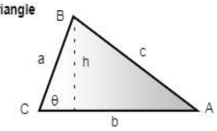
ASIAN DEVELOPMENT FOUNDATION COLLEGE
Tacloban City

Part of:
Plane and Solid Geometry by RTFVerterra © October 2003

PLANE GEOMETRY

PLANE AREAS

Triangle



Given base b and altitude h

$A = \frac{1}{2}bh$

Given two sides a and b and included angle θ:

$A = \frac{1}{2}ab\sin\theta$

Given three sides a, b, and c: (Hero's Formula)

$A = \sqrt{s(s-a)(s-b)(s-c)}$

$s = \frac{a+b+c}{2}$

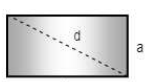
The area under this condition can also be solved by finding one angle using cosine law and apply the formula for two sides and included angle.

Given three angles A, B, and C and one side a:

$A = \frac{a^2\sin B\sin C}{2\sin A}$

The area under this condition can also be solved by finding one side using sine law and apply the formula for two sides and included angle.

Rectangle

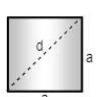


Area, $A = ab$

Perimeter, $P = 2(a + b)$

Diagonal, $d = \sqrt{a^2 + b^2}$

Square

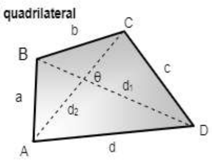


Area, $A = a^2$

Perimeter, $P = 4a$

Diagonal, $d = a\sqrt{2}$

General quadrilateral



Given diagonals d_1 and d_2 and included angle θ:

$A = \frac{1}{2}d_1 \times d_2 \times \sin\theta$

Given four sides a, b, c, d, and sum of two opposite angles:

$A = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\theta}$

$s = \frac{a+b+c+d}{2}$

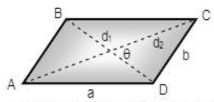
$\theta = \frac{\angle A + \angle C}{2}$ or $\theta = \frac{\angle B + \angle D}{2}$

Given four sides a, b, c, d, and two opposite angles B and D:

Divide the area into two triangles

$A = \frac{1}{2}ab\sin B + \frac{1}{2}cd\sin D$

Parallelogram



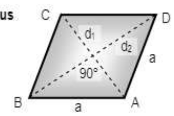
Given diagonals d_1 and d_2 and included angle θ:

$A = \frac{1}{2}d_1 \times d_2 \times \sin\theta$

Given two sides a and b and one angle A:

$A = ab\sin A$

Rhombus



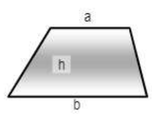
Given diagonals d_1 and d_2:

$A = \frac{1}{2}d_1 \times d_2$

Given side a and one angle A:

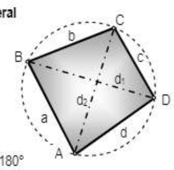
$A = a^2\sin A$

Trapezoid



$A = \frac{a+b}{2}h$

Cyclic Quadrilateral



A cyclic quadrilateral is a quadrilateral whose vertices lie on the circumference of a circle.

$\angle A + \angle C = 180°$

$\angle B + \angle D = 180°$

$\text{Area} = \sqrt{(s-a)(s-b)(s-c)(s-d)}$

$s = \frac{a+b+c+d}{2}$

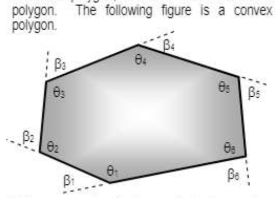
Ptolemy's theorem

"For any cyclic quadrilateral, the product of the diagonals equals the sum of the products of the opposite sides"

$d_1 \times d_2 = ac + bd$

POLYGONS

There are two basic types of polygons, a convex and a concave polygon. A convex polygon is one in which no side, when extended, will pass inside the polygon, otherwise it called concave polygon. The following figure is a convex polygon.



Polygons are classified according to the number of sides. The following are some names of polygons.

3 sides	=	triangle
4 sides	=	quadrangle or quadrilateral
5 sides	=	pentagon
6 sides	=	hexagon
7 sides	=	heptagon or septagon
8 sides	=	octagon
9 sides	=	nonagon
10 sides	=	decagon
11 sides	=	undecagon
12 sides	=	dodecagon
15 sides	=	quindecagon
16 sides	=	hexadecagon

Sum of interior angles

The sum of interior angles θ of a polygon of n sides is:

Sum, $\Sigma\theta = (n-2) \times 180°$

Sum of exterior angles

The sum of exterior angles β is equal to 360°.

$\Sigma\beta = 360°$

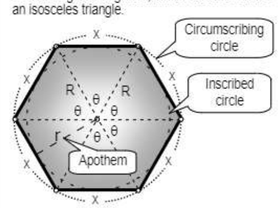
Number of diagonals, D

The diagonal of a polygon is the line segment joining two non-adjacent sides. The number of diagonals is given by:

$D = \frac{n}{2}(n-3)$

Regular polygons

Polygons whose sides are equal are called equilateral polygons. Polygons with equal interior angles are called equiangular polygons. Polygons that are both equilateral and equiangular are called regular polygons. The area of a regular polygon can be found by considering one segment, which has the form of an isosceles triangle.



$x =$ side

$\theta =$ angle subtended by the side from the center

$R =$ radius of circumscribing circle

$r =$ radius of inscribed circle, also called the apothem

$n =$ number of sides

$\theta = 360° / n$

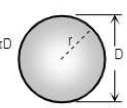
Area, $A = \frac{1}{2}R^2\sin\theta \times n = \frac{1}{2}x\,r \times n$

Perimeter, $P = n \times x$

Interior angle $= \frac{n-2}{n} \times 180°$

Exterior angle $= 360° / n$

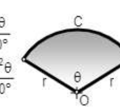
Circle



Circumference $= 2\pi r = \pi D$

Area, $A = \pi r^2 = \frac{\pi}{4}D^2$

Sector of a circle



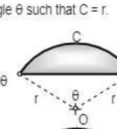
Arc $C = r \times \theta_{radians} = \frac{\pi r\theta}{180°}$

Area $= \frac{1}{2}r^2\,\theta_{radians} = \frac{\pi r^2\theta}{360°}$

Area $= \frac{1}{2}C \times r$

Note: 1 radian is the angle θ such that C = r.

Segment of a circle

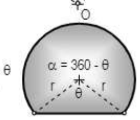


Area $= A_{sector} - A_{triangle}$

Area $= \frac{1}{2}r^2\,\theta_r - \frac{1}{2}r^2\sin\theta$

Area $= \frac{1}{2}r^2(\theta_r - \sin\theta)$

$\theta_r =$ angle in radians

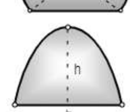


Area $= A_{sector} + A_{triangle}$

Area $= \frac{1}{2}r^2\,\alpha_r + \frac{1}{2}r^2\sin\theta$

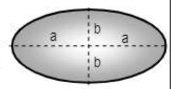
Area $= \frac{1}{2}r^2(\alpha_r + \sin\theta)$

Parabolic segment



Area $= \frac{2}{3}bh$

Ellipse



Area $= \pi\,a\,b$

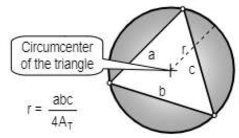
Perimeter, P

$P = 2\pi\sqrt{\frac{a^2+b^2}{2}}$

RADIUS OF CIRCLES

Circle circumscribed about a triangle (Cicumcircle)

A circle is circumscribed about a triangle if it passes through the vertices of the triangle.

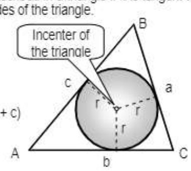


$r = \frac{abc}{4A_T}$

$A_T =$ area of the triangle

Circle inscribed in a triangle (Incircle)

A circle is inscribed in a triangle if it is tangent to the three sides of the triangle.

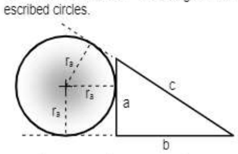


$r = \frac{A_T}{s}$

$s = \frac{1}{2}(a + b + c)$

Circles escribed about a triangle (Excircles)

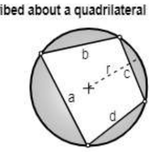
A circle is escribed about a triangle if it is tangent to one side and to the prolongation of the other two sides. A triangle has three escribed circles.



$r_a = \frac{A_T}{s-a}$; $r_c = \frac{A_T}{s-c}$; $r_b = \frac{A_T}{s-b}$

Circle circumscribed about a quadrilateral

A circle is circumscribed about a quadrilateral if it passes through the vertices of the quadrilateral.



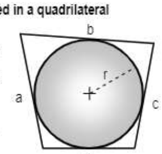
$r = \frac{\sqrt{(ab+cd)(ac+bd)(ad+bc)}}{4A_{quad}}$

$A_{quad} = \sqrt{(s-a)(s-b)(s-c)(s-d)}$

$s = \frac{1}{2}(a + b + c + d)$

Circle incribed in a quadrilateral

A circle is inscribed in a quadrilateral if it is tangent to the three sides of the quadrilateral.



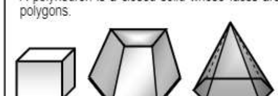
$r = \frac{A_{quad}}{s}$; $s = \frac{1}{2}(a + b + c + d)$

$A_{quad} = \sqrt{abcd}$

SOLID GEOMETRY

POLYHEDRONS

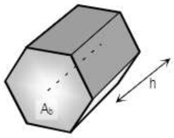
A polyhedron is a closed solid whose faces are polygons.



PRISM

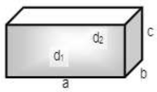
A prism is a polyhedron whose bases are equal polygons in parallel planes and whose sides are parallelograms.

Prisms are classified according to their bases. Thus, a hexagonal prism is one whose base is a hexagon, and a regular hexagonal prism has a base of a regular hexagon. The **axis** of a prism is the line joining the centroids of the bases. A **right prism** is one whose axis is perpendicular to the base. The **height** "h" of a prism is the distance between the bases.



$\text{Volume} = A_b \times h$

Rectangular parallelepiped



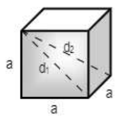
$\text{Volume} = A_b \times h = abc$
Lateral area, $A_L = 2(ac + bc)$
Total surface area, $A_S = 2(ab + bc + ac)$

Face diagonal, $d_1 = \sqrt{a^2 + c^2}$

Space diagonal, $d_2 = \sqrt{a^2 + b^2 + c^2}$

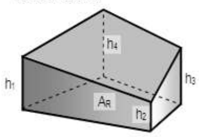
Cube (Regular hexahedron)

$\text{Volume} = A_b \times h = a^3$
Lateral area, $A_L = 4a^2$
Total surface area
$A_S = 6a^2$
Face diagonal
$d_1 = a\sqrt{2}$
Space diagonal
$d_2 = a\sqrt{3}$



Truncated prism

A_R = area of the right section
n = number of sides



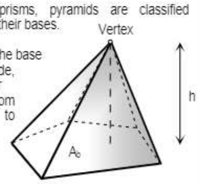
$\text{Volume} = A_R \dfrac{\Sigma h}{n}$

PYRAMIDS

A pyramid is a polyhedron with a polygonal base and triangular faces that meet at a common point called the vertex.

Similar to prisms, pyramids are classified according to their bases.

A_b = area of the base
h = altitude, perpendicular distance from the vertex to the base

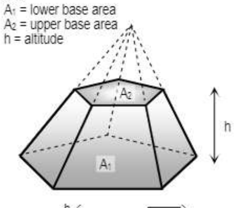


$\text{Volume} = \dfrac{1}{3} A_b \times h$

Frustum of pyramid

A frustum of a pyramid is the volume included between the base and a cutting plane parallel to the base.

A_1 = lower base area
A_2 = upper base area
h = altitude

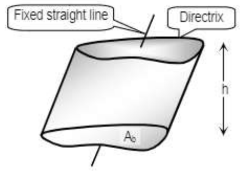


$\text{Volume} = \dfrac{h}{3}\left(A_1 + A_2 + \sqrt{A_1 A_2}\right)$

CYLINDERS

A cylinder is the surface generated by a straight line intersecting and moving along a closed plane curve, the directrix, while remaining parallel to a fixed straight line that is not on or parallel to the plane of the directrix.

Like prisms, cylinders are classified according to their bases.

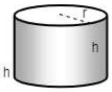


$\text{Volume} = A_b \times h$

Right circular cylinder

$\text{Volume} = A_b \times h = \pi r^2 h$

Lateral area, A_L
A_L = Base perimeter × h
$A_L = 2\pi r h$

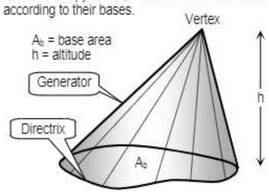


CONE

A cone is the surface generated by a straight line, the generator, passing through a fixed point, the vertex, and moving along a fixed curve, the directrix.

Similar to pyramids, cones are classified according to their bases.

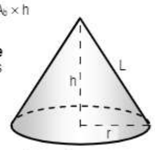
A_b = base area
h = altitude



$\text{Volume} = \dfrac{1}{3} A_b \times h$

Right circular cone

r = base radius
h = altitude



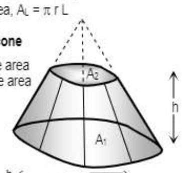
L = slant height = $\sqrt{r^2 + h^2}$

$\text{Volume} = \dfrac{1}{3} A_b \times h = \dfrac{1}{3}\pi r^2 h$

Lateral area, $A_L = \pi r L$

Frustum of a cone

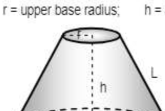
A_1 = lower base area
A_2 = upper base area
h = altitude



$\text{Volume} = \dfrac{h}{3}\left(A_1 + A_2 + \sqrt{A_1 A_2}\right)$

Frustum of right circular cone

R = lower base radius
r = upper base radius; h = altitude



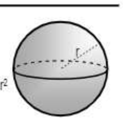
L = slant height = $\sqrt{h^2 + (R - r)^2}$

$\text{Volume} = \dfrac{\pi h}{3}\left(R^2 + r^2 + Rr\right)$

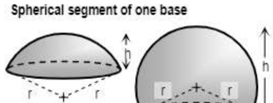
Lateral area = $\pi (R + r) L$

SPHERE

$\text{Volume} = \dfrac{4}{3}\pi r^3$
Surface area, $A_s = 4\pi r^2$



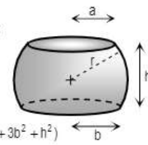
Spherical segment of one base



$A_{zone} = 2\pi r h$

$\text{Volume} = \dfrac{\pi h^2}{3}(3r - h)$

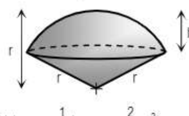
Spherical segment of two bases



$A_s = 2\pi r h$

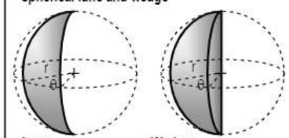
$\text{Volume} = \dfrac{\pi h}{6}(3a^2 + 3b^2 + h^2)$

Spherical cone or spherical sector



$\text{Volume} = \dfrac{1}{3} A_{zone} r = \dfrac{2}{3}\pi r^2 h$

Spherical lune and wedge



Lune		Wedge

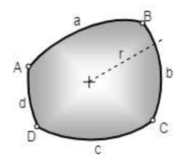
$\dfrac{A_{lune}}{\theta} = \dfrac{4\pi r^2}{360°}$		$A_{lune} = \dfrac{\pi r^2 \theta}{90°}$

$\dfrac{V_{wedge}}{\theta} = \dfrac{\frac{4}{3}\pi r^3}{360°}$		$V_{wedge} = \dfrac{\pi r^3 \theta}{270°}$

Spherical polygons

A spherical polygon is a polygon on the surface of a sphere whose sides are arcs of great circles.

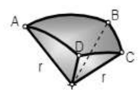
n = number of sides; r = radius of sphere
E = spherical excess



$\text{Area} = \dfrac{\pi r^2 E}{180°}$

E = sum of angles – (n – 2)180°

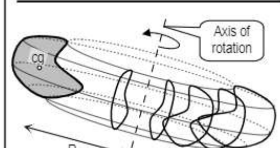
Spherical pyramid



r = radius of sphere
E = spherical excess of the polygon
E = sum of angles – (n – 2)180°

$\text{Volume} = \dfrac{\pi r^3 E}{540°}$

SOLID OF REVOLUTION



First proposition of Pappus

The surface area generated by a surface of revolution equals the product of the length of the generating arc and the distance traveled by its centroid.

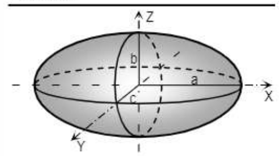
$A_s = L \times 2\pi R$

Second proposition of Pappus

The volume area generated by a solid of revolution equals the product of the generating area and the distance traveled by its centroid.

$\text{Volume} = A \times 2\pi R$

ELLIPSOID



$\text{Volume} = \dfrac{4}{3}\pi abc$

Prolate spheroid

Prolate spheroid is formed by revolving the ellipse about its major (X) axis. Thus from the figure above, c = b, then,

$\text{Volume} = \dfrac{4}{3}\pi a b^2$

$A_s = 2\pi b^2 + 2\pi ab\,\dfrac{\text{arcsine}}{e}$

$e = \sqrt{a^2 - b^2}\,/a$

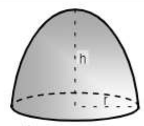
Oblate spheroid

Prolate spheroid is formed by revolving the ellipse about its minor (Z) axis. Thus from the figure above, c = a, then,

$\text{Volume} = \dfrac{4}{3}\pi a^2 b$

$A_s = 2\pi a^2 + \dfrac{\pi b^2}{e}\ln\dfrac{1+e}{1-e}$

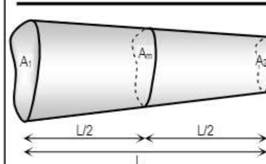
PARABOLOID OF REVOLUTION



$\text{Volume} = \dfrac{1}{2}\pi r^2 h$

$A_L = \dfrac{4\pi r}{3h^2}\left[\left(\dfrac{r^2}{4} + h^2\right)^{3/2} - \left(\dfrac{r}{2}\right)^3\right]$

PRISMOIDAL RULE

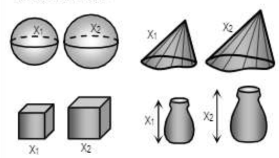


$\text{Volume} = \dfrac{L}{6}\left[A_1 + 4A_m + A_2\right]$

The prismoidal rule gives precise values of volume for regular solid such as pyramids, cones, frustums of pyramids or cones, spheres, and prismoids.

SIMILAR SOLIDS

Two solids are similar if any two corresponding sides or planes are proportional. All spheres, cubes are similar.



For all similar solids:

$\dfrac{As_1}{As_2} = \left(\dfrac{x_1}{x_2}\right)^2$ and $\dfrac{V_1}{V_2} = \left(\dfrac{x_1}{x_2}\right)^3$

Where A_s is the surface, total area, or any corresponding area. The dimension x may be the height, base diameter, diagonal, or any corresponding dimension.