COMP9024: Data Structures and Algorithms

Week Eight: Sortings and Sets

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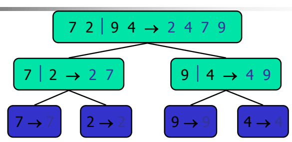
Session 1, 2015

http://www.cse.unsw.edu.au/~cs9024

Outline

- Merge Sort
- Quick Sort
- Bucket-Sort
- Radix Sort
- Sorting Lower Bound
- Union-Find Partition Structures

Merge Sort



Divide-and-Conquer

- Divide-and conquer is a general algorithm design paradigm:
  - Divide: divide the input data S in two disjoint subsets S_1 and S_2
  - Recur: solve the subproblems associated with S_1 and S_2
  - Conquer: combine the solutions for S_1 and S_2 into a solution for S
- The base case for the recursion are subproblems of size 0 or 1
- Merge-sort is a sorting algorithm based on the divide-and-conquer paradigm
- Like heap-sort
  - It uses a comparator
  - It has $O(n \log n)$ running time
- Unlike heap-sort
  - It does not use an auxiliary priority queue
  - It accesses data in a sequential manner (suitable to sort data on a disk)

Merge-Sort

- Merge-sort on an input sequence S with n elements consists of three steps:
  - Divide: partition S into two sequences S_1 and S_2 of about $n/2$ elements each
  - Recur: recursively sort S_1 and S_2
  - Conquer: merge S_1 and S_2 into a unique sorted sequence

Algorithm ***mergeSort(S, C)***
  Input sequence S with n elements, comparator C
  Output sequence S sort according to C

```
{
  if ( S.size() > 1 )
  {
    (S1, S2) = partition(S, n/2);
    mergeSort(S1, C);
    mergeSort(S2, C);
    S = merge(S1, S2);
  }
}
```


Merging Two Sorted Sequences

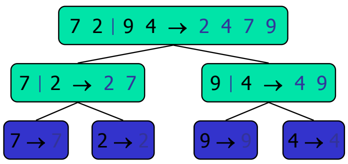
- The conquer step of merge-sort consists of merging two sorted sequences A and B into a sorted sequence S containing the union of the elements of A and B
- Merging two sorted sequences, each with $n/2$ elements and implemented by means of a doubly linked list, takes $O(n)$ time

Algorithm ***merge(A, B)***
  Input sequences A and B with $n/2$ elements each
  Output sorted sequence of $A \cup B$

```
{
  S = empty sequence;
  while ( ¬A.isEmpty() ∧ ¬B.isEmpty() )
    if ( A.first().element() < B.first().element() )
      S.insertLast(A.remove(A.first()));
    else
      S.insertLast(B.remove(B.first()));
  while ( ¬A.isEmpty()
    S.insertLast(A.remove(A.first()));
  while ( ¬B.isEmpty() )
    S.insertLast(B.remove(B.first()));
  return S;
}
```

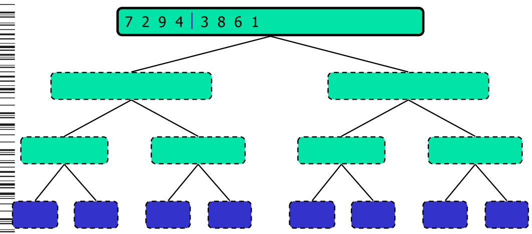
Merge-Sort Tree

- An execution of merge-sort is depicted by a binary tree
  - each node represents a recursive call of merge-sort and stores
    - unsorted sequence before the execution and its partition
    - sorted sequence at the end of the execution
  - the root is the initial call
  - the leaves are calls on subsequences of size 0 or 1



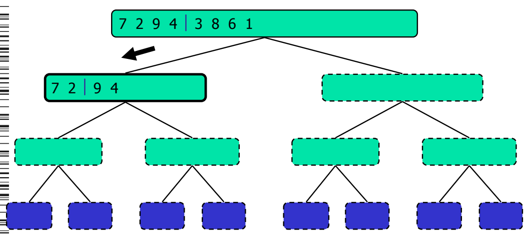
Execution Example (1/10)

- Partition



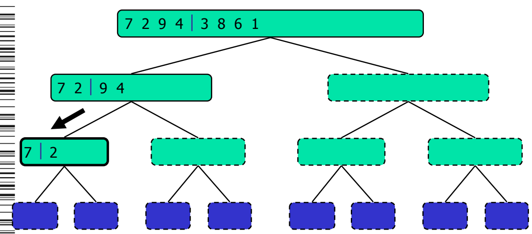
Execution Example (2/10)

- Recursive call, partition



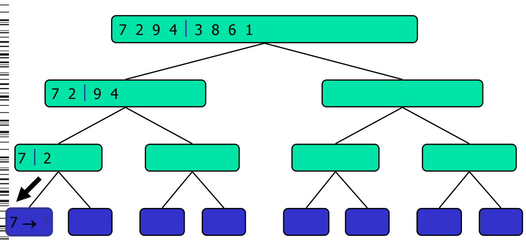
Execution Example (3/10)

- Recursive call, partition



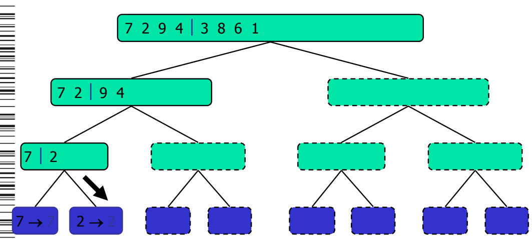
Execution Example (4/10)

- Recursive call, base case



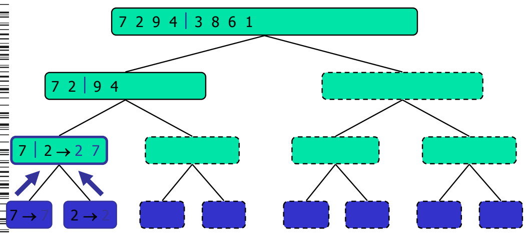
Execution Example (5/10)

- Recursive call, base case



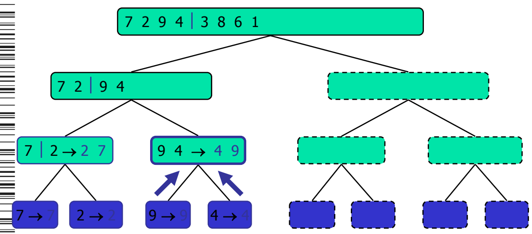
Execution Example (6/10)

- Merge



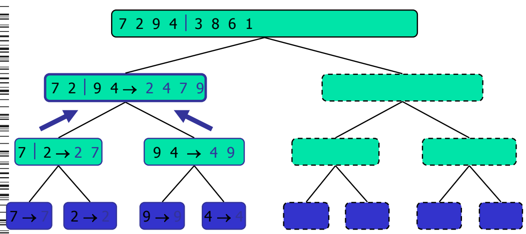
Execution Example (7/10)

- Recursive call, …, base case, merge



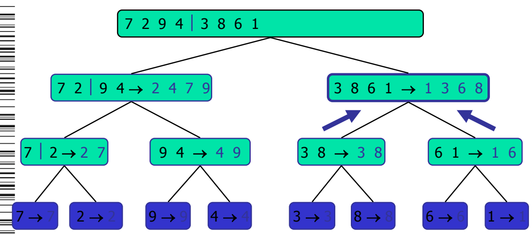
Execution Example (8/10)

- Merge



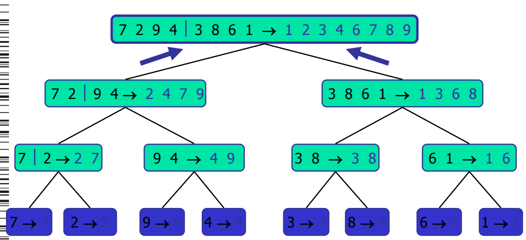
Execution Example (9/10)

- Recursive call, …, merge, merge



Execution Example (10/10)

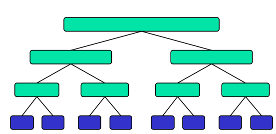
- Merge



Analysis of Merge-Sort

- The height h of the merge-sort tree is $O(\log n)$
  - at each recursive call we divide in half the sequence,
- The overall amount of work done at the nodes of depth i is $O(n)$
  - we partition and merge 2^i sequences of size $n/2^i$
  - we make 2^{i+1} recursive calls
- Thus, the total running time of merge-sort is $O(n \log n)$

depth	#seqs	size
0	1	n
1	2	$n/2$
i	2^i	$n/2^i$
…	…	…



Summary of Sorting Algorithms

Algorithm	Time	Notes
selection-sort	$O(n^2)$	▪ slow ▪ in-place ▪ for small data sets (< 1K)
insertion-sort	$O(n^2)$	▪ slow ▪ in-place ▪ for small data sets (< 1K)
heap-sort	$O(n \log n)$	fast in-place for large data sets (1K — 1M)
merge-sort	$O(n \log n)$	fast sequential data access for huge data sets (> 1M)

Nonrecursive Merge-Sort

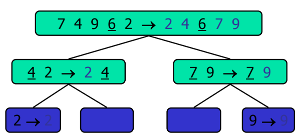
merge runs of length 2, then 4, then 8, and so on

```
public static void mergeSort(Object[] orig, Comparator c) { // nonrecursive
  Object[] in = new Object[orig.length]; // make a new temporary array
  System.arraycopy(orig,0,in,0,in.length); // copy the input
  Object[] out = new Object[in.length]; // output array
  Object[] temp; // temp array reference used for swapping
  int n = in.length;
  for (int i=1; i < n; i*=2) { // each iteration sorts all length-2*i runs
    for (int j=0; j < n; j+=2*i)  // each iteration merges two length-i pairs
      merge(in,out,c,j,i); // merge from in to out two length-i runs at j
    temp = in; in = out; out = temp; // swap arrays for next iteration
  }
  // the "in" array contains the sorted array, so re-copy it
  System.arraycopy(in,0,orig,0,in.length);
}
```

merge two runs in the in array to the out array

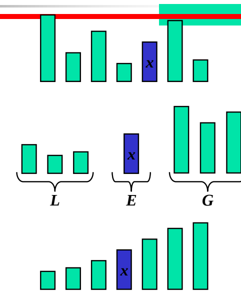
```
protected static void merge(Object[] in, Object[] out, Comparator c, int start,
    int inc) { // merge in[start..start+inc-1] and in[start+inc..start+2*inc-1]
  int x = start; // index into run #1
  int end1 = Math.min(start+inc, in.length); // boundary for run #1
  int end2 = Math.min(start+2*inc, in.length); // boundary for run #2
  int y = start+inc; // index into run #2 (could be beyond array boundary)
  int z = start; // index into the out array
  while ((x < end1) && (y < end2))
    if (c.compare(in[x],in[y]) <= 0) out[z++] = in[x++];
    else out[z++] = in[y++];
  if (x < end1) // first run didn't finish
    System.arraycopy(in, x, out, z, end1 - x);
  else if (y < end2) // second run didn't finish
    System.arraycopy(in, y, out, z, end2 - y);
}
```


Quick-Sort



Quick-Sort

- Quick-sort is a randomized sorting algorithm based on the divide-and-conquer paradigm:
  - Divide: pick a random element x (called pivot) and partition S into
    - L elements less than x
    - E elements equal x
    - G elements greater than x
  - Recur: sort L and G
  - Conquer: join L, E and G



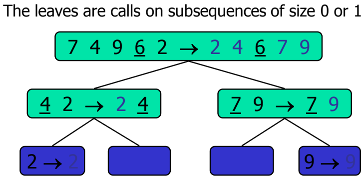
Partition

- We partition an input sequence as follows:
  - We remove, in turn, each element y from S and
  - We insert y into L, E or G, depending on the result of the comparison with the pivot x
- Each insertion and removal is at the beginning or at the end of a sequence, and hence takes $O(1)$ time
- Thus, the partition step of quick-sort takes $O(n)$ time

Algorithm *partition*(S, p)
 Input sequence S, position p of pivot
 Output subsequences L, E, G of the elements of S less than, equal to, or greater than the pivot, resp.
 { L, E, G = empty sequences;
  $x = S.remove(p)$;
  while ($\neg S.isEmpty()$)
   { $y = S.remove(S.first())$;
    if ($y < x$)
     $L.insertLast(y)$;
    else if ($y = x$)
     $E.insertLast(y)$;
    else // $y > x$
     $G.insertLast(y)$;}
  return L, E, G;
 }

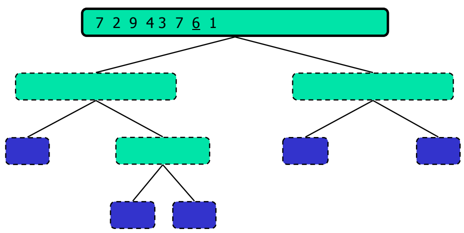
Quick-Sort Tree

- An execution of quick-sort is depicted by a binary tree
  - Each node represents a recursive call of quick-sort and stores
    - Unsorted sequence before the execution and its pivot
    - Sorted sequence at the end of the execution
  - The root is the initial call
  - The leaves are calls on subsequences of size 0 or 1



Execution Example (1/7)

- Pivot selection



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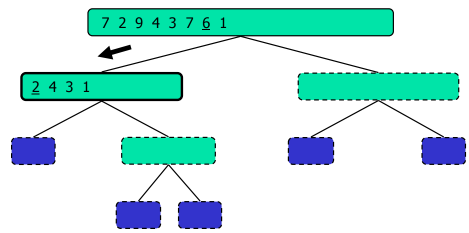
Execution Example (2/7)

- Partition, recursive call, pivot selection



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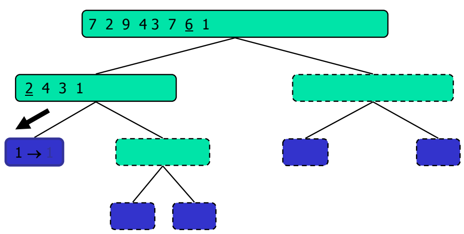
Execution Example (3/7)

- Partition, recursive call, base case



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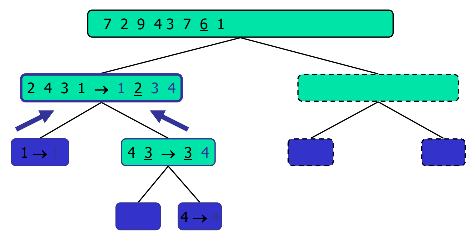
Execution Example (4/7)

- Recursive call, …, base case, join



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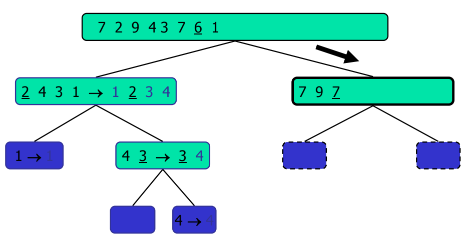
Execution Example (5/7)

- Recursive call, pivot selection



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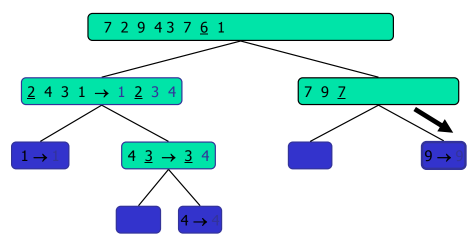
Execution Example (6/7)

- Partition, …, recursive call, base case



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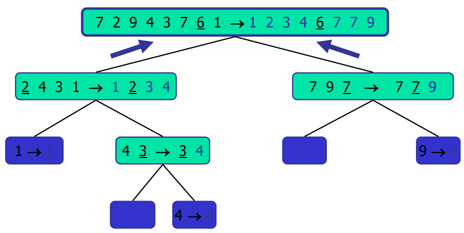
Execution Example (7/7)

- Join, join

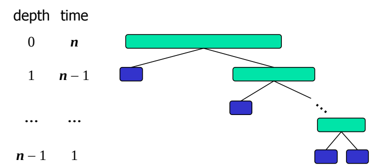


Worst-case Running Time

- The worst case for quick-sort occurs when the pivot is the unique minimum or maximum element
- One of L and G has size $n-1$ and the other has size 0
- The running time is proportional to the sum

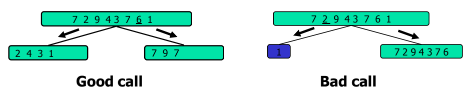
$$n + (n-1) + \ldots + 2 + 1$$

- Thus, the worst-case running time of quick-sort is $O(n^2)$

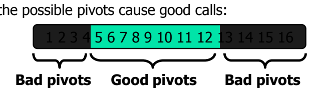


Expected Running Time (1/2)

- Consider a recursive call of quick-sort on a sequence of size s
  - **Good call:** the sizes of L and G are each less than $3s/4$
  - **Bad call:** one of L and G has size greater than $3s/4$

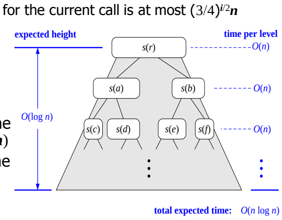


- A call is good with probability $1/2$
  - 1/2 of the possible pivots cause good calls:



Expected Running Time (2/2)

- Probabilistic Fact: The expected number of coin tosses required in order to get k heads is $2k$
- For a node of depth i, we expect
  - $i/2$ ancestors are good calls
  - The size of the input sequence for the current call is at most $(3/4)^{i/2}n$
- Therefore, we have
  - For a node of depth $2\log_{4/3}n$, the expected input size is one
  - The expected height of the quick-sort tree is $O(\log n)$
- The amount or work done at the nodes of the same depth is $O(n)$
- Thus, the expected running time of quick-sort is $O(n \log n)$



In-Place Quick-Sort

- Quick-sort can be implemented to run in-place
- In the partition step, we use replace operations to rearrange the elements of the input sequence such that
  - the elements less than the pivot have rank less than h
  - the elements equal to the pivot have rank between h and k
  - the elements greater than the pivot have rank greater than k
- The recursive calls consider
  - elements with rank less than h
  - elements with rank greater than k

Algorithm *inPlaceQuickSort*(S, l, r)
  Input sequence S, ranks l and r
  Output sequence S with the elements of rank between l and r rearranged in increasing order
{ **if** ($l \geq r$)
    return;
  i = a random integer between l and r;
  $x = S.elemAtRank(i)$;
  $(h, k) = inPlacePartition(x)$;
  $inPlaceQuickSort(S, l, h-1)$;
  $inPlaceQuickSort(S, k+1, r)$;
}

In-Place Partitioning

- Perform the partition using two indices to split S into L and E U G (a similar method can split E U G into E and G).

j k
3 2 5 1 0 7 3 5 9 2 7 9 8 9 7 **6** 9 (pivot = 6)

- Repeat until j and k cross:
  - Scan j to the right until finding an element $\geq$ x.
  - Scan k to the left until finding an element < x.
  - Swap elements at indices j and k

j k
3 2 5 1 0 $\boxed{7}$ 3 5 9 $\boxed{2}$ 7 9 8 9 7 **6** 9

Summary of Sorting Algorithms

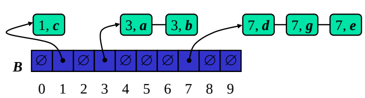
Algorithm	Time	Notes
selection-sort	$O(n^2)$	▪ in-place ▪ slow (good for small inputs)
insertion-sort	$O(n^2)$	▪ in-place ▪ slow (good for small inputs)
quick-sort	$O(n \log n)$ expected	in-place, randomized fastest (good for large inputs)
heap-sort	$O(n \log n)$	in-place fast (good for large inputs)
merge-sort	$O(n \log n)$	sequential data access fast (good for huge inputs)

Java Implementation

```
public static void quickSort (Object[] S, Comparator c) {
   if (S.length < 2) return; // the array is already sorted in this case
   quickSortStep(S, c, 0, S.length-1); // recursive sort method
 }
 private static void quickSortStep (Object[] S, Comparator c,
                        int leftBound, int rightBound ) {
   if (leftBound >= rightBound) return; // the indices have crossed
   Object temp;  // temp object used for swapping
   Object pivot = S[rightBound];
   int leftIndex = leftBound;     // will scan rightward
   int rightIndex = rightBound-1; // will scan leftward
   while (leftIndex <= rightIndex) { // scan right until larger than the pivot
     while ( (leftIndex <= rightIndex) && (c.compare(S[leftIndex], pivot)<=0) )
       leftIndex++;
     // scan leftward to find an element smaller than the pivot
     while ( (rightIndex >= leftIndex) && (c.compare(S[rightIndex], pivot)>=0))
       rightIndex--;
     if (leftIndex < rightIndex) { // both elements were found
       temp = S[rightIndex];
      S[rightIndex] = S[leftIndex]; // swap these elements
      S[leftIndex] = temp;
     }
   } // the loop continues until the indices cross
   temp = S[rightBound]; // swap pivot with the element at leftIndex
   S[rightBound] = S[leftIndex];
   S[leftIndex] = temp; // the pivot is now at leftIndex, so recurse
   quickSortStep(S, c, leftBound, leftIndex-1);
   quickSortStep(S, c, leftIndex+1, rightBound);
 }
```

only works for distinct elements

Bucket-Sort and Radix-Sort



Bucket-Sort

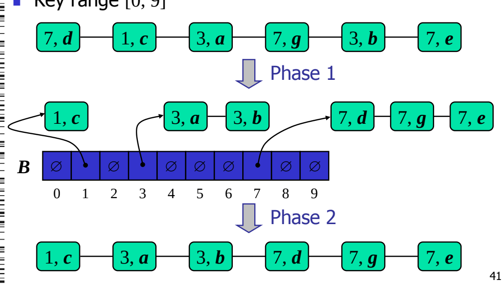
- Let be S be a sequence of n (key, element) entries with keys in the range $[0, N - 1]$
- Bucket-sort uses the keys as indices into an auxiliary array B of sequences (buckets)
  - Phase 1: Empty sequence S by moving each entry (k, o) into its bucket $B[k]$
  - Phase 2: For $i = 0, \ldots, N - 1$, move the entries of bucket $B[i]$ to the end of sequence S
- Analysis:
  - Phase 1 takes $O(n)$ time
  - Phase 2 takes $O(n + N)$ time

Bucket-sort takes $O(n + N)$ time

Algorithm *bucketSort(S, N)*
  Input sequence S of (key, element) items with keys in the range $[0, N - 1]$
  Output sequence S sorted by increasing keys
 { B = array of N empty sequences;
  while ($\neg S.isEmpty()$)
   { $f = S.first()$;
    $(k, o) = S.remove(f)$;
    $B[k].insertLast((k, o))$; }
  for ($i = 0$; i++; $i <= N - 1$)
   while ($\neg B[i].isEmpty()$)
    { $f = B[i].first()$;
     $(k, o) = B[i].remove(f)$;
     $S.insertLast((k, o))$; }
 }

Example

- Key range $[0, 9]$



Properties and Extensions

- Key-type Property
  - The keys are used as indices into an array and cannot be arbitrary objects
  - No external comparator
- **Stable** Sort Property
  - The relative order of any two items with the same key is preserved after the execution of the algorithm

Extensions
- Integer keys in the range $[a, b]$
  - Put entry (k, o) into bucket $B[k - a]$
- String keys from a set D of possible strings, where D has constant size (e.g., names of the 50 U.S. states)
  - Sort D and compute the rank $r(k)$ of each string k of D in the sorted sequence
  - Put entry (k, o) into bucket $B[r(k)]$

Lexicographic Order

- A d-tuple is a sequence of d keys $(k_1, k_2, \ldots, k_d)$, where key k_i is said to be the i-th dimension of the tuple
- Example:
  - The Cartesian coordinates of a point in space are a 3-tuple
- The lexicographic order of two d-tuples is recursively defined as follows

$$(x_1, x_2, \ldots, x_d) < (y_1, y_2, \ldots, y_d)$$
$$\Leftrightarrow$$
$$x_1 < y_1 \lor x_1 = y_1 \land (x_2, \ldots, x_d) < (y_2, \ldots, y_d)$$

I.e., the tuples are compared by the first dimension, then by the second dimension, etc.

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Lexicographic-Sort

- Let C_i be the comparator that compares two tuples by their i-th dimension
- Let ***stableSort***(S, C) be a stable sorting algorithm that uses comparator C
- Lexicographic-sort sorts a sequence of d-tuples in lexicographic order by executing d times algorithm ***stableSort***, one per dimension
- Lexicographic-sort runs in $O(dT(n))$ time, where $T(n)$ is the running time of ***stableSort***

Algorithm ***lexicographicSort***(S)
  Input sequence S of d-tuples
  Output sequence S sorted in lexicographic order
  { **for** ($i = d$; $i >= 1$; i--)
    stableSort(S, C_i);
  }

Example:

(7,4,6) (5,1,5) (2,4,6) (2, 1, 4) (3, 2, 4)

(2, 1, 4) (3, 2, 4) (5,1,5) (7,4,6) (2,4,6)

(2, 1, 4) (5,1,5) (3, 2, 4) (7,4,6) (2,4,6)

(2, 1, 4) (2,4,6) (3, 2, 4) (5,1,5) (7,4,6)

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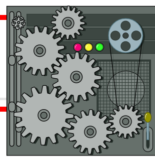
Radix-Sort



- Radix-sort is a specialization of lexicographic-sort that uses bucket-sort as the stable sorting algorithm in each dimension
- Radix-sort is applicable to tuples where the keys in each dimension i are integers in the range $[0, N-1]$
- Radix-sort runs in time $O(d(n + N))$

Algorithm ***radixSort***(S, N)
  Input sequence S of d-tuples such that $(0, \ldots, 0) \le (x_1, \ldots, x_d)$ and $(x_1, \ldots, x_d) \le (N-1, \ldots, N-1)$ for each tuple $(x_1, \ldots, x_d)$ in S
  Output sequence S sorted in lexicographic order
  { **for** ($i = d$; $i>=1$; i--)
    bucketSort(S, N);
  }

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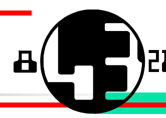
Radix-Sort for Binary Numbers



- Consider a sequence of n b-bit integers
  $x = x_{b-1} \ldots x_1 x_0$
- We represent each element as a b-tuple of integers in the range $[0, 1]$ and apply radix-sort with $N = 2$
- This application of the radix-sort algorithm runs in $O(bn)$ time
- For example, we can sort a sequence of 32-bit integers in linear time

Algorithm ***binaryRadixSort***(S)
  Input sequence S of b-bit integers
  Output sequence S sorted
  replace each element x of S with the item $(0, x)$
  { **for** ($i = 0$; $i <= b-1$; i++)
    { replace the key k of each item (k, x) of S with bit x_i of x;
      bucketSort$(S, 2)$;
    }
  }

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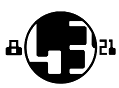
Example



- Sorting a sequence of 4-bit integers

1001	0010	1001	1001	0001
0010	1110	1101	0001	0010
1101	1001	0001	0010	1001
0001	1101	0010	1101	1101
1110	0001	1110	1110	1110

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Sorting Lower Bound

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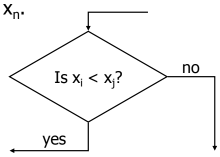
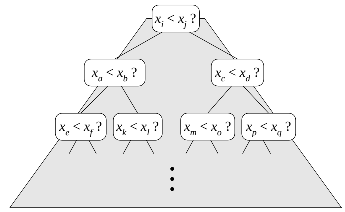
Comparison-Based Sorting

- Many sorting algorithms are comparison based.
  - They sort by making comparisons between pairs of objects
  - Examples: bubble-sort, selection-sort, insertion-sort, heap-sort, merge-sort, quick-sort, ...
- Let us therefore derive a lower bound on the running time of any algorithm that uses comparisons to sort n elements, $x_1, x_2, \ldots, x_n$.



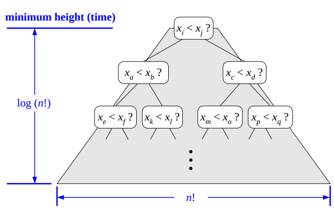
Counting Comparisons

- Let us just count comparisons then.
- Each possible run of the algorithm corresponds to a root-to-leaf path in a **decision tree**

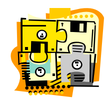


Decision Tree Height

- The height of this decision tree is a lower bound on the running time
- Every possible input permutation must lead to a separate leaf output.
  - If not, some input ...4...5... would have same output ordering as ...5...4..., which would be wrong.
- Since there are n!=1*2*...*n leaves, the height is at least log (n!)



The Lower Bound



- Any comparison-based sorting algorithms takes at least log (n!) time
- Therefore, any such algorithm takes time at least

$$\log\,(n!) \geq \log \left(\frac{n}{2}\right)^{\frac{n}{2}} = (n/2)\log(n/2).$$

- That is, any comparison-based sorting algorithm must run in Ω(n log n) time.

Selection

The Selection Problem

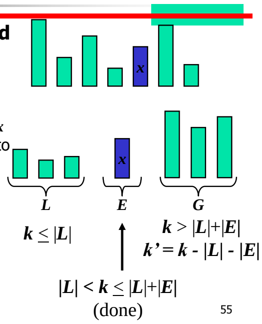
- Given an integer k and n elements $x_1, x_2, \ldots, x_n$, taken from a total order, find the k-th smallest element in this set.
- Of course, we can sort the set in O(n log n) time and then index the k-th element.

k=3 7 4 9 6 2 → 2 4 6 7 9

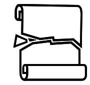
- Can we solve the selection problem faster?

Quick-Select

- Quick-select is a **randomized** selection algorithm based on the prune-and-search paradigm:
  - Prune: pick a random element x (called pivot) and partition S into
    - L elements less than x
    - E elements equal x
    - G elements greater than x
  - Search: depending on k, either answer is in E, or we need to recurse in either L or G



Partition

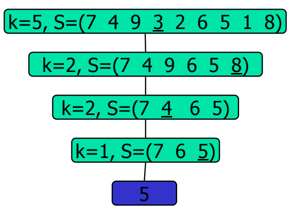


- We partition an input sequence as in the quick-sort algorithm:
  - We remove, in turn, each element y from S and
  - We insert y into L, E or G, depending on the result of the comparison with the pivot x
- Each insertion and removal is at the beginning or at the end of a sequence, and hence takes $O(1)$ time
- Thus, the partition step of quick-select takes $O(n)$ time

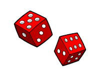
```
Algorithm partition(S, p)
  Input sequence S, position p of pivot
  Output subsequences L, E, G of the
  elements of S less than, equal to,
  or greater than the pivot, resp.
  { L, E, G = empty sequences;
    x = S.remove(p);
  while ( ¬S.isEmpty() )
    { y = S.remove(S.first());
      if ( y < x )
        L.insertLast(y);
      else if ( y = x )
        E.insertLast(y);
      else // y > x
        G.insertLast(y); }
  return L, E, G;
  }
```

Quick-Select Visualization

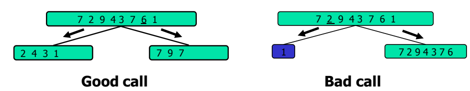
- An execution of quick-select can be visualized by a recursion path
  - Each node represents a recursive call of quick-select, and stores k and the remaining sequence



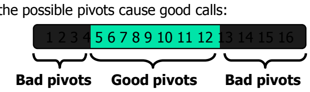
Expected Running Time (1/2)



- Consider a recursive call of quick-select on a sequence of size s
  - **Good call:** the sizes of L and G are each less than $3s/4$
  - **Bad call:** one of L and G has size greater than $3s/4$



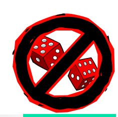
- A call is good with probability $1/2$
  - 1/2 of the possible pivots cause good calls:



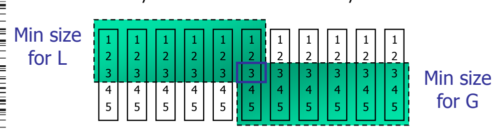
Expected Running Time (2/2)

- Probabilistic Fact #1: The expected number of coin tosses required in order to get one head is two
- Probabilistic Fact #2: Expectation is a linear function:
  - $E(X + Y) = E(X) + E(Y)$
  - $E(cX) = cE(X)$
- Let T(n) denote the expected running time of quick-select.
- By Fact #2,
  - $T(n) \leq T(3n/4) + bn*$(expected # of calls before a good call)
- By Fact #1,
  - $T(n) \leq T(3n/4) + 2bn$
- That is, T(n) is a geometric series:
  - $T(n) \leq 2bn + 2b(3/4)n + 2b(3/4)^2n + 2b(3/4)^3n + \ldots$
- So T(n) is O(n).
- We can solve the selection problem in O(n) expected time.

Deterministic Selection



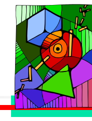
- We can do selection in O(n) worst-case time.
- Main idea: recursively use the selection algorithm itself to find a good pivot for quick-select:
  - Divide S into n/5 sets of 5 each
  - Find a median in each set
  - Recursively find the median of the "baby" medians.



- See Exercise C-11.31 for details of analysis.

Sets

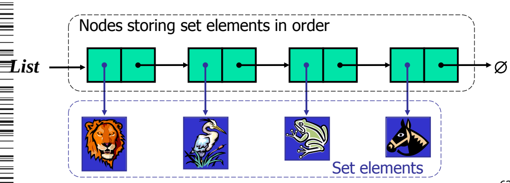
Set Operations



- We represent a set by the sorted sequence of its elements
- By specializing the auxliliary methods he generic merge algorithm can be used to perform basic set operations:
  - union
  - intersection
  - subtraction
- The running time of an operation on sets *A* and *B* should be at most $O(n_A + n_B)$
- Set union:
  - *aIsLess(a, S)*
    S.insertFirst(a)
  - *bIsLess(b, S)*
    S.insertLast(b)
  - *bothAreEqual(a, b, S)*
    S. insertLast(a)
- Set intersection:
  - *aIsLess(a, S)*
    { *do nothing* }
  - *bIsLess(b, S)*
    { *do nothing* }
  - *bothAreEqual(a, b, S)*
    S. insertLast(a)

Storing a Set in a List

- We can implement a set with a list
- Elements are stored sorted according to some canonical ordering
- The space used is $O(n)$

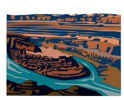


Generic Merging

- Generalized merge of two sorted lists *A* and *B*
- Template method genericMerge
- Auxiliary methods
  - aIsLess
  - bIsLess
  - bothAreEqual
- Runs in $O(n_A + n_B)$ time provided the auxiliary methods run in $O(1)$ time

```
Algorithm genericMerge(A, B)
  {  S = empty sequence;
     while ( ¬A.isEmpty() ∧ ¬B.isEmpty() )
       { a = A.first().element();  b ← B.first().element(); }
         if  ( a < b )
           { aIsLess(a, S);  A.remove(A.first()); }
         else if  ( b < a )
           { bIsLess(b, S);  B.remove(B.first()); }
         else  // b = a
           { bothAreEqual(a, b, S);
             A.remove(A.first());  B.remove(B.first()); }
     while ( ¬A.isEmpty() )
       { aIsLess(a, S);  A.remove(A.first()); }
     while ( ¬B.isEmpty() )
       { bIsLess(b, S);  B.remove(B.first()); }
     return S; }
```


Using Generic Merge for Set Operations



- Any of the set operations can be implemented using a generic merge
- For example:
  - For intersection: only copy elements that are duplicated in both list
  - For union: copy every element from both lists except for the duplicates
- All methods run in linear time.

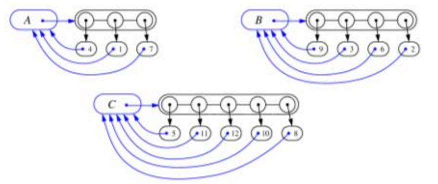
Union-Find Partition Structures

Partitions with Union-Find Operations

- makeSet(x): Create a singleton set containing the element x and return the position storing x in this set.
- union(A,B): Return the set A U B, destroying the old A and B.
- find(p): Return the set containing the element in position p.

List-based Implementation

- Each set is stored in a sequence represented with a linked-list
- Each node should store an object containing the element and a reference to the set name

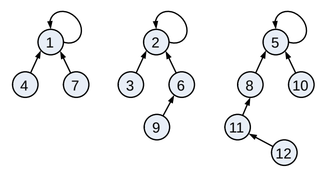


Analysis of List-based Representation

- When doing a union, always move elements from the smaller set to the larger set
  - Each time an element is moved it goes to a set of size at least double its old set
  - Thus, an element can be moved at most O(log n) times
- Total time needed to do n unions and finds is O(n log n).

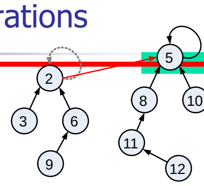
Tree-based Implementation

- Each element is stored in a node, which contains a pointer to a set name
- A node v whose set pointer points back to v is also a set name
- Each set is a tree, rooted at a node with a self-referencing set pointer
- For example: The sets "1", "2", and "5":

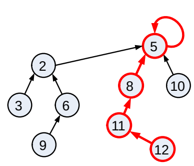


Union-Find Operations

- To do a union, simply make the root of one tree point to the root of the other

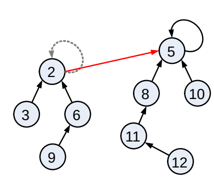


- To do a find, follow set-name pointers from the starting node until reaching a node whose set-name pointer refers back to itself



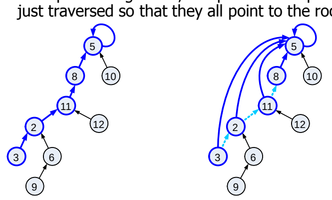
Union-Find Heuristic 1

- Union by size:
  - When performing a union, make the root of smaller tree point to the root of the larger
- Implies O(n log n) time for performing n union-find operations:
  - Each time we follow a pointer, we are going to a subtree of size at least double the size of the previous subtree
  - Thus, we will follow at most O(log n) pointers for any find.



Union-Find Heuristic 2

- Path compression:
  - After performing a find, compress all the pointers on the path just traversed so that they all point to the root



- Implies O(n log* n) time for performing n union-find operations:
  - Proof is somewhat involved… (and not in the book)

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Proof of log* n Amortized Time

- For each node v that is a root
  - define $n(v)$ to be the size of the subtree rooted at v (including v)
  - identified a set with the root of its associated tree.
- We update the size field of v each time a set is unioned into v. Thus, if v is not a root, then $n(v)$ is the largest the subtree rooted at v can be, which occurs just before we union v into some other node whose size is at least as large as v's.
- For any node v, then, define the ***rank*** of v, which we denote as $r(v)$, as $r(v) = [\log n(v)]$:
- Thus, $n(v) \geq 2^{r(v)}$.
- Also, since there are at most n nodes in the tree of v, $r(v) = [\log n]$, for each node v.

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Proof of log* n Amortized Time (2)

- For each node v with parent w:
  - $r(v) > r(w)$
- **Claim:** *There are at most $n/2^s$ nodes of rank s.*
- **Proof:**
  - Since $r(v) < r(w)$, for any node v with parent w, ranks are monotonically increasing as we follow parent pointers up any tree.
  - Thus, if $r(v) = r(w)$ for two nodes v and w, then the nodes counted in $n(v)$ must be separate and distinct from the nodes counted in $n(w)$.
  - If a node v is of rank s, then $n(v) \geq 2^s$.
  - Therefore, since there are at most n nodes total, there can be at most $n/2^s$ that are of rank s.

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Proof of log* n Amortized Time (3)

- Definition: Tower of two's function:
  - $t(i) = 2^{t(i-1)}$
- Nodes v and u are in the same rank group g if
  - $g = \log^*(r(v)) = \log^*(r(u))$:
- Since the largest rank is log n, the largest rank group is
  - $\log^*(\log n) = (\log^* n)-1$

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Proof of log* n Amortized Time (4)

- Charge 1 cyber-dollar per pointer hop during a find:
  - If w is the root or if w is in a different rank group than v, then charge the find operation one cyber-dollar.
  - Otherwise (w is not a root and v and w are in the same rank group), charge the node v one cyber-dollar.
- Since there are most $(\log^* n)-1$ rank groups, this rule guarantees that any find operation is charged at most $\log^* n$ cyber-dollars.

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Proof of log* n Amortized Time (5)

- After we charge a node v then v will get a new parent, which is a node higher up in v's tree.
- The rank of v's new parent will be greater than the rank of v's old parent w.
- Thus, any node v can be charged at most the number of different ranks that are in v's rank group.
- If v is in rank group $g > 0$, then v can be charged at most $t(g)-t(g-1)$ times before v has a parent in a higher rank group (and from that point on, v will never be charged again). In other words, the total number, C, of cyber-dollars that can ever be charged to nodes can be bound as

$$C \leq \sum_{g=1}^{\log^* n-1} n(g)\cdot(t(g)-t(g-1))$$

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Proof of log* n Amortized Time (end)

- Bounding $n(g)$:

$$n(g) \le \sum_{s=t(g-1)+1}^{t(g)} \frac{n}{2^s}$$

$$= \frac{n}{2^{t(g-1)+1}} \sum_{s=0}^{t(g)-t(g-1)-1} \frac{1}{2^s}$$

$$< \frac{n}{2^{t(g-1)+1}} \cdot 2$$

$$= \frac{n}{2^{t(g-1)}}$$

$$= \frac{n}{t(g)}$$

- Returning to C:

$$C < \sum_{g=1}^{\log^* n-1} \frac{n}{t(g)} \cdot (t(g) - t(g-1))$$

$$\le \sum_{g=1}^{\log^* n-1} \frac{n}{t(g)} \cdot t(g)$$

$$= \sum_{g=1}^{\log^* n-1} n$$

$$\le n \log^* n$$

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References

1. Chapter 11, Data Structures and Algorithms by Goodrich and Tamassia.

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