EUCLIDEAN GREEN'S FUNCTIONS AND WIGHTMAN DISTRIBUTIONS

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I. INTRODUCTION

The use of Euclidean methods in relativistic quantum field theory has a long history. The idea to pass to imaginary times or energies in order to replace the indefinite Minkowski metric by the Euclidean metric appeared first in Dyson's work on renormalization [Dy 1]. Fradkin [Fr 1], Nakano[Na 1], Wick [Wi 1] and most forcefully Schwinger [Sc 1, 2] proposed to consider the Green's functions of a field theory, continued to imaginary times, and to study their properties as solutions of certain differential equations or as the n-point functions of Euclidean field operators. It was Symanzik [Sy 1, 2] who first advocated a purely Euclidean approach to quantum field theory. He realized that given a formal Lagrangian density the construction of the Green's functions at imaginary times - called Euclidean Green's functions or Schwinger functions - might be simpler than the direct construction of Wightman distributions. It is only in the Euclidean context that Feynman's famous history integral [Fe 1, 2] can be given a mathematically rigorous meaning, see e.g. ref. [Ne 4] for the case of boson theories and [OS 1, 2], [Oz 1] for theories involving bosons and fermions. In his work, Symanzik developed many of the concepts and ideas which these days belong to the basic instruments of the Euclidean approach to field theory. More recently Nelson's mathematically rigorous formulation of Euclidean bose field theories in terms of Markov fields [Ne 1-4], his reconstruction theorem and Guerra's first application of Nelson's scheme [Gu 1] paved the way for all the various recent applications of Euclidean methods in constructive field theory.

Both in Nelson's and in Symanzik's work and also in most other work on Euclidean field theory, Euclidean field operators play an essential role. But the existence of such Euclidean field operators is not automatically guaranteed by what one usually knows or assumes about the relativistic theories. It only follows from the so-called Nelson-Symanzik positivity condition on the Schwinger functions, which is expected to hold in theories describing scalar bosons only, but

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not necessarily in cases where fermions are involved.

It is therefore natural to try to work with the Schwinger functions only and to study their relations to the relativistic theory. Since the Schwinger functions are not necessarily the vacuum expectations of a (Euclidean) field, one does not have a "Euclidean field theory" anymore, but only a "Euclidean formulation of (relativistic!) field theory".

It is the purpose of these lectures to define and to study the Schwinger functions within the framework of the Wightman axioms. The main task will be to determine under which conditions Schwinger functions are indeed the Euclidean Green's functions of a well defined Wightman theory.

The plan of these lectures is as follows: First we shall briefly review some of the main definitions and results of axiomatic quantum field theory and give a precise definition of the Schwinger functions. Second we will inquire which properties of the Schwinger functions can be derived as consequences of the Wightman axioms. Finally we will show how to reconstruct the Wightman distributions from Schwinger functions $\mathfrak{S}_n$ satisfying the following conditions

(E0) A distribution property

(E1) Euclidean covariance

(E2) Positivity

(E3) Symmetry

(E4) Cluster property.

These conditions (E0) - (E4) are <u>necessary and sufficient</u> for the existence of Wightman distributions satisfying all the Wightman axioms. In detail the connection is as follows.

<u>Euclidean</u>		<u>Relativistic</u>
[Distribution Covariance Positivity]	⟷	[Distribution Covariance Positivity Spectrum]
[ ] + Symmetry	⟷	[ ] + Locality
[ ] + Cluster	⟷	[ ] + Cluster

For applications in constructive field theory the distribution property (E0) does not seem to be appropriate. We therefore introduce a second distribution property (E0') and prove that (E0'), (E1) - (E4) are still sufficient for the reconstruction of the Wightman theory. The above chart holds again with all the arrows pointing to the right only. (E0') is a simple temperedness condition with a restriction on the order of the distributions $\mathfrak{S}_n$ (the Euclidean Green's functions), for large n. It is this second result which I think will turn out to be useful in constructive field theory; the equivalence result, though more general, seems to be of purely esthetical interest.

Remark: As discovered first by Schrader and Simon, there is a mistake in the proof of a technical lemma in the original paper on the axioms for Euclidean Green's functions (ref. [OS 3], lemma 8.8). As a consequence, the temperedness condition of ref. [OS 3]seems to be too weak to allow for the reconstruction of the Wightman theory.

In these lectures we restrict our attention to theories involving one neutral scalar field only; the generalization to finitely or countably many fields of general transformation properties requires only minor and mostly obvious modifications, see Osterwalder-Schrader [OS 3, 4]. Most of the material covered in these lectures is taken from references [OS 3, 4].

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II. WIGHTMAN DISTRIBUTIONS, WIGHTMAN FUNCTIONS, EUCLIDEAN GREEN'S FUNCTIONS

In this section we state the Wightman axioms in terms of the Wightman distributions and summarize some of the classical results of axiomatic field theory. For details see the monographs of Jost [Jo 1] and of Streater and Wightman [SW 1], and references given there.

Let $\varphi(x)$ be a neutral scalar field obeying all the Wightman axioms. Then its vacuum expectation values or Wightman distributions

$$\mathcal{W}_n(\underline{x}) = \mathcal{W}_n(x_1, \dots x_n) = (\Omega, \varphi(x_1) \dots \varphi(x_n)\Omega)$$

have the following properties:

Distribution property: For each n, $\mathcal{W}_n$ is a tempered distribution:

(W0) $\mathcal{W}_n(\underline{x}) \in \mathcal{S}'(\mathbb{R}^{4n})$; $\mathcal{W}_0 = 1$

Notation: $\underline{x} = (x_1, \cdots x_n) \in \mathbb{R}^{4n}$, $x_i = (x_i^0, \vec{x}_i) \in \mathbb{R}^4$

Relativistic covariance: For each n, $\mathcal{W}_n$ is Poincaré invariant:

(W1) $\mathcal{W}_n(\underline{x}) = \mathcal{W}_n(\underline{\Lambda}x + a)$

for all $(a, \Lambda) \in \mathcal{P}_+^\uparrow$,

where $\underline{\Lambda}x + a = (\Lambda x_1 + a, \ldots \Lambda x_n + a)$.

Positivity: For all finite sequences $f_0, f_1, \cdots f_N$ of test functions, $f_0 \in \mathbb{C}$, $f_n \in \mathcal{S}(\mathbb{R}^{4n})$, $n = 1, \ldots N$,

(W2) $\sum_{n,m} \mathcal{W}_{n+m}(f_n^* \times f_m) \geq 0$,

where $f_n^* \times f_m$ is defined by $(f_n^* \times f_m)(\underline{x}, \underline{y}) = f_n^*(\underline{x})\, f_m(\underline{y})$

and $f_n^*(\underline{x}) = f_n^*(x_1, \ldots x_n) = \bar{f}_n(x_n, \ldots x_1) \equiv \bar{f}_n(\overleftarrow{\underline{x}})$.

Locality: For any n and k = 1, ... n - 1

(W3) $\mathcal{W}_n(x_1, \ldots x_k, x_{k+1}, \ldots x_n) = \mathcal{W}_n(x_1, \ldots x_{k+1}, x_k, \ldots x_n)$

if $(x_k - x_{k+1})^2 < 0$.

Cluster Property: For any spacelike a and k = 1, ... n - 1,

$\underline{x} = x_1, \ldots x_k, \underline{y} = y_1, \ldots y_{n-k}$,

(W4) $\lim_{\lambda \to \infty} \mathcal{W}_n(\underline{x}, \underline{y} + \lambda a) = \mathcal{W}_k(\underline{x})\, \mathcal{W}_{n-k}(\underline{y})$.

Spectral condition: Using the translation invariance of the $\mathcal{W}_n$'s we conclude that there exist distributions $W_{n-1} \in \mathcal{S}'(\mathbb{R}^{4(n-1)})$, such that $\mathcal{W}_n(\underline{x}) = W_{n-1}(\underline{\xi})$ where $\underline{\xi} = (\xi_1, \ldots \xi_{n-1})$ and $\xi_k = x_{k+1} - x_k$. Then

(W5) $\mathrm{supp}\, \widetilde{W}_{n-1} \subset \bar{V}_+^{n-1} \equiv \{ \underline{q} \mid q_i \in \bar{V}_+ \ , i = 1, \ldots n-1 \}$,

where $\widetilde{W}_{n-1}(\underline{q}) = (2\pi)^{-4(n-1)} \int \exp[-i \sum q_k \xi_k]\, W_{n-1}(\underline{\xi})\, d^{4(n-1)}\xi$

is the Fourier transform of W_{n-1} , $\bar{V}_+$ is the closed forward light cone, and $q_k \xi_k = q_k^0 \xi_k^0 - \vec{q}_k \vec{\xi}_k$.

Remark: Equations (W1) - (W5) have to be interpreted in distributional sense.

From a given set of Wightman distributions satisfying (W0) - (W5) we can reconstruct the physical Hilbert space $\mathcal{H}$, the vacuum vector $\Omega \in \mathcal{H}$, the field operators $\varphi(x)$ and a unitary representation $\mathcal{U}(a, \Lambda)$ of $\mathcal{P}_+^\uparrow$ in $\mathcal{H}$. This is the Wightman reconstruction theorem; see Wightman [Wg 1].

Let us now introduce vector valued distributions formally defined by

$$\Psi_n(x_1, \underline{\xi}) = \varphi(x_1) \cdots \varphi(x_n)\, \Omega ,$$

where $\underline{\xi} = (\xi_1, \ldots \xi_{n-1})$ and $\xi_k = x_{k+1} - x_k$ for k =1 , ...n-1. Ψ_n is a tempered distribution with values in $\mathcal{H}$. The scalar product of two vectors $\Psi_n(f)$ and $\Psi_m(g)$ for $f \in \mathcal{S}(\mathbb{R}^{4n})$, $g \in \mathcal{S}(\mathbb{R}^{4m})$ is

$$(\Psi_n(f), \Psi_m(g)) = \int \bar{f}(x,\underline{\xi})\, g(x',\underline{\xi}')\, W_{n+m-1}(-\overleftarrow{\underline{\xi}}, -x+x', \underline{\xi}')\, dx\,dx'\,d\underline{\xi}\,d\underline{\xi}'.$$

The Fourier transform of Ψ_n is defined as usual by $\widetilde{\Psi}_n(\tilde{f}) = \Psi_n(f)$, where $\tilde{f}$ is the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^{4n})$.

<u>Theorem 1:</u> For n = 1,2,.., the support of $\widetilde{\Psi}_n(\underline{q})$ is contained in $\overline{V}_+^n$.

For a proof we just take a test function $f \in \mathcal{S}(\mathbb{R}^{4n})$ with $\tilde{f}(\underline{q}) = 0$ for $\underline{q} \in \overline{V}_+^n$, write down the norm $\|\Psi_n(f)\|$ in terms of the distribution $\widetilde{W}_{2n-1}$ and show that it is zero, using the spectrum condition.

We can now study the Laplace transform $\Psi_n(z,\underline{\zeta})$ of $\Psi_n(\underline{q})$. The Laplace transforms of tempered distributions have been extensively studied, e.g. in references [Sz 1, Li 1, Vl 1, SW 1]. The particular theorem we quote below is due to Bros, Epstein and Glaser [BEG].

<u>Theorem 2</u>: Let $\widetilde{F}(\underline{q})$ be a distribution in $\mathcal{S}'(\mathbb{R}^{4n})$ with support in $\overline{V}_+^n$. Then there exists an analytic function $G(\underline{z})$, regular in the forward tube $\mathcal{T}_+^n = \{\underline{z} \mid \underline{z} = \underline{x} + i\underline{y} \in \mathbb{C}^{4n},\ \underline{y} \in V_+^n\}$, so that

a) the Fourier transform F of $\widetilde{F}$ is the boundary value of G:

$$\lim_{\lambda \downarrow 0} \int G(\underline{x} + i\lambda\underline{y})\, f(\underline{x})\, d^{4n}x = F(f)$$

for all $f \in \mathcal{S}(\mathbb{R}^{4n})$, $\underline{y} \in V_+^n$,

b) For some positive constants c, α, β and for all $\underline{z} \in \mathcal{T}_+^n$,

(1) $$|G(\underline{z})| \le c\,(1+|\underline{z}|)^{\alpha}\left[1 + \left(\min_{1\le j\le n}(y_j^0 - |\vec{y}_j|)\right)^{-\beta}\right]$$

The function $G(\underline{z})$ is called the Laplace transform of $\widetilde{F}$ and can be written (heuristically) as

$$G(\underline{z}) = \int \exp(i \textstyle\sum z_i q_i)\, \widetilde{F}(\underline{q})\, d^{4n}q .$$

Theorem 2 can be immediately generalized to hold for distributions $\widetilde{F}$ with values in some Banach space. Applied to our vector valued distributions $\widetilde{\Psi}_n$ it shows that the Laplace transform

$$\Psi_n(z,\underline{\zeta}) = \int \exp\left[i z q_0 + i \textstyle\sum \zeta_i q_i\right] \widetilde{\Psi}_n(q_0, q_1, \dots q_{n-1})\, d^{4n}q$$

is an analytic vector valued function, regular in $\mathcal{T}_+^n$, with boundary value $\Psi_n(x,\underline{\xi})$.

Applying Theorem 2 to the distributions $\widetilde{W}_n(\underline{q})$ we obtain the Wightman functions $W_n(\underline{\zeta})$, which are analytic functions for $\underline{\zeta} \in \mathcal{T}_+^n$. Obviously we have

(2) $$W_{n-1}(\underline{\zeta}) = (\Omega, \Psi_n(z,\underline{\zeta}))\ ;$$

translation invariance of Ω implies that the right hand side of (2) does not depend on z. The following lemma is also simple to prove.

Lemma 3 (see [Jo 1], p. 74): Let $(z,\underline{\zeta}) \in \mathcal{T}_+^n$, $(z',\underline{\zeta}') \in \mathcal{T}_+^m$. Then

$$\left(\Psi_n(z,\underline{\zeta}),\ \Psi_m(z',\underline{\zeta}')\right) = W_{n+m-1}(-\overleftarrow{\overline{\underline{\zeta}}},\ -\bar{z}+z',\ \underline{\zeta}') \ .$$

Using relativistic invariance (W1) and the Bargmann-Hall-Wightman theorem, we obtain a single valued analytic extension of the Wightman functions $W_n(\underline{\zeta})$ - but not of the vectors $\Psi_n(\underline{\zeta})$ - into the extended tube $\mathcal{T}^n_{+,ext} = \{\underline{\zeta} \mid \Lambda\underline{\zeta} \in \mathcal{T}_+^n$ for some $\Lambda \in L_+(\mathbb{C})\}$, where $L_+(\mathbb{C})$ denotes the set of complex Lorentz transformations with determinant 1. The functions $\mathcal{W}_n(\underline{z})$, defined by $\mathcal{W}_n(\underline{z}) = W_{n-1}(\underline{\zeta})$, where $\zeta_k = z_{k+1} - z_k$, are analytic in $\mathfrak{S}^n_{ext} \equiv \{\underline{z} \mid \underline{\zeta} \in \mathcal{T}^n_{+,ext}\}$ and have as boundary values the distributions $\mathcal{W}_n(\underline{x})$. Finally using locality (W3), we obtain a single valued analytic extension of $\mathcal{W}_n(\underline{z})$ into the set $\mathfrak{S}^n_{ext,perm} \equiv \{\underline{z} \mid (z_{\pi(1)},\dots z_{\pi(n)}) \in \mathfrak{S}^n_{ext}$ for some permutation $\pi\}$. We denote this extension again by $\mathcal{W}_n(\underline{z})$. It is invariant under the inhomogeneous complex Lorentz group $iL_+(\mathbb{C})$ and under permutations of the arguments $z_1, \dots, z_n$.

The set $\mathfrak{S}^n_{ext,perm}$ contains $\mathcal{E}^n = \{\underline{z} \mid \underline{z} \in \mathbb{C}^n,\ \mathrm{Re}\, z_k^0 = 0,\ \mathrm{Im}\,\vec{z}_k = 0,\ z_k \neq z_\ell$ for all $1 \le k < \ell \le n\}$. Points in $\mathcal{E}^n$ are called Euclidean points (of non-coinciding arguments).

Definition: The restriction of the Wightman function $\mathcal{W}_n(\underline{z})$ to $\mathcal{E}^{(n)}$ is called the n-point Euclidean Green's function or Schwinger function. We set $\mathfrak{S}_0 = \mathcal{W}_0 = 1$ and

$$\mathfrak{S}_n(\underline{x}) = \mathfrak{S}_n(x_1,\dots x_n) = \mathcal{W}_n((ix_1^0,\vec{x}_1),\dots(ix_n^0,\vec{x}_n)) \ ,$$

for $\underline{x} \in \Omega^n = \{\underline{x} \mid x_i \neq x_j$ for all $1 \le i < j \le n\}$.

From the invariance properties of the Wightman functions we immediately derive the following lemma.

Lemma 4: The Schwinger functions $\mathfrak{S}_n(\underline{x})$ are invariant under the inhomogeneous proper Euclidean group iSO_4 and under permutations of the arguments $x_1, \dots x_n$.

Let us now introduce the difference variable Schwinger functions $S_{n-1}(\underline{\xi})$, defined by

$$(3) \qquad S_{n-1}(\underline{\xi}) = W_{n-1}((i\xi_1^0,\vec{\xi}_1),\dots(i\xi_{n-1}^0,\vec{\xi}_{n-1})) = \mathfrak{S}_n(\underline{x})$$

where $\xi_k = x_{k+1} - x_k$ and $\underline{x} \in \Omega^n$. We also define the real analytic vector valued functions $\Psi_n^E(\underline{\xi})$ by

$$\Psi_n^E(x,\underline{\xi}) = \Psi_n((ix^0,\vec{x}),(i\xi_1^0,\vec{\xi}_1),\dots(i\xi_{n-1}^0,\vec{\xi}_{n-1})) \quad ,$$

for $x^0>0$, $\xi_k^0>0$, $1 \le k \le n-1$. Define $\vartheta\xi$ by $(\vartheta\xi)_k = (-\xi_k^0, \vec{\xi}_k)$.
Lemma 3 and the above definitions lead immediately to

Lemma 5. Let all of x^0, ξ_k^0, x'^0, $\xi_k'^0$ be positive. Then

$$(4) \qquad \left(\Psi_n^E(x,\xi),\ \Psi_m^E(x',\xi')\right) = S_{n+m-1}(-\overleftarrow{\vartheta\xi}, -\vartheta x + x', \xi') .$$

Lemma 5 of course yields a positivity property for the Schwinger functions, see (E2) below.

The cluster property (W4) implies that for any two vectors Φ and Φ' in $\mathcal{H}$, and spacelike a,

$$(5) \quad \lim_{\lambda\to\infty} (\Phi, U(\lambda a, \mathbb{1})\Phi') = (\Phi, \Omega)(\Omega, \Phi') .$$

Lemma 6. For $0 < x_1^0 < \ldots < x_n^0$, $0 < y_1^0 < \ldots < y_m^0$, and $a = (0, \vec{a})$

$$\lim_{\lambda\to\infty} \mathfrak{S}_{n+m}(\vartheta x_n, \ldots \vartheta x_1, y_1+\lambda a, \ldots y_m+\lambda a) = \mathfrak{S}_n(\vartheta x_n, \ldots \vartheta x_1)\, \mathfrak{S}_m(y_1, \ldots y_m) .$$

The lemma follows essentially from substituting $\Phi = \Psi_n^E(\xi)$, $\Phi' = \Psi_m^E(\xi')$, $\xi_k = x_{k+1} - x_k$, $\xi'_k = y_{k+1} - y_k$ in equation (5).

The Schwinger functions are real analytic functions, but they also define distributions. Before we investigate that aspect we have to stop for a mathematical digression. We shall collect some simple lemmas on distributions and Laplace transforms without giving proofs. They are either contained in refs. [SW 1, Vl 1, OS 3] or they are simple consequences of things established there.

Distributions and Laplace Transforms

By $\mathbb{R}_\pm$ we denote the open half intervals $(0, \pm\infty)$, by $\overline{\mathbb{R}}_\pm$ their closure. Let $\mathcal{S}(\mathbb{R}_\pm)$ denote the space of functions $f \in \mathcal{S}(\mathbb{R})$ with $supp f$ in $\mathbb{R}_\pm$, given the induced topology. $\mathcal{S}(\overline{\mathbb{R}}_\pm)$ is the set of all functions, defined on $\overline{\mathbb{R}}_+$, C^∞ on $\mathbb{R}_+$, whose derivatives all have a continuous extension to $\overline{\mathbb{R}}_+$ and are of fast decrease at infinity. The topology on $\mathcal{S}(\overline{\mathbb{R}}_+)$ is defined by the seminorms

$$|g|_{m,+} = \sup_{\substack{x>0 \\ \alpha\le m}} (1+x)^m |g^{(\alpha)}(x)| .$$

Lemma 7. The space $\mathcal{S}(\overline{\mathbb{R}}_+)$ is isomorphic to the topological quotient space $\mathcal{S}(\mathbb{R})/\mathcal{S}(\mathbb{R}_-)$.

The main point of this lemma is that any element in $\mathcal{S}(\overline{\mathbb{R}}_+)$ is the restriction to $\overline{\mathbb{R}}_+$ of some element in $\mathcal{S}(\mathbb{R})$.

The dual space of $\mathcal{S}(\mathbb{R})/\mathcal{S}(\mathbb{R}_-)$ is the polar of $\mathcal{S}(\mathbb{R}_-)$, which is the set of distributions $T \in \mathcal{S}'(\mathbb{R})$ with $supp\, T \in \overline{\mathbb{R}}_+$. Hence by Lemma 7 a distribution in $\mathcal{S}'(\overline{\mathbb{R}}_+)$ can be identified with a distribution in $\mathcal{S}'(\mathbb{R})$

with support in $\overline{\mathbb{R}}_+$. For $f \in \mathcal{S}(\mathbb{R}_+)$ we define $\check{f}$ by

$$\check{f}(p) = \int e^{-px} f(x)\, dx \restriction \overline{\mathbb{R}}_+ .$$

<u>Lemma 8</u>. $f \to \check{f}$ is a continuous map of $\mathcal{S}(\mathbb{R}_+)$ into $\mathcal{S}(\overline{\mathbb{R}}_+)$ whose range is dense and whose kernel is zero.

Let $T \in \mathcal{S}'(\mathbb{R})$ with $\operatorname{supp} T \subset \overline{\mathbb{R}}_+$. Then by Lemma 7, T also defines a distribution in $\mathcal{S}'(\overline{\mathbb{R}}_+)$, again denoted by T, and we conclude that there are contants c and m, such that for all $f \in \mathcal{S}(\mathbb{R}_+)$

(6) $$|T(\check{f})| \leq c\, |\check{f}|_{m,+} \equiv c\, |f|'_m .$$

On the other hand we can use the Laplace transform S of T to define a distribution in $\mathcal{S}'(\mathbb{R}_+)$. For $x > 0$ we define

$$S(x) = \int e^{-xp}\, T(p)\, dp ,$$

and for $f \in \mathcal{S}(\mathbb{R}_+)$ we set

(7) $$S(f) = \int f(x)\, S(x)\, dx .$$

(More precisely we first define $S(f)$ for elements in $\mathcal{S}(\mathbb{R}_+)$ which have compact support, then prove continuity of S using a bound on $S(x)$ similar to (1), and finally extend S to all of $\mathcal{S}(\mathbb{R}_+)$.)

The next theorem contains the main result of this mathematical digression.

<u>Theorem 9</u>: Let T be a distribution in $\mathcal{S}'(\mathbb{R})$ with $\operatorname{supp} T \subset \overline{\mathbb{R}}_+$ and define S as in (7). Then for all $f \in \mathcal{S}(\mathbb{R}_+)$

(8) $$S(f) = T(\check{f})$$

(9) $$|S(f)| \leq c\, |f|'_m$$

for some constants c, m depending on T only. Conversely if S is a distribution in $\mathcal{S}'(\mathbb{R}_+)$, satisfying (9) for some c, m, then there exists a unique distribution $T \in \mathcal{S}'(\mathbb{R})$ with support in $\overline{\mathbb{R}}_+$, such that (8) holds.

In our applications we will use a multivariable version of Theorem 9, which due to the nuclear theorem is easy to prove. Let us just introduce the necessary notations and definitions.

By $\mathbb{R}_+^{4n}$ we denote the set $\{\underline{\xi} \mid \xi_k^0 > 0 , k = 1, \dots n\}$, by $\overline{\mathbb{R}}_+^{4n}$ its closure. For $f \in \mathcal{S}(\mathbb{R}_+^{4n})$ we define $\check{f}$ by

$$\check{f}(q) = \int \exp\left[-\sum_{k=1}^{n}(q_k^0 \xi_k^0 + i\vec{q}_k \vec{\xi}_k)\right] f(\underline{\xi})\, d^{4n}\xi \restriction \overline{\mathbb{R}}_+^{4n} \quad ,$$

and we introduce a set of norms on $\mathcal{S}(\mathbb{R}_+^{4n})$ by

$$(10) \qquad |f|'_m = |\check{f}|_{m,+} = \sup_{\substack{\underline{q} \in \mathbb{R}_+^{4n} \\ |\alpha| \leq m}} (1+|\underline{q}|)^m \, |\check{f}^{(\alpha)}(\underline{q})| \, .$$

For $T \in \mathcal{S}'(\mathbb{R}^{4n})$ with $\operatorname{supp} T \subset \overline{\mathbb{R}}_+^{4n}$ we can again define S (formally) by

$$S(\underline{\xi}) = \int \exp\left[-\sum_{k=1}^{n}(\xi_k^0 q_k^0 + i\vec{\xi}_k \vec{q}_k)\right] T(\underline{q})\, d^{4n}q \restriction \mathbb{R}_+^{4n} \quad ,$$

(Laplace transform with respect to the q_k^0 variables, Fourier transform in the distributional sense for the $\vec{q}_k$ variables). For $f \in \mathcal{S}(\mathbb{R}_+^{4n})$ $S(f) = \int f(\underline{\xi})\, S(\underline{\xi})\, d^{4n}\xi$ again defines a distribution in $\mathcal{S}'(\mathbb{R}_+^{4n})$. We leave it as an exercise to formulate the multivariable version of Theorem 9 for S and T defined as above.

For later use we introduce two more sub-spaces of $\mathcal{S}(\mathbb{R}^{4n})$.

$$\mathcal{S}_0(\mathbb{R}^{4n}) = \{ f \mid f \in \mathcal{S}(\mathbb{R}^{4n}),\ f^{(\alpha)}(\underline{x}) = 0 \quad \text{if} \quad x_i = x_k \\ \text{for some } i \neq k, \text{ all } \alpha \}$$

$$\mathcal{S}(\mathbb{R}_<^{4n}) = \{ f \mid f \in \mathcal{S}(\mathbb{R}^{4n}),\ \operatorname{supp} f \subset \mathbb{R}_<^{4n} \} ,$$

where $\mathbb{R}_<^{4n} = \{ \underline{x} \mid 0 < x_1^0 < \ldots < x_n^0 \}$.

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Now we are prepared to study the distributional aspects of the Schwinger functions. For $\xi_k^0 > 0$, $S_n(\underline{\xi})$ is just the Fourier-Laplace transform of $\widetilde{W}_n(\underline{q})$, see eq. (3), and the multivariable version of Theorem 9 applies. We obtain

<u>Lemma 10</u>. a) The difference variable Schwinger functions $S_n(\underline{\xi})$ define distributions in $\mathcal{S}'(\mathbb{R}_+^{4n})$ through

$$S_n(f) = \int f(\underline{\xi})\, S_n(\underline{\xi})\, d^{4n}\xi$$

for $f \in \mathcal{S}(\mathbb{R}_+^{4n})$. Furthermore $S_n(f) = \widetilde{W}_n(\check{f})$, and for some c and m

$$|S_n(f)| \leq c\, |f|'_m \, .$$

b) The Schwinger functions $\mathfrak{S}_n(\underline{x})$ define distributions in $\mathcal{S}'_0(\mathbb{R}^{4n})$.

We remark that a) implies that $\mathfrak{S}_n(\underline{x})$ defines a distribution in

$\mathcal{S}'(\mathbb{R}^{4n}_{<})$. In order to obtain b), an additional geometrical argument is necessary, see [OS 3, Si 1].

In the following theorem we collect all the properties of the Schwinger functions derived so far and state our main equivalence result.

Proposition I: The Schwinger functions associated to a Wightman theory have the following properties:

Distribution property: For each $n \geqslant 1$

$$\mathfrak{S}_n(\underline{x}) \in \mathcal{S}'_0(\mathbb{R}^{4n}) \quad ; \quad \mathfrak{S}_0 = 1 .$$

(E0) S_n defines an element in $\mathcal{S}'(\mathbb{R}^{4n}_{+})$ and is continuous with respect to some $|\cdot|'_m$ -norm.

Euclidean Covariance: For each $n \geqslant 1$ and all $(a, R) \in iSO_4$,

(E1) $\mathfrak{S}_n(\underline{x}) = \mathfrak{S}_n(\underline{R}x + a)$.

Positivity: For all finite sequences $f_0, f_1, \dots f_N$ of test functions $f_n \in \mathcal{S}(\mathbb{R}^{4n}_{<})$,

(E2) $\sum_{n,m} \mathfrak{S}_{n+m}(\Theta f_n^* \times f_m) \geqslant 0$,

where $\Theta f_n(\underline{x}) = f_n(\underline{\vartheta x})$.

Symmetry: For all permutations π,

(E3) $\mathfrak{S}_n(x_1, \dots x_n) = \mathfrak{S}_n(x_{\pi(1)}, \dots x_{\pi(n)})$.

Cluster Property: For all n, m, $f \in \mathcal{S}(\mathbb{R}^{4n}_{<})$, $g \in \mathcal{S}(\mathbb{R}^{4m}_{<})$,

$a = (0, \vec{a}) \in \mathbb{R}^4$,

(E4) $\lim_{\lambda\to\infty} \mathfrak{S}_{n+m}(\Theta f^* \times g_{\lambda a}) = \mathfrak{S}_n(\Theta f^*)\, \mathfrak{S}_m(g)$,

where $g_{\lambda a}$ is defined by $g_{\lambda a}(\underline{x}) = g(\underline{x} + \lambda a)$.

Conversely, Schwinger "functions" obeying (E0) - (E4) are the Schwinger functions associated with a unique Wightman theory.

Proof: The derivation of (E0) - (E4) from the Wightman axioms follows from Lemmas 4, 5, 6, and 10. Because $\int \mathfrak{S}_n(\underline{x}) f(\underline{x})\, dx$, $f \in \mathcal{S}_0(\mathbb{R}^{4n})$, can be defined as an ordinary integral, covariance, positivity, symmetry, and cluster properties hold in the distributional sense if they hold pointwise.

For a proof of the converse statement it suffices to assume instead of (E0) that S_n is in the algebraic dual of $\mathcal{S}(\mathbb{R}_+^{4n})$ and that it is continuous with respect to some $|\cdot|'_m$ -norm. Then the rest of (E0) follows from Lemma 8,(E1) and (E3). By the multivariable version of theorem 9, (E0) implies that there exist distributions $\widetilde{W}(\underline{q}) \in \mathcal{S}'(\mathbb{R}^{4n})$ with $\operatorname{supp}\widetilde{W} \subset \overline{\mathbb{R}}_+^{4n}$, such that (in the distributional sense) for $\underline{\xi} \in \mathbb{R}_+^{4n}$,

(11) $$S_n(\underline{\xi}) = \int \exp\left[-\sum_{k=1}^{n}(\xi_k^0 q_k^0 + i\vec{\xi}_k\vec{q}_k)\right]\widetilde{W}_n(\underline{q})\, d^{4n}q .$$

We define $\widetilde{W}_n(\underline{q})$ to be the Fourier transform of the difference variable Wightman distributions of our theory. From (E1) we conclude that the $\widetilde{W}_n(\underline{q})$ are Lorentz invariant distributions, see Nelson [Ne 2] and hence have support in $\overline{V}_+^n$. Positivity (W2) and the cluster property (W4) follow easily from the corresponding conditions (E2) and (E4). Locality (W3) finally follows from symmetry (E3) and all the other axioms already established, see ref. [Jo 1], p. 83. In the next section, these arguments will be discussed in more detail.

The norms $|\cdot|'_m$ which appear in (E0)-see eq.(6) for the definition - might be difficult to deal with in constructive field theory. We now introduce another distribution property (E0'):

> There is a Schwartz norm $|\cdot|_S$ on $\mathcal{S}(\mathbb{R}_+^4)$ and some $L>0$, such that for all n and for all $f_k \in \mathcal{S}(\mathbb{R}_+^4)$, k=1, ... n,
>
> (E0') $$|S_n(f_1 \times f_2 \times \dots \times f_n)| \le (n!)^L \prod_{k=1}^{n} |f_k|_S .$$

Our main result is the following proposition.

Proposition II: Schwinger functions satisfying (E0'), (E1) - (E4) determine a unique Wightman theory (whose Schwinger functions they are).

III. RECONSTRUCTING THE WIGHTMAN THEORY

In this section we start from a set of Schwinger functions $\mathfrak{S}_n$, satisfying (E0'), (E1) - (E4) and reconstruct the Wightman theory belonging to it. This will prove proposition II, but it will also shed some more light on the proof of proposition I.

Let $\underline{\mathcal{S}}_<$ be the vector space consisting of sequences $\underline{f} = (f_0, f_1, \dots)$

where $f_0 \in \mathbb{C}$, $f_n \in \mathcal{S}(\mathbb{R}^{4n}_<)$, for $1 \le n \le N$, and $f_n = 0$ for $n > N$, some finite N. For $\underline{f}$, $\underline{g} \in \underline{\mathcal{S}}_<$ let

$$\langle \underline{f}, \underline{g} \rangle = \sum_{n,m} \mathfrak{S}_{n+m} (\Theta f_n^* \times g_m)$$

By (E2), $\langle \, , \rangle$ is a positive semi-definite inner product. Set $\mathcal{N} = \{\underline{f} \mid \underline{f} \in \underline{\mathcal{S}}_<, \|\underline{f}\|^2 = \langle \underline{f}, \underline{f} \rangle = 0\}$. Then the completion of $\underline{\mathcal{S}}_< / \mathcal{N}$ defines a Hilbert space $\mathcal{H}$, which will turn out to be the physical Hilbert space. Denoting by Φ the natural injection of $\underline{\mathcal{S}}_<$ into $\mathcal{H}$, we obtain $(\Phi(\underline{f}), \Phi(\underline{g})) = \langle \underline{f}, \underline{g} \rangle$ for $\underline{f}, \underline{g} \in \underline{\mathcal{S}}_<$. We set $\Omega = \Phi((1, 0, 0 \ldots))$. Let $f \in \mathcal{S}(\mathbb{R}^{4n}_<)$, some n. Then if $\underline{f}$ is of the form $f_n = f$, $f_k = 0$ for $k \neq n$, we write $\Phi(\underline{f}) = \Phi_n(f) = \int \Phi_n(\underline{x}) f(\underline{x}) d^{4n}x$. We also define $\Psi_n^E(x_1, \underline{\xi}) = \Phi_n(\underline{x})$, where $\xi_k = x_{k+1} - x_k$, and Ψ_n^E is a vector valued distribution in $\mathcal{S}'(\mathbb{R}^{4n}_+)$. By (E2), positivity, the scalar product of two such vectors is again given by eq. (4) in Lemma 5.

In the following we will analytically extend the "functions" $\Psi_n^E(x_1, \underline{\xi})$ in the zero components $x_1^0, \xi_1^0, \ldots \xi_{n-1}^0$ of their arguments, after smearing them in the remaining variables. We define for $g \in \mathcal{S}(\mathbb{R}^{3n})$, $h \in \mathcal{S}(\mathbb{R}^{3m})$,

(12) $\quad \Psi_n^E(x_1^0, \underline{\xi}^0 | g) = \int \Psi_n^E(x_1, \underline{\xi})\, g(\vec{x}_1, \underline{\vec{\xi}})\, d\vec{x}_1 d\vec{\xi}$, and

(13) $\quad S_{n+m-1}(\overleftarrow{\underline{\xi}}^0, x^0 + x'^0, \underline{\xi}'^0 | gh) = (\Psi_n^E(x^0, \underline{\xi}^0|g) \;,\; \Psi_m^E(x'^0, \underline{\xi}'^0|h))$

Let $\mathbb{C}_+ = \{z \mid \mathrm{Re}\, z > 0\}$ and $\mathbb{C}_+^k = (\mathbb{C}_+)^k$.

Theorem 11: For fixed gh, the distributions $S_{n+m-1}(\underline{\xi}^0|gh)$ are restrictions to the product of positive real half axes of functions $S_{n+m-1}(\underline{\zeta}^0|gh)$, analytic in $\mathbb{C}_+^{n+m-1}$. There are vector valued functions $\Psi_n^E(z^0, \underline{\zeta}^0|g)$ analytic in $\mathbb{C}_+^n$, such that

$$S_{n+m-1}(\overleftarrow{\overline{\underline{\zeta}}}^0, \bar{z}^0 + z'^0, \underline{\zeta}'^0 | gh) =$$

(14) $$(\Psi_n^E(z^0, \underline{\zeta}^0|g) \,,\; \Psi_m^E(z'^0, \underline{\zeta}'^0|h)) \; .$$

Furthermore $S_{n+m-1}(\underline{\zeta}^0|gh)$ satisfies for $\underline{\zeta}^0 \in \mathbb{C}_+^{n+m-1}$

(15) $\quad |S_{n+m-1}(\underline{\zeta}^0|gh)| \le c|g|_s |h|_s (1 + |\underline{\zeta}^0|)^a (1 + [\min_k \mathrm{Re}\, \zeta_k^0]^{-1})^b$

for some Schwartz norms $|\cdot|_s$ and some constants a, b, c depending on n + m - 1.

Remarks: (1) By standard arguments, (15) implies that $S_{n+m-1}(\underline{\xi}^\circ | g h)$ is the Laplace transform of some distribution in $\mathcal{S}'$ with support in $\overline{\mathbb{R}}_+^{n+m-1}$, hence $S_{n+m-1}(\underline{\xi})$ is of the form (11), and again $\widetilde{W}_n(\underline{q})$ is the Fourier transform of the difference variable Wightman distribution. Note that in the proof of proposition I this fact was a direct consequence of the $|\cdot|'_m$-continuity assumption for the S_n 's . Positivity (W2) follows easily from (13).

(2) The proof of Theorem 11 as presented below is due to Osterwalder-Schrader [OS 4]. The construction of the analytic continuation of S_n was also found simultaneously and independently by Glaser [Gl 1,2].

Proof of Theorem 11: A. Constructing the Hamiltonian

For $t \geqslant 0$ we define $\hat{T}_t : \underline{\mathcal{S}}_< \to \underline{\mathcal{S}}_<$ by

$$(\hat{T}_t \underline{f})_n(\underline{x}) = f_n(x_1^0 - t, \vec{x}_1, \ldots x_n^0 - t, \vec{x}_n)$$

By the distribution property of $\mathfrak{S}_n$, for $\underline{f}, \underline{g} \in \underline{\mathcal{S}}_<$, $t \to \langle \underline{f}, \hat{T}_t \underline{g} \rangle$ is continuous and for some c, m, depending on $\underline{f}$,

$$(16) \qquad |\langle \underline{f}, \hat{T}_t \underline{f} \rangle| \leq c(1 + t^m) \quad .$$

Furthermore for $t \geqslant 0$, $s \geqslant 0$, $\hat{T}_t \hat{T}_s = \hat{T}_{t+s}$, and $\langle \underline{f}, \hat{T}_t \underline{g} \rangle = \langle \hat{T}_t \underline{f}, \underline{g} \rangle$.
By the Schwarz inequality

$$(17) \qquad |\langle \underline{f}, \hat{T}_t \underline{f} \rangle| \leq \|\underline{f}\| \, \|\hat{T}_t \underline{f}\| = \langle \underline{f}, \hat{T}_{2t} \underline{f} \rangle^{1/2}$$

for any $\underline{f} \in \underline{\mathcal{S}}_<$ with $\|\underline{f}\| = 1$. Iterating (17) and substituting (16) we obtain

$$|\langle \underline{f}, \hat{T}_t \underline{f} \rangle| \leq \langle \underline{f}, \hat{T}_{2^n t} \underline{f} \rangle^{2^{-n}} \leq \left[c(1 + (2^n t)^m) \right]^{2^{-n}} \longrightarrow 1$$

as $n \to \infty$. As $\hat{T}_t$ leaves the set $\mathcal{N}$ of norm zero vectors invariant, we may define a semigroup T_t in $\mathcal{H}$ by $T_t \Phi(\underline{f}) = \Phi(\hat{T}_t \underline{f})$ and extend it by continuity to all of $\mathcal{H}$. As for all $\Phi, \Phi' \in \mathcal{H}$, $\|T_t \Phi\| \leq \|\Phi\|$ and $(T_t \Phi, \Phi') = (\Phi, T_t \Phi')$, we conclude that T_t , $t \geqslant 0$, is a weakly continuous one parameter semigroup of selfadjoint contractions on $\mathcal{H}$ and $T_t = e^{-tH}$, where H is a selfadjoint positive operator. H is the Hamiltonian.

B. The Analytic Continuation

We will use the holomorphic semigroup $e^{-\tau H}$, $\mathrm{Re}\,\tau > 0$, to construct the analytic continuation in the time variables of the Schwinger functions. For the following arguments the space variables will play no role, so we shall drop them completely and write $S_{n-1}(\underline{\xi})$ and

$\Psi^E_n(x,\underline{\xi})$ instead of $S_{n-1}(\underline{\xi}^\circ | gh)$ and $\Psi^E_n(x^\circ_, \underline{\xi}^\circ | h)$ resp., where $\underline{\xi}$ now stands for the n - 1 time variables $\xi^\circ_1, \dots \xi^\circ_{n-1}$.

Sandwiching $e^{-\tau H}$ between two vectors Ψ^E_n and Ψ^E_m we find that for $\tau = t + is$, $t > 0$,

$$(\Psi^E_n(x,\underline{\xi}), \ e^{-\tau H}\Psi^E_m(x',\underline{\xi}')) = S_{n+m-1}(\overleftarrow{\underline{\xi}}, x+x'+t, \underline{\xi}' \mid s)$$

is a distribution in $\underline{\xi}$, x + x' + t, $\underline{\xi}'$ and s which satisfies the Cauchy-Riemann equations in t and s. It follows (see e.g. Vladimirov [Vl 1], p. 31) that

$$S_{n+m-1}(\overleftarrow{\underline{\xi}}, x+x'+t, \underline{\xi}' | s) = S_{n+m-1}(\overleftarrow{\underline{\xi}}, x+x'+\tau, \underline{\xi}')$$

is a distribution in the $\underline{\xi}$, $\underline{\xi}'$ variables and a function of x + x' + τ= z, analytic in the right half plane $\mathbb{C}_+ = \{z \mid \mathrm{Re}\, z > 0\}$, if properly smeared in the other variables. For n + m - 1 = k fixed and m = 0, 1, ... k, the "functions" $S_{n+m-1}(\overleftarrow{\underline{\xi}}, z, \underline{\xi}')$ are all analytic continuations of the same distribution S_k and we are in the situation of an edge of the wedge theorem for flat tubes. More precisely, if we define $\hat{S}_{n+m-1}(\overleftarrow{\underline{u}}, w, \underline{u}') = S_{n+m-1}(\overleftarrow{\underline{\xi}}, z, \underline{\xi}')$,where $\xi_k = e^{u_k}$, $z = e^w = e^{u+iv}$, then the $\hat{S}_{n+m-1}$ are analytic functions in w for $|\mathrm{Im}\, w| < \pi/2$ and distributions in the u-variables, and for $\mathrm{Im}\, w \to 0$, they all coincide. The Malgrange-Zerner theorem (see Epstein [Ep 1]) tells us now that there is a function $\hat{S}_k(w_1, \cdots w_k)$, analytic in $\{\underline{w} \mid \sum_{\ell=1}^{k} |\mathrm{Im}\, w_\ell| < \pi/2\}$, which continues all the $\hat{S}_{n+m-1}(\overleftarrow{\underline{u}}, w, \underline{u}')$. Equivalently we may say that the $S_{n+m-1}(\overleftarrow{\underline{\xi}}, z, \underline{\xi}')$ are all restrictions of a function $S_k(\zeta_1, \dots \zeta_k) = S_k(\underline{\zeta})$, k = n + m - 1, which is analytic in

$$C^{(1)}_k = \{\underline{\zeta} \mid \zeta_\ell \in \mathbb{C}_+ \, , \sum_{\ell=1}^{k} |\arg \zeta_\ell| < \pi/2\}.$$

We claim that the functions $S_k(\underline{\zeta})$ can be interpreted as the scalar products of two vectors in $\mathcal{H}$. Let us define $\mathcal{D}^{(1)}_n \subset \mathbb{C}^n_+$ by

$$\mathcal{D}^{(1)}_n = \{(x, \underline{\zeta}) \mid x > 0 \, , (\overleftarrow{\underline{\zeta}}, 2x, \underline{\zeta}) \in C^{(1)}_{2n-1}\}. \tag{18}$$

<u>Lemma 12</u>: There are vector valued functions $\Psi^E_n(x,\underline{\zeta}) : \mathcal{D}^{(1)}_n \longrightarrow \mathcal{H}$, analytic in $\underline{\zeta}$, such that

$$(\Psi^E_n(x,\underline{\zeta}), \Psi^E_m(x',\underline{\zeta}')) = S_{n+m-1}(\overleftarrow{\underline{\zeta}}, x+x', \underline{\zeta}'). \tag{19}$$

<u>Proof</u>: For $(\overset{\circ}{x}, \underline{\xi}) \in \mathcal{D}^{(1)}_n$ we choose a "polydisc" $P = \{(\overset{\circ}{x}, \underline{\zeta}) \mid |\zeta_k - \overset{\circ}{\xi}_k| < r_k, k = 1, \dots n-1\}$, centered at some real point $(\overset{\circ}{x}, \overset{\circ}{\underline{\xi}})$ and containing the

point $(\mathring{x}, \underline{\zeta})$, with r_k small enough, so that $P \subset \mathcal{D}_n^{(1)}$. (Note that the first variable $\mathring{x}$ is always kept fixed and real.) Then the Taylor series expansion of $S_{2n-1}(\overleftarrow{\underline{\zeta}}, \mathring{x}+\mathring{x}', \underline{\zeta}')$ around the point $(\overleftarrow{\mathring{\underline{\xi}}}, \mathring{x}+\mathring{x}', \mathring{\underline{\xi}}')$ is convergent for $(\mathring{x}, \underline{\zeta}) \in P$, $(\mathring{x}, \underline{\zeta}') \in P'$. Now let $\chi_\nu(x, \underline{\xi})$ be a C_0^∞ approximation to the δ-function $\delta(x - \mathring{x}) \cdot \prod_{i=1}^{n-1} \delta(\xi_k - \mathring{\xi}_k)$ such that $\operatorname{supp} \chi_\nu \subset \mathbb{R}_+^n$, $\chi_\nu \geqslant 0$ and $\int \chi_\nu \, dx \, d\underline{\xi} = 1$.

Furthermore define

$$f_{\nu\mu}(x, \underline{\xi}) = \sum_{|k| \leq \mu} \frac{(\mathring{\underline{\xi}} - \underline{\zeta})^{\underline{k}}}{\underline{k}!} \frac{\partial^{|k|}}{\partial \underline{\xi}^{\underline{k}}} \chi_\nu(x, \underline{\xi}) \quad ,$$

where $\underline{k} = (k_1, \ldots k_{n-1})$, $\underline{k}! = \prod_i k_i!$, $|k| = \sum_i k_i$, $\underline{x}^{\underline{k}} = \prod_i x_i^{k_i}$,

$$\frac{\partial^{|k|}}{\partial \underline{\xi}^{\underline{k}}} = \prod_i \frac{\partial^{k_i}}{\partial \xi_i^{k_i}}$$

Then

$$\Psi^E_{n,\nu\mu}(\mathring{x}, \underline{\zeta}) = \int \Psi^E_n(x, \underline{\xi}) \, f_{\nu\mu}(x, \underline{\xi}) \, dx \, d^{n-1}\xi$$

are vectors in $\mathcal{H}$, which for fixed ν, μ depend analytically on $\underline{\zeta}$. Using eq. (13) and the fact that $S_{n+m-1}(\overleftarrow{\underline{\xi}}, x+x', \underline{\xi}')$ is a real analytic function we easily check that $\Psi^E_n(\mathring{x}, \underline{\zeta}) = \lim_{\mu\to\infty} \lim_{\nu\to\infty} \Psi^E_{n,\nu\mu}(\mathring{x}, \underline{\zeta})$ exist and satisfy equation (19).

Lemma 12 enables us to construct an analytic extension of $S_k(\underline{\zeta})$ to a larger domain $C_k^{(2)}$. For $(x, \underline{\zeta}) \in \mathcal{D}_n^{(1)}$ and $(x', \underline{\zeta}') \in \mathcal{D}_m^{(1)}$ we define for $\tau \in \mathbb{C}_+$

$$(20) \qquad S_{n+m-1}(\overleftarrow{\underline{\zeta}}, x+x'+\tau, \underline{\zeta}') = \left(\Psi^E_n(x, \underline{\zeta}), e^{-\tau H} \Psi^E_m(x', \underline{\zeta}') \right) .$$

With $n + m - 1 = k$ fixed and $n = 0, \ldots. k$, eq. (20) yields an analytic extension of S_k to the domain

$$(21) \qquad \hat{C}_k^{(2)} = \bigcup_n \left\{ (\overleftarrow{\underline{\zeta}}, x+x'+\tau, \underline{\zeta}') \mid (x, \underline{\zeta}) \in \mathcal{D}_n^{(1)}, (x', \underline{\zeta}') \in \mathcal{D}_m^{(1)}, \tau \in \mathbb{C}_+, n+m-1 = k \right\} .$$

In terms of the variables $w_k = u_k + i v_k = \ln \zeta_k$, the domain $\hat{C}_k^{(2)}$ is a tube. By the tube theorem, see e.g. Vladimirov [Vl 1], p. 154, $S_k(\underline{\zeta})$ can be analytically extended to the envelope of holomorphy $C_k^{(2)}$ of $\hat{C}_k^{(2)}$, which is just the convex hull of $\hat{C}_k^{(2)}$ in the variables w_k. As in (18) we define

$$(22) \qquad \mathcal{D}_k^{(2)} = \left\{ (x, \underline{\zeta}) \mid x > 0, (\overleftarrow{\underline{\zeta}}, 2x, \underline{\zeta}) \in C_{2k-1}^{(2)} \right\}$$

and we prove Lemma 12 with $\mathcal{D}_k^{(2)}$ replacing $\mathcal{D}_k^{(1)}$ as before.

Repeating this procedure N times, we end up with an analytic extension of $S_k(\underline{\zeta})$ to a domain $C_k^{(N)}$, and vector valued functions $\mathcal{U}_n^{\equiv}(x,\underline{\zeta})$ defined on a domain $\mathcal{D}_k^{(N)}$, such that equation (19) holds. The first part of Theorem 11 follows if we can show that $\bigcup_N C_k^{(N)} = \mathbb{C}_+^k$ or equivalently $\bigcup_N \mathcal{D}_n^{(N)} = \mathbb{R}_+ \times \mathbb{C}_+^{n-1}$. We will prove a stronger result:

Lemma 13: For all N = 3, 4, ..., n = 1, 2, ...

(A) $\mathcal{D}_n^{(N)}$ contains the set

$$\{(x,\underline{\zeta}) \mid x \in \mathbb{R}_+ ,\ |\arg \zeta_r| < \tfrac{\pi}{2}\,(1 - 2^{-N}\sum_{t=0}^{r-1}\tbinom{N}{t})\ ,\ 1 \le r \le n\}$$

(B) $C_n^{(N)}$ contains the sets

$$\{\underline{\zeta} \mid |\arg \zeta_r| < \tfrac{\pi}{2}\,(1-2^{-N}\sum_{t=0}^{r-1}\tbinom{N}{t})\ ),\ 1 \le r \le n\} \quad \text{, and}$$

$$\{\underline{\zeta} \mid |\arg \zeta_r| < \max\{0, \tfrac{\pi}{2}(1-2^{-N/2}\gamma_n)\},\ 1 \le r \le n\}\ ,$$

where $\gamma_n = \max\limits_{N>1} 2^{-N/2}\sum_{t=0}^{n-1}\binom{N}{t}$, and $\sum_{t=0}^{r-1}\binom{N}{t} \equiv 2^N$ for $r > N$.

Proof: By construction the regions $C_k^{(N)}$ and $\mathcal{D}_k^{(N)}$ become tubes after the transformation $\zeta_i = e^{w_i}$. We define $c_k^{(N)}$ and $d_k^{(N)}$ to be the bases of these tubes:

$$c_k^{(N)} = \{\underline{v} \mid v_i = \operatorname{Im} w_i\ ,\ 1 \le i \le k\ ,\ (e^{w_1},\dots e^{w_k}) \in C_k^{(N)}\}$$
$$d_k^{(N)} = \{(0,\underline{v}) \mid v_i = \operatorname{Im} w_i\ ,\ 1 \le i \le k-1\ ,\ (e^{u_0}, e^{w_1},\dots e^{w_{k-1}}) \in \mathcal{D}_k^{(N)}\}$$

Note that $c_k^{(N)}$ and $d_k^{(N)}$ are subsets of $[-\pi/2\ ,\ \pi/2\ ]^n$. From the definitions of $C_k^{(N)}$ and $\mathcal{D}_k^{(N)}$ we find (see (21), (22))

$$c_k^{(N)} = \text{convex hull of } \{\underline{v} \mid \underline{v} = (-\overleftarrow{\underline{v}}', v, \underline{v}'') \text{ for } (0,\underline{v}') \in d_n^{(N)}\ , \tag{23}$$
$$(0,v'') \in d_m^{(N)}\ ,\ v \in (-\pi/2, \pi/2),\ n+m-1 = k\},$$

$$d_k^{(N)} = \{(0,\underline{v}) \mid (-\overleftarrow{\underline{v}}, 0, \underline{v}) \in c_{2k-1}^{(N)}\}\ . \tag{24}$$

Equations (23), (24) together with $c_k^{(0)} = (0, \dots 0)$ define the sets $c_k^{(N)}$, $d_k^{(N)}$ for all k and N inductively. Observe that all the sets $c_k^{(N)}$ and $d_k^{(N)}$ are convex. Moreover if $(v_1, \dots v_k) \in c_k^{(N)}$ (or $d_k^{(N)}$), then the whole hyperrectangle with corners $(\pm v_1, \dots \pm v_k)$ is also contained in $c_k^{(N)}$ ($d_k^{(N)}$ resp.).

For a proof of Lemma 13 we first construct a function $h(r,N)$, $r = 1,2,\dots,$ $N = 1, 2, \dots,$ such that for all $t \geq 1$, $s \geq 0$,

$$w(t,s,N) \equiv (\underbrace{0,0,\dots 0}_{t}, h(1,N), \dots h(s,N)) \in d_{t+s}^{(N)} \tag{25}$$

We choose $h(r,0) = 0$ for $r = 1, 2, \dots$. Obviously $w(t,s,0) \in d_{t+s}^{(0)}$.

Suppose now we have already constructed $h(r,N)$ for all $r \geq 1$ and $N = 0, 1, \dots L$, such that (25) holds. Then by (23) the following points are contained in $C^{(L+1)}_{2(s+t)-1}$, for all $|\alpha| < \pi/2$:

$$(-h(s,L), -h(s-1,L), \dots -h(2,L), -h(1,L), \underbrace{0, \dots 0}_{2t-1}, \alpha, h(1,L), \dots h(s-1,L))$$

and

$$(-h(s-1,L), \dots\dots -h(1,L), -\alpha, \underbrace{0, \dots 0}_{2t-1}, h(1,L), h(2,L), \dots h(s,L))$$

Because $C^{(L+1)}_{2(s+t)-1}$ is convex, it also contains the point

$$\Big(-\tfrac{1}{2}[h(s,L)+h(s-1,L)], \dots -\tfrac{1}{2}[h(2,L)+h(1,L)], -\tfrac{1}{2}[h(1,L)+\alpha], \underbrace{0, \dots 0}_{2t-1}, \tfrac{1}{2}[h(1,L)+\alpha], \dots \tfrac{1}{2}[h(s,L)+h(s-1,L)]\Big)$$

This means that with

$$\begin{aligned} & h(r,0) = 0 , \qquad r = 1,2,\dots \\ (26) \qquad & h(1,L+1) = \tfrac{1}{2}[h(1,L) + \alpha] \\ & h(r,L+1) = \tfrac{1}{2}[h(r,L) + h(r-1,L)] , \qquad r = 2,3,\dots \end{aligned}$$

the points $w(t, s, L+1)$, defined by (25), are again in $d^{(L+1)}_{t+s}$. We take (26) as inductive definition of $h(r,N)$. A simple calculation shows that the solution of (26) is

$$h(r,N) = \alpha\Big[1 - 2^{-N}\sum_{t=0}^{r-1}\binom{N}{t}\Big] ,$$

where we define $\sum_{t=0}^{s}\binom{N}{t} = 2^N$ for $s \geq N$.

Because α can be chosen arbitrarily close to $\pi/2$, the points $w_0(1,n-1,N) = (0, v_1, v_2, \dots v_{n-1})$ with $|v_r| < \frac{\pi}{2}\Big[1 - 2^{-N}\sum_{t=0}^{r-1}\binom{N}{t}\Big]$ are in $d^{(N)}_k$, which proves part (A) of Lemma 13. For part (B) we just use (A) and the equation $S_n(\underline{\zeta}) = (\Omega, \Psi^E_{n+1}(x, \underline{\zeta}))$.

C. Estimating $S_k(\underline{\xi})$

We only sketch the main ideas which lead to the estimate (15). All the detailed arguments will be in [OS 4]. In particular we shall continue neglecting the space variables completely, which of course we should not do in a complete derivation of inequality (15).

In a first step we use (E0') to conclude that there are integers α, β, and γ, such that for $\xi_k > 0$, $k = 1, 2, \dots n$, and all n,

$$(27) \qquad |S_n(\xi_1, \dots \xi_n)| \leq (\alpha n)^{\beta n} \prod_{i=1}^{n}\big[(1+\xi_i)(1+\xi_i^{-1})\big]^{\gamma} .$$

Now let $\varepsilon > 0$ and define

(28) $\quad S_{n,\varepsilon}(\underline{\zeta}) = \prod_{i=1}^{n} \left[(1+\zeta_i+\varepsilon)(1+\varepsilon^{-1})\right]^{-\gamma} S_n(\underline{\zeta}+\underline{\varepsilon})$

and

$$\Psi^{\varepsilon}_{n,\varepsilon}(x,\underline{\zeta}) = \left[(1+2x+\varepsilon)(1+\varepsilon^{-1})\right]^{-\gamma/2} \prod_{i=1}^{n-1} \left[(1+\zeta_i+\varepsilon)(1+\varepsilon^{-1})\right]^{-\gamma} \Psi^{\varepsilon}_{n}(x+\varepsilon/2, \underline{\zeta}+\underline{\varepsilon}),$$

where $\underline{\zeta}+\underline{\varepsilon}$ is defined by $(\underline{\zeta}+\underline{\varepsilon})_k = \zeta_k+\varepsilon$.
Both $S_{n,\varepsilon}(\underline{\zeta})$ and $\Psi^{\varepsilon}_{n+1,\varepsilon}(x,\underline{\zeta})$ are defined and analytic for $x \in \mathbb{R}_+$, $\underline{\zeta} \in \mathbb{C}^n_+$. By (28) we have for $z = x + iy$,

$$S_{n+m-1,\varepsilon}(\overleftarrow{\underline{\zeta}}, 2z, \underline{\zeta}') = \tag{29}$$
$$\left(\frac{1+2x+\varepsilon}{1+2z+\varepsilon}\right)^{\gamma} \left(\Psi^{\varepsilon}_{n,\varepsilon}(x,\underline{\zeta}),\ e^{-2iyH}\ \Psi^{\varepsilon}_{m,\varepsilon}(x,\underline{\zeta}')\right).$$

We claim that for $\underline{\zeta} \in C^{(N)}_n$,

(30) $\quad |S_{n,\varepsilon}(\underline{\zeta})| \le (\alpha n)^{\beta n}\ 2^{\beta n N}$.

We prove (30) by induction. First we notice that for N = 0, (30) is just ineq. (27). Now assume we have verified ineq. (30) for N = 1, 2, ... L and all n. Then for $(x,\underline{\zeta}) \in \mathcal{D}^{(L)}_n$, $(x,\underline{\zeta}') \in \mathcal{D}^{(L)}_m$, $k = n + m - 1$, by (29),

$$\begin{aligned}|S_{n+m-1,\varepsilon}(\overleftarrow{\underline{\zeta}}, 2(x+iy), \underline{\zeta}')| &\le \|\Psi^{\varepsilon}_{n,\varepsilon}(x,\underline{\zeta})\| \cdot \|\Psi^{\varepsilon}_{m,\varepsilon}(x,\underline{\zeta}')\| \\ &= \left[S_{2n-1,\varepsilon}(\overleftarrow{\underline{\zeta}}, 2x, \underline{\zeta})\ \ S_{2m-1,\varepsilon}(\overleftarrow{\underline{\zeta}}', 2x, \underline{\zeta}')\right]^{1/2} \\ &\le \left[\alpha(2n-1)\right]^{\beta(n-1/2)} \left[\alpha(2m-1)\right]^{\beta(m-1/2)}\ 2^{\beta k L} \\ &\le (\alpha k)^{\beta k}\ 2^{\beta k(L+1)} .\end{aligned}$$

As $C^{(L+1)}_k$ is just the envelop of holomorphy of the region $\bigcup_{n+m-1=k} \{(\overleftarrow{\underline{\zeta}}, 2(x+iy), \underline{\zeta}') \mid (x,\underline{\zeta}) \in \mathcal{D}^{(L)}_n ,\ (x,\underline{\zeta}') \in \mathcal{D}^{(L)}_m\}$ we can use the maximum principle (see e.g. [Vl 1], p. 178) to conclude that (30) holds for N = L + 1. This proves ineq. (30) for all n and N.

The next step is to eliminate N from the right hand side of ineq. (30). In order to do that we choose and fix $\underline{\zeta} \in \mathbb{C}^n_+$ and determine $N = N(\underline{\zeta})$ so that $\underline{\zeta} \in C^{(N)}_n$. Let N_n be such that $2^{-N_n/2}\gamma_n < 1/2$. Then by Lemma 13(B) $\underline{\zeta} \in C^{(N_n)}_n$ if $|\arg \zeta_r| < \pi/4$ for r = 1, ... n and we may set $N(\underline{\zeta}) = N_n$ for such $\underline{\zeta}$. The dependence on n of N_n will be unimportant. Now take a $\underline{\zeta} \in \mathbb{C}^n_+$ with $\max_{1\le r\le n} |\arg \zeta_r| = |\arg \zeta_s| \ge \pi/4$ for some $1 \le s \le n$.
Then for $\xi = \operatorname{Re}\zeta_s$, $\eta = |\operatorname{Im}\zeta_s|$,

$$|\arg \zeta_s| = \operatorname{arc\,tg}\frac{\eta}{\xi} = \frac{\pi}{2} - \operatorname{arc\,tg}\frac{\xi}{\eta} < \frac{\pi}{2}\left(1 - \frac{\xi}{2\eta}\right) .$$

(use that $\operatorname{arc\,tg} x > \frac{\pi}{4} x$ for $x < 1$.)
We define $N(\underline{\zeta}) = \lceil M \rceil$, where

$$2^{-M/2} \gamma_n = \frac{\xi}{2\eta} \quad , \text{ or } \quad M = c_n + \frac{2}{\ln 2} \ln \eta \xi^{-1} \quad ,$$

for some constant c_n depending on n only. In the following the meaning of c_n might change from line to line, but it will always be an n-dependent constant. With this choice of $N(\underline{\zeta})$ we find that for $1 \leq r \leq n$

$$|\arg \zeta_r| \leq |\arg \zeta_s| \leq \frac{\pi}{2} \left(1 - 2^{-N(\zeta)/2} \gamma_n\right) \quad ,$$

hence $\underline{\zeta} \in C_n^{(N(\zeta))}$, by Lemma 13(B). Now we substitute $N(\underline{\zeta})$ in ineq. (30). This gives

(31) $$| S_{n,\varepsilon}(\underline{\zeta}) | \leq (\alpha n)^{\beta n} 2^{\beta n N_n} = c_n \,, \quad \text{for } \max_r |\arg \zeta_r| < \pi/4 \,,$$

and

(32) $$| S_{n,\varepsilon}(\underline{\zeta}) | \leq c_n \, e^{2\beta n \cdot \ln \eta \xi^{-1}} = c_n \, (\eta \xi^{-1})^{2\beta n} \,,$$

for $\max |\arg \zeta_r| = \arg(\xi + i\eta) \geq \pi/4$. Combining (31) and (32) we finally obtain for some c_n

(33) $$| S_{n,\varepsilon}(\underline{\zeta}) | \leq c_n \left[\left(1 + |\max_k \operatorname{Im} \zeta_k|\right)\left(1 + (\min_k \operatorname{Re} \zeta_k)^{-1}\right)\right]^{2\beta n} .$$

Now we combine (33) with (28) and obtain by choosing $\varepsilon = \frac{1}{2} \min_k \operatorname{Re} \zeta_k$

$$\begin{aligned} | S_n(\underline{\zeta}) | &= \left| \prod_{i=1}^{n} \left[(1+\zeta_i)(1+\varepsilon^{-1})\right]^{\gamma} S_{n,\varepsilon}(\underline{\zeta} - \underline{\varepsilon}) \right| \\ &\leq c_n \left| \prod_{i=1}^{n} \left[(1+\zeta_i)(1+\varepsilon^{-1})\right]^{\gamma} \right| \cdot \left[(1 + |\max_k \operatorname{Im} \zeta_k|)(1+\varepsilon^{-1})\right]^{2\beta n} \\ &\leq c \left(1 + |\underline{\zeta}|\right)^{a} \left(1 + (\min_k \operatorname{Re} \zeta_k)^{-1}\right)^{b} \end{aligned}$$

for some constants a, b, c, depending on n. This proves (15), if we neglect the space variables.

By standard arguments, theorem 11 implies that there are distributions $\widetilde{W}_n(\underline{q})$ in $\mathcal{S}'(\mathbb{R}^{4n})$ with supp $\widetilde{W}_n \subset \overline{\mathbb{R}}_+^{4n}$, such that

$$S_n(\underline{\xi}) = \int \exp\left[- \sum_{k=1}^{n} (\xi_k^0 q_k^0 + i \vec{\xi}_k \vec{q}_k)\right] \widetilde{W}_n(\underline{q}) \, d^{4n} q \quad .$$

We define the Wightman distributions by

(34) $$\mathcal{W}_{n+1}(\underline{x}) = \int \exp\left[i \sum_k q_k (x_{k+1} - x_k)\right] \widetilde{W}_n(\underline{q}) \, d^{4n} q \quad ,$$

and we have to verify (W0) - (W5).

Verification of the Wightman Axioms

(W0) The distribution property (E0) follows from (34) and

(W1) Invariance of $\mathcal{W}_n$ under translations and space rotations follows immediately from (34) and the corresponding properties of S_n. To prove invariance under Lorentz rotations we need only show that $X_{0j}\tilde{W}(\underline{q}) = 0$, where $X_{0j} = \sum_{k=1}^{n}(q_k^0 \frac{\partial}{\partial q_k^j} + q_k^j \frac{\partial}{\partial q_k^0})$, j = 1, 2, 3, are the infinitesimal generators of Lorentz rotations, see [Ne 2]. But with $Y_{0j} = \sum_{k=1}^{n}(\xi_k^0 \frac{\partial}{\partial \xi_k^j} - \xi_k^j \frac{\partial}{\partial \xi_k^0})$, the infinitesmal generators of rotations in Euclidean space, we conclude from (E1) that $0 = Y_{0j}S_n(\underline{\xi}) = i\int \exp[-\Sigma(\xi_k^0 q_k^0 + i\vec{\xi}_k\vec{q}_k)]\, X_{0j}\tilde{W}_n(\underline{q})\,d^{4n}\underline{q}$. Because the kernel of Fourier and Laplace transforms is zero, $X_{0j}\tilde{W}_n(\underline{q}) = 0$, j = 1, 2, 3.

(W2) Positivity follows simply from Lemma 12 and the fact that the distribution $W_n(\underline{\xi})$ can be obtained as boundary value of the analytic continuation of the Schwinger functions S_n.

(W5) Lorentz covariance of $\tilde{W}_n(\underline{q})$ and $\operatorname{supp}\tilde{W}_n \subset \overline{\mathbb{R}}_+^{4n}$ imply that $\operatorname{supp}\tilde{W}_n \subset \overline{V}_+^n$.

(W4) The cluster property (E4) implies that $\lim_{\lambda\to\infty}(\Phi, \mathcal{U}(a,\mathbb{1})\Phi') = (\Phi,\Omega)(\Omega,\Phi')$ for any a = $(0, \vec{a})$ and arbitrary vectors $\Phi, \Phi' \in \mathcal{H}$. This immediately gives the cluster property (W4) for the Wightman distributions.

(W3) Locality follows from the symmetry of the Schwinger functions and a theorem in Jost [Jo 1], p. 83.

This ends the proof of proposition II.

Remark: 1) It has become clear from our proof, that condition (E0') was picked somewhat at random; other, similar conditions would do the same job. Work in progress even suggests, that (E0') can be replaced by a similar condition on the Schwinger functions $\mathfrak{S}_n$ rather than on S_n, see [OS 4]. This would probably be the most convenient form for applications.

2) For the Schwinger functions of $P(\varphi)_2$ models with small coupling constant, all the axioms (E0'), (E1) - (E4) follow easily from the cluster expansion estimates of Glimm, Jaffe and Spencer [GJS 1]. In particular (E0') is a consequence of corollary 1.1.9 in [GJS 1], which implies that

(35) $\qquad |\mathfrak{S}_n(f)| < c^n\, n!\, \|f\|_2 \quad ,$

for f having support in the product of unit lattice squares $\Delta_{i_1} \times \dots \times \Delta_{i_n} = \Delta_{\underline{i}}$. (We use the notation of [GJS 1].) For, let $h \in \mathcal{S}(\mathbb{R}_+^{2(n-1)})$ and let $\chi_{\underline{j}}(\underline{\xi})$ be the characteristic function of $\Delta_{j_1} \times \dots \times \Delta_{j_{n-1}}$ then

$$|S_{n-1}(h)| \leq \sum_{\underline{j}} |S_{n-1}(h\chi_{\underline{j}})|$$

Furthermore for any $g \in \mathcal{S}(\mathbb{R}^2)$, $\operatorname{supp} g \in \Delta_0$, $\int g(x)\, d^2x = 1$, $\xi_k = x_{k+1} - x_k$,

(36)
$$\begin{aligned} S_{n-1}(h\chi_j) &= \int \mathfrak{S}_n(\underline{x})\, g(x_1)\, h(\underline{\xi})\, \chi_{\underline{j}}(\underline{\xi})\, d^{2n}x \\ &= \sum_{\underline{i}} \int \mathfrak{S}_n(\underline{x})\, g(x_1)\, h(\underline{\xi})\, \chi_{\underline{j}}(\underline{\xi})\, \chi_{\underline{i}}(\underline{x})\, d^{2n}x \, . \end{aligned}$$

Observe, that $g(x_1)\chi_{\underline{j}}(\underline{\xi})\chi_{\underline{i}}(\underline{x})$ is different from zero only if $x_1 \in \Delta_0 \cap \Delta_{i_1}$; $x_2 - x_1 \in \Delta_{j_1}$, $x_2 \in \Delta_{i_2}$, ... $x_n - x_{n-1} \in \Delta_{j_{n-1}}$, $x_n \in \Delta_{i_n}$. This implies that for fixed $\underline{j}$, not more than 4^n values of $\underline{i}$ give non-vanishing contributions to $\sum_{\underline{i}}$ in (36). Hence we obtain using (35)

$$\begin{aligned} |S_{n-1}(h\chi_j)| &\leq 4^n \max_{\underline{i}} \left| \int \mathfrak{S}_n(\underline{x})\, g(x_1)\, h(\underline{\xi})\, \chi_{\underline{j}}(\underline{\xi})\, \chi_{\underline{i}}(\underline{x})\, d^{2n}x \right| \\ &\leq c_1^n\, n!\, \max_{\underline{i}} \left(\int |g(x_1)\, h(\underline{\xi})\, \chi_{\underline{j}}(\underline{\xi})\, \chi_{\underline{i}}(\underline{x})|^2\, d^{2n}x \right)^{1/2} \\ &\leq c_1^n\, n!\, \|g\|_2\, \|h\chi_{\underline{j}}\|_2 \, . \end{aligned}$$

In the last step we have used the fact that the substitution $\underline{x} \to (x_1, \underline{\xi})$ has Jacobean 1. Hence for $h = h_1 \times h_2 \times \dots h_{n-1}$

$$\begin{aligned} |S_{n-1}(h)| &\leq c_2^n\, n! \sum_{\underline{j}} \|h\chi_{\underline{j}}\|_2 \\ &- c_2^n\, n! \prod_k \Big(\sum_j \|h_k \chi_j\|_2 \Big) \end{aligned}$$

and this gives immediately (E0').

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