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Emergent Optimal Hedges

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Minimum-Variance Hedge

Optimal hedge for some covariance matrix *of model parameters*

- $U = U(\text{model}) = U(x) = U(x_-) + \delta \cdot dx$
  - so U has a variance, $\delta^T \sigma \delta$, where σ is the covariance matrix of x
- M candidate hedge instruments, with value $V_1 \ldots V_M$
- $V_i(x) = V_i(x_-) + J_i \cdot dx$ defines the matrix J
- For some hedge h, $U - \Sigma_i V_i h_i$ has minimum variance:

  - $h = -(J\sigma J^T)^{-1} J\sigma\delta$

- We need to know the model's covariance matrix σ as well as the (easier to measure) covariance matrix $J\sigma J^T$ of hedge instrument values

Understanding Underdetermined Search

Calibration is constrained optimization

- N parameters of a model means N degrees of freedom
- M calibration instruments
  - N<M in stationary models with very limited fitting power
  - Much more often we have N>>M
- So there is an (N-M)-dimensional manifold of solutions, and it is the calibrator's job to pick one
- Classical "bootstrap" methods accomplish this by reducing N
  - bootstrap is only defined with N=M
  - so we add constraints, such as interpolation rules between instrument maturities, and use them to describe our model in terms of a less-rich model with M parameters
  - these constraints are nonphysical, and often of low quality
  - the problem *is intrinsically underdetermined*, even if we ignore the fact
- Since we must choose one solution, let's try to choose the right one

Understanding Underdetermined Search (2)

Iterated Quadratic Programming

- We can't optimize without a figure of merit G(x)
  - we seek the valid solution (on the manifold of x reproducing market prices) with minimum G
- Most likely G has a global optimum (minimum) at x_0
  - and has a quadratic form near that minimum
- If G is a quadratic form, then it defines a metric
  - we seek the valid solution *closest to* x_0
  - $G(x_0+s) = s^T W s$ for some positive definite W
- If V is linear, then this is the *quadratic programming problem*
  - solution is very fast, even for large N, if W is sparse
- For nonlinear V, there is an iterative extension of Newton's method
  - linearize V around prior point x_-
  - solve QP problem and take the resulting QP step
  - iterate to convergence

Using Underdetermined Search

Iterated Quadratic Programming

- We control the parametrization (x)
  - favor parametrizations which make V linear
    - never calibrate a term structure of reversion rate
- We control the attractor x_0
  - we can make it sensible, smooth, and somewhat stationary
  - most likely we will give the user some control, with <M parameters
- We control the metric W
  - we can penalize non-smooth or non-stationary solutions
  - subject to the usual trade-off between smoothness and locality
  - build W from couplings, so it is naturally sparse

- $s = -W^{-1}J^T(JW^{-1}J^T)^{-1}f$

Using Underdetermined Search (2)

Efficiency of Calibrations

- Bootstrap calibration takes ~8 evaluations for each new constraint; total calibration time ~8M
- Fully-determined exact fit (M parameters) takes many more evaluations and also gradient computations
- Overdetermined fit (least squares) takes vastly more evaluations and gradients
- Underdetermined fit takes ~6 evaluations for common problems, but also requires ~2 gradient computations
  - so time ~(6+2N)M, where N can be very large
  - unless we can analytically compute the Jacobian
  - this computation need not be very exact
- In practice underdetermined fit is as fast as non-optimized bootstrap, much faster than other methods

Response Functions

Sufficient statistics of a calibration

- Bump one of the constraints (by a small amount), and measure the change in x
- The Law of Response Functions: *If you don't know what they look like, they look bad*
- Crucial for every aspect of calibration
  - if response to this artificial bump is bad, response to real fluctuations may be as well
  - response function dictates calibration instrument risk (chain rule)
    - hedge in one instrument is response function dotted with portfolio gradient δ
- Underdetermined response function is the QP step for this small bump
  - $s = -W^{-1}J^T(JW^{-1}J^T)^{-1}e_j$ where e_j is a unit basis vector
  - note the base model is the starting point of recalibration: $x_0 = x_-$
  - the hedge is

    - $H = -(JW^{-1}J^T)^{-1}JW^{-1}\delta$

UBS

Conclusion

When does an optimal hedge emerge from calibration?

- $W^{-1=}\sigma$

UBS