Finite temperature and chemical potential

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Some old method to heat kernel like regularization! The content of the details in residue theory is shown below. The details of are shown below.

$$
\begin{aligned}
Im\left[s_1\right] &= \left[\left(\mu^2-M^2-\omega_l^2\right)^2+\left(2\omega_l\mu\right)^2\right]^{\frac{1}{4}}\sin\left(\frac{1}{2}\arctan\left(\frac{2\omega_l\mu}{\mu^2-M^2-\omega_l^2}\right)\right)\\
&\overset{|\sin\frac{\alpha}{2}|=\sqrt{\frac{1-\cos\alpha}{2}}}{=} \left[\left(\mu^2-M^2-\omega_l^2\right)^2+\left(2\omega_l\mu\right)^2\right]^{\frac{1}{4}}\frac{1}{\sqrt{2}}\sqrt{1-\cos\left(\arctan\left(\frac{2\omega_l\mu}{\mu^2-M^2-\omega_l^2}\right)\right)}\\
&\overset{\cos(\arctan x)=\frac{1}{\sqrt{1+x^2}}}{=} \left[\left(\mu^2-M^2-\omega_l^2\right)^2+\left(2\omega_l\mu\right)^2\right]^{\frac{1}{4}}\frac{1}{\sqrt{2}}\sqrt{1-\frac{1}{\sqrt{1+\left(\frac{2\omega_l\mu}{\mu^2-M^2-\omega_l^2}\right)^2}}}\\
&= \frac{1}{\sqrt{2}}\left\{\left[\left(\mu^2-M^2-\omega_l^2\right)^2+\left(2\omega_l\mu\right)^2\right]^{\frac{1}{2}}-\left(\mu^2-M^2-\omega_l^2\right)\right\}^{\frac{1}{2}},\ When\ \mu^2-M^2-\omega_l^2>0\&\mu\omega_l\geq 0,\\
Im\left[s_1\right] &= \left[\left(\mu^2-M^2-\omega_l^2\right)^2+\left(2\omega_l\mu\right)^2\right]^{\frac{1}{4}}\sin\left(\frac{\pi}{4}\right)\\
&= \sqrt{\omega_l\mu},\ When\ \mu^2-M^2-\omega_l^2=0\&\mu\omega_l>0,\\
Im\left[s_1\right] &= \left[\left(\mu^2-M^2-\omega_l^2\right)^2+\left(2\omega_l\mu\right)^2\right]^{\frac{1}{4}}\sin\left(\frac{\pi}{2}+\frac{1}{2}\arctan\left(\frac{2\omega_l\mu}{\mu^2-M^2-\omega_l^2}\right)\right)\\
&= \left[\left(\mu^2-M^2-\omega_l^2\right)^2+\left(2\omega_l\mu\right)^2\right]^{\frac{1}{4}}\cos\left(\frac{1}{2}\arctan\left(\frac{2\omega_l\mu}{\mu^2-M^2-\omega_l^2}\right)\right)\\
&\overset{|\cos\frac{\alpha}{2}|=\sqrt{\frac{1+\cos\alpha}{2}}}{=} \left[\left(\mu^2-M^2-\omega_l^2\right)^2+\left(2\omega_l\mu\right)^2\right]^{\frac{1}{4}}\frac{1}{\sqrt{2}}\sqrt{1+\cos\left(\arctan\left(\frac{2\omega_l\mu}{\mu^2-M^2-\omega_l^2}\right)\right)}\\
&\overset{\cos(\arctan x)=\frac{1}{\sqrt{1+x^2}}}{=} \left[\left(\mu^2-M^2-\omega_l^2\right)^2+\left(2\omega_l\mu\right)^2\right]^{\frac{1}{4}}\frac{1}{\sqrt{2}}\sqrt{1+\frac{1}{\sqrt{1+\left(\frac{2\omega_l\mu}{\mu^2-M^2-\omega_l^2}\right)^2}}}\\
&= \frac{1}{\sqrt{2}}\left\{\left[\left(\mu^2-M^2-\omega_l^2\right)^2+\left(2\omega_l\mu\right)^2\right]^{\frac{1}{2}}-\left(\mu^2-M^2-\omega_l^2\right)\right\}^{\frac{1}{2}},\ When\ \mu^2-M^2-\omega_l^2<0,\\
Im\left[s_1\right] &= \left[\left(\mu^2-M^2-\omega_l^2\right)^2+\left(2\omega_l\mu\right)^2\right]^{\frac{1}{4}}\sin\left(\frac{3\pi}{4}\right)\\
&= \sqrt{\omega_l\mu},\ When\ \mu^2-M^2-\omega_l^2=0\&\mu\omega_l\leq 0,\\
Im\left[s_1\right] &= \left[\left(\mu^2-M^2-\omega_l^2\right)^2+\left(2\omega_l\mu\right)^2\right]^{\frac{1}{4}}\sin\left(\pi+\frac{1}{2}\arctan\left(\frac{2\omega_l\mu}{\mu^2-M^2-\omega_l^2}\right)\right)\\
&= -\left[\left(\mu^2-M^2-\omega_l^2\right)^2+\left(2\omega_l\mu\right)^2\right]^{\frac{1}{4}}\sin\left(\frac{1}{2}\arctan\left(\frac{2\omega_l\mu}{\mu^2-M^2-\omega_l^2}\right)\right)\\
&\overset{|\sin\frac{\alpha}{2}|=\sqrt{\frac{1-\cos\alpha}{2}}}{=} \left[\left(\mu^2-M^2-\omega_l^2\right)^2+\left(2\omega_l\mu\right)^2\right]^{\frac{1}{4}}\frac{1}{\sqrt{2}}\sqrt{1-\cos\left(\arctan\left(\frac{2\omega_l\mu}{\mu^2-M^2-\omega_l^2}\right)\right)}\\
&\overset{\cos(\arctan x)=\frac{1}{\sqrt{1+x^2}}}{=} \left[\left(\mu^2-M^2-\omega_l^2\right)^2+\left(2\omega_l\mu\right)^2\right]^{\frac{1}{4}}\frac{1}{\sqrt{2}}\sqrt{1-\frac{1}{\sqrt{1+\left(\frac{2\omega_l\mu}{\mu^2-M^2-\omega_l^2}\right)^2}}}\\
&= \frac{1}{\sqrt{2}}\left\{\left[\left(\mu^2-M^2-\omega_l^2\right)^2+\left(2\omega_l\mu\right)^2\right]^{\frac{1}{2}}-\left(\mu^2-M^2-\omega_l^2\right)\right\}^{\frac{1}{2}},\ When\ \mu^2-M^2-\omega_l^2>0\&\mu\omega_l\leq 0,
\end{aligned} \tag{1}
$$

The details of are shown below.

$$
\begin{aligned}
&T\sum_{l=-\infty}^{\infty} f\left(p_0=\widetilde{\omega}_l=\omega_l+i\mu\right) \\
&\underset{p_0=\omega_l+i\mu,\beta=\frac{1}{T}}{\overset{the\ residue\ theorem}{=}} \frac{T}{2\pi i}\oint_{\mathcal{L}} dp_0 f\left(p_0\right)\times\frac{\beta}{2}\frac{e^{\frac{i\beta}{2}(p_0-i\mu)}-e^{-\frac{i\beta}{2}(p_0-i\mu)}}{e^{\frac{i\beta}{2}(p_0-i\mu)}+e^{-\frac{i\beta}{2}(p_0-i\mu)}} \\
&=\frac{1}{2\pi i}\frac{1}{2}\int_{-\infty+i(\mu-\epsilon)}^{+\infty+i(\mu-\epsilon)} dp_0 f\left(p_0\right)\times\frac{e^{\frac{i\beta}{2}(p_0-i\mu)}-e^{-\frac{i\beta}{2}(p_0-i\mu)}}{e^{\frac{i\beta}{2}(p_0-i\mu)}+e^{-\frac{i\beta}{2}(p_0-i\mu)}} \\
&+\frac{1}{2\pi i}\frac{1}{2}\int_{+\infty+i(\mu+\epsilon)}^{-\infty+i(\mu+\epsilon)} dp_0 f\left(p_0\right)\times\frac{e^{\frac{i\beta}{2}(p_0-i\mu)}-e^{-\frac{i\beta}{2}(p_0-i\mu)}}{e^{\frac{i\beta}{2}(p_0-i\mu)}+e^{-\frac{i\beta}{2}(p_0-i\mu)}} \\
&=\frac{1}{2\pi i}\int_{-\infty+i(\mu-\epsilon)}^{+\infty+i(\mu-\epsilon)} dp_0 f\left(p_0\right)\times[\frac{1}{2}-\frac{1}{e^{i\beta(p_0-i\mu)}+1}] \\
&-\frac{1}{2\pi i}\int_{+\infty+i(\mu+\epsilon)}^{-\infty+i(\mu+\epsilon)} dp_0 f\left(p_0\right)\times[\frac{1}{2}-\frac{1}{e^{-i\beta(p_0-i\mu)}+1}] \\
&=-\frac{1}{2\pi i}\int_{-\infty+i(\mu+\epsilon)}^{+\infty+i(\mu+\epsilon)} dp_0 f\left(p_0\right)\frac{1}{e^{-i\beta(p_0-i\mu)}+1}-\frac{1}{2\pi i}\int_{-\infty+i(\mu-\epsilon)}^{+\infty+i(\mu-\epsilon)} dp_0 f\left(p_0\right)\frac{1}{e^{i\beta(p_0-i\mu)}+1} \\
&+\frac{1}{2\pi i}\int_{-\infty+i(\mu-\epsilon)}^{+\infty+i(\mu-\epsilon)} dp_0 f\left(p_0\right)\frac{1}{2}+\frac{1}{2\pi i}\int_{-\infty+i(\mu+\epsilon)}^{+\infty+i(\mu+\epsilon)} dp_0 f\left(p_0\right)\frac{1}{2} \\
&=-\frac{1}{2\pi i}\int_{-\infty+i(\mu+\epsilon)}^{+\infty+i(\mu+\epsilon)} dp_0 f\left(p_0\right)\frac{1}{e^{-i\beta(p_0-i\mu)}+1}-\frac{1}{2\pi i}\int_{-\infty+i(\mu-\epsilon)}^{+\infty+i(\mu-\epsilon)} dp_0 f\left(p_0\right)\frac{1}{e^{i\beta(p_0-i\mu)}+1} \\
&+\frac{1}{2\pi i}\int_{-\infty+i\mu}^{+\infty+i\mu} dp_0 f\left(p_0\right) \\
=&-\frac{1}{2\pi i}\int_{-\infty+i(\mu+\epsilon)}^{+\infty+i(\mu+\epsilon)} dp_0 f\left(p_0\right)\frac{1}{e^{-i\beta(p_0-i\mu)}+1} \\
&-\frac{1}{2\pi i}\int_{-\infty+i(\mu-\epsilon)}^{+\infty+i(\mu-\epsilon)} dp_0 f\left(p_0\right)\frac{1}{e^{i\beta(p_0-i\mu)}+1} \\
&+\frac{1}{2\pi i}\oint_{R_C} dp_0 f\left(p_0\right)+\frac{1}{2\pi i}\int_{-\infty}^{+\infty} dp_0 f\left(p_0\right)
\end{aligned} \tag{2}
$$

The details here is over !