A Dynamic Model for Correlation

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Preliminary version

Introductory Remarks

- If there are lessons to be learned from the financial crises in terms of modelling then one of the lessons is certainly that correlations of financial systems are neither constant nor just time-dependent but rather state- or even path- dependent.
- Example: In times of financial distress two stocks with a mediocre correlation can have a correlation close to one when investors pull out of risky (equity) investments
- As a result, in times of crises, investment managers find themselves less diversified than originally anticipated
- The hedging of a derivative book can be quite challenging during times of distress:

Introductory Remarks

Hedging of a multi-asset derivatives book in financial distress

- Assume a bank has sold a multi-asset derivative structure where the payoff depends on SX5E and N225 with price $C_t\left(t, SX5E_t, N225_t\right)$ .Note that options prices are generally non-linear and in most cases structure that get sold observe

$$\frac{\partial^2\left(-C_t\left(t, SX5E, N225_t\right)\right)}{\partial\left(SX5E_t\right)\ \partial\left(N225_t\right)} < 0$$

This implies that the option delta on SX5E changes when N225 changes, e.g.

$$\frac{\partial\left(-\Delta_{SX5E}\right)}{\partial\left(N225_t\right)} < 0$$

Hence SX5E exposure changes and requires re-hedging when N225 changes even though the market in London is closed and SX5E does not move.

Introductory Remarks

Hedging of a multi-asset derivatives book in financial distress

- Assume an exotic trader in London leaves work after having hedged all her first order risk. While still asleep the market in Asia opens and N225 going down (during the crises).
- Her exotic book goes long delta on SX5E in this case, e.g. $\left(-\Delta_{SX5E}\right) > 0$
- She needs to sell SX5E futures immediately at the London open.
- However because $correlation\left(SX5E, N225\right) > 0$,SX5E is likely to open “down” from yesterday’s close inducing a loss…
- Hence: Whenever correlations exceed expectations there is be a loss

Introductory Remarks

Hedging of a multi-asset derivatives book in financial distress

- The objective of this talk:

- Formulate a model that reflects the increased hedging costs in case of downside at inception of the trade in terms of better premiums and better hedge-deltas as one goes into a downward scenario

Introductory Remarks

Hedging of a multi-asset derivatives book in financial distress

- Or putting it another way:

- The objective of this talk is to formulate a model that makes her sleep better even in times of crises (:-)

A Dynamic Model for correlation

- Take the DAX for example $DAX_t \equiv S_t^{(0)} \equiv \sum_{i=1}^{N=30} \alpha_i \, S_t^{(i)}$

  Options on the DAX are traded as well as options on all its components.

  This creates an ideal testing ground for all sort of hypothesis testing on the correlation structure between the individual stocks

A Dynamic Model for Correlation

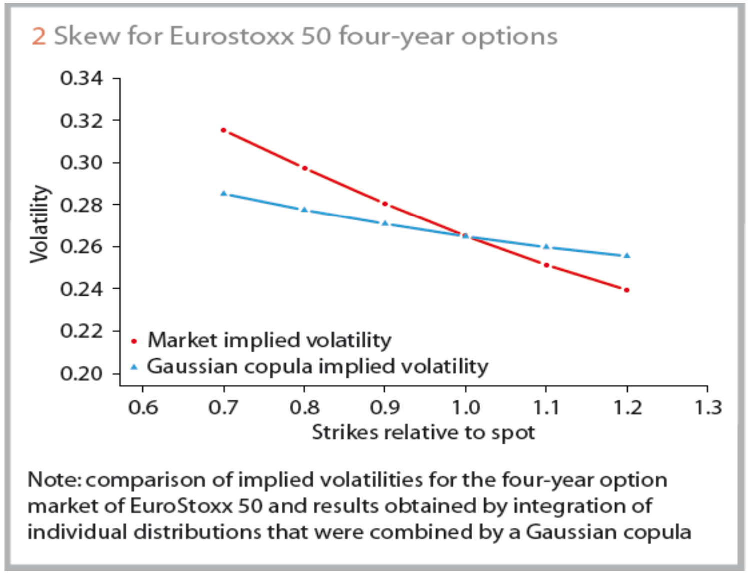
- Combine the individual (option implied) asset price distributions via a Gaussian copula and calculate

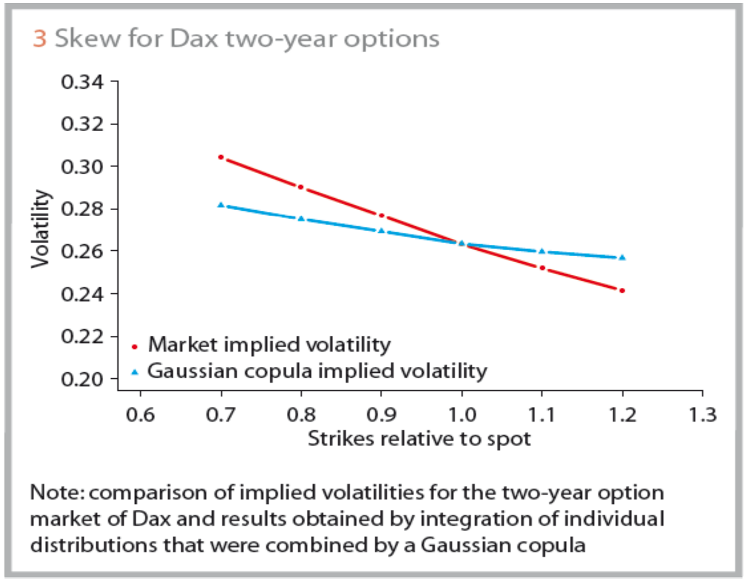
$$\hat{C}_t(T,K) = Df \int_0^{\infty} \prod_{i=1}^{N} dS_T^{(i)} \ \rho_t\left(S_T^{(1)},\ldots,S_T^{(N)}\right)\left(\sum_{j=1}^{N} \alpha_j \ S_T^{(j)} - K\right)^{+}$$

Consistency would require

$$C_t(T,K) = \hat{C}_t(T,K) \quad \forall \, T,K$$

A Dynamic Model for Correlation





Note: Individual stock skews only explain ca. 50% of the index skew!

A Dynamic Model for Correlation

- Important statement of this talk:

Market is encoding a non-trivial copula structure between its assets !

A Dynamic model for Correlation

- Objective of this talk:

Explicitly construct a set coupled stochastic differential equations for $S_t^{(i)}$ that reconcile index option and stock-option markets by construction

A Dynamic Model for Correlation

A one parameter family of correlation matrices

- Let $\rho_{ij}^{down}, \rho_{ij}^{up}, \rho_{ij}^{centre}$ valid correlation matrices. Then

$$\hat{\rho}_{ij} \equiv \alpha \; \rho_{ij}^{down} + (1-\alpha)\rho_{ij}^{up}$$

is a valid correlation matrix for $\alpha \in R_0^+$ or setting $\alpha = \frac{1}{1+u^2} \quad u \in R$

$$\hat{\rho}_{ij}(u) \equiv \begin{cases} \dfrac{\rho_{ij}^{down} + u^2 \rho_{ij}^{centre}}{1+u^2} & u \le u^* \\ \\ \dfrac{\rho_{ij}^{centre} + u^2}{1+u^2} & u > u^* \end{cases}$$

The Local Correlation Model (LCM)

- Assume a economy of N assets $S_t^{(i)}:\ i=1,..,N$ where the filtration is generated by an n-dimensional Brownian motion $W_t = \left(\omega_t^{(1)},...,\omega_t^{(N)}\right)$ and

$$d < \omega_t^{(i)}, \omega_t^{(j)} >_t = \rho_{ij}\left(t,.\right) dt$$

and

$$\frac{dS_t^{(i)}}{S_t^{(i)}} = \mu_t^{(i)}\, dt + \sigma_i\left(t, S_t^{(i)}\right) dw_t^{(i)} \qquad i=1,...,N \qquad (*)$$

How does the local volatility $\sigma_i\left(t, S_t^{(i)}\right)$ need to be constructed to achieve consistency with $C_t^{(i)}(T,K)$?

The Local Correlation Model (LCM)

- We extend the above model by introducing $S_t^{(0)} = \sum_{i=1}^{N} \alpha_i \, S_t^{(i)} \quad or \quad dS_t^{(0)} = \sum_{i=1}^{N} \alpha_i \, dS_t^{(i)}$

  Note also $S_t^{(0)}$ has a Dupire local vol associated with $\sigma_0\left(t, S_t^{(0)}\right)$ according to Eq.(*)

  Note that a necessary condition for matching single stock-skews and index skews is

$$E^Q\left(\left(S_t^{(0)}\right)^2 \sigma_0^2\left(t, S_t^{(0)}\right) | F_t\right) = E^Q\left(\sum_{i,j=1}^{N} \alpha_i \alpha_j \, S_t^{(i)} S_t^{(j)} \sigma_i\left(t, S_t^{(i)}\right) \sigma_j\left(t, S_t^{(j)}\right) \rho_{ij}(t,.) \,|\, F_t \wedge S_t^0 = \sum_{i=1}^{N} \alpha_i S_t^{(i)}\right)$$

  Assume one could construct a local correlation $\rho_{ij}(t,.)$

  such that

$$\left(\sum_{i=1}^{N} \alpha_i S_t^{(i)}\right)^2 \sigma_0^2\left(t, S_t^{(0)}\right) = \sum_{i,j=1}^{N} \alpha_i \alpha_j \, S_t^{(i)} S_t^{(j)} \sigma_i\left(t, S_t^{(i)}\right) \sigma_j\left(t, S_t^{(j)}\right) \rho_{ij}(t,.)$$

  is matched path-by-path for all t

The Local Correlation Model (LCM)

- LCM condition:

Assume $S_t^{(i)} \quad i=1,..,N$ obey the local vol dynamics of Eq.(*) and there exists a correlation matrix ρ_{ij}^{down} such that $\rho_{ij}(t,.) \geq \rho_{ij}^{down} \quad \forall i,j,t$

and the following (sufficient) condition is fulfilled

$$\sum_{i,j=1}^{N} \alpha_i \alpha_j S_t^{(i)} S_t^{(j)} \sigma_i\left(t, S_t^{(i)}\right) \sigma_j\left(t, S_t^{(j)}\right) \rho_{ij}^{down} \leq \left(\sum_{i=1}^{N} \alpha_i S_t^{(i)}\right)^2 \sigma_0^2\left(t, S_t^{(0)}\right) \leq \sum_{i,j=1}^{N} \alpha_i \alpha_j S_t^{(i)} S_t^{(j)} \sigma_i\left(t, S_t^{(i)}\right) \sigma_j\left(t, S_t^{(j)}\right)$$

The Local Correlation Model (LCM)

- LCM model:

  Assume the LCM condition holds. Then there exists a (no-dispersion) arbitrage setting that is given by

$$\rho_{ij}\left(t,.\right)=\hat{\rho}_{ij}\left(u^*\right)$$

and

$$u^*=\begin{cases}\sqrt{\left(-\dfrac{\mathrm{cov}_{LB}-\sigma_0^2\left(t,S_t^{(0)}\right)\left(S_t^{(0)}\right)^2}{\mathrm{cov}_0-\sigma_0^2\left(t,S_t^{(0)}\right)\left(S_t^{(0)}\right)^2}\right)} & if \quad \mathrm{cov}_0-\sigma_0^2\left(t,S_t^{(0)}\right)\left(S_t^{(0)}\right)^2>0\\ \sqrt{\left(-\dfrac{\mathrm{cov}_{LB}-\sigma_0^2\left(t,S_t^{(0)}\right)\left(S_t^{(0)}\right)^2}{\mathrm{cov}_{UB}-\sigma_0^2\left(t,S_t^{(0)}\right)\left(S_t^{(0)}\right)^2}\right)} & otherwise\end{cases}$$

The Local Correlation Model (LCM)

$$\mathrm{cov}_{LB} \equiv \sum_{i,j=1}^{N} \alpha_i \alpha_j \; \rho_{ij}^{down} \; S_t^{(i)} S_t^{(j)} \sigma_i\left(t, S_t^{(i)}\right) \sigma_j\left(t, S_t^{(j)}\right)$$

$$\mathrm{cov}_0 \equiv \sum_{i,j=1}^{N} \alpha_i \alpha_j \; \rho_{ij}^{centre} \; S_t^{(i)} S_t^{(j)} \sigma_i\left(t, S_t^{(i)}\right) \sigma_j\left(t, S_t^{(j)}\right)$$

$$\mathrm{cov}_{UP} \equiv \sum_{i,j=1}^{N} \alpha_i \alpha_j \; S_t^{(i)} S_t^{(j)} \sigma_i\left(t, S_t^{(i)}\right) \sigma_j\left(t, S_t^{(j)}\right)$$

The Local Correlation Model (LCM) in steeply skewed markets

- Do some work here, sorry

The Local Correlation Model (LCM)

- Show that options on $S_T^{(0)}$ can be replicated two different ways:
  - (i) delta hedging $S_t^{(0)}$
  - (ii) delta hedging into the individual components $S_t^{(i)}$

$$\begin{aligned}
\left(S_T^{(0)}-K\right)^+ &= C_t^{(0)}(T,K)+\int_0^T dC_t^{(0)}(T,K) \\
&= C_t^{(0)}(T,K)+\int_0^T \frac{\partial C_u^{(0)}(T,K)}{\partial S_u^{(0)}}\left(dS_u^{(0)}-\mu_u^{(0)}S_u^{(0)}\right)du \\
&\quad +\int_0^T \frac{1}{2}\frac{\partial^2 C_u^{(0)}(T,K)}{\partial^2 S_u^{(0)}} d<S_u^{(0)},S_u^{(0)}>_u+\left(\frac{\partial C_u^{(0)}(T,K)}{\partial u}+\frac{\partial C_u^{(0)}(T,K)}{\partial S_u^{(0)}}\mu_u^{(0)}S_u^{(0)}\right)du \\
&= C_t^{(0)}(T,K)+\int_0^T \frac{\partial C_u^{(0)}(T,K)}{\partial S_u^{(0)}}\left(dS_u^{(0)}-\mu_u^{(0)}S_u^{(0)}\right)du
\end{aligned}$$

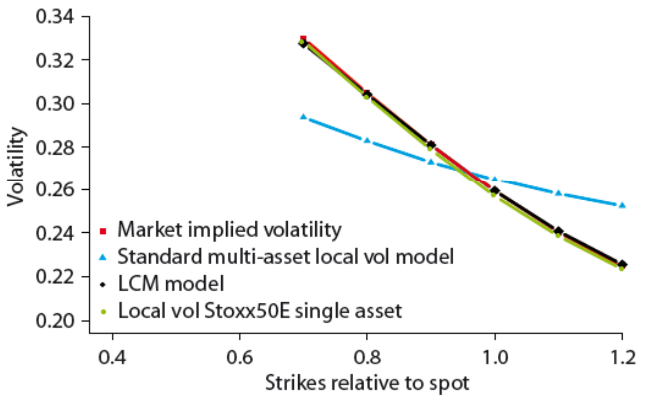
Note that a similar argument yields

$$\left(S_T^{(0)}-K\right)^+ = C_t^{(0)}(T,K)+\int_0^T \frac{\partial C_u^{(0)}(T,K)}{\partial S_u^{(0)}}\left(\sum_{i=1}^N \alpha_i dS_u^{(i)}-\mu_u^{(i)}S_u^{(i)}\right)du$$

The Local Correlation Model (LCM)

Simulation results

4 Skew comparison



Note: the calculation compares the market implied volatility quotes for two-year options with the (standard) multi-asset local volatility model as well as the LCM. Whereas the LCM tracks the market distribution quite closely, multi-asset local vol deviates from the market substantially. To identify possible sources of numerical errors, we included EuroStoxx 50 in the same simulation as a 'single' asset as well

A. Average instantaneous correlations of two-year options between the constituents of EuroStoxx 50 along the simulated paths in the LCM

Strike in terms of spot	Average correlation (%)
0.7	58.3
0.8	56.1
0.9	53.8
1.0	50.6
1.1	46.9
1.2	43.5

Note: no fitting or calibration procedure was required to obtain these results

The Local Correlation Model (LCM)
Some mechanics of the model

- Pre-prepare a table of $\hat{\rho}_{ij}(u)$ (see slide 19) for about 50-100 values of u together with the Cholesky decompositions

- During the simulation calculate $\mathrm{cov}_{LB}, \mathrm{cov}_0, \mathrm{cov}_{UP}$ (see slide 25) as well as u^* (slide 24) at each time step
- And retrieve the appropriate correlation $\hat{\rho}_{ij}(u^*)$ and Cholesky decomposition from the table
- Simulate the next time step according to slide 20 for using $\rho_{ij}(t,.) = \hat{\rho}_{ij}(u^*)$

The Local Correlation Model (LCM)

Non-uniqueness: Chewing Gum effect

- Consider the chewing gum effect
- Is the solution unique? Consider for example a different family of correlation matrices defined by

$$\hat{\rho}_{ij}(u) \equiv \begin{cases} \dfrac{\rho_{ij}^{down} + u^2\ \xi_i \xi_j\ \rho_{ij}^{up}}{\sqrt{\left(1+\xi_i^2 u^2\right)\left(1+\xi_j^2 u^2\right)}} & i \neq j \in \{1,\ldots,N\} \\ 1 & i = j \end{cases}$$

- Study worst-off options $\left(K - Min\left(\dfrac{S_T^{(0)}}{S_0^{(0)}},\ldots,\dfrac{S_T^{(N)}}{S_0^{(N)}}\right)\right)^+$

C. Price of worst-of put		
Strike in terms of spot	Worst-of put price: set 1	Worst-of put price: set 2
0.6	0.306	0.463
0.7	0.391	0.560
0.8	0.482	0.657
0.9	0.575	0.754

Summary

- Financial Markets are encoding a non-trivial copula structure between the components of Equity indices
- LCM explicitly constructs a local state-dependent correlation matrix that reconciles index option markets and single stock option markets in a arbitrage-free setting by construction
- LCM solution is not unique (Chewing Gum effect). Worst-of options are ideal candidates to further complete the market and to provide new clues about the joint dynamics of the system
- It is possible to link the “correlation skew” to systemic risk of a portfolio. This could provide new opportunities to monitor, mark and risk-manage systemic risks in portfolios

$$
\begin{aligned}
&\left(S_T^0 - K\right)^+ = C_t^0(T,K) + \int_t^T dC_u^0(T,K) \\
&= C_t^0(T,K) + \int_t^T \frac{\partial C_u^0(T,K)}{\partial S_u^0}\left(dS_u^0 - \mu_u^0 S_u^0 du\right) \\
&+ \int_t^T \frac{1}{2}\frac{\partial^2 C_u^0(T,K)}{\partial^2 S_u^0} d\left\langle S_u^0, S_u^0\right\rangle + \left(\frac{\partial C_u(T,K)}{\partial u} + \frac{\partial C_u^0(T,K)}{\partial S_u^0}\mu_u^0 S_u^0\right)du \\
&= C_t^0(T,K) + \int_t^T \frac{\partial C_u^0(T,K)}{\partial S_u^0}\left(dS_u^0 - \mu_u^0 S_u^0 du\right)
\end{aligned}
$$

$$
\left(S_T^0 - K\right)^+ = C_t^0(T,K) + \int_t^T \frac{\partial C_t^0(T,K)}{\partial S_t^0}\left(\sum_{i=1}^n \alpha_i\left(dS_t^i - \mu_t^i S_t^i dt\right)\right)
$$