From Spot Volatilities
to Implied Volatilities

Julien Guyon

SOCIETE GENERALE
Corporate & Investment Banking
Global Markets Quantitative Research

Global Derivatives 2011, Paris
April 14th, 2011

Joint work with Pierre Henry-Labordère

julien.guyon@sgcib.com
pierre.henry-labordere@sgcib.com

Motivation A link between spot and implied volatilities Previous results Our new estimates of implied volatilities Numerical experiments Conclusion
00
00
000

00000
0

00000
000
0

0000
0

Outline

- Motivation
- A link between spot volatilities and implied volatilities
- Previous results
- Our new estimates of implied volatilities
- Numerical experiments
- Conclusion

Motivation

- One knows how to infer the local volatility function from the implied volatility surface (Dupire, 1994):

$$\sigma(T, K)^2 = \frac{\frac{\partial C}{\partial T}}{\frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}}$$

- How to invert this formula?
    - Classical solution: solve numerically the forward equation for call prices.
    - Alternative solutions?

Motivation **A link between spot and implied volatilities** Previous results Our new estimates of implied volatilities Numerical experiments Conclusion

A link between spot volatilities and implied volatilities

A link between spot volatilities and implied volatilities

- Let σ_t be the spot volatility of the T-forward price f_{tT} of some asset S_t:

$$df_{tT} = \sigma_t f_{tT} dW_t^T$$

- **No assumption is made on σ_t**: it is a general stochastic process. W^T is a $(\mathcal{F}_t)$-Brownian motion under the T-forward measure $\mathbb{P}^T$, where $(\mathcal{F}_t)$ denotes the filtration generated by the processes σ_t and f_{tT}.

- Let B denote the Black price:

$$
\begin{aligned}
B(t, f; \sigma) &= fN\left(d_+\left(f, \sigma^2(T-t), K\right)\right) - KN\left(d_-\left(f, \sigma^2(T-t), K\right)\right) \\
d_\pm(f, v, K) &= \frac{\ln f/K}{\sqrt{v}} \pm \frac{1}{2}\sqrt{v}
\end{aligned}
$$

Motivation   **A link between spot and implied volatilities**   Previous results   Our new estimates of implied volatilities   Numerical experiments   Conclusion

A link between spot volatilities and implied volatilities

- $(S_T - K)_+ = (f_{TT} - K)_+ = B(T, f_{TT}; \sigma)$ for any σ, so by Itô's formula,

$$\mathbb{E}^T \left[(S_T - K)_+ \right] - B(0, f_{0T}; \sigma) = \mathbb{E}^T \left[\int_0^T \left(\partial_t B + \frac{1}{2} \sigma_t^2 f_{tT}^2 \partial_f^2 B \right) (t, f_{tT}; \sigma) dt \right]$$

- Besides, $\partial_t B + \frac{1}{2} \sigma^2 f^2 \partial_f^2 B = 0$.

- $\Rightarrow \sigma_{\mathrm{BS}}(T, K)$ is the value of the constant parameter σ such that

$$\mathbb{E}^T \left[\int_0^T \left(\sigma_t^2 - \sigma^2 \right) f_{tT}^2 \partial_f^2 B(t, f_{tT}; \sigma) dt \right] = 0 \tag{1}$$

- The time integral is the P&L generated between inception and maturity by a delta-hedged position using constant volatility σ.

- $\Rightarrow \sigma_{\mathrm{BS}}(T, K)$ obeys the self-consistence equation:[1]

$$\sigma_{\mathrm{BS}}(T, K)^2 = \frac{\mathbb{E}^T \left[\int_0^T \sigma_t^2 f_{tT}^2 \partial_f^2 B(t, f_{tT}; \sigma_{\mathrm{BS}}(T, K)) dt \right]}{\mathbb{E}^T \left[\int_0^T f_{tT}^2 \partial_f^2 B(t, f_{tT}; \sigma_{\mathrm{BS}}(T, K)) dt \right]} \tag{2}$$

---

[1]As far as we know, this representation of implied volatility was first used by Dupire in the late 1990's [2].

Motivation   A link between spot and implied volatilities   Previous results   Our new estimates of implied volatilities   Numerical experiments   Conclusion
○○   ○○○○○   ○○○○○   ○○○○
●○   ○   ○○○   ○
○○○   ○

Comparison with Gatheral's formula

Comparison with Gatheral's formula

- Gatheral [3] states that

$$\sigma_{\mathrm{BS}}(T,K)^2 = \frac{1}{T} \int_0^T \frac{\mathbb{E}^T \left[\sigma_t^2 f_{tT}^2 \partial_f^2 B^v(t, f_{tT}) \right]}{\mathbb{E}^T \left[f_{tT}^2 \partial_f^2 B^v(t, f_{tT}) \right]} dt$$

- B^v slightly differs from B: it is the Black price of the call option with time-dependent deterministic volatility $\sqrt{v(t)}$, where v is the so-called "forward implied variance" function, implicitly defined by

$$v(t) = \frac{\mathbb{E}^T \left[\sigma_t^2 f_{tT}^2 \partial_f^2 B^v(t, f_{tT}) \right]}{\mathbb{E}^T \left[f_{tT}^2 \partial_f^2 B^v(t, f_{tT}) \right]}$$

Motivation   A link between spot and implied volatilities   Previous results   Our new estimates of implied volatilities   Numerical experiments   Conclusion

Comparison with Gatheral's formula

- Both expressions for $\sigma_{\mathrm{BS}}(T,K)$ can be seen as special cases of a more general result: For any positive function $w(t)$,

$$\sigma_{\mathrm{BS}}(T,K)^2 = \frac{1}{T}\int_0^T w(t)dt \Longleftrightarrow \int_0^T \mathbb{E}^T\left[\left(\sigma_t^2 - w(t)\right) f_{tT}^2 \partial_f^2 B^w(t, f_{tT})\right] dt = 0$$

- The implied forward variance v is the only function w such that, the time integrand on the r.h.s. is zero for each time slice t.

- The implied volatility $\sigma_{\mathrm{BS}}(T,K)$ is the only constant function w such that the equation holds.

Motivation **A link between spot and implied volatilities** Previous results Our new estimates of implied volatilities Numerical experiments Conclusion

Our general strategy for estimating implied volatilities

Our general strategy for estimating implied volatilities

- Let $\sigma_{\mathsf{loc}}^2(t, f_{tT}) = \mathbb{E}^T\left[\sigma_t^2 \,|\, f_{tT}\right]$ be the local variance of the forward.
- Since

$$\mathbb{E}^T\left[\sigma_t^2 f_{tT}^2 \partial_f^2 B(t, f_{tT}; \sigma)\right] = \mathbb{E}^T\left[\sigma_{\mathsf{loc}}^2(t, f_{tT}) f_{tT}^2 \partial_f^2 B(t, f_{tT}; \sigma)\right]$$

  one can replace σ_t by $\sigma_{\mathsf{loc}}(t, f_{tT})$ in (1)-(2).

- We will build estimates of the implied volatility by estimating the **fixed point** of the mapping

$$\sigma \mapsto \sqrt{\frac{\mathbb{E}^T\left[\int_0^T \sigma_{\mathsf{loc}}^2(t, f_{tT}) f_{tT}^2 \partial_f^2 B(t, f_{tT}; \sigma) dt\right]}{\mathbb{E}^T\left[\int_0^T f_{tT}^2 \partial_f^2 B(t, f_{tT}; \sigma) dt\right]}} = \sqrt{\int_0^T \int_0^\infty \sigma_{\mathsf{loc}}^2(t, f) \, q_\sigma(t, f) \, dt df}$$

$$q_\sigma(t, f) = \frac{f^2 \partial_f^2 B(t, f; \sigma) p(t, f)}{\int_0^T \int_0^\infty f'^2 \partial_f^2 B(t', f'; \sigma) p(t', f') df' dt'}$$

Motivation | **A link between spot and implied volatilities** | Previous results | Our new estimates of implied volatilities | Numerical experiments | Conclusion

Our general strategy for estimating implied volatilities

- Note that

$$f^2 \partial_f^2 B(t,f;\sigma) p(t,f) = \frac{f}{\sigma\sqrt{2\pi(T-t)}} \exp\left(-\frac{d_+(f,\sigma^2(T-t),K)^2}{2}\right) p(t,f).$$

- $p(t,f)$, hence $q_\sigma(t,f)$, is not known explicitly.
- Our suggestion: **approximate the fixed point using an estimate $\hat{p}(t,f)$ of $p(t,f)$.**
  - Of course, $\hat{p}(t,f)$ also leads to an estimate of the forward price of the call option, by numerical integration of the payoff against $\hat{p}(t,f)$.
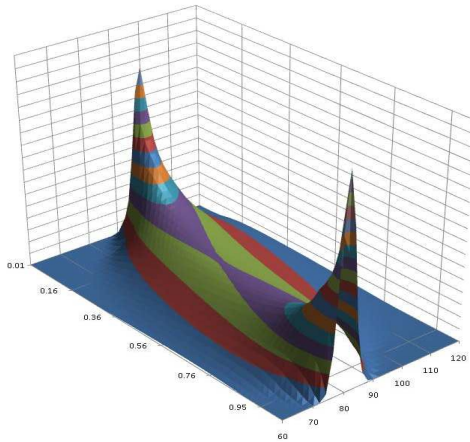 - But our method proves much more accurate, as we directly average σ_{loc}^2, rather than the payoff.

Motivation   A link between spot and implied volatilities   Previous results   Our new estimates of implied volatilities   Numerical experiments   Conclusion

Our general strategy for estimating implied volatilities

Figure: Graph of $q_\sigma(t, f)$ when the forward is lognormal; $\sigma = 30\%$, $f_0 = 100$, $K = 80$, $T = 1$.

Motivation   A link between spot and implied volatilities   **Previous results**   Our new estimates of implied volatilities   Numerical experiments   Conclusion
         OO   ●○○○○   OOOOO
         OO   ○   OOO   OOOO
         OOO   ○   ○

The effective path estimator

Previous results

- Several authors have already suggested estimators of the implied volatility based on approximations of $q_\sigma(t, f)$.

- $p(0, f) = \delta_{f_0}(f)$ and $\partial_f^2 B(T, f; \sigma) = \delta_K(f)$.

- $\Rightarrow$ **Double Dirac Property:** $q_\sigma(t, f)$ reduces to a multiple of a Dirac mass at $f = f_0$ when $t = 0$, and to a multiple of a Dirac mass at $f = K$ when $t = T$.

- Inspired by this property, Gatheral [3], then Reghai [6] have suggested approximating $q_\sigma(t, f)$ by $\hat{q}_\sigma^{\mathrm{EP}}(t, f) = \frac{1}{T}\delta_{\mathrm{EP}(t)}(f)$ where $\delta_{\mathrm{EP}(t)}$ is a Dirac mass that singles out one particular **effective path** $\mathrm{EP}(t)$ with fixed end points f_0 at $t = 0$ and K at $t = T$.

- The authors call this approximation the "most likely path" technique.

Motivation   A link between spot and implied volatilities   **Previous results**   Our new estimates of implied volatilities   Numerical experiments   Conclusion

The effective path estimator

Examples of effective paths

- Common examples are

$$\mathrm{EP}(t) = f_0 + \frac{t}{T}\left(K - f_0\right), \qquad \mathrm{EP}(t) = f_0 \exp\left(\frac{t}{T}\ln\frac{K}{f_0}\right)$$

- Alternatively, one may use

$$
\begin{aligned}
\mathrm{EP}(t) &= f_0 \exp\left(\frac{\nu(t)}{\nu(T)}\ln\frac{K}{f_0} + \frac{1}{2}\nu(T)\frac{\nu(t)}{\nu(T)}\left(1 - \frac{\nu(t)}{\nu(T)}\right)\right) \\
\nu(t) &= \int_0^t \sigma(s)^2 ds, \qquad \sigma(t) = \sigma_{\mathsf{loc}}\left(t, \mathrm{EP}(t)\right)
\end{aligned}
$$

$\mathrm{EP}(t)$ is the expected value of a lognormal forward $\bar{f}_t$ having volatility $\sigma_{\mathsf{loc}}\left(t, \mathrm{EP}(t)\right)$, given $\bar{f}_T = K$:

$$\mathrm{EP}(t) = \mathbb{E}^T[\bar{f}_t \,|\, \bar{f}_T = K], \qquad d\bar{f}_t = \sigma_{\mathsf{loc}}\left(t, \mathrm{EP}(t)\right)\bar{f}_t dW_t^T$$

One can build this effective path using a fixed point algorithm. Very similar to Reghai.

Motivation   A link between spot and implied volatilities   **Previous results**   Our new estimates of implied volatilities   Numerical experiments   Conclusion

The effective path estimator

Properties of the effective path approximation

- The estimated density $\hat{q}_\sigma^{\mathrm{EP}}(t, f)$ does not depend on σ.
- The estimated implied volatility squared reads explicitly

$$\hat{\sigma}_{\mathrm{BS}}^{\mathrm{EP}}(T, K)^2 = \frac{1}{T} \int_0^T \sigma_{\mathsf{loc}}^2(t, \mathrm{EP}(t)) \; dt$$

- One estimates the implied volatility squared by the time average of the square of the local volatility of the forward, taken at some effective path.
- The average of the local volatility over all possible values for the forward has been purely and simply replaced by the value of the local volatility at some effective path.

Motivation   A link between spot and implied volatilities   **Previous results**   Our new estimates of implied volatilities   Numerical experiments   Conclusion
                    ○○   ○○○○●   ○○○○○○
                    ○○   ○   ○○○
                    ○○○   ○

The effective path estimator

Some drawbacks of the effective path estimator

A first drawback. Imagine that for each time t the local volatility is U-shaped with a minimum at $f = f_0$, and that $K = f_0$. For the usual choices (2), the effective path is constant at f_0, so $\hat{\sigma}_{\text{BS}}^{\text{EP}}(T, f_0)^2$ is just the time average of $\sigma_{\text{loc}}^2(t, f_0)$, the minimum value of σ_{loc}^2 at time t. Hence

$$
\begin{aligned}
\sigma_{\text{BS}}(T, f_0)^2 &= \frac{1}{T} \int_0^T \frac{\mathbb{E}^T\left[\sigma_{\text{loc}}^2(t, f_{tT}) f_{tT}^2 \partial_f^2 B^v(t, f_{tT})\right]}{\mathbb{E}^T\left[f_{tT}^2 \partial_f^2 B^v(t, f_{tT})\right]} dt \\
&\geq \frac{1}{T} \int_0^T \frac{\mathbb{E}^T\left[\sigma_{\text{loc}}^2(t, f_0) f_{tT}^2 \partial_f^2 B^v(t, f_{tT})\right]}{\mathbb{E}^T\left[f_{tT}^2 \partial_f^2 B^v(t, f_{tT})\right]} dt \\
&= \frac{1}{T} \int_0^T \sigma_{\text{loc}}^2(t, f_0) \; dt = \hat{\sigma}_{\text{BS}}^{\text{EP}}(T, f_0)^2
\end{aligned}
$$

$\hat{\sigma}_{\text{BS}}^{\text{EP}}(T, f_0)$ surely underestimates $\sigma_{\text{BS}}(T, f_0)$.

The effective path estimator

Some drawbacks of the effective path estimator

A second drawback. Assume $\mathrm{EP}\,(t) = f_0 \exp\left(\frac{t}{T}\ln\frac{K}{f_0}\right)$:

$$\hat{\sigma}_{\mathrm{BS}}^{\mathrm{EP}}(T,K)^2 = \int_0^1 \sigma_{\mathrm{loc}}^2\left(uT, f_0\exp\left(u\ln\frac{K}{f_0}\right)\right)\,du$$

$\Rightarrow$ Under a mild domination assumption:

$$\hat{\sigma}_{\mathrm{BS}}^{\mathrm{EP}}(0,K)^2 = \int_0^1 \sigma_{\mathrm{loc}}^2\left(0, f_0\exp\left(u\ln\frac{K}{f_0}\right)\right)\,du$$

This is not in line with Berestycki-Busca-Florent:

$$\sigma_{\mathrm{BS}}(0,K)^{-1} = \int_0^1 \sigma_{\mathrm{loc}}^{-1}\left(0, f_0\exp\left(u\ln\frac{K}{f_0}\right)\right)\,du$$

The conditional distribution estimator

The conditional distribution estimator

- In [6], Reghai suggests another heuristic approximation for the implied volatility (CD stands for conditional distribution):

$$\sigma_{\mathrm{BS}}(T, K)^2 \simeq \hat{\sigma}_{\mathrm{BS}}^{\mathrm{CD}}(T, K)^2 = \frac{1}{T} \int_0^T \mathbb{E}^T \left[\sigma_{\mathrm{loc}}^2(t, f_{tT}) \,|\, f_{TT} = K \right] \, dt$$

- This boils down to estimating $q_\sigma(t, f)$ by $\frac{1}{T}$ times the pdf of f_{tT} conditional on $f_{TT} = K$ under $\mathbb{P}^T$:

$$\hat{q}^{\mathrm{CD}}(t, f) = \frac{1}{T} p(t, f \,|\, f_{TT} = K)$$

- Computation of $\hat{q}^{\mathrm{CD}}$ cannot be performed exactly. We suggest to use

$$\mathrm{Law}\,(f_{tT} | f_{TT} = K) \simeq \mathrm{Law}\,(\bar{f}_t | \bar{f}_T = K) = \mathrm{Law}\,(f_0 \exp(G_t))$$

where $\mathrm{Law}(G_t) = \mathcal{N}\left(\frac{\nu(t)}{\nu(T)} \ln \frac{K}{f_0}, \nu(T) \frac{\nu(t)}{\nu(T)} \left(1 - \frac{\nu(t)}{\nu(T)} \right) \right)$. One can then approximate $\mathbb{E}^T \left[\sigma_{\mathrm{loc}}^2(t, f_{tT}) \,|\, f_{TT} = K \right]$ thanks to a Hermite quadrature.

Our new estimates of implied volatilities

- $\hat{q}^{\mathrm{CD}}$ is a very natural candidate among all the functions satisfying the Double Dirac Property.
- However, this property is stable by multiplication.
- It is not clear why $\hat{q}^{\mathrm{CD}}$ would be a good proxy for q, in the sense that it would produce a good proxy for the implied volatility.

$\Rightarrow$ We will build two new estimates $\hat{q}_\sigma^{\mathrm{LN}}(t, f)$ and $\hat{q}_\sigma^{\mathrm{HKE}}(t, f)$ of

$$q_\sigma(t, f) = \frac{f^2 \partial_f^2 B(t, f; \sigma) p(t, f)}{\int_0^T \int_0^\infty f'^2 \partial_f^2 B(t', f'; \sigma) p(t', f') df' dt'} \tag{3}$$

based on estimations of the pdf $p(t, f)$ of f_{tT}.

- In the first estimate, we replace $p(t, f)$ by a **lognormal** density.
- In the second one, we replace $p(t, f)$ by its **heat kernel expansion**.

Motivation | A link between spot and implied volatilities | Previous results | **Our new estimates of implied volatilities** | Numerical experiments | Conclusion

The lognormal estimator

The lognormal estimator

- The estimate $\hat{q}_\sigma^{\mathrm{LN}}$ is defined by (3), where $p(t, f)$ is simply replaced by a lognormal density

$$\hat{p}^{\mathrm{LN}}(t, f) = \frac{1}{f\sqrt{2\pi v_t^{\mathrm{LN}}}} \exp\left(-\frac{d_+(f, v_t^{\mathrm{LN}}, f_0)^2}{2}\right)$$

where

$$v_t^{\mathrm{LN}} = \int_0^t \sigma^{\mathrm{LN}}(u)^2 du$$

for some deterministic function of time $\sigma^{\mathrm{LN}}(t)$, the Black volatility.

- If the forward has volatility $\sigma^{\mathrm{LN}}(t)$, $\hat{p}^{\mathrm{LN}}(t, f) = p(t, f)$, $\hat{q}_\sigma^{\mathrm{LN}} = q_\sigma$ (and $\sigma_{\mathrm{BS}}^2(T, K)$ is simply v_T^{LN}/T, independently of K).

- Provided $\sigma_{\mathrm{loc}}(t, f)$ has little dependence on f, it can be well approximated by some $\sigma^{\mathrm{LN}}(t)$ and we expect $\hat{\sigma}_{\mathrm{BS}}^{\mathrm{LN}}(T, K) \simeq \sigma_{\mathrm{BS}}(T, K)$.

Motivation   A link between spot and implied volatilities   Previous results   **Our new estimates of implied volatilities**   Numerical experiments   Conclusion
OO   OOOOO   O●OOO   OOOO
OO   O   OOO   O
OOO   O

The lognormal estimator

Some straightforward computations lead to

$$\hat{\sigma}_{\mathrm{BS}}^{\mathrm{LN}}(T,K)^2 = \frac{\int_0^T a(t) \int_{\mathbb{R}} \sigma_{\mathsf{loc}}^2\left(t, f^{\mathrm{LN}}(t,z)\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz dt}{\int_0^T a(t) dt}$$

where

$$f^{\mathrm{LN}}(t,z) = f_0 \exp\left(\frac{v_t^{\mathrm{LN}}}{v_t^{\mathrm{LN}} + \hat{\sigma}_{\mathrm{BS}}^{\mathrm{LN}}(T,K)^2(T-t)} \ln\frac{K}{f_0} + \frac{\sqrt{v_t^{\mathrm{LN}}}\hat{\sigma}_{\mathrm{BS}}^{\mathrm{LN}}(T,K)\sqrt{T-t}}{\sqrt{v_t^{\mathrm{LN}} + \hat{\sigma}_{\mathrm{BS}}^{\mathrm{LN}}(T,K)^2(T-t)}} z\right)$$

$$a(t) = \frac{1}{\sqrt{v_t^{\mathrm{LN}} + \hat{\sigma}_{\mathrm{BS}}^{\mathrm{LN}}(T,K)^2(T-t)}} \exp\left(-\frac{d_+\left(f_0, v_t^{\mathrm{LN}} + \hat{\sigma}_{\mathrm{BS}}^{\mathrm{LN}}(T,K)^2(T-t), K\right)^2}{2}\right)$$

Motivation   A link between spot and implied volatilities   Previous results   **Our new estimates of implied volatilities**   Numerical experiments   Conclusion

The lognormal estimator

- One has

$$a(t) = \frac{\sqrt{2\pi}}{f_0} \mathbb{E}^T \left[\left(f_{tT}^{\mathrm{LN}} \right)^2 \partial_f^2 B(t, f_{tT}^{\mathrm{LN}}; \hat{\sigma}_{\mathrm{BS}}^{\mathrm{LN}}(T, K)) \right]$$

where f_{tT}^{LN} has lognormal dynamics $df_{tT}^{\mathrm{LN}} = \sigma^{\mathrm{LN}}(t) f_{tT}^{\mathrm{LN}} dW_t^T$.

- Ex: pick $\sigma^{\mathrm{LN}}(t) = \hat{\sigma}_{\mathrm{BS}}^{\mathrm{LN}}(T, K)$. Then $\left(f_{tT}^{\mathrm{LN}} \right)^2 \partial_f^2 B \left(t, f_{tT}^{\mathrm{LN}}; \hat{\sigma}_{\mathrm{BS}}^{\mathrm{LN}}(T, K) \right)$ is the Black-Scholes Gamma notional "ΓS^2", it is a $\mathbb{P}^T$-martingale, $a(t)$ does not depend on t and

$$
\begin{aligned}
\hat{\sigma}_{\mathrm{BS}}^{\mathrm{LN}}(T, K)^2 &= \frac{1}{T} \int_0^T \int_{\mathbb{R}} \sigma_{\mathrm{loc}}^2 \left(t, f_0 \exp \left(\frac{t}{T} \ln \frac{K}{f_0} + \hat{\sigma}_{\mathrm{BS}}^{\mathrm{LN}}(T, K) \sqrt{T} \sqrt{\frac{t}{T} \left(1 - \frac{t}{T} \right)} z \right) \right) \\
&\quad \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{z^2}{2} \right) dz\, dt.
\end{aligned}
$$

- This is exactly what one would get at first order in $\delta\sigma_{\mathrm{loc}}$ when one expands the local volatility around some constant value $\bar{\sigma}$, provided that in the r.h.s. $\hat{\sigma}_{\mathrm{BS}}^{\mathrm{LN}}(T, K)$ is replaced by $\bar{\sigma}$.

Motivation   A link between spot and implied volatilities   Previous results   **Our new estimates of implied volatilities**   Numerical experiments   Conclusion

The lognormal estimator

- Equation (2) improves the effective path estimator $\hat{\sigma}_{\mathrm{BS}}^{\mathrm{EP}}(T, K)^2$: **we do not only integrate $\sigma_{\mathbf{loc}}^2$ along the effective path**

$$\mathrm{EP}^{\mathrm{LN}}(t) = f^{\mathrm{LN}}(t, 0) = f_0 \exp \left(\frac{v_t^{\mathrm{LN}}}{v_t^{\mathrm{LN}} + \hat{\sigma}_{\mathrm{BS}}^{\mathrm{LN}}(T, K)^2(T-t)} \ln \frac{K}{f_0} \right)$$

but also around it.

- The fluctuation around $\mathrm{EP}^{\mathrm{LN}}(t)$ is given by a (time-changed) Brownian bridge. It is **Gaussian**, with standard deviation

$$\mathrm{STDEV}^{\mathrm{LN}}(t) = \frac{\sqrt{v_t^{\mathrm{LN}}}\,\hat{\sigma}_{\mathrm{BS}}^{\mathrm{LN}}(T, K)\sqrt{(T-t)}}{\sqrt{v_t^{\mathrm{LN}} + \hat{\sigma}_{\mathrm{BS}}^{\mathrm{LN}}(T, K)^2(T-t)}}$$

- In the effective path estimator, this standard deviation is taken to be zero, or, equivalently, the Gaussian pdf $\frac{1}{\sqrt{2\pi}} \exp \left(-\frac{1}{2}z^2 \right)$ is replaced by a Dirac mass at point $z = 0$.

Motivation | A link between spot and implied volatilities | Previous results | **Our new estimates of implied volatilities** | Numerical experiments | Conclusion

The lognormal estimator

- We do not give the same weight dt/T to each infinitesimal interval $[t, t + dt]$.
- Rather, we attach a varying weight $a(t)dt$ which is an estimate of $\mathbb{E}^T \left[f_{tT}^2 \partial_f^2 B(t, f_{tT}; \sigma_{\mathrm{BS}}(T, K)) \right] dt$, and depends on the particular function $\sigma^{\mathrm{LN}}(t)$ chosen.
- Because of the shape of $q_\sigma(t, f)$, a natural choice for $\sigma^{\mathrm{LN}}(t)$ is $\sigma_{\mathrm{loc}}(t, \mathrm{EP}(t))$ for some convenient effective path.

Motivation   A link between spot and implied volatilities   Previous results   **Our new estimates of implied volatilities**   Numerical experiments   Conclusion

The Heat Kernel Expansion estimator

The Heat Kernel Expansion estimator

The estimate $\hat{q}_\sigma^{\mathrm{HKE}}$ is defined by (3), where $p(t,f)$ is replaced by its expansion at first order in the potential Q, the so-called heat kernel expansion

$$\hat{p}^{\mathrm{HKE}}(t,f) = \frac{1}{f}\frac{\sigma_{\mathsf{loc}}(t,f_0)^{1/2}}{\sigma_{\mathsf{loc}}(t,f)^{3/2}}\sqrt{\frac{f_0}{f}}e^{\int_{f_0}^{f}\frac{df'}{C(t,f')}\int_{f_0}^{f'}\partial_t\frac{1}{C(t,f'')}df''}P_0(t,s)\left(1+t\overline{Q}(t,s)+\cdots\right)$$

$$C(t,f) = f\sigma_{\mathsf{loc}}(t,f), \qquad s = \int_{f_0}^{f}\frac{df'}{C(t,f')}$$

$$P_0(t,s) = \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{s^2}{2t}\right)$$

$$\overline{Q}(t,s) = \int_0^1 du \int_{\mathbb{R}} dz\, Q\left(ut, us+\sqrt{t}\sqrt{u(1-u)}z\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}z^2\right)$$

$$Q(t,s) = -\frac{1}{2}\left(\partial_s b + b^2\right)(t,s) - \int_0^s \partial_t b(t,s')ds'$$

$$b(t,s) = \int_{f_0}^{f}\partial_t \frac{1}{C(t,f')}df' - \frac{1}{2}\partial_f C(t,f)$$

Motivation   A link between spot and implied volatilities   Previous results   **Our new estimates of implied volatilities**   Numerical experiments   Conclusion

The Heat Kernel Expansion estimator

After some calculations, we get:

$$\hat{\sigma}_{\mathrm{BS}}^{\mathrm{HKE}}(T,K)^2 = \frac{\int_0^T \int_{\mathbb{R}} \sigma_{\mathsf{loc}}\left(t, f_0 e^{x(t,z)}\right)^{1/2} w(t,z) dz dt}{\int_0^T \int_{\mathbb{R}} \sigma_{\mathsf{loc}}\left(t, f_0 e^{x(t,z)}\right)^{-3/2} w(t,z) dz dt}$$

with

$$w(t,z) = \pi\left(t, x(t,z)\right) \exp\left(-\frac{1}{2}z^2\right)\left(1 + t\overline{Q}\left(t, \int_0^{x(t,z)} \frac{dv}{\sigma_{\mathsf{loc}}\left(t, f_0 e^v\right)}\right)\right)$$

$$x(t,z) = \frac{t}{T}\ln\frac{K}{f_0} + \hat{\sigma}_{\mathrm{BS}}^{\mathrm{HKE}}(T,K)\sqrt{T}\sqrt{\frac{t}{T}\left(1 - \frac{t}{T}\right)}z$$

$$\pi(t,x) = \sigma_{\mathsf{loc}}(t,f_0)^{1/2} \exp\left(\lambda(t,x) - \mu(t,x)/2\right)$$

$$\lambda(t,x) = \int_0^x \frac{dv}{\sigma_{\mathsf{loc}}\left(t, f_0 e^v\right)} \int_0^v \partial_t \frac{dw}{\sigma_{\mathsf{loc}}(t, f_0 e^w)}$$

$$\mu(t,x) = -\frac{1}{4}\hat{\sigma}_{\mathrm{BS}}^{\mathrm{HKE}}(T,K)^2 t + \frac{1}{t}\left(\varepsilon(t,x)^2 + \frac{2x\varepsilon(t,x)}{\hat{\sigma}_{\mathrm{BS}}^{\mathrm{HKE}}(T,K)}\right)$$

$$\varepsilon(t,x) = \int_0^x \frac{dv}{\sigma_{\mathsf{loc}}\left(t, f_0 e^v\right)} - \frac{x}{\hat{\sigma}_{\mathrm{BS}}^{\mathrm{HKE}}(T,K)}$$

Motivation   A link between spot and implied volatilities   Previous results   **Our new estimates of implied volatilities**   Numerical experiments   Conclusion
OO   OOOOO   OOOO
OO   O   OO●   O
OOO   O

The Heat Kernel Expansion estimator

- Similarly to the log-normal estimator, we do not only integrate the local volatility along the effective path

$$\mathrm{EP}^{\mathrm{HKE}}(t) = f_0 \exp(x(t,0)) = f_0 \exp\left(\frac{t}{T}\ln\frac{K}{f_0}\right)$$

  but also around it.

- The fluctuation around $\mathrm{EP}^{\mathrm{HKE}}(t)$ is **not** Gaussian.
- Two spatial integrations are needed:
  - $\sigma_{\mathrm{loc}}^{1/2}$ in the numerator,
  - $\sigma_{\mathrm{loc}}^{-3/2}$ in the denominator.

Motivation   A link between spot and implied volatilities   Previous results   **Our new estimates of implied volatilities**   Numerical experiments   Conclusion

Another version of the lognormal and heat kernel expansion estimates

Another version of the lognormal and heat kernel expansion estimates

- Recall Gatheral's representation of implied volatility:

$$\sigma_{\mathrm{BS}}(T,K)^2 = \frac{1}{T}\int_0^T \int_0^\infty \sigma_{\mathsf{loc}}^2(t,f)h_t(f) \; df\,dt$$

$$h_t(f) = \frac{f^2 \partial_f^2 B^v(t,f)p(t,f)}{\int_0^\infty f'^2 \partial_f^2 B^v(t',f')p(t',f')df'}$$

- We can replace $p(t,f)$ by our estimate $\hat{p}^{\mathrm{LN}}(t,f)$ or $\hat{p}^{\mathrm{HKE}}(t,f)$ into the above formula to get an estimate $\hat{h}_t(f)$ of $h_t(f)$, and then get an estimate

$$\hat{\sigma}_{\mathrm{BS}}(T,K)^2 = \frac{1}{T}\int_0^T \int_0^\infty \sigma_{\mathsf{loc}}^2(t,f)\hat{h}_t(f) \; df\,dt$$

- However, $\hat{h}_t(f)$ is not known explicitly: it depends on v, and v is unknown: we are precisely looking for $\frac{1}{T}\int_0^T v(t)dt = \sigma_{\mathrm{BS}}(T,K)^2$
  $\Rightarrow$ Also replace $v(t)$ by some $\hat{v}(t)$ such that $\hat{v}(0) = \sigma_{\mathsf{loc}}^2(0,f_0)$ and
  $\hat{v}(T) = \sigma_{\mathsf{loc}}^2(T,K)$. For instance, one may pick $\hat{v}(t) = \sigma_{\mathsf{loc}}^2(t,\mathrm{EP}(t))$ for some convenient effective path.

Motivation A link between spot and implied volatilities Previous results Our new estimates of implied volatilities **Numerical experiments** Conclusion

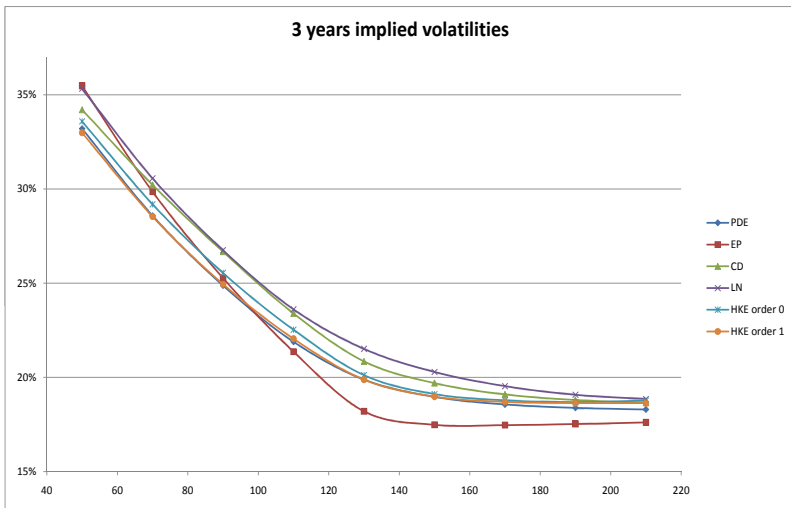Numerical comparison of estimators

Numerical experiments

- We have picked a parametric local volatility $f \mapsto \sigma_{\text{loc}}(t, f)$. To give the "effective path estimator" a chance to be competitive, we deliberately picked a smooth local volatility.

- We have fitted the (time-dependent) parameters to the implied volatility surface of the Eurostoxx 50, as of November 28, 2007.

- To compute the EP estimator, we have used the effective path $\text{EP}(t) = f_0 \exp\left(\frac{t}{T} \ln \frac{K}{f_0}\right)$. For the lognormal estimator, we have used $\sigma^{\text{LN}}(t) = \sigma_{\text{loc}}(t, \text{EP}(t))$.

Motivation   A link between spot and implied volatilities   Previous results   Our new estimates of implied volatilities   **Numerical experiments**   Conclusion

Numerical comparison of estimators

3 months implied volatilities

Motivation   A link between spot and implied volatilities   Previous results   Our new estimates of implied volatilities   **Numerical experiments**   Conclusion

Numerical comparison of estimators

Motivation   A link between spot and implied volatilities   Previous results   Our new estimates of implied volatilities   **Numerical experiments**   Conclusion

Numerical comparison of estimators

Motivation   A link between spot and implied volatilities   Previous results   Our new estimates of implied volatilities   **Numerical experiments**   Conclusion
      oo   ooooo   ooooo
      oo   o   ooo
      ooo   o
      ●
Computational complexity

Computational complexity

- Complexity of PDE solver: $n_K \times n_T$.
  When one needs to compute a single implied volatility for a strike K and a maturity T, one has to compute a whole set of $n_K \times n_T$ call prices.
- Complexity of new HKE estimator at order 1: $n_I \times n_L^3 \times n_H^2$ where
  - n_I: number of iterations in the fixed point algorithm (3 or 4)
  - n_L: number of operations needed to estimate integrals on finite intervals, e.g., the number of points in a Legendre integration
  - n_H: number of operations needed to estimate Gaussian integrals of the type $\int_{\mathbb{R}} \varphi(z) \exp(-z^2/2)dz$, e.g., the number of points in a Hermite integration.
- Complexity of new HKE estimator at order 0: $n_I \times n_L^2 \times n_H$.

Conclusion

- We have suggested new ways of computing implied volatilities from spot volatilities, based on the fact that **the implied volatility is the fixed point of a function mapping a volatility σ to a weighted quadratic average of the effective local volatility**, with a weight $q_\sigma(t, f)$ depending on σ.

- In previous works (Gatheral, Reghai), some heuristic $\hat{q}(t, f)$'s, independent of σ, have been suggested.

- Here, our estimates $\hat{q}_\sigma(t, f)$ are **explicitly built as small deviations from the true weight $q_\sigma(t, f)$**, by replacing the exact pdf of the forward by an estimated pdf:
  - a lognormal density, or
  - the HKE of the exact pdf of the forward, at first order in the potential

Conclusion

- The HKE estimator proves to be **remarkably accurate**, even for long maturities, while the accuracy of other estimators is not sufficient for practical trading purposes.
- Computing the local volatility on an effective path only, such as in Gatheral, is too crude.
- Averaging the local volatility using Gaussian fluctuations around an effective path, like in our LN estimator, is not enough.
- **We need to average the local volatility using non-Gaussian fluctuations around an effective path** to get accurate results.

In terms of computational time:

- Our new HKE estimator cannot beat a PDE solver, if one wants to compute a whole implied volatility surface.
- However, if one needs to compute implied volatilities for only a few strikes and maturities, it is competitive. Dropping the order 1 in the HKE expansion makes our estimator slightly less accurate, but much faster.

Some references

📄 Berestycki H., Busca J., Florent I., *Asymptotics and calibration of local volatility models*, Quantitative Finance, volume 2, 61–69, 2002.

📄 Dupire B., *A new approach for understanding the impact of volatility on option prices*, slides of a presentation at the Risk Conference, October 30th, 1998.

📄 Gatheral J., *The volatility surface, a practitioner's guide*, Wiley, 2006.

📄 Henry-Labordère P., *Analysis, Geometry, and Modeling in Finance*, Chapman & Hall, CRC Financial Mathematics Series, 2008.

📄 Lee R., *Implied volatility: Statics, dynamics, and probabilistic interpretation*, in Ricardo Baeza-Yates, Joseph Glaz, Henryk Gzyl, Jürgen Hüsler and José Luis Palacios, eds, Recent Advances in Applied Probability. Berlin: Springer-Verlag, 2005.

📄 Reghai A., *The hybrid most likely path*, Risk Magazine, April 2006.