Proving a Sublinear Bound on $\sum_{t=1}^{T} \|d_t\|^2$
Additional Material

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1 What is this

This is an additional material containing the proof of sublinear bound on $\sum_{t=1}^{T} \|d_t\|^2$. However, we completed it just before the report submission and haven't yet verified it with the instructor. It is not a part of the report yet we included it, since it may give some ideas to someone looking to extend our work. We would request the instructor to check its correctness, if he gets time.

2 Formal Problem Description

The primary objective of our project is to design an online algorithm for SIM learning having sublinear regret bound and polynomial time complexity. Initially we'll assume noiseless data and realisable setting. Formally, problem setting is as follows:

- The data is a collection of $(\mathbf{x}_t, y_t)$ pair, where $\mathbf{x}_t \in \mathbb{R}^d$ and $y_t \in \mathbb{R}$. The data arrives sequentially.
- Adversary is stochastic.
- No noise in data
- The setting is realizable i.e. $\exists u^*, \mathbf{w}^*$ s.t $u^*(\mathbf{w}^*.x_i) = y_i \; \forall i$
- $\|\mathbf{x}_t\| \leq \mathbf{X}$, $\|\mathbf{w}_t \leq \mathbf{W}$, $\|\mathbf{w}^* \leq \mathbf{W}$, and both u_t and u^* are bounded, smooth, and 1-Lipschitz.

3 Proposed Algorithm: FTL-SLISOTRON

The algorithm we propose is a simple FTL style extension of of offline SLISOTRON algorithm. Basically, we run SLISOTRON for all $t-1$ points to obtain a $\mathbf{w}_t$ and u_t, which are used to predict $\hat{y^t} = u^t(\mathbf{w}^t\mathbf{x}^t)$. It is given as Algorithm 1. Note that time complexity of SLISOTRON is polynomial in m, the no. of training points. So, FTL-SLISOTRON's time complexity will be polynomial in T.

Algorithm 1: FTL-SLISOTRON

$\mathbf{w}^0 \leftarrow 0$
for *t = 1, 2, ... , T* **do**
  Split the $t-1$ points as held out set $H = \{x_\tau, y_\tau\}_{\tau \in \{1,2,..,m\}}$
  and training set $S_t = \{x_\tau, y_\tau\}_{\tau \in \{m,m+1,..,t-1\}}$, where $m = 0.2\lfloor t-1 \rfloor$
  Receive $\mathbf{x}_t$
  $u^t, \mathbf{w}_t \leftarrow SLISOTRON(S_t, H)$
  Predict $\hat{y}_t = u^t(\mathbf{w}_t \cdot \mathbf{x}_t)$
  Receive y_t
  Incurr loss $l(y_t, \hat{y}_t) = (y_t - \hat{y}_t)^2$
end

4 Regret Bound

After defining the algorithm, we'll now define the notion of regret. Since our adversary is stochastic so we'll obtain a definitional expression of Expected Regret bound with high probability.

4.1 Notion of Regret

We have $\hat{y_t} = h^t(\mathbf{x_t}) = u^t(\mathbf{w}^t\mathbf{x}_t)$ and $\tilde{y_t} = h^*(\mathbf{x_t}) = u^*(\mathbf{w}^*.\mathbf{x}_t)$. Let $l(\hat{y_t}, y_t) = (y_t - \hat{y_t})^2$. Now, since setting is realizable, $\tilde{y_t} = y_t$. Hence the Regret R_T is defined as

$$\begin{aligned}R_T &= \sum_{t=1}^{T}[l(\hat{y_t}, y_t) - l(\tilde{y_t}, y_t)] \\ &= \sum_{t=1}^{T}[l(\hat{y_t}, y_t) - 0] \\ &= \sum_{t=1}^{T}(y_t - \hat{y_t})^2\end{aligned}$$

4.2 Expected Regret

Now, since our adversary is stochastic, we'll come up with an expression for a bound on Expected Regret. We use $\epsilon(h)$ as defined in [Kakade et al., 2011],

$$\epsilon(h) = \mathbb{E}_{(x,y)}[(h(x) - u^*(\mathbf{w}^*.x))^2] = \mathbb{E}_{(x,y)}[(h(x) - y)^2]$$

We define $Z^t = \epsilon(h^t) - l(h^t(\mathbf{x}_t), y_t)$. We'll now show that Z^t is an MDS sequence.

$$\mathbb{E}_{(x,y)}[Z^t|\mathcal{H}^t] = \mathbb{E}_{(x,y)}[\epsilon(h^t) - l(h^t(\mathbf{x}_t), y_t)|\mathcal{H}^t]$$

$$\implies \mathbb{E}_{(x,y)}[Z^t|\mathcal{H}^t] = \epsilon(h^t) - \mathbb{E}_{(x,y)}[l(h^t(\mathbf{x}_t), y_t)|\mathcal{H}^t] = 0$$

Hence, Z^t is an MDS sequence. Next we'll come up with an expression of the regret bound. As we stated in Section 2, $h^t(\mathbf{x}^t)$ is bounded, and hence Z^t is also bounded i.e. $|Z^t| \leq B\ \forall t$. So, using Azuma Hoeffding's inequality, with probability $1 - \delta$

$$|\frac{1}{T}[\sum_{t=1}^{T}(\epsilon(h^t) - l(h^t(\mathbf{x}_t), y_t))]| \leq O(B\sqrt{\frac{lg(1/\delta)}{T}})$$

$$\begin{aligned}\sum_{t=1}^{T}\epsilon(h^t) &\leq \sum_{t=1}^{T} l(h^t(\mathbf{x}_t), y_t) + O(B\sqrt{lg(1/\delta)}\sqrt{T}), \quad \text{(w.p. } 1-\delta) \\ \Rightarrow R_s &\leq \sum_{t=1}^{T} l(h^t(\mathbf{x}_t), y_t) + O(B\sqrt{lg(1/\delta)}\sqrt{T}), \quad \text{(w.p. } 1-\delta)\end{aligned}$$

R_s is the Stochastic Regret. We need to compute worst case bound for first term of RHS to get bound on stochastic regret. We can also find a high probability bound on RHS and then take a union bound on the probability. Rest of the sections are dedicated to solve this bound.

5 Bounding the Loss of FTL-SLISOTRON

For bounding the loss of FTL-SLISOTRON, we need to prove two things:

- Given $t-1$ training points, FTL-SLISOTRON indeed learns a perfect 1-Lipschitz function u_t and a corresponding $\mathbf{w}_t$ on these points.

- After fitting on $t-1$ points it predicts a value on $\mathbf{x}_t$, and incurs a loss. Cumulative sum of these losses should be sublinear.

SLISOTRON is able to learn the best predictor on training points with a very high probability. Hence it should be able to to get the exact fit on training points.

Now we'll give a proof for the regret bound, but it'll stricly be valid only for noiseless case i.e. realisable setting and exact fit for $t-1$ points. Our SLISOTRON-FTL is able to learn u^t, and $\mathbf{w}^t$ such that at step t

$$u^t(w^t \cdot \mathbf{x}_i) = y_i, \forall i = 1, 2, ..., t-1$$

Assume $\|\mathbf{x}\| \leq \mathbf{X}$, $\|\mathbf{w}\| \leq \mathbf{W}$

At iteration t, let $\mathbf{x}_i = argmin_{\mathbf{x}_j} \|\mathbf{x}_t - \mathbf{x}_j\|$ where $j = 1, 2, ..., t-1$, then from 1-Lipschitz constraint for u^t and u^*, (Note that LIR step in SLISOTRON ensures that u^t is Lipschitz [Kakade et al., 2011]) we have,

$$|u^t(\mathbf{w}^t \cdot \mathbf{x}_t) - u^t(\mathbf{w}^t \cdot \mathbf{x}_i)| \leq |\mathbf{w}^t \cdot \mathbf{x}_t - \mathbf{w}^t \cdot \mathbf{x}_i|$$
$$|u^*(\mathbf{w}^* \cdot \mathbf{x}_t) - u^*(\mathbf{w}^* \cdot \mathbf{x}_i)| \leq |\mathbf{w}^* \cdot \mathbf{x}_t - \mathbf{w}^* \cdot \mathbf{x}_i|$$

And since, $|a| \leq |c_1|, |b| \leq |c_2| \implies |a-b| \leq |c_1| + |c_2|$, hence

$$|u^t(\mathbf{w}^t \cdot \mathbf{x}_t) - u^*(\mathbf{w}^* \cdot \mathbf{x}_i)| \leq |\mathbf{w}^t \cdot \mathbf{d}_t| + |\mathbf{w}^* \cdot \mathbf{d}_t|$$
$$|u^t(\mathbf{w}^t \cdot \mathbf{x}_t) - u^*(\mathbf{w}^* \cdot \mathbf{x}_i)| \leq \|\mathbf{w}^t\|\|\mathbf{d}_t\| + \|\mathbf{w}^*\|\|\mathbf{d}_t\| \quad \text{(Cauchy-Schwatz)}$$

$$|u^t(\mathbf{w}^t \cdot \mathbf{x}_t) - u^*(\mathbf{w}^* \cdot \mathbf{x}_i)| \leq 2\|\mathbf{W}\|\|\mathbf{d}_t\| \quad \text{where} \quad \mathbf{d}_t = \mathbf{x}_t - \mathbf{x}_i$$

We have,

$$|u^t(\mathbf{w}^t \cdot \mathbf{x}_t) - u^*(\mathbf{w}^* \cdot \mathbf{x}_t)| \leq 2\|\mathbf{W}\| \cdot \|\mathbf{d}_t\|$$

Now, taking the sum over T time steps, we have,

$$\sum_{t=1}^{T} \left(u^t(\mathbf{w}^t \cdot \mathbf{x}_t) - u^*(\mathbf{w}^* \cdot \mathbf{x}_t) \right)^2 \leq 4\|\mathbf{W}\|^2 \sum_{t=1}^{T} \|\mathbf{d}_t\|^2$$

Here $\|.\|$ is the l2 norm. Now we aim to bound the summation on the RHS i.e. $\sum_{t=1}^{T} \|\mathbf{d}_t\|^2$. Assume that our adversary is stochastic, and that all the points : $\mathbf{x}_t$ (s) are coming i.i.d. Assume $\|\mathbf{x}^t\|_2 \leq 1$ Then at time T_0 we have $T_0 - 1$ points in a unit ball. Now as $\mathbf{x}_{T_0}$ arrives, assume an ϵ size ball around it. Then for any $i \in 1, 2, ..., T_0 - 1$, we have:

$$p(\mathbf{x}_i \text{ lies inside } \epsilon \text{ ball} | \mathbf{x}_{T_0}) = \epsilon^d \quad \text{(ratio of volumes of balls of dimension d)}$$

Here for now we are ignoring the case when ϵ ball isn't completely contained in unit ball (we'll relax this assumption later). Hence

$$p(\text{none of the } T_0 - 1 \text{ points lie inside } \epsilon \text{ ball} | \mathbf{x}_{T_0}) = (1 - \epsilon^d)^{T_0 - 1}$$

This will give

$$p(\text{At least one of the } T_0 - 1 \text{ points lie inside } \epsilon \text{ ball} | \mathbf{x}_{T_0}) = 1 - (1 - \epsilon^d)^{T_0 - 1}$$

If at least one point lies inside the epsilon ball of $\mathbf{x}_{T_0}$ then $\|d_{T_0}\|^2 \leq \epsilon^2$. So,

$$\|d_{T_0}\|^2 \leq \epsilon^2 \quad \text{w.p. } 1 - (1 - \epsilon^d)^{T_0 - 1}$$

Since all the points are i.i.d. so this will be valid for any point t coming after T_0 so:

$$\|d_t\|^2 \leq \epsilon^2 \quad \text{w.p. } 1 - (1 - \epsilon^d)^{t-1} \quad (\text{for } t \geq T_0)$$

Summing over all the values from T_0 to T, we get:

$$\sum_{t=T_0}^{T} \|d_t\|^2 \leq (T - T_0)\epsilon^2 \quad \text{w.p. } 1 - \sum_{t=T_0}^{T} (1 - \epsilon^d)^{t-1} \quad \text{(applying union bound)}$$

And for $t = 1, 2, ..., T_0 - 1$ points, we can say that no two points can be further than the diameter so

$$\sum_{t=1}^{T_0} \|d_t\|^2 \leq 2T_0$$

Hence the final sum comes to be:

$$\sum_{t=1}^{T} \|d_t\|^2 \leq 2T_0 + (T - T_0)\epsilon^2 \quad \text{w.p. } 1 - \beta \quad \text{, where } \beta = \sum_{t=T_0}^{T} (1 - \epsilon^d)^{t-1}$$

So, for getting a sublinear Regret bound we have to satisfy the following:

1. T_0 is sublinear in T. Let $T_0 = T^a$, where $a < 1$.
2. $\epsilon^2 T$ is sublinear. For that take $\epsilon = T^{-b}$, where $b > 0$
3. As $T \to \infty$, $\beta \to 0 \Rightarrow \log(\beta) \to -\infty$.

(1) and (2) are satisfied by setting them so we'll set such a and b that (3) is satisfied. At $T \to \infty$,

$$\lim_{T\to\infty} \beta = \lim_{T\to\infty} \sum_{t=T_0}^{T} (1 - \epsilon^d)^{t-1} = \frac{(1 - \epsilon^d)^{T_0 - 1}}{\epsilon^d} \quad \text{(infinite sum of G.P.)}$$

$$\begin{aligned}
\lim_{T\to\infty} \log(\beta) &= \lim_{T\to\infty} (T_0 - 1)\log(1 - \epsilon^d) - d\log(\epsilon) \\
&= \lim_{T\to\infty} (T^a - 1)\log(1 - T^{-bd}) - d\log(T^{-b}) \quad \text{(Putting values from (1) and (2))} \\
&= \lim_{T\to\infty} -\frac{(T^a - 1}{T^{bd}}) + db\log(T) \\
&= \lim_{T\to\infty} -T^{a-bd} + db\log(T)
\end{aligned}$$

We want this limit to tend to $-\infty$ which happens when $a - bd > 0$. Take $a - bd = 0.1$ and $a = 0.5$

$$a - bd > 0 \Rightarrow b < a/d$$

Take $a - bd = 0.1$ then we have $b = \frac{a-0.1}{d}$. Take $a = 0.5$ then we get $b = \frac{0.4}{d}$. Hence the bound on sum becomes:

$$\begin{aligned}
\sum_{t=1}^{T} \|d_t\|_2^2 &\leq 2T^{0.5} + (T - T^{0.5})T^{\frac{-0.8}{d}} \quad \text{w.p. } 1 - \beta \\
\sum_{t=1}^{T} \|d_t\|_2^2 &= \mathcal{O}(T^{0.5} + T^{1-\frac{0.8}{d}}) \quad \text{w.p. } 1 - \beta
\end{aligned}$$

This results in a sublinear loss at a very high probability of $1 - \beta$. Taking its Union Bound with the expression of expected regret gives:

$R_s \leq \mathcal{O}(\|\mathbf{W}\|_2^2(T^{0.5} + T^{1-\frac{0.8}{d}})) + O(B\sqrt{lg(1/\delta)}\sqrt{T}), \quad (\text{w.p. } 1 - \delta - \beta)$

So if this analysis is correct then we are getting a sublinear bound with high probability. However it becomes worse as the dimension d increases.

References

[Kakade et al., 2011] Kakade, S. M., Kalai, A. T., Kanade, V., and Shamir, O. (2011). Efficient learning of generalized linear and single index models with isotonic regression. *Advances in Neural Information Processing Systems*.