}_{\pi}^{i}(q; -P) \right]_{sr} \frac{\partial K_{tu}^{rs}(q, k; P)}{\partial P_{\mu}} \left[ \chi_{\pi}^{j}(k; P) \right]_{ut} .$$

$$(21)$$

Substituting (7) on the l.h.s. of (19), (20) on the right, and equating residues at the pion pole, one obtains

$$\delta^{ij} r_A P_\mu = Z_2 \int_q^{\Lambda} \operatorname{tr} \left[ \frac{\tau^i}{2} \gamma_5 \gamma_\mu S(q_+) \Gamma_\pi^j(q; P) S(q_-) \right] . \tag{22}$$

The factor of  $Z_2$  on the right-hand-side (r.h.s.) is necessary to ensure that  $r_A$  is independent of the renormalisation point, regularisation mass-scale and gauge parameter: recall that  $Z_2 \equiv 1$  at one-loop in Landau gauge.

The renormalised axial-vector vacuum polarisation is

$$\Pi_{\text{w}\mu\nu}^{ij}(P) = \delta^{ij}(Z_3^{\text{w}} - 1)(\delta_{\mu\nu}P^2 - P_{\mu}P_{\nu}) - Z_2 g_{\text{w}}^2 \int_{q}^{\Lambda} \text{tr}\left[\frac{\tau^i}{2}\gamma_5 \gamma_{\mu} \chi_{5\nu}^j(q; P)\right] (23)$$

where  $Z_3^{\rm w}$  is the weak-boson wave-function renormalisation constant and  $g_{\rm w}$  is the electroweak coupling. The pion leptonic decay constant,  $f_{\pi}$ , is obtained from the pion-pole contribution to this vacuum polarisation:

$$\Pi_{w\mu\nu}^{ij}(P) = \delta^{ij} \left( \delta_{\mu\nu} P^2 - P_{\mu} P_{\nu} \right) \left[ \Pi(P^2) + g_w^2 f_{\pi}^2 \frac{1}{P^2 + m_{\pi}^2} \right], \tag{24}$$

where  $\Pi(P^2)$  is regular as  $P^2 \to -m_\pi^2$ . Substituting (24) in the l.h.s. of (23), (7) on the right, projecting with  $(\delta_{\mu\nu}P^2 - 4P_\mu P_\nu)$ , and equating pole residues one obtains

$$r_A = f_\pi \,; \tag{25}$$

i.e., the residue of the pion pole in the axial-vector vertex is the pion decay constant.

The relationship, in the chiral limit, between the normalisation of the pion BSA and  $f_{\pi}$  has often been discussed. Consider that if one chooses to normalise  $\Gamma^{j}_{\pi}$  such that  $E_{\pi}(0;0) = B(0)$ , and defines the BSA so normalised as  $\Gamma^{j}_{\pi N_{\pi}}(k;P)$ , then the r.h.s. of (21), evaluated with  $\Gamma^{j}_{\pi} \to \Gamma^{j}_{\pi N_{\pi}}$ , is equal to  $2P_{\mu}N_{\pi}^{2}$ , where  $N_{\pi}$  is a dimensioned constant. Using (10)-(13) it is clear that in the chiral limit

$$N_{\pi} = f_{\pi} \,. \tag{26}$$

However, in model studies to date, this result is not obtained unless one assumes  $A(k^2) \equiv 1$ . It follows that any kernel which leads, via (3), to  $A(k^2) \equiv 1$  must also yield,  $F_{\pi} \equiv 0 \equiv G_{\pi} \equiv H_{\pi}$ , if it preserves the AV-WTI. In realistic model studies, where  $A(k^2) \not\equiv 1$ , the difference between the values of  $N_{\pi}$  and  $f_{\pi}$  is an artefact of neglecting  $F_{\pi}$ ,  $G_{\pi}$  and  $H_{\pi}$  in (2) [1].

To determine  $r_P$ , one rewrites (15) as

$$\frac{1}{Z_4} \left[ \Gamma_5^j(k; P) \right]_{tu} = \left[ \gamma_5 \frac{\tau^j}{2} \right]_{tu} + \int_q^{\Lambda} \left[ S(q_+) \frac{\tau^j}{2} \gamma_5 S(q_-) \right]_{sr} M_{tu}^{rs}(q, k; P) . \tag{27}$$

Substituting (17) in the l.h.s. of (27), (20) on the right, and equating residues at the pion pole, one obtains

$$i\delta^{ij}r_P = Z_4 \int_q^{\Lambda} \operatorname{tr}\left[\frac{1}{2}\tau^i \gamma_5 S(q_+) \Gamma_{\pi}^j(q; P) S(q_-)\right]. \tag{28}$$

The factor  $Z_4$  on the r.h.s. depends on the gauge parameter, the regularisation mass-scale and the renormalisation point. This dependence is exactly

<sup>&</sup>lt;sup>6</sup> For example, Refs. [7,14]. As shown in Ref. [15], and contrary to the suggestion in Ref. [14], the integral appearing in the calculation of  $\pi^0 \to \gamma \gamma$  does not provide an additional "definition" of  $f_{\pi}$ ; it must yield 1/2, independent of the details of the model, in order to be consistent with the anomalous Ward-Takahashi identity for the isosinglet axial-vector vertex.

that required to ensure that: 1)  $r_P$  is finite in the limit  $\Lambda \to \infty$ ; 2)  $r_P$  is gauge-parameter independent; and 3) the renormalisation point dependence of  $r_P$  is just such as to ensure that the r.h.s. of (18) is renormalisation point independent. This is obvious at one-loop order, especially in Landau-gauge where  $Z_2 \equiv 1$  and hence  $Z_4 = Z_m$ .

In the chiral limit, using (2), (10)-(13) and (25), (28) yields

$$r_P^0 = \frac{1}{f_\pi} \langle \bar{q}q \rangle_\mu^0, \ \langle \bar{q}q \rangle_\mu^0 \equiv Z_4(\mu, \Lambda) N_c \int_q^{\Lambda} \operatorname{tr}_D \left[ S_{\hat{m}=0}(q) \right]; \tag{29}$$

 $\langle \bar{q}q \rangle_{\mu}^{0}$  is the chiral-limit vacuum quark condensate. It is renormalisation-point dependent but independent of the gauge parameter and the regularisation mass-scale. (29) demonstrates that the chiral-limit residue of the pion pole in the pseudoscalar vertex is  $\langle \bar{q}q \rangle_{\mu}^{0}/f_{\pi}$ .

Using (25) and (29), (18) yields

$$f_{\pi}^2 m_{\pi}^2 = 2 m_{(\mu)} \langle \bar{q}q \rangle_{\mu}^0 + \mathcal{O}(\hat{m}^2) \,.$$
 (30)

In general, (18) is the statement

$$f_{\pi}^2 m_{\pi}^2 = 2 m_{(\mu)} \langle \bar{q}q \rangle_{\mu}^{\hat{m}},$$
 (31)

where we have introduced the notation  $\langle \bar{q}q \rangle_{\mu}^{\hat{m}} \equiv f_{\pi}r_{P}$ . This result is qualitatively equivalent to that obtained in Ref. [16]. It is exact, with what is commonly known as the Gell-Mann–Oakes–Renner relation, (30), an obvious corollary. The extension of (31) to  $SU(N_{f} \geq 3)$ , and  $m_{u} \neq m_{d}$ , is relatively straightforward: (22) and (28) remain correct, apart from obvious modifications associated with the flavour-dependence of the Schwinger functions, and possible flavour-dependence of the renormalisation scheme; and the analogues of (30) provide a good approximation for  $SU(N_{f} = 3)$ , when  $\hat{m}_{f} \lesssim \Lambda_{\rm QCD}$ .

As remarked above, the quark-antiquark scattering kernel appears in each of (1), (3), (5) and (15), and any practical, quantitative calculation of pion (and other hadron) observables will involve a direct or implicit truncation of K. Our discussion indicates the crucial nature of chiral symmetry and its dynamical breakdown in connection with the pion. This entails that in developing a tractable truncation for quantitative calculations of pion observables, it is

 $<sup>\</sup>overline{{}^{7}}$  We emphasise that, for  $\hat{m} \neq 0$ , the r.h.s. of (18) is *not* a difference of vacuum quark condensates; a phenomenological assumption often employed.

necessary to ensure that (14) is preserved. The simplest truncation to do this is the "rainbow-ladder approximation", in which

$$K_{tu}^{rs}(q,k;P) = -g^2 D_{\mu\nu}(k-q) \left(\gamma_{\mu} \frac{\lambda^a}{2}\right)_{tr} \left(\gamma_{\nu} \frac{\lambda^a}{2}\right)_{su}$$
(32)

in (1), (5) and (15), and  $Z_1\Gamma_{\nu}(q,p) = \gamma_{\nu}$  in (3), with a model form for  $D_{\mu\nu}(k)$  chosen consistent with Refs. [10]. This truncation has been used extensively [1] and provides a quantitatively reliable description of many pseudoscalar, vector and axial-vector meson observables without fine-tuning.

We note that in "rainbow-ladder approximation"  $\partial_{\mu}^{P}K(q,k;P) \equiv 0$  in (21) and it is a simple exercise to demonstrate explicitly the chiral limit equality:  $N_{\pi} = f_{\pi}$ . One method [2] is to use (5) to eliminate  $Z_{2}(\tau^{i}/2)\gamma_{5}\gamma_{\mu}$  on the r.h.s. of (22) in favour of the regular-part of the axial-vector vertex in (7). Integration by parts in the expression for  $N_{\pi}^{2}$ , neglecting surface terms that vanish in a translationally invariant regularisation scheme, then yields the expected result. In this truncation, using a straightforward modification of the method employed in Ref. [4], it is also a simple exercise to confirm (10)-(13) from a direct comparison of (3), in the chiral limit, with (5) and (7). To elucidate the results derived herein, a detailed numerical study of a QCD-based model using this truncation is underway. As an example of the results to expect, neglecting  $F_{\pi}$ ,  $G_{\pi}$  and  $H_{\pi}$  in solving the pion BSE leads to a reduction in the eigenvalue of  $\approx 40\%$ ; i.e., an underestimation of  $m_{\pi}$  by 20%.

A systematic extension of "rainbow-ladder approximation" is introduced in Ref. [17]. Therein it is shown that higher order contributions to the kernel cancel approximately, order-by-order, so that "rainbow-ladder approximation", as defined herein, provides a good estimate of the properties of the pion (and other flavour-octet pseudoscalar and vector mesons); i.e., for these mesons, there are no large corrections to the rainbow-ladder results from contributions to the kernel that are neglected in this truncation.

Herein the primary results are: (10)-(13) and their consequences, (25), (26), (29) and (31), which entails (30), and their derivation in a model-independent manner using the DSEs. In any model study, if one employs a truncation that violates (14), then, in general, these results will not be recovered and one may arrive at erroneous conclusions.

To conclude, our analysis indicates that understanding the "dichotomy" of the pion as both a Goldstone mode and a bound state is straightforward. It also unambiguously identifies the root cause of the defect in model studies that require fine-tuning to describe the pion as a nearly-massless bound state of very-massive constituents. Models that do not preserve the axial-vector Ward-Takahashi identity, (14), although perhaps valid in the regime where explicit

chiral symmetry breaking effects dominate, should not be expected to provide generally robust insight into the qualitative features of pion observables.

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