# Analyse First order or Crossover

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When $\mu' \leq R_B$, the equation only have one solution, hence the multisolution zone must be $\mu' > R_B$. (There exists one possibility that both in $\mu' \leq R_B$ and $\mu' > R_B$ has at least one solution, but this case will not happen in CEP, so we ignore) When $\mu' > R_B$, the equation becomes

$$\begin{aligned} R_B =& m + \frac{8}{3\pi}\frac{\alpha_{IR}}{m_G^2}\int_{\mu'^2-R_B^2}^{\Lambda^2}\frac{R_B s^{\frac{1}{2}}}{\sqrt{s+R_B^2}}ds \\ =& m + \frac{8}{3\pi}\frac{\alpha_{IR}R_B}{m_G^2}\left[R_B^2\ln(\sqrt{x+R_B^2}-\sqrt{x})+\sqrt{x(x+R_B^2)}\right]|_{\mu'^2-R_B^2}^{\Lambda^2} \\ =& m + \frac{8}{3\pi}\frac{\alpha_{IR}R_B}{m_G^2}\left[R_B^2\ln\frac{\sqrt{\Lambda^2+R_B^2}-\Lambda}{\mu'-\sqrt{\mu'^2-R_B^2}}+\Lambda\sqrt{\Lambda^2+R_B^2}-\mu'\sqrt{\mu'^2-R_B^2}\right] \end{aligned} \tag{1}$$

When the first order transition happens, we will get the picture as below:

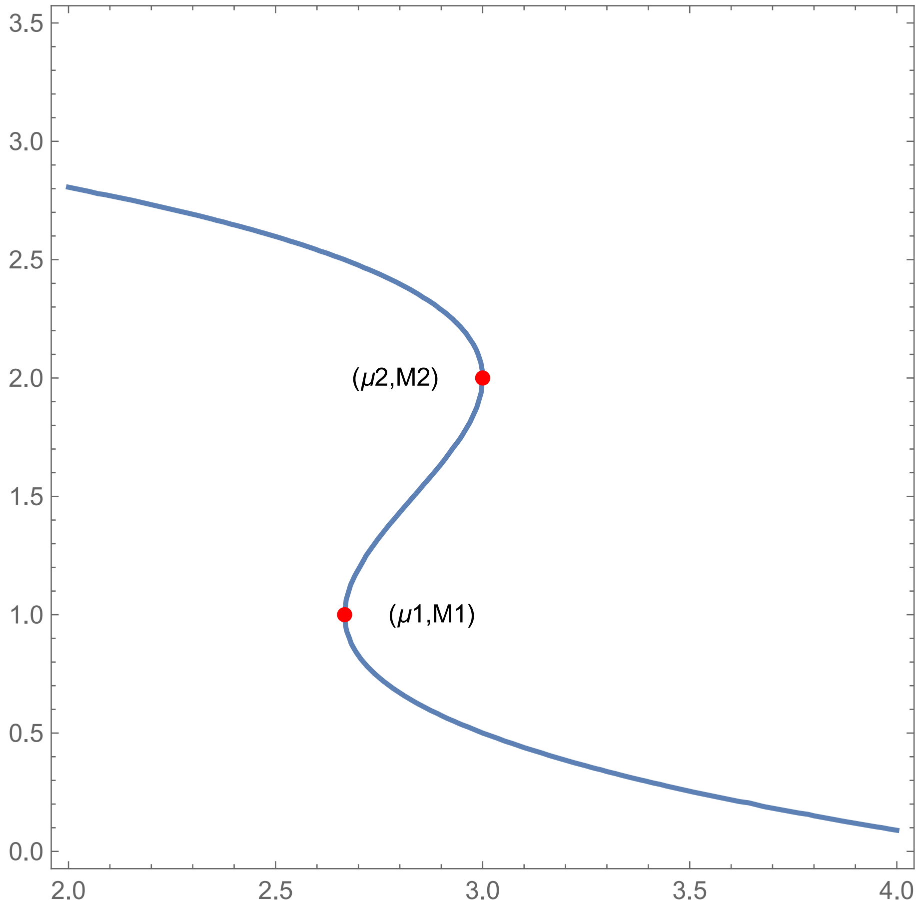


Figure 1: There exists two points $(\mu_1, R_{B1}), (\mu_2, R_{B2})$; If the transition is crossover, we could not get these two points.

Define $\frac{8}{3\pi}\frac{\alpha_{IR}}{m_G^2} = k$. At these two points, the following relationship must be satisfied:

- $$F(x) = m + kx\left[x^2\ln\frac{\sqrt{\Lambda^2+x^2}-\Lambda}{\mu'-\sqrt{\mu'^2-x^2}}+\Lambda\sqrt{\Lambda^2+x^2}-\mu'\sqrt{\mu'^2-x^2}\right] - x = 0 \tag{2}$$

- $$F'(x) = \frac{k\Lambda(3x^2+\Lambda^2)}{\sqrt{x^2+\Lambda^2}} - k\mu'\sqrt{\mu'^2-x^2} + 3kx^2\ln\frac{\sqrt{\Lambda^2+x^2}-\Lambda}{\mu'-\sqrt{\mu'^2-x^2}} - 1 = 0 \tag{3}$$

Solve the simultaneous equations and eliminate the logarithm part, we could get

$$\begin{aligned} &\frac{3}{2}m - kx\mu'\sqrt{\mu'^2-x^2} - x + \frac{kx\Lambda^3}{\sqrt{\Lambda^2+x^2}} = 0 \\ &\Leftrightarrow \mu'\sqrt{\mu'^2-x^2} = \frac{3}{2}\frac{m}{kx} - \frac{1}{k} + \frac{\Lambda^3}{\sqrt{\Lambda^2+x^2}} \equiv \frac{t(x)}{2} \\ &\Leftrightarrow \mu' = \sqrt{\frac{x^2+\sqrt{x^4+t^2(x)}}{2}} \end{aligned} \tag{4}$$

where $t(x) = \frac{3m}{kx} - \frac{2}{k} + \frac{2\Lambda^3}{\sqrt{\Lambda^2+x^2}}$ Substitute this equation into (3), we could get

$$\begin{aligned} &\frac{kx^2\Lambda}{\sqrt{x^2+\Lambda^2}} - \frac{1}{2}\frac{m}{x} + kx^2\ln\frac{\sqrt{\Lambda^2+x^2}-\Lambda}{\mu'-\sqrt{\mu'^2-x^2}} = 0 \\ &\Leftrightarrow \frac{kx^2\Lambda}{\sqrt{x^2+\Lambda^2}} - \frac{1}{2}\frac{m}{x} + kx^2\ln\frac{\sqrt{\Lambda^2+x^2}-\Lambda}{(\sqrt{x^4+t^2}-t)^{\frac{1}{2}}} = 0 \\ &\Leftrightarrow \frac{\Lambda}{\sqrt{x^2+\Lambda^2}} - \frac{1}{2}\frac{m}{kx^3} + \ln(\sqrt{\Lambda^2+x^2}-\Lambda) - \frac{1}{2}\ln(\sqrt{x^4+t^2}-t) \\ &= \frac{\Lambda}{\sqrt{x^2+\Lambda^2}} - \frac{1}{2}\frac{m}{kx^3} + \ln\frac{\sqrt{\Lambda^2+x^2}-\Lambda}{(\sqrt{x^4+t^2}-t)^{\frac{1}{2}}} = 0 \equiv F(x) \end{aligned} \tag{5}$$

If first order transition, the equation above must have two solutions; if crossover, the equation do not have any solution (in $x < M_0$).
So the CEP is:

- $$F(x) = \frac{\Lambda}{\sqrt{x^2+\Lambda^2}} - \frac{1}{2}\frac{m}{kx^3} + \ln\frac{\sqrt{\Lambda^2+x^2}-\Lambda}{(\sqrt{x^4+t^2}-t)^{\frac{1}{2}}} = 0 \tag{6}$$

- $$xF'(x) = \frac{3m}{2kx^3} + \frac{xt'}{2\sqrt{x^4+t^2}} + \frac{\Lambda^3}{(x^2+\Lambda^2)^{\frac{3}{2}}} - \frac{t}{\sqrt{x^4+t^2}} = 0 \tag{7}$$

  where $t' = \frac{dt}{dx} = -\frac{3m}{kx^2} - \frac{2x\Lambda^3}{(x^2+\Lambda^2)^{\frac{3}{2}}}$.

We have use a fact that if $F(x) = 0, F'(x) = 0$, then $x^nF(x) = 0, x^nF'(x) = 0$.
Solve the Equation 7, we could get

$$\frac{3m}{2kx^3} + \frac{1}{\sqrt{x^4+t^2}}(\frac{xt'}{2} - t) + \frac{\Lambda^3}{(x^2+\Lambda^2)^{\frac{3}{2}}} = 0 \tag{8}$$

This equation is not a simple equation, we could numerically get a solution $x_0 = x(m, k, \Lambda)$, and take into Equation 6. If $F(x_0) > 0$, first order transition; If $F(x_0) \leq 0$, crossover.