Contents

1 The quark-quark contact interaction and the Rainbow-Ladder truncation

We use the following expression for the contact (momentum-independent) vector $\otimes$ vector quark-quark interaction:

$$g^2 D_{\mu\nu}(p-q) = \delta_{\mu\nu}\frac{4\pi\alpha_{\rm IR}}{m_{\rm G}^2}, \tag{1}$$

where $m_{\rm G} = 0.8\,\text{GeV}$ is a gluon mass-scale and the fitted parameter $\alpha_{\rm IR} = 0.93\pi$ is the zero-momentum value of a running-coupling constant in QCD.

We embed Eq. (1) in a rainbow-ladder truncation of the DSEs, which is the leading order in the most widely used, global-symmetry-preserving truncation scheme. This means

$$\Gamma^a_\nu(q,p) = \frac{\lambda^a}{2}\gamma_\nu \tag{2}$$

in the gap equation and in the subsequent construction of the Bethe-Salpeter kernels.

N.B. This work is based on two first references which are [1] and [2].

1.1 Dressed-quark propagator

Using Eqs. (1) and (2), the gap equation, expressed in Euclidean space, for a quark

$$S^{-1}(p) = i\gamma\cdot p + m + \Sigma(p), \tag{3}$$

where

$$\Sigma(p) = \int \frac{d^4q}{(2\pi)^4} g^2 D_{\mu\nu}(p-q)\gamma_\mu \frac{\lambda^a}{2} S(q)\Gamma^a_\nu(q,p), \tag{4}$$

becomes

$$\begin{aligned} S^{-1}(p) &= i\gamma\cdot p + m + \Sigma(p) \\ &= i\gamma\cdot p + m + \int \frac{d^4q}{(2\pi)^4} g^2 D_{\mu\nu}(p-q)\gamma_\mu \frac{\lambda^a}{2} S(q)\Gamma^a_\nu(q,p) \\ &= i\gamma\cdot p + m + \int \frac{d^4q}{(2\pi)^4} \delta_{\mu\nu}\frac{4\pi\alpha_{\rm IR}}{m_{\rm G}^2}\gamma_\mu \frac{\lambda^a}{2} S(q)\frac{\lambda^a}{2}\gamma_\nu \\ &= i\gamma\cdot p + m + \frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\int \frac{d^4q}{(2\pi)^4}\gamma_\mu S(q)\gamma_\mu, \end{aligned} \tag{5}$$

where m is the quark's current-mass and $\sum_{a=1}^{8}\left(\frac{\lambda^a}{2}\frac{\lambda^a}{2}\right) = C_F\mathbf{1} = \frac{4}{3}\mathbf{1}$. Our Euclidean metric conventions are detailed in Appendix B.

At zero temperature and chemical potential the most general Poincaré covariant solution of this gap equation involves two scalar functions. There are three common, equivalent expressions

$$S(p) = \frac{1}{i\gamma\cdot pA(p^2) + B(p^2)} = \frac{Z(p^2)}{i\gamma\cdot p + M(p^2)} = -i\gamma\cdot p\,\sigma_V(p^2) + \sigma_S(p^2). \tag{6}$$

In the second form, $Z(p^2)$ is called the wave-function renormalisation and $M(p^2)$ is the dressed-quark mass function.

If one introduces the first expression of the quark propagator in Eq. (5), one obtains the following

$$\begin{aligned} i\gamma\cdot pA(p^2) + B(p^2) &= i\gamma\cdot p + m + \frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\mu\frac{1}{i\gamma\cdot qA(q^2)+B(q^2)}\gamma_\mu \\ &= i\gamma\cdot p + m + \frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\mu\frac{-i\gamma\cdot qA(q^2)+B(q^2)}{q^2A^2(q^2)+B^2(q^2)}\gamma_\mu. \end{aligned} \tag{7}$$

If one multiples Eq. (7) by $(-i\gamma \cdot p)$

$$
\begin{aligned}
(-i\gamma \cdot p)(i\gamma \cdot p)A(p^2) + (-i\gamma \cdot p)B(p^2) =& (-i\gamma \cdot p)(i\gamma \cdot p) + (-i\gamma \cdot p)m \\
&+ \frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\int \frac{d^4q}{(2\pi)^4}(-i\gamma \cdot p)\gamma_\mu \frac{(-i\gamma \cdot q)A(q^2)}{q^2A^2(q^2)+B^2(q^2)}\gamma_\mu \\
&+ \frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\int \frac{d^4q}{(2\pi)^4}(-i\gamma \cdot p)\gamma_\mu \frac{B(q^2)}{q^2A^2(q^2)+B^2(q^2)}\gamma_\mu
\end{aligned} \tag{8}
$$

and subsequently evaluates a matrix trace over spinor (Dirac) indices, then one finds

$$
\begin{aligned}
-4(p\cdot p)A(p^2) &= -4(p\cdot p) + \frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\int \frac{d^4q}{(2\pi)^4} 8(p\cdot q)\frac{A(q^2)}{q^2A^2(q^2)+B^2(q^2)}, \\
p^2A(p^2) &= p^2 - \frac{32\pi}{3}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\int \frac{d^4q}{(2\pi)^4}(p\cdot q)\frac{A(q^2)}{q^2A^2(q^2)+B^2(q^2)},
\end{aligned} \tag{9}
$$

It is straightforward to show that $\int d^4q\,(p\cdot q)\,\mathrm{Function}(\mathrm{p}^2,\mathrm{q}^2) = 0$ and so

$$
A(p^2) = 1. \tag{10}
$$

If, on the other hand, one multiples Eq. (7) by $\mathbf{I}_D$, uses Eq. (10) and subsequently evaluates a trace over Dirac indices, then

$$
B(p^2) = m + \frac{64\pi}{3}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\int \frac{d^4q}{(2\pi)^4}\frac{B(q^2)}{q^2+B^2(q^2)}. \tag{11}
$$

Since the integral here is p^2-independent then a solution at one value of p^2 must be the solution at all values; viz., any nonzero solution must be of the form

$$
B(p^2) = \text{constant} = M. \tag{12}
$$

Using this result, Eq. (11) becomes

$$
\begin{aligned}
M &= m + M\frac{64\pi}{3}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2+M^2} \\
M &= m + M\frac{64\pi}{3}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\frac{1}{(2\pi)^4}\int q^3\sin^2\theta_1\sin\theta_2\, dq d\theta_1 d\theta_2 d\phi \frac{1}{q^2+M^2} \\
M &= m + M\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\frac{2^6\pi}{3}\frac{2\pi^2}{2^4\pi^4 2}\int_0^\infty ds s\,\frac{1}{s+M^2} \\
M &= m + M\frac{4}{3\pi}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\int_0^\infty ds s\,\frac{1}{s+M^2}
\end{aligned} \tag{13}
$$

where $d^4q = q^3\sin^2\theta_1\sin\theta_2\, dq d\theta_1 d\theta_2 d\phi$ and $s = q^2$ with $ds = 2qdq$[1].

[1] Volume of n-dimensional unit sphere

$$
\Omega_n = \frac{2\,\pi^{n/2}}{\Gamma(\frac{n}{2})} \tag{14}
$$

A heat-kernel-like regularization procedure is useful in hadron physics applications; viz., one writes

$$\frac{1}{s+M^2} = \int_0^\infty d\tau e^{-\tau(s+M^2)} \to \int_{\tau_{uv}^2}^{\tau_{ir}^2} d\tau e^{-\tau(s+M^2)} = \frac{e^{-\tau_{uv}^2(s+M^2)} - e^{-\tau_{ir}^2(s+M^2)}}{s+M^2}, \tag{15}$$

where $\tau_{ir,uv}$ are, respectively, infrared and ultraviolet regulators. Since Eqs. (1) and (2) do not define a renormalisable theory, then $\Lambda_{uv} = 1/\tau_{uv}$ cannot be removed but instead plays a dynamical role, setting the scale of all dimensioned quantities. Using Eq. (15) the gap equation becomes

$$\begin{aligned} M &= m + M\frac{4}{3\pi}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\int_0^\infty ds s\,\frac{1}{s+M^2} \\ &= m + M\frac{4}{3\pi}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\int_0^\infty ds s\int_0^\infty d\tau e^{-\tau(s+M^2)} \\ &= m + M\frac{4}{3\pi}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\int_0^\infty ds s\int_{\tau_{uv}^2}^{\tau_{ir}^2} d\tau e^{-\tau(s+M^2)} \\ &= m + M\frac{4}{3\pi}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\int_{\tau_{uv}^2}^{\tau_{ir}^2} d\tau e^{-\tau M^2}\int_0^\infty ds s\, e^{-\tau s} \\ &= m + M\frac{4}{3\pi}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\int_{\tau_{uv}^2}^{\tau_{ir}^2} d\tau\frac{e^{-\tau M^2}}{\tau^2} \\ &= m + M\frac{4}{3\pi}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\int_{M^2\tau_{uv}^2}^{M^2\tau_{ir}^2}\frac{dt}{M^2}M^4\frac{e^{-t}}{t^2} \\ &= m + M\frac{4}{3\pi}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}M^2\left[\int_{M^2\tau_{uv}^2}^\infty dt\frac{e^{-t}}{t^2} - \int_{M^2\tau_{ir}^2}^\infty dt\frac{e^{-t}}{t^2}\right] \\ &= m + M\frac{4\alpha_{\rm IR}}{3\pi m_{\rm G}^2}\mathcal{C}^{iu}(M^2), \end{aligned} \tag{16}$$

where $\mathcal{C}^{iu}(M^2) = M^2\left[\Gamma(-1, M^2\tau_{uv}^2) - \Gamma(-1, M^2\tau_{ir}^2)\right]$, with $\Gamma(\alpha, y)$ being the incomplete gamma-function

$$\Gamma(\alpha, y) = \int_y^\infty t^{a-1}e^{-t}. \tag{17}$$

One may set $\Lambda_{ir} = 0$. However, in hadron physics phenomenology, a finite value of $\tau_{ir} = \frac{1}{\Lambda_{ir}}$ implements confinement by ensuring the absence of quark production thresholds in S-matrix amplitudes. Subsequently, Λ_{uv} defines the only mass-scale in a nonrenormalisable model. Hence one can set $\Lambda_{uv} = 1$ and hereafter merely interpret all other mass-scales as being expressed in units of Λ_{uv}, in which case $\mathcal{C}^{01}(M^2) = M^2\Gamma(-1, M^2)$.

Now, if one considers the chiral limit, $m = 0$, one has

$$M = M\frac{4\alpha_{\rm IR}}{3\pi\hat{m}_{\rm G}^2}\mathcal{C}^{01}(M^2) \tag{18}$$

where $\hat{m}_{\rm G} = m_{\rm G}/\Lambda_{uv}$. One solution of Eq. (18) is $M = 0$. This is the result that connects smoothly with perturbation theory: one begins with no mass and no mass is generated.

Suppose, on the other hand, that $M \neq 0$. Then the expression makes sense if, and only if, the following equation has a solution

$$1 = \frac{4\alpha_{\rm IR}}{3\pi\hat{m}_{\rm G}^2}\mathcal{C}^{iu}(M^2), \tag{19}$$

	m_u	m_s	m_s/m_u	M_0	M_u	M_s	M_s/M_u
Ref. [2]	0.007	0.17	24.3	0.36	0.37	0.53	1.43
jsegovia	0.007	0.17	24.2857	0.3576	0.3674	0.5330	1.4507
Ref. [1]	0.007	-	-	0.358	0.368	-	-
jsegovia	0.007	-	-	0.3578	0.3677	-	-

Table 1. Computed dressed-quark masses. All results obtained with $\alpha_{\rm IR} = 0.93\pi$ and, in GeV, $\Lambda_{ir} = 0.24$ and $\Lambda_{uv} = 0.905$.

The $\mathcal{C}^{iu}(M^2)$ is a monotonically decreasing function with maximum value 1 for $M = 0$. Consequently $\exists M \neq 0$ if, and only if

$$\frac{\alpha_{\rm IR}}{\pi} > \frac{3}{4}\hat{m}_{\rm G}. \tag{20}$$

Although one began with a model of massless fermions, the interaction alone has provided those fermions with mass. This is the phenomenon of dynamical chiral symmetry breaking; namely, the generation of mass from nothing.

Table 1 shows the computed dressed quark properties with the formula

$$M = m + M\frac{4\alpha_{\rm IR}}{3\pi m_{\rm G}^2}\mathcal{C}^{iu}(M^2), \tag{21}$$

where m is the current quark mass, and $m_{\rm G} = 0.8\,\text{GeV}$, $\alpha_{\rm IR} = 0.93\pi$, $\Lambda_{ir} = 0.24\,\text{GeV}$ and $\Lambda_{uv} = 0.905\,\text{GeV}$ are parameters.

1.2 Mesons and diquarks: Formulating the bound-state problem

The bound-state problem for hadrons characterised by two valence-fermions may be studied using the homogeneous Bethe-Salpeter equation

$$[\Gamma(k;P)]_{tu} = \int \frac{d^4q}{(2\pi)^4}[\chi(q;P)]_{sr}K^{rs}_{tu}(q,k;P), \tag{22}$$

where Γ is the bound-state's Bethe-Salpeter amplitude and $\chi(q;P) = S(q+P)\Gamma S(q)$ is its Bethe-Salpeter wave-function where S is the dressed-quark propagator. The r, s, t and u represent color, flavor and spinor indices, and K is the relevant fermion-fermion scattering kernel. This equation possesses solutions on that discrete set of P^2-values for which bound-states exist.

We have already mentioned that for ground-state, charged pseudoscalar- and vector-mesons constitued from a valence-quark and -antiquark with equal current-mass, the rainbow ladder truncation of the Bethe-Salpeter and gap equations provides a good approximation. This means $\Gamma^a_\nu = \frac{\lambda^a}{2}\gamma_\nu$ in both Eq. (3) and the construction of K in Eq. (22), so that one works with

$$\begin{aligned} S^{-1}(p) &= i\gamma\cdot p + m + \int\frac{d^4q}{(2\pi)^4}g^2D_{\mu\nu}(p-q)\gamma_\mu\frac{\lambda^a}{2}S(q)\frac{\lambda^a}{2}\gamma_\nu, \\ \Gamma(k;P) &= -\int\frac{d^4q}{(2\pi)^4}g^2D_{\mu\nu}(p-q)\gamma_\mu\frac{\lambda^a}{2}S(q+P)\Gamma(q;P)S(q)\frac{\lambda^a}{2}\gamma_\nu. \end{aligned} \tag{23}$$

In this truncation, color-antitriplet quark-quark correlations (diquarks) are described by an homogeneous Bethe-Salpeter equation that is readily inferred from the second equation in (23)

$$\Gamma_{qq}(k;P)H^c = -\int\frac{d^4q}{(2\pi)^4}g^2D_{\mu\nu}(p-q)\gamma_\mu\frac{\lambda^a}{2}S(q+P)\Gamma_{qq}(q;P)H^c[S(-q)]^T\left[\frac{\lambda^a}{2}\right]^T[\gamma_\nu]^T\,, \tag{24}$$

where $c = 1, 2, 3$ is a color label and $\{H^c\}$ are defined as follows

$$\left\{H^1 = i\lambda^7, H^2 = -i\lambda^5, H^3 = i\lambda^2\right\}, \tag{25}$$

then $\epsilon_{c_1c_2c_3} = (H^{c_3})_{c_1c_2}$ is the Levi-Civita tensor expressed via the antisymmetric Gell-Mann matrices. Using the properties of the Dirac and Gell-Mann matrices, it is straightforward to show that

$$\Gamma_{qq}(k;P)C^\dagger = -\frac{1}{2}\int \frac{d^4q}{(2\pi)^4} g^2 D_{\mu\nu}(p-q)\gamma_\mu \frac{\lambda^a}{2} S(q+P)\Gamma_{qq}(q;P)C^\dagger S(q)\frac{\lambda^a}{2}\gamma_\nu, \tag{26}$$

which explicates the observation that an interaction which binds mesons also generates strong diquark correlations in the color-$\bar{3}$ channel. It follows moreover that one may obtain the mass and Bethe-Salpeter amplitude for a diquark with spin-parity J^P from the equation for a J^{-P}-meson in which the only change is a halving of the interaction strength[2]. The flipping of the sign in parity occurs because fermions and antifermions have opposite parity[3].

Using the interaction given in Eq. (1), the homogeneous Bethe-Salpeter equation (BSE) for a pseudoscalar meson is

$$\Gamma_{(q\bar{q},0^-)}(k;P) = -\frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\int \frac{d^4q}{(2\pi)^4}\gamma_\mu S(q+P)\Gamma_{(q\bar{q},0^-)}(q;P)S(q)\gamma_\mu. \tag{27}$$

Because the integrand does not depend on the external relative momentum, k, a symmetry preserving regularization of Eq. (27) yields solutions that are independent of k. This is the defining characteristic of a point-like composite particle. The solution for a pseudoscalar meson can be written as

$$\Gamma_{(q\bar{q},0^-)}(P) = i\gamma_5 E_{(q\bar{q},0^-)}(P) + \frac{1}{M}\gamma_5\gamma\cdot P F_{(q\bar{q},0^-)}(P), \tag{28}$$

where $M = 2\mu$ with μ the reduced mass of the system. Now it is straightforward to write the Bethe-Salpeter equation for a $J^P = 0^+$ diquark

$$\Gamma^C_{(qq,0^+)}(P) = -\frac{8\pi}{3}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\int \frac{d^4q}{(2\pi)^4}\gamma_\mu S(q+P)\Gamma^C_{(qq,0^+)}(P)S(q)\gamma_\mu. \tag{29}$$

where

$$\Gamma^C_{(qq,0^+)}(P) = \Gamma_{(qq,0^+)}(P)C^\dagger = i\gamma_5 E_{(qq,0^+)}(P) + \frac{1}{M}\gamma_5\gamma\cdot P F_{(qq,0^+)}(P). \tag{30}$$

In the same manner, for a vector meson one has[4]

$$\begin{aligned}\Gamma_{\alpha\,(q\bar{q},1^-)}(k;P) &= -\frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int \frac{d^4q}{(2\pi)^4}\gamma_\mu S(q+P)\Gamma_{\alpha\,(q\bar{q},1^-)}(q;P)S(q)\gamma_\mu,\\ \Gamma_{\mu\,(q\bar{q},1^-)}(P) &= \gamma_\mu^\perp E_{(q\bar{q},1^-)}(P).\end{aligned} \tag{31}$$

[2] Take into account that now one has $\frac{\lambda^a}{2}\lambda^b\frac{\lambda^a}{2}$.

[3] We note that the rainbow-ladder truncation usually generates asymptotic diquark states. Such states are not observed and their appearance is an artifact of the truncation. Nevertheless, studies with kernels that do not produce diquark bound states, do support a physical interpretation of the masses, $m_{(qq)_{J^P}}$, obtained using rainbow-ladder truncation; viz., the quantity $l_{(qq)_{J^P}} = 1/m_{(qq)_{J^P}}$ may be interpreted as a range over which the diquark correlation can propagate before fragmentation.

[4] Actually, the most general form of the vector Bethe-Salpeter equation is $\Gamma_{\mu\,(q\bar{q},1^-)} = \gamma_\mu^\perp E_{(q\bar{q},1^-)}(P) + \frac{1}{M}\sigma_{\mu\nu}P_\nu F_{(q\bar{q},1^-)}(P)$. In the Rainbow-ladder truncation we obtain $F_{(q\bar{q},1^-)} = 0$, but it should be kept in mind that this is an artifact of the Rainbow-Ladder truncation.

And for the $J^P = 1^+$ diquark one has

$$\begin{aligned}
\Gamma^C_{\alpha\,(qq,1^+)}(P) &= -\frac{8\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int \frac{d^4q}{(2\pi)^4}\gamma_\mu S(q+P)\Gamma^C_{\alpha\,(qq,1^+)}(P)S(q)\gamma_\mu, \\
\Gamma^C_{\mu\,(qq,1^+)}(P) &= \Gamma_{\mu\,(qq,1^+)}C^\dagger = \gamma^\perp_\mu E_{(qq,1^+)}(P).
\end{aligned} \tag{32}$$

where

$$\begin{aligned}
\gamma^\perp_\mu &= \gamma_\mu - \frac{\gamma\cdot P}{P^2}P_\mu, \\
\gamma^{||}_\mu &= \frac{\gamma\cdot P}{P^2}P_\mu,
\end{aligned} \tag{33}$$

and $\gamma^\perp_\mu + \gamma^{||}_\mu = \gamma_\mu$, so $P_\mu\gamma^\perp_\mu = 0$ and $P_\mu\gamma^{||}_\mu = \gamma\cdot P$.

N.B. Taking into account that

$$k\cdot P = 0 \Leftarrow \begin{cases} (k-\frac{P}{2})^2 = k^2 + \frac{P^2}{4} - 2k\cdot P = -m^2 \\ (k+\frac{P}{2})^2 = k^2 + \frac{P^2}{4} + 2k\cdot P = -m^2 \end{cases} \tag{34}$$

one obtains

$$\begin{aligned}
\gamma^\perp_\mu k_\mu &= \gamma\cdot k, \\
\gamma^{||}_\mu k_\mu &= 0.
\end{aligned} \tag{35}$$

1.3 Ward-Takahashi identities

No study of low-energy hadron observables is meaningful unless it ensures expressly that the axial-vector Ward-Takahashi identity is satisfied. Without this it is impossible to preserve the pattern of chiral symmetry breaking in QCD and hence a veracious understanding of hadron mass splittings is not achievable.

1.3.1 Axial-vector Ward-Takahashi identity

The axial-vector Ward-Takahashi identity, which expresses chiral symmetry and its breaking pattern is

$$P_\mu \Gamma^{fg}_{5\mu}(k;P) + i\left[m_f(\zeta) + m_g(\zeta)\right]\Gamma^{fg}_5(k;P) = S^{-1}_f(k_+)i\gamma_5 + i\gamma_5 S^{-1}_g(k_-), \tag{36}$$

where $P = p_1 + p_2$ is the total momentum entering the vertex and k is the relative momentum between the amputated quark legs. To be explicit, $k = (1-\eta)p_1 + \eta p_2$, with $\eta \in [0,1]$, and hence $k_+ = p_1 = k + \eta P$, $k_- = p_2 = k - (1-\eta)P$.

The chiral limit, $m = 0$, of the axial-vector identity states ($k_+ = k + P$)

$$P_\mu \Gamma_{5\mu}(k_+, k) = S^{-1}(k_+)i\gamma_5 + i\gamma_5 S^{-1}(k), \tag{37}$$

where $\Gamma_{5\mu}(k_+, k)$ is the axial-vector vertex, which is determined by

$$\Gamma_{5\mu}(k_+, k) = \gamma_5\gamma_\mu - \frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int \frac{d^4q}{(2\pi)^4}\gamma_\alpha S(q_+)\Gamma_{5\mu}(q_+, q)S(q)\gamma_\alpha. \tag{38}$$

One must therefore implement a regularisation of this inhomogeneous BSE that maintains Eq. (37). To see what this entails, contract Eq. (38) with P_μ and use Eq. (37) within the integrand

$$
\begin{aligned}
P_\mu \Gamma_{5\mu}(k_+,k) &= P_\mu \gamma_5 \gamma_\mu - \frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int \frac{d^4q}{(2\pi)^4}\gamma_\alpha S(q_+) P_\mu \Gamma_{5\mu}(q_+,q) S(q)\gamma_\alpha \\
&= \gamma_5 \gamma\cdot P - \frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int \frac{d^4q}{(2\pi)^4}\gamma_\alpha S(q_+)[S^{-1}(q_+)i\gamma_5 + i\gamma_5 S^{-1}(q)]S(q)\gamma_\alpha \\
&= \gamma_5 \gamma\cdot P - \frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int \frac{d^4q}{(2\pi)^4}\gamma_\alpha [i\gamma_5 S(q) + S(q_+)i\gamma_5]\gamma_\alpha \\
&= \gamma_5 \gamma\cdot P + \frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int \frac{d^4q}{(2\pi)^4} i\gamma_5\gamma_\alpha S(q)\gamma_\alpha + \frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int \frac{d^4q}{(2\pi)^4}\gamma_\alpha S(q_+)\gamma_\alpha i\gamma_5
\end{aligned}
\tag{39}
$$

now, using a translational invariant regularization of the integral $\int_q^\Lambda = \int^\Lambda \frac{d^4q}{4\pi^2}$ or with is the same $\frac{1}{4\pi^2}\int_q^\Lambda = \int \frac{d^4q}{(2\pi)^4}$, the equation above is

$$
P_\mu \Gamma_{5\mu}(k_+,k) = \gamma_5\gamma\cdot P + \frac{4}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_q^\Lambda i\gamma_5\gamma_\alpha S(q)\gamma_\alpha + \frac{4}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_q^\Lambda \gamma_\alpha S(q_+)\gamma_\alpha i\gamma_5 \tag{40}
$$

and taking into account that in a translational invariant regularization the two integrals above cannot depend on the total momentum entering the vertex, P, we have

$$
P_\mu \Gamma_{5\mu}(k_+,k) = \gamma_5\gamma\cdot P + \frac{4}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_q^\Lambda i\gamma_5\gamma_\alpha S(q)\gamma_\alpha + \frac{4}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_q^\Lambda \gamma_\alpha S(q)\gamma_\alpha i\gamma_5 \tag{41}
$$

and taking into account the expression for the dressed-quark propagator[5]

$$
\begin{aligned}
P_\mu \Gamma_{5\mu}(k_+,k) &= \gamma_5\gamma\cdot P + i\gamma_5(S^{-1}(p) - i\gamma\cdot p) + (S^{-1}(p) - i\gamma\cdot p)i\gamma_5 \\
&= \gamma_5\gamma\cdot P + i\gamma_5 S^{-1}(p) + S^{-1}(p)i\gamma_5.
\end{aligned}
\tag{43}
$$

The Eq. (43) coincides with Eq. (37) if, and only if, $P = 0$. The condition $P = 0$ is not equivalent to $P^2 = 0$. Indeed, in a Poincaré covariant theory $P = 0$ is strictly impossible: a massless particle light-like.

I want to enforce the axial-vector Ward-Takahashi identity

$$
\begin{aligned}
P_\mu \Gamma_{5\mu}(k_+,k) &= \gamma_5\gamma\cdot P + S^{-1}(p)i\gamma_5 + i\gamma_5 S^{-1}(p) \\
&= \gamma_5\gamma\cdot P + (i\gamma\cdot p + M)i\gamma_5 + i\gamma_5(i\gamma\cdot p + M) \\
&= \gamma_5\gamma\cdot P + 2i\gamma_5 M \\
&= \gamma_5\gamma\cdot P + \frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int \frac{d^4q}{(2\pi)^4} i\gamma_5\gamma_\alpha S(q)\gamma_\alpha + \frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int \frac{d^4q}{(2\pi)^4}\gamma_\alpha S(q_+)\gamma_\alpha i\gamma_5,
\end{aligned}
\tag{44}
$$

[5]The dressed-quark propagator is given as

$$
\begin{aligned}
S^{-1}(p) &= i\gamma\cdot p + m + \frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int \frac{d^4q}{(2\pi)^4}\gamma_\mu S(q)\gamma_\mu \\
&= i\gamma\cdot p + m + \frac{4}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_q^\Lambda \gamma_\mu S(q)\gamma_\mu,
\end{aligned}
\tag{42}
$$

when a translational invariant regularization is performed.

which means

$$\begin{aligned}
2i\gamma_5 M &= \frac{16\pi}{3}\frac{\alpha_{\mathrm{IR}}}{m_G^2}\int \frac{d^4q}{(2\pi)^4} i\gamma_5\gamma_\alpha S(q)\gamma_\alpha + \frac{16\pi}{3}\frac{\alpha_{\mathrm{IR}}}{m_G^2}\int \frac{d^4q}{(2\pi)^4}\gamma_\alpha S(q_+)\gamma_\alpha i\gamma_5 \\
&= \frac{4}{3\pi}\frac{\alpha_{\mathrm{IR}}}{m_G^2}\int_q^\Lambda i\gamma_5\gamma_\alpha S(q)\gamma_\alpha + \frac{4}{3\pi}\frac{\alpha_{\mathrm{IR}}}{m_G^2}\int_q^\Lambda \gamma_\alpha S(q_+)\gamma_\alpha i\gamma_5 \\
&= \frac{4}{3\pi}\frac{\alpha_{\mathrm{IR}}}{m_G^2}\int_q^\Lambda i\gamma_5\gamma_\alpha \frac{-i\gamma\cdot q + M}{q^2+M^2}\gamma_\alpha + \frac{4}{3\pi}\frac{\alpha_{\mathrm{IR}}}{m_G^2}\int_q^\Lambda \gamma_\alpha \frac{-i\gamma\cdot(q+P)+M}{(q+P)^2+M^2}\gamma_\alpha i\gamma_5 \\
&= \frac{4}{3\pi}\frac{\alpha_{\mathrm{IR}}}{m_G^2}\int_q^\Lambda i\gamma_5\gamma_\alpha \left[\frac{-i\gamma\cdot q + M}{q^2+M^2} + \frac{i\gamma\cdot(q+P)+M}{(q+P)^2+M^2}\right]\gamma_\alpha,
\end{aligned} \tag{45}$$

and now take the trace

- First equation

$$\begin{aligned}
8M &= 16\frac{4}{3\pi}\frac{\alpha_{\mathrm{IR}}}{m_G^2}\int_q^\Lambda \left[\frac{M}{q^2+M^2} + \frac{M}{(q+P)^2+M^2}\right] \\
M &= M\frac{8}{3\pi}\frac{\alpha_{\mathrm{IR}}}{m_G^2}\int_q^\Lambda \left[\frac{1}{q^2+M^2} + \frac{1}{(q+P)^2+M^2}\right].
\end{aligned} \tag{46}$$

- Second equation (multiplying by $(\gamma\cdot P)$)

$$\begin{aligned}
0 &= \frac{4}{3\pi}\frac{\alpha_{\mathrm{IR}}}{m_G^2}\int_q^\Lambda \frac{8i(P\cdot q)}{q^2+M^2} - \frac{8i(P\cdot(q+P))}{(q+P)^2+M^2} \\
0 &= \int_q^\Lambda \frac{P\cdot q}{q^2+M^2} - \frac{P\cdot(q+P)}{(q+P)^2+M^2}.
\end{aligned} \tag{47}$$

Analyzing the integrands using a Feynman parametrisation, beginning with the Eq. (46)

$$
\begin{aligned}
M &= M\frac{8}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_q^\Lambda\left[\frac{1}{q^2+M^2}+\frac{1}{(q+P)^2+M^2}\right]\\
&= M\frac{8}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_q^\Lambda\frac{(q+P)^2+M^2+q^2+M^2}{(q^2+M^2)((q+P)^2+M^2)}\\
&= M\frac{8}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_q^\Lambda\frac{2q^2+2M^2+P^2+2q\cdot P}{(q^2+M^2)((q+P)^2+M^2)}\\
&= M\frac{8}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_q^\Lambda\int_0^1 d\alpha\,\frac{2q^2+2M^2+P^2+2q\cdot P}{\left[(1-\alpha)(q^2+M^2)+\alpha((q+P)^2+M^2)\right]^2}\\
&= M\frac{8}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_q^\Lambda\int_0^1 d\alpha\,\frac{2q^2+2M^2+P^2+2q\cdot P}{\left[(q+\alpha P)^2+M^2+P^2\alpha(1-\alpha)\right]^2}\\
&= M\frac{8}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_q^\Lambda\int_0^1 d\alpha\,\frac{2(q^2+M^2)+2(1-2\alpha)q\cdot P+(1-2\alpha+2\alpha^2)P^2}{\left[q^2+M^2+P^2\alpha(1-\alpha)\right]^2}\\
&= M\frac{8}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_q^\Lambda\int_0^1 d\alpha\,\frac{2(q^2+M^2)+(1-2\alpha+2\alpha^2)P^2}{\left[q^2+M^2+P^2\alpha(1-\alpha)\right]^2}\\
&\overset{P^2=m^2=0}{=} M\frac{8}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_q^\Lambda\int_0^1 d\alpha\,\frac{2(q^2+M^2)}{\left[q^2+M^2\right]^2}\\
&= M\frac{8}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_q^\Lambda\frac{2(q^2+M^2)}{\left[q^2+M^2\right]^2}\\
&= M\frac{16}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_q^\Lambda\frac{1}{q^2+M^2},
\end{aligned}
\tag{48}
$$

where we have done a linear transformation $q\to q-\alpha P$ and so $q^2\to q^2+\alpha^2P^2-2\alpha q\cdot P$. The last equation is just the chiral-limit of the gap equation (13). Hence it requires nothing new of the regularisation scheme.

On the other hand, the equation

$$
\begin{aligned}
0 &= \int_q^\Lambda\frac{P\cdot q}{q^2+M^2}-\frac{P\cdot(q+P)}{(q+P)^2+M^2}\\
&= \int_q^\Lambda\frac{(P\cdot q)((q+P)^2+M^2)-(P\cdot(q+P))(q^2+M^2)}{[q^2+M^2][(q+P)^2+M^2]}\\
&= \int_q^\Lambda\frac{P^2\left(-q^2-M^2+q\cdot P+\frac{2(q\cdot P)^2}{P^2}\right)}{[q^2+M^2][(q+P)^2+M^2]},
\end{aligned}
\tag{49}
$$

knowing that

$$
\begin{aligned}
&(P\cdot q)[(q+P)^2+M^2]-[P\cdot(q+P)][q^2+M^2]=\\
&=(P\cdot q)(q^2+P^2+2q\cdot P+M^2)-(P\cdot q)(q^2+M^2)-P^2(q^2+M^2)\\
&=(P\cdot q)(P^2+2q\cdot P)-P^2(q^2+M^2)\\
&=P^2\left(-q^2-M^2+q\cdot P+\frac{2(q\cdot P)^2}{P^2}\right).
\end{aligned}
\tag{50}
$$

Now, if one makes a Feynman parametrisation and taking into account that the integral of $(q\cdot P)$ is zero (as above) and the integral of $(q\cdot P)^2$ is equal to $\frac{1}{4}(q^2P^2)$

$$\begin{aligned}
0 = & \int_q^\Lambda \frac{P^2\left(q^2+M^2-q\cdot P-\frac{2(q\cdot P)^2}{P^2}\right)}{[q^2+M^2][(q+P)^2+M^2]} \\
= & \int_q^\Lambda \int_0^1 d\alpha\, \frac{(q-\alpha P)^2+M^2-(q-\alpha P)\cdot P-\frac{2}{P^2}((q-\alpha P)\cdot P)^2}{\left[q^2+M^2+P^2\alpha(1-\alpha)\right]^2} \\
= & \int_q^\Lambda \int_0^1 d\alpha\, \frac{(q-\alpha P)^2+M^2-(q-\alpha P)\cdot P-\frac{2}{P^2}((q\cdot P-\alpha P^2)^2}{\left[q^2+M^2+P^2\alpha(1-\alpha)\right]^2} \\
= & \int_q^\Lambda \int_0^1 d\alpha\, \frac{(q-\alpha P)^2+M^2-(q-\alpha P)\cdot P-\frac{2}{P^2}((q\cdot P)^2+\alpha^2P^4-2\alpha P^2(q\cdot P))}{\left[q^2+M^2+P^2\alpha(1-\alpha)\right]^2} \\
= & \int_q^\Lambda \int_0^1 d\alpha\, \frac{\left(\frac{1}{2}q^2+M^2+\alpha(1-\alpha)P^2\right)}{\left[q^2+M^2+P^2\alpha(1-\alpha)\right]^2} \\
\overset{P^2=m^2=0}{=} & \int_q^\Lambda \int_0^1 d\alpha\, \frac{\frac{1}{2}q^2+M^2}{\left[q^2+M^2\right]^2} \\
= & \int_q^\Lambda \frac{\frac{1}{2}q^2+M^2}{\left[q^2+M^2\right]^2}
\end{aligned} \tag{51}$$

This is a fine thing because it says that the axial-vector Ward-Takahashi identity is satisfied if, and only if, the model is regularised in such a manner as to ensure there are no quadratic or logarithmic divergences. In such circumstances a change in integration variables is permitted. Hence, one must enforce Eq. (51) whenever possible in the following. In doing so, one is redefining the model in such a way as to enforce the axial-vector Ward-Takahashi identity.

Taking into account the following expressions

$$\begin{aligned}
\int \frac{d^4q}{(2\pi)^4}\int_0^1 d\alpha\, \frac{1}{q^2+\Delta} &= \frac{1}{16\pi^2}\int_0^1 d\alpha\, \mathcal{C}^{iu}(\Delta) \\
\int \frac{d^4q}{(2\pi)^4}\int_0^1 d\alpha\, \frac{\Delta}{(q^2+\Delta)^2} &= \frac{1}{16\pi^2}\int_0^1 d\alpha\, \mathcal{C}_1^{iu}(\Delta) \\
\int \frac{d^4q}{(2\pi)^4}\int_0^1 d\alpha\, \frac{1}{(q^2+\Delta)^2} &= \frac{1}{16\pi^2}\int_0^1 d\alpha\, \overline{\mathcal{C}}_1^{iu}(\Delta)
\end{aligned} \tag{52}$$

with

$$\begin{aligned}
&\mathcal{C}^{iu}(z) = z\left(\Gamma(-1,z\tau_{uv}^2)-\Gamma(-1,z\tau_{ir}^2)\right) \\
&\mathcal{C}_1^{iu}(z) = -z\frac{d}{dz}\mathcal{C}^{iu}(z) = z\left[\Gamma(0,z\tau_{uv}^2)-\Gamma(0,z\tau_{ir}^2)\right] \\
&\overline{\mathcal{C}}_1^{iu} = \mathcal{C}_1^{iu}(z)/z
\end{aligned} \tag{53}$$

One obtains that

$$
\begin{aligned}
0 &= \int_q^\Lambda \int_0^1 d\alpha \, \frac{\left(\frac{1}{2}q^2 + M^2 + \alpha(1-\alpha)P^2\right)}{\left[q^2 + M^2 + P^2\alpha(1-\alpha)\right]^2} \\
&= \int_q^\Lambda \int_0^1 d\alpha \, \frac{\left(\frac{1}{2}q^2 + \omega\right)}{\left[q^2 + \omega\right]^2} \\
&= \int_q^\Lambda \int_0^1 d\alpha \, \frac{\left(\frac{1}{2}q^2 + \frac{1}{2}\omega\right)}{\left[q^2 + \omega\right]^2} + \frac{\frac{1}{2}\omega}{\left[q^2 + \omega\right]^2} \\
&= \int_q^\Lambda \int_0^1 d\alpha \, \frac{\left(q^2 + \omega\right)}{\left[q^2 + \omega\right]^2} + \frac{\omega}{\left[q^2 + \omega\right]^2} \\
&= \int_q^\Lambda \int_0^1 d\alpha \, \frac{1}{\left[q^2 + \omega\right]} + \frac{\omega}{\left[q^2 + \omega\right]^2} \\
&= \int_0^1 d\alpha \, \mathcal{C}^{iu}(\omega) + \mathcal{C}_1^{iu}(\omega)
\end{aligned}
\tag{54}
$$

with $\omega(M^2, \alpha, P^2) = M^2 + \alpha(1-\alpha)P^2$.

1.3.2 Vector Ward-Takahashi identity

As a completeness, we discuss here the vector Ward-Takahashi identity. This can be expressed as

$$
P_\mu i\Gamma_\mu^\gamma(k_+, k) = S^{-1}(k_+) - S^{-1}(k), \tag{55}
$$

where $\Gamma_\mu^\gamma(k_+, k)$ is the dressed-quark photon vertex, is crucial for a sensible study of a bound state's electromagnetic form factors. The vertex must be dressed at a level consistent with the truncation used to compute the bound state's Bethe-Salpeter or Faddeev amplitude. Herein, this means the vertex should be determined from the following inhomogeneous Bethe-Salpeter equation

$$
\Gamma_\mu(P) = \gamma_\mu - \frac{16\pi}{3}\frac{\alpha_{\mathrm{IR}}}{m_G^2}\int \frac{d^4q}{(2\pi)^4}\gamma_\alpha S(q+P)\Gamma_\mu(P)S(q)\gamma_\alpha. \tag{56}
$$

Owing to the momentum-independent nature of the interaction kernel, the general form of the solution is

$$
\Gamma_\mu(P) = \gamma_\mu^\perp P_\perp(P^2) + \gamma_\mu^{||}P_{||}(P^2). \tag{57}
$$

Now, multiplying Eq. (56) by $P_\mu i$ and inserting Eq. (57) into the integrands

$$
\begin{aligned}
P_\mu i\Gamma_\mu(P) &= P_\mu i\gamma_\mu - \frac{16\pi}{3}\frac{\alpha_{\mathrm{IR}}}{m_G^2}\int \frac{d^4q}{(2\pi)^4}\gamma_\alpha S(q+P)P_\mu i\Gamma_\mu(P)S(q)\gamma_\alpha \\
&= P_\mu i\gamma_\mu - \frac{16\pi}{3}\frac{\alpha_{\mathrm{IR}}}{m_G^2}\int \frac{d^4q}{(2\pi)^4}\gamma_\alpha S(q+P)[S^{-1}(q+P) - S^{-1}(q)]S(q)\gamma_\alpha \\
&= P_\mu i\gamma_\mu - \frac{16\pi}{3}\frac{\alpha_{\mathrm{IR}}}{m_G^2}\int \frac{d^4q}{(2\pi)^4}\gamma_\alpha[S(q) - S(q+P)]\gamma_\alpha \\
&= P_\mu i\gamma_\mu - \frac{4}{3\pi}\frac{\alpha_{\mathrm{IR}}}{m_G^2}\int_q^\Lambda \gamma_\alpha S(q)\gamma_\alpha + \frac{4}{3\pi}\frac{\alpha_{\mathrm{IR}}}{m_G^2}\int_q^\Lambda \gamma_\alpha S(q+P)\gamma_\alpha \\
&= P_\mu i\gamma_\mu - (S^{-1}(p) - i\gamma\cdot p) + (S^{-1}(p) - i\gamma\cdot p) \\
&= P_\mu i\gamma_\mu,
\end{aligned}
\tag{58}
$$

and so

$$
\begin{aligned}
&P_\mu i\Gamma_\mu(P) = P_\mu i\gamma_\mu \\
&P_\mu i(\gamma_\mu^\perp P_\perp(P^2) + \gamma_\mu^{||} P_{||}(P^2)) = P_\mu i\gamma_\mu \\
&i\gamma\cdot P P_{||}(P^2) = i\gamma\cdot P,
\end{aligned}
\tag{59}
$$

arriving that

$$
P_{||}(P^2) = 1. \tag{60}
$$

On the other hand, if one multiplies Eq. (56) by k_μ

$$
\begin{aligned}
k_\mu\Gamma_\mu(P) &= k_\mu\gamma_\mu - \frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\alpha S(q+P)k_\mu\Gamma_\mu(P)S(q)\gamma_\alpha \\
k_\mu[\gamma_\mu^\perp P_\perp(P^2) + \gamma_\mu^{||}] &= k_\mu\gamma_\mu - \frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\alpha S(q+P)k_\mu[\gamma_\mu^\perp P_\perp(P^2) + \gamma_\mu^{||}]S(q)\gamma_\alpha \\
\gamma\cdot k\, P_\perp(P^2) &= \gamma\cdot k - P_\perp(P^2)\frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\alpha S(q+P)\gamma\cdot k S(q)\gamma_\alpha,
\end{aligned}
\tag{61}
$$

Now, if one multiples Eq. (152) by $(\gamma\cdot k)$

$$
\begin{aligned}
&(\gamma\cdot k)(\gamma\cdot k)P_\perp(P^2) = (\gamma\cdot k)(\gamma\cdot k) - P_\perp(P^2)\frac{4}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_q^\Lambda(\gamma\cdot k)\gamma_\alpha S(q+P)(\gamma\cdot k)S(q)\gamma_\alpha \\
&(\gamma\cdot k)(\gamma\cdot k)P_\perp(P^2) =(\gamma\cdot k)(\gamma\cdot k) \\
&\qquad\qquad - P_\perp(P^2)\frac{4}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_q^\Lambda(\gamma\cdot k)\gamma_\alpha\frac{-i\gamma\cdot(q+P)+M}{(q+P)^2+M^2}(\gamma\cdot k)\frac{-i\gamma\cdot q+M}{q^2+M^2}\gamma_\alpha \\
&4k^2P_\perp(P^2) = 4k^2 - P_\perp(P^2)\frac{4}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_q^\Lambda\frac{16(k\cdot q)(k\cdot(q+P)) - 8k^2q\cdot(q+P) - 8k^2M^2}{[q^2+M^2][(q+P)^2+M^2]} \\
&P_\perp(P^2) = 1 - P_\perp(P^2)\frac{1}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_q^\Lambda\frac{8}{k^2}\frac{2(k\cdot q)(k\cdot(q+P)) - k^2q\cdot(q+P) - k^2M^2}{[q^2+M^2][(q+P)^2+M^2]} \\
&P_\perp(P^2) = 1 - P_\perp(P^2)\frac{1}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_q^\Lambda\frac{8}{k^2}\frac{2(k\cdot q)^2 + 2(k\cdot q)(k\cdot P)) - k^2q^2 - k^2q\cdot P - k^2M^2}{[q^2+M^2][(q+P)^2+M^2]} \\
&P_\perp(P^2) = 1 - P_\perp(P^2)\frac{1}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_q^\Lambda\frac{8}{k^2}\frac{2(k\cdot q)^2 - k^2q^2 - k^2q\cdot P - k^2M^2}{[q^2+M^2][(q+P)^2+M^2]}
\end{aligned}
\tag{62}
$$

where $k\cdot P = 0$, and so

$$
P_\perp(P^2) = \frac{1}{1+K_\gamma(P^2)} \tag{63}
$$

where

$$\begin{aligned}
K_\gamma(P^2) &= \frac{1}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_q^\Lambda \frac{8}{k^2}\frac{2(k\cdot q)^2 - k^2q^2 - k^2 q\cdot P - k^2M^2}{[q^2+M^2][(q+P)^2+M^2]} \\
&= \frac{1}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_q^\Lambda \int_0^1 d\alpha\, \frac{8}{k^2}\frac{2(k\cdot q)^2 - k^2q^2 - k^2 q\cdot P - k^2M^2}{[(1-\alpha)(q^2+M^2)+\alpha((q+P)^2+M^2)]^2} \\
&= \frac{1}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_q^\Lambda \int_0^1 d\alpha\, \frac{-4\left[(q^2+\omega)+\omega-4P^2\alpha(1-\alpha)\right]}{(q^2+\omega)^2} \\
&= -\frac{4}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_q^\Lambda \int_0^1 d\alpha\, \frac{1}{(q^2+\omega)} + \frac{\omega}{(q^2+\omega)^2} - \frac{4P^2\alpha(1-\alpha)}{(q^2+\omega)^2} \\
&= -\frac{4}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}(4\pi^2)\int \frac{d^4q}{(2\pi)^4}\int_0^1 d\alpha\, \frac{1}{(q^2+\omega)} + \frac{\omega}{(q^2+\omega)^2} - \frac{4P^2\alpha(1-\alpha)}{(q^2+\omega)^2} \\
&= -\frac{4}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}(4\pi^2)(-4)\int \frac{d^4q}{(2\pi)^4}\int_0^1 d\alpha\, P^2\alpha(1-\alpha)\frac{1}{(q^2+\omega)^2} \\
&= -\frac{4}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}(4\pi^2)(-4)\frac{1}{16\pi^2}\int_0^1 d\alpha\, P^2\alpha(1-\alpha)\bar{\mathcal{C}}_1^{iu}(\omega) \\
&= \frac{4}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_0^1 d\alpha\, P^2\alpha(1-\alpha)\bar{\mathcal{C}}_1^{iu}(\omega(M^2,\alpha,P^2))
\end{aligned} \tag{64}$$

where

$$\begin{aligned}
&\frac{8}{k^2}\left[2(k\cdot q)^2 - k^2q^2 - k^2q\cdot P - k^2M^2\right] = \\
&= \frac{8}{k^2}\left[2(k\cdot(q-\alpha P))^2 - k^2(q-\alpha P)^2 - k^2((q-\alpha P)\cdot P) - k^2M^2\right] \\
&= \frac{8}{k^2}\left[2(k\cdot q - \alpha k\cdot P)^2 - k^2(q^2 - 2\alpha q\cdot P + \alpha^2P^2) - k^2(q\cdot P - \alpha P^2) - k^2M^2\right] \\
&= \frac{8}{k^2}\left\{2(k\cdot q)^2 - k^2\left[q^2+M^2+P^2\alpha(\alpha-1)+q\cdot P(1-2\alpha)\right]\right\} \\
&= \frac{8}{k^2}\left\{\frac{1}{2}k^2q^2 - k^2\left[q^2+M^2+P^2\alpha(\alpha-1)\right]\right\} \\
&= -4\left[q^2+2M^2-2P^2\alpha(1-\alpha)\right] \\
&= -4\left[q^2+2M^2+2P^2\alpha(1-\alpha)-4P^2\alpha(1-\alpha)\right] \\
&= -4\left[(q^2+\omega)+\omega-4P^2\alpha(1-\alpha)\right]
\end{aligned} \tag{65}$$

Plainly,

$$P_T(P^2=0) = 1, \tag{66}$$

so that at $P^2 = 0$ in the rainbow-ladder truncation treatment of the interaction, the dressed-quark-photon vertex is equal to the bare vertex.

However, this is not true for $P^2 \neq 0$. In fact, the transverse part of the dressed-quark-photon vertex will display a pole at $P^2 < 0$ for which

$$1 + K_\gamma(P^2) = 0. \tag{67}$$

This is just the model's Bethe-Salpeter equation for the ground state vector meson as we shall see later. If one draws the function that dress the transverse part of the quark photon vertex, $P_T(P^2)$, over P^2, the pole associated with the ground state vector meson is clear. This is accompained by a minimun at spacelike P^2. The minimun arises because, in an internally consistent computation, spectral strength in the 1^{--} channel is shifted to the ρ-meson pole. One cannot simultaneously satisfy the Ward-Takahashi identity $P_T(P^2 = 0) = 1$ and exhibit the ρ pole unless the dressing fuction is depleted for $P^2 > 0$. The precise location and depth of the minimun are model dependent, but its existence is model independent. Another important feature is the behaviour at large spacelike P^2, namely $P_T(P^2) = 1^-$ as $P^2 \to \infty$. This is the statement that a dressed quark is pointlike to a large-P^2 probe. The same is true in QCD up to the logarithmic corrections, which are characteristic of an asymptotically free theory.

1.4 Eigenvalue problem of Mesons and diquarks

We present here explicit forms of the homogeneous rainbow-ladder- Bethe-Salpeter equations for ground state $J^P = 0^-, 0^+, 1^-, 1^+$ mesons and diquarks, obtained using the interaction and regularisation scheme described above.

1.4.1 Pseudoscalar mesons and scalar diquarks

The explicit form of the equation

$$\Gamma_{(q\bar{q},0^-)}(P) = -\frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\mu S(q+P)\Gamma_{(q\bar{q},0^-)}(P)S(q)\gamma_\mu \tag{68}$$

with

$$\Gamma_{(q\bar{q},0^-)}(P) = i\gamma_5 E_{(q\bar{q},0^-)}(P) + \frac{1}{M}\gamma_5\gamma\cdot P F_{(q\bar{q},0^-)}(P), \tag{69}$$

is given by

$$\begin{bmatrix} E_{(q\bar{q},0^-)}(P) \\ F_{(q\bar{q},0^-)}(P) \end{bmatrix} = \frac{4}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\begin{bmatrix} \mathcal{K}_{EE}^{0^-} & \mathcal{K}_{EF}^{0^-} \\ \mathcal{K}_{FE}^{0^-} & \mathcal{K}_{FF}^{0^-} \end{bmatrix}\begin{bmatrix} E_{(q\bar{q},0^-)}(P) \\ F_{(q\bar{q},0^-)}(P) \end{bmatrix}, \tag{70}$$

where

$$\begin{aligned}
\mathcal{K}_{EE}^{\pi} &= \int_0^1 d\alpha\left[\mathcal{C}^{iu}(\omega(M^2,\alpha,P^2)) - 2\alpha(1-\alpha)P^2\bar{\mathcal{C}}_1^{iu}(\omega(M^2,\alpha,P^2))\right], \\
\mathcal{K}_{EF}^{\pi} &= P^2\int_0^1 d\alpha\,\bar{\mathcal{C}}_1^{iu}(\omega(M^2,\alpha,P^2)), \\
\mathcal{K}_{FE}^{\pi} &= \frac{1}{2}M^2\int_0^1 d\alpha\,\bar{\mathcal{C}}_1^{iu}(\omega(M^2,\alpha,P^2)), \\
\mathcal{K}_{FF}^{\pi} &= -2\mathcal{K}_{FE}^{\pi}.
\end{aligned} \tag{71}$$

How do we obtain the expression above?, see below

$$
\begin{aligned}
&\gamma_5\left[iE_{(q\bar{q},0^-)}(P)+\frac{1}{M}\gamma\cdot PF_{(q\bar{q},0^-)}(P)\right]= \\
&=-\frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\mu S(q+P)\gamma_5\left[iE_{(q\bar{q},0^-)}(P)+\frac{1}{M}\gamma\cdot PF_{(q\bar{q},0^-)}(P)\right]S(q)\gamma_\mu \\
&=-\frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\mu\frac{-i\gamma\cdot(q+P)+M}{(q+P)^2+M^2}\gamma_5\left[iE_{(q\bar{q},0^-)}(P)+\frac{1}{M}\gamma\cdot PF_{(q\bar{q},0^-)}(P)\right]\frac{-i\gamma\cdot q+M}{q^2+M^2}\gamma_\mu \\
&=\gamma_5\frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\mu\frac{i\gamma\cdot(q+P)+M}{(q+P)^2+M^2}\left[iE_{(q\bar{q},0^-)}(P)+\frac{1}{M}\gamma\cdot PF_{(q\bar{q},0^-)}(P)\right]\frac{-i\gamma\cdot q+M}{q^2+M^2}\gamma_\mu
\end{aligned}
\tag{72}
$$

now, one can do the trace

$$
\begin{aligned}
&4iE_{(q\bar{q},0^-)}(P)=\frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\times \\
&\times\int\frac{d^4q}{(2\pi)^4}\frac{16iE_{(q\bar{q},0^-)}(P)q\cdot(q+P)+16iF_{(q\bar{q},0^-)}(P)P\cdot(q+P)+16iM^2E_{(q\bar{q},0^-)}(P)-16iF_{(q\bar{q},0^-)}(P)P\cdot q}{[q^2+M^2][(q+P)^2+M^2]} \\
&E_{(q\bar{q},0^-)}(P)=\frac{64\pi}{3}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\times \\
&\times\int\frac{d^4q}{(2\pi)^4}\frac{E_{(q\bar{q},0^-)}(P)q\cdot(q+P)+F_{(q\bar{q},0^-)}(P)P\cdot(q+P)+M^2E_{(q\bar{q},0^-)}(P)-F_{(q\bar{q},0^-)}(P)P\cdot q}{[q^2+M^2][(q+P)^2+M^2]}
\end{aligned}
\tag{73}
$$

and so for $E_{(q\bar{q},0^-)}(P)$ one obtains

$$
E_{(q\bar{q},0^-)}(P)=\frac{4}{3\pi}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\left[\mathcal{K}^\pi_{EE}E_{(q\bar{q},0^-)}(P)+\mathcal{K}^\pi_{EF}F_{(q\bar{q},0^-)}(P)\right] \tag{74}
$$

where

$$
\begin{aligned}
\mathcal{K}^\pi_{EE}&=16\pi^2\int\frac{d^4q}{(2\pi)^4}\frac{q\cdot(q+P)+M^2}{[q^2+M^2][(q+P)^2+M^2]} \\
\mathcal{K}^\pi_{EF}&=16\pi^2\int\frac{d^4q}{(2\pi)^4}\frac{P\cdot(q+P)-P\cdot q}{[q^2+M^2][(q+P)^2+M^2]}
\end{aligned}
\tag{75}
$$

with

$$
\begin{aligned}
\mathcal{K}^\pi_{EE}&=16\pi^2\int\frac{d^4q}{(2\pi)^4}\frac{q\cdot(q+P)+M^2}{[q^2+M^2][(q+P)^2+M^2]} \\
&=16\pi^2\int\frac{d^4q}{(2\pi)^4}\int_0^1 d\alpha\frac{q^2+M^2-P^2\alpha(1-\alpha)}{(q^2+\omega)^2} \\
&=16\pi^2\int\frac{d^4q}{(2\pi)^4}\int_0^1 d\alpha\frac{q^2+\omega-2P^2\alpha(1-\alpha)}{(q^2+\omega)^2} \\
&=16\pi^2\int\frac{d^4q}{(2\pi)^4}\int_0^1 d\alpha\frac{1}{(q^2+\omega)}-\frac{2P^2\alpha(1-\alpha)}{(q^2+\omega)^2} \\
&=\int_0^1 d\alpha\left[\mathcal{C}^{iu}(\omega)-2\alpha(1-\alpha)P^2\bar{\mathcal{C}}^{iu}_1(\omega)\right]
\end{aligned}
\tag{76}
$$

and

$$
\begin{aligned}
\mathcal{K}^{\pi}_{EF} &= 16\pi^2 \int \frac{d^4q}{(2\pi)^4} \frac{P\cdot(q+P) - P\cdot q}{[q^2+M^2][(q+P)^2+M^2]} \\
&= 16\pi^2 \int \frac{d^4q}{(2\pi)^4} \frac{P^2}{[q^2+M^2][(q+P)^2+M^2]} \\
&= P^2 \int_0^1 d\alpha\, \bar{\mathcal{C}}_1^{\,iu}(\omega)
\end{aligned} \tag{77}
$$

To demonstrate the another equation one must multiply Eq. (72) by $(\gamma \cdot P)$

$$
\begin{aligned}
&-\gamma_5 \left[i(\gamma\cdot P)E_{(q\bar{q},0^-)}(P) + \frac{1}{M}(\gamma\cdot P)(\gamma\cdot P)F_{(q\bar{q},0^-)}(P)\right] = \\
&= -\gamma_5 \frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\int \frac{d^4q}{(2\pi)^4}(\gamma\cdot P)\gamma_\mu \frac{i\gamma\cdot(q+P)+M}{(q+P)^2+M^2}\left[iE_{(q\bar{q},0^-)}(P) + \frac{1}{M}\gamma\cdot P F_{(q\bar{q},0^-)}(P)\right]\frac{-i\gamma\cdot q+M}{q^2+M^2}\gamma_\mu
\end{aligned} \tag{78}
$$

now, one can do the trace

$$
\begin{aligned}
&\frac{4}{M}P^2 F_{(q\bar{q},0^-)}(P) = \frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\times \\
&\times \int \frac{d^4q}{(2\pi)^4} \frac{8ME(P\cdot(q+P)) + (F/M)[-16(P\cdot q)(P\cdot(q+P)) + 8P^2(q\cdot(q+P))] - 8ME(P\cdot q) - 8P^2FM}{[q^2+M^2][(q+P)^2+M^2]} \\
&F_{(q\bar{q},0^-)}(P) = \frac{32\pi}{3}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\times \\
&\times \int \frac{d^4q}{(2\pi)^4}\frac{1}{P^2} \frac{M^2E(P\cdot(q+P)) + F[-2(P\cdot q)(P\cdot(q+P)) + P^2(q\cdot(q+P))] - M^2E(P\cdot q) - P^2FM^2}{[q^2+M^2][(q+P)^2+M^2]}
\end{aligned} \tag{79}
$$

and so for $F_{(q\bar{q},0^-)}(P)$ one obtains

$$
F_{(q\bar{q},0^-)}(P) = \frac{4}{3\pi}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\left[\mathcal{K}^{\pi}_{FE}E_{(q\bar{q},0^-)} + \mathcal{K}^{\pi}_{FF}F_{(q\bar{q},0^-)}\right] \tag{80}
$$

where

$$
\begin{aligned}
\mathcal{K}^{\pi}_{FE} &= 8\pi^2 \int \frac{d^4q}{(2\pi)^4}\frac{1}{P^2}\frac{M^2[(P\cdot(q+P)) - (P\cdot q)]}{[q^2+M^2][(q+P)^2+M^2]} \\
&= 8\pi^2 \int \frac{d^4q}{(2\pi)^4}\int_0^1 d\alpha\, \frac{M^2}{(q^2+\omega)^2} \\
&= \frac{M^2}{2}\int_0^1 d\alpha\, \bar{\mathcal{C}}_1^{iu}(\omega),
\end{aligned} \tag{81}
$$

and

$$
\begin{aligned}
\mathcal{K}_{FF}^{\pi} &= 8\pi^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{P^2} \frac{-2(P\cdot q)(P\cdot(q+P)) + P^2(q\cdot(q+P)) - P^2M^2}{[q^2+M^2][(q+P)^2+M^2]} \\
&= 8\pi^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{P^2} \frac{\frac{1}{2}q^2P^2 - M^2P^2 + P^4\alpha(1-\alpha)}{(q^2+\omega)^2} \\
&= 8\pi^2 \int \frac{d^4q}{(2\pi)^4} \frac{\frac{1}{2}q^2 - M^2 + P^2\alpha(1-\alpha)}{(q^2+\omega)^2} \\
&= 8\pi^2 \int \frac{d^4q}{(2\pi)^4} \frac{\frac{1}{2}q^2 + \frac{1}{2}\omega + \frac{1}{2}\omega - 2M^2}{(q^2+\omega)^2} \\
&= -M^2 \int_0^1 d\alpha\, \bar{\mathcal{C}}_1^{iu}(\omega) \\
&= -2\mathcal{K}_{FE}^{\pi}
\end{aligned}
\tag{82}
$$

It follows immediately that the explicit form of Eq. (29) is

$$
\begin{bmatrix} E_{(qq,0^+)}(P) \\ F_{(qq,0^+)}(P) \end{bmatrix} = \frac{2}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2} \begin{bmatrix} \mathcal{K}_{EE}^{\pi} & \mathcal{K}_{EF}^{\pi} \\ \mathcal{K}_{FE}^{\pi} & \mathcal{K}_{FF}^{\pi} \end{bmatrix} \begin{bmatrix} E_{(qq,0^+)}(P) \\ F_{(qq,0^+)}(P) \end{bmatrix}, \tag{83}
$$

Eqs. (70) and (83) are eigenvalue problems: they each have a solution at a single value of $P^2 < 0$, at which point the eigenvector describes the on-shell Bethe-Salpeter amplitude.

In the computation of observables, one must use the canonically-normalised Bethe-Salpeter amplitude[6]; i.e., Γ_π is rescaled so that

$$
\mathcal{N}^2\, P_\mu = N_c {\rm tr}_D \int \frac{d^4q}{(2\pi)^4} \Gamma_\pi(-P) \frac{\partial}{\partial P_\mu} S(q+P)\Gamma_\pi(P)S(q), \tag{84}
$$

where $N_c = 3$. Eq. (84) can be rewritten in the form

$$
\begin{aligned}
\mathcal{N}^2\, P_\mu &= \frac{\partial}{\partial P_\mu}\left[3{\rm tr}_D \int \frac{d^4q}{(2\pi)^4} \Gamma_\pi(-K)S(q+P)\Gamma_\pi(K)S(q)\right]_{K=P} \\
&= 2P_\mu \left[\frac{d}{dP^2} 3{\rm tr}_D \int \frac{d^4q}{(2\pi)^4} \Gamma_\pi(-K)S(q+P)\Gamma_\pi(K)S(q)\right]_{K=P},
\end{aligned}
\tag{85}
$$

using

$$
\frac{d}{dP^2} = \frac{1}{2P^2} P_\mu \frac{\partial}{\partial P_\mu}. \tag{86}
$$

And so one has

$$
\begin{aligned}
\mathcal{N}^2 &= 2\left[\frac{d}{dP^2}\Pi(K,P)\right]_{K=P}, \\
\Pi(K,P) &= 3{\rm tr}_D \int \frac{d^4q}{(2\pi)^4} \Gamma_\pi(-K)S(q+P)\Gamma_\pi(K)S(q).
\end{aligned}
\tag{87}
$$

[6]This normalisation guarantees that the pion's electromagnetic form factor is unity at zero momentum transfer. Another view is the following: one must ensure that in the computation of the mass pole of a meson one obtains a residue equal one which tells you about the probability of the state

Now one has to evaluate

$$\begin{aligned}
\Pi(K,P) &= 3\mathrm{tr}_D \int \frac{d^4q}{(2\pi)^4}\Gamma_\pi(-K)S(q+P)\Gamma_\pi(K)S(q) \\
&= 3\mathrm{tr}_D \int \frac{d^4q}{(2\pi)^4}\gamma_5[iE_\pi(-K) - \frac{1}{M}(\gamma\cdot K)F_\pi(-K)]\frac{-i\gamma\cdot(q+P)+M}{(q+P)^2+M^2}\times \\
&\times \gamma_5[iE_\pi(K) + \frac{1}{M}(\gamma\cdot K)F_\pi(K)]\frac{-i\gamma\cdot q+M}{q^2+M^2} \\
&= 3\mathrm{tr}_D \int \frac{d^4q}{(2\pi)^4}[iE_\pi(-K) + \frac{1}{M}(\gamma\cdot K)F_\pi(-K)]\frac{i\gamma\cdot(q+P)+M}{(q+P)^2+M^2}\times \\
&\times [iE_\pi(K) + \frac{1}{M}(\gamma\cdot K)F_\pi(K)]\frac{-i\gamma\cdot q+M}{q^2+M^2}
\end{aligned} \tag{88}$$

with the following terms

- The $E_\pi^2(P)$ term:

$$\begin{aligned}
&-12\int \frac{d^4q}{(2\pi)^4}E_\pi(-K)E_\pi(K)\frac{((q\cdot(q+P))+M^2)}{[q^2+M^2][(q+P)^2+M^2]} \\
&= -12\int \frac{d^4q}{(2\pi)^4}E_\pi(-K)E_\pi(K)\frac{q^2+q\cdot P+M^2}{[q^2+M^2][(q+P)^2+M^2]} \\
&= -12\int \frac{d^4q}{(2\pi)^4}\int_0^1 d\alpha\, E_\pi(-K)E_\pi(K)\frac{q^2+M^2+P^2\alpha(\alpha-1)}{(q^2+M^2+P^2\alpha(1-\alpha))^2}
\end{aligned} \tag{89}$$

now if one takes the derivate and thereafter sets $K=P$ and implements the chiral limit ($P^2=0$), then

$$\begin{aligned}
\mathcal{N}^2_{E_\pi E_\pi}(P^2) &= -24\int \frac{d^4q}{(2\pi)^4}\int_0^1 d\alpha\, E_\pi(-K)E_\pi(K)\frac{\alpha(\alpha-1)[3q^2+3M^2+P^2\alpha(\alpha-1)]}{(q^2+M^2+P^2\alpha(1-\alpha)^3} \\
&= -24\int \frac{d^4q}{(2\pi)^4}\int_0^1 d\alpha\, E_\pi(-K)E_\pi(K)\frac{\alpha(\alpha-1)[3q^2+3M^2+3P^2\alpha(1-\alpha)+4P^2\alpha(\alpha-1)]}{(q^2+M^2+P^2\alpha(1-\alpha))^3} \\
&= -24\int \frac{d^4q}{(2\pi)^4}\int_0^1 d\alpha\, E_\pi(-K)E_\pi(K)\frac{3\alpha(\alpha-1)}{(q^2+M^2+P^2\alpha(1-\alpha))^2} + \frac{4P^2\alpha^2(\alpha-1)^2}{(q^2+M^2+P^2\alpha(1-\alpha))^3} \\
&= -24\int \frac{d^4q}{(2\pi)^4}\int_0^1 d\alpha\, E_\pi(-P)E_\pi(P)\frac{3\alpha(\alpha-1)}{(q^2+M^2)^2} \\
&= 12\, E_\pi^2(P)\int \frac{d^4q}{(2\pi)^4}\frac{1}{(q^2+M^2)^2} \\
&= \frac{2^2\, 3\, 2\pi^2}{2^5\pi^4}\, E_\pi^2(P)\frac{1}{M^2}\int_0^\infty dss\frac{M^2}{(s+M^2)^2} = \frac{3}{4\pi^2}E_\pi^2(P)\frac{1}{M^2}\mathcal{C}_1(M^2,\tau_{ir}^2,\tau_{uv}^2)
\end{aligned} \tag{90}$$

- The $E_\pi(P)F_\pi(P)$ term

$$
\begin{aligned}
&3\int \frac{d^4q}{(2\pi)^4}E_\pi(K)F_\pi(K)\frac{-8K\cdot(q+P)+8K\cdot q}{(q^2+M^2)((q+P)^2+M^2)}\\
&=3\int \frac{d^4q}{(2\pi)^4}E_\pi(K)F_\pi(K)\frac{-8K\cdot P}{(q^2+M^2)((q+P)^2+M^2)}\\
&=3\int \frac{d^4q}{(2\pi)^4}\int_0^1 d\alpha\, E_\pi(K)F_\pi(K)\frac{-8K\cdot P}{[q^2+M^2+P^2\alpha(1-\alpha)]^2}\\
&=-24\int \frac{d^4q}{(2\pi)^4}\int_0^1 d\alpha\, E_\pi(K)F_\pi(K)\frac{K\cdot P}{[q^2+M^2+P^2\alpha(1-\alpha)]^2}\\
&=-24\int \frac{d^4q}{(2\pi)^4}\int_0^1 d\alpha\, E_\pi(K)F_\pi(K)\left[\frac{\frac{1}{2P^2}K\cdot P}{[q^2+M^2+P^2\alpha(1-\alpha)]^2}-\frac{2\alpha(1-\alpha)K\cdot P}{[M^2+q^2+P^2\alpha(1-\alpha)]^3}\right]\\
&=-24\int \frac{d^4q}{(2\pi)^4}\int_0^1 d\alpha\, E_\pi(K)F_\pi(K)\left[\frac{\frac{1}{2}}{[q^2+M^2+P^2\alpha(1-\alpha)]^2}-\frac{2\alpha(1-\alpha)P^2}{[M^2+q^2+P^2\alpha(1-\alpha)]^3}\right]\\
&=-12E_\pi(K)F_\pi(K)\int \frac{d^4q}{(2\pi)^4}\int_0^1 d\alpha\,\frac{1}{(q^2+M^2)^2}\\
&=-12E_\pi(K)F_\pi(K)\int \frac{d^4q}{(2\pi)^4}\frac{1}{(q^2+M^2)^2}\\
&=\frac{3}{4\pi^2}(-1)E_\pi(P)F_\pi(P)\frac{1}{M^2}\int_0^\infty dss\frac{M^2}{(s+M^2)^2}\\
&=\frac{3}{4\pi^2}(-1)E_\pi(P)F_\pi(P)\frac{1}{M^2}\mathcal{C}_1(M^2,\tau_{ir}^2,\tau_{uv}^2)
\end{aligned}
\tag{91}
$$

- The $F_\pi^2(P)$ term

$$
\begin{aligned}
&3F_\pi^2(K)\int \frac{d^4q}{(2\pi)^4}\frac{8(K\cdot q)(K\cdot(q+P))-4K^2q\cdot(q+P)+4K^2M^2}{M^2}\frac{1}{(q^2+M^2)((q+P)^2+M^2)}\\
&=3F_\pi^2(K)\int \frac{d^4q}{(2\pi)^4}\frac{8[(K\cdot q)^2+K\cdot P]-4K^2q^2-4K^2q\cdot P+4K^2M^2}{M^2}\frac{1}{(q^2+M^2)((q+P)^2+M^2)}\\
&=3F_\pi^2(K)\int \frac{d^4q}{(2\pi)^4}\int_0^1 d\alpha\frac{-2K^2q^2-2\alpha^2K^2P^2-4\alpha K^2P^2+4K^2M^2}{M^2[q^2+M^2+P^2\alpha(1-\alpha)]^2}\\
&=3F_\pi^2(K)\int \frac{d^4q}{(2\pi)^4}\int_0^1 d\alpha\left[\frac{-2\alpha^2K^2-4\alpha K^2}{M^2[q^2+M^2+P^2\alpha(1-\alpha)]^2}\right.\\
&\left.+\frac{[-2\alpha(1-\alpha)][-2K^2q^2-2\alpha^2K^2P^2-4\alpha K^2P^2+4K^2M^2]}{M^2[q^2+M^2+P^2\alpha(1-\alpha)]^3}\right]\\
&=0
\end{aligned}
\tag{92}
$$

And finally

$$
\mathcal{N}^2(P)=\frac{3}{4\pi^2}\frac{1}{M^2}\mathcal{C}_1(M^2,\tau_{ir}^2,\tau_{uv}^2)]E_\pi(P)(E_\pi(P)-2F_\pi(P))
\tag{93}
$$

with

$$E_\pi^c = \frac{E_\pi}{\mathcal{N}}$$
$$F_\pi^c = \frac{F_\pi}{\mathcal{N}} \tag{94}$$

N.B. one needs to know that

$$\begin{aligned}
0 &= \frac{\partial}{\partial P_\mu}\left[S(q+P)S^{-1}(q+P)\right] \\
&= \left[\frac{\partial}{\partial P_\mu}S(q+P)\right]S^{-1}(q+P) + S(q+P)\left[\frac{\partial}{\partial P_\mu}S^{-1}(q+P)\right] \\
&= \left[\frac{\partial}{\partial P_\mu}S(q+P)\right] + S(q+P)\left[\frac{\partial}{\partial P_\mu}S^{-1}(q+P)\right]S(q+P) \\
&= \left[\frac{\partial}{\partial P_\mu}S(q+P)\right] + S(q+P)\left[\frac{\partial}{\partial P_\mu}(i\gamma\cdot(q+P)+M)\right]S(q+P) \\
&= \left[\frac{\partial}{\partial P_\mu}S(q+P)\right] + S(q+P)i\gamma_\mu S(q+P)
\end{aligned} \tag{95}$$

and so

$$\left[\frac{\partial}{\partial P_\mu}S(q+P)\right] = -S(q+P)i\gamma_\mu S(q+P) \tag{96}$$

1.4.2 Scalar mesons and pseudoscalar diquarks

The Bethe-Salpeter amplitude for a scalar meson is

$$\Gamma_{(q\bar{q},0^+)}(P) = \mathbf{I}_D E_{(q\bar{q},0^+)}(P). \tag{97}$$

In this case dressing the vertex does not generate new covariants because a momentum-independent interaction cannot generate a Bethe-Salpeter amplitude that depends on the relative momentum. Inserting Eq. (97) into the Eq. (23) yields the following BSE

$$\begin{aligned}
\Gamma_{(q\bar{q},0^+)}(P) &= -\frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\mu S(q+P)\Gamma_{(q\bar{q},0^+)}(P)S(q)\gamma_\mu \\
\mathbf{I}_D E_{(q\bar{q},0^+)}(P) &= -\frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\mu S(q+P)\mathbf{I}_D E_{(q\bar{q},0^+)}(P)S(q)\gamma_\mu \\
\mathbf{I}_D &= -\frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\mu S(q+P)\mathbf{I}_D S(q)\gamma_\mu \\
\mathbf{I}_D &= -\frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\mu\frac{-i\gamma\cdot(q+P)+M}{(q+P)^2+M^2}\mathbf{I}_D\frac{-i\gamma\cdot q+M}{q^2+M^2}\gamma_\mu \\
4 &= -\frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\frac{-16q\cdot(q+P)+16M^2}{[q^2+M^2][(q+P)^2+M^2]} \\
1 &= \frac{64\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\frac{q^2+q\cdot P-M^2}{[q^2+M^2][(q+P)^2+M^2]},
\end{aligned} \tag{98}$$

Now consider the equation obtained before

$$M = M\frac{32\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\left[\frac{1}{q^2+M^2}+\frac{1}{(q+P)^2+M^2}\right], \tag{99}$$

if one sets $P^2 = -4M^2$ in that chiral limit identity, then one finds

$$\begin{aligned}
M &= M\frac{32\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\left[\frac{1}{q^2+M^2}+\frac{1}{(q+P)^2+M^2}\right]\\
&= M\frac{32\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\frac{(q+P)^2+q^2+2M^2}{[q^2+M^2][(q+P)^2+M^2]}\\
&= M\frac{64\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\frac{q^2+q\cdot P-M^2}{[q^2+M^2][(q+P)^2+M^2]},
\end{aligned} \tag{100}$$

which is equivalent to Eq. (98). Hence, for $m = 0$ yields

$$m_{(q\bar{q},0^+)} = 2M, \tag{101}$$

since

$$P^2 = -m^2_{(q\bar{q},0^+)} = -4M^2. \tag{102}$$

For general values of the current-quark mass, using our symmetry-preserving regularisation prescription, Eq. (98) can be written

$$1 + K^\sigma(-m^2_{(q\bar{q},0^+)}) = 0, \tag{103}$$

with

$$\begin{aligned}
K^\sigma(P^2) &= \frac{64\pi}{3}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\int\frac{d^4q}{(2\pi)^4}\frac{-q^2-q\cdot P+M^2}{[q^2+M^2][(q+P)^2+M^2]}\\
&= \frac{64\pi}{3}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\int\frac{d^4q}{(2\pi)^4}\int_0^1 d\alpha\frac{-q^2-\omega+2\omega}{(q^2+\omega)^2}\\
&= \frac{4}{3\pi}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\int_0^1 d\alpha\,\left[2\mathcal{C}_1^{iu}(\omega)-\mathcal{C}^{iu}(\omega)\right]
\end{aligned} \tag{104}$$

The sign is different in the Reference that I follow. I obtain correct values. It follows that in the rainbow-ladder truncation the mass of a pseudoscalar diquark is determined by

$$1 + \frac{1}{2}K^\sigma(-m^2_{(qq,0^-)}) = 0. \tag{105}$$

The canonical normalisation conditions are

$$\begin{aligned}
\frac{1}{E^2_{(q\bar{q},0^+)}} &= -\frac{9}{2}\frac{m_{\rm G}^2}{4\pi\alpha_{\rm IR}}\frac{d}{dP^2}K^\sigma(P^2)\bigg|_{P^2=-m^2_{(q\bar{q},0^+)}},\\
\frac{1}{E^2_{(qq,0^-)}} &= -3\frac{m_{\rm G}^2}{4\pi\alpha_{\rm IR}}\frac{d}{dP^2}K^\sigma(P^2)\bigg|_{P^2=-m^2_{(qq,0^-)}}.
\end{aligned} \tag{106}$$

1.4.3 Vector mesons and axial-vector diquarks

The vector Bethe-Salpeter amplitude has a particularly simple form if one uses the quark-quark contact interaction and in the rainbow-ladder truncation; viz.,

$$\Gamma_\mu^{(q\bar{q},1^-)}(P) = \gamma_\mu^\perp E_{(q\bar{q},1^-)}(P). \tag{107}$$

Hence the explicit form of Eq. (23) for the ground-state vector meson, whose solution yields its mass-squared, is

$$
\begin{aligned}
\Gamma_{\alpha}^{(q\bar{q},1^{-})}(P) &= -\frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\mu S(q+P)\Gamma_{\alpha}^{(q\bar{q},1^{-})}(P)S(q)\gamma_\mu \\
\gamma_\alpha^\perp E_{(q\bar{q},1^-)}(P) &= -\frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\mu S(q+P)\gamma_\alpha^\perp E_{(q\bar{q},1^-)}(P)S(q)\gamma_\mu \\
k_\alpha\gamma_\alpha^\perp E_{(q\bar{q},1^-)}(P) &= -\frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\mu S(q+P)k_\alpha\gamma_\alpha^\perp E_{(q\bar{q},1^-)}(P)S(q)\gamma_\mu \\
(\gamma\cdot k)E_{(q\bar{q},1^-)}(P) &= -\frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\mu S(q+P)(\gamma\cdot k)E_{(q\bar{q},1^-)}(P)S(q)\gamma_\mu \\
(\gamma\cdot k)E_{(q\bar{q},1^-)}(P) &= -E_{(q\bar{q},1^-)}(P)\frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\mu S(q+P)(\gamma\cdot k)S(q)\gamma_\mu
\end{aligned}
\tag{108}
$$

If one follows the development of the vector Ward-Takahashi identity, one arrives in this case at

$$
1+K^\rho(-m^2_{(q\bar{q},1^-)})=0, \tag{109}
$$

with

$$
K^\rho(P^2)=\frac{4}{3\pi}\frac{\alpha_{\rm IR}}{m_{\rm G}^2}\int_0^1 d\alpha P^2\alpha(1-\alpha)\bar{\mathcal{C}}_1^{iu}(\omega(M^2,\alpha,P^2)). \tag{110}
$$

The BSE for the axial-vector diquark again follows immediately; viz,

$$
1+\frac{1}{2}K^\rho(-m^2_{(qq,1^+)})=0. \tag{111}
$$

The canonical normalisation conditions are readily expressed; viz.,

$$
\begin{aligned}
\frac{1}{E^2_{(q\bar{q},1^-)}} &= -9\frac{m_{\rm G}^2}{4\pi\alpha_{\rm IR}}\frac{d}{dP^2}K^\rho(P^2)\bigg|_{P^2=-m^2_{(q\bar{q},1^-)}}, \\
\frac{1}{E^2_{(qq,1^+)}} &= -6\frac{m_{\rm G}^2}{4\pi\alpha_{\rm IR}}\frac{d}{dP^2}K^\rho(P^2)\bigg|_{P^2=-m^2_{(qq,1^+)}}.
\end{aligned}
\tag{112}
$$

1.4.4 Axial-vector mesons and vector diquarks

Again owing to the simplicity of the interaction, the Bethe-Salpeter amplitude for an axial-vector meson is

$$
\Gamma_\mu^{(q\bar{q},1^+)}(P)=\gamma_5\gamma_\mu^\perp E_{(q\bar{q},1^+)}(P). \tag{113}
$$

As with scalar mesons, dressing the vertex does not generate new covariants. Inserting Eq. (113) into Eq. (23)

$$
\begin{aligned}
\Gamma_\alpha^{(q\bar{q},1^+)}(P) &= -\frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\mu S(q+P)\Gamma_\alpha^{(q\bar{q},1^+)}(P)S(q)\gamma_\mu \\
\gamma_5\gamma_\alpha^\perp E_{(q\bar{q},1^+)}(P) &= -\frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\mu S(q+P)\gamma_5\gamma_\alpha^\perp E_{(q\bar{q},1^+)}(P)S(q)\gamma_\mu \\
\gamma_5 k_\alpha\gamma_\alpha^\perp E_{(q\bar{q},1^+)}(P) &= -\frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\mu S(q+P)\gamma_5 k_\alpha\gamma_\alpha^\perp E_{(q\bar{q},1^+)}(P)S(q)\gamma_\mu \\
\gamma_5(\gamma\cdot k)E_{(q\bar{q},1^+)}(P) &= -E_{(q\bar{q},1^+)}(P)\frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\mu S(q+P)\gamma_5(\gamma\cdot k)S(q)\gamma_\mu \\
\gamma_5(\gamma\cdot k)E_{(q\bar{q},1^+)}(P) &= -E_{(q\bar{q},1^+)}(P)\frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\mu\frac{-i\gamma\cdot(q+P)+M}{(q+P)^2+M^2}\gamma_5(\gamma\cdot k)\frac{-i\gamma\cdot q+M}{q^2+M^2}\gamma_\mu \\
\gamma_5(\gamma\cdot k)E_{(q\bar{q},1^+)}(P) &= \gamma_5 E_{(q\bar{q},1^+)}(P)\frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\mu\frac{i\gamma\cdot(q+P)+M}{(q+P)^2+M^2}(\gamma\cdot k)\frac{-i\gamma\cdot q+M}{q^2+M^2}\gamma_\mu \\
\gamma_5(\gamma\cdot k)(\gamma\cdot k)E_{(q\bar{q},1^+)}(P) &= \gamma_5 E_{(q\bar{q},1^+)}(P)\frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}(\gamma\cdot k)\gamma_\mu\frac{i\gamma\cdot(q+P)+M}{(q+P)^2+M^2}(\gamma\cdot k)\frac{-i\gamma\cdot q+M}{q^2+M^2}\gamma_\mu \\
4k^2 &= \frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\frac{-16(k\cdot q)(k\cdot(q+P))+8k^2q\cdot(q+P)-8k^2M^2}{[q^2+M^2][(q+P)^2+M^2]} \\
1 &= \frac{32\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\frac{1}{k^2}\frac{-2(k\cdot q)(k\cdot(q+P))+k^2q\cdot(q+P)-k^2M^2}{[q^2+M^2][(q+P)^2+M^2]}
\end{aligned}
\tag{114}
$$

yields the following BSE:

$$1+K^{a_1}(-m^2_{(q\bar{q},1^+)})=0, \tag{115}$$

with

$$
\begin{aligned}
K^{a_1}(P^2) &= -\frac{32\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\frac{1}{k^2}\frac{-2(k\cdot q)(k\cdot(q+P))+k^2q\cdot(q+P)-k^2M^2}{[q^2+M^2][(q+P)^2+M^2]} \\
&= -\frac{32\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\int_0^1 d\alpha\,\frac{\frac{1}{2}q^2-\omega}{(q^2+\omega)^2} \\
&= -\frac{32\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\int_0^1 d\alpha\,\frac{\frac{1}{2}q^2+\frac{1}{2}\omega-\frac{1}{2}\omega-\omega}{(q^2+\omega)^2} \\
&= -\frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\int_0^1 d\alpha\left[\frac{1}{(q^2+\omega)}-\frac{\omega}{(q^2+\omega)^2}-\frac{2\omega}{(q^2+\omega)^2}\right] \\
&= -\frac{1}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_0^1 d\alpha\,\left[\mathcal{C}^{iu}(\omega)-\mathcal{C}_1^{iu}(\omega)-2\mathcal{C}_1^{iu}(\omega)\right] \\
&= \frac{4}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_0^1 d\alpha\,\mathcal{C}_1^{iu}(\omega)
\end{aligned}
\tag{116}
$$

The sign is different in the Reference that I follow. I obtain correct values. It follows that the vector-diquark mass is determined by

$$1+\frac{1}{2}K^{a_1}(-m_{(qq,1^-)})=0. \tag{117}$$

Ref.	m	M	E_π	F_π	E_π^c	F_π^c	$\kappa_\pi^{1/3}$	f_π	m_π
Ref. [1]	0	0.358	-	-	3.568	0.459	0.241	0.100	0
Ref. [1]	0.007	0.368	-	-	3.639	0.481	0.243	0.101	0.140
jsegovia	0	0.357826	0.991851	0.125401	3.56557	0.457951	0.241180	0.10358	0.005184
jsegovia	0.007	0.367690	0.991385	0.130981	3.63695	0.480511	0.244466	0.102467	0.139595

Table 2. Results obtained with (in GeV) $m_G = 0.132$, $\Lambda = 0.24$ and $\Lambda = 0.905$, which yield a root-mean-square relative error of 13% in comparison with our specified goals for the observables

	Computed masses of mesons			
	0^-	0^+	1^-	1^+
Ref. [1]	0.14	0.74	0.93	1.08
jsegovia	0.14	0.74	0.93	1.08
	Computed masses of diquarks			
	0^+	0^-	1^+	1^-
Ref. [1]	0.78	0.93	1.06	1.16
jsegovia	0.82	0.93	1.06	1.16

Table 3. Diquark masses (GeV) computed using our contact-interaction DSE kernel, which produces a momentum-independent dressed-quark mass $M = 0.37\,\text{GeV}$ from a current quark mass of $m = 7\,\text{MeV}$.

The canonical normalisation conditions are

$$\begin{aligned}
\frac{1}{E^2_{(q\bar{q},1^+)}} &= -9\frac{m_G^2}{4\pi\alpha_{\text{IR}}}\frac{d}{dP^2}K^{a_1}(P^2)\Big|_{P^2=-m^2_{(q\bar{q},1^+)}}, \\
\frac{1}{E^2_{(qq,1^+)}} &= -6\frac{m_G^2}{4\pi\alpha_{\text{IR}}}\frac{d}{dP^2}K^{\rho}(P^2)\Big|_{P^2=-m^2_{(qq,1^+)}}.
\end{aligned} \tag{118}$$

1.4.5 Numerical results

We present here our numerical results about the computation of masses and Bethe Salpeter amplitudes of mesons which have been discussed in the previous sections. As completeness we also show results for other physical observables related with them.

Tables 2 and 3 show the results. Below one can see how to calculate decay constant and in-meson condensates

$$\begin{aligned}
f_\pi &= \frac{1}{M}\frac{3}{2\pi^2}(E_\pi^c - 2F_\pi^c)\mathcal{K}^\pi_{FE}(P^2=-m_\pi^2) \\
\kappa_\pi &= f_\pi\frac{3}{4\pi^2}(E_\pi^c\mathcal{K}^\pi_{EE}(P^2=-m_\pi^2) + F_\pi^c\mathcal{K}^\pi_{EF}(P^2=-m_\pi^2)) \\
f_\rho &= -\frac{9}{2}\frac{E_\rho^c}{m_\rho}K^\rho(P^2=-m_\rho^2)
\end{aligned} \tag{119}$$

1.4.6 Meson and diquark radial excitations

In quantum mechanichs the radial wave function for a bound-state's first radial excitation possesses a single zero. A similar feature is expressed in quantum field theory: namely, in a fully covariant approach a single zero is seen in the relative-momentum dependence of the leading Tchebychev moment of the dominant Dirac structure in the bound state amplitude for a meson's first radial excitation [3]. The existence of radial excitations is therefore very obvious evidence against the possibility that the interaction between quarks is momentum-independent: a bound-state amplitude that is independent of the relative momentum cannot exibit a single zero. One may also express this differently; namely, if the location of the zero is at k_0^2, then a momentum-independent interaction can only produce reliable results for phenomena that probe momentum scales $k^2 \ll k_0^2$. In QCD, $k_0 \sim M$ and hence this criterion is equivalent to that noted in Ref. [4].

Herein, however, we skirt this difficulty inserting a zero by hand into the kernels. This means that we identify the Bethe-Salpeter equation (BSE) for a radial excitation as the form of that of a ground state obtained with the independent momentum quark-quark contact interaction but inserting into the integrand a factor

$$1 - d_{\mathcal{F}} q^2, \tag{120}$$

which forces a zero into the kernel at $q^2 = 1/d_{\mathcal{F}}$, where $d_{\mathcal{F}}$ is a parameter. It is plain that the presence of this zero has the effect of reducing the coupling in the BSE and hence it increases the bound-state's mass. Although this may not be as transparent with a more sophisticated interaction, a qualitatively equivalent mechanism is always responsible for the elevated values of the masses of radial excitations.

To illustrate our procedure, consider the BSE for the vector meson, in which the following replacement is made

$$K^{\rho}(P^2) \to K^{\rho^*} = \frac{4}{3\pi}\frac{\alpha_{IR}}{m_G^2}\int_0^1 d\alpha\, \alpha(1-\alpha)\overline{\mathcal{F}}_1^{iu}\left(\omega(M^2,\alpha,P^2)\right), \tag{121}$$

where

$$\begin{aligned} \mathcal{F}^{iu} &= \mathcal{C}^{iu} - d_{\mathcal{F}}\mathcal{D}^{iu}, \\ \mathcal{D}^{iu}\left(\omega(M^2,\alpha,P^2)\right) &= \int_0^\infty ds\, s\,\frac{s}{s+\omega} \to \int_{\tau_{uv}^2}^{\tau_{ir}^2} d\tau\,\frac{2}{\tau^3}\exp\left[-\tau\omega(M^2,\alpha,P^2)\right], \end{aligned} \tag{122}$$

$\mathcal{F}_1^{iu}(z) = -z(d/dz)\mathcal{F}^{iu}(z)$, and $\overline{\mathcal{F}}_1^{iu}(z) = \mathcal{F}_1^{iu}/z$.

Regarding the location of the zero, motivated by extant studies in the pseudoscalar channel [3], we choose $1/d_{\mathcal{F}} = M^2$. The position of the zero in the leading Tchebychev moment of an excited state in a given channel is an indication of that state's size. Hence it is an oversimplification to place the zero at the same location in each channel, one can move the zero to a 20% of its location to estimate a theoretical error.

N.B. The procedure is a little more involve in the pseudoscalar channel owing to the axial-vector Ward-Takahashi identity and the richer structure of the Bethe-Salpeter amplitude

$$\begin{aligned} \mathcal{K}_{EE}^{\pi^*} &= \int_0^1 d\alpha\left[\mathcal{F}^{iu}(\omega(M^2,\alpha,P^2)) - 2\alpha(1-\alpha)P^2\overline{\mathcal{F}}_1^{iu}(\omega(M^2,\alpha,P^2))\right], \\ \mathcal{K}_{EF}^{\pi^*} &= P^2\int_0^1 d\alpha\,\overline{\mathcal{F}}_1^{iu}(\omega(M^2,\alpha,P^2)), \\ \mathcal{K}_{FE}^{\pi^*} &= \frac{1}{2}M^2\int_0^1 d\alpha\,\overline{\mathcal{F}}_1^{iu}(\omega(M^2,\alpha,P^2)) - \frac{1}{2}M_0^2\int_0^1 d\alpha\,\overline{\mathcal{F}}_1^{iu}(\omega(M^2,\alpha,P^2)), \\ \mathcal{K}_{FF}^{\pi^*} &= -2\mathcal{K}_{FE}^{\pi}. \end{aligned} \tag{123}$$

1.5 Spectrum of Baryons

We compute the masses of light-quark baryons using a Faddeev equation built from the contact quark-quark interaction and the diquark correlations discussed quantitatively above.

1.5.1 General structure of the nucleon and Δ Faddeev equation

The nucleon is represented by a Faddeev amplitude

$$\Psi = \Psi_1 + \Psi_2 + \Psi_3, \tag{124}$$

where the subscript identifies the bystander (spectator) quark and, e.g., $\Psi_{1,2}$ are obtained from Ψ_3 by a cyclic permutation of all the quark labels. We employ the simplest realistic representation of Ψ. The spin- and isospin-1/2 nucleon is a sum of scalar and axial-vector diquark correlations

$$\Psi_3^N(p_i,\alpha_i,\tau_i) = \mathcal{N}_3^{(qq,0^+)} + \mathcal{N}_3^{(qq,1^+)}, \tag{125}$$

with (p_i,α_i,τ_i) the momentum, spin and isospin labels of the quarks constituting the bound state, and $P = p_1+p_2+p_3$ the system's total momentum.

It is conceivable that pseudoscalar and vector diquarks could play a role in the ground-state nucleon's Faddeev amplitude. However, they have parity opposite to that of the nucleon and hence can only appear in concert with nonzero quark angular momentum. Since one expects the ground-state nucleon to possess the minimum possible amount of quark orbital angular momentum and these diquark correlations are significantly more massive than the scalar and axial-vector (Table 3), they can safely be ignored in computing properties of the ground state.

The scalar diquark piece in Eq. (125) is

$$\mathcal{N}_3^{(qq,0^+)}(p_i,\alpha_i,\tau_i) = \left[\Gamma^{(qq,0^+)}\left(\frac{1}{2}p_{[12]};K\right)\right]_{\alpha_1\alpha_2}^{\tau_1\tau_2} \Delta^{(qq,0^+)}(K)[\mathcal{S}(l;P)u(P)]_{\alpha_3}^{\tau_3}, \tag{126}$$

where: the spinor satisfies

$$(i\gamma\cdot P + M)u(P) = 0 = \bar{u}(P)(i\gamma\cdot P+M), \tag{127}$$

with M the mass obtained by solving the Faddeev equation, and it is also a spinor is isospin space with $\varphi_+ = \text{col}(1,0)$ for the proton and $\varphi_- = \text{col}(0,1)$ for the neutron; $K = p_1+p_2 = p_{\{12\}}$, $p_{[12]} = p_1 - p_2$, $l = (-p_{\{12\}} + 2p_3)/3$;

$$\Delta^{(qq,0^+)}(K) = \frac{1}{K^2 + m_{(qq,0^+)}^2} \tag{128}$$

is a propagator for the scalar diquark formed from quarks 1 and 2, with $m_{(qq,0^+)}$ the mass-scale associated with this correlation, and $\Gamma^{(qq,0^+)}$ is the canonically-normalised Bethe-Salpeter amplitude describing their relative momentum correlation; and $\mathcal{S}$, a 4×4 Dirac matrix, describes the relative quark-diquark momentum correlation. The color antisymmetry of Ψ_3 is implicit in $\Gamma^{(qq,J^P)}$.

The axial-vector component in Eq. (125) is

$$\mathcal{N}^{(qq,1^+)}(p_i,\alpha_i,\tau_i) = \left[t^i\Gamma_\mu^{(qq,1^+)}\left(\frac{1}{2}p_{[12]};K\right)\right]_{\alpha_1\alpha_2}^{\tau_1\tau_2} \Delta_{\mu\nu}^{(qq,1^+)}(K)[\mathcal{A}_\nu^i(l;P)u(P)]_{\alpha_3}^{\tau_3}, \tag{129}$$

where the symmetric isospin-triplet matrices are

$$t^+ = \frac{1}{\sqrt{2}}(\tau^0+\tau^3), \quad t^0 = \tau^1, \quad t^- = \frac{1}{\sqrt{2}}(\tau^0-\tau^3), \tag{130}$$

and the other elements in Eq. (129) are straightforward generalisation of those in Eq. (126) with, e.g.,

$$\Delta_{\mu\nu}^{(qq,1^+)}(K) = \frac{1}{K^2+m_{(qq,1^+)}^2}\left(\delta_{\mu\nu} + \frac{K_\mu K_\nu}{m_{(qq,1^+)}^2}\right). \tag{131}$$

Since it is not possible to combine an isospin-0 diquark with an isospin-1/2 to obtain isospin-3/2, the spin- and isospin-3/2 Δ contains only an axial-vector diquark component

$$\Psi_3^{\Delta}(p_i, \alpha_i, \tau_i) = \mathcal{D}_3^{(qq,1^+)}. \tag{132}$$

Understanding the structure of the Δ is plainly far simpler than in the nucleon since, whilst the general form of the Faddeev amplitude for a spin- and isospin-3/2 can be complicated, isospin symmetry means that one can focus on the Δ^{++}, with its simple flavor structure, because all the charge states are degenerate:

$$\mathcal{D}_3^{(qq,1^+)} = \left[t^+ \Gamma_\mu^{(qq,1^+)}\left(\frac{1}{2}p_{[12]}; K\right)\right]_{\alpha_1\alpha_2}^{\tau_1\tau_2} \Delta_{\mu\nu}^{(qq,1^+)}(K)[\mathcal{D}_{\nu\rho}(l;P)u_\rho(P)\varphi_+]_{\alpha_3}^{\tau_3}, \tag{133}$$

where $u_\rho(P)$ is a Rarita-Schwinger spinor.

The general forms of the matrices $\mathcal{S}(l;P)$, $\mathcal{A}_\nu^i(l;P)$ and $\mathcal{D}_{\nu\rho}(l;P)$, which describe the momentum-space correlation between the quark and diquark in the nucleon and Δ, respectively, are described in Refs. [5, 6]. The requirement that $\mathcal{S}(l;P)$ represent a positive energy nucleon entails

$$\mathcal{S}(l;P) = s_1(l;P)\mathbf{I}_D + (i\gamma\cdot\hat{l} - \hat{l}\cdot\hat{P}I_D)s_2(l;P), \tag{134}$$

where $(\mathbf{I}_D)_{rs} = \delta_{rs}$, $\hat{l}^2 = 1$, $\hat{P}^2 = -1$. In the nucleon rest frame, $s_{1,2}$ describe, respectively, the upper, lower component of the bound-state nucleon's spinor. Placing the same constraint on the axial-vector component, one has

$$\mathcal{A}_\nu^i(l;P) = \sum_{n=1}^{6} p_n^i(l;P)\gamma_5 A_\nu^n(l;P), \quad i = +, 0, -, \tag{135}$$

where ($\hat{l}_\nu^\perp = \hat{l}_\nu + \hat{l}\cdot P\hat{P}_\nu$, $\gamma_\nu^\perp = \gamma_\nu + \gamma\cdot\hat{P}\hat{P}_\nu$)

$$\begin{aligned}
A_\nu^1 &= \gamma\cdot\hat{l}^\perp\hat{P}_\nu,\\
A_\nu^2 &= -i\hat{P}_\nu,\\
A_\nu^3 &= \gamma\cdot\hat{l}^\perp\hat{l}_\nu^\perp,\\
A_\nu^4 &= i\hat{l}_\mu^\perp,\\
A_\nu^5 &= \gamma_\nu^\perp - A_\nu^3,\\
A_\nu^6 &= i\gamma_\nu^\perp\gamma\cdot\hat{l}^\perp - A_\nu^4.
\end{aligned} \tag{136}$$

Finally, requiring also that $\mathcal{D}_{\nu\rho}(l;P)$ be an eigenfunction of $\Lambda_+(P)$, one obtains

$$\mathcal{D}_{\nu\rho}(l;P) = \mathcal{S}^\Delta(l;P)\delta_{\nu\rho} + \gamma_5\mathcal{A}_\nu^\Delta(l;P)l_\rho^\perp, \tag{137}$$

with $\mathcal{S}^\Delta$ and $\mathcal{A}_\nu^\Delta$ given by obvious analogues of Eqs. (134) and (135), respectively.

One can now write the Faddeev equation satisfied by Ψ_3 as

$$\begin{bmatrix}\mathcal{S}(k;P)u(P)\\ \mathcal{A}_\mu^i(k;P)u(P)\end{bmatrix} = -4\int\frac{d^4l}{(2\pi)^4}\mathcal{M}(k,l;P)\begin{bmatrix}\mathcal{S}(l;P)u(P)\\ \mathcal{A}_\nu^j(l;P)u(P)\end{bmatrix}. \tag{138}$$

The kernel in equation above is

$$\mathcal{M}(k,l;P) = \begin{bmatrix}\mathcal{M}_{00} & (\mathcal{M}_{01})_\nu^j\\ (\mathcal{M}_{10})_\mu^i & (\mathcal{M}_{11})_{\mu\nu}^{ij}\end{bmatrix}, \tag{139}$$

and defining $l_q = l + \frac{P}{3}$, $k_q = k + \frac{P}{3}$, $l_{qq} = -l + \frac{2}{3}P$, $k_{qq} = -k + \frac{2}{3}P$, the explicit expressions of the different terms are

$$\begin{aligned}
\mathcal{M}_{00} &= \Gamma^{(qq,0^+)}(k_q - \frac{1}{2}l_{qq}; l_{qq})\, S^T(l_{qq}-k_q)\, \bar{\Gamma}^{(qq,0^+)}(l_q - \frac{1}{2}k_{qq}; -k_{qq})\, S(l_q)\, \Delta^{(qq,0^+)}(l_{qq}),\\
(\mathcal{M}_{01})^j_\nu &= t^j \Gamma^{(qq,1^+)}_\mu(k_q - \frac{1}{2}l_{qq}; l_{qq})\, S^T(l_{qq}-k_q)\, \bar{\Gamma}^{(qq,0^+)}(l_q - \frac{1}{2}k_{qq}; -k_{qq})\, S(l_q)\, \Delta^{(qq,1^+)}_{\mu\nu}(l_{qq}),\\
(\mathcal{M}_{10})^i_\mu &= \Gamma^{(qq,0^+)}(k_q - \frac{1}{2}l_{qq}; l_{qq})\, S^T(l_{qq}-k_q)\, (t^i)^T \bar{\Gamma}^{(qq,1^+)}_\mu(l_q - \frac{1}{2}k_{qq}; -k_{qq})\, S(l_q)\, \Delta^{(qq,0^+)}(l_{qq}),\\
(\mathcal{M}_{11})^{ij}_{\mu\nu} &= t^j \Gamma^{(qq,1^+)}_\rho(k_q - \frac{1}{2}l_{qq}; l_{qq})\, S^T(l_{qq}-k_q)\, (t^i)^T \bar{\Gamma}^{(qq,1^+)}_\mu(l_q - \frac{1}{2}k_{qq}; -k_{qq})\, S(l_q)\, \Delta^{(qq,1^+)}_{\rho\nu}(l_{qq}),
\end{aligned} \tag{140}$$

N.B. The Bethe-Salpeter amplitude of the scalar diquark has an implicit isospin matrix.

The Δ's Faddeev equation is

$$\mathcal{D}_{\lambda\rho}(k;P)u_\rho(P) = -4\int \frac{d^4l}{(2\pi)^4}\mathcal{M}^\Delta_{\lambda\mu}(k,l;P)\mathcal{D}_{\mu\sigma}(l;P)u_\sigma(P), \tag{141}$$

with

$$\mathcal{M}^\Delta_{\lambda\mu} = t^+\Gamma^{(qq,1^+)}_\sigma(k_q - \frac{1}{2}l_{qq}; l_{qq})S^T(l_{qq}-k_q)(t^+)^T\bar{\Gamma}^{(qq,1^+)}_\lambda(l_q - \frac{1}{2}k_{qq}; -k_{qq})S(l_q)\Delta^{(qq,1^+)}_{\sigma\nu}(l_{qq}). \tag{142}$$

1.5.2 Ground-State Δ

With a momentum-independent kernel, the Faddeev amplitude cannot depend on relative momentum. Hence, Eq. (137) becomes

$$\begin{aligned}
\mathcal{D}_{\nu\rho}(l;P)u_\rho(P;r) &= \left[\mathcal{S}^\Delta(l;P)\delta_{\nu\rho}\right]u_\rho(P,r)\\
&= \left[s(l;P)\mathbf{I}_D\delta_{\nu\rho}\right]u_\rho(P,r)\\
&= f^\Delta(P)\mathbf{I}_D u_\nu(P,r).
\end{aligned} \tag{143}$$

Using Eq. (143), Eq. (141) can be written

$$\begin{aligned}
\mathcal{D}_{\lambda\rho}(k;P)u_\rho(P,r) &= 4\int \frac{d^4l}{(2\pi)^4}\mathcal{M}^\Delta_{\lambda\mu}(k,l;P)\mathcal{D}_{\mu\sigma}(l;P)u_\sigma(P,r)\\
f^\Delta(P)u_\lambda(P,r) &= 4\frac{g^2_\Delta}{M}\int \frac{d^4l}{(2\pi)^4}\mathcal{M}^\Delta_{\lambda\mu}(l;P)f^\Delta(P)u_\mu(P,r)\\
f^\Delta(P)u_\mu(P,r) &= 4\frac{g^2_\Delta}{M}\int \frac{d^4l}{(2\pi)^4}\mathcal{M}^\Delta_{\mu\nu}(l;P)f^\Delta(P)u_\nu(P,r),
\end{aligned} \tag{144}$$

with $K = -l + P$, $K^2 = -m_{(qq,1^+)^2}$, $P^2 = -m^2_\Delta$ and

$$\mathcal{M}^\Delta_{\mu\nu}(l;P) = 2i\Gamma^{(qq,1^+)}_\rho(K)i\bar{\Gamma}^{(qq,1^+)}_\mu(-K)S(l)\Delta^{(qq,1^+)}_{\rho\nu}(K). \tag{145}$$

See Fig. **??** to clarify the dependency on the momenta. The factor 2 comes from the isospin matrices:

Taking into account that

$$u = \begin{pmatrix}1\\0\end{pmatrix}, \quad d = \begin{pmatrix}0\\1\end{pmatrix}. \tag{146}$$

and also

$$
\begin{aligned}
t^{+} &= \frac{1}{\sqrt{2}}(\tau^0 + \tau^3) = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \\
t^{0} &= \tau^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
t^{-} &= \frac{1}{\sqrt{2}}(\tau^0 - \tau^3) = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2} \end{pmatrix}
\end{aligned}
\tag{147}
$$

one arrives at

$$
\begin{aligned}
\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2
\end{aligned}
\tag{148}
$$

N.B. How does one knows the flavor composition of a scalar or axial-vector diquarks: In $SU(2)$

$$
\begin{aligned}
|u\rangle &= \left|\frac{1}{2}, +\frac{1}{2}\right\rangle \\
|d\rangle &= \left|\frac{1}{2}, -\frac{1}{2}\right\rangle
\end{aligned}
\tag{149}
$$

and so

$$
\begin{aligned}
|1, +1\rangle &= |uu\rangle \\
|1, 0\rangle &= \frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle) \\
|1, -1\rangle &= |dd\rangle \\
&\text{and} \\
|0, 0\rangle &= \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle)
\end{aligned}
\tag{150}
$$

At this point, one post-multiples by $\bar{u}_\beta(P; r)$ and sums over the polarization index to obtain, Eq. (236),

$$
\begin{aligned}
f^{\Delta}(P) u_\mu(P, r) \bar{u}_\beta(P; r) &= 4 \frac{g_\Delta^2}{M} \int \frac{d^4 l}{(2\pi)^4} \mathcal{M}^{\Delta}_{\mu\nu}(l; P) f^{\Delta}(P) u_\nu(P, r) \bar{u}_\beta(P; r) \\
u_\mu(P, r) \bar{u}_\beta(P; r) &= 4 \frac{g_\Delta^2}{M} \int \frac{d^4 l}{(2\pi)^4} \mathcal{M}^{\Delta}_{\mu\nu}(l; P) u_\nu(P, r) \bar{u}_\beta(P; r) \\
\left[\frac{1}{2m_\Delta} \sum_{r=-3/2}^{3/2} u_\mu(P, r) \bar{u}_\beta(P; r) \right] &= 4 \frac{g_\Delta^2}{M} \int \frac{d^4 l}{(2\pi)^4} \mathcal{M}^{\Delta}_{\mu\nu}(l; P) \left[\frac{1}{2m_\Delta} \sum_{r=-3/2}^{3/2} u_\nu(P, r) \bar{u}_\beta(P; r) \right] \\
\Lambda_+(P) R_{\mu\beta}(P) &= 4 \frac{g_\Delta^2}{M} \int \frac{d^4 l}{(2\pi)^4} \mathcal{M}^{\Delta}_{\mu\nu}(l; P) \Lambda_+(P) R_{\nu\beta}(P),
\end{aligned}
\tag{151}
$$

Now, one has to contract with $\delta_{\mu\beta}$

$$\Lambda_+(P)R_{\mu\beta}(P)\delta_{\mu\beta} = 4\frac{g_\Delta^2}{M}\int \frac{d^4l}{(2\pi)^4}\mathcal{M}^\Delta_{\mu\nu}(l;P)\Lambda_+(P)R_{\nu\beta}(P)\delta_{\mu\beta}, \tag{152}$$

and take the trace:

- Left hand side of the Eq. (152)

$$\begin{aligned}\Lambda_+(P)R_{\mu\beta}(P)\delta_{\mu\beta} &= \left[\frac{1}{2m_\Delta}(-i\gamma\cdot P + m_\Delta)\right]\left[\delta_{\mu\beta}\mathbf{I}_D - \frac{1}{3}\gamma_\mu\gamma_\beta + \frac{2}{3}\hat{P}_\mu\hat{P}_\beta\mathbf{I}_D - i\frac{1}{3}[\hat{P}_\mu\gamma_\beta - \hat{P}_\beta\gamma_\mu]\right]\delta_{\mu\beta}\\ &= \frac{1}{2m_\Delta}\left[16m_\Delta - \frac{16m_\Delta}{3} + \frac{8m_\Delta\hat{P}^2}{3}\right]\\ &= \frac{1}{2}\left(16 - \frac{16}{3} - \frac{8}{3}\right) = 4,\end{aligned} \tag{153}$$

and so one has

$$1 = \frac{g_\Delta^2}{M}\,\mathrm{tr}_D\int \frac{d^4l}{(2\pi)^4}\mathcal{M}^\Delta_{\mu\nu}(l;P)\Lambda_+(P)R_{\nu\mu}(P). \tag{154}$$

- Right hand side of the Eq. (152)

$$\begin{aligned}&\frac{g_\Delta^2}{M}\,\mathrm{tr}_D\int \frac{d^4l}{(2\pi)^4}\mathcal{M}^\Delta_{\mu\nu}(l;P)\Lambda_+(P)R_{\nu\mu}(P) =\\ &= \frac{g_\Delta^2}{M}\,\mathrm{tr}_D\int \frac{d^4l}{(2\pi)^4}\left[2i\Gamma_\rho^{(qq,1^+)}(K)i\bar{\Gamma}_\mu^{(qq,1^+)}(-K)S(l)\Delta_{\rho\nu}^{(qq,1^+)}(K)\right]\left[\frac{1}{2m_\Delta}(-i\gamma\cdot P + m_\Delta)\right]\\ &\qquad\left[\delta_{\nu\mu}\mathbf{I}_D - \frac{1}{3}\gamma_\nu\gamma_\mu + \frac{2}{3}\hat{P}_\nu\hat{P}_\mu\mathbf{I}_D - i\frac{1}{3}[\hat{P}_\nu\gamma_\mu - \hat{P}_\mu\gamma_\nu]\right]\end{aligned} \tag{155}$$

Now, one needs to know

$$\begin{aligned}\bar{\Gamma}_\mu^{(qq,1^+)}(-K) &= C^\dagger\left[\Gamma_\mu^{(qq,1^+)}(-K)\right]^T C\\ &= C^\dagger\left[\gamma_\mu^\perp(-K)E_{(qq,1^+)}(-K)\right]^T C\\ &= C^\dagger\left[\left(\gamma_\mu - \frac{\gamma\cdot K}{K^2}K_\mu\right)E_{(qq,1^+)}(-K)\right]^T C\\ &= C^\dagger\left[\left(\gamma_\mu^T - \frac{\gamma^T\cdot K}{K^2}K_\mu\right)E_{(qq,1^+)}(-K)\right] C\\ &= \left(-\gamma_\mu + \frac{\gamma\cdot K}{K^2}K_\mu\right)E_{(qq,1^+)}(-K)\\ &= -\Gamma_\mu^{(qq,1^+)}(-K)\end{aligned} \tag{156}$$

and also that $\hat{P} = \frac{P}{m_\Delta}$ such that $\hat{P} = \frac{P^2}{m_\Delta^2} = \frac{-m_\Delta^2}{m_\Delta^2} = -1$.

$$
\begin{aligned}
&= \frac{g_\Delta^2}{M} \operatorname{tr}_D \int \frac{d^4 l}{(2\pi)^4} \times \\
&\left\{ \left[2i \left(\gamma_\rho^\perp(K) E_{(qq,1^+)}(K)\right) i \left(-\gamma_\mu^\perp(-K) E_{(qq,1^+)}(-K)\right) \left(\frac{-i\gamma \cdot l + M}{l^2 + M^2}\right) \left(\frac{1}{K^2 + m_{(qq,1^+)}^2} \left(\delta_{\rho\nu} + \frac{K_\rho K_\nu}{m_{(qq,1^+)}^2}\right)\right)\right] \right. \\
&\left[\frac{1}{2m_\Delta}(-i\gamma \cdot P + m_\Delta)\right] \\
&\left. \left[\delta_{\nu\mu} \mathbf{I}_D - \frac{1}{3}\gamma_\nu \gamma_\mu + \frac{2}{3m_\Delta^2} P_\nu P_\mu \mathbf{I}_D - i\frac{1}{3m_\Delta}[P_\nu \gamma_\mu - P_\mu \gamma_\nu]\right] \right\}
\end{aligned}
\tag{157}
$$

and finally[7]

$$
\begin{aligned}
=&\frac{8}{3} \frac{g_\Delta^2}{M m_\Delta^3} \frac{E_{(qq,1^+)}^2}{m_{(qq,1^+)}^2} \int \frac{d^4 l}{(2\pi)^4} \frac{1}{(K^2 + m_{(qq,1^+)}^2)(l^2 + M^2)} \left\{ (-l \cdot P) \left[3 m_{(qq,1^+)}^2 m_\Delta^2 + (K \cdot P)^2\right] \right. \\
&\left. + m_\Delta \left[2 m_\Delta (l \cdot K)(K \cdot P) + 3M \left(m_{(qq,1^+)}^2 m_\Delta^2 + (K \cdot P)^2\right)\right] \right\}.
\end{aligned}
\tag{158}
$$

Now, with the aid of a Feynman parametrisation, the right hand side becomes

$$
\begin{aligned}
&\frac{8}{3} \frac{g_\Delta^2}{M m_\Delta^3} \frac{E_{(qq,1^+)}^2}{m_{(qq,1^+)}^2} \int \frac{d^4 l}{(2\pi)^4} \frac{1}{(K^2 + m_{(qq,1^+)}^2)(l^2 + M^2)} \left\{ (-l \cdot P) \left[3 m_{(qq,1^+)}^2 m_\Delta^2 + (K \cdot P)^2\right] \right. \\
&\left. + m_\Delta \left[2 m_\Delta (l \cdot K)(K \cdot P) + 3M \left(m_{(qq,1^+)}^2 m_\Delta^2 + (K \cdot P)^2\right)\right] \right\} \\
&= \frac{8}{3} \frac{g_\Delta^2}{M m_\Delta^3} \frac{E_{(qq,1^+)}^2}{m_{(qq,1^+)}^2} \int \frac{d^4 l}{(2\pi)^4} \int_0^1 d\alpha \frac{1}{[\alpha(K^2 + m_{(qq,1^+)}^2) + (1-\alpha)(l^2 + M^2)]^2} \times \\
&\times \left\{ (-l \cdot P) \left[3 m_{(qq,1^+)}^2 m_\Delta^2 + (K \cdot P)^2\right] + m_\Delta \left[2 m_\Delta (l \cdot K)(K \cdot P) + 3M \left(m_{(qq,1^+)}^2 m_\Delta^2 + (K \cdot P)^2\right)\right] \right\} \\
&= \frac{8}{3} \frac{g_\Delta^2}{M m_\Delta^3} \frac{E_{(qq,1^+)}^2}{m_{(qq,1^+)}^2} \int \frac{d^4 l}{(2\pi)^4} \int_0^1 d\alpha \frac{1}{[(l - \alpha P)^2 + \sigma_\Delta(\alpha, M, m_{(qq,1^+)}, m_\Delta)]^2} \times \\
&\times \left\{ (-l \cdot P) \left[3 m_{(qq,1^+)}^2 m_\Delta^2 + (K \cdot P)^2\right] + m_\Delta \left[2 m_\Delta (l \cdot K)(K \cdot P) + 3M \left(m_{(qq,1^+)}^2 m_\Delta^2 + (K \cdot P)^2\right)\right] \right\},
\end{aligned}
\tag{159}
$$

with

$$
\sigma_\Delta(\alpha, M, m_{(qq,1^+)}, m_\Delta) = (1-\alpha)M^2 + \alpha m_{(qq,1^+)}^2 - m_\Delta^2 \alpha(1-\alpha), \tag{160}
$$

[7]N.B. taking into account ($T \equiv$ transversal operator) that $T^2 = T$ and so $\gamma_\rho^\perp T_{\rho\nu} = \gamma_\rho T_{\rho\nu}$. In the other hand, one has that $P_\mu u_\mu(P, r) = 0$ and so $\gamma_\mu^T R_{\nu\mu} = \gamma_\mu R_{\nu\mu}$.

N.B. We have in the denominator the following

$$
\begin{aligned}
\alpha(K^2+m^2_{(qq,1^+)})+(1-\alpha)(l^2+M^2) &= l^2-\alpha l^2+(1-\alpha)M^2+\alpha[(-l+P)^2+m^2_{(qq,1^+)}]\\
&= l^2-\alpha l^2+(1-\alpha)M^2+\alpha l^2+\alpha P^2-2\alpha(l\cdot P)+\alpha m^2_{(qq,1^+)}\\
&= l^2-\alpha l^2+(1-\alpha)M^2+\alpha l^2+\alpha P^2-2\alpha(l\cdot P)+\alpha^2P^2-\alpha^2P^2+\alpha m^2_{(qq,1^+)}\\
&= (l-\alpha P)^2+(1-\alpha)M^2+\alpha m^2_{(qq,1^+)}+\alpha P^2(1-\alpha)\\
&= (l-\alpha P)^2+(1-\alpha)M^2+\alpha m^2_{(qq,1^+)}-m^2_\Delta\alpha(1-\alpha)
\end{aligned}
\tag{161}
$$

We employ a symmetry-preserving regularisation scheme. Hence the shift $l\to l'+\alpha P$ is permitted, whereafter $O(4)$-invariance entails $l'\cdot P=0$ so that one may set

$$
\begin{aligned}
&l\cdot P=(l'+\alpha P)\cdot P=\alpha P^2,\\
&K\cdot P=(-l+P)\cdot P=((-l'-\alpha P)+P)\cdot P=(1-\alpha)P^2,\\
&l\cdot K=(l'+\alpha P)\cdot(-l'+P(1-\alpha))=-l'^2+\alpha(1-\alpha)P^2,
\end{aligned}
\tag{162}
$$

where in the last equation we eliminate the term in l'^2, arriving at

$$
\begin{aligned}
1 &= \frac{8}{3}\frac{g^2_\Delta}{Mm^3_\Delta}\frac{E^2_{(qq,1^+)}}{m^2_{(qq,1^+)}}\int\frac{d^4l'}{(2\pi)^4}\int_0^1 d\alpha\frac{3m^3_\Delta\left[\left(m^2_{(qq,1^+)}+(1-\alpha)^2m^2_\Delta\right)(\alpha m_\Delta+M)\right]}{[l'^2+\sigma_\Delta(\alpha,M,m_{(qq,1^+)},m_\Delta)]^2}\\
&= 8\frac{g^2_\Delta}{M}\frac{E^2_{(qq,1^+)}}{m^2_{(qq,1^+)}}\int\frac{d^4l'}{(2\pi)^4}\int_0^1 d\alpha\frac{\left(m^2_{(qq,1^+)}+(1-\alpha)^2m^2_\Delta\right)(\alpha m_\Delta+M)}{[l'^2+\sigma_\Delta(\alpha,M,m_{(qq,1^+)},m_\Delta)]^2}\\
&= \frac{g^2_\Delta}{M}\frac{E^2_{(qq,1^+)}}{m^2_{(qq,1^+)}}\frac{1}{2\pi^2}\int_0^1 d\alpha\left(m^2_{(qq,1^+)}+(1-\alpha)^2m^2_\Delta\right)(\alpha m_\Delta+M)\,\overline{\mathcal{C}}^{iu}_1(\sigma_\Delta(\alpha,M,m_{(qq,1^+)},m_\Delta))
\end{aligned}
\tag{163}
$$

1.5.3 Ground-State Nucleon

Taking into account that in the Faddeev equation of the nucleon, the quark exchanged between the diquarks is represented as

$$\frac{g_N^2}{M},\tag{164}$$

the nucleon's Faddeev amplitude simplifies and can be written in terms of

$$
\begin{aligned}
&\mathcal{S}(P)=s(P)\mathbf{I}_D,\\
&\mathcal{A}^i_\mu(P)=a^i_1(P)i\gamma_5\gamma_\mu+a^i_2(P)\gamma_5\hat{P}_\mu,\ i=+,0.
\end{aligned}
\tag{165}
$$

where $i=+,0$ is due to we are taking into account the flavor content of a proton which is uud ($[uu]=+$, $[ud]=0$, $[dd]=-$).

Now, the nucleon's Faddeev Equation is given by

$$\begin{bmatrix} \mathcal{S}(P)u(P,s) \\ \mathcal{A}^i_\mu(P)u(P,s) \end{bmatrix} = -4\int \frac{d^4l}{(2\pi)^4} \begin{bmatrix} \mathcal{M}_{00} & (\mathcal{M}_{01})^j_\nu \\ (\mathcal{M}_{10})^i_\mu & (\mathcal{M}_{11})^{ij}_{\mu\nu} \end{bmatrix} \begin{bmatrix} \mathcal{S}(l;P)u(P,s) \\ \mathcal{A}^j_\nu(l;P)u(P,s) \end{bmatrix}$$
$$\begin{bmatrix} s(P)\mathbf{I}_D \\ a_1^+(P)i\gamma_5\gamma_\mu \\ a_1^0(P)i\gamma_5\gamma_\mu \\ a_2^+(P)\gamma_5\hat{P}_\mu \\ a_2^0(P)\gamma_5\hat{P}_\mu \end{bmatrix} u(P,s) = -4\int \frac{d^4l}{(2\pi)^4} \begin{bmatrix} \mathcal{M}_{00} & (\mathcal{M}_{01})^j_\nu \\ (\mathcal{M}_{10})^i_\mu & (\mathcal{M}_{11})^{ij}_{\mu\nu} \end{bmatrix} \begin{bmatrix} s(P)\mathbf{I}_D \\ a_1^+(P)i\gamma_5\gamma_\nu \\ a_1^0(P)i\gamma_5\gamma_\nu \\ a_2^+(P)\gamma_5\hat{P}_\nu \\ a_2^0(P)\gamma_5\hat{P}_\nu \end{bmatrix} u(P,s). \tag{166}$$

The explicit expressions of the different terms are

$$\begin{aligned} \mathcal{M}_{00} &= \frac{g_N^2}{M}\, t^0\, \Gamma^{(qq,0^+)}(k_{0^+})\, (t^0)^T\, \bar{\Gamma}^{(qq,0^+)}(-k_{0^+})\, S(l)\, \Delta^{(qq,0^+)}(k_{0^+}), \\ (\mathcal{M}_{01})^j_\nu &= \frac{g_N^2}{M}\, t^j\, \Gamma^{(qq,1^+)}_\mu(k_{1^+})\, (t^0)^T\, \bar{\Gamma}^{(qq,0^+)}(-k_0^+)\, S(l)\, \Delta^{(qq,1^+)}_{\mu\nu}(k_{1^+}), \\ (\mathcal{M}_{10})^i_\mu &= \frac{g_N^2}{M}\, t^0\, \Gamma^{(qq,0^+)}(k_{0^+})\, (t^i)^T\, \bar{\Gamma}^{(qq,1^+)}_\mu(-k_{1^+})\, S(l)\, \Delta^{(qq,0^+)}(k_{0^+}), \\ (\mathcal{M}_{11})^{ij}_{\mu\nu} &= \frac{g_N^2}{M}\, t^j\, \Gamma^{(qq,1^+)}_\rho(k_{1^+})\, (t^i)^T\, \bar{\Gamma}^{(qq,1^+)}_\mu(-k_1^+)\, S(l)\, \Delta^{(qq,1^+)}_{\rho\nu}(k_{1^+}), \end{aligned} \tag{167}$$

where

$$\begin{aligned} & l\cdot P = \alpha P^2, \\ & k_{0^+}\cdot P = (1-\alpha)P^2, \\ & l\cdot k_{0^+} = \alpha(1-\alpha)P^2, \\ & k_{1^+}\cdot P = (1-\alpha)P^2, \\ & l\cdot k_{1^+} = \alpha(1-\alpha)P^2, \\ & k_{0^+}\cdot k_{1^+} = (-l+P)\cdot(-l+P) \\ & \qquad = ((-l'-\alpha P)+P)\cdot((-l'-\alpha P)+P) \\ & \qquad = (-l'+P(1-\alpha))^2 \\ & \qquad = l'^2+(1-\alpha)^2P^2-2(1-\alpha)l'\cdot P \\ & \qquad = (1-\alpha)^2P^2, \end{aligned} \tag{168}$$

and also

$$\begin{aligned} P^2 &= -m_N^2, \\ k_{0^+} &= -m^2_{(qq,0^+)}, \\ k_{1^+} &= -m^2_{(qq,1^+)}. \end{aligned} \tag{169}$$

At this point, one post-multiples by $\bar{u}(P,s)$ and sums over the polarization index to obtain

$$\begin{bmatrix} s(P)\mathbf{I}_D \\ a_1^+(P)i\gamma_5\gamma_\mu \\ a_1^0(P)i\gamma_5\gamma_\mu \\ a_2^+(P)\gamma_5\hat{P}_\mu \\ a_2^0(P)\gamma_5\hat{P}_\mu \end{bmatrix} \Lambda_+(P) = -4\int \frac{d^4l}{(2\pi)^4} \begin{bmatrix} \mathcal{M}_{00} & (\mathcal{M}_{01})^j_\nu \\ (\mathcal{M}_{10})^i_\mu & (\mathcal{M}_{11})^{ij}_{\mu\nu} \end{bmatrix} \begin{bmatrix} s(P)\mathbf{I}_D \\ a_1^+(P)i\gamma_5\gamma_\nu \\ a_1^0(P)i\gamma_5\gamma_\nu \\ a_2^+(P)\gamma_5\hat{P}_\nu \\ a_2^0(P)\gamma_5\hat{P}_\nu \end{bmatrix} \Lambda_+(P) \tag{170}$$

Therefore, as an example, the first row of the matrix has the following form

$$s(P)\mathbf{I}_D\Lambda_+(P) = -4\int\frac{d^4l}{(2\pi)^4}\Big[\mathcal{M}_{00}s(P)\mathbf{I}_D + i(\mathcal{M}_{01})^+_\nu\gamma_5\gamma_\nu a^+_1(P)$$
$$+i(\mathcal{M}_{01})^0_\nu\gamma_5\gamma_\nu a^0_1(P) + (\mathcal{M}_{01})^+_\nu\gamma_5\hat{P}_\nu a^+_2(P) + (\mathcal{M}_{01})^0_\nu\gamma_5\hat{P}_\nu a^0_2(P)\Big]\Lambda_+(P) \tag{171}$$

- The term K^{00}_{ss}

$$\begin{aligned} s(P)\mathbf{I}_D\Lambda_+(P) &= -4\int\frac{d^4l}{(2\pi)^4}\mathcal{M}_{00}s(P)\mathbf{I}_D\Lambda_+(P) \\ 2s(P) &= -4s(P)\int\frac{d^4l}{(2\pi)^4}\left[\frac{g_N^2}{M}\Gamma^{(qq,0^+)}(k_{0^+})\,\bar{\Gamma}^{(qq,0^+)}(-k_{0^+})\,S(l)\,\Delta^{(qq,0^+)}(k_{0^+})\right]\mathbf{I}_D\Lambda_+(P) \end{aligned} \tag{172}$$

where

$$\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}^T\begin{pmatrix}1\\0\end{pmatrix} = 1 \tag{173}$$

and so

$$K^{00}_{ss} = -2\frac{g_N^2}{M}\int\frac{d^4l}{(2\pi)^4}\left[\Gamma^{(qq,0^+)}(k_{0^+})\,\bar{\Gamma}^{(qq,0^+)}(-k_{0^+})\,S(l)\,\Delta^{(qq,0^+)}(k_{0^+})\right]\mathbf{I}_D\Lambda_+(P) \tag{174}$$

if one knows that

$$\begin{aligned} \Gamma^{(qq,0^+)}(K) &= i\gamma_5E_{(qq,0^+)}(K) + \frac{1}{M}\gamma_5(\gamma\cdot K)F_{(qq,0^+)}(K), \\ \bar{\Gamma}^{(qq,0^+)}(K) &= C^\dagger[\Gamma^{(qq,0^+)}(K)]^TC \\ &= C^\dagger\left[i\gamma_5E_{(qq,0^+)}(K) + \frac{1}{M}\gamma_5(\gamma\cdot K)F_{(qq,0^+)}(K)\right]^TC \\ &= C^\dagger\left[i\gamma_5E_{(qq,0^+)}(K) + \frac{1}{M}(\gamma^T\cdot K)\gamma_5F_{(qq,0^+)}(K)\right]C \\ &= \left[i\gamma_5E_{(qq,0^+)}(K) + \frac{1}{M}\gamma_5(\gamma\cdot K)F_{(qq,0^+)}(K)\right] \\ &= \Gamma^{(qq,0^+)}(K) \end{aligned} \tag{175}$$

we have

$$\begin{aligned} K^{00}_{ss} &= -2\frac{g_N^2}{M}\int\frac{d^4l}{(2\pi)^4}\left[\Gamma^{(qq,0^+)}(k_{0^+})\,\bar{\Gamma}^{(qq,0^+)}(-k_{0^+})\,S(l)\,\Delta^{(qq,0^+)}(k_{0^+})\right]\mathbf{I}_D\Lambda_+(P) \\ &= -2\frac{g_N^2}{M}\int\frac{d^4l}{(2\pi)^4}\Bigg\{\gamma_5\left[iE_{(qq,0^+)}(k_{0^+}) + \frac{1}{M}(\gamma\cdot k_{0^+})F_{(qq,0^+)}(k_{0^+})\right]\times \\ &\quad\gamma_5\left[iE_{(qq,0^+)}(-k_{0^+}) - \frac{1}{M}(\gamma\cdot k_{0^+})F_{(qq,0^+)}(-k_{0^+})\right]\frac{-i\gamma\cdot l + M}{l^2+M^2}\frac{1}{k^2_{0^+} + m^2_{(qq,0^+)}}\Bigg\}\times \\ &\quad\mathbf{I}_D\frac{1}{2m_N}(-i\gamma\cdot P + m_N) \end{aligned} \tag{176}$$

now we have four terms:

1. The term with $E^2_{(qq,0^+)}$

$$
\begin{aligned}
&-2\frac{g_N^2}{M}\frac{-2E^2_{(qq,0^+)}}{M^2 m_N}\int\frac{d^4l}{(2\pi)^4}\frac{-M^2(l\cdot P)+M^3 m_N}{(k_{0^+}^2+m^2_{(qq,0^+)})(l^2+M^2)}\\
&=4\frac{g_N^2}{Mm_N}E^2_{(qq,0^+)}\int\frac{d^4l}{(2\pi)^4}\int_0^1 d\alpha\,\frac{m_N(M+\alpha m_N)}{[l^2+\sigma_N^0]^2}\\
&=c_N E^2_{(qq,0^+)}\int_0^1 d\alpha\,(M+\alpha m_N)\overline{\mathcal{C}}_1^{iu}(\sigma_N^0)
\end{aligned}
\tag{177}
$$

2. The term with $E_{(qq,0^+)}F_{(qq,0^+)}$

$$
\begin{aligned}
&-2\frac{g_N^2}{M}\frac{-2E_{(qq,0^+)}F_{(qq,0^+)}}{M^2 m_N}\int\frac{d^4l}{(2\pi)^4}\frac{2Mm_N(l\cdot k_{0^+})+2M^2(k_{0^+}\cdot P)}{(k_{0^+}^2+m^2_{(qq,0^+)})(l^2+M^2)}\\
&=4\frac{g_N^2}{M}\frac{E_{(qq,0^+)}F_{(qq,0^+)}}{M^2 m_N}\int\frac{d^4l}{(2\pi)^4}\int_0^1 d\alpha\,\frac{-2Mm_N^3\alpha(1-\alpha)-2M^2m_N^2(1-\alpha)}{[l^2+\sigma_N^0]^2}\\
&=-2c_N E_{(qq,0^+)}F_{(qq,0^+)}\frac{m_N}{M}\int_0^1 d\alpha\,(1-\alpha)(\alpha m_N+M)\overline{\mathcal{C}}_1^{iu}(\sigma_N^0)
\end{aligned}
\tag{178}
$$

3. The term with $F^2_{(qq,0^+)}$

$$
\begin{aligned}
&-2\frac{g_N^2}{M}\frac{-2F^2_{(qq,0^+)}}{M^2 m_N}\int\frac{d^4l}{(2\pi)^4}\frac{-m^2_{(qq,0^+)}(l\cdot P)+Mm^2_{(qq,0^+)}m_N}{(k_{0^+}^2+m^2_{(qq,0^+)})(l^2+M^2)}\\
&=4\frac{g_N^2}{M}\frac{m^2_{(qq,0^+)}}{M^2}F^2_{(qq,0^+)}\int\frac{d^4l}{(2\pi)^4}\int_0^1 d\alpha\,\frac{(\alpha m_N+M)}{[l^2+\sigma_N^0]^2}\\
&=c_N\frac{m^2_{(qq,0^+)}}{M^2}F^2_{(qq,0^+)}\int_0^1 d\alpha\,(\alpha m_N+M)\overline{\mathcal{C}}_1^{iu}(\sigma_N^0)
\end{aligned}
\tag{179}
$$

Taking into account that

$$
\begin{aligned}
c_N &= \frac{g_N^2}{4\pi^2 M}\\
\sigma_N^0 &= \sigma_N(\alpha,M,m_{(qq,0^+)},m_N)=(1-\alpha)M^2+\alpha m_{(qq,0^+)}-\alpha(1-\alpha)m_N^2\\
\sigma_N^1 &= \sigma_N(\alpha,M,m_{(qq,1^+)},m_N)=(1-\alpha)M^2+\alpha m_{(qq,1^+)}-\alpha(1-\alpha)m_N^2
\end{aligned}
\tag{180}
$$

- The term $-\sqrt{2}K^{01}_{sa_1}$

$$
\begin{aligned}
&s(P)\mathbf{I}_D\Lambda_+(P)=-4\int\frac{d^4l}{(2\pi)^4}(\mathcal{M}_{01})^+_\nu i\gamma_5\gamma_\nu\Lambda_+(P)a_1^+(P)\\
&2s(P)=-4ia_1^+(P)\int\frac{d^4l}{(2\pi)^4}\left[\frac{g_N^2}{M}\,t^+\,\Gamma_\mu^{(qq,1^+)}(k_{1^+})\,(t^0)^T\,\bar{\Gamma}^{(qq,0^+)}(-k_{0^+})\,S(l)\,\Delta_{\mu\nu}^{(qq,1^+)}(k_{1^+})\right]\gamma_5\gamma_\nu\Lambda_+(P).
\end{aligned}
\tag{181}
$$

Taking into account

$$
\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}\sqrt{2} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}^T\begin{pmatrix}0\\ 1\end{pmatrix}=-\sqrt{2}
\tag{182}
$$

we have

$$K^{01}_{sa_1} = 2i\frac{g_N^2}{M}\int\frac{d^4l}{(2\pi)^4}\left[\Gamma^{(qq,1^+)}_{\mu}(k_{1^+})\,\bar{\Gamma}^{(qq,0^+)}(-k_{0^+})\,S(l)\,\Delta^{(qq,1^+)}_{\mu\nu}(k_{1^+})\right]\gamma_5\gamma_\nu\Lambda_+(P) \tag{183}$$

Now if one writes

$$K^{01}_{sa_1} = K^{01}_{s_E a_1} + K^{10}_{s_F a_1} \tag{184}$$

we have

$$\begin{aligned}
K^{01}_{s_E a_1} &= -2i\frac{g_N^2}{M}2i\frac{E_{(qq,1^+)}E_{(qq,E_{0^+})}}{m^2_{(qq,1^+)}}\int\frac{d^4l}{(2\pi)^4}\int_0^1 d\alpha\,\frac{m^2_{(qq,1^+)}(3M+\alpha m_N)+2\alpha m_N^3(1-\alpha)^2}{(l^2+\sigma_N^1)^2}\\
&= c_N\frac{E_{(qq,1^+)}E_{(qq,E_{0^+})}}{m^2_{(qq,1^+)}}\int_0^1 d\alpha\,\left[m^2_{(qq,1^+)}(3M+\alpha m_N)+2\alpha m_N^3(1-\alpha)^2\right]\overline{\mathcal{C}}^{iu}_1(\sigma_N^1)\\
K^{01}_{s_F a_1} &= -2i\frac{g_N^2}{M}-2i\frac{E_{(qq,1^+)}F_{(qq,E_{0^+})}}{m^2_{(qq,1^+)}}\frac{m_N}{M}\int\frac{d^4l}{(2\pi)^4}\int_0^1 d\alpha\,(1-\alpha)\frac{m^2_{(qq,1^+)}(M+3\alpha m_N)+2Mm_N^2(1-\alpha)^2}{(l^2+\sigma_N^1)^2}\\
&= -c_N\frac{E_{(qq,1^+)}F_{(qq,E_{0^+})}}{m^2_{(qq,1^+)}}\frac{m_N}{M}\int_0^1 d\alpha\,\left[(1-\alpha)(m^2_{(qq,1^+)}(M+3\alpha m_N)+2Mm_N^2(1-\alpha)^2)\right]\overline{\mathcal{C}}^{iu}_1(\sigma_N^1)
\end{aligned} \tag{185}$$

- The term $-\sqrt{2}K^{01}_{sa_2}$

$$\begin{aligned}
s(P)\mathbf{I}_D\Lambda_+(P) &= -4\int\frac{d^4l}{(2\pi)^4}(\mathcal{M}_{01})^+_\nu\gamma_5\hat{P}_\nu\Lambda_+(P)a_2^+(P)\\
2s(P) &= -4a_2^+(P)\int\frac{d^4l}{(2\pi)^4}\left[\frac{g_N^2}{M}\,t^+\,\Gamma^{(qq,1^+)}_{\mu}(k_{1^+})\,(t^0)^T\,\bar{\Gamma}^{(qq,0^+)}(-k_{0^+})\,S(l)\,\Delta^{(qq,1^+)}_{\mu\nu}(k_{1^+})\right]\gamma_5\hat{P}_\nu\Lambda_+(P)
\end{aligned} \tag{186}$$

and so

$$K^{01}_{sa_2} = -2\frac{g_N^2}{M}\int\frac{d^4l}{(2\pi)^4}\left[\Gamma^{(qq,1^+)}_{\mu}(k_{1^+})\,\bar{\Gamma}^{(qq,0^+)}(-k_{0^+})\,S(l)\,\Delta^{(qq,1^+)}_{\mu\nu}(k_{1^+})\right]\gamma_5\hat{P}_\nu\Lambda_+(P) \tag{187}$$

Now if one writes

$$K^{01}_{sa_2} = K^{01}_{s_E a_2} + K^{10}_{s_F a_2} \tag{188}$$

we have

$$\begin{aligned}
K^{01}_{s_E a_2} &= -2\frac{g_N^2}{M}(-2)\frac{E_{(qq,1^+)}E_{(qq,E_{0^+})}}{m^2_{(qq,1^+)}}\int\frac{d^4l}{(2\pi)^4}\int_0^1 d\alpha\,\frac{(\alpha m_N-M)((1-\alpha)^2m_N^2-m^2_{(qq,1^+)})}{(l^2+\sigma_N^1)^2}\\
&= c_N\frac{E_{(qq,1^+)}E_{(qq,E_{0^+})}}{m^2_{(qq,1^+)}}\int_0^1 d\alpha\,(\alpha m_N-M)((1-\alpha)^2m_N^2-m^2_{(qq,1^+)})\overline{\mathcal{C}}^{iu}_1(\sigma_N^1)\\
K^{01}_{s_F a_2} &= c_N\frac{E_{(qq,1^+)}F_{(qq,E_{0^+})}}{m^2_{(qq,1^+)}}\frac{m_N}{M}\int_0^1 d\alpha\,\left[(1-\alpha)(\alpha m_N-M)((1-\alpha)^2m_N^2-m^2_{(qq,1^+)}\right]\overline{\mathcal{C}}^{iu}_1(\sigma_N^1)
\end{aligned} \tag{189}$$

- The terms $K^{10}_{a_1 s}$ and $K^{10}_{a_2 s}$

$$
\begin{aligned}
a_1^0(P) i\gamma_5\gamma_\mu \Lambda_+(P) &= -4\int \frac{d^4l}{(2\pi)^4} (\mathcal{M}_{10})^0_\mu s(P) I_D \Lambda_+(P) \\
a_2^0(P) \gamma_5 \hat{P}_\mu \Lambda_+(P) &= -4\int \frac{d^4l}{(2\pi)^4} (\mathcal{M}_{10})^0_\mu s(P) I_D \Lambda_+(P)
\end{aligned}
\tag{190}
$$

or what is the same

$$
\begin{aligned}
a_1^0(P) i\gamma_5\gamma_\mu \Lambda_+(P) &= -4\int \frac{d^4l}{(2\pi)^4} \left[\frac{g_N^2}{M}\, t^0\, \Gamma^{(qq,0^+)}(k_{0^+})\, (t^0)^T\, \bar{\Gamma}^{(qq,1^+)}_\mu(-k_{1^+})\, S(l)\, \Delta^{(qq,0^+)}(k_{0^+})\right] s(P) I_D \Lambda_+(P) \\
a_2^0(P) \gamma_5 \hat{P}_\mu \Lambda_+(P) &= -4\int \frac{d^4l}{(2\pi)^4} \left[\frac{g_N^2}{M}\, t^0\, \Gamma^{(qq,0^+)}(k_{0^+})\, (t^0)^T\, \bar{\Gamma}^{(qq,1^+)}_\mu(-k_{1^+})\, S(l)\, \Delta^{(qq,0^+)}(k_{0^+})\right] s(P) I_D \Lambda_+(P)
\end{aligned}
\tag{191}
$$

N.B. Flavor terms

$$
\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \tag{192}
$$

This terms must be solved at the same time because the left hand side of the equations above is zero when one proceed to take trace as in the terms before. So one needs to define a function which is a linear combination of the Dirac structure of both terms

$$
\Gamma^{aux}_\mu(P) = b_1 \left[i\gamma_5\gamma_\mu\right] + b_2 \left[\gamma_5 \hat{P}_\mu\right] \tag{193}
$$

in fact we stablished a system of coupled linear equations

$$
\begin{aligned}
\mathrm{tr}_D \left[\Gamma^{aux}_\mu(P)\Lambda_+(P)\left(\frac{1}{2} i\gamma_5\gamma_\mu\right)\right] &= 4b_1 + b_2 \\
\mathrm{tr}_D \left[\Gamma^{aux}_\mu(P)\Lambda_+(P)\left(-\frac{1}{2} \gamma_5\hat{P}_\mu\right)\right] &= b_1 + b_2
\end{aligned}
\tag{194}
$$

sucht that

$$
\begin{aligned}
T_1 &= 4b_1 + b_2 \\
T_2 &= b_1 + b_2
\end{aligned}
\tag{195}
$$

and so $b_1 = \frac{1}{3}(T_1 - T_2)$ and $b_2 = \frac{1}{3}(-T_1 + 4T_2)$. Now we have

$$
\begin{aligned}
T_1 &= -4\int \frac{d^4l}{(2\pi)^4} \left[\frac{g_N^2}{M}\, t^0\, \Gamma^{(qq,0^+)}(k_{0^+})\, (t^0)^T\, \bar{\Gamma}^{(qq,1^+)}_\mu(-k_{1^+})\, S(l)\, \Delta^{(qq,0^+)}(k_{0^+})\right] s(P) I_D \Lambda_+(P) \left(\frac{1}{2} i\gamma_5\gamma_\mu\right) \\
&= (-4)\frac{g_N^2}{M}\int \frac{d^4l}{(2\pi)^4}\int_0^1 d\alpha \times \\
&\times (-1)\frac{E_{(qq,1^+)}\left(3E_{(qq,0^+)} M m^2_{(qq,1^+)} + F_{(qq,0^+)} m_N (m^2_{(qq,1^+} + 2m_N^2(1-\alpha)^2)(\alpha-1)\right)(M + m_N\alpha)}{M m^2_{(qq,1^+)}[\alpha(k^2_{0^+} + m^2_{(qq,0^+)}) + (1-\alpha)(l^2 + M^2)]^2} \\
T_2 &= -4\int \frac{d^4l}{(2\pi)^4} \left[\frac{g_N^2}{M}\, t^0\, \Gamma^{(qq,0^+)}(k_{0^+})\, (t^0)^T\, \bar{\Gamma}^{(qq,1^+)}_\mu(-k_{1^+})\, S(l)\, \Delta^{(qq,0^+)}(k_{0^+})\right] s(P) I_D \Lambda_+(P) \left(-\frac{1}{2} \gamma_5\hat{P}_\mu\right) \\
&= (-4)\frac{g_N^2}{M}\int \frac{d^4l}{(2\pi)^4}\int_0^1 d\alpha \times \\
&\times \frac{E_{(qq,1^+)}\left(E_{(qq,0^+)} M - F_{(qq,0^+)} m_N(\alpha-1)\right)(-m^2_{(qq,1^+)} + m_N^2(1-\alpha)^2)(M + m_N\alpha)}{M m^2_{(qq,1^+)}[\alpha(k^2_{0^+} + m^2_{(qq,0^+)}) + (1-\alpha)(l^2 + M^2)]^2}
\end{aligned}
\tag{196}
$$

and so

$$
\begin{aligned}
K^{10}_{a_1 s_E} &= \frac{c_N}{3}\frac{E_{(qq,1^+)}E_{(qq,0^+)}}{m^2_{(qq,1^+)}}\int_0^1 d\alpha\,\overline{\mathcal{C}}^{iu}_1(\sigma^0_N)(\alpha m_N+M)(2m^2_{(qq,1^+)}+(1-\alpha)^2m^2_N)\\
K^{10}_{a_1 s_F} &= -\frac{c_N}{3}\frac{E_{(qq,1^+)}F_{(qq,0^+)}}{m^2_{(qq,1^+)}}\frac{m_N}{M}\int_0^1 d\alpha\,\overline{\mathcal{C}}^{iu}_1(\sigma^0_N)(1-\alpha)(5m^2_{(qq,1^+)}-2m^2_N(1-\alpha)^2)(\alpha m_N+M)\\
K^{10}_{a_2 s_E} &= \frac{c_N}{3}\frac{E_{(qq,1^+)}E_{(qq,0^+)}}{m^2_{(qq,1^+)}}\int_0^1 d\alpha\,\overline{\mathcal{C}}^{iu}_1(\sigma^0_N)(\alpha m_N+M)(m^2_{(qq,1^+)}-4m^2_N(1-\alpha)^2)\\
K^{10}_{a_2 s_F} &= \frac{c_N}{3}\frac{E_{(qq,1^+)}F_{(qq,0^+)}}{m^2_{(qq,1^+)}}\frac{m_N}{M}\int_0^1 d\alpha\,\overline{\mathcal{C}}^{iu}_1(\sigma^0_N)(1-\alpha)(2m^2_{(qq,1^+)}+(1-\alpha)^2m^2_N)(\alpha m_N+M)
\end{aligned}
\tag{197}
$$

with

$$
\begin{aligned}
K^{10}_{a_1 s} &= K^{10}_{a_1 s_E}+K^{10}_{a_1 s_F}\\
K^{10}_{a_2 s} &= K^{10}_{a_2 s_E}+K^{10}_{a_2 s_F}
\end{aligned}
\tag{198}
$$

- The terms $K^{11}_{a_1a_1}$, $K^{11}_{a_1a_2}$, $K^{11}_{a_2a_1}$ and $K^{11}_{a_2a_2}$

$$
\begin{aligned}
a^0_1(P)i\gamma_5\gamma_\mu\Lambda_+(P) &= -4\int\frac{d^4l}{(2\pi)^4}(\mathcal{M}_{11})^{00}_{\mu\nu}a^0_1(P)i\gamma_5\gamma_\nu\Lambda_+(P)\\
a^0_1(P)i\gamma_5\gamma_\mu\Lambda_+(P) &= -4\int\frac{d^4l}{(2\pi)^4}(\mathcal{M}_{11})^{00}_{\mu\nu}a^0_2(P)\gamma_5\hat{P}_\nu\Lambda_+(P)\\
a^0_2(P)\gamma_5\hat{P}_\mu\Lambda_+(P) &= -4\int\frac{d^4l}{(2\pi)^4}(\mathcal{M}_{11})^{00}_{\mu\nu}a^0_1(P)i\gamma_5\gamma_\nu\Lambda_+(P)\\
a^0_2(P)\gamma_5\hat{P}_\mu\Lambda_+(P) &= -4\int\frac{d^4l}{(2\pi)^4}(\mathcal{M}_{11})^{00}_{\mu\nu}a^0_2(P)\gamma_5\hat{P}_\nu\Lambda_+(P)
\end{aligned}
\tag{199}
$$

or what is the same

$$
\begin{aligned}
a^0_1(P)i\gamma_5\gamma_\mu\Lambda_+(P) &= -4\int\frac{d^4l}{(2\pi)^4}\left[\frac{g^2_N}{M}\,t^0\,\Gamma^{(qq,1^+)}_\rho(k_{1^+})\,(t^0)^T\,\bar{\Gamma}^{(qq,1^+)}_\mu(-k^+_1)\,S(l)\,\Delta^{(qq,1^+)}_{\rho\nu}(k_{1^+})\right]a^0_1(P)i\gamma_5\gamma_\nu\Lambda_+(P)\\
a^0_1(P)i\gamma_5\gamma_\mu\Lambda_+(P) &= -4\int\frac{d^4l}{(2\pi)^4}\left[\frac{g^2_N}{M}\,t^0\,\Gamma^{(qq,1^+)}_\rho(k_{1^+})\,(t^0)^T\,\bar{\Gamma}^{(qq,1^+)}_\mu(-k^+_1)\,S(l)\,\Delta^{(qq,1^+)}_{\rho\nu}(k_{1^+})\right]a^0_2(P)\gamma_5\hat{P}_\nu\Lambda_+(P)\\
a^0_2(P)\gamma_5\hat{P}_\mu\Lambda_+(P) &= -4\int\frac{d^4l}{(2\pi)^4}\left[\frac{g^2_N}{M}\,t^0\,\Gamma^{(qq,1^+)}_\rho(k_{1^+})\,(t^0)^T\,\bar{\Gamma}^{(qq,1^+)}_\mu(-k^+_1)\,S(l)\,\Delta^{(qq,1^+)}_{\rho\nu}(k_{1^+})\right]a^0_1(P)i\gamma_5\gamma_\nu\Lambda_+(P)\\
a^0_2(P)\gamma_5\hat{P}_\mu\Lambda_+(P) &= -4\int\frac{d^4l}{(2\pi)^4}\left[\frac{g^2_N}{M}\,t^0\,\Gamma^{(qq,1^+)}_\rho(k_{1^+})\,(t^0)^T\,\bar{\Gamma}^{(qq,1^+)}_\mu(-k^+_1)\,S(l)\,\Delta^{(qq,1^+)}_{\rho\nu}(k_{1^+})\right]a^0_2(P)\gamma_5\hat{P}_\nu\Lambda_+(P)
\end{aligned}
\tag{200}
$$

we stablished a system of coupled linear equations

$$
\begin{aligned}
\mathrm{tr}_D\left[\Gamma^{aux}_\mu(P)\Lambda_+(P)\left(\frac{1}{2}i\gamma_5\gamma_\mu\right)\right] &= 4b_1+b_2\\
\mathrm{tr}_D\left[\Gamma^{aux}_\mu(P)\Lambda_+(P)\left(-\frac{1}{2}\gamma_5\hat{P}_\mu\right)\right] &= b_1+b_2
\end{aligned}
\tag{201}
$$

sucht that

$$\begin{aligned} T_1 &= 4b_1 + b_2 \\ T_2 &= b_1 + b_2 \end{aligned} \tag{202}$$

and so $b_1 = \frac{1}{3}(T_1 - T_2)$ and $b_2 = \frac{1}{3}(-T_1 + 4T_2)$. Now we have

$$\begin{aligned} T_1 &= -4 \int \frac{d^4l}{(2\pi)^4} \left[\frac{g_N^2}{M}\, t^0\, \Gamma_\rho^{(qq,1^+)}(k_{1^+})\, (t^0)^T\, \bar{\Gamma}_\mu^{(qq,1^+)}(-k_1^+)\, S(l)\, \Delta_{\rho\nu}^{(qq,1^+)}(k_{1^+}) \right] \Gamma_\nu^{aux}(P) \Lambda_+(P) \left(\frac{i}{2} \gamma_5 \gamma_\nu \right) \\ &= -4 \frac{g_N^2}{M} \frac{E^2_{(qq,1^+)}}{m^2_{(qq,1^+)}} \int \frac{d^4l}{(2\pi)^4} \int_0^1 d\alpha \times \\ &\times \frac{b_2(m_{(qq,1^+)} - m_N^2(1-\alpha)^2)(M - m_N\alpha) + b_1(3Mm^2_{(qq,1^+)} + m_N(m^2_{(qq,1^+)} + 2\alpha m_N^2(1-\alpha)^2)}{[(1-\alpha)(k^2_{(qq,1^+)} + m^2_{(qq,1^+)}) + \alpha(l^2 + M^2)]^2} \\ T_2 &= -4 \int \frac{d^4l}{(2\pi)^4} \left[\frac{g_N^2}{M}\, t^0\, \Gamma_\rho^{(qq,1^+)}(k_{1^+})\, (t^0)^T\, \bar{\Gamma}_\mu^{(qq,1^+)}(-k_1^+)\, S(l)\, \Delta_{\rho\nu}^{(qq,1^+)}(k_{1^+}) \right] \Gamma_\nu^{aux}(P) \Lambda_+(P) \left(-\frac{1}{2} \gamma_5 \hat{P}_\nu \right) \\ &= -4 \frac{g_N^2}{M} \frac{E^2_{(qq,1^+)}}{m^2_{(qq,1^+)}} \int \frac{d^4l}{(2\pi)^4} \int_0^1 d\alpha\, \frac{(m^2_{(qq,1^+)} + m_N^2(1-\alpha)^2)(b_1 M - b_2 M + 3b_1 m_N \alpha + b_2 m_N \alpha)}{[(1-\alpha)(k^2_{(qq,1^+)} + m^2_{(qq,1^+)}) + \alpha(l^2 + M^2)]^2} \end{aligned} \tag{203}$$

and so

$$\begin{aligned} b_1 &= \frac{1}{3}(T_1 - T_2) = K^{11}_{a_1a_1} b_1 + K^{11}_{a_1a_2} b_2 \\ b_2 &= \frac{1}{3}(-T_1 + 4T_2) = K^{11}_{a_2a_1} b_1 + K^{11}_{a_2a_2} b_2 \end{aligned} \tag{204}$$

with

$$\begin{aligned} K^{11}_{a_1a_1} &= -\frac{c_N}{3} \frac{E^2_{(qq,1^+)}}{m^2_{(qq,1^+)}} \int_0^1 d\alpha\, \overline{\mathcal{C}}_1^{iu}(\sigma_N^1) \left[2m^2_{(qq,1^+)}(M - \alpha m_N) + (1-\alpha)^2 m_N^2 (M + 5\alpha m_N) \right] \\ K^{11}_{a_1a_2} &= -\frac{2c_N}{3} \frac{E^2_{(qq,1^+)}}{m^2_{(qq,1^+)}} \int_0^1 d\alpha\, \overline{\mathcal{C}}_1^{iu}(\sigma_N^1) \left[(-m^2_{(qq,1^+)} - m_N^2(1-\alpha)^2)(m_N\alpha - M) \right] \\ K^{11}_{a_2a_1} &= -\frac{c_N}{3} \frac{E^2_{(qq,1^+)}}{m^2_{(qq,1^+)}} \int_0^1 d\alpha\, \overline{\mathcal{C}}_1^{iu}(\sigma_N^1) \left[m^2_{(qq,1^+)}(11\alpha m_N + M) - 2(1-\alpha)^2 m_N^2 (7m_N\alpha + 2M) \right] \\ K^{11}_{a_2a_1} &= -\frac{5c_N}{3} \frac{E^2_{(qq,1^+)}}{m^2_{(qq,1^+)}} \int_0^1 d\alpha\, \overline{\mathcal{C}}_1^{iu}(\sigma_N^1) \left[(m^2_{(qq,1^+)} - m_N^2(1-\alpha)^2)(\alpha m_N - M) \right] \end{aligned} \tag{205}$$

It is important to note that the eigenvectors exhibit the pattern

$$a_i^+ = -\sqrt{2}\, a_i^0, \quad i = 1, 2. \tag{206}$$

The kernel for the Ropper resonance has the same form but there is one change; namely, the functions $\mathcal{C}^{iu}$ are replaced by functions $\mathcal{F}^{iu} = \mathcal{C}^{iu} - d_{\mathcal{F}} \mathcal{D}^{iu}$, where

$$\begin{aligned} \mathcal{D}^{iu}\left(\omega(M^2, \alpha, P^2)\right) &= \int_0^\infty ds\, s\, \frac{s}{s + \omega} \\ &\to \int_{\tau_{uv}^2}^{\tau_{ir}^2} d\tau\, \frac{2}{\tau^3} \exp\left[-\tau\omega(M^2, \alpha, P^2)\right], \end{aligned} \tag{207}$$

$\mathcal{F}_1^{iu}(z) = -z(d/dz)\mathcal{F}^{iu}(z)$, and $\overline{\mathcal{F}}_1^{iu}(z) = \mathcal{F}_1^{iu}/z$. This has the effect of inserting a zero $q^2 = 1/d_{\mathcal{F}}$ in the amplitude for the nucleon's excitation, which then has the structure of a radial excitation of the bystander quark with respect to the diquark core. (For the Roper resonance one has $d_{\mathcal{F}}^{1/2} M = 0.62$).

1.5.4 Radial excitations and parity partners of the baryons

Turning to radial excitations we note that, in analogy with mesons, the leading Tchebychev moment of bound-state amplitude for a baryon's first radial excitation should possess a single zero. Hence it is possible to estimate masses for these states by employing the expedient described above with the zero located as prescribed before.

Following this reasoning, the Faddeev equation for the first radial excitation of each baryon is simply obtained by making the following replacement throuhout the different equations

$$\overline{\mathcal{C}}_1(\sigma) \to \overline{\mathcal{F}}_1(\sigma). \tag{208}$$

In a more general setting one might imagine that a baryon's first radial excitation could be an admixture of two components: one with a zero in the Faddeev amplitude, describing a radial excitation of the quark-diquark system, and the other with a zero in the diquark's Bethe Salpeter amplitude, which represents an internal excitation of the diquark. The procedure in the approach for baryons can conceivably distinguish between these components via a mixing term whose strength is $\propto E_{(fg)_{J^P}} E_{(fg)^*_{J^P}}$, where the latter is the excited diquark's amplitude. Owing to orthogonality of the two-body ground- and first-radially-excited states, we anticipate that this mixing term is negligible. Under this assumption, a baryon's first radial excitation is redominantly a radial excitation of the quark-diquark system.

The case of parity partners is more complicated. One must typically reanalyse each isospin/flavor channel because of the altered Dirac structure of the diquark correlations that are involved. For example, the nucleon's parity partner is composed of pseudoscalar and vector diquark correlations and its Faddev amplitude must change sign under a parity transformation. These alterations lead to changes in the locations of the γ_5 matrices in the Faddeev equation and also a reduction in the number of terms because the pseudoscalr diquark does not possess a $F(P)$-component. The Faddeev equation for the $\Delta(\frac{3}{2})^-$ is written in Ref. [2], as one can see from there, the Faddeev kernels that one obtains via this reanalysis have similar forms to those for the positive parity states but sign changes are introduced and, throught, one has the replacements

$$E_{(fg)_{J^P}} \to E_{(fg)_{J^{-P}}}, \quad m_{(fg)_{J^P}} \to m_{(fg)_{J^{-P}}}. \tag{209}$$

Equations for the radial excitations of the parity partners are readily obtained by using Eq. (208).

Conceptual	
Quark-quark interaction	$g^2 D_{\mu\nu}(p-q) = \delta_{\mu\nu}\frac{4\pi\alpha_{\rm IR}}{m_{\rm G}^2}$
Rainbow-ladder truncation	$\Gamma_\nu^a(q,p) = \frac{\lambda^a}{2}\gamma_\nu$
Dressed quark propagator	$S^{-1}(p) = i\gamma\cdot p + M$
Constituent quark mass	$M = m + M\frac{4\alpha_{\rm IR}}{3\pi m_{\rm G}^2}\,\mathcal{C}^{iu}(M^2)$
Axial-vector WTI in χ-limit	$P_\mu\Gamma_{5\mu}(k_+,k) = S^{-1}(k_+)i\gamma_5 + i\gamma_5 S^{-1}(k)$
Axial-vector vertex	$\Gamma_{5\mu}(k_+,k) = \gamma_5\gamma_\mu - \frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\alpha S(q_+)\Gamma_{5\mu}(q_+,q)S(q)\gamma_\alpha$
Corollaries of AV-WTI	$M = M\frac{8}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_q^\Lambda\left[\frac{1}{q^2+M^2}+\frac{1}{(q+P)^2+M^2}\right]$
	$0 = \int_q^\Lambda \frac{P\cdot q}{q^2+M^2} - \frac{P\cdot(q+P)}{(q+P)^2+M^2}$
$P^2 = m^2 = 0$	$M = M\frac{16}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_q^\Lambda \frac{1}{q^2+M^2}$
$P^2 = m^2 = 0$	$0 = \int_q^\Lambda \frac{\frac{1}{2}q^2+M^2}{[q^2+M^2]^2}$
	$0 = \int_0^1 d\alpha\,\mathcal{C}^{iu}(\omega) + \mathcal{C}_1^{iu}(\omega)$
Vector WTI	$P_\mu i\Gamma_\mu^\gamma(k_+,k) = S^{-1}(k_+) - S^{-1}(k)$
Vector vertex	$\Gamma_\mu(P) = \gamma_\mu - \frac{16\pi}{3}\frac{\alpha_{\rm IR}}{m_G^2}\int\frac{d^4q}{(2\pi)^4}\gamma_\alpha S(q+P)\Gamma_\mu(P)S(q)\gamma_\alpha$
Corollaries of V-WTI	$P_L(P^2) = 1$
	$P_T(P^2) = \frac{1}{1+K_\gamma(P^2)}$
	$K_\gamma(P^2) = \frac{4}{3\pi}\frac{\alpha_{\rm IR}}{m_G^2}\int_0^1 d\alpha\, P^2\alpha(1-\alpha)\bar{\mathcal{C}}_1^{iu}(\omega(M^2,\alpha,P^2))$

A Relevant expressions and relations

We write here the most relevant results and relations which are using to derive other expressions.

Technical	
Expressions	$\frac{1}{X^n} \to \frac{1}{(n-1)!} \int_{\tau_{uv}^2}^{\tau_{IR}^2} d\tau \tau^{n-1} \exp(-\tau X)$
	$\int_{\tau_{uv}^2}^{\tau_{IR}^2} dss \frac{1}{q^2+M^2} = \mathcal{C}(M^2; \tau_{ir}^2, \tau_{uv}^2)$
	$\int_{\tau_{uv}^2}^{\tau_{IR}^2} dss \frac{M^2}{q^2+M^2} = \mathcal{C}_1(M^2; \tau_{ir}^2, \tau_{uv}^2)$
Transversal gamma	$\gamma_\mu^T = \gamma_\mu - \frac{\gamma \cdot P}{P^2} P_\mu$
Longitudinal gamma	$\gamma_\mu^L = \frac{\gamma \cdot P}{P^2} P_\mu$
Relations	$\gamma_\mu^T + \gamma_\mu^L = \gamma_\mu$
	$P_\mu \gamma_\mu^T = 0,\ P_\mu \gamma_\mu^L = \gamma \cdot P$
	$\gamma_\mu^T k_\mu = \gamma \cdot k,\ \gamma_\mu^L k_\mu = 0$
Some useful integrals	$\int \frac{d^4q}{(2\pi)^4} (q \cdot P) \mathrm{F}(q^2, P^2) = 0$
	$\int \frac{d^4q}{(2\pi)^4} q_\alpha q_\beta \mathrm{F}(q^2) = \frac{1}{4} \int \frac{d^4q}{(2\pi)^4} q^2 \mathrm{F}(q^2)$
	$\int \frac{d^4q}{(2\pi)^4} q_\alpha q_\beta q_\mu q_\nu \mathrm{F}(q^2) = \frac{1}{24} \int \frac{d^4q}{(2\pi)^4} q^4 (\delta_{\alpha\beta}\delta_{\mu\nu} + \delta_{\alpha\mu}\delta_{\beta\nu} + \delta_{\alpha\nu}\delta_{\beta\mu}) \mathrm{F}(q^2)$
Integral	$\frac{1}{4\pi^2} \int_q^\Lambda = \int \frac{d^4q}{(2\pi)^4}$
Other view of propagator	$S^{-1}(p) = i\gamma \cdot p + m + \frac{16\pi}{3} \frac{\alpha_{\mathrm{IR}}}{m_G^2} \int \frac{d^4q}{(2\pi)^4} \gamma_\mu S(q) \gamma_\mu$
Feynman parametrization	$\frac{1}{AB} = \int_0^1 d\alpha \frac{1}{[\alpha A + (1-\alpha)B]^2}$
Linear transformation	$q \to q - \alpha P$
Omega	$\omega(M^2, \alpha, P^2) = M^2 + P^2 \alpha(1-\alpha)$
Integrals	$\int \frac{d^4q}{(2\pi)^4} \int_0^1 d\alpha \frac{1}{q^2+\Delta} = \frac{1}{16\pi^2} \int_0^1 d\alpha\, \mathcal{C}^{iu}(\Delta)$
	$\int \frac{d^4q}{(2\pi)^4} \int_0^1 d\alpha \frac{\Delta}{(q^2+\Delta)^2} = \frac{1}{16\pi^2} \int_0^1 d\alpha\, \mathcal{C}_1^{iu}(\Delta)$
	$\int \frac{d^4q}{(2\pi)^4} \int_0^1 d\alpha \frac{1}{(q^2+\Delta)^2} = \frac{1}{16\pi^2} \int_0^1 d\alpha\, \overline{\mathcal{C}}_1^{iu}(\Delta)$
Special functions	$\mathcal{C}_0^{iu}(z) = z \left(\Gamma(-1, z\tau_{uv}^2) - \Gamma(-1, z\tau_{ir}^2) \right)$
	$\mathcal{C}_1^{iu}(z) = -z \frac{d}{dz} \mathcal{C}_0^{iu}(z) = z \left[\Gamma(0, z\tau_{uv}^2) - \Gamma(0, z\tau_{ir}^2) \right]$
	$\overline{\mathcal{C}}_1^{iu} = \mathcal{C}_1^{iu}(z)/z$
	$\mathcal{C}_2^{iu}(z) = \frac{z^2}{2} \frac{d^2}{dz^2} \mathcal{C}_0^{iu}(z) = \frac{z}{2} \left(e^{-z\tau_{uv}^2} - e^{-z\tau_{ir}^2} \right)$
	$\overline{\mathcal{C}}_2^{iu} = \mathcal{C}_2^{iu}(z)/z^2$
Derivative	$\frac{d}{dP^2} = \frac{1}{2P^2} P_\mu \frac{\partial}{\partial P_\mu}$

B Euclidean metric

B.1 The metric tensor

The Euclidean conventions used herein are

$$a \cdot b = a_\mu b_\nu \delta_{\mu\nu} = \sum_{\mu=1}^{4} a_\mu b_\mu, \tag{210}$$

where $\delta_{\mu\nu}$ is the Kronecker delta and the metric tensor. Hence, a space-like vector, Q_μ, has $Q^2 > 0$.

B.2 The Dirac matrices

The Dirac matrices are Hermitian, $\gamma_\mu^\dagger = \gamma_\mu$, and defined by the algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}. \tag{211}$$

We use

$$\begin{aligned} \sigma_{\mu\nu} &= \frac{i}{2}[\gamma_\mu, \gamma_\nu], \\ \gamma_5 &= -\gamma_1\gamma_2\gamma_3\gamma_4, \end{aligned} \tag{212}$$

so that

$$\mathrm{tr}[\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] = -4\epsilon_{\mu\nu\rho\sigma},\ \epsilon_{1234} = 1. \tag{213}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric Levi-Civita tensor in 4 dimensions. The Dirac-like representation of these matrices used here is (Chiral representation)

$$\vec{\gamma} = -i\vec{\gamma}_M = \begin{pmatrix} 0 & -i\vec{\sigma} \\ i\vec{\sigma} & 0 \end{pmatrix}, \quad \gamma_4 = \gamma_M^0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \tag{214}$$

or with is the same

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \gamma_3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \quad \gamma_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{215}$$

with

$$\gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{216}$$

where the 2×2 Pauli matrices are

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{217}$$

B.2.1 Traces

We next list some rules and relations which are extremely useful in evaluating the trace of a product of γ-matrices:

1. For any two $n \times n$ matrices U and V
$$\mathrm{tr}(UV) = \mathrm{tr}(VU). \tag{218}$$
2. If $(\gamma_\mu \gamma_\nu \ldots \gamma_\alpha \gamma_\beta)$ contains an odd number of γ-matrices, then
$$\mathrm{tr}(\gamma_\mu \gamma_\nu \ldots \gamma_\alpha \gamma_\beta) = 0. \tag{219}$$
3. For a product of an even number of γ-matrices
$$\begin{aligned} \mathrm{tr}(\gamma_\mu \gamma_\nu) &= 4\delta_{\mu\nu}, \\ \mathrm{tr}(\sigma_{\mu\nu}) &= 0, \\ \mathrm{tr}(\gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta) &= 4(\delta_{\mu\nu}\delta_{\alpha\beta} - \delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha}). \end{aligned} \tag{220}$$

B.3 Minkowski $\Leftrightarrow$ Euclidean

One may obtain the Euclidean version of any Minkowski space expression by using the following transcription rules:

Configuration space	Momentum space
(1) $\int^M d^4x^M \to -i \int^E d^4x^E$	(1) $\int^M d^4k^M \to i \int^E d^4k^E$
(2) $\not\partial \to i\gamma^E \cdot \partial^E$	(2) $\not k \to -i\gamma^E \cdot k^E$
(3) $\not A \to -i\gamma^E \cdot A^E$	(3) $\not A \to -i\gamma^E \cdot A^E$
(4) $A_\mu B^\mu \to -A^E \cdot B^E$	(4) $k_\mu q^\mu \to -k^E \cdot q^E$
(5) $x^\mu \partial_\mu \to x^E \cdot \partial^E$	(5) $k_\mu x^\mu \to -k^E \cdot x^E$

where, as usual, $\not A$ represents $g_{\mu\nu}\gamma^\mu_M A^\nu_M$. These transcription rules can be used as a blind implementation of an analytic continuation in the time variable, $x^0 : x^0 \to -ix^4$ with $\vec{x}^M \to \vec{x}^E$ and the same for the momentum k

These rules are legitimate in perturbation theory. When one begins with Euclidean space the reverse is also true. Take care when you are treating nonperturbatively because there may be differences.

B.4 Dirac spinors

A positive energy spinor satisfies
$$(i\gamma \cdot P + M)u(P,s) = 0, \quad \bar{u}(P,s)(i\gamma \cdot P + M) = 0 \tag{221}$$
where $s = \pm$ is the spin label. It is normalised
$$\bar{u}(P,s)u(P,s) = 2M, \tag{222}$$
and may be expressed explicitly
$$\begin{aligned} u(P,s) &= \sqrt{M - i\mathcal{E}} \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{P}}{M - i\mathcal{E}}\chi_s \end{pmatrix}, \\ \bar{u}(P,s) &= \sqrt{M + i\mathcal{E}} \left(\chi_s^\dagger, -\chi_s^\dagger \frac{\vec{\sigma}\cdot\vec{P}}{M + i\mathcal{E}} \right), \end{aligned} \tag{223}$$

with $\mathcal{E} = i\sqrt{\vec{P}^2 + M^2}$, $\bar{u}(P,s) = u^\dagger(P,s)\gamma_4$ and

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{224}$$

The spinor can be used to construct a positive energy projection operator

$$\Lambda_+(P) = \frac{1}{2M}\sum_{s=\pm} u(P,s)\bar{u}(P,s) = \frac{1}{2M}(-i\gamma \cdot P + M). \tag{225}$$

check with particles of physics Alfredo Valcarce.

$$\begin{aligned}
\Lambda_+(P) &= \frac{1}{2M}\sum_{s=\pm} u(P,s)\bar{u}(P,s) \\
&= \frac{1}{2M}\left[u(P,+)\bar{u}(P,+) + u(P,-)\bar{u}(P,-)\right] \\
&= \frac{1}{2M}\sqrt{M - i\mathcal{E}}\sqrt{M + i\mathcal{E}}\left[\begin{pmatrix} \chi_+ \\ \frac{\vec{\sigma}\cdot\vec{P}}{M - i\mathcal{E}}\chi_+ \end{pmatrix}\left(\chi_+^\dagger, -\chi_+^\dagger\frac{\vec{\sigma}\cdot\vec{P}}{M + i\mathcal{E}}\right) + \begin{pmatrix} \chi_- \\ \frac{\vec{\sigma}\cdot\vec{P}}{M - i\mathcal{E}}\chi_- \end{pmatrix}\left(\chi_-^\dagger, -\chi_-^\dagger\frac{\vec{\sigma}\cdot\vec{P}}{M + i\mathcal{E}}\right)\right] \\
&= \frac{1}{2M}\sqrt{M - i\mathcal{E}}\sqrt{M + i\mathcal{E}}\left[\begin{pmatrix} \chi_+\chi_+^\dagger & -\chi_+\chi_+^\dagger\frac{\vec{\sigma}\cdot\vec{P}}{M + i\mathcal{E}} \\ \frac{\vec{\sigma}\cdot\vec{P}}{M - i\mathcal{E}}\chi_+\chi_+^\dagger & -\frac{\vec{\sigma}\cdot\vec{P}}{M - i\mathcal{E}}\chi_+\chi_+^\dagger\frac{\vec{\sigma}\cdot\vec{P}}{M + i\mathcal{E}} \end{pmatrix} + \begin{pmatrix} \chi_-\chi_-^\dagger & -\chi_-\chi_-^\dagger\frac{\vec{\sigma}\cdot\vec{P}}{M + i\mathcal{E}} \\ \frac{\vec{\sigma}\cdot\vec{P}}{M - i\mathcal{E}}\chi_-\chi_-^\dagger & -\frac{\vec{\sigma}\cdot\vec{P}}{M - i\mathcal{E}}\chi_-\chi_-^\dagger\frac{\vec{\sigma}\cdot\vec{P}}{M + i\mathcal{E}} \end{pmatrix}\right] \\
&= \frac{1}{2M}\sqrt{M - i\mathcal{E}}\sqrt{M + i\mathcal{E}}\left[\begin{pmatrix} 1 & -\frac{\vec{\sigma}\cdot\vec{P}}{M + i\mathcal{E}} \\ \frac{\vec{\sigma}\cdot\vec{P}}{M - i\mathcal{E}} & -\frac{\vec{\sigma}\cdot\vec{P}}{M - i\mathcal{E}}\frac{\vec{\sigma}\cdot\vec{P}}{M + i\mathcal{E}} \end{pmatrix} + \begin{pmatrix} 1 & -\frac{\vec{\sigma}\cdot\vec{P}}{M + i\mathcal{E}} \\ \frac{\vec{\sigma}\cdot\vec{P}}{M - i\mathcal{E}} & -\frac{\vec{\sigma}\cdot\vec{P}}{M - i\mathcal{E}}\frac{\vec{\sigma}\cdot\vec{P}}{M + i\mathcal{E}} \end{pmatrix}\right] \\
&= \frac{2}{2M}\sqrt{M - i\mathcal{E}}\sqrt{M + i\mathcal{E}}\begin{pmatrix} 1 & -\frac{\vec{\sigma}\cdot\vec{P}}{M + i\mathcal{E}} \\ \frac{\vec{\sigma}\cdot\vec{P}}{M - i\mathcal{E}} & -\frac{\vec{\sigma}\cdot\vec{P}}{M - i\mathcal{E}}\frac{\vec{\sigma}\cdot\vec{P}}{M + i\mathcal{E}} \end{pmatrix} \\
&= \text{pp. 361, Quarks and Leptons, Halzen and Martin (Mikowski space)} \\
&= \frac{1}{2M}(-i\gamma \cdot P + M)
\end{aligned} \tag{226}$$

A negative energy spinor satisfies

$$(i\gamma \cdot P - M)v(P,s) = 0, \quad \bar{v}(P,s)(i\gamma \cdot P - M) = 0. \tag{227}$$

Therefore, following the same steps as above, we have

$$\bar{v}(P,s)v(P,s) = 2M, \tag{228}$$

and may be expressed explicitly

$$\begin{aligned}
v(P,s) &= \sqrt{M - i\mathcal{E}}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{P}}{M - i\mathcal{E}}\chi_s' \\ \chi_s' \end{pmatrix}, \\
\bar{v}(P,s) &= \sqrt{M + i\mathcal{E}}\left(\chi_s'^\dagger\frac{\vec{\sigma}\cdot\vec{P}}{M + i\mathcal{E}}, -\chi_s'^\dagger\right),
\end{aligned} \tag{229}$$

with $\mathcal{E} = i\sqrt{\vec{P}^2 + M^2}$, $\bar{v}(P,s) = v^\dagger(P,s)\gamma_4$ and

$$\chi_+' = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \chi_-' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{230}$$

Now, the spinor can be used to construct a negative energy projection operator as

$$\Lambda_-(\vec{P}) = -\frac{1}{2M}\sum_{s=\pm} v(P,s)\bar{v}(\vec{P},s) = \frac{1}{2M}(-i\gamma\cdot P - M) \tag{231}$$

A charge-conjugated Bethe-Salpeter amplitude is obtained via

$$\bar{\Gamma}(k;P) = C^{\dagger}\Gamma(-k,P)^{T}C, \tag{232}$$

where 'T' denotes a transposing of all matrix indices and $C = \gamma_2\gamma_4$ is the charge conjugation matrix, $C^{\dagger} = -C$. We note that

$$C^{\dagger}\gamma_{\mu}^{T}C = -\gamma_{\mu}, \quad [C,\gamma_5] = 0. \tag{233}$$

In describing the Δ resonance we employ a Rarita-Schwinger spinor to unambiguously represent a covariant spin-3/2 field. The positive energy spinor is defined by the following equations

$$\begin{aligned}(i\gamma\cdot P + M)u_{\mu}(P;r) &= 0,\\ \gamma_{\mu}u_{\mu}(P;r) &= 0,\\ P_{\mu}u_{\mu}(P;r) &= 0.\end{aligned} \tag{234}$$

where $r = -3/2, -1/2, 1/2, 3/2$. It is normalised

$$\bar{u}_{\mu}(P;r')u_{\mu}(P;r) = 2M, \tag{235}$$

and satisfies a completeness relation

$$\frac{1}{2M}\sum_{r=-3/2}^{3/2} u_{\mu}(P;r)\bar{u}_{\nu}(P;r) = \Lambda_+(P)R_{\mu\nu}, \tag{236}$$

where

$$R_{\mu\nu} = \delta_{\mu\nu}\mathbf{I}_D - \frac{1}{3}\gamma_{\mu}\gamma_{\nu} + \frac{2}{3}\hat{P}_{\mu}\hat{P}_{\nu}\mathbf{I}_D - i\frac{1}{3}[\hat{P}_{\mu}\gamma_{\nu} - \hat{P}_{\nu}\gamma_{\mu}] \tag{237}$$

with $\hat{P}^2 = -1$, which is very useful in simplifying the positive energy $\Delta's$ Faddeev equation.

C Color Group SU(3)

The Hermitian Gell-Mann matrices are

$$\begin{aligned}
&\lambda_1 = \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}, \quad \lambda_2 = \begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}, \quad \lambda_3 = \begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},\\
&\lambda_4 = \begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}, \quad \lambda_5 = \begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix}, \quad \lambda_6 = \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},\\
&\lambda_7 = \begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}, \quad \lambda_8 = \sqrt{\frac{1}{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix},
\end{aligned} \tag{238}$$

are the generator of SU(3). Usually they also appear as $T_a = \frac{\lambda_a}{2}$. The λ_a matrices satisfy

$$
\begin{aligned}
&\mathrm{tr}(\lambda_a \lambda_b) = 2\delta_{ab}, \\
&[\lambda_a, \lambda_b] = 2i f_{abc} \lambda_c, \\
&\mathrm{tr}(t_a t_b) = T_F \delta_{ab} = \frac{\delta_{ab}}{2}, \\
&[t_a, t_b] = f_{abc} t_c = -C_{abc} t_c
\end{aligned}
\tag{239}
$$

Note that for SU(N), the Casimir invariants C_A, C_F and T_F take the values $C_A = N$, $C_F = (N^2-1)/2N$ and $T_F = 1/2$.

D Examples of David J. Wilson

Using the properties of our baryon spinors, the electromagnetic current for a nucleon can be written in the following form

$$
J_\mu(P_i, P_f) = ie\Lambda_+(P_f)\left[F_1\gamma_\mu + F_2\frac{\sigma_{\mu\nu}Q_\nu}{2m_N}\right]\Lambda_+(P_i). \tag{240}
$$

The vertex involves two momenta, expressed through the ingoing and outgoing momenta P_i, P_f and photon momentum $Q = P_f - P_i$. Since the particle is on-shell, $P_i^2 = P_f^2 = -m_N^2$. Also, one has

$$
\begin{aligned}
Q \cdot P_i &= -\frac{Q^2}{2}, \\
Q \cdot P_f &= +\frac{Q^2}{2}, \\
P_i \cdot P_f &= -m_N^2 - \frac{Q^2}{2}.
\end{aligned}
\tag{241}
$$

The electromagnetic form factors are extracted from Lorentz-invariant traces of the current J_μ. The index μ can be contracted with the momenta $P = (P_i + P_f)/2$, Q or a gamma matrix:

$$
\begin{aligned}
\mathrm{tr}_D\{P_\mu J_\mu(P,Q)\} &= -\frac{(4m_N^2 + Q^2)(4m_N^2 F_1 - Q^2 F_2)}{8m_N^3}, \\
\mathrm{tr}_D\{\gamma_\mu J_\mu(P,Q)\} &= 2iF_1 - \frac{i(2F_1 + 3F_2)Q^2}{2m_N^2},
\end{aligned}
\tag{242}
$$

where we have used the relations between momenta to arrive at the expressions above. After a litle bit algebra, the projection operators that give, respectively, the F_1 and F_2 form factors are

$$
\begin{aligned}
P_{1,\mu} &= -\frac{m_N^2(12m_N P_\mu - i(4m_N^2 + Q^2)\gamma_\mu)}{(4m_N^2 + Q^2)^2}, \\
P_{2,\mu} &= -\frac{4\left[2m_N^3(2m_N^2 - Q^2)P_\mu - i m_N^4(4m_N^2 + Q^2)\gamma_\mu\right]}{Q^2(4m_N^2 + Q^2)^2}
\end{aligned}
\tag{243}
$$

There are many terms in the complete expression for the nucleon elastic electromagnetic form factors: according to one enumeration scheme [7], 11 each for F_1 and F_2. Hence, we choose only to list one pair as an example. The procedure is the same in all cases.

$$
\begin{aligned}
F_{1,\mathcal{I}_s^2}(Q^2) &= \mathrm{tr}_D\{P_{1,\mu}\left[\Lambda_+(P_f)\mathcal{I}^2_{s,\mu}\Lambda_+(P_i)\right]\} \\
&= \mathrm{tr}_D\left\{P_{1,\mu}\left[\Lambda_+(P_f)\left(s^2\int\frac{d^4l}{(2\pi)^4}S(l)\Delta^{(qq,0^+)}(-l_{-f})e_{[ud]}\Gamma_\mu^{(qq,0^+)}(-l_{-f},-l_{-i})\Delta^{(qq,0^+)}(-l_{-i})\right)\Lambda_+(P_i)\right]\right\} \\
F_{2,\mathcal{I}_s^2}(Q^2) &= \mathrm{tr}_D\{P_{2,\mu}\left[\Lambda_+(P_f)\mathcal{I}^2_{s,\mu}\Lambda_+(P_i)\right]\} \\
&= \mathrm{tr}_D\left\{P_{2,\mu}\left[\Lambda_+(P_f)\left(s^2\int\frac{d^4l}{(2\pi)^4}S(l)\Delta^{(qq,0^+)}(-l_{-f})e_{[ud]}\Gamma_\mu^{(qq,0^+)}(-l_{-f},-l_{-i})\Delta^{(qq,0^+)}(-l_{-i})\right)\Lambda_+(P_i)\right]\right\}
\end{aligned} \tag{244}
$$

where

$$
\begin{aligned}
S(l) &= \frac{-i\gamma\cdot l+M}{l^2+M^2}, \\
\Delta^{(qq,0^+)}(K) &= \frac{1}{K^2+m_{(qq,0^+)}}, \\
\Gamma_\mu^{(qq,0^+)}(-l_{-f},-l_{-i}) &= -(l_{-f}+l_{-i})F_{(qq,0^+)}(Q^2) \\
&= (K_{0^+i}+K_{0^+f})F_{(qq,0^+)}(Q^2) = (2K_{0^+i}+Q)F_{(qq,0^+)}(Q^2),
\end{aligned} \tag{245}
$$

and also

$$
\begin{aligned}
K_{0^+i} &= -l_{-i} = -(l-P_i) = -l+P_i, \\
K_{0^+f} &= -l_{-f} = -(l-P_f) = -l+P_f = -l+P_i+Q = K_{0^+i}+Q,
\end{aligned} \tag{246}
$$

The following expression providing an accurate interpolation on the domain $Q^2\in[-m_\rho^2,10]\,\mathrm{GeV}^2$, m_ρ is the ρ meson's mass,

$$
F_{(qq,0^+)}(Q^2) \overset{\text{interpolation}}{=} \frac{1+0.25Q^2+0.027Q^4}{1+1.27Q^2+0.13Q^4}. \tag{247}
$$

In the Eq. (244) appears the following denominator:

$$
\frac{1}{l^2+M^2}\,\frac{1}{l_{-f}^2+m_{(qq,0^+)}}\,\frac{1}{l_{-i}^2+m_{(qq,0^+)}^2} = \int_0^1 dx\,dy\,2x\frac{1}{[l'^2+\omega(x,y,M,m_{(qq,0^+)},m_N,Q^2)]^3}, \tag{248}
$$

in which a Feynman parametrization of three denominators has been performed to give only one denominator. The function omega is

$$
\omega(x,y,M,m_{(qq,0^+)},m_N,Q^2) = M^2(1-x)+x(m_{(qq,0^+)}^2-(1-x)m_N^2)+x^2y(1-y)Q^2 \tag{249}
$$

and also the following shift in the momentum-integration variable

$$
l\to l'+x(P_i+yQ) \tag{250}
$$

On the other hand, the trace is given by

$$
\begin{aligned}
&\mathrm{tr}_D\left\{P_{1,\mu}\left[\Lambda_+(P_f)\left((-i\gamma\cdot l+M)(-1)(l_{-f}+l_{-i})\right)\Lambda_+(P_i)\right]\right\} = \\
&= \frac{1}{(4m_N^2+Q^2)^2}\times \\
&\times\Big[(4m_N^2+Q^2)\big((Mm_N-l\cdot P_i)(K_{0^+i}\cdot P_f+K_{0^+f}\cdot P_f)+(Mm_N-l\cdot P_f)(K_{0^+i}\cdot P_i+K_{0^+f}\cdot P_i)\big) \\
&+(4m_N^2+Q^2)(K_{0^+i}\cdot l+K_{0^+f}\cdot l)(m_N^2+P_f\cdot P_i) \\
&+12m_N(K_{0^+i}\cdot P+K_{0^+f}\cdot P)\big(m_N(l\cdot P_f+l\cdot P_i)+M(-m_N^2+P_f\cdot P_i)\big)\Big]
\end{aligned} \tag{251}
$$

and

$$
\begin{aligned}
&\mathrm{tr}_D \left\{P_{2,\mu}\left[\Lambda_+(P_f)\left((-i\gamma\cdot l+M)(-1)(l_{-f}+l_{-i})\right)\Lambda_+(P_i)\right]\right\}= \\
&=\frac{1}{(4m_N^2Q+Q^3)^2}4m_N\times \\
&\times\Big[m_N(4m_N^2+Q^2)\left((Mm_N-l\cdot P_i)(K_{0^+i}\cdot P_f+K_{0^+f}\cdot P_f)+(Mm_N-l\cdot P_f)(K_{0^+i}\cdot P_i+K_{0^+f}\cdot P_i)\right) \\
&-2(2m_N^2-Q^2)(K_{0^+i}\cdot P+K_{0^+f}\cdot P)\left(-m_N(l\cdot P_f+l\cdot P_i)+M(m_N^2-P_f\cdot P_i)\right) \\
&+m_N(4m_N^2+Q^2)(K_{0^+i}\cdot l+K_{0^+f}\cdot l)(m_N^2+P_f\cdot P_i)\Big]
\end{aligned} \tag{252}
$$

that taking into account

$$
\begin{aligned}
K_{0^+i}&=-m_{(qq,0^+)}, \\
Q\cdot K_{0^+i}&=-\frac{1}{2}Q^2, \\
Q\cdot K_{0^+f}&=\frac{1}{2}Q^2, \\
K_{0^+i}\cdot K_{0^+f}&=-m_{(qq,0^+)}^2-\frac{1}{2}Q^2.
\end{aligned} \tag{253}
$$

and their corollaries

$$
\begin{aligned}
P_i\cdot K_{0^+i}&=-l'\cdot P_i-(1-x)m_N^2+\frac{1}{2}xyQ^2 \\
P_i\cdot K_{0^+f}&=-l'\cdot P_i-(1-x)m_N^2-\frac{1}{2}(1-xy)Q^2 \\
l\cdot P_i&=l'\cdot P_i-xm_N^2-\frac{1}{2}xyQ^2 \\
l\cdot Q&=l'\cdot Q-\frac{1}{2}x(1-2y)Q^2 \\
l\cdot K_{0^+i}&=-l'^2+(1-2x)l'\cdot P_i-2xyl'\cdot Q-x(1-x)m_N^2-\frac{1}{2}xy(1-2x(1-y))Q^2 \\
l\cdot K_{0^+f}&=l\cdot K_{0^+i}+l\cdot Q
\end{aligned} \tag{254}
$$

and subsequently $O(4)$ invariance, we arrive at

$$
\begin{aligned}
&F_{1,\mathcal{I}_s^2}(Q^2) = s^2 e_{[ud]} F_{(qq,0^+)} \int \frac{d^4 l'}{(2\pi)^4} \int_0^1 dx\, dy\, 2x \frac{1}{[l'^2 + \omega(x,y,M,m_{(qq,0^+)},m_N,Q^2)]^3} \times \\
&\times \frac{1}{2(4m_N^2+Q^2)^2} \Big\{ (4m_N^2+Q^2)\big[2l^{'2}Q^2 + 4m_N(Q^2+4m_N^2(1-x))(xm_N+M) \\
&-Q^2xy\left(8Mm_N + (8m_N^2+Q^2)x\right) + 2Q^4x^2y^2\big] + 8(-8m_N^2+Q^2)(l'\cdot P_i)(l'\cdot P_i) + 4(-8m_N^2+Q^2)(l'\cdot P_i)(l'\cdot Q)\Big\} \\
&= s^2 e_{[ud]} F_{(qq,0^+)} \int \frac{d^4 l'}{(2\pi)^4} \int_0^1 dx\, dy\, 2x \frac{1}{Q^2+4m_N^2} \frac{1}{[l'^2 + \omega(x,y,M,m_{(qq,0^+)},m_N,Q^2)]^3} \times \\
&\times \left\{ 2m_N(xm_N+M)\left(4m_N^2(1-x) + (1-2xy)Q^2\right) - \frac{1}{2}x^2y(1+2y)Q^4 + \frac{1}{4}(3Q^2+8m_N^2)l^{'2} \right\} \\
&= s^2 e_{[ud]} F_{(qq,0^+)} \frac{1}{16\pi^2} \int_0^1 dx\, dy\, 2x \frac{1}{Q^2+4m_N^2} \times \\
&\times \left\{ \left[2m_N(xm_N+M)\left(4m_N^2(1-x)+(1-2xy)Q^2\right) - \frac{1}{2}x^2y(1+2y)Q^4 \right] \overline{\mathcal{C}}_2^{iu}(\omega) \right. \\
&\left. + \frac{1}{4}(3Q^2+8m_N^2)(\overline{\mathcal{C}}_1^{iu}(\omega) - \omega\overline{\mathcal{C}}_2^{iu}(\omega)) \right\}
\end{aligned}
\tag{255}
$$

and

$$
\begin{aligned}
&F_{2,\mathcal{I}_s^2}(Q^2) = s^2 e_{[ud]} F_{(qq,0^+)} \int \frac{d^4 l'}{(2\pi)^4} \int_0^1 dx\, dy\, 2x \frac{1}{[l'^2 + \omega(x,y,M,m_{(qq,0^+)},m_N,Q^2)]^3} \times \\
&\times \frac{1}{(4m_N^2+Q^2)^2} 2m_N \Big\{ (4m_N^2+Q^2)\big[2l^{'2}m_N - (Q^2+4m_N^2(1-x))(xm_N+M) + Q^2xy(2M+m_Nx) \\
&+2m_NQ^2x^2y^2\big] + 24m_N(l'\cdot P_i)(l'\cdot P_i) + 12m_N(l'\cdot P_i)(l'\cdot Q)\Big\} \\
&= -s^2 e_{[ud]} F_{(qq,0^+)} \int \frac{d^4 l'}{(2\pi)^4} \int_0^1 dx\, dy\, 2x 2m_N \frac{1}{(Q^2+4m_N^2)} \frac{1}{[l'^2 + \omega(x,y,M,m_{(qq,0^+)},m_N,Q^2)]^3} \times \\
&\times \left\{ 4m_N^2(xm_N+M)(1-x) + \left[(1-2xy)M + x(1-xy)(1+2y)m_N\right]Q^2 - m_N^2 l'^2 \right\} \\
&= -s^2 e_{[ud]} F_{(qq,0^+)} \frac{1}{16\pi^2} \int_0^1 dx\, dy\, 2x 2m_N \frac{1}{(Q^2+4m_N^2)} \times \\
&\times \left\{ 4m_N^2(xm_N+M)(1-x) + \left[(1-2xy)M + x(1-xy)(1+2y)m_N\right]Q^2\overline{\mathcal{C}}_2^{iu}(\omega) - m_N^2(\overline{\mathcal{C}}_1^{iu}(\omega) - \omega\overline{\mathcal{C}}_2^{iu}(\omega)) \right\}
\end{aligned}
\tag{256}
$$

D.1 Another issue

First one

$$\int d^D k\, k^\mu k^\nu f(k^2) = \frac{g^{\mu\nu}}{D} \int d^D k\, k^2 f(k^2) \tag{257}$$

proof:

$$\begin{aligned}
&\int d^D k\, k^\mu k^\nu f(k^2) = \frac{g^{\mu\nu}}{cte} \int d^D k\, k^2 f(k^2) \\
&\int d^D k\, g^{\mu\nu} k^\mu k^\nu f(k^2) = \frac{g^{\mu\nu} g^{\mu\nu}}{cte} \int d^D k\, k^2 f(k^2) \\
&\int d^D k\, k^2 f(k^2) = \frac{D\mathrm{I}_D}{cte} \int d^D k\, k^2 f(k^2) \\
&\Rightarrow cte = D.
\end{aligned} \tag{258}$$

Second one

$$\int d^D k\, k^\mu k^\nu k^\rho k^\sigma f(k^2) = \frac{g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}}{D(D+2)} \int d^D k\, k^4 f(k^2) \tag{259}$$

proof:

$$\begin{aligned}
&\int d^D k\, k^\mu k^\nu k^\rho k^\sigma f(k^2) = \frac{g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}}{cte} \int d^D k\, k^4 f(k^2) \\
&g^{\mu\nu} g^{\rho\sigma} \int d^D k\, k^\mu k^\nu k^\rho k^\sigma f(k^2) = g^{\mu\nu} g^{\rho\sigma} \frac{g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}}{cte} \int d^D k\, k^4 f(k^2) \\
&\int d^D k\, k^4 f(k^2) = \frac{D^2 + D + D}{cte} \int d^D k\, k^4 f(k^2) \\
&\Rightarrow cte = D(D+2)
\end{aligned} \tag{260}$$

Cases (Euclidean space):

$$
\begin{aligned}
\int d^4l\,(l\cdot p^i)^4 f(l^2) &= \int d^4l\, l_\mu p^i_\mu l_\nu p^i_\nu l_\rho p^i_\rho l_\sigma p^i_\sigma \\
&= \frac{1}{24}\int d^4l\, l^4 f(l^2)\, p^i_\mu p^i_\nu p^i_\rho p^i_\sigma\,(\delta_{\mu\nu}\delta_{\rho\sigma}+\delta_{\mu\rho}\delta_{\nu\sigma}+\delta_{\mu\sigma}\delta_{\nu\rho}) \\
&= \frac{3}{24}\int d^4l\, l^4 p_i^4 f(l^2) \\
\int d^4l\,(l\cdot p^i)^3(l\cdot p^j) f(l^2) &= \int d^4l\, l_\mu p^i_\mu l_\nu p^i_\nu l_\rho p^i_\rho l_\sigma p^j_\sigma \\
&= \frac{1}{24}\int d^4l\, l^4 f(l^2)\, p^i_\mu p^i_\nu p^i_\rho p^j_\sigma\,(\delta_{\mu\nu}\delta_{\rho\sigma}+\delta_{\mu\rho}\delta_{\nu\sigma}+\delta_{\mu\sigma}\delta_{\nu\rho}) \\
&= \frac{3}{24}\int d^4l\, l^4 f(l^2)\,[p_i^2(p_i\cdot p_j)] \\
\int d^4l\,(l\cdot p^i)^2(l\cdot p^j)^2 f(l^2) &= \int d^4l\, l_\mu p^i_\mu l_\nu p^i_\nu l_\rho p^j_\rho l_\sigma p^j_\sigma \\
&= \frac{1}{24}\int d^4l\, l^4 f(l^2)\, p^i_\mu p^i_\nu p^j_\rho p^j_\sigma\,(\delta_{\mu\nu}\delta_{\rho\sigma}+\delta_{\mu\rho}\delta_{\nu\sigma}+\delta_{\mu\sigma}\delta_{\nu\rho}) \\
&= \frac{1}{24}\int d^4l\, l^4 f(l^2)\,[p_i^2 p_j^2+2(p_i\cdot p_j)^2] \\
\int d^4l\,(l\cdot p^i)^2(l\cdot p^j)(l\cdot p^k) f(l^2) &= \int d^4l\, l_\mu p^i_\mu l_\nu p^i_\nu l_\rho p^j_\rho l_\sigma p^k_\sigma \\
&= \frac{1}{24}\int d^4l\, l^4 f(l^2)\, p^i_\mu p^i_\nu p^j_\rho p^k_\sigma\,(\delta_{\mu\nu}\delta_{\rho\sigma}+\delta_{\mu\rho}\delta_{\nu\sigma}+\delta_{\mu\sigma}\delta_{\nu\rho}) \\
&= \frac{1}{24}\int d^4l\, l^4 f(l^2)\,[p_i^2(p_j\cdot p_k)+2(p_i\cdot p_j)(p_i\cdot p_k)] \\
\int d^4l\,(l\cdot p^i)(l\cdot p^j)(l\cdot p^k)(l\cdot p^s) f(l^2) &= \int d^4l\, l_\mu p^i_\mu l_\nu p^j_\nu l_\rho p^k_\rho l_\sigma p^s_\sigma \\
&= \frac{1}{24}\int d^4l\, l^4 f(l^2)\, p^i_\mu p^j_\nu p^k_\rho p^s_\sigma\,(\delta_{\mu\nu}\delta_{\rho\sigma}+\delta_{\mu\rho}\delta_{\nu\sigma}+\delta_{\mu\sigma}\delta_{\nu\rho}) \\
&= \frac{1}{24}\int d^4l\, l^4 f(l^2)\,[(p_i\cdot p_j)(p_k\cdot p_s)+(p_i\cdot p_k)(p_j\cdot p_s)+(p_i\cdot p_s)(p_k\cdot p_j)]
\end{aligned}
\tag{261}
$$

Third one

$$
\begin{aligned}
&\int d^Dk\, k^\mu k^\nu k^\rho k^\sigma k^\alpha k^\beta f(k^2) = \frac{g^{\mu\nu}g^{\rho\sigma}g^{\alpha\beta}+g^{\mu\rho}g^{\sigma\alpha}g^{\beta\nu}+g^{\mu\sigma}g^{\alpha\beta}g^{\nu\rho}+g^{\mu\alpha}g^{\beta\nu}g^{\rho\sigma}+g^{\mu\beta}g^{\nu\rho}g^{\sigma\alpha}}{cte}\int d^Dk\, k^6 f(k^2) \\
&g^{\mu\nu}g^{\rho\sigma}g^{\alpha\beta}\int d^Dk\, k^\mu k^\nu k^\rho k^\sigma k^\alpha k^\beta f(k^2) = \\
&g^{\mu\nu}g^{\rho\sigma}g^{\alpha\beta}\frac{g^{\mu\nu}g^{\rho\sigma}g^{\alpha\beta}+g^{\mu\rho}g^{\sigma\alpha}g^{\beta\nu}+g^{\mu\sigma}g^{\alpha\beta}g^{\nu\rho}+g^{\mu\alpha}g^{\beta\nu}g^{\rho\sigma}+g^{\mu\beta}g^{\nu\rho}g^{\sigma\alpha}}{cte}\int d^Dk\, k^6 f(k^2) \\
&\int d^Dk\, k^6 f(k^2) = \frac{D^3+4D}{cte}\int d^Dk\, k^6 f(k^2) \\
&\Rightarrow cte = D^3+4D
\end{aligned}
\tag{262}
$$

Forth one

$$\int d^D k\, k^{\mu_1} \dots k^{\mu_{2n+1}} f(k^2) = 0. \tag{263}$$

D.2 Demonstration that both diagrams are the same in the limit $Q^2 = 0$

We have calculated the following contributions:

$$\begin{aligned}
\Pi^1_{\mu,\alpha\beta}(P,Q) &= \int \frac{d^4 l}{(2\pi)^4} \mathbf{I}_D \left[S(l_{+f}) \left(i\gamma^T_\mu P_T(Q^2) \right) S(l_{+i}) \right] \mathbf{I}_D \Delta^{(qq,1^+)}_{\alpha\beta}(-l), \\
\Pi^2_{\mu,\alpha\beta}(P,Q) &= \int \frac{d^4 l}{(2\pi)^4} \mathbf{I}_D S(l) \left[\Delta^{(qq,1^+)}_{\alpha\rho}(-l_{-f}) \left(\sum_{j=1}^{3} T^j_{\mu,\rho\sigma}(-l_{-f},-l_{-i}) F^{(qq,1^+)}_j \right) \Delta^{(qq,1^+)}_{\sigma\beta}(-l_{-i}) \right] \mathbf{I}_D
\end{aligned} \tag{264}$$

- Diagram d_1:

$$\begin{aligned}
\Pi^1_{\mu,\alpha\beta}(P,Q) &= \int \frac{d^4 l}{(2\pi)^4} \mathbf{I}_D \left[S(l_{+f}) \left(i\gamma^T_\mu P_T(Q^2) \right) S(l_{+i}) \right] \mathbf{I}_D \Delta^{(qq,1^+)}_{\alpha\beta}(-l) \\
&= \int \frac{d^4 l}{(2\pi)^4} \left[S(l+P_f) i\Gamma^\gamma_\mu(Q^2) S(l+P_i) \right] \Delta^{(qq,1^+)}_{\alpha\beta}(-l) \\
&= \int \frac{d^4 l}{(2\pi)^4} \left[S(l+P_i+Q) i\Gamma^\gamma_\mu(Q^2) S(l+P_i) \right] \Delta^{(qq,1^+)}_{\alpha\beta}(-l) \\
&\overset{Q^2=0}{=} \int \frac{d^4 l}{(2\pi)^4} \left[S(l+P_i) \frac{\partial}{\partial (l+P_i)_\mu} S^{-1}(l+P_i) S(l+P_i) \right] \Delta^{(qq,1^+)}_{\alpha\beta}(-l) \\
&= \int \frac{d^4 l'}{(2\pi)^4} S(l') \frac{\partial}{\partial l'_\mu} S^{-1}(l') S(l') \Delta^{(qq,1^+)}_{\alpha\beta}(-l'+P_i) \\
&= \int \frac{d^4 l'}{(2\pi)^4} S(l') \frac{\partial}{\partial l'_\mu} \Delta^{(qq,1^+)}_{\alpha\beta}(-l'+P_i)
\end{aligned} \tag{265}$$

- Diagram d_2:

$$
\begin{aligned}
\Pi^2_{\mu,\alpha\beta}(P,Q) &= \int \frac{d^4l}{(2\pi)^4}\mathbf{I}_D S(l)\left[\Delta^{(qq,1^+)}_{\alpha\rho}(-l_{-f})\left(\sum_{j=1}^{3}T^j_{\mu,\rho\sigma}(-l_{-f},-l_{-i})F^{(qq,1^+)}_j\right)\Delta^{(qq,1^+)}_{\sigma\beta}(-l_{-i})\right]\mathbf{I}_D \\
&= \int \frac{d^4l}{(2\pi)^4}S(l)\left[\Delta^{(qq,1^+)}_{\alpha\rho}(k_f)\left(\sum_{j=1}^{3}T^j_{\mu,\rho\sigma}(k_f,k_i)F^{(qq,1^+)}_j\right)\Delta^{(qq,1^+)}_{\sigma\beta}(k_i)\right] \\
&\overset{Q^2=0}{=} \int \frac{d^4l}{(2\pi)^4}S(l)\left[\Delta^{(qq,1^+)}_{\alpha\rho}(k_i)T^1_{\mu,\rho\sigma}(k_i)\Delta^{(qq,1^+)}_{\sigma\beta}(k_i)\right] \\
&= \int \frac{d^4l}{(2\pi)^4}S(l)\left[\Delta^{(qq,1^+)}_{\alpha\rho}(k_i)2k_{i\mu}\mathcal{P}^T_{\rho\xi}(k_i)\mathcal{P}^T_{\xi\sigma}(k_i)\Delta^{(qq,1^+)}_{\sigma\beta}(k_i)\right] \\
&= \int \frac{d^4l}{(2\pi)^4}S(l)\left[\Delta^{(qq,1^+)}_{\alpha\rho}(k_i)2k_{i\mu}\mathcal{P}^T_{\rho\sigma}(k_i)\Delta^{(qq,1^+)}_{\sigma\beta}(k_i)\right] \\
&= \int \frac{d^4l}{(2\pi)^4}S(l)\frac{\partial(k_i^2+m^2_{(qq,1^+)})}{\partial k_{i\mu}}\frac{1}{k_i^2+m^2_{(qq,1^+)}}\Delta^{(qq,1^+)}_{\alpha\beta}(k_i) \\
&= \int \frac{d^4l}{(2\pi)^4}S(l)\frac{\partial}{\partial k_{i\mu}}\Delta^{(qq,1^+)}_{\alpha\beta}(k_i)
\end{aligned}
\tag{266}
$$

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