Volatility and Correlation Workshop (Part I)

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Volatility

Volatility : some definitions

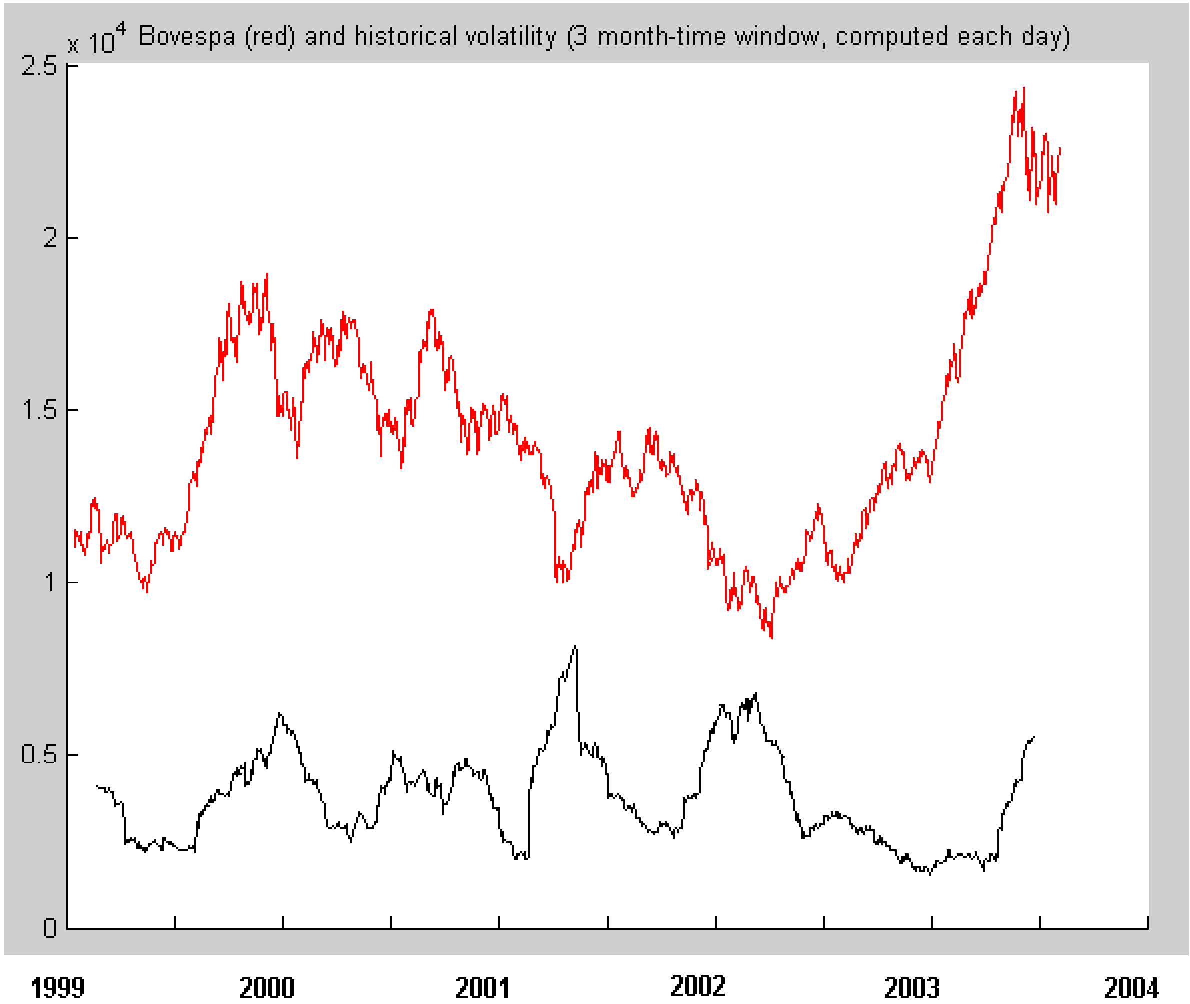
Historical volatility :

annualized standard deviation of the logreturns; measure of uncertainty/activity

Implied volatility :

measure of the option price given by the market

Historical volatility



Historical Volatility Estimation

- Textbook Method: annualized SD of $x_{t_i} \equiv \ln \dfrac{S_{t_i}}{S_{t_{i-1}}}$

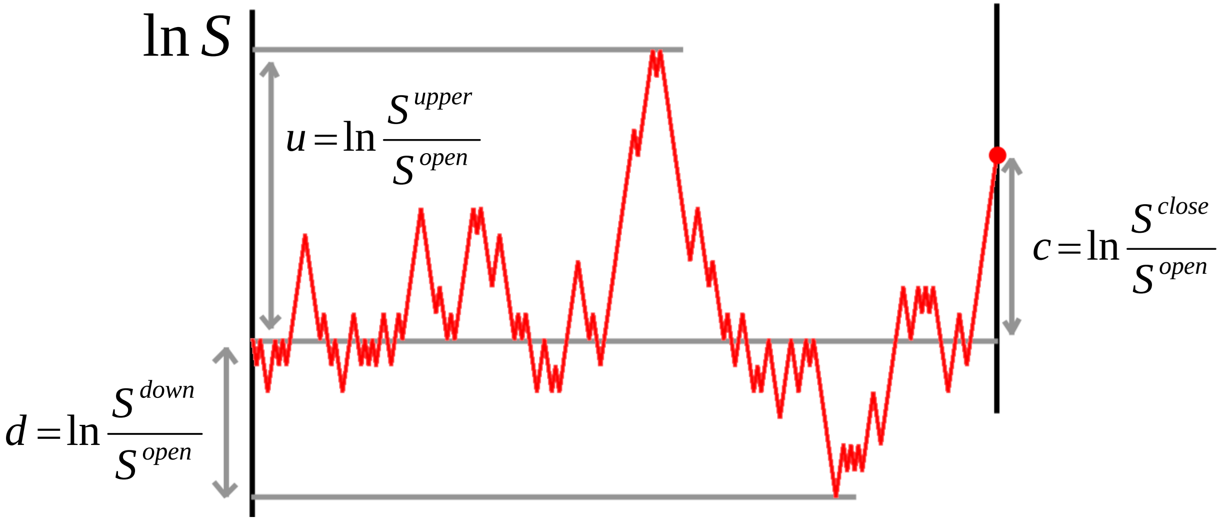
$$\sigma = \sqrt{\frac{252}{n-1}\sum_{i=1}^{n}(x_{t_i} - \overline{x})^2}$$

- Better Method: subtract RN drift instead of realized drift

- Textbook method slightly underestimates volatility

Estimates based on High/Low

- Commonly available information: open, close, high, low



- Captures valuable volatility information
- Parkinson estimate: $\sigma_P^2 = \dfrac{1}{4n\ln 2}\sum_{t=1}^{n}(u_t - d_t)^2$
- Garman-Klass estimate: $\sigma_{GK}^2 = \dfrac{0.5}{n}\sum_{t=1}^{n}(u_t - d_t)^2 - \dfrac{0.39}{n}\sum_{t=1}^{n} c_t^2$

GARCH Estimation

GARCH estimates a process for the instantaneous volatility

h_t is the variance of the next period return

GARCH(1,1):

$$h_t = \omega + \alpha\, r_{t-1}^2 + \beta\, h_{t-1} \text{ (estimated by maximum likelihood)}$$

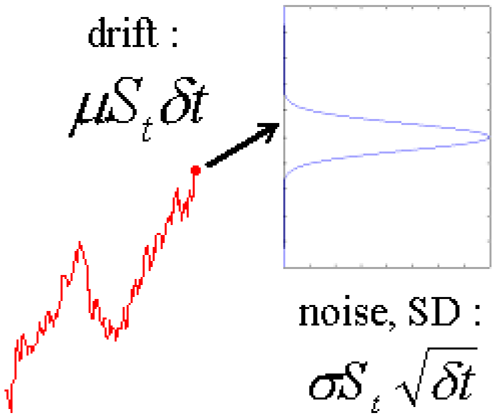
$$\alpha + \beta = 1 \text{(RiskMetrics)}: h_t = (1-\beta)\sum_{j=0}^{\infty} \beta^j\, r_{t-j-1}^2 \equiv \sigma_{\beta,t}^2$$

$$\alpha + \beta \neq 1\ (<1) \text{ with } \sigma^2 = \frac{\omega}{1-\alpha-\beta} : h_t = (1-\frac{\alpha}{1-\beta})\,\sigma^2 + (\frac{\alpha}{1-\beta})\,\sigma_{\beta,t}^2$$

Black-Scholes Model

If instantaneous volatility is constant :

$$\frac{dS}{S} = \mu dt + \sigma dW$$



Then call prices are given by :

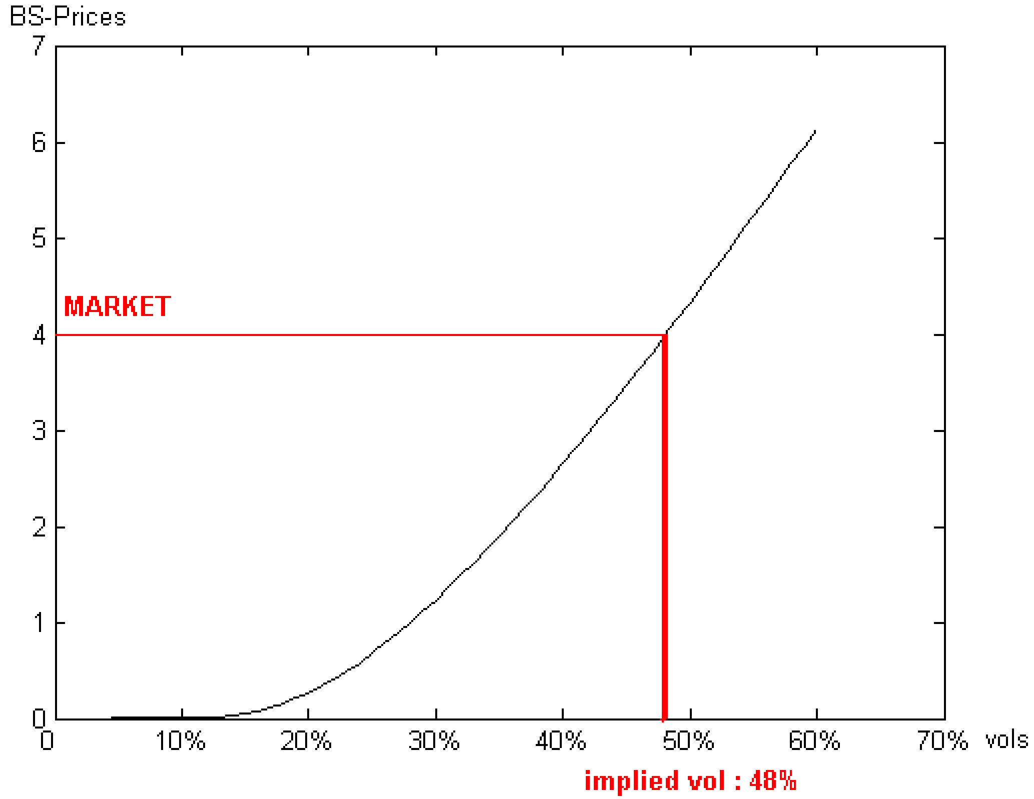
$$C_{BS} = S_0 N\left(\frac{1}{\sigma\sqrt{T}} \ln\left(\frac{S_0 \exp(rT)}{K}\right) + \frac{1}{2}\sigma\sqrt{T}\right)$$

$$- K \exp(-rT) N\left(\frac{1}{\sigma\sqrt{T}} \ln\left(\frac{S_0 \exp(rT)}{K}\right) - \frac{1}{2}\sigma\sqrt{T}\right)$$

No drift in the formula, only the interest rate r due to the hedging argument.

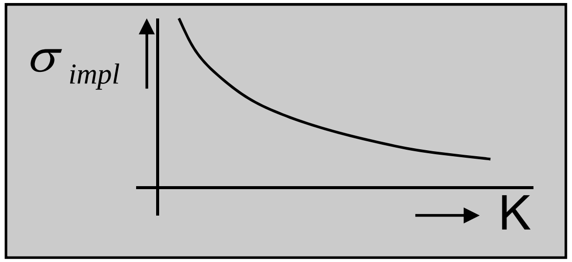
Implied volatility

Input of the Black-Scholes formula which makes it fit the market price :



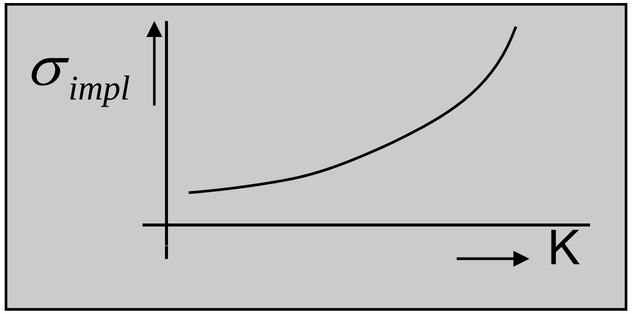
Market Skews

Dominating fact since 1987 crash: strong negative skew on Equity Markets

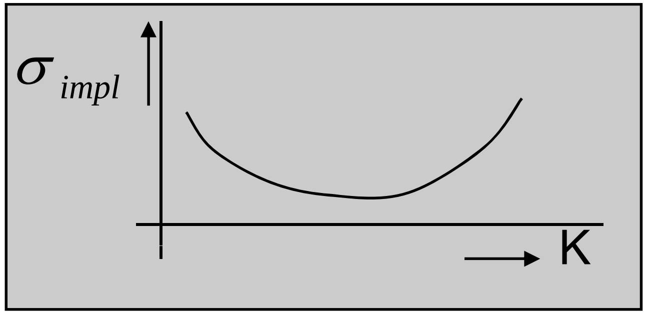


Not a general phenomenon

Gold:



FX:



We focus on Equity Markets

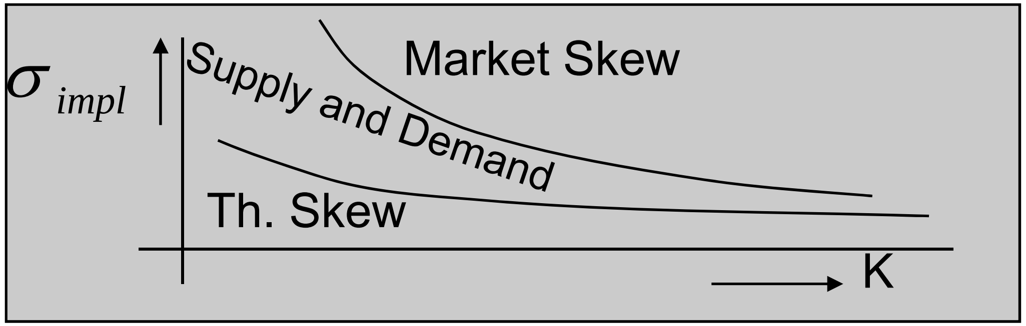
Skews

- Volatility Skew: slope of implied volatility as a function of Strike
- Link with Skewness (asymmetry) of the Risk Neutral density function φ ?

Moments	Statistics	Finance
1	Expectation	FWD price
2	Variance	Level of implied vol
3	Skewness	Slope of implied vol
4	Kurtosis	Convexity of implied vol

Why Volatility Skews?

- Market prices governed by
  - a) Anticipated dynamics (future behavior of volatility or jumps)
  - b) Supply and Demand

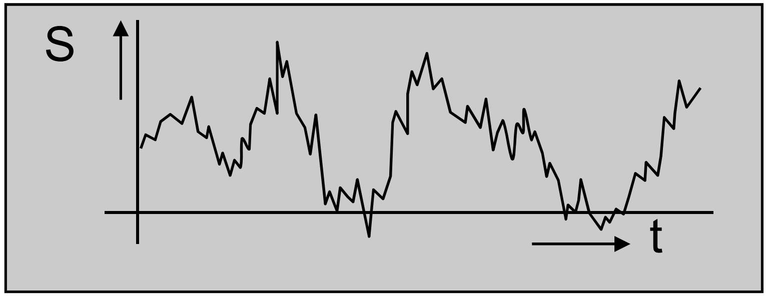


- To " arbitrage" European options, estimate a) to capture risk premium b)
- To "arbitrage" (or correctly price) exotics, find Risk Neutral dynamics calibrated to the market

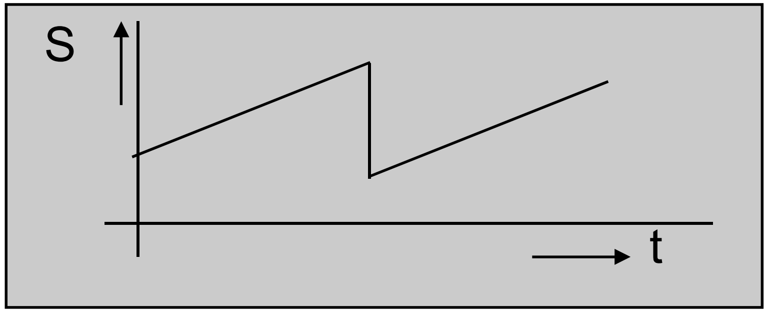
Modeling Uncertainty

Main ingredients for spot modeling

- Many small shocks: Brownian Motion (continuous prices)

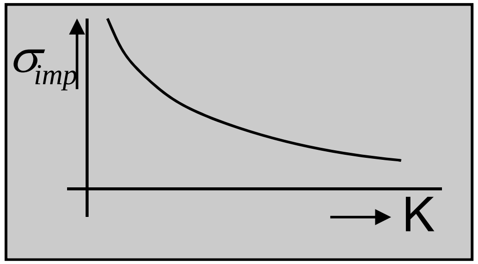


- A few big shocks: Poisson process (jumps)



2 mechanisms to produce Skews (1)

- To obtain downward sloping implied volatilities

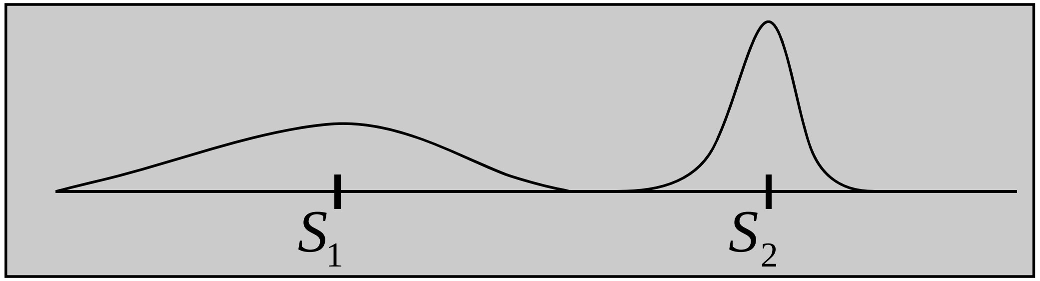


  - a) Negative link between prices and volatility
    - Deterministic dependency (Local Volatility Model)
    - Or negative correlation (Stochastic volatility Model)

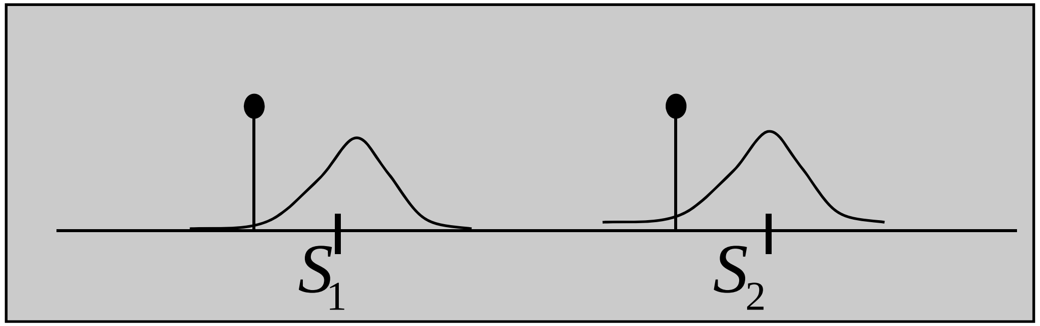
  - b) Downward jumps

2 mechanisms to produce Skews (2)

– a) Negative link between prices and volatility



– b) Downward jumps



Model Requirements

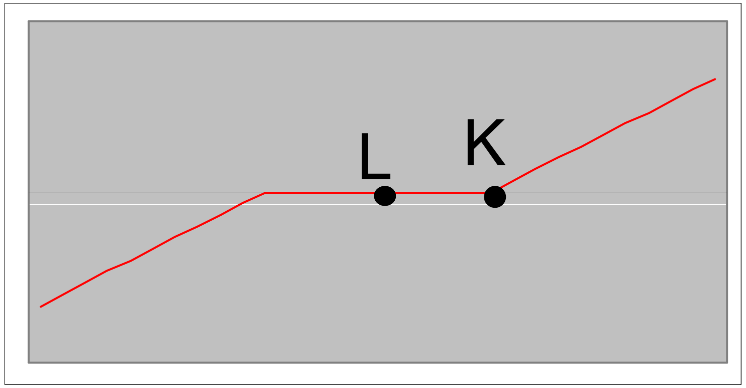
- Has to fit static/current data:
  - Spot Price
  - Interest Rate Structure
  - Implied Volatility Surface
- Should fit dynamics of:
  - Spot Price (Realistic Dynamics)
  - Volatility surface when prices move
  - Interest Rates (possibly)
- Has to be
  - Understandable
  - In line with the actual hedge
  - Easy to implement

Beyond initial vol surface fitting

- Need to have proper dynamics of implied volatility
  - Future skews determine the price of Barriers and OTM Cliquets
  - Moves of the ATM implied vol determine the Δ of European options
- Calibrating to the current vol surface do not impose these dynamics

Barrier options as Skew trades

- In Black-Scholes, a Call option of strike K extinguished at L can be statically replicated by a Risk Reversal



- Value of Risk Reversal at L is 0 for any level of (flat) vol
- Pb: In the real world, value of Risk Reversal at L depends on the Skew

A Brief History of Volatility

A Brief History of Volatility (1)

- $dS_t = \sigma \, dW_t^Q$: Bachelier 1900
- $\frac{dS_t}{S_t} = r \, dt + \sigma \, dW_t^Q$: Black-Scholes 1973
- $\frac{dS_t}{S_t} = r_t \, dt + \sigma(t) \, dW_t^Q$: Merton 1973
- $\frac{dS_t}{S_t} = (r - \lambda k) \, dt + \sigma \, dW_t^Q + dq$: Merton 1976
- $\begin{cases} \frac{dS_t}{S_t} = r \, dt + \sigma_t \, dW_t^Q \\ d\sigma_t^2 = a(\sigma_\infty^2 - \sigma_t^2)dt + \xi \, \sigma^\alpha \, dZ_t \end{cases}$: Hull&White 1987

A Brief History of Volatility (2)

$$\frac{dS_t}{S_t} = \sigma_t \, dW_t^Q$$

$$d\sigma_t^2 = 2\frac{\partial^2 L_T(t)}{\partial T^2} dt + \alpha \, dZ_t^Q$$

Dupire 1992, arbitrage model which fits term structure of volatility given by log contracts.

$$\frac{dS_t}{S_t} = r(t)\, dt + \sigma(S,t)\, dW_t^Q$$

$$\sigma^2(K,T) = 2\frac{\dfrac{\partial C}{\partial T} + rK\dfrac{\partial C}{\partial K}}{K^2 \dfrac{\partial^2 C_{K,T}}{\partial K^2}}$$

Dupire 1993, minimal model to fit current volatility surface

A Brief History of Volatility (3)

$$\begin{cases} \dfrac{dS_t}{S_t} = r\,dt + \sigma_t dW_t \\ d\sigma_t^2 = b(\sigma_\infty^2 - \sigma_t^2)dt + \beta\sigma_t\,dZ_t \end{cases}$$

Heston 1993,
semi-analytical formulae.

$$dV_{K,T} = \alpha_{K,T}\,dt + b_{K,T}\,dZ_t^Q$$

$$V_{K,T}\ : \text{instantaneous forward variance conditional to } S_T = K$$

Dupire 1996 (UTV),
Derman 1997,
stochastic volatility model which fits current volatility surface HJM treatment.

A Brief History of Volatility (4)

- Bates 1996, Heston + Jumps:

$$\begin{cases} \dfrac{dS_t}{S_t} = r\,dt + \sigma_t\,dZ_t + dq \\ d\sigma_t^2 = b(\sigma_\infty^2 - \sigma_t^2)dt + \beta\sigma_t\,dW_t \end{cases}$$

- Local volatility + stochastic volatility:
  - Markov specification of UTV
  - Reech Capital Model: f is quadratic
  - SABR: f is a power function

$$\frac{dS_t}{S_t} = r\,dt + \sigma_t\,f\left(S,t\right)dZ_t^Q$$

A Brief History of Volatility (5)

- Lévy Processes
- Stochastic clock:
  - VG (Variance Gamma) Model:
    - BM taken at random time g(t)
  - CGMY model:
    - same, with integrated square root process
- Jumps in volatility (Duffie, Pan & Singleton)
- Path dependent volatility
- Implied volatility modelling
- Incorporate stochastic interest rates
- n dimensional dynamics of volatility
- n assets stochastic correlation matrix

Local Volatility Model

From Simple to Complex

- How to extend Black-Scholes to make it compatible with market option prices?
  - Exotics are hedged with Europeans.
  - A model for pricing complex options has to price simple options correctly.

Black-Scholes assumption

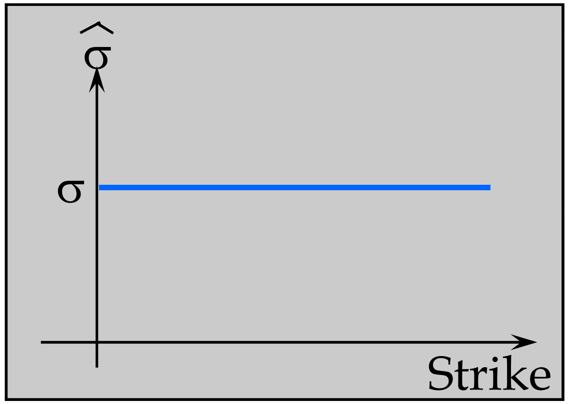
- BS assumes constant volatility

  => same implied vols for all options.

$$\frac{dS}{S} = \mu \ dt + \sigma \ dW$$

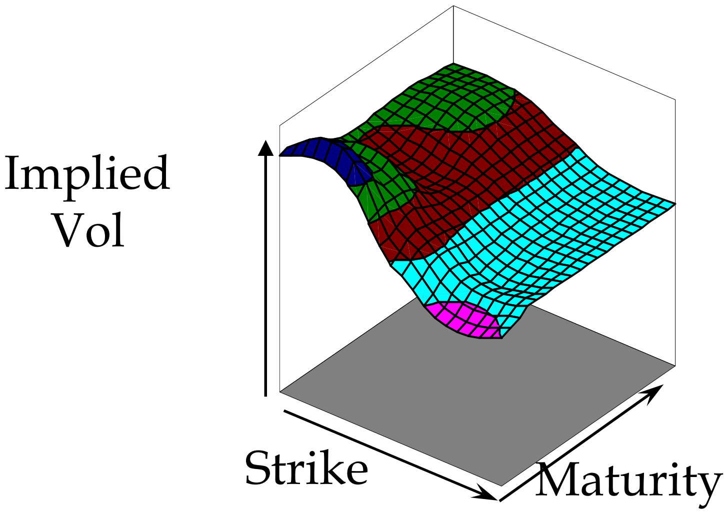
(instantaneous vol)

CALL PRICES

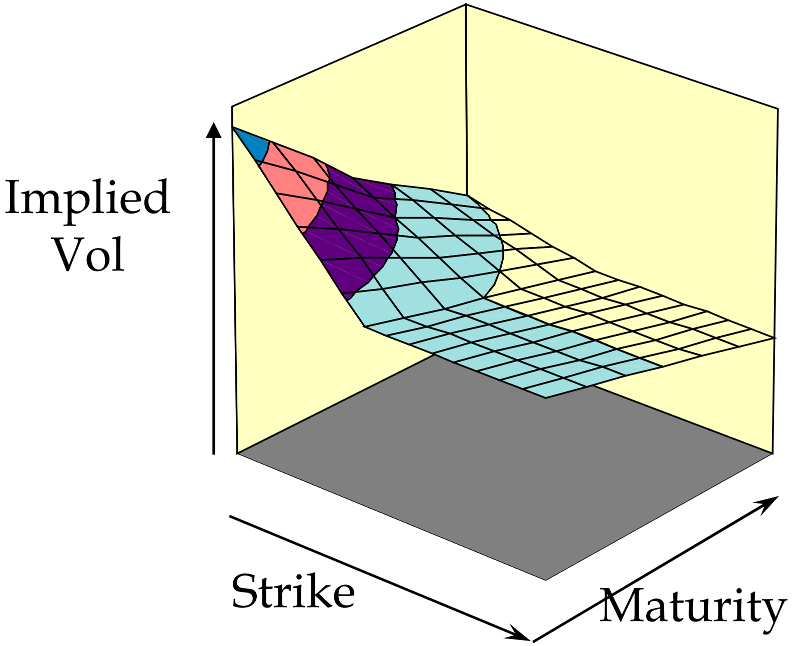


Black-Scholes assumption

- In practice, highly varying.



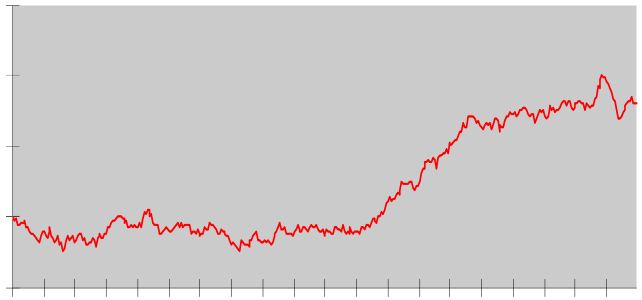
Nikkei



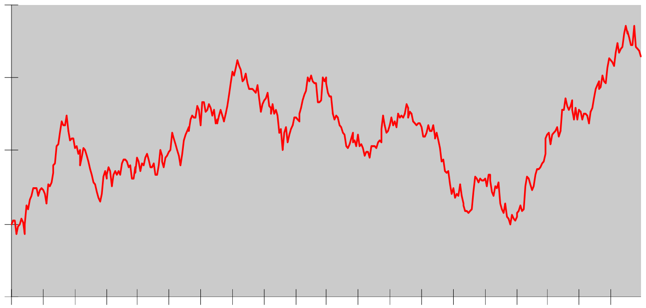
Japanese Government Bonds

Modeling Problems

- Problem: one model per option.
  - for C1 (strike 130) $\sigma = 10\%$



–for C2 (strike 80) $\sigma = 20\%$



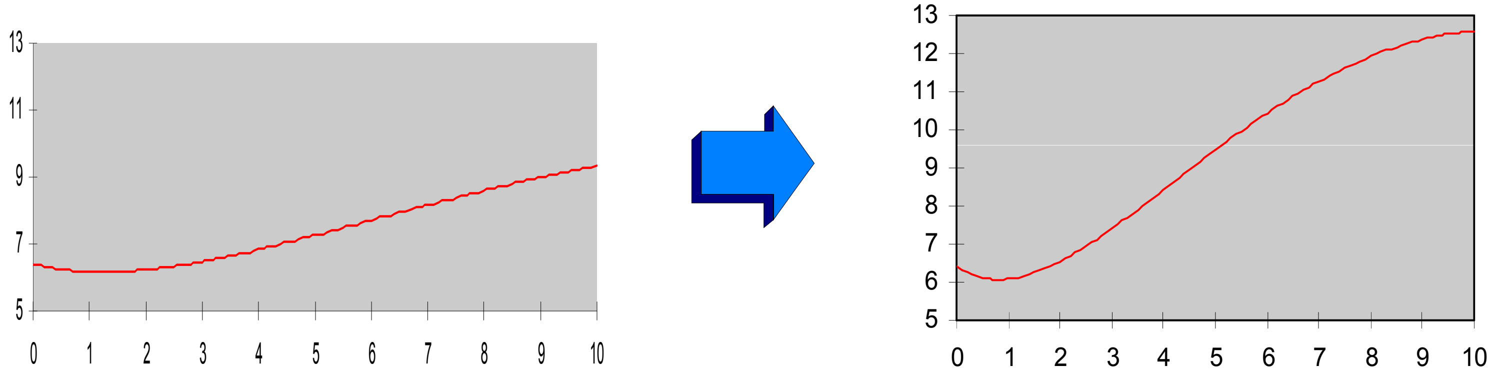
One Single Model

- We know that a model with $\sigma(S,t)$ would generate smiles.
  - Can we find $\sigma(S,t)$ which fits market smiles?
  - Are there several solutions?

ANSWER: One and only one way to do it.

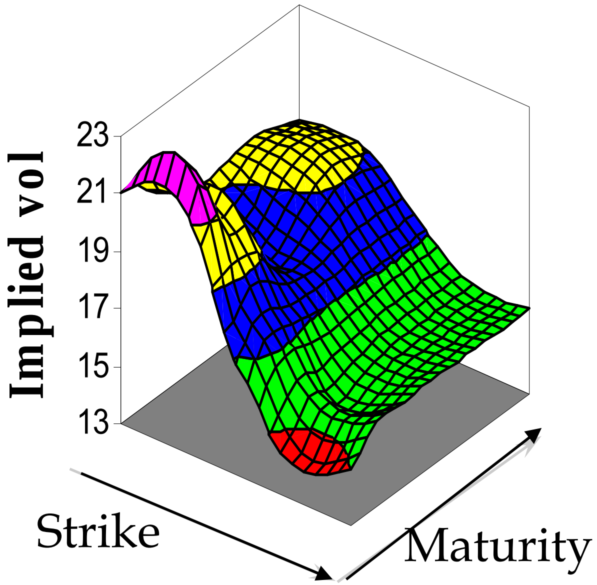
Interest rate analogy

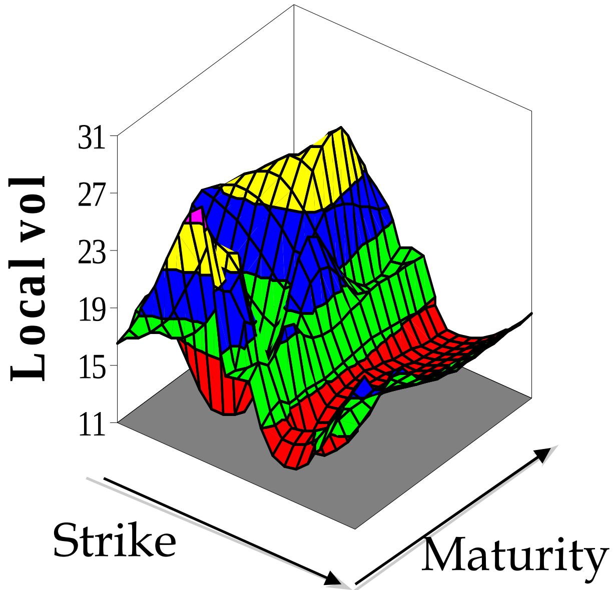
- From the current Yield Curve, one can compute an Instantaneous Forward Rate.



  - Would be realized in a world of certainty,
  - Are not realized in real world,
  - Have to be taken into account for pricing.

Volatility





Dream: from Implied Vols read Local (Instantaneous Forward) Vols

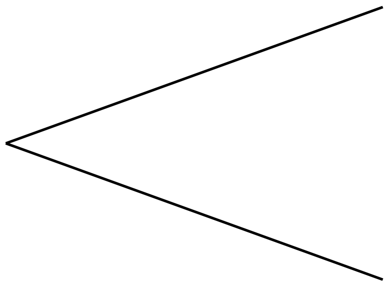
How to make it real?

Discretization

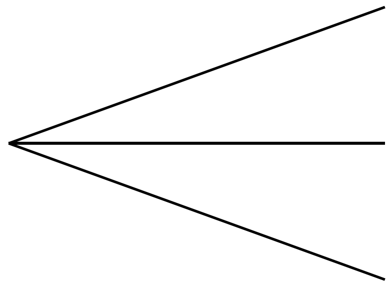
- Two approaches:
  - to build a tree that matches European options,
  - to seek the continuous time process that matches European options and discretize it.

Tree Geometry

Binomial

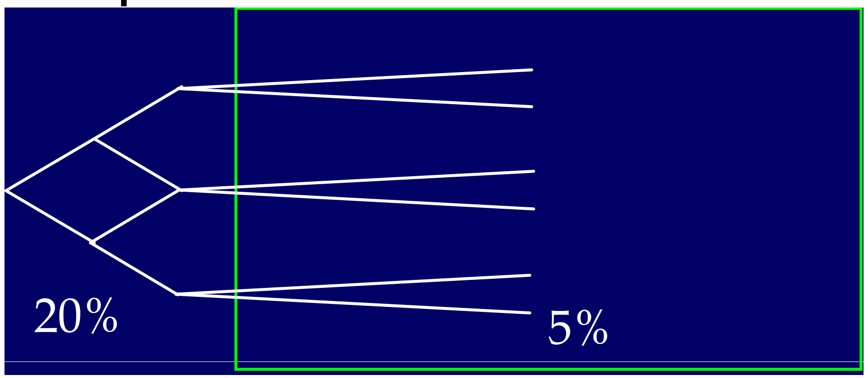


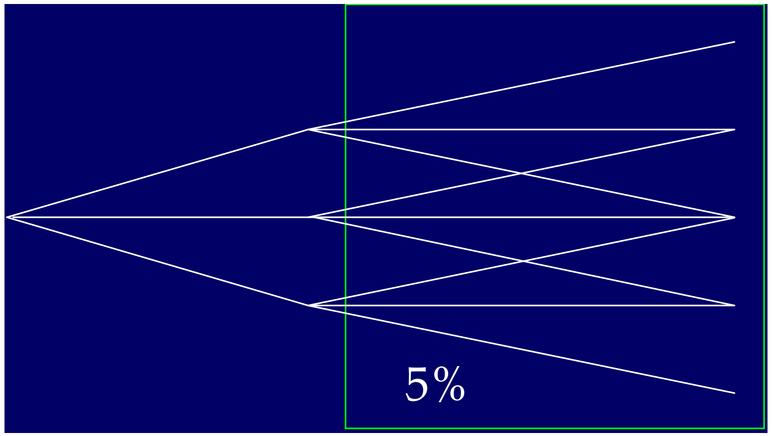
Trinomial



To discretize σ*(S,t)* TRINOMIAL is more adapted

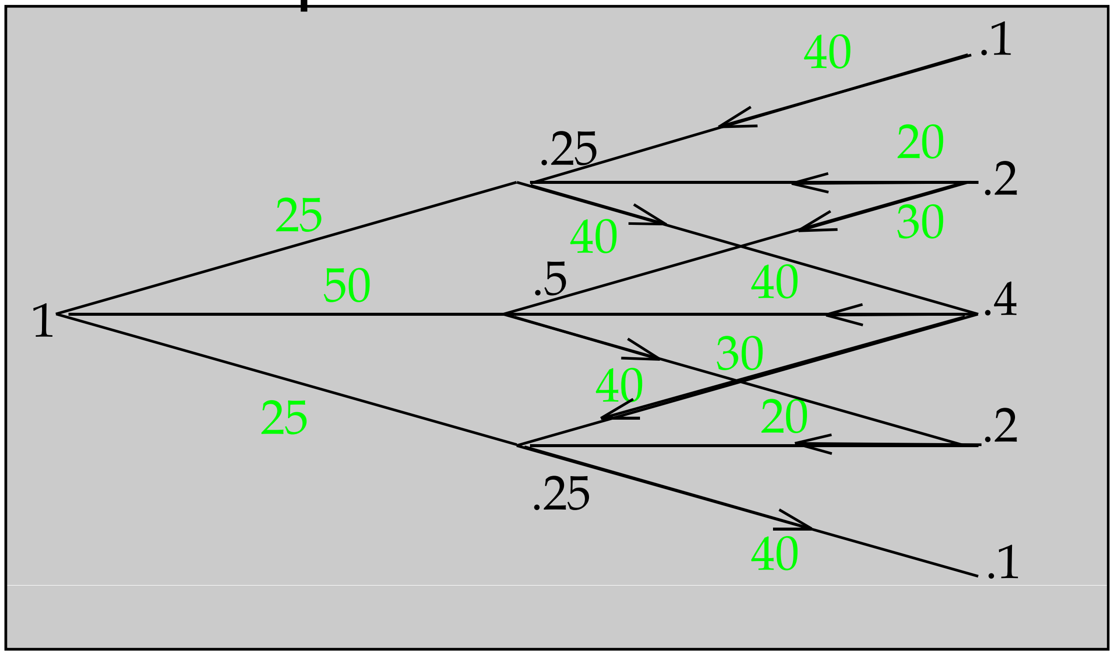
Example:





Tango Tree

- Rules to compute connections
  - price correctly Arrow-Debreu associated with nodes
  - respect local risk-neutral drift
- Example

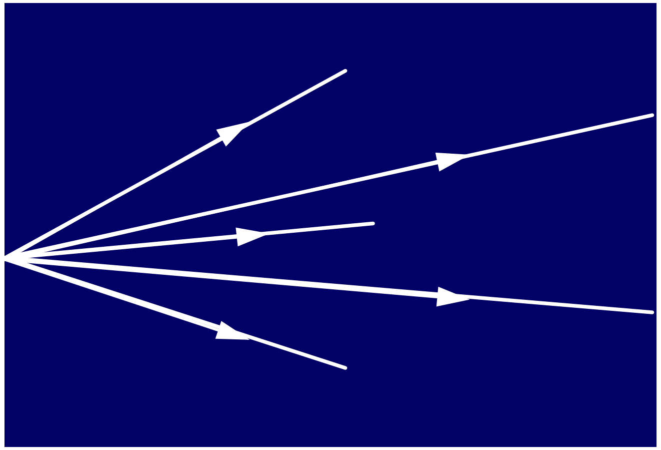


Continuous Time Approach

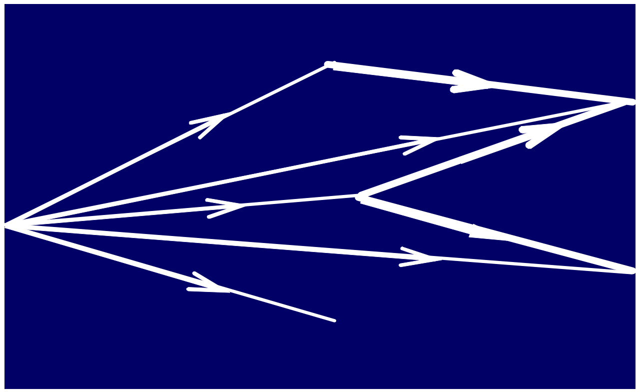
$$
\begin{tikzcd}
\text{Call Prices} \arrow[rr, leftrightarrow] \arrow[d, leftrightarrow] & & \text{Exotics} \arrow[d, leftrightarrow] \\
\text{Distributions} \arrow[rr, leftrightarrow, "?"] & & \text{Process}
\end{tikzcd}
$$

Distributions - Diffusion

Distributions

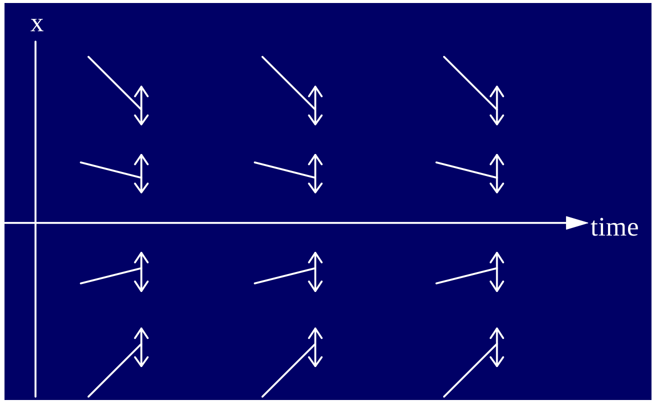


Process

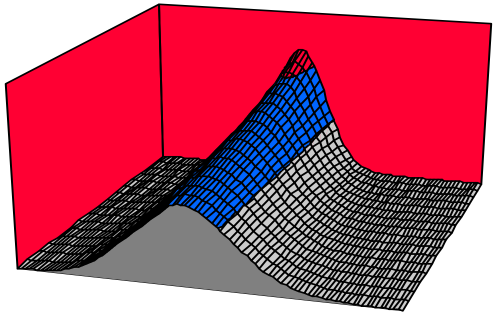


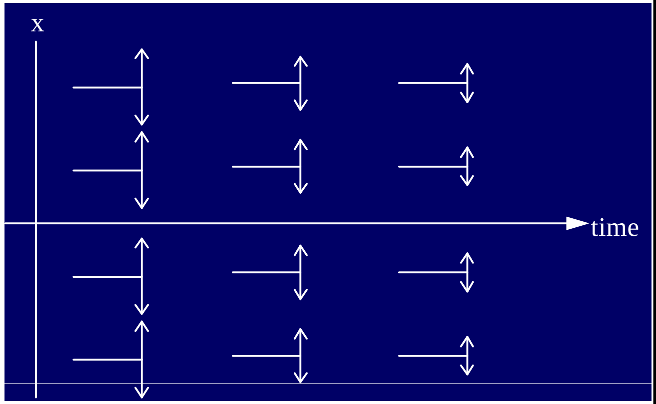
Distributions - Diffusion

- Two different diffusions may generate the same distributions



$$dx = -\lambda\, x\, dt + \sigma\, dW_t$$

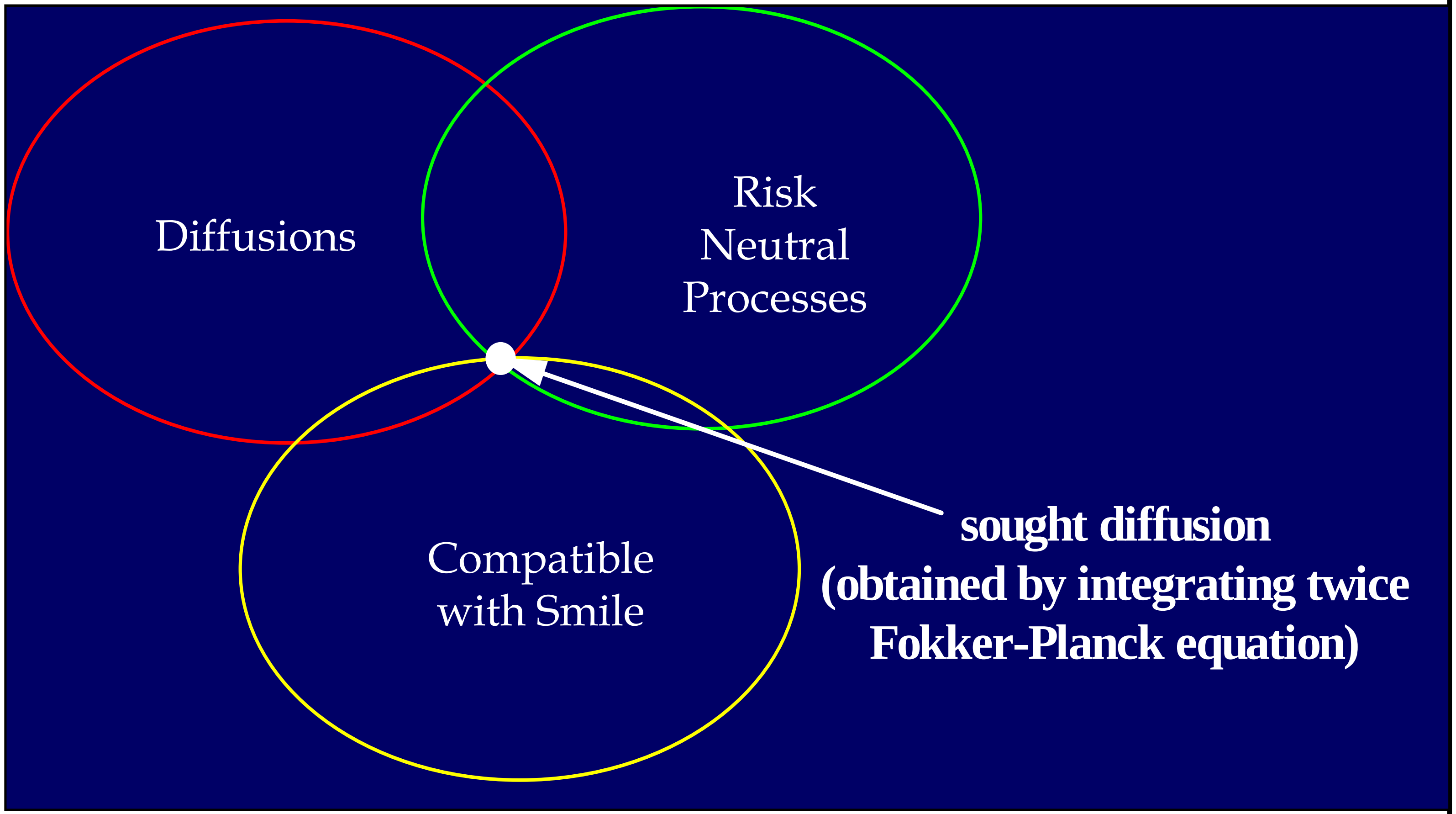




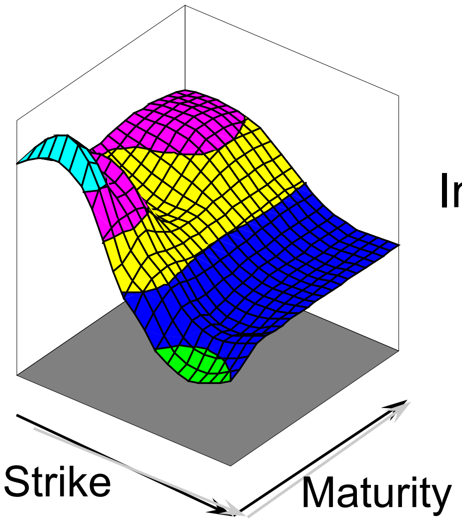
$$dx = b(t)\, dW_t$$

The Risk-Neutral Solution

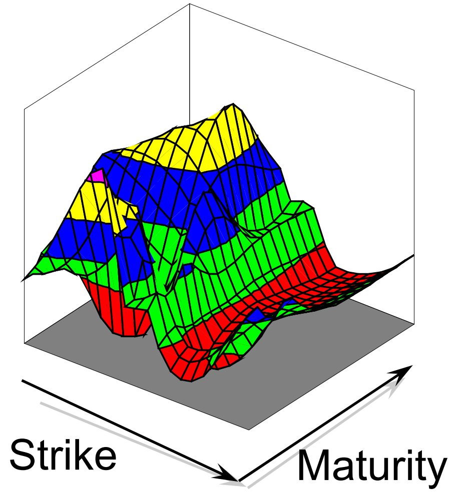
But if drift imposed (by risk-neutrality), uniqueness of the solution

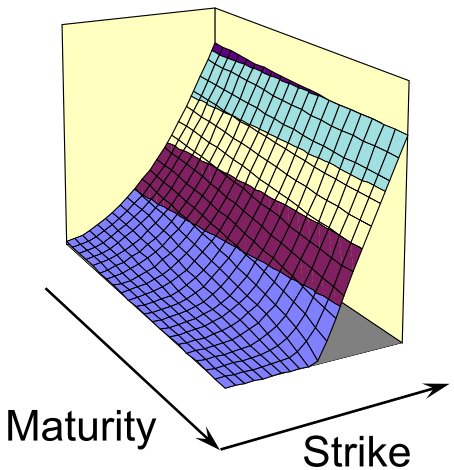


Continuous Time Analysis

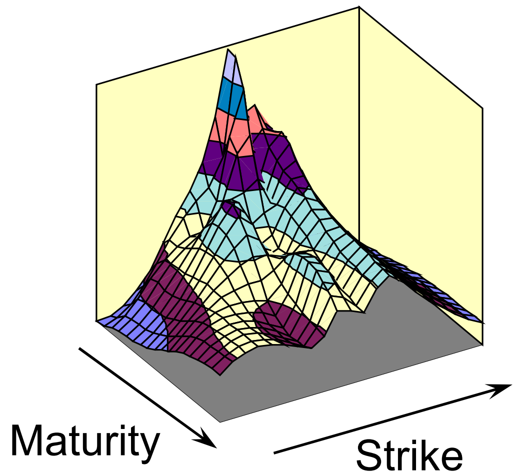


Implied Volatility ⟶ Local Volatility

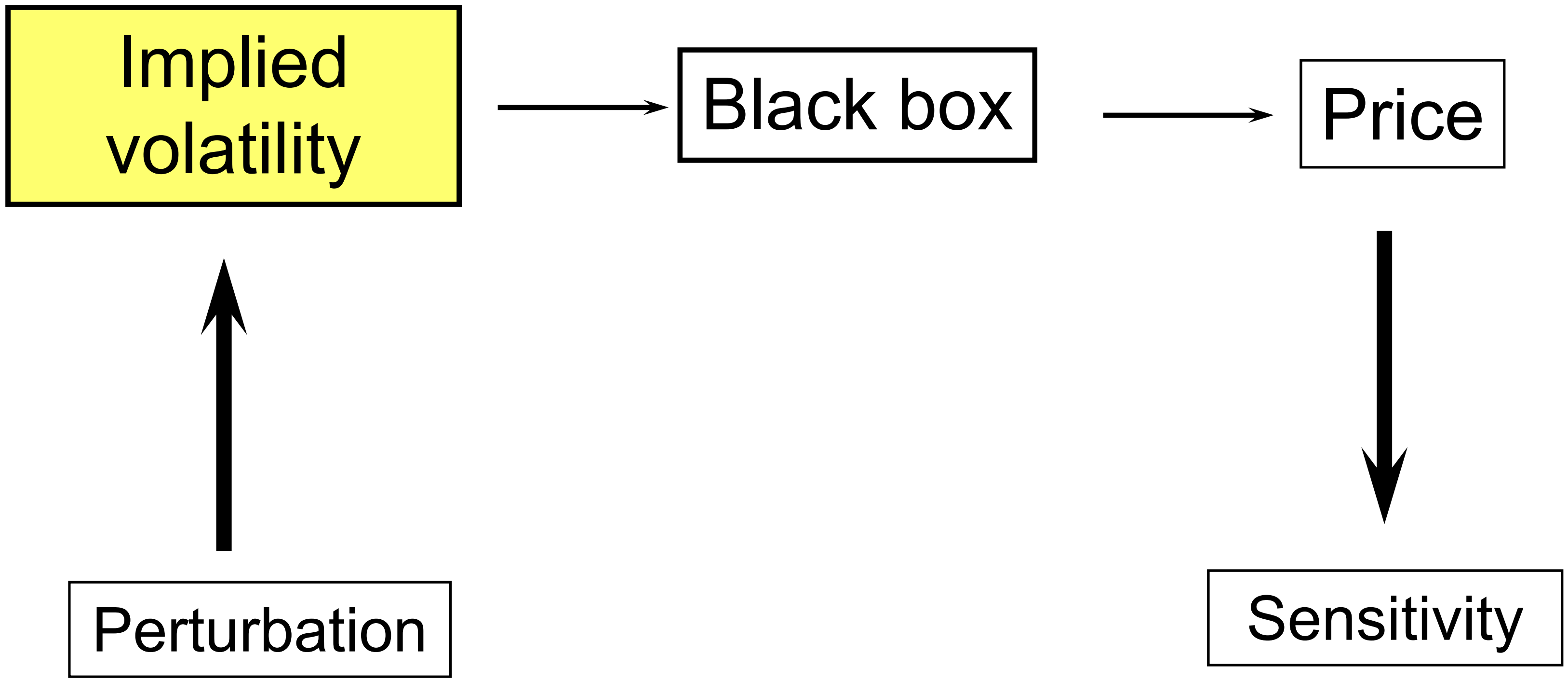




Call Prices ⟶ Densities



Implication : risk management



Forward Equations (1)

- BWD Equation:

  price of one option $C(K_0, T_0)$ for different (S, t)

- FWD Equation:

  price of all options $C(K, T)$ for current (S_0, t_0)

- Advantage of FWD equation:
  - If local volatilities known, fast computation of implied volatility surface,
  - If current implied volatility surface known, extraction of local volatilities,
  - Understanding of forward volatilities and how to lock them.

Forward Equations (2)

- Several ways to obtain them:
  - Fokker-Planck equation:
    - Integrate twice Kolmogorov Forward Equation
  - Tanaka formula:
    - Expectation of local time
  - Replication
    - Replication portfolio gives a much more financial insight

Fokker-Planck

- Assume $dx = b(x,t)dW$
- Fokker-Planck Equation: $\dfrac{\partial \varphi}{\partial T} = \dfrac{1}{2}\dfrac{\partial^2 (b^2 \varphi)}{\partial K^2}$

where $\varphi(x,t;K,T)$ is the transition density

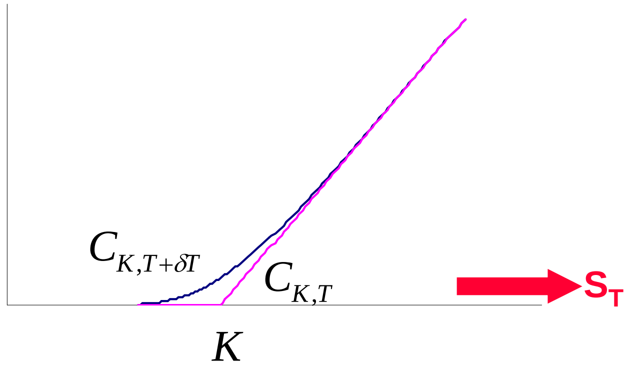
As $\varphi = \dfrac{\partial^2 C}{\partial K^2}$

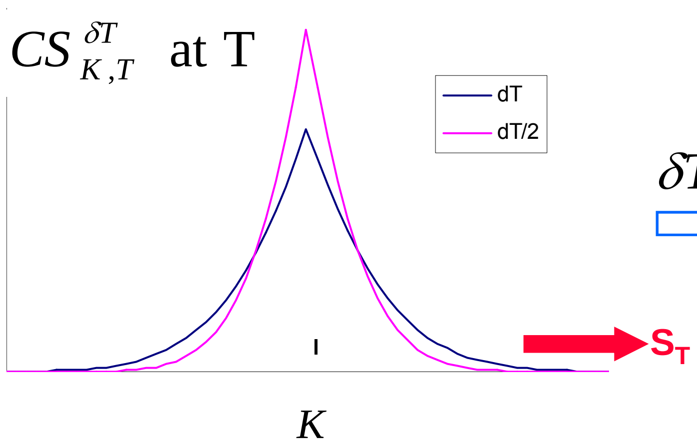
$$\frac{\partial^2\left(\frac{\partial C}{\partial T}\right)}{\partial K^2} = \frac{\partial\left(\frac{\partial^2 C}{\partial K^2}\right)}{\partial T} = \frac{1}{2}\frac{\partial^2\left(b^2 \frac{\partial^2 C}{\partial K^2}\right)}{\partial K^2}$$

- Integrating twice w.r.t. K: $\dfrac{\partial C}{\partial T} = \dfrac{b^2(K,T)}{2}\dfrac{\partial^2 C}{\partial K^2}$

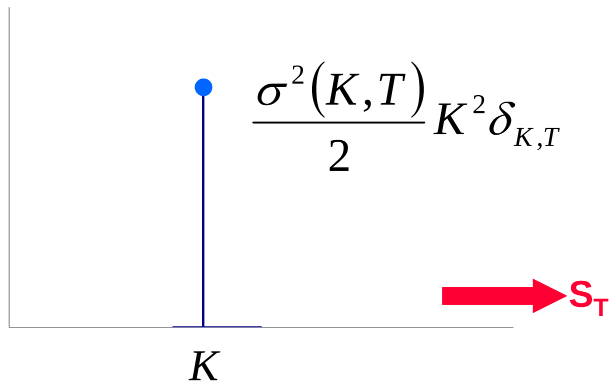
FWD Equation: dS/S = σ(S,t) dW

Define $CS_{K,T}^{\delta T} \equiv \dfrac{C_{K,T+\delta T} - C_{K,T}}{\delta T}$



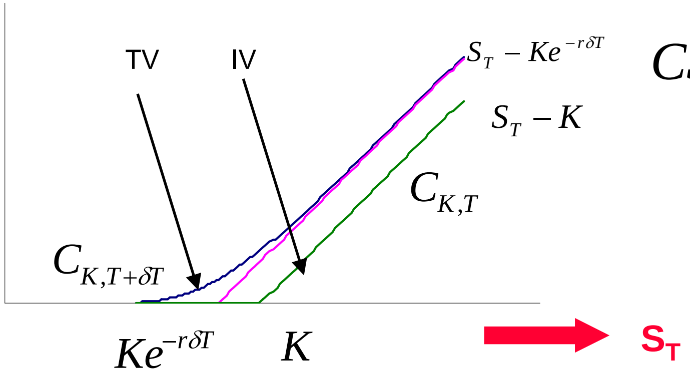


$\delta T \to 0$



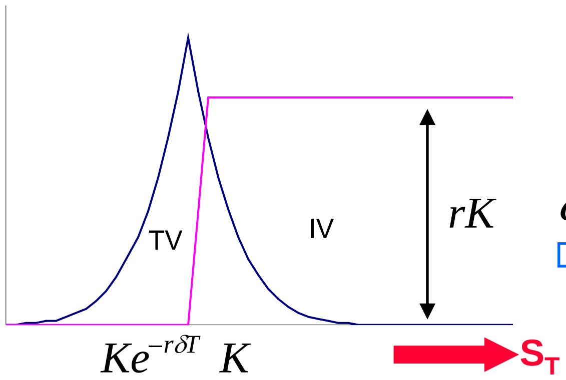
Equating prices at t_0: $\boxed{\dfrac{\partial C}{\partial T} = \dfrac{\sigma^2(K,T)}{2} K^2 \dfrac{\partial^2 C}{\partial K^2}}$

FWD Equation: dS/S = r dt + σ(S,t) dW

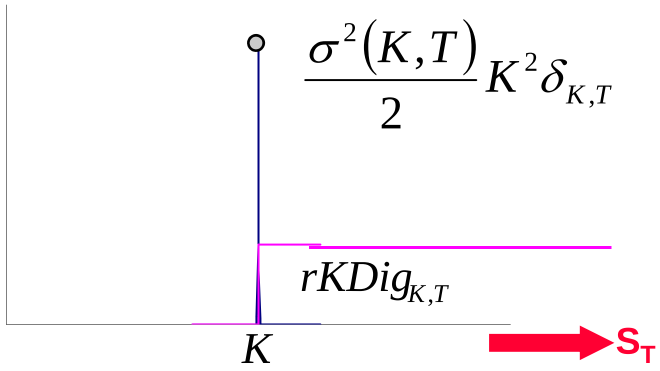


$CS^{\delta T}_{K,T}$ at T = Time Value + Intrinsic Value

(Strike Convexity) (Interest on Strike)



$\delta T \to 0$



Equating prices at t_0: $\boxed{\frac{\partial C}{\partial T} = \frac{\sigma^2(K,T)}{2} K^2 \frac{\partial^2 C}{\partial K^2} - rK \frac{\partial C}{\partial K}}$

FWD Equation: $dS/S = r_t\, dt + \sigma_t\, dW$

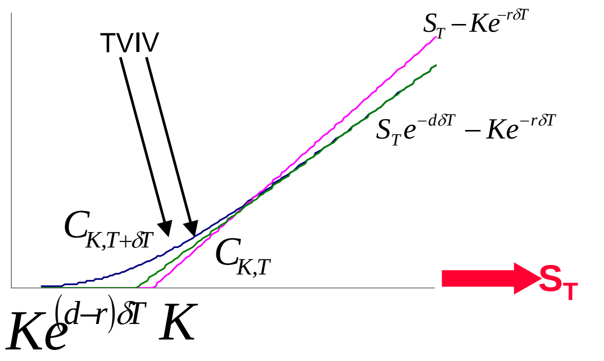
- The limit of Calendar Spreads gives at T :

$$\frac{1}{2}\sigma_T^2 K^2 \delta_{K,T} + r_T K Dig_{K,T}$$

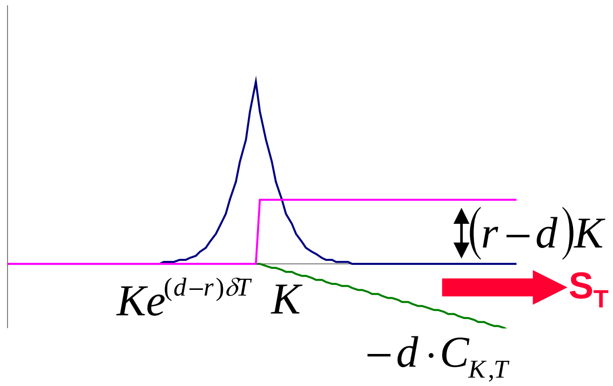
- Equating prices at t_0 :

$$\boxed{\frac{\partial C}{\partial T} = \frac{K^2}{2} E\left[\sigma_T^2 \middle| S_T = K\right]\frac{\partial^2 C}{\partial K^2} - KE\left[r_T \middle| S_T > K\right]\frac{\partial C}{\partial K}}$$

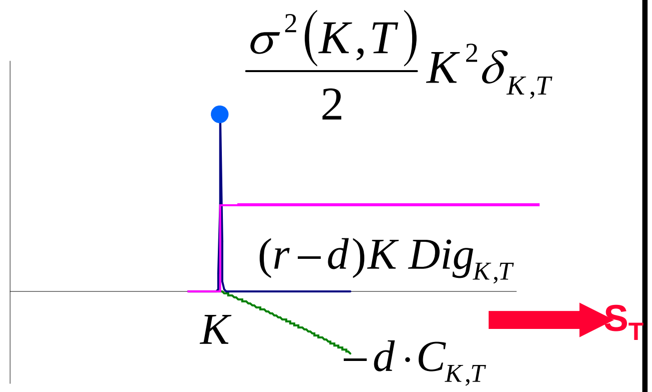
FWD Equation: dS/S = (r-d) dt + σ(S,t) dW



$$CS^{\delta T}_{K,T} \text{ at T} = \text{TV + Interests on K} \\ \text{– Dividends on S}$$



$\delta T \to 0$



Equating prices at t_0: $\boxed{\dfrac{\partial C}{\partial T} = \dfrac{\sigma^2(K,T)}{2} K^2 \dfrac{\partial^2 C}{\partial K^2} - (r-d)K \dfrac{\partial C}{\partial K} - d \cdot C}$

Stripping Formula

$$\frac{\partial C}{\partial T} = \frac{\sigma^2(K,T)K^2}{2}\frac{\partial^2 C}{\partial K^2} - (r-d)K\frac{\partial C}{\partial K} - d \cdot C$$

– If $\sigma(K,T)$ known, quick computation of all $C_{K,T}(S_0,t_0)$ today,

– If all $C_{K,T}(S_0,t_0)$ known:

$$\sigma(K,T) = \sqrt{2\frac{\frac{\partial C}{\partial T} + (r-d)K\frac{\partial C}{\partial K} + dC}{K^2\frac{\partial^2 C}{\partial K^2}}}$$

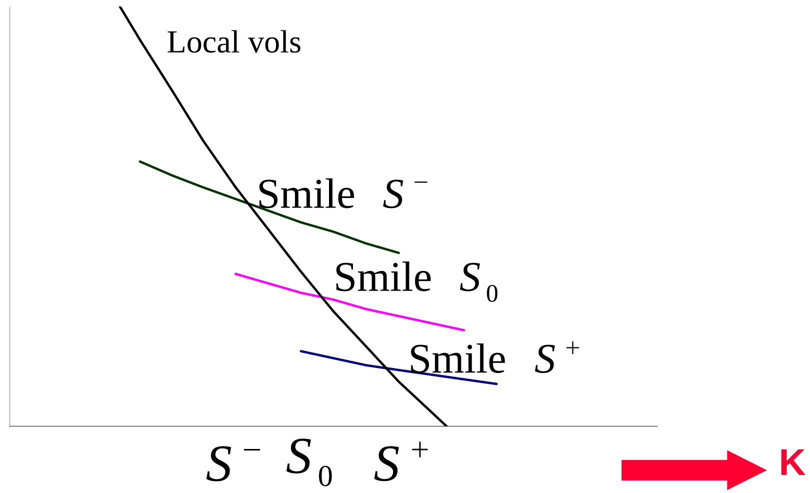
Local volatilities extracted from vanilla prices

and used to price exotics.

Smile dynamics: Local Vol Model (1)

- Consider, for one maturity, the smiles associated to 3 initial spot values

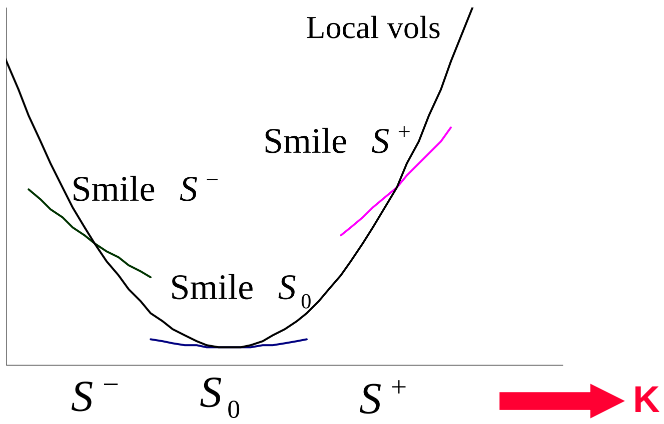
Skew case



– ATM short term implied follows the local vols
– Similar skews

Smile dynamics: Local Vol Model (2)

- Pure Smile case



  - ATM short term implied follows the local vols
  - Skew can change sign

Summary of LVM Properties

Σ_0 is the initial volatility surface

- $\sigma(S,t)$ compatible with $\Sigma_0 \Leftrightarrow \sigma =$ local vol
- $\sigma(\omega)$ compatible with $\Sigma_0 \quad \Leftrightarrow E\left[\sigma_T^2 \middle| S_T = K\right] = \sigma_{local}^2(K,T)$
- $\hat{\sigma}_{K,T}$ deterministic function of (S,t), no jumps

$\Leftrightarrow$ future smile = FWD smile from local vol

LVM Implementation

LVM Implementation

First step: extract local vols from option prices

- Obtain smooth implied vols by inter/extrapolation and strip them into local vols, or
- Fit a model and strip it into local vols, or
- Parameterize local vols and find best fit to market prices.

Second step: use of local vols for pricing.

According to the products:

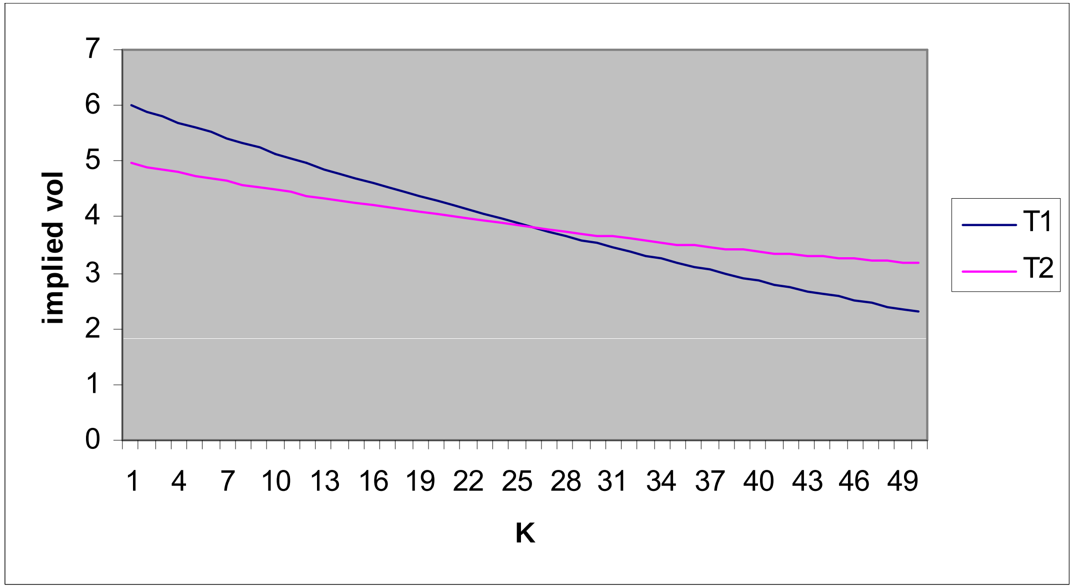
- Finite differences
- Monte Carlo simulations

Obtaining Local Volatilities

Smooth Implied vols I

Simple method:

- Cubic spline interpolation in Strikes & Maturities of BS vols
- May lead to arbitrage, for instance extrapolation in K for 2 maturities.



Inter/extrapolate carelessly
from arbitrage free price may create arbitrage

Roger Lee's Moments Formula

- Volatility extrapolation requires attention
- $K_1 < K_2 \;\Rightarrow\; C_{K_2}\left(\hat{\sigma}_{K_2}\right) \leq C_{K_1}\left(\hat{\sigma}_{K_1}\right) \;\Rightarrow\; \hat{\sigma}$ cannot increase too fast
- Roger Lee's Moment Formula:

$$\limsup_{K\infty} \frac{T\hat{\sigma}^2(K)}{\ln K} = \beta \in [0,2] \text{ with } \frac{1}{2\beta} + \frac{\beta}{8} - \frac{1}{2} = \sup\left\{p : \mathbf{E} S_T^{1+p} < \infty\right\}$$

$\beta = 0 \;\Leftrightarrow$ every moment of *S* is finite

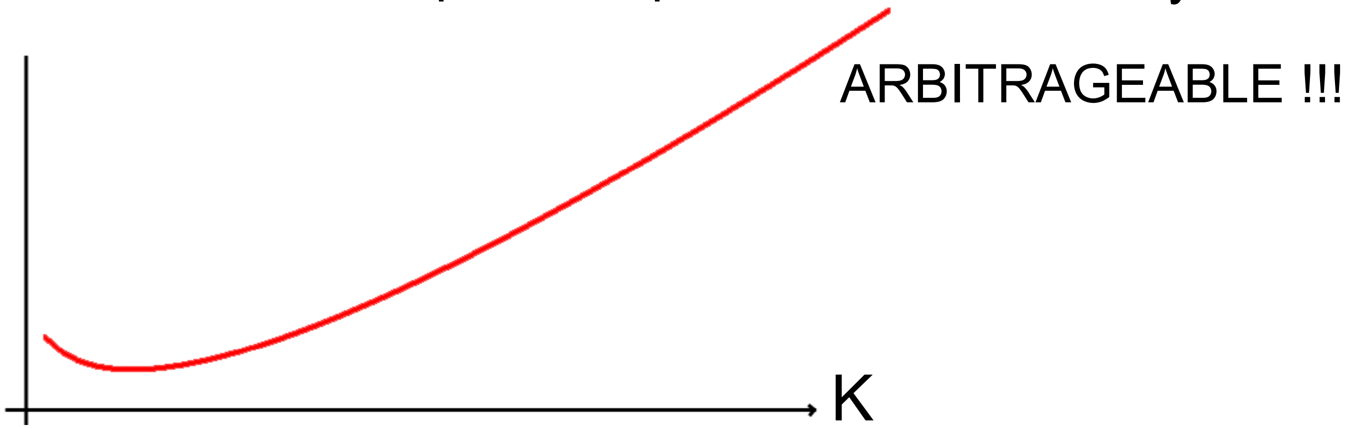
Benaim-Friz Sharpening

- f is the density of the returns X
- If $\ln f$ is regularly varying and $\exists \varepsilon > 0 : \mathbf{E}\left[e^{(1+\varepsilon)X}\right] < \infty$ then

$$\frac{\hat{\sigma}^2(K)}{\ln K} \sim \Psi\left(-1 - \frac{\ln f}{\ln K}\right) \quad \text{where } \Psi(x) = 2 - 4\left(\sqrt{x^2 + x} - x\right)$$

$$\sim -\frac{\ln K}{2 \ln f(K)} \quad \text{if this tends to zero}$$

- In particular DO NOT extrapolate implied volatilities linearly!



Implied vol extrapolation

In summary

- Do not extrapolate flatly
- Do not extrapolate linearly

Instead,

- Extrapolate implied variance expressed in log moneyness (e.g. hyperbolic SVI)

Fit to CDS (lump mass at 0) imposes the slope at 0

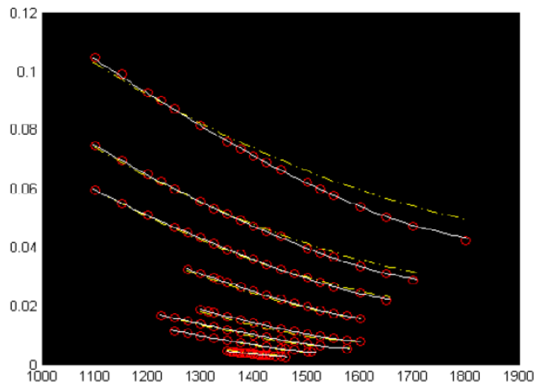
Building a good implied vol surface

To ensure :

- Accurate fit
- Smooth surface
- No arbitrage

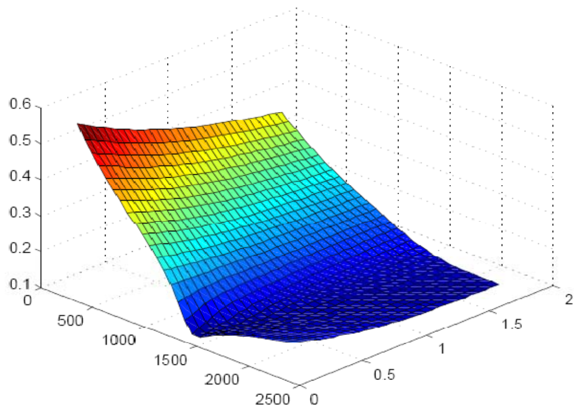
1. Calibrate to a good base model (Heston for instance)
2. C^∞ non parametric strike interpolation of the residual (Market implied – Heston implied)
3. Smooth maturity interpolation of the residual

S&P 500 (May 9, 2008)

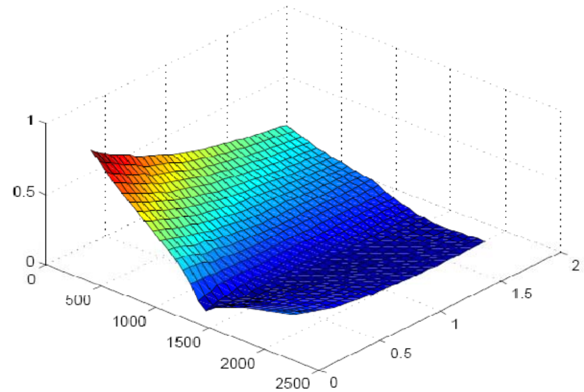


Implied variance for different maturities

- Bubbles : market
- Dashed line : Heston
- Solid line : Heston + residuals



Implied volatility



Local volatility

Model fitting

- Models provide smooth and arbitrage free prices
- Calibrate a model to the market prices and convert the model prices/volatilities into local volatilities (best fit, not perfect fit)
- Examples:

  - mix of lognormals

  - Heston

  - slices of VG

Implied Vols → Local Vols

The formula : $$\sigma^2(K,T) = 2\frac{\dfrac{\partial C}{\partial T} + rK\dfrac{\partial C}{\partial K}}{K^2\dfrac{\partial^2 C}{\partial K^2}}$$

Behaves badly far from the money

Better express as function of implied vol $\hat{\sigma}$:

$$\sigma^2(K,T) = \frac{2T\hat{\sigma}\dfrac{\partial\hat{\sigma}}{\partial T} + \hat{\sigma}^2}{\left(1 - x\dfrac{1}{\hat{\sigma}}\dfrac{\partial\hat{\sigma}}{\partial T}\right)^2 + T\dfrac{1}{\hat{\sigma}}\dfrac{\partial^2\hat{\sigma}}{\partial x^2} - \dfrac{1}{4}T^2\hat{\sigma}^2\dfrac{\partial^2\hat{\sigma}}{\partial x^2}}$$

Where $x \equiv \ln\dfrac{K}{FWD}$

Parametric Local Vols

Alternative approach

- Parameterize local vols over each maturity interval.
- For a set of parameters θ
  - Compute option price from the FWD PDE:

    $$C_i(\theta) \rightarrow \hat{\sigma}_i(\theta)$$

  - Compute quadratic error term

    $$Error(\theta) \equiv \sum w_i (\hat{\sigma}_i(\theta) - \hat{\sigma}_i)^2$$

- Minimize Error(θ) globally or by bootstrap from short maturities.

Local volatility parametrization

- Global functional form (bad)
- Local in time and price (includes double cubic splines on a time/price grid)
- Maturity slices
  - a) time independent function of price between two maturity dates
  - b) function of price at maturity dates + time interpolation

Calibration

- Bootstrap calibration solves maturity by maturity starting from the first one
- Global calibration solves a higher dimensional problem
- The error function is computed with the forward PDE. It should include regularization term in case of bootstrap
- Popular calibration algorithm is Levenberg-Marquardt

Getting Local Vols

- Interpolation is suitable for Index options.

- Parametric form is better for individual stocks, notably to handle discrete dividends because of the arbitrage:

  $$C_{K,T_D^-} = C_{K-D,T}$$

  for a dividend D falling at time T_D.

- Other methods :
  - Entropy optimization
  - Tichonov regularization

Pricing with Local Volatilities

Numerical Techniques

- Barrier, American type options

Finite Difference applied to BWD PDE :

$$\frac{\partial C}{\partial t} = r\left(C - \frac{\partial C}{\partial S}.S \right) - \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2(S,t)$$

- Path dependent options :

Monte Carlo simulation applied to SDE

$$dS_t = rSdt + \sigma(S,t)SdW_t$$

Second order Milstein scheme better than first order Euler scheme

Finite differences

- Discretization of the PDE:

$$\frac{\partial C}{\partial t} = r\left(C - \frac{\partial C}{\partial S}.S\right) - \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2(S,t)$$

by implicit or Crank-Nicholson methods

In the presence of kinks, start with a few steps of implicit or analytical valuation

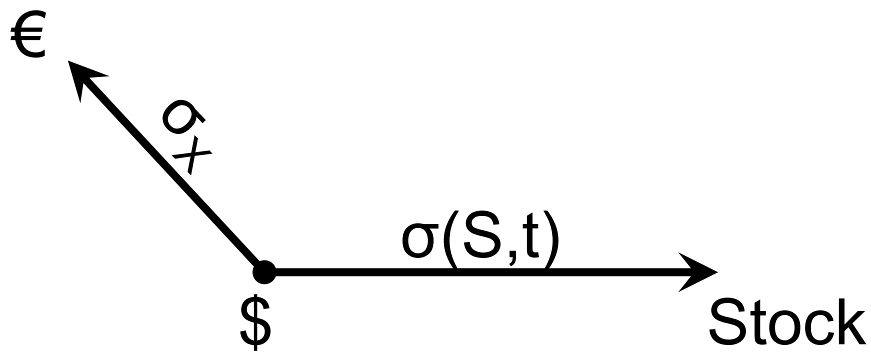
With high drift/volatility ratio: upwind or streamline diffusion schemes

LVM Quanto

S: Price of a stock in \$: $$\frac{dS}{S} = r_{\$}(t)dt + \sigma(S,t)dW_t$$

X: Price of a € in \$: $$\frac{dX}{X} = \left(r_{€}(t) - r_{\$}(t)\right)dt + \sigma_X dZ_t$$

Option quantoed in €: $\left(S_T - K\right)^+_{€}$



LVM Quanto

Price at t=0 in € = $e^{-\int_0^T r_€(t)\,dt} E^{Q_€}\left[(S_T - K)^+\right]$

Dynamics of S under $Q_€$: Girsanov Theorem ➔

$$E^{Q_€}[dW] = \langle dW, \sigma_X dZ \rangle = \rho\sigma_X dt$$

$$\frac{dS}{S} = \left(r_\$(t) + \rho\sigma_X \sigma(S,t)\right)dt + \sigma(S,t)dW^€$$

$$W^€ : Q_€ \ BM$$

- Add covariance term in drift of S
- Discount at € rate

Stochastic Volatility Models

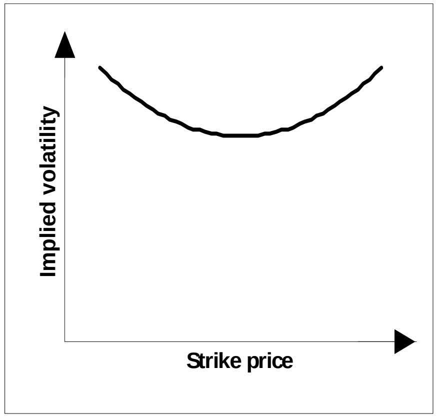
Hull & White

- Stochastic volatility model **Hull&White (87)**

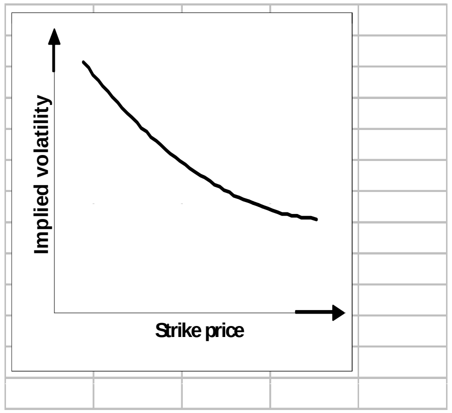
$$\frac{dS_t}{S_t} = r\,dt + \sigma_t\,dW_t^P$$

$$d\sigma_t = \alpha\,dt + \beta\,dZ_t^P$$

- Incomplete model, depends on risk premium
- Does not fit market smile



$\rho_{Z,W} = 0$



$\rho_{Z,W} < 0$

Heston Model

$$\begin{cases} \dfrac{dS}{S} = \mu\, dt + \sqrt{v}\, dW \\ dv = \kappa\left(v_\infty - v\right)dt + \eta\sqrt{v}dZ \quad \left\langle dW, dZ\right\rangle = \rho\, dt \end{cases}$$

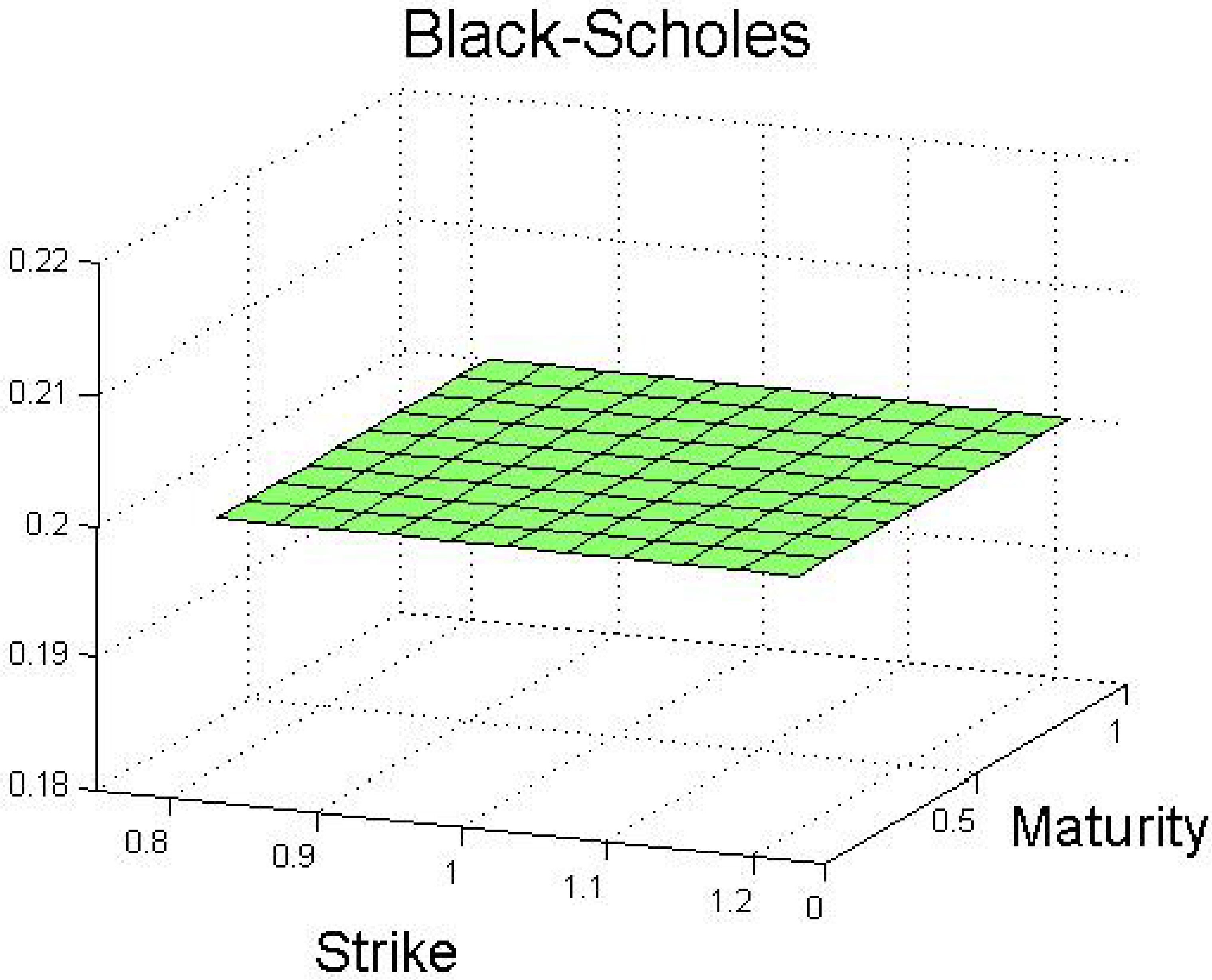
Solved by Fourier transform:

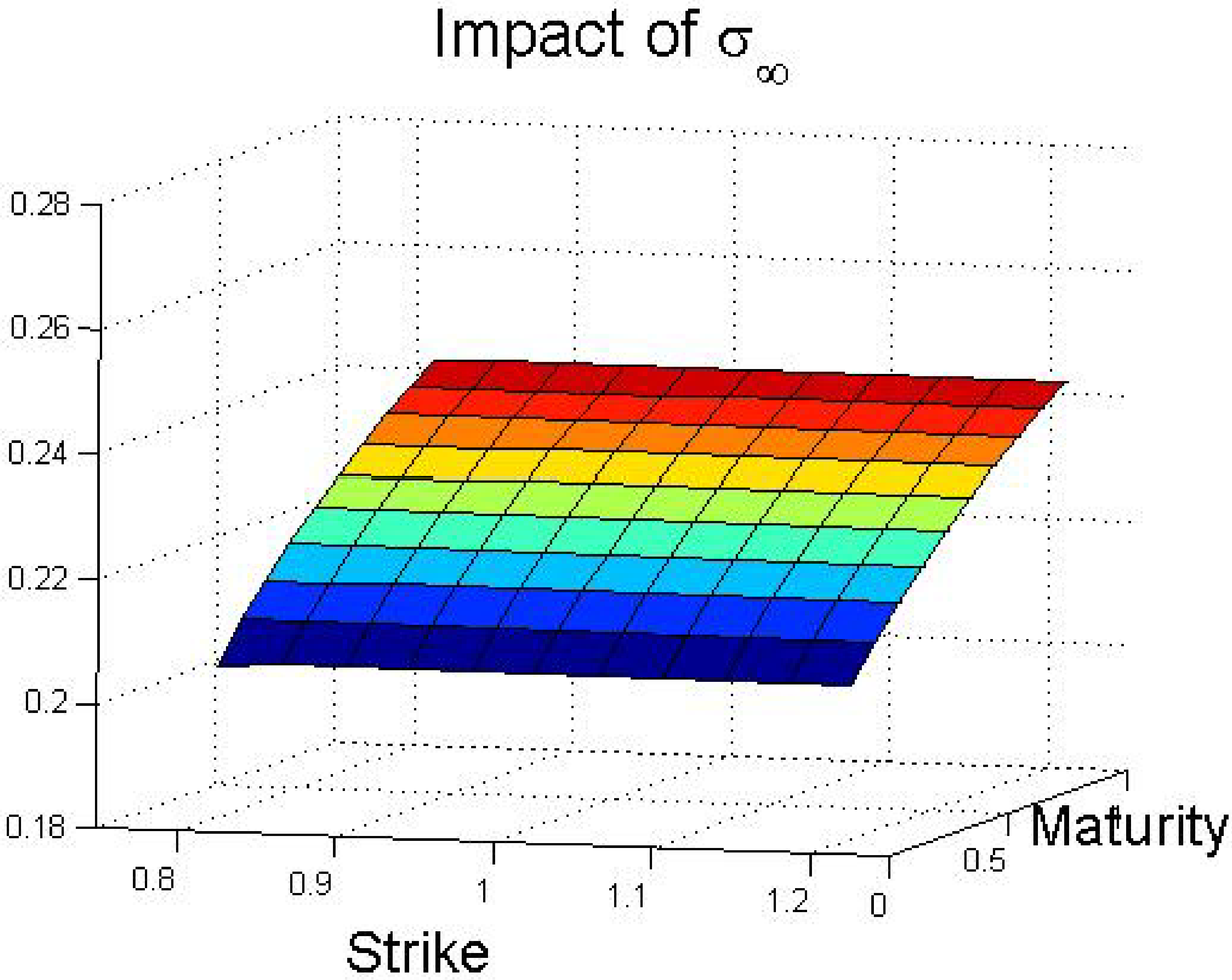
$$x \equiv \ln\frac{FWD}{K} \quad \tau = T - t$$

$$C_{K,T}\left(x, v, \tau\right) = e^x P_1\left(x, v, \tau\right) - P_0\left(x, v, \tau\right)$$

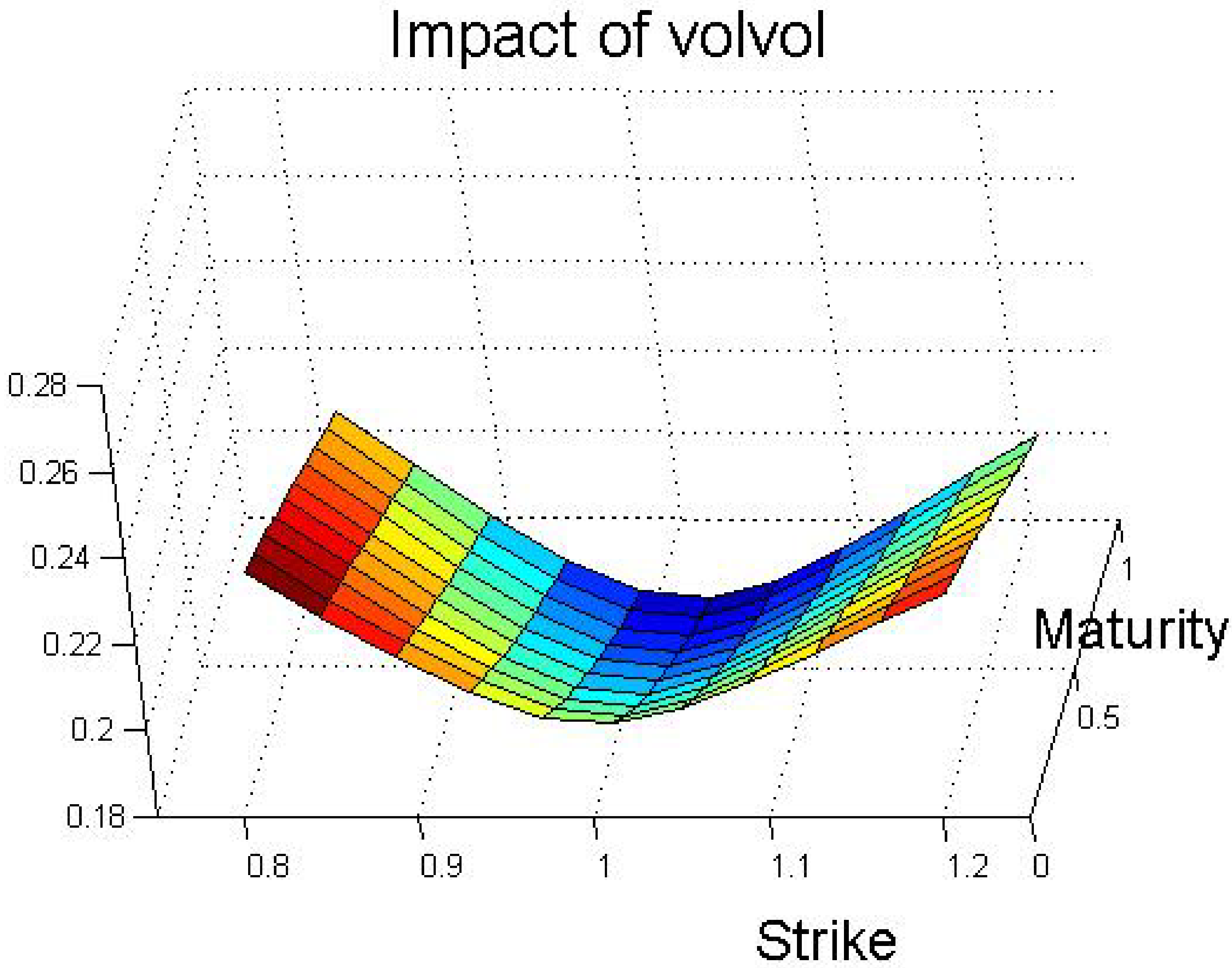
Role of parameters

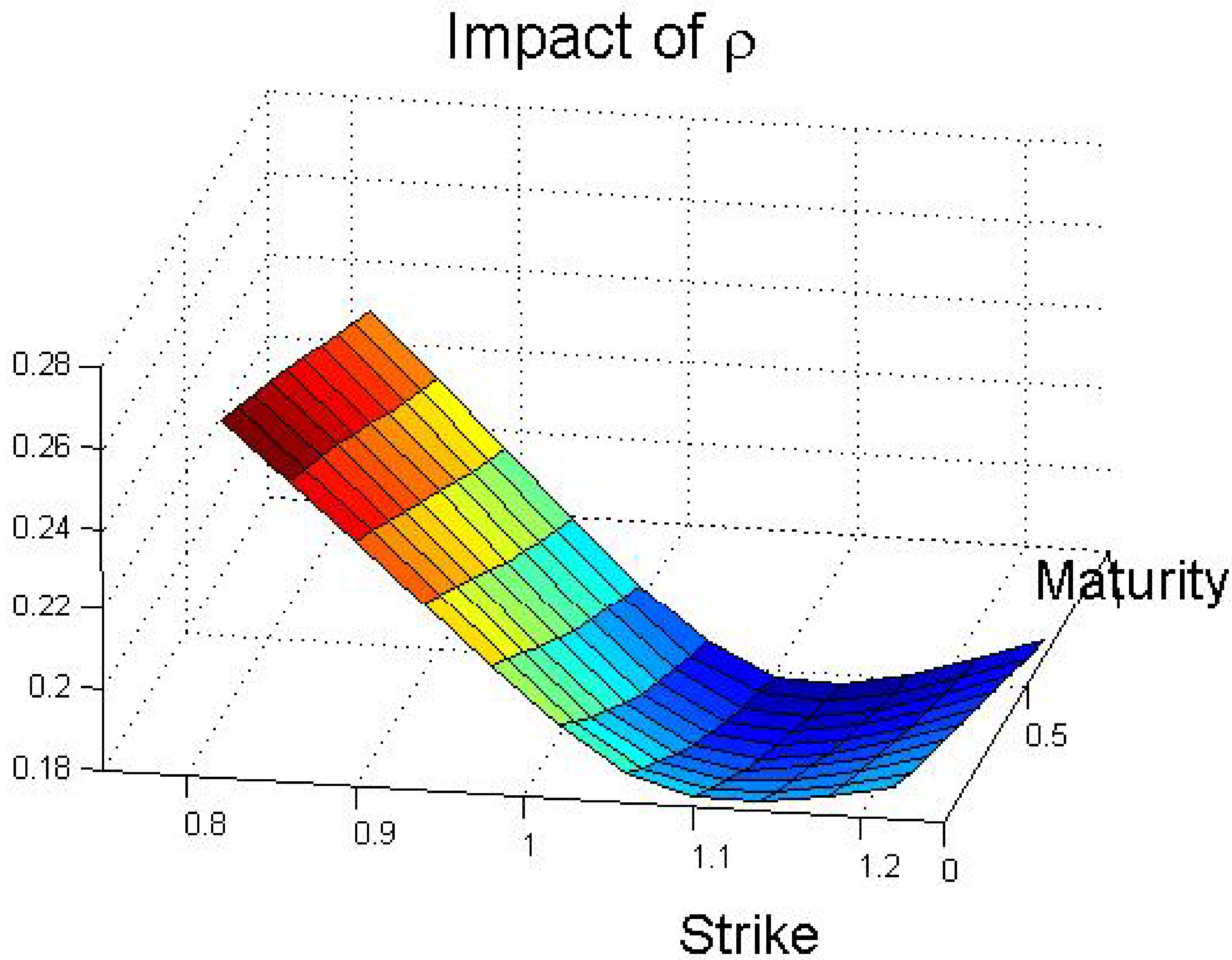
- Correlation gives the short term skew
- Mean reversion level determines the long term value of volatility
- Mean reversion strength
  - Determine the term structure of volatility
  - Dampens the skew for longer maturities
- Volvol gives convexity to implied vol
- Functional dependency on S has a similar effect to correlation


Black-Scholes
0.22
0.21
0.2
0.19
0.18
0.8
0.9
1
1.1
1.2
Strike
0
0.5
1
1
Maturity

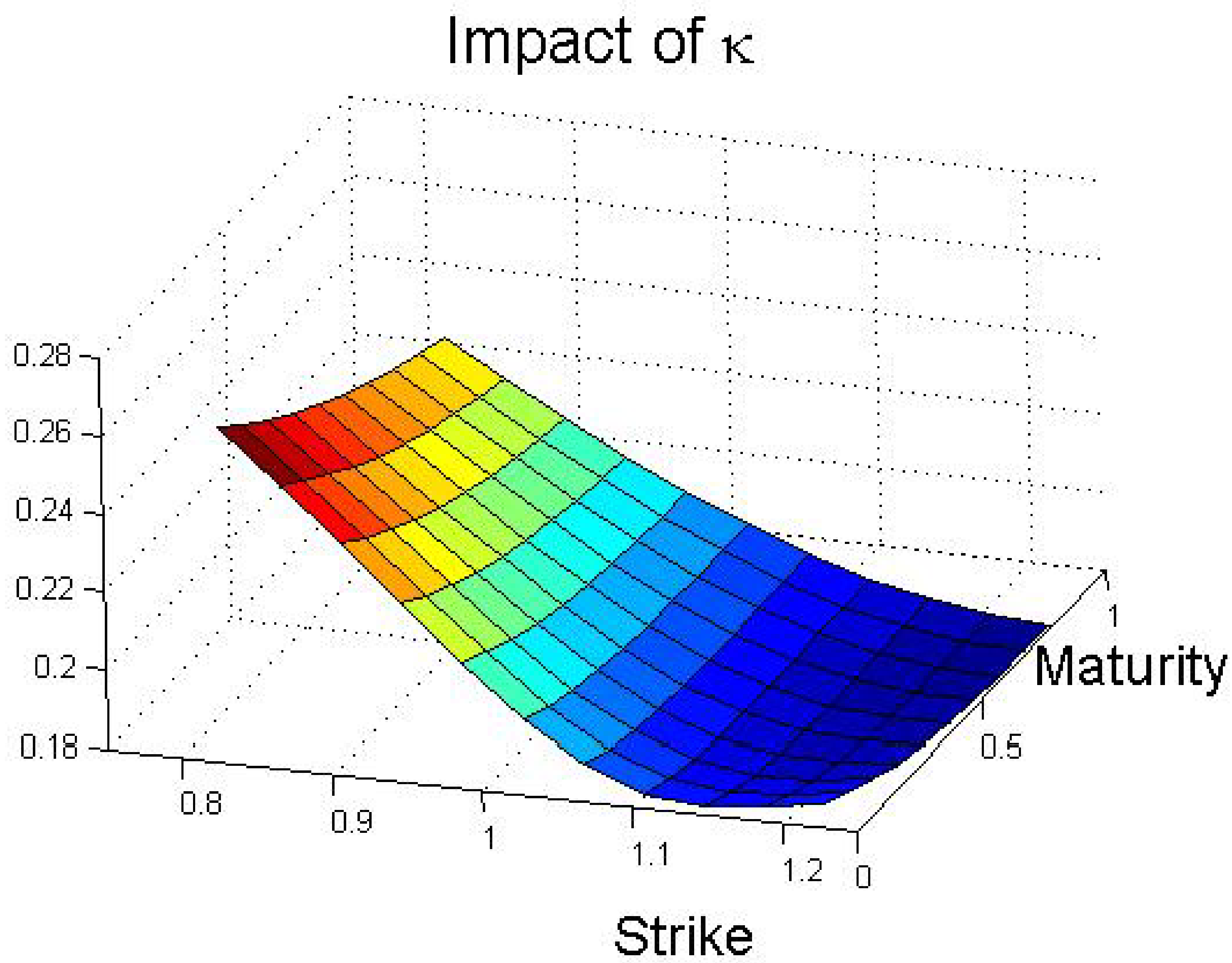


Impact of σ_∞
0.28
0.26
0.24
0.22
0.2
0.18
0.8
0.9
1
1.1
1.2
0
0.5
1
Strike
Maturity


Impact of volvol
0.28
0.26
0.24
0.22
0.2
0.18
0.8
0.9
1
1.1
1.2
0
0.5
1
Maturity
Strike



Impact of ρ
0.28
0.26
0.24
0.22
0.2
0.18
0.8
0.9
1
1.1
1.2
Strike
0
0.5
Maturity



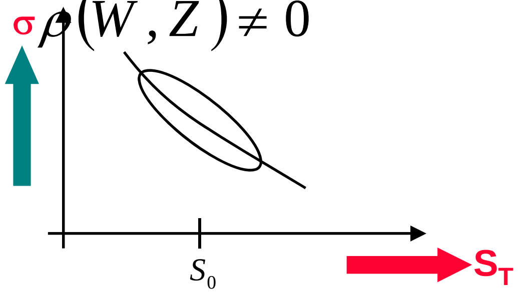
Impact of κ
0.28
0.26
0.24
0.22
0.2
0.18
0.8
0.9
1
1.1
1.2
Strike
0
0.5
1
Maturity

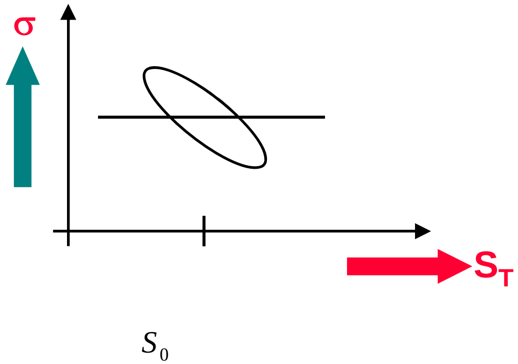
Spot dependency

2 ways to generate skew in a stochastic vol model

1) $\sigma_t = x_t f(S,t), \rho(W,Z) = 0$

2) $\rho(W,Z) \neq 0$





-Mostly equivalent: similar (S_t,σ_t) patterns, similar future evolutions

-1) more flexible (and arbitrary!) than 2)

-For short horizons: stoch vol model ⇔ local vol model + independent noise on vol.

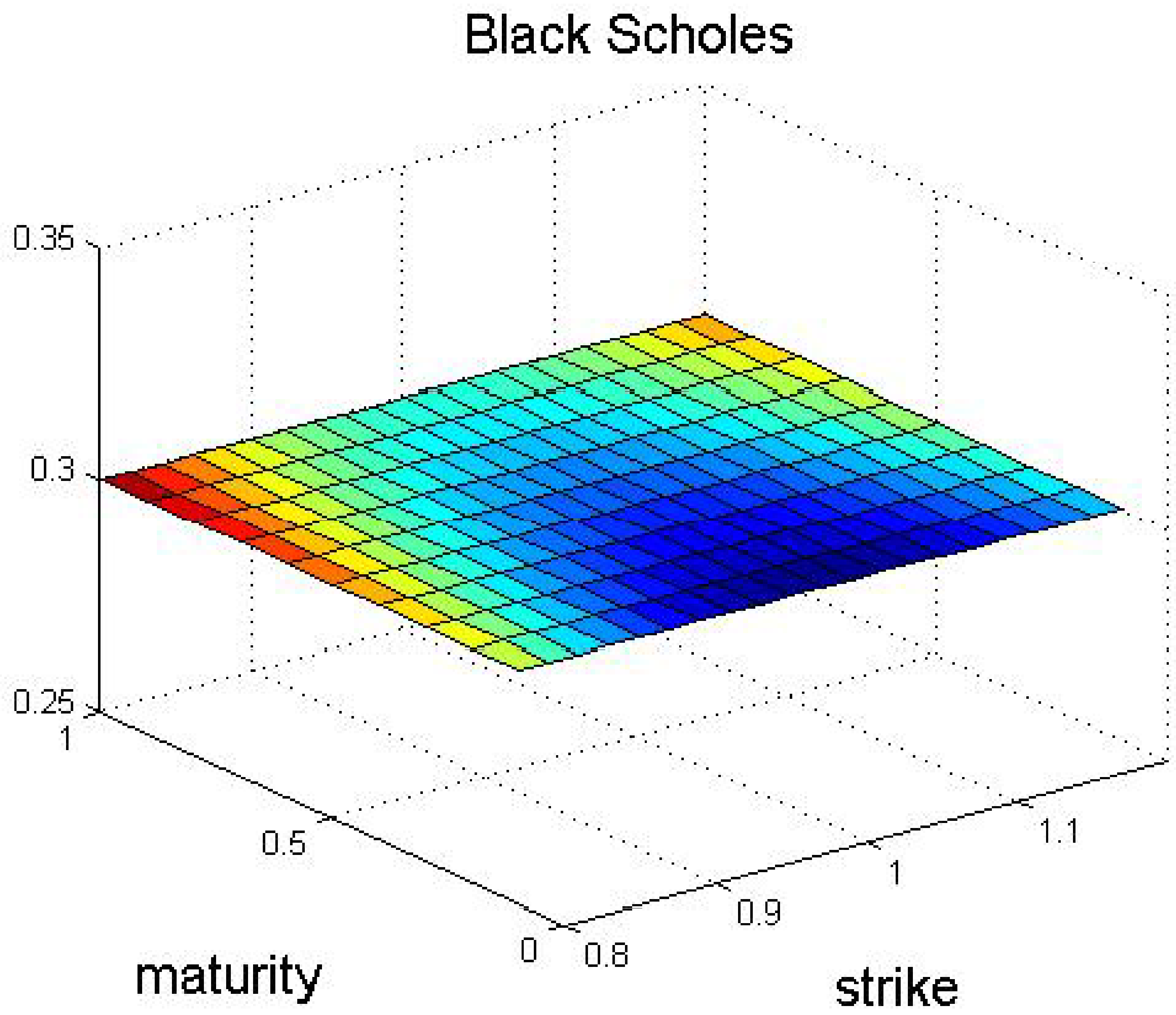
SABR model

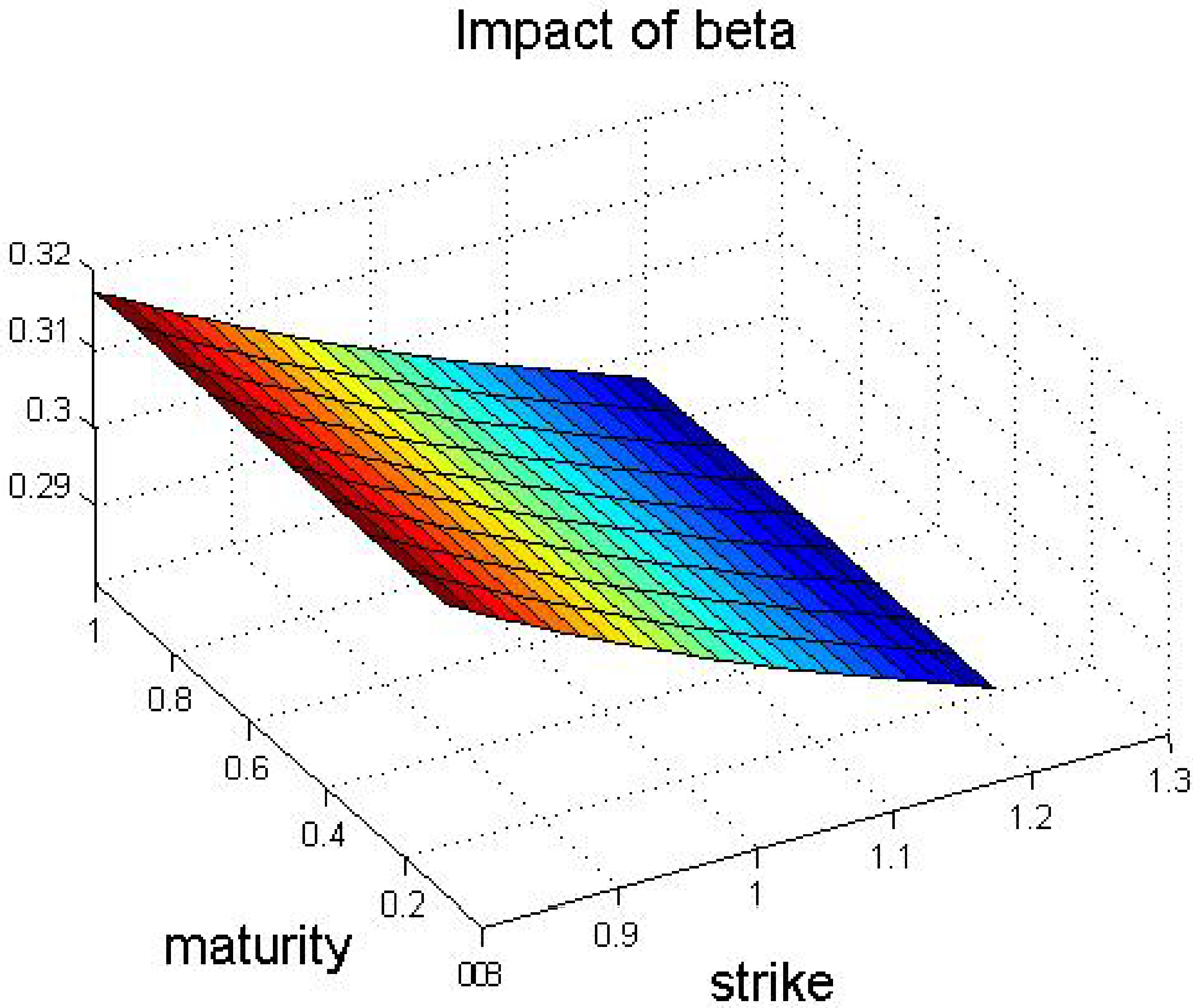
- F: Forward price

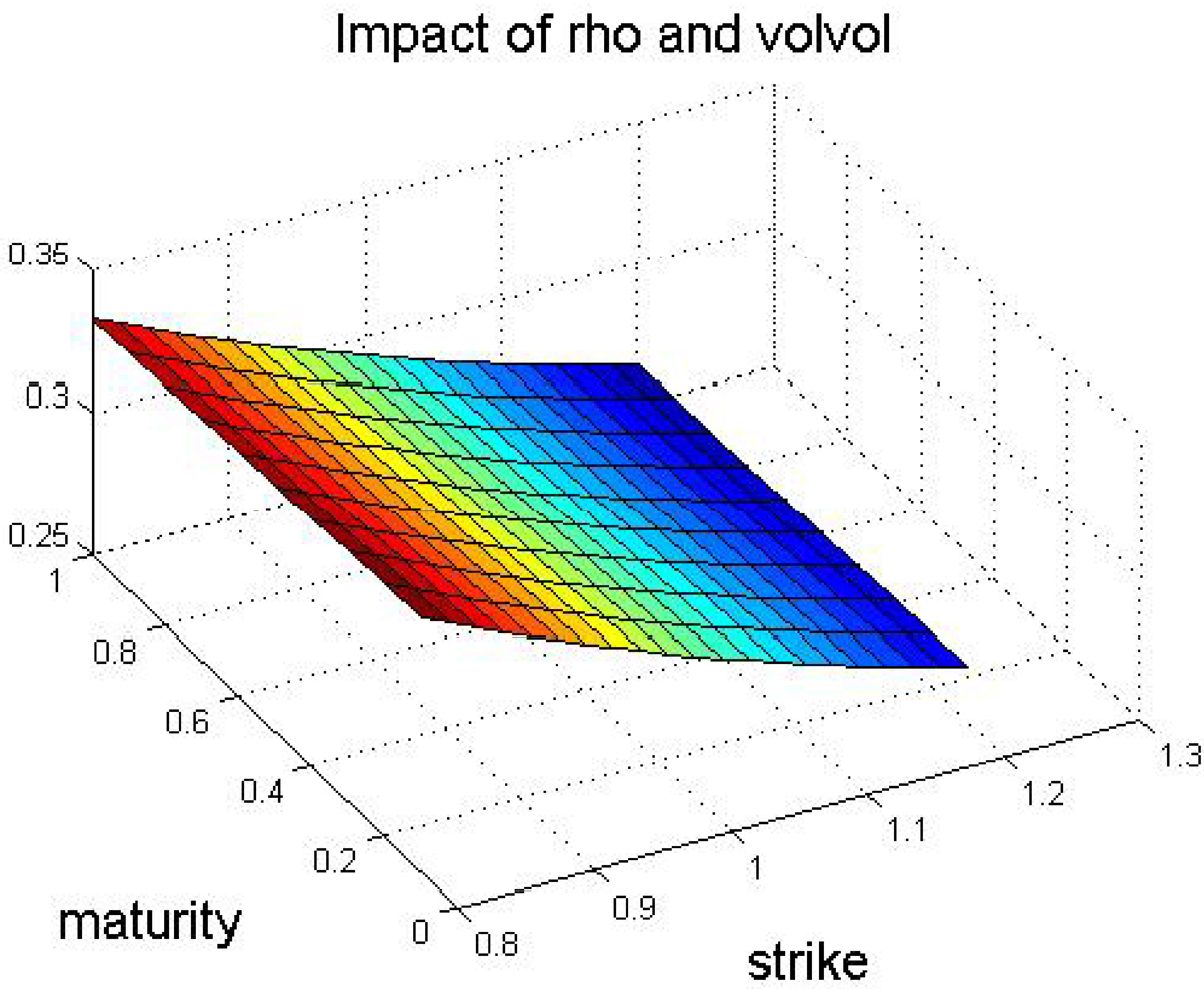
$$dF = F^{\beta}\sigma_t dW$$

$$\frac{d\sigma}{\sigma} = \alpha\, dZ$$

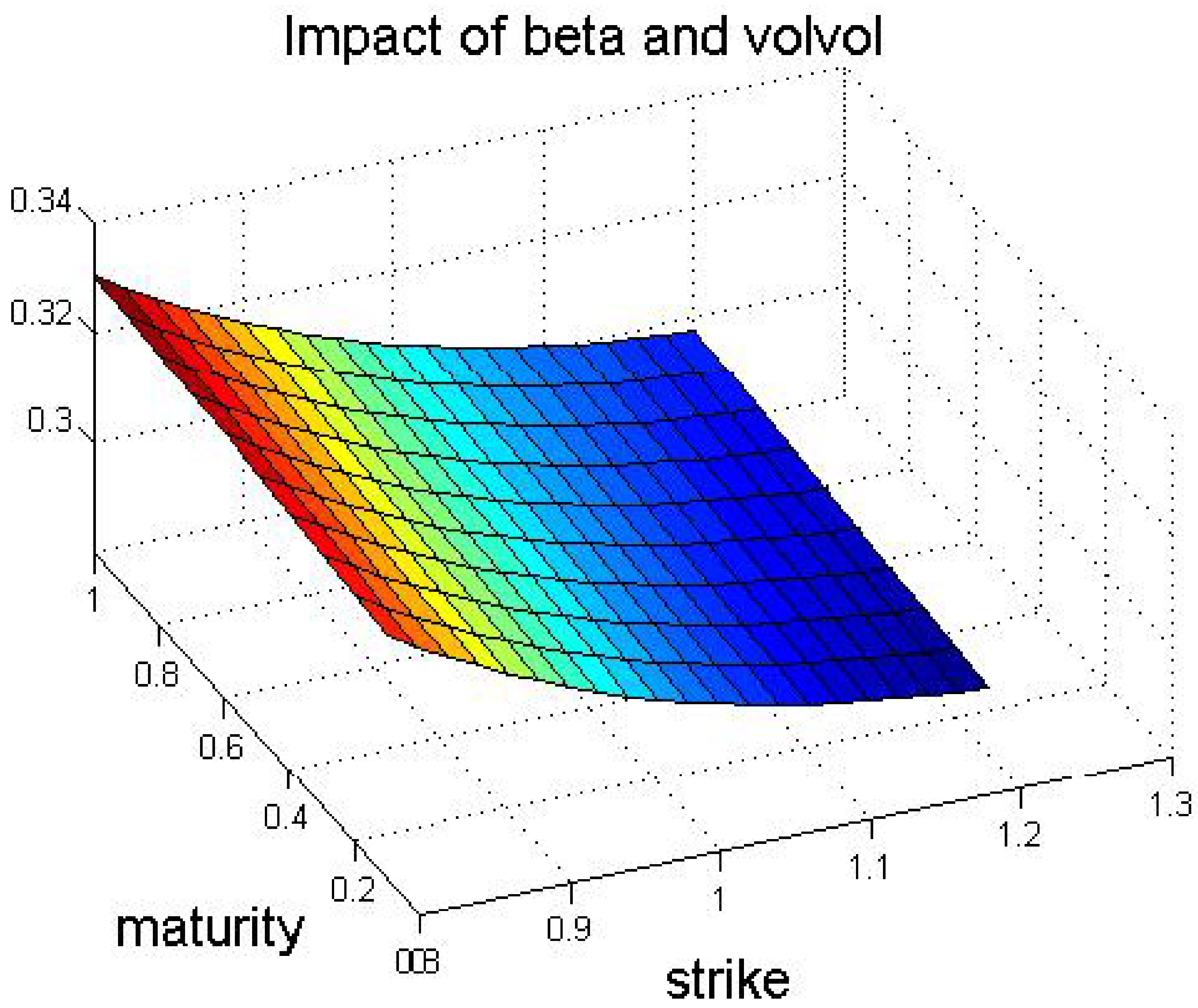
- With correlation ρ


Black Scholes
0.35
0.3
0.25
1
0.5
0
maturity
0.8
0.9
1
1.1
strike

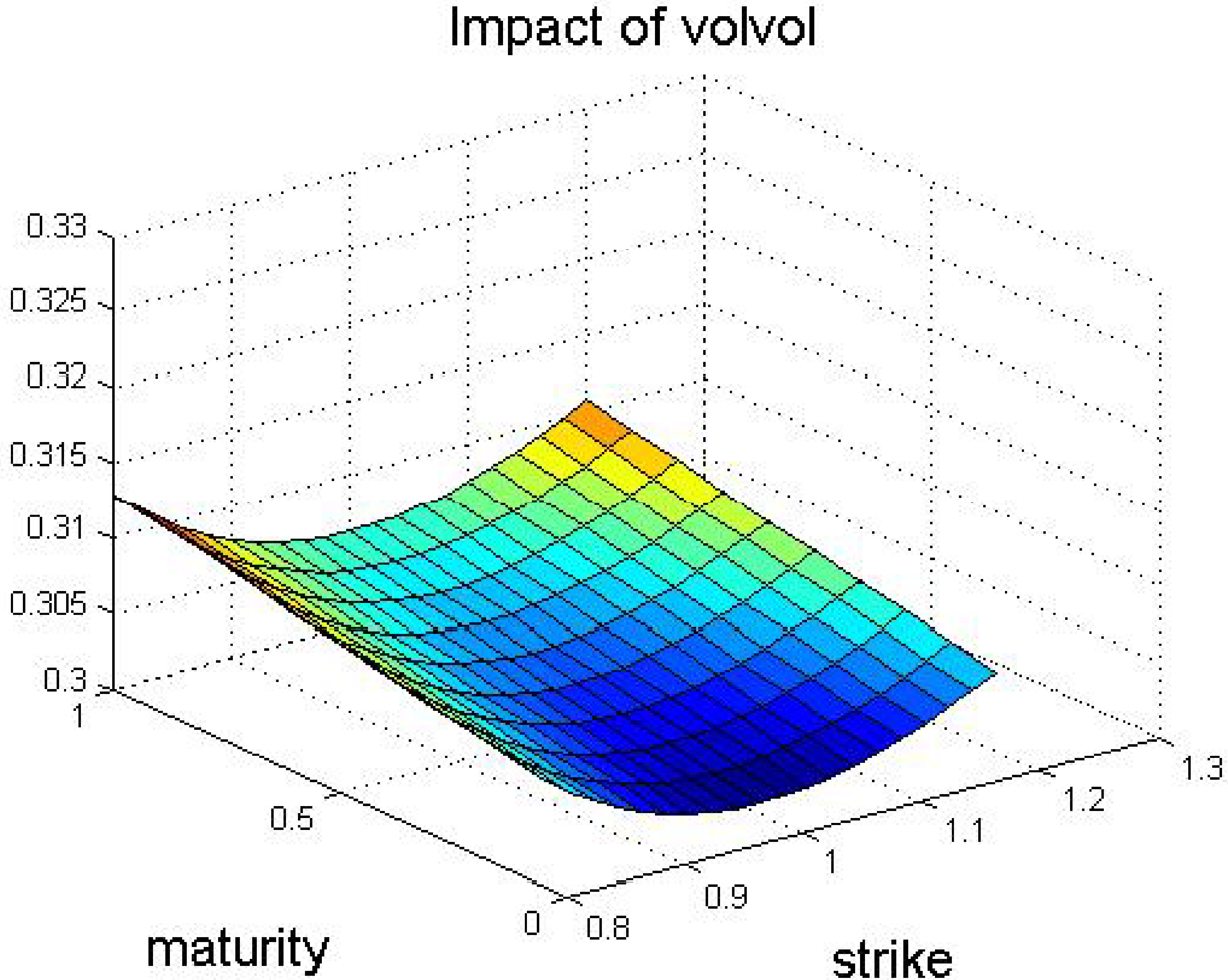

Impact of beta
0.32
0.31
0.3
0.29
1
0.8
0.6
0.4
0.2
maturity
0.8
0.9
1
1.1
1.2
1.3
strike



Impact of rho and volvol
0.35
0.3
0.25
1
0.8
0.6
0.4
0.2
0
maturity
0.8
0.9
1
1.1
1.2
1.3
strike



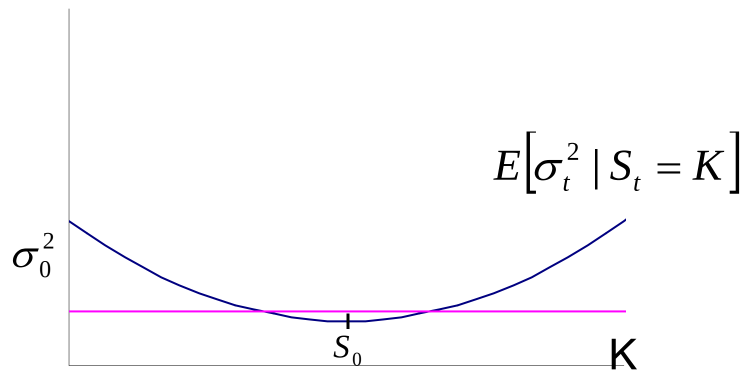
Impact of beta and volvol
0.34
0.32
0.3
1
0.8
0.6
0.4
0.2
maturity
0.8
0.9
1
1.1
1.2
1.3
strike


Impact of volvol
0.33
0.325
0.32
0.315
0.31
0.305
0.3
1
0.5
0
maturity
0.8
0.9
1
1.1
1.2
1.3
strike

Convexity Bias

$$\begin{cases} dS = \sigma_t dW \\ d\sigma_t^2 = \alpha dZ \\ \rho(W, Z) = 0 \end{cases} \quad \Rightarrow \quad E\left[\sigma_t^2 \mid S_t = K\right] = \sigma_0^2 \,?$$

NO! only $E\left[\sigma_t^2\right] = \sigma_0^2$



σ_t likely to be high if $S_t >> S_0$ or $S_t << S_0$

Impact on Models

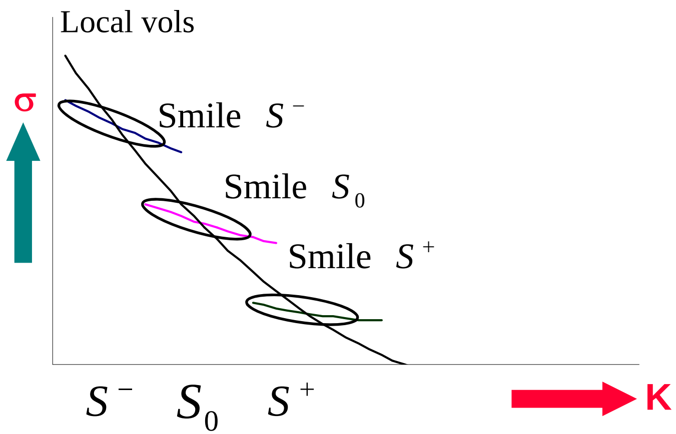
- Risk Neutral drift for instantaneous forward variance
- Markov Model:

$$\frac{dS}{S} = f(S,t)\sigma_t dW$$ fits initial smile with local vols $\sigma(S,t)$

$$\Leftrightarrow f(S,t) = \frac{\sigma^2(S,t)}{E[\sigma_t^2 \mid S_t = S]}$$

Smile dynamics: Stoch Vol Model (1)

Skew case (r<0)



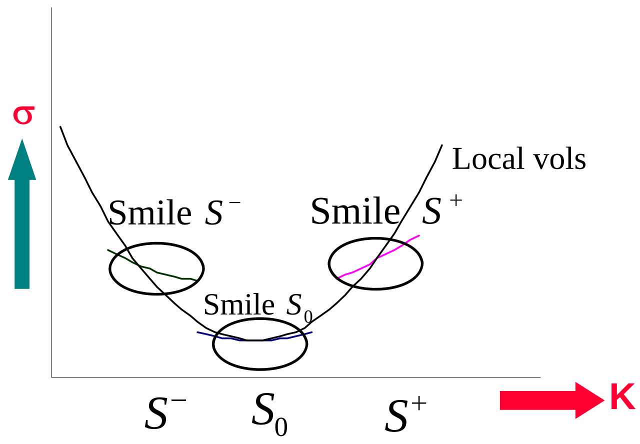
- ATM short term implied still follows the local vols

$$\left(E\left[\sigma^2{}_T \middle| S_T = K\right] = \sigma^2(K,T)\right)$$

- Similar skews as local vol model for short horizons
- Common mistake when computing the smile for another spot: just change S_0 forgetting the conditioning on σ : if S : $S_0 \rightarrow S^+$ where is the new σ ?

Smile dynamics: Stoch Vol Model (2)

- Pure smile case (r=0)



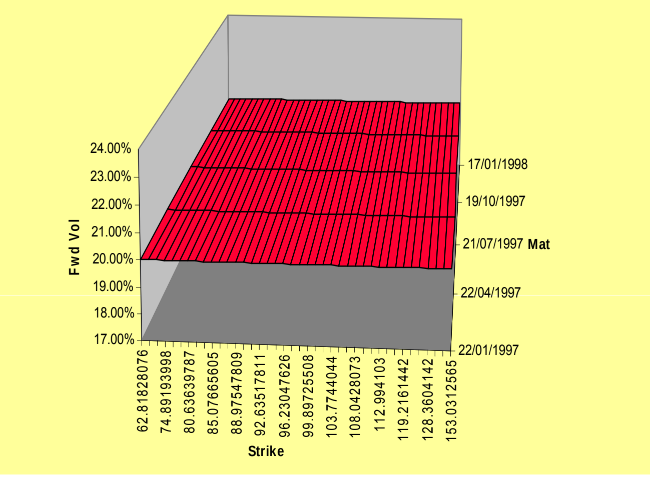
- ATM short term implied follows the local vols
- Future skews quite flat, different from local vol model
- Again, do not forget conditioning of vol by S

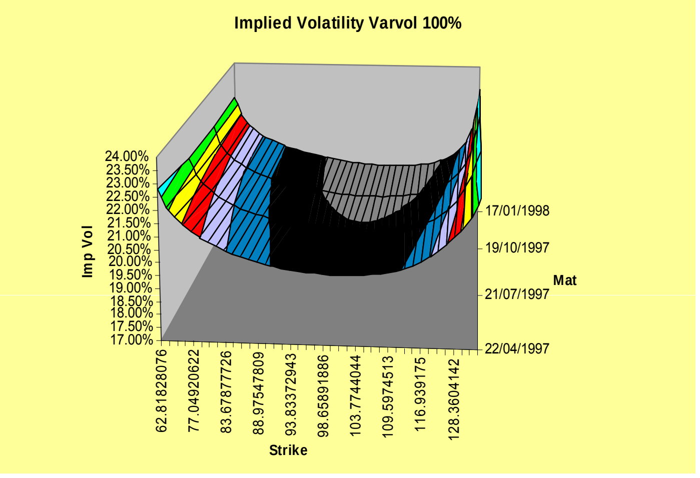
Naïve Markov Approach

- Let M be a Q-martingale, with $M_0 = 1$
- Naïve approach $v(S_t, t) = V_{S,t}(S_0, t_0) M_t$

Problem: this model generates a H&W type smile

Example: flat initial smile $V_{K,T}(S_0, t_0) = V_0$





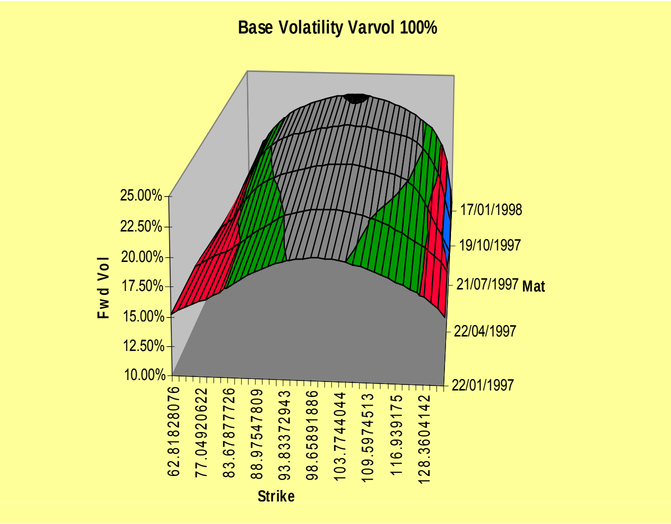
Correct approach

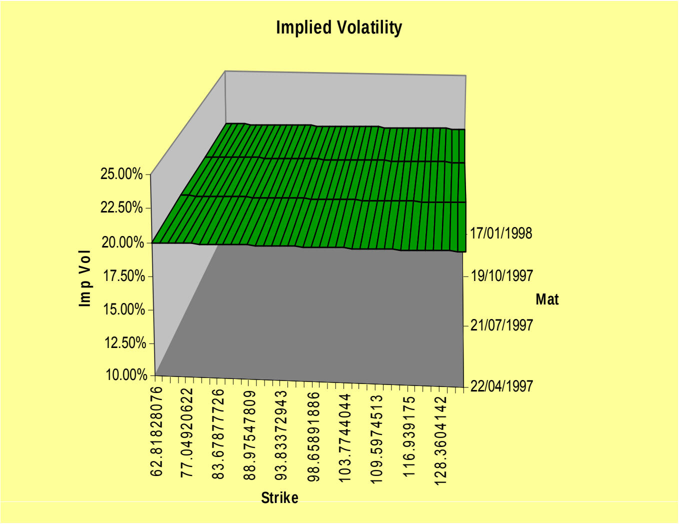
- Cure: correct for change of numeraire bias

it must respect: $E^{K,T}[v] = V_{K,T}$

- Then we have: $v(S,t) = V_{S,t}(S_0,t_0)\dfrac{M_t}{E^Q[M_t | S_t = S]} = f(S,t)M_t$

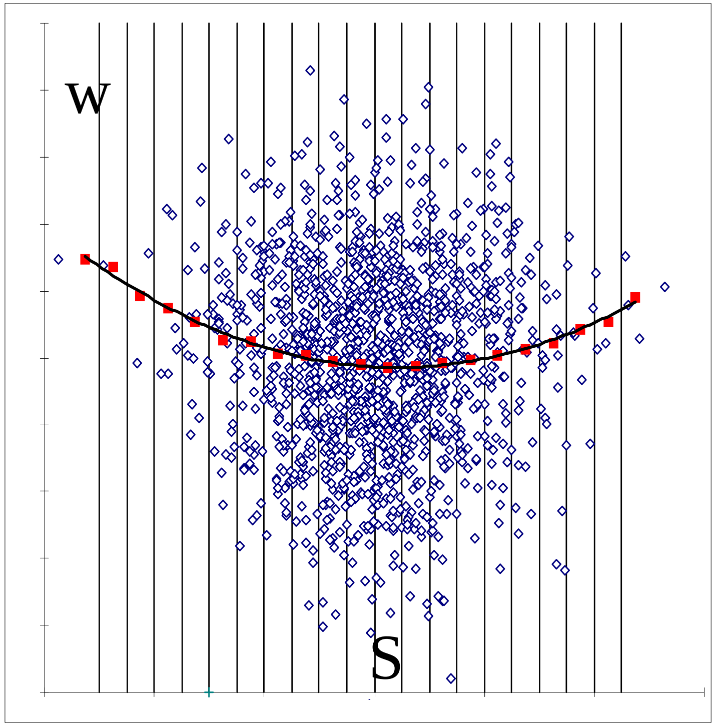
Example: $M_t = \exp\left(-\dfrac{b^2}{2}t + bW_t\right)$





Computation of conditional expectation

- Run N paths for (S, W) up to time t
- Rank the N paths according to S values
- Group them in p buckets (similar values of S within the same buckets)
- Compute average of W on each bucket



Fixed point methodology

$V_{S_0,0}(S,t) \longrightarrow$ Pricing of Europeans with previous model $\longrightarrow \hat{\sigma}^2(K,T)$

As mentioned earlier, $\hat{\sigma}^2(K,T) \neq \hat{\sigma}^2_{market}(K,T)$

γ is to be chosen by the user. We decide to calibrate only the $V_{S_0,0}(S,t)$ term, i.e.:

$$dS_t = \sqrt{v_{S,t}}\, dZ_t^Q \qquad v_{S,t} = f(S,t)\, e^{-\frac{1}{2}\gamma^2 t + \gamma W_t}$$

$f(S,t)\,? \longrightarrow$ Pricing of europeans with this model $\longrightarrow \hat{\sigma}^2_{market}(K,T)$

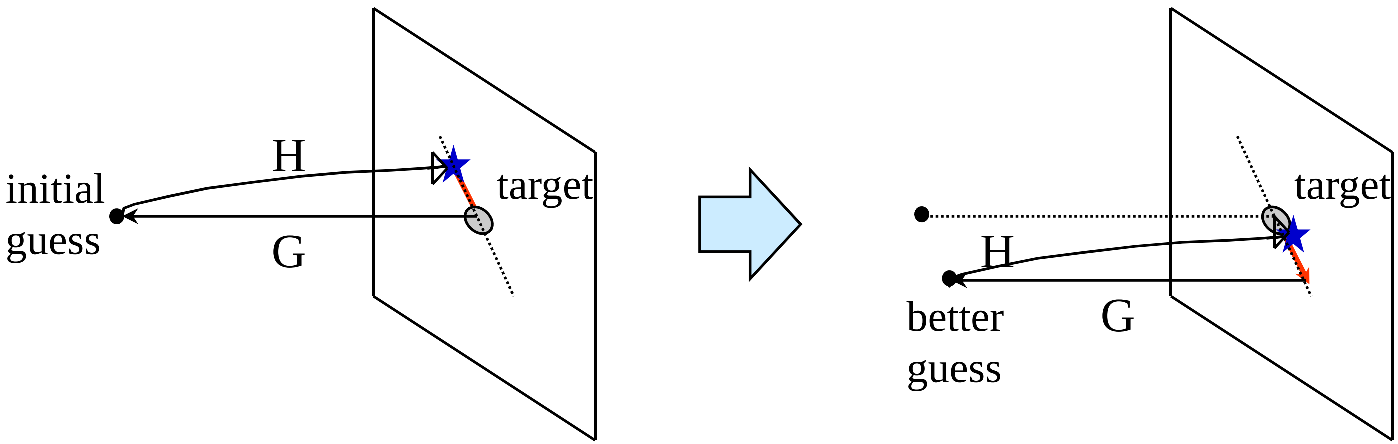
Fixed point methodology (2)

$f(S,t)$? $\longrightarrow$ $\boxed{\text{Pricing of europeans with this model}}$ $\longrightarrow$ $\hat{\sigma}^2_{market}(K,T)$

In mathematical terms: $\hat{\sigma}^2_{market} = H(f)$ Question: $H^{-1}(\hat{\sigma}^2_{market})$?

We do not know H^{-1} but we know

G = Dupire's formula $\approx H^{-1}$



Fixed point methodology (3)

Applying this principle, we build the following sequence:

$$\begin{cases} \hat{\sigma}_n = H\left(f_n\right) \\ \text{modified target } \bar{\hat{\sigma}}_n = \hat{\sigma}_{market} - \lambda(\hat{\sigma}_n - \hat{\sigma}_{market}) \\ f_{n+1} = G\left(\bar{\hat{\sigma}}_n\right) \end{cases}$$

Under appropriate conditions (usually satisfied),

$$\hat{\sigma}_n \underset{n\to\infty}{\longrightarrow} \hat{\sigma}_{market}$$

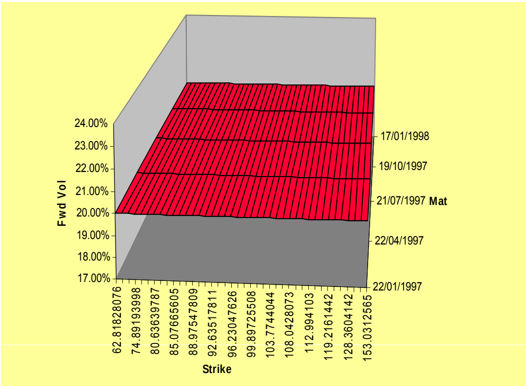
$$f_n \underset{n\to\infty}{\longrightarrow} \bar{f}$$

We finally price any security in the following model:

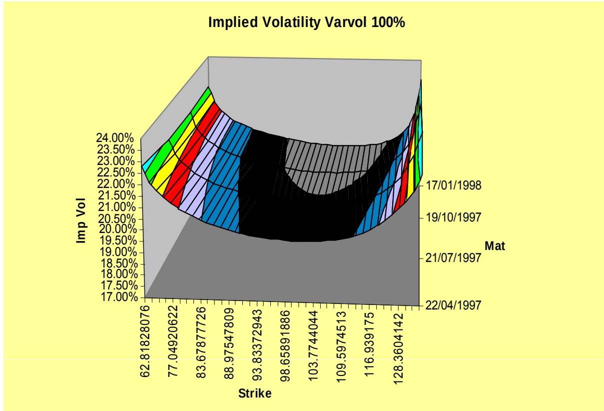
$$dS_t = \sqrt{v_{S,t}}\, dZ_t^Q \qquad v_{S,t} = \bar{f}\left(S,t\right) e^{-\frac{1}{2}\beta^2 t + \beta W_t}$$

Fixed point results

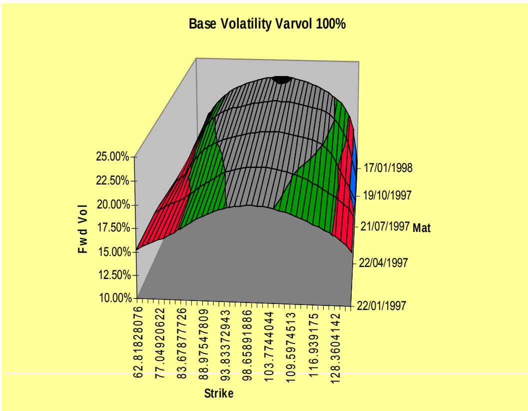
initial flat local vol



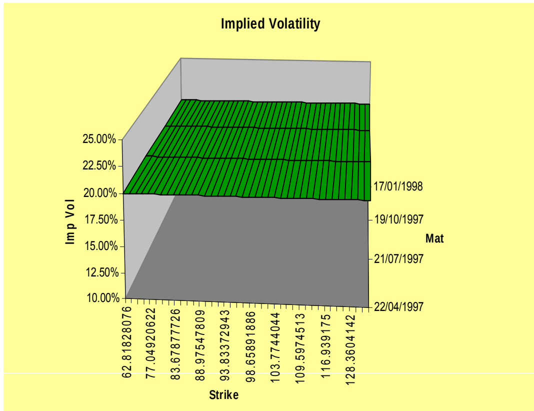
convex implied vol



final concave local vol



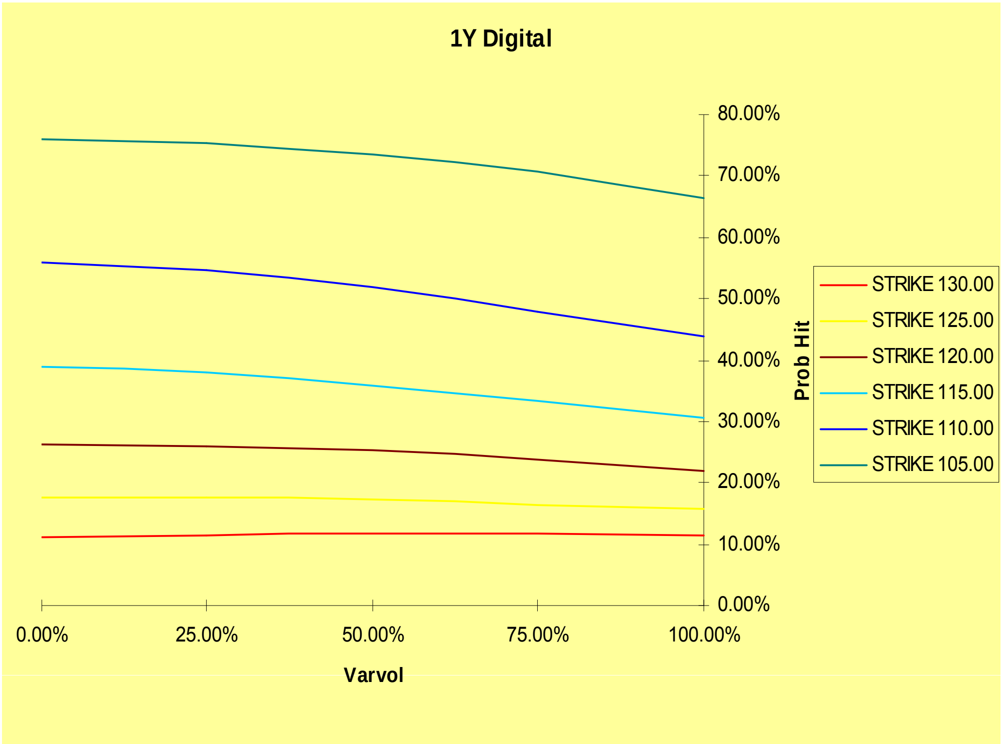
target: flat implied vol



Options pricing

For each b:

- Recalibrate the model
- Reprice the option (1Y american digital here)



Conclusion

- We can get a model which fits market smiles and produces realistic behaviors by either:
  - Extracting forward volatilities and risk-neutralizing their dynamics
  - Or calibrating to current market smile thanks to forward induction
- Exotics are priced in line with Europeans
- The hedging portfolio of Europeans is determined by perturbation (Superbuckets)

Market Models of Implied Volatility

Market Model of Implied Volatility

- Implied volatilities are directly observable
- Can we model directly their dynamics? $(r=0)$

$$\begin{cases} \dfrac{dS}{S} = \sigma \, dW_1 \\ \dfrac{d\hat{\sigma}}{\hat{\sigma}} = \alpha \, dt + u_1 dW_1 + u_2 dW_2 \end{cases}$$

where $\hat{\sigma}$ is the implied volatility of a given $C_{K,T}$

- Condition on $\hat{\sigma}$ dynamics?

Drift Condition

- Apply Ito's lemma to $C(S, \hat{\sigma}, t)$
- Cancel the drift term
- Rewrite derivatives of $C(S, \hat{\sigma}, t)$

gives the condition that the drift α of $\frac{d\hat{\sigma}}{\hat{\sigma}}$ must satisfy.

For short T,

$$\hat{\sigma}^2 = (\sigma + u_1 X)^2 + (u_2 X)^2$$ (Short Skew Condition :SSC)

where $X = \ln K - \ln S$

Local Volatility Model Case

σ det. function of $(S,t) \Rightarrow \hat{\sigma}$ det. function of (S,t)

and $\dfrac{d\hat{\sigma}}{\hat{\sigma}} = \alpha\, dt + \dfrac{\hat{\sigma}_S \sigma S}{\hat{\sigma}} dW_1$: $u_1 = \dfrac{\hat{\sigma}_S \sigma S}{\hat{\sigma}}$, $u_2 = 0$

SSC: $\hat{\sigma} = \sigma + u_1 X = \sigma\left(1 + \dfrac{\hat{\sigma}_S}{\hat{\sigma}} SX\right) \Rightarrow \sigma = \dfrac{\hat{\sigma}}{1 + \dfrac{\hat{\sigma}_S}{\hat{\sigma}} SX}$

solved by $\hat{\sigma} = \dfrac{X}{\int_S^K \dfrac{du}{u\sigma(u,0)}}$

"Dual" Equation

The stripping formula $\sigma^2 = 2\dfrac{C_T}{K^2 C_{KK}}$

can be expressed in terms of $\hat{\sigma}$:

When $T \to 0$ $\quad \sigma = \dfrac{\hat{\sigma}}{1 + \dfrac{\hat{\sigma}_K}{\hat{\sigma}} KX}$

solved by $\hat{\sigma} = \dfrac{X}{\displaystyle\int_S^K \frac{du}{u\sigma(u,0)}}$

Large Deviation Interpretation

The important quantity is $\int_S^K \frac{du}{u\sigma(u,0)}$

If $dx = a(x)dW$ then $y(x) = \int_{x_0}^x \frac{du}{a(u)}$ satisfies:

$dy = \mu dt + dW$ and $x_t = K \Leftrightarrow W_t = y(K)$

$$\hat{\sigma} \;/\; \int_S^K \frac{du}{u\hat{\sigma}} = \int_S^K \frac{du}{u\sigma(u,0)} \Rightarrow \hat{\sigma} = \frac{\ln K - \ln S}{\int_S^K \frac{du}{u\sigma(u,0)}}$$

Strengths and Limitations

+ : automatically compatible with current prices

— : in general, non Markov model $\longrightarrow$ inefficient implementation

care needed to incorporate mean reversion

need for nD model in general

Barrier Options

Introduction

- This talk aims at providing a better understanding of
  - The limitation of Black-Scholes when applied to barrier options
  - The smile approach for pricing barrier options
  - The extension to stochastic volatility and jump models and their impact on barrier prices
  - Optimal hedges in vega and gamma

Limitation of the Black-Scholes assumptions

- Black-Scholes model assumes constant instantaneous volatility

$$\frac{dS}{S} = \mu \; dt + \sigma \; dW$$

(instantaneous vol)

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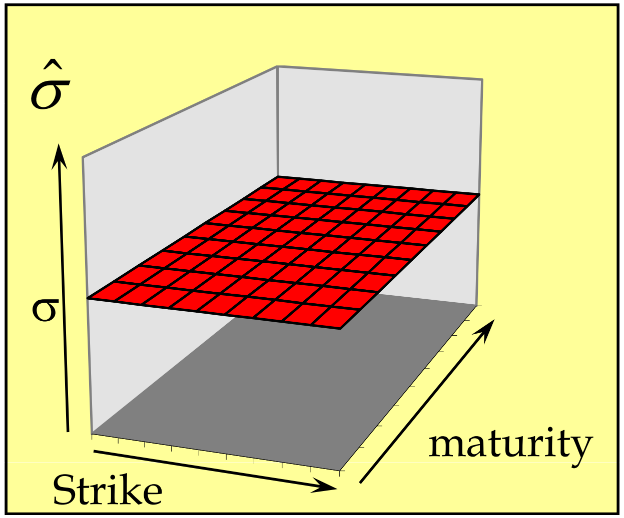
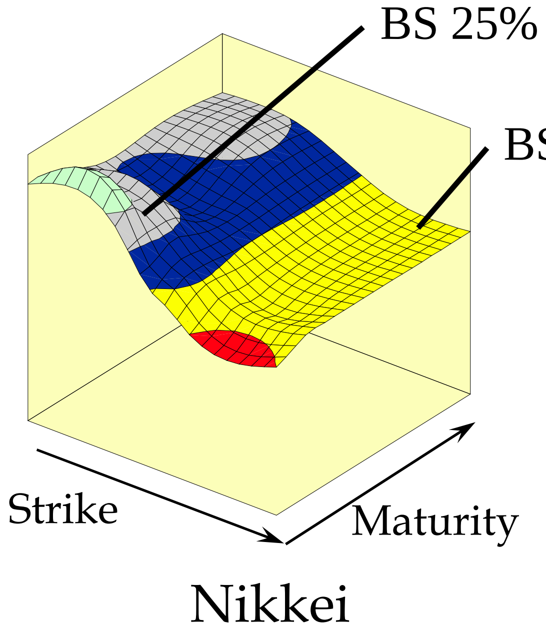
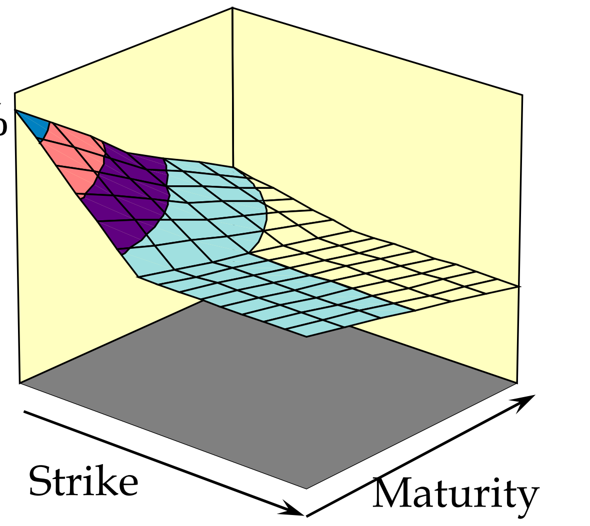
implied vols



Real world

- In practice, implied volatility highly varies with strike:





<u>Problem</u>: barriers do not depend on only one volatility!
=> need for one model consistent with all option prices

The smile model

•<u>Black-Scholes</u>:

$$\frac{dS}{S} = \sigma \, dW_t$$

•<u>Merton</u>:

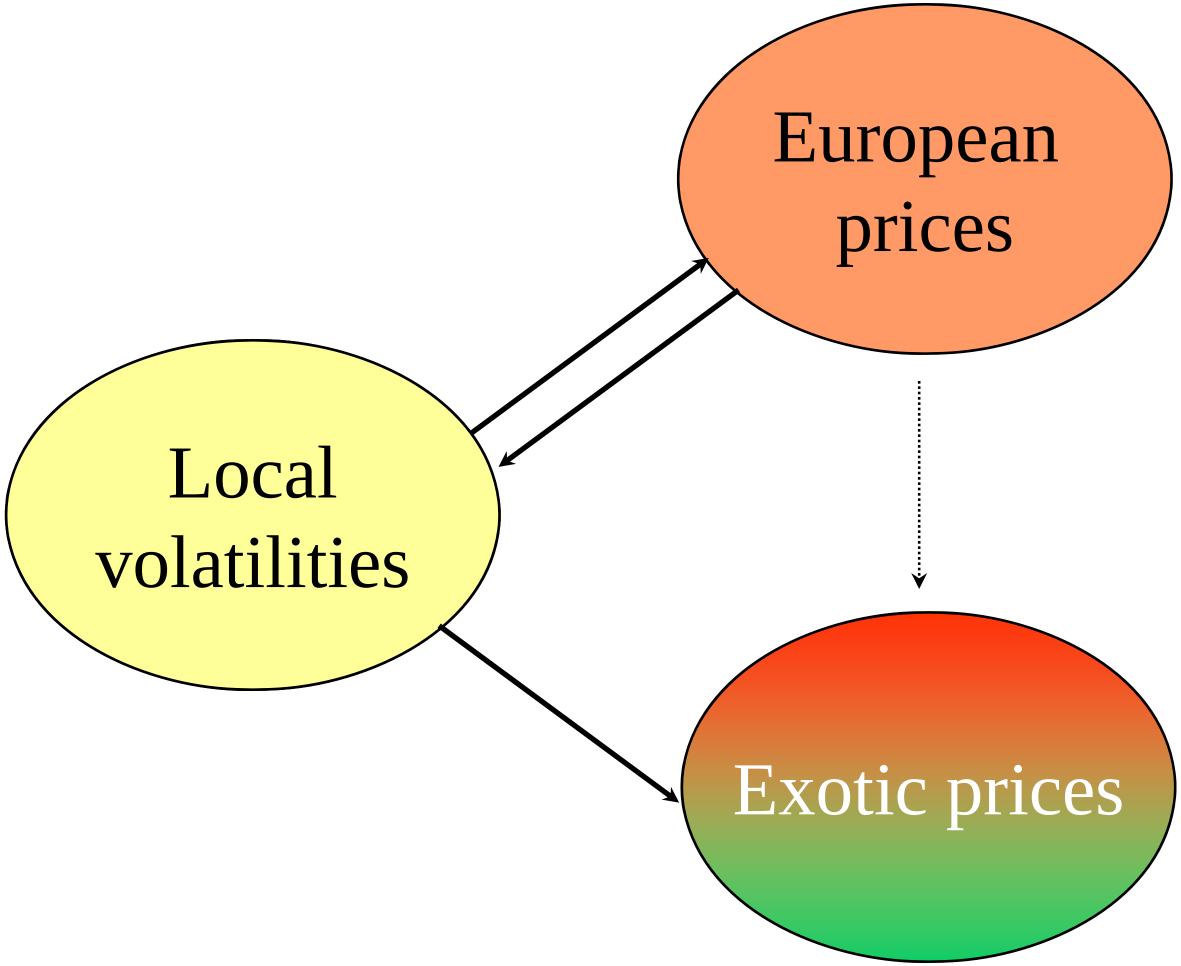
$$\frac{dS}{S} = \sigma(t) \, dW_t$$

• <u>Simplest extension consistent with smile</u>:

$$dS = \sigma(S,t) \, dW_t$$

$\sigma(S,t)$ is called “local volatility”

From simple to complex

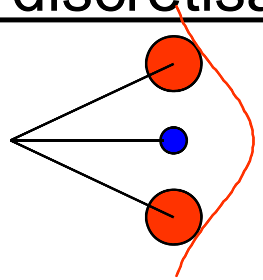


Barrier option pricing within the smile model

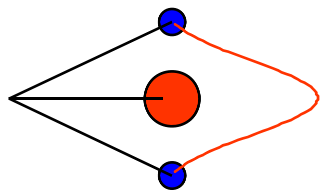
- solve generalised Black-Scholes's PDE

$$\frac{\partial}{\partial t}C(S,t)+\mu_S(S,t)\frac{\partial}{\partial S}C(S,t)+\frac{1}{2}\sigma^2(S,t)\frac{\partial^2}{\partial S^2}C(S,t)=rC(S,t)$$

- with the relevant boundary conditions depending on the product (e.g. value = 0 for knock-out options)
- Explicit discretisation: recombining trinomial tree



High volatility:
weights concentrated on boundary nodes



Low volatility:
weights concentrated on central nodes

Preferred solution: semi-explicit (Crank-Nicholson) grid

Unconditionally stable

Converges in O (Δt2) (Trinomial: O (Δt))

Monte-Carlo implementation

Stochastic differential equation:

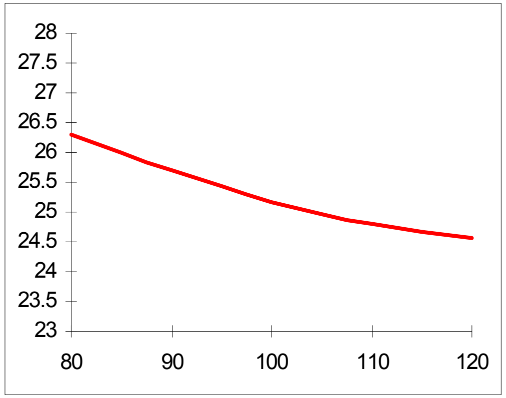
$$dS = \alpha dt + \sigma\left(S,t\right)dW$$

Simplest scheme: Euler discretisation

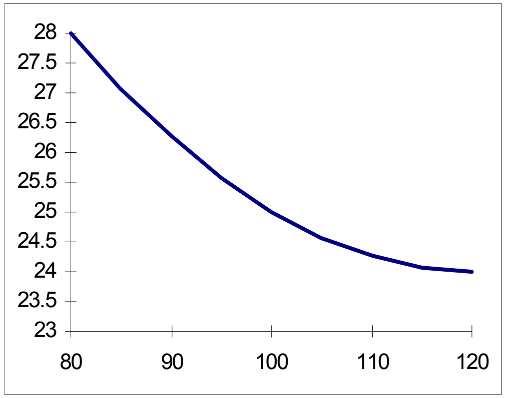
$$S_{n+1} = S_n + \alpha\delta t + \sigma\left(S_n,t_n\right)\delta W$$

Typical example: decreasing smile

Example:



implied volatility

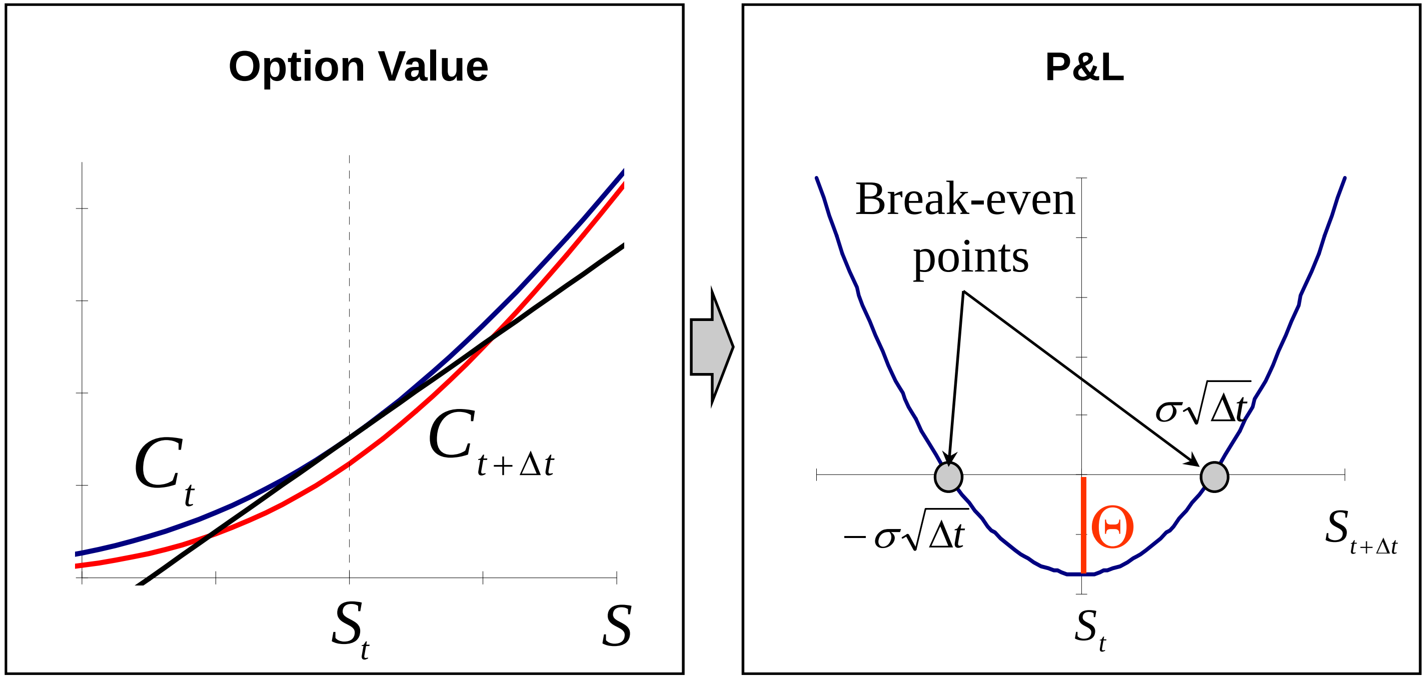


local volatility

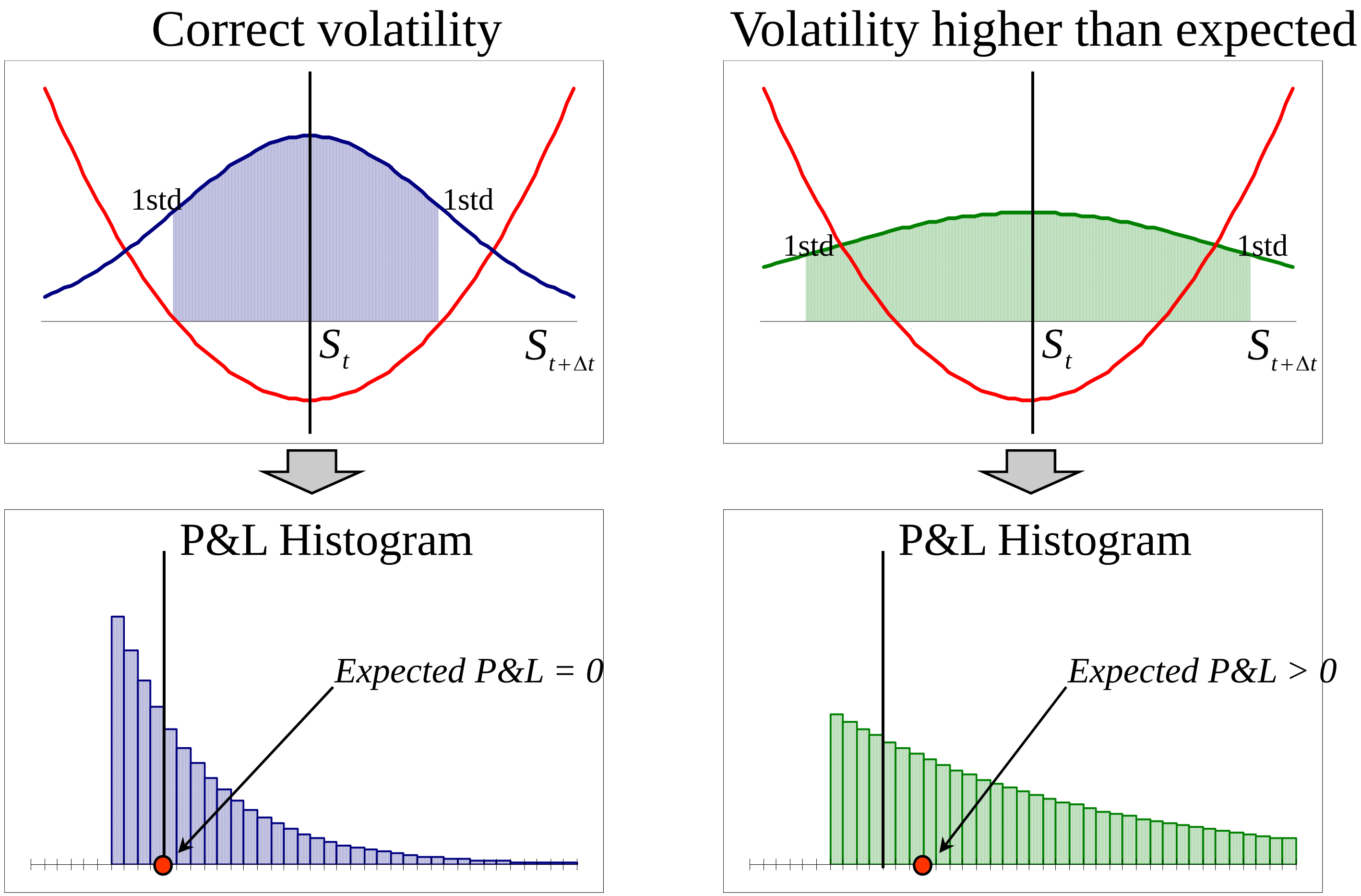
How to measure the impact of volatility on the barrier option?

P&L of a delta hedged option (1)

P&L of delta-hedged option position over Δt:



P&L of a delta hedged option (2)



Ito: When $\Delta t \to 0$, spot dependency disappears

Black-Scholes PDE

$Let\ \sigma_0$ be the Black-Scholes volatility

P&L is a balance between gain from G and loss from Θ:

$$P\&L_{(t,t+dt)} = \left(\frac{\sigma^2}{2}\Gamma_0 + \Theta_0\right)dt$$

From Black-Scholes PDE:

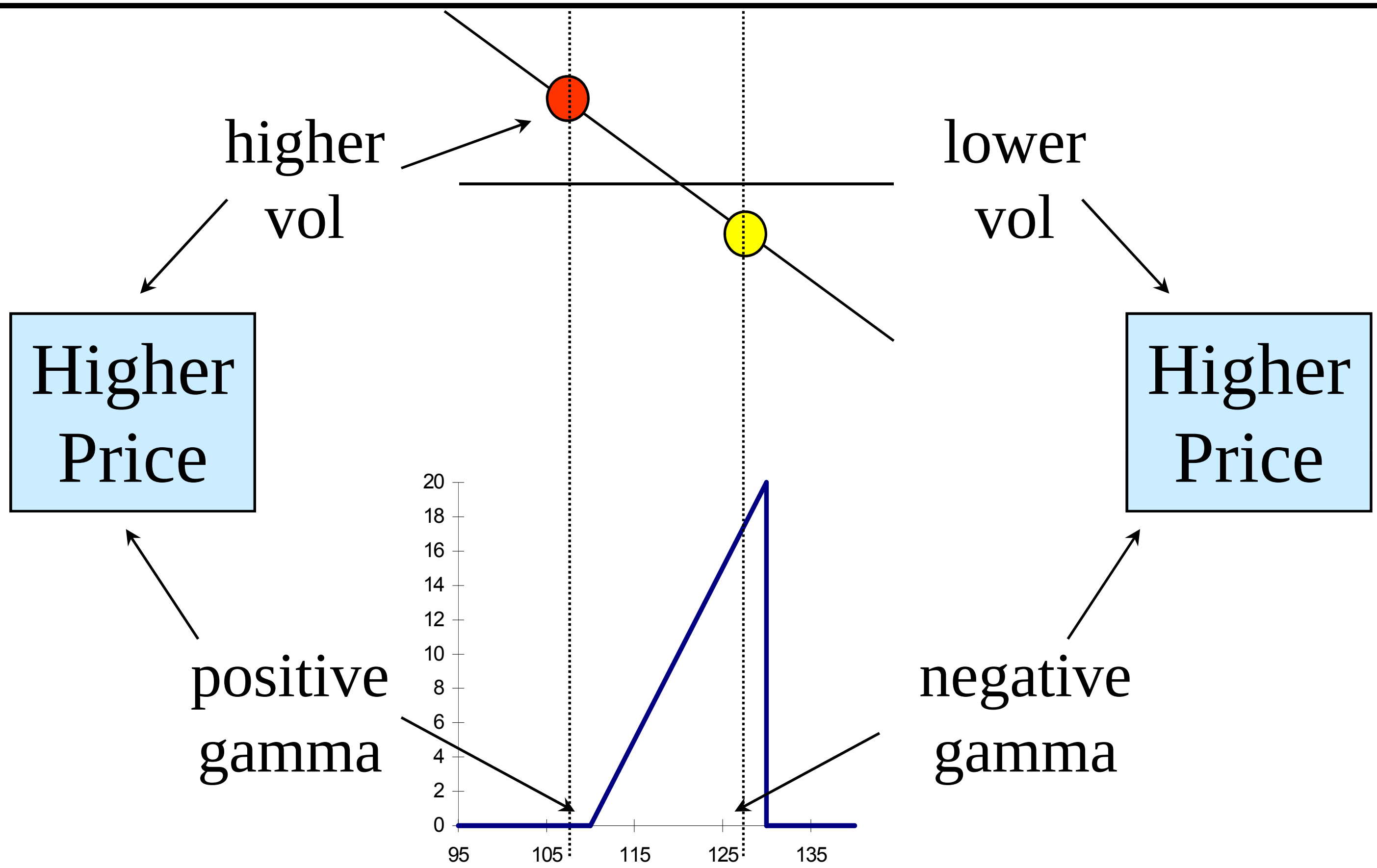
$$\Theta_0 = -\frac{\sigma_0^2}{2}\Gamma_0$$

=> discrepancy if *s* different from expected:

$$\text{gain over dt} = \frac{1}{2}\left(\sigma^2 - \sigma_0^2\right)\Gamma_0 dt$$

$$\left.\begin{array}{l} \bullet\ \sigma > \sigma_0 : \text{Profit} \\ \bullet\ \sigma < \sigma_0 : \text{Loss} \end{array}\right\} \text{Magnified by } \Gamma_0$$

Impact of local vol on barrier option

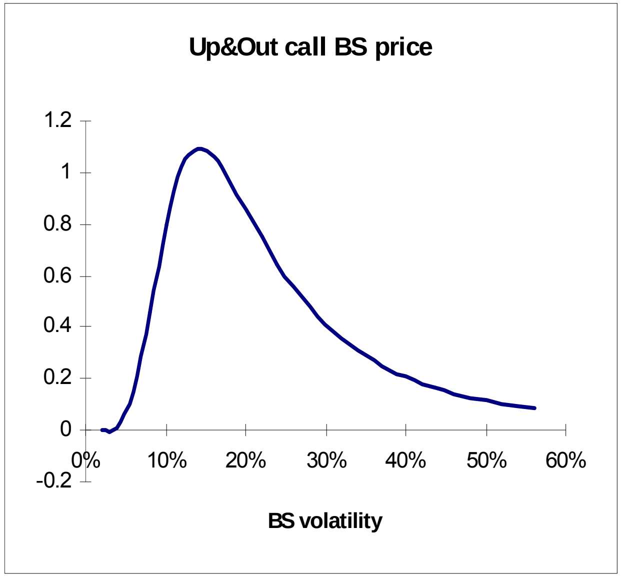


Conclusion: instead of being compensated (like for europeans), vol differences double up price difference!

Black-Scholes price as a function of vol

- <u>Example</u>: Up & Out call

S0 = 100 K=110 B=130



Low vol=>small probability of ending in the money
high vol=>high probability to lose the option

<u>Questions</u>:

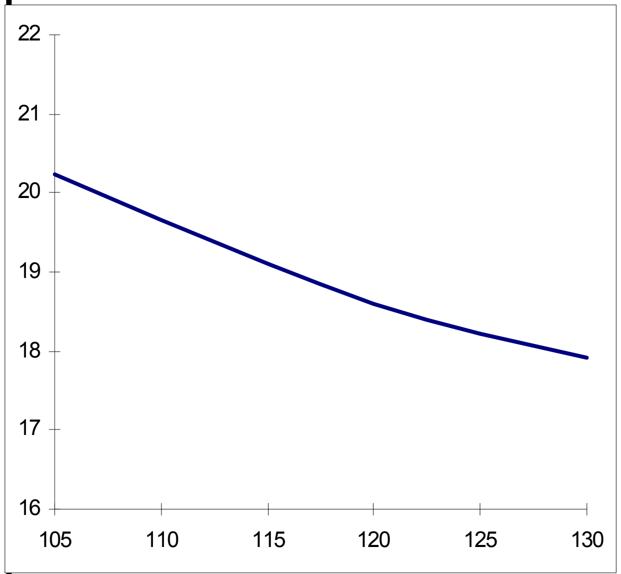
1) if $\hat{\sigma}_{110} = 25\%$, $\hat{\sigma}_{130} = 20\%$, which one should we use?

2) Could the fair price be higher than any of the points on the curve?

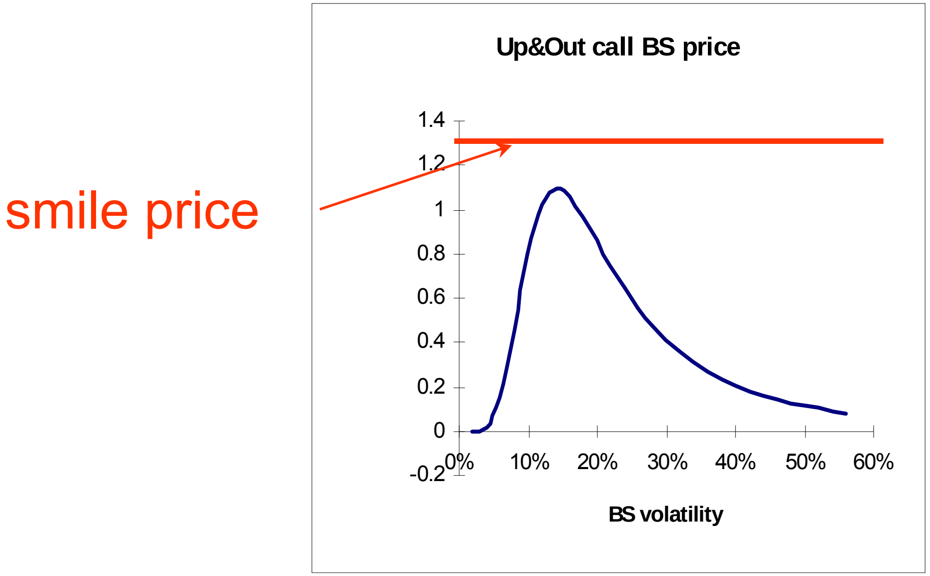
Typical case

If we come back to our Up&Out call

With a smile like this:



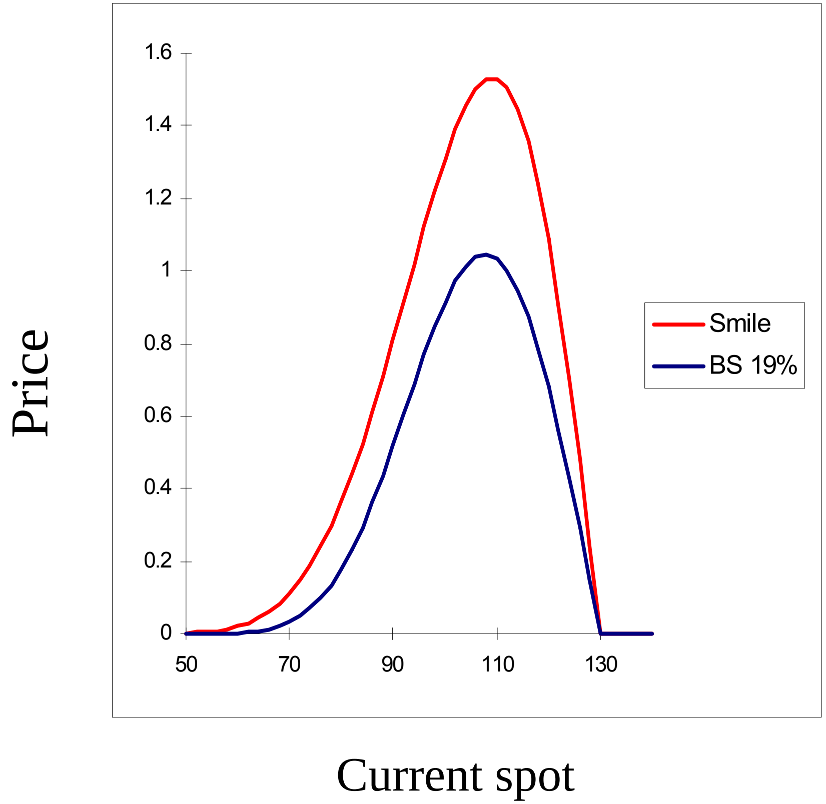
With get the following price:



<u>Conclusion</u>: We cannot produce a correct price with BS

Typical case (2)

Black-Scholes and Smile will lead to fairly different profiles and thus different hedges



Stochastic volatility

So far, volatility was a deterministic function of spot and time:

$$dS = \sigma(S, t)\, dW_t$$

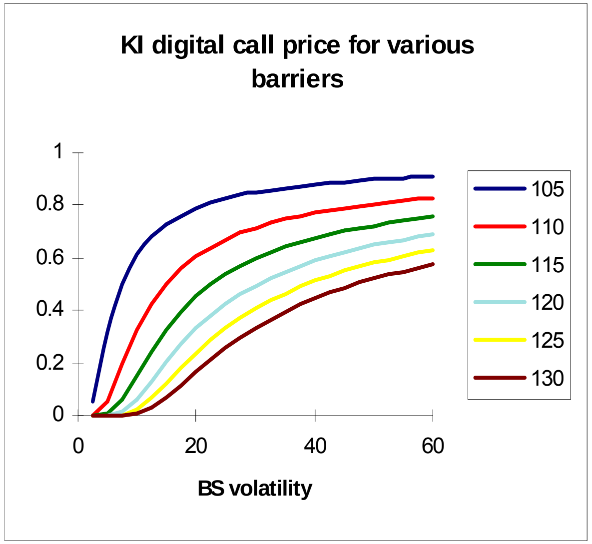
In a stochastic volatility model, volatility will have its own brownian:

$$d\sigma_t = \alpha dt + \beta dZ_t^P$$

Z_t and W_t might be correlated

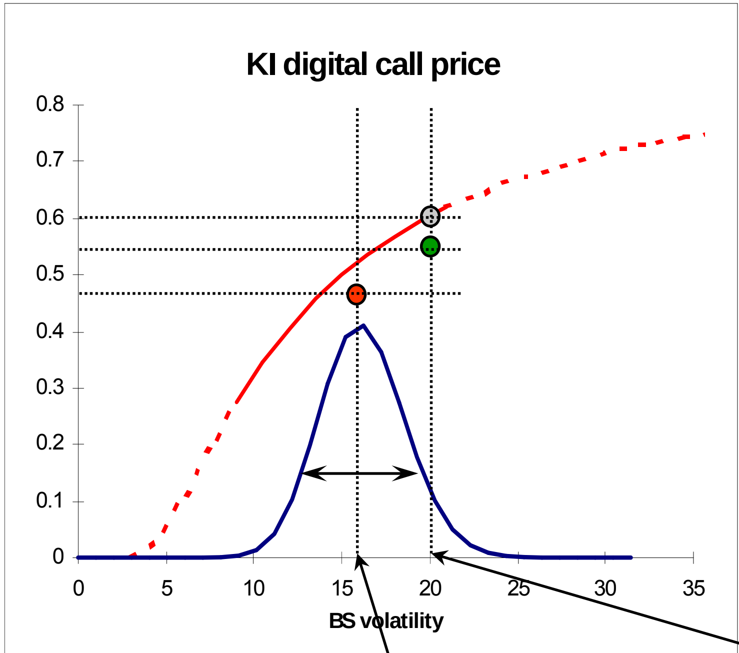
Impact on barrier prices: intuition

• Sensitivity to stochastic volatility will be linked to convexity of the option with respect to volatility.
• We can get an idea of the impact by looking at Black-Scholes price as a function of vol:



Price being generally concave as a function of vol, price should be lower with stochastic vol.
BUT european prices should go down too
=> after calibration, impact of vovol is not obvious.

Impact on barrier prices(2)



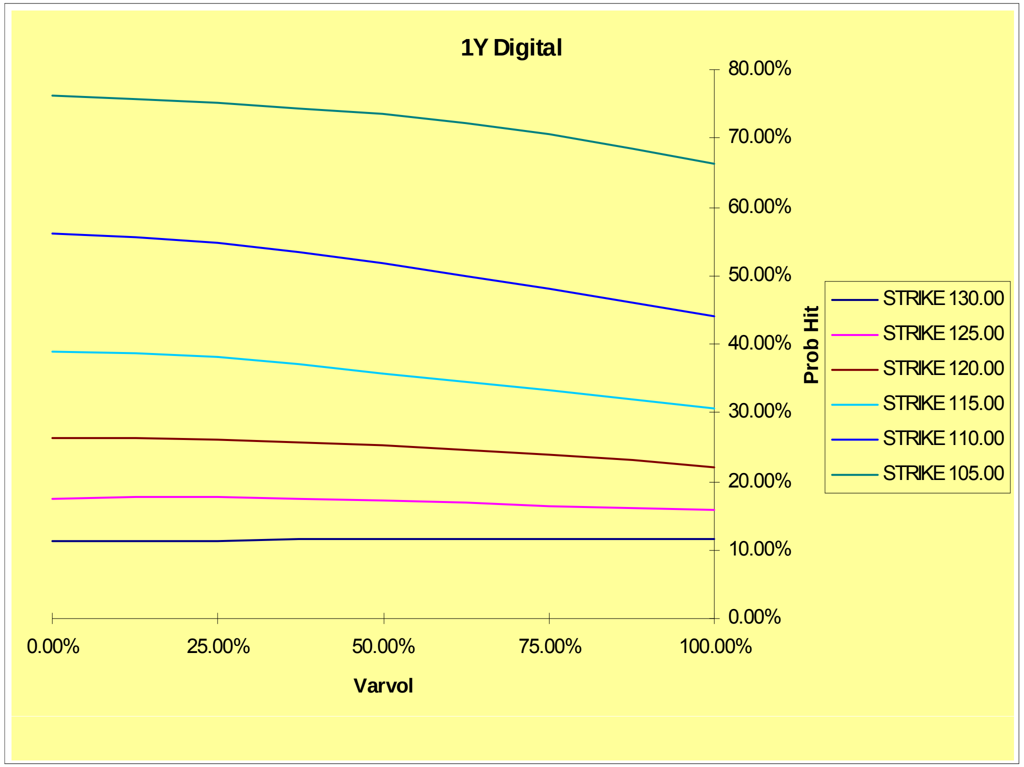
- Black-Scholes price
- Stoch vol price
  - no calibration
- Stoch vol price
  with calibration

Initial Black-Scholes volatility

Corrected expected volatility level after fixed point calibration (local vol has to be readjusted downwards except at the money where it goes slightly upwards)

Impact on barrier prices: numerical results

Current spot : 100
maturity: 1Y
no rates



Conclusion

- A model to price complex options has to price simple options correctly.
- The simplest model to achieve it is a spot/time dependent volatility: the smile model.
- More sophisticated models display stochastic volatility.
- Those models affect barrier option prices substantially.
- They allow to decompose volatility risk through strikes and maturities

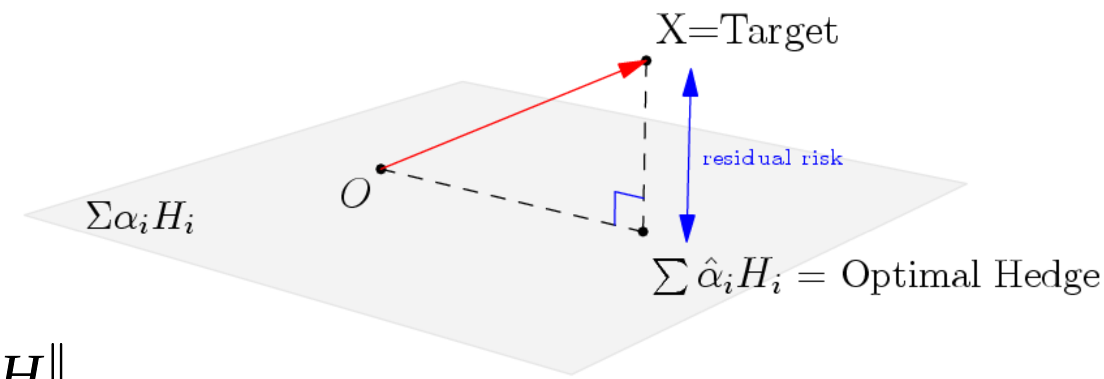
Hedging

The Geometry of Hedging

- Risk measured as $SD[PL_T]$
- Target X, hedge H $\qquad PL_t = X_t - H_t$

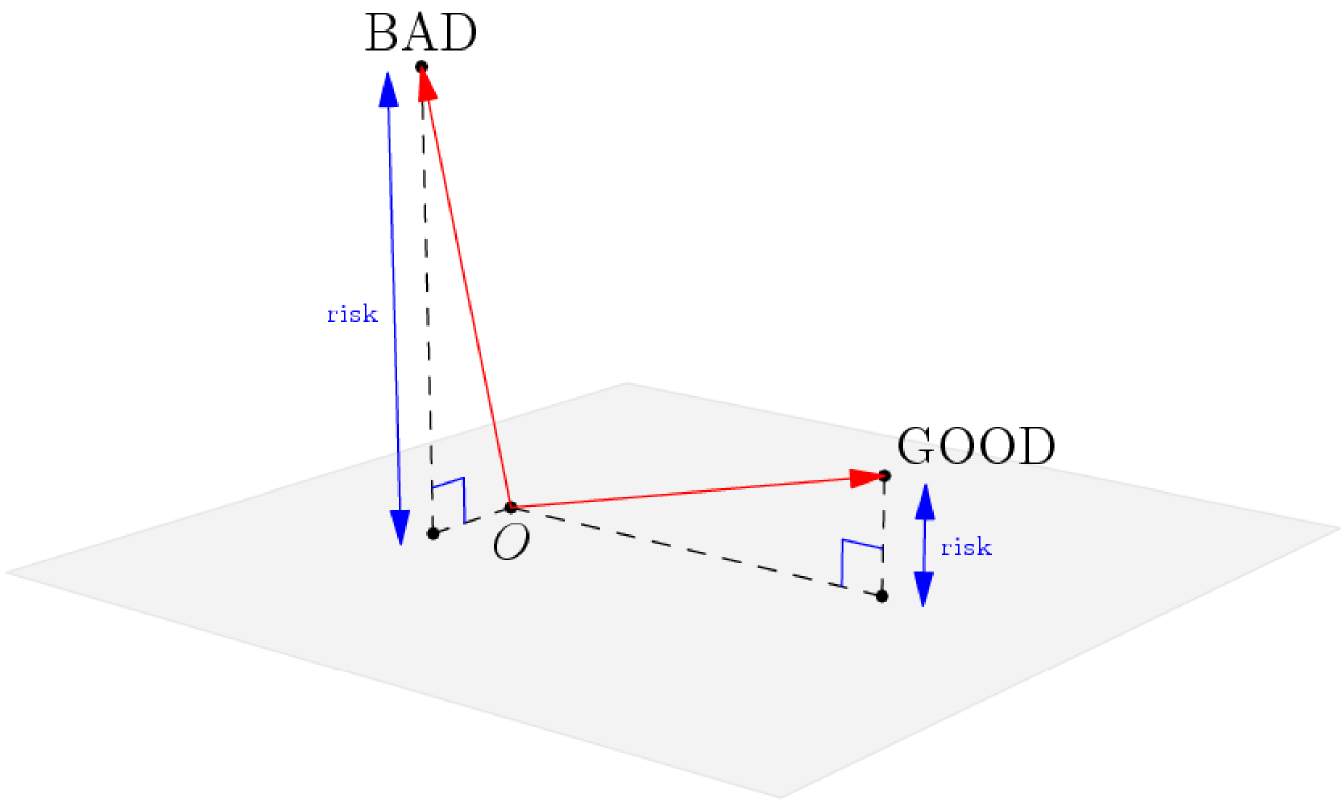
$$Risk = \sqrt{\mathrm{var}[X_T - H_T]} = \|X - H\|_{\perp}$$

- Risk is an L^2 norm, with general properties of orthogonal projections
- Optimal Hedge: $\hat{H}$



$$\|X - \hat{H}\| = \inf_{H \in \mathrm{H}} \|X - H\|$$

The Geometry of Hedging



Where does Tracking Error come from?

- Mainly because reality does not follow a model
- But even within a model:

    1) because trading is discrete in time

    2) because the model is incomplete

$$
\begin{aligned}
Risk^2 = Var[PL_T] &= E[\langle PL \rangle_T] \\
&= E\left[\int_0^T (dPL)^2\right] \\
&= \int_0^T E\left[(dPL)^2\right] \\
&= \int_0^T Var[dPL]
\end{aligned}
$$

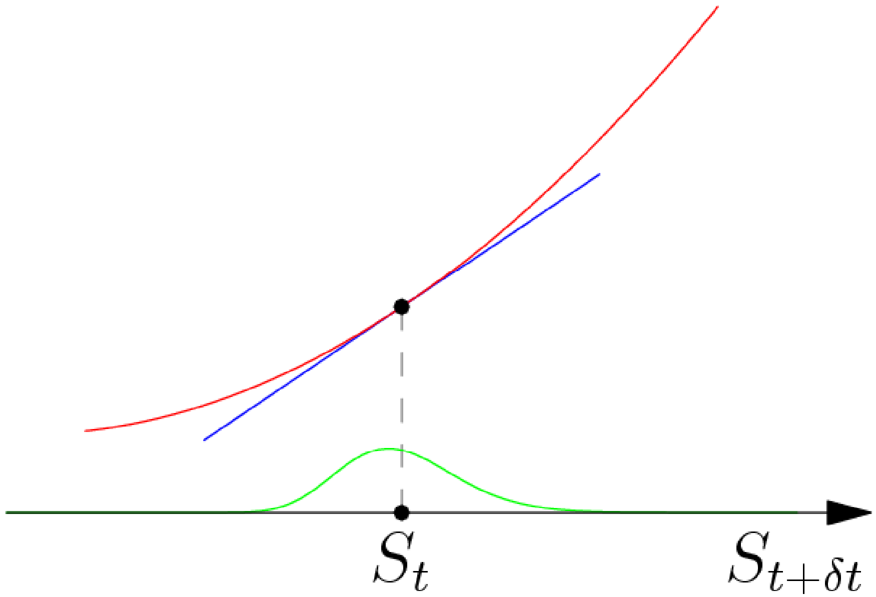
$\Longrightarrow$ We analyze $Var[\delta PL]$

Black-Scholes, no hedge

$$X_T = X_0 + \int_0^T \Delta_t dS_t$$

$$\delta PL_t = dX_t = \Delta_t dS_t$$

$$Var[dX_T] = \Delta_t^2 (dS)^2$$
$$= \Delta_t^2 \sigma^2 S^2 dt$$

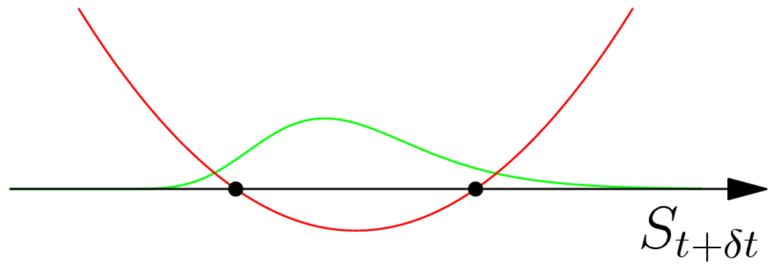


$$Var[X_T] = \sigma^2 E\left[\int_0^T \Delta_t^2 S^2 dt\right]$$

Black-Scholes, Δ-hedge every δt

$$\delta PL_t = \frac{\Gamma}{2}\left((\delta S)^2 - \sigma^2 S^2 \delta t\right)$$

$$Var[\delta PL] = \frac{1}{2}\Gamma^2 S^4 \sigma^4 \delta t^2$$



$$Var[PL_t] = \frac{\sigma^4}{2}\delta t\, E\left[\sum \Gamma^2 S^4 \delta t\right]$$

$$\sim \frac{\sigma^4}{2} E\left[\int_0^T \Gamma^2 S^4 dt\right]\delta t$$

$\Rightarrow \Gamma$-norm

Incomplete Model

Stochastic Vol

$$\begin{cases} dS = \sigma_t dW^1 \\ d\sigma_t = \alpha dt + u_1 dW^1 + u_2 dW^2 \end{cases}$$

$$W^1 \perp W^2$$

$$\Rightarrow C(S,\sigma,t)$$

$$dC = \underbrace{\left(C_t + \alpha C_\sigma + \frac{\sigma^2}{2} C_{SS} + u_1 \sigma_t C_{S\sigma} + \frac{1}{2} C_{\sigma\sigma} \left(u_1^2 + u_2^2 \right) \right)}_{=0 \Rightarrow PDE} dt + \underbrace{\left(C_S + \frac{C_\sigma u_1}{\sigma} \right)}_{\min Var\, \Delta\, hedge} dS + \underbrace{C_\sigma u_2 dW^2}_{residual\, risk}$$

$$dPL = u_2 C_\sigma dW^2$$

$$PL_T = \int_0^T u_2 C_\sigma dW^2$$

$$V\left[TE_T\right] = E\left[\int_0^T u_2^2 C_\sigma^2 dt\right]$$

δt Δ hedging $\Rightarrow \Gamma$ risk

$\sigma(\omega)$ model $\Rightarrow$ Vega risk

Link with MCV

In Monte-Carlo simulations, Multiple Control Variates
Target Y, hedging instruments H_i
Path $\omega_j \to \left(X\left(\omega_j\right), H_1\left(\omega_j\right), \ldots, H_n\left(\omega_j\right)\right)$

Multiple regression of $X\left(\omega_j\right)$ on $\left(H_i\left(\omega_j\right)\right)_i$

$$\Rightarrow X = \sum_{i=1}^{n} \alpha_i H_i + \varepsilon \qquad \hat{X} = X^{MC} + \sum \alpha_i \left(\overline{H}_i - H_i^{MC}\right)$$

This projection corresponds to

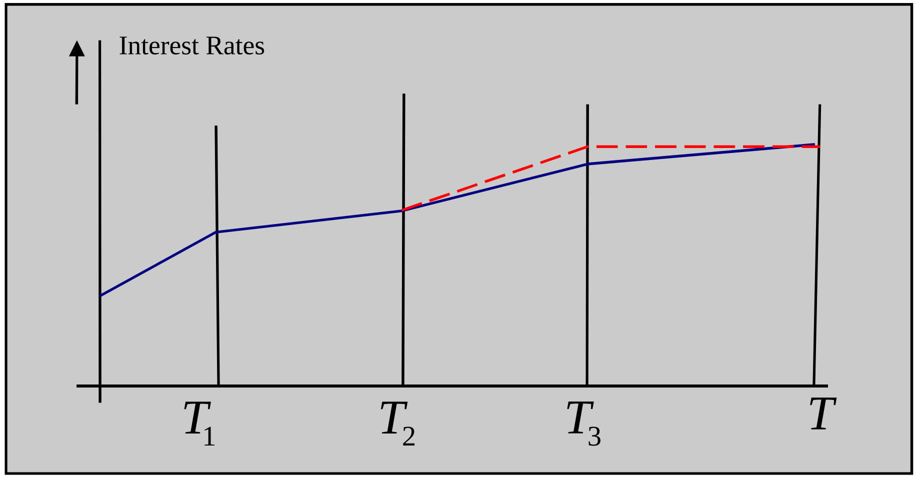
- Static position in options
- <u>No</u> dynamic hedging

Vega hedge

- Smile prices can be very different from Black-Scholes prices
- If the market smile disappears tomorrow, you might get hurt
- <u>Conclusion</u>: you have to set up a vega hedge composed of European options
- <u>Question</u>: which European options and which proportions?

Bucket Hedging For Interest Rates

- Valuate PF with initial YC
- Bump rate for one maturity



- Revalue and Compute sensitivity w.r.t. this maturity
- Repeat for all maturities
- Compute hedging PF: immunised against any move of YC today

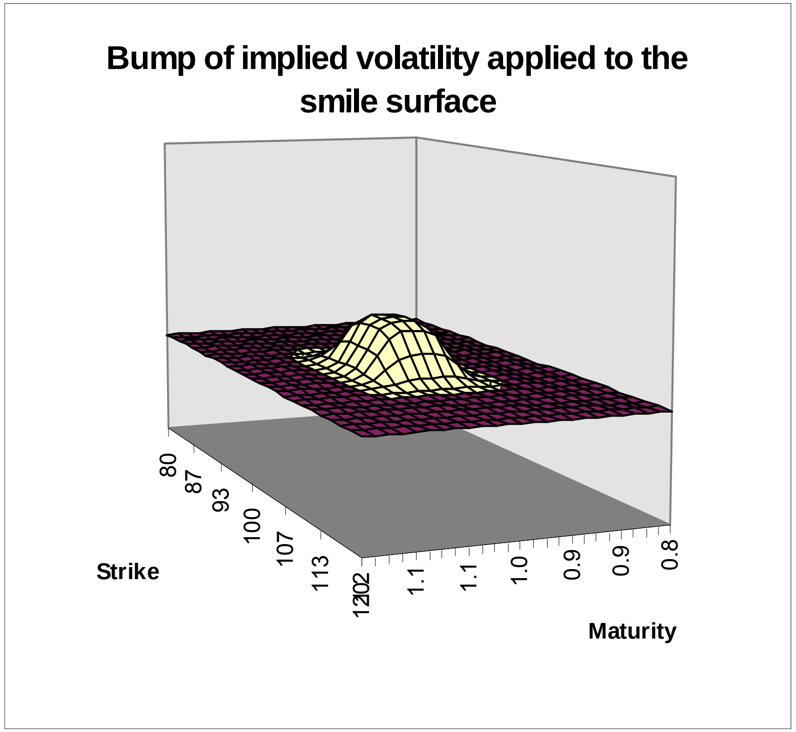
Volatility: Superbucket Hedging

- Extension of interest rate bucket hedging: bump implied volatilities around one Strike and one Maturity
- Revalue PF with a model fully calibrated to the vol surface (like LVM)
- Compute sensitivities, or local Vegas, to decompose volatility risk through Strikes and Maturities
- Compute «Superbucket Hedge »: PF of European options with same sensitivities

Superbuckets

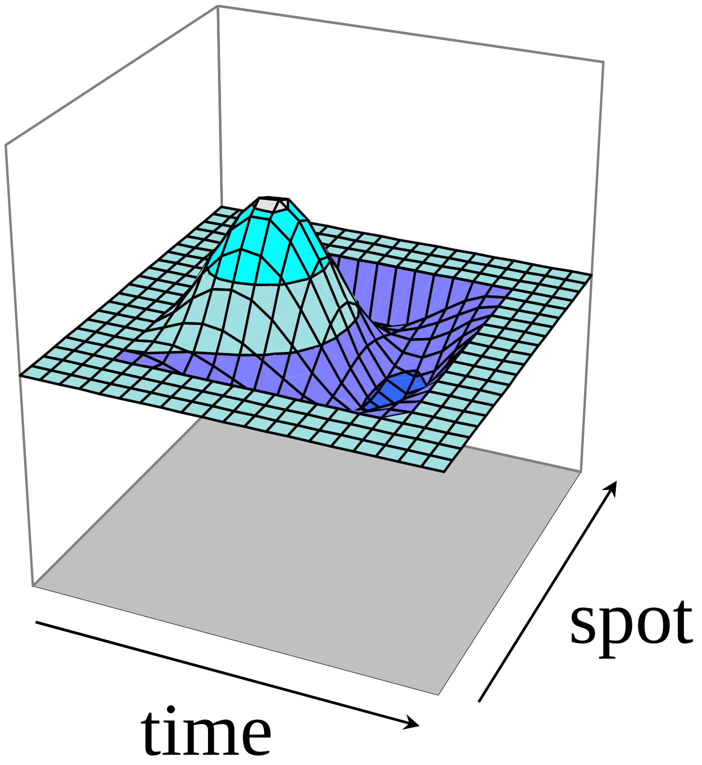
Price depends on all points of implied vol surface

We want to compute a sensitivity to all points

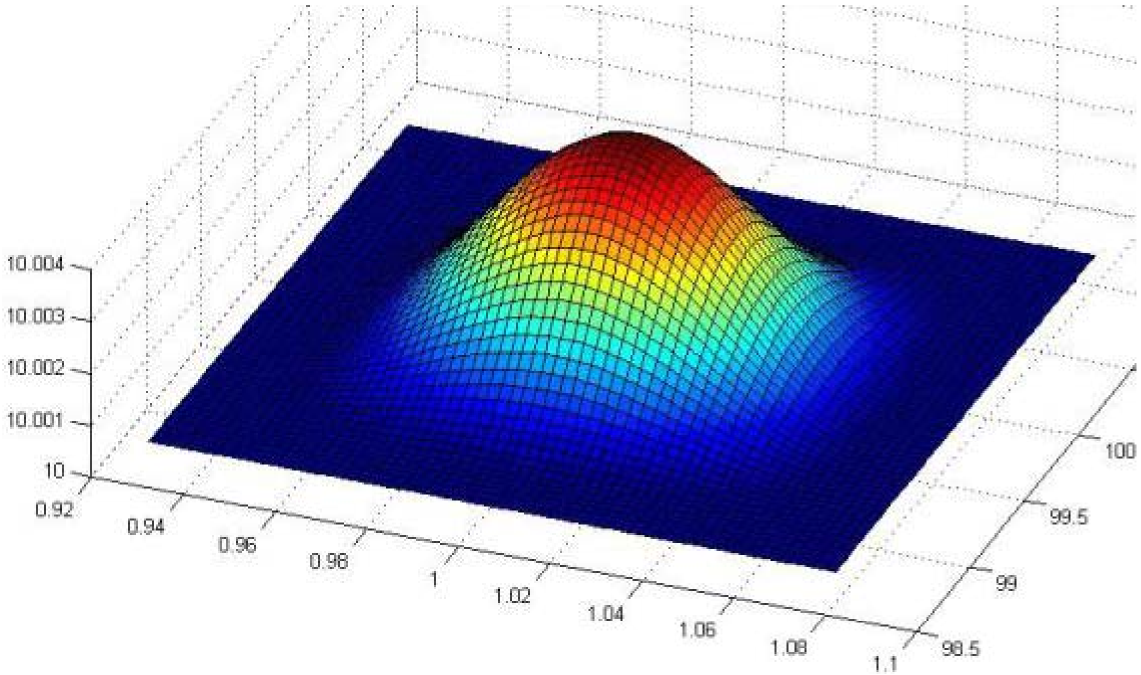


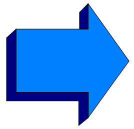
Superbuckets (2)

Applied previous bump on implied volatility generates the following bump in local volatility:



Bump in Implied Volatility

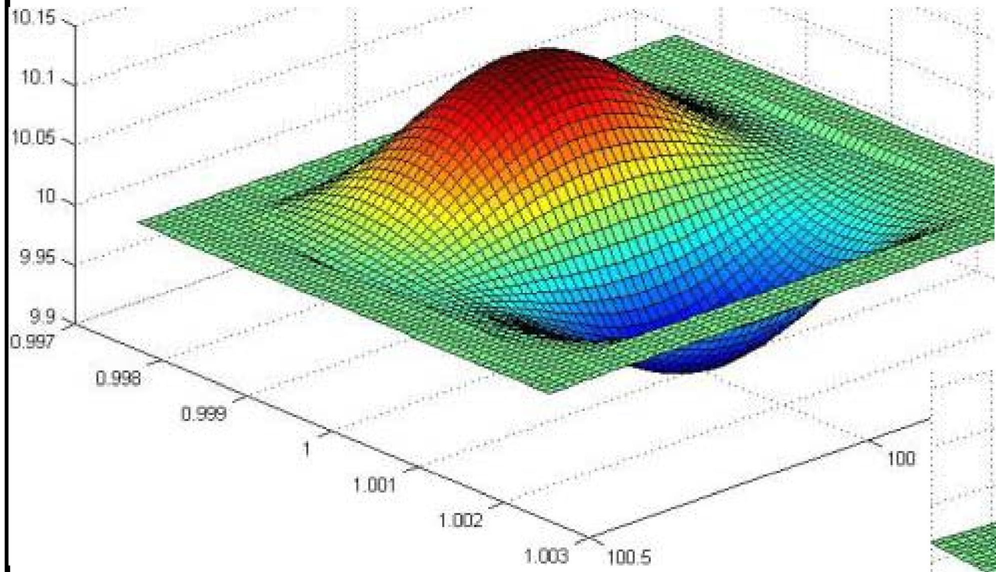




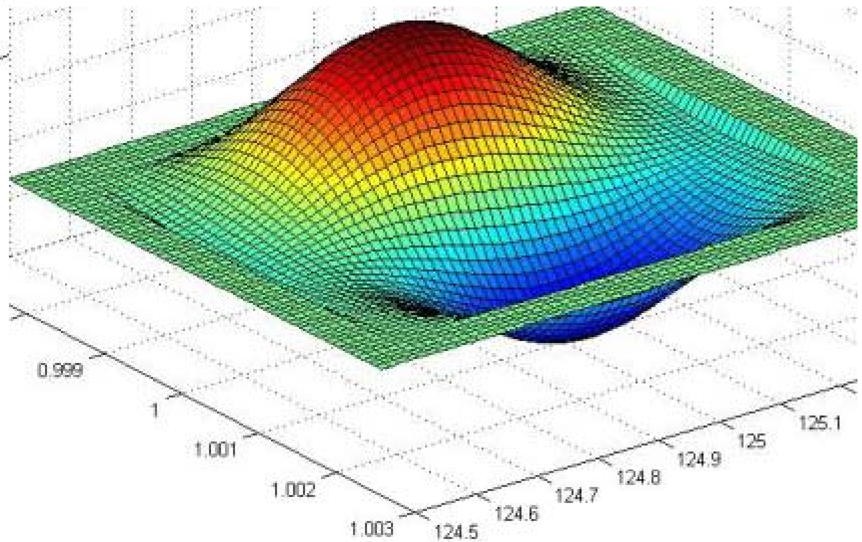
?

Bumped Local Volatility
small time step

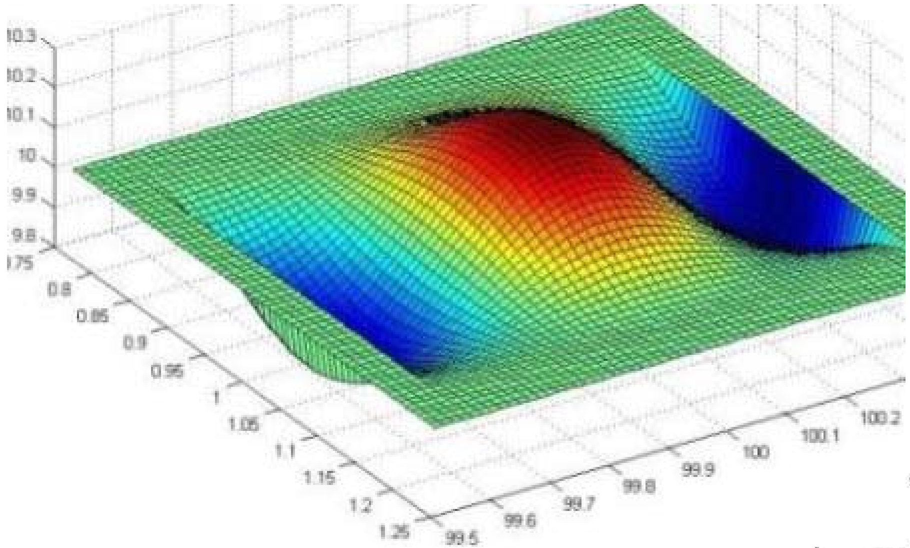
ATM



OTM

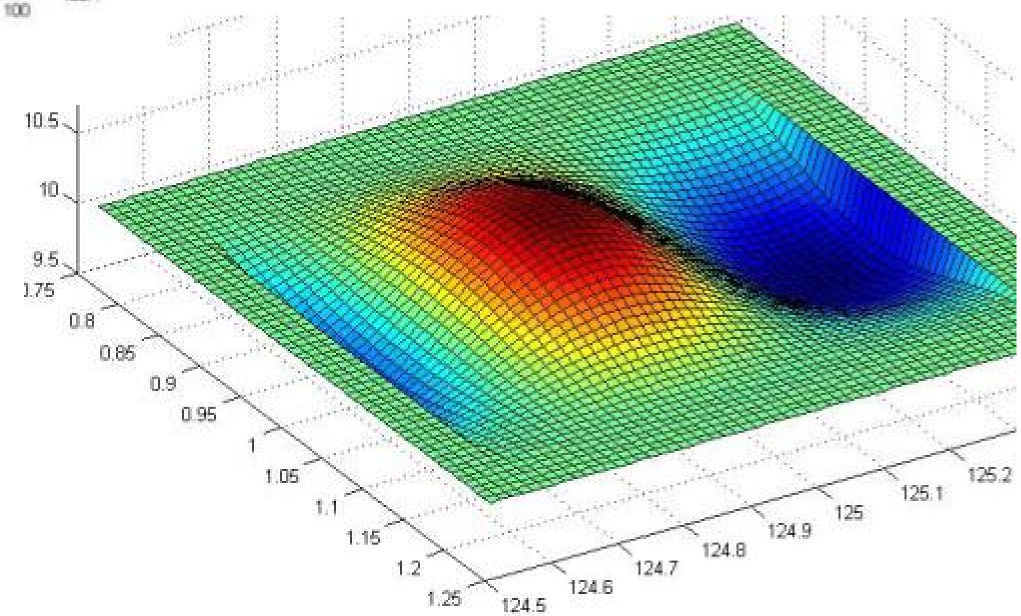


Bumped Local Volatility large time step

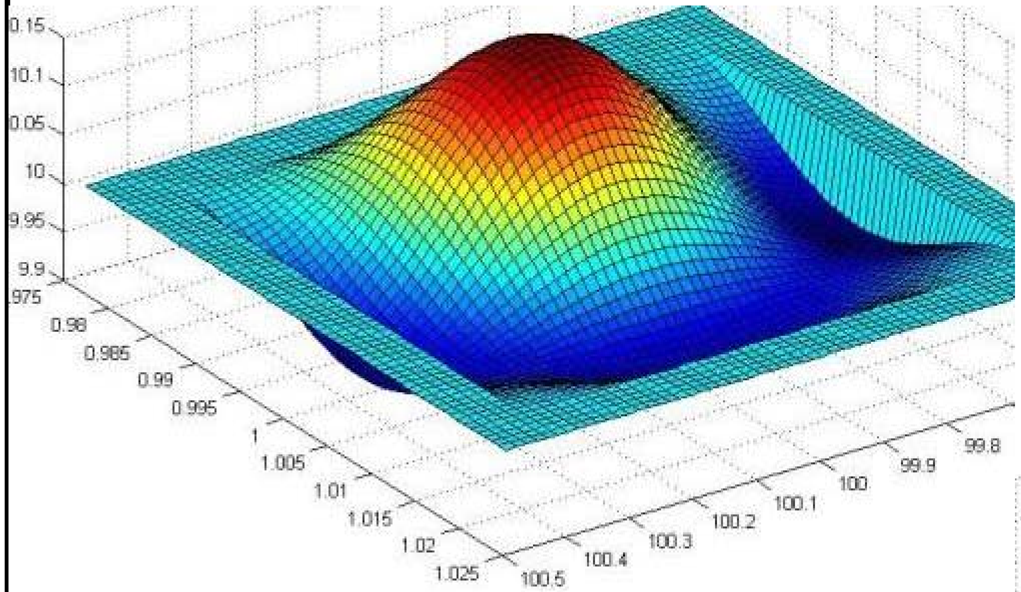


ATM

OTM

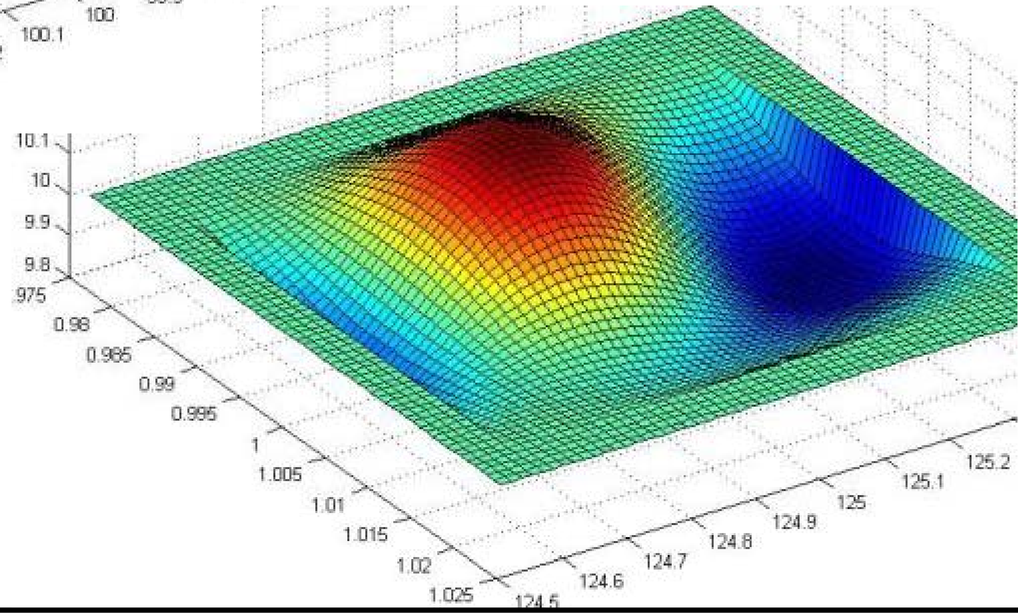


Bumped Local Volatility average time step



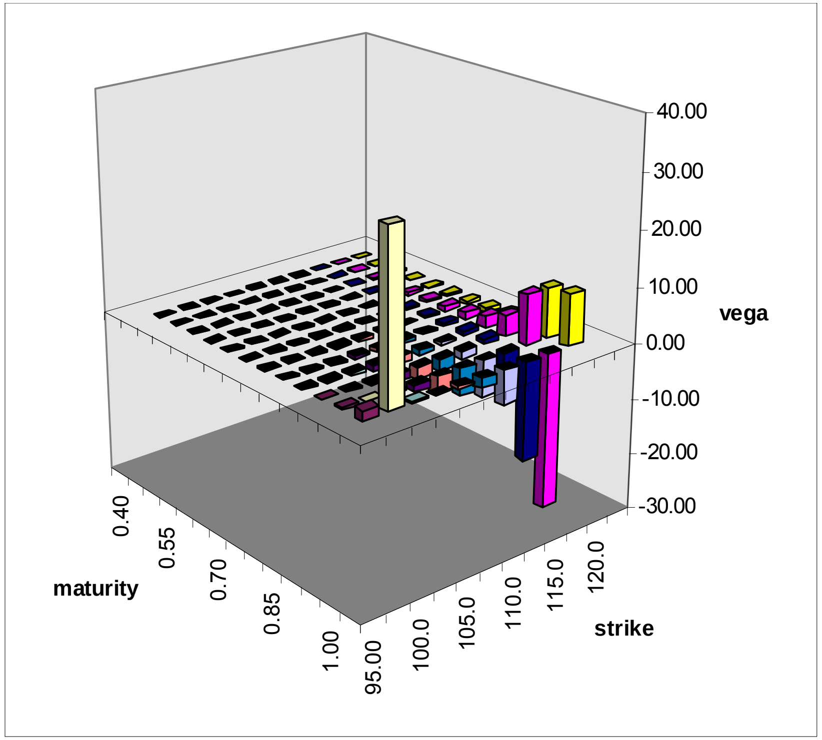
ATM

OTM



Superbuckets (3)

Finally, we have the decomposition of the vega risk throughout strikes and maturities



<u>Example</u>: up & out call 1y, strike 100, barrier 120, spot 100, no rates

Superhedge 1

- Interpretation of Superbuckets as hedging PF: "Superhedge"
- Superhedge as tangent PF: first order hedge because Superbucket is a gradient
- Superhedge as projection of PF on European options
- Superbucket allows for risk aggregation

Superhedge 2

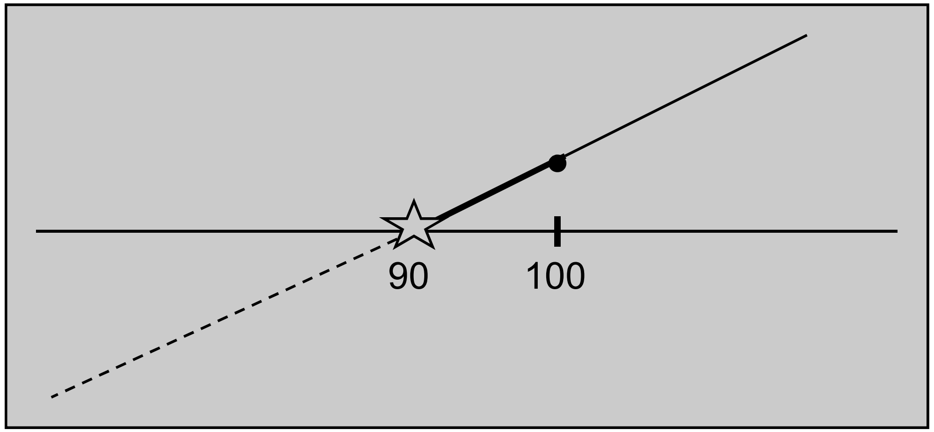
- Perfect hedge w.r.t. <u>any</u> <u>instantaneous</u> vol surface move
- Dynamic volatility rehedging may have a cost (hedge against moves not authorized by LVM)
- To account for this second order effect, need to have a fuller model (e.g. stochastic vol)

Limitation

- Superhedge not stable with time
- For instance, for barriers options:
  - Ignores what happens beyond barrier level
  - Unstable hedge: vanishes when close to barrier level

Static Hedging

- Example with barrier options: Assume no rates, no jumps



$S_0 = 100$

Down & Out Call strike 90, barrier 90

Equivalent to 10 + Future with Stop Loss at 90

Price = 10 independent of vol or model

Barrier Static Hedging

Down & Out Call Strike K, Barrier L, r=0 :

- With BS: $DOC_{K,L} = C_K - \frac{K}{L} P_{L^2/K}$

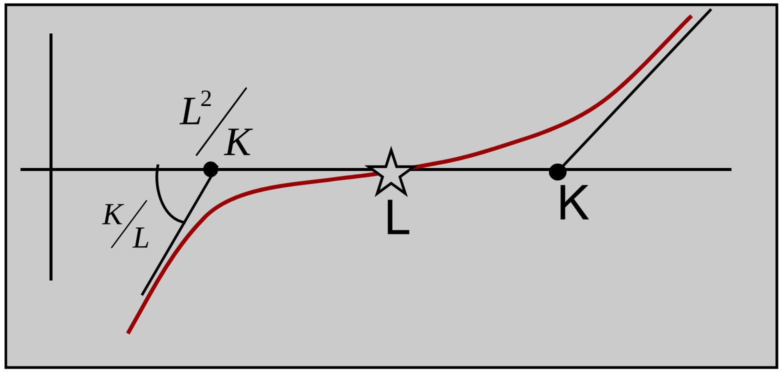
If $S_t = L$,unwind hedge, at 0 cost

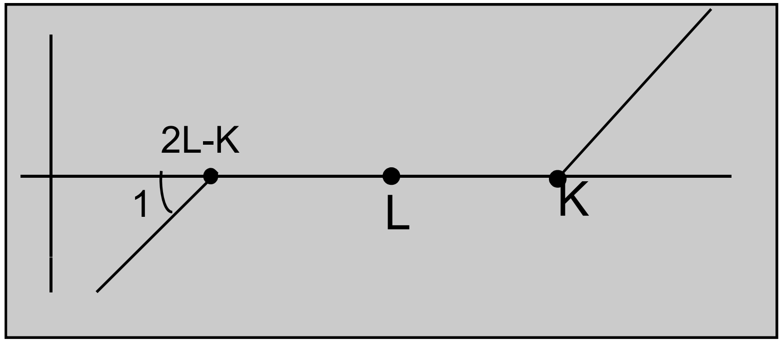
If not touched, IV's are equal

- With normal model

$$DOC_{K,L} = C_K - P_{2L-K}$$

$$dS = \sigma dW$$

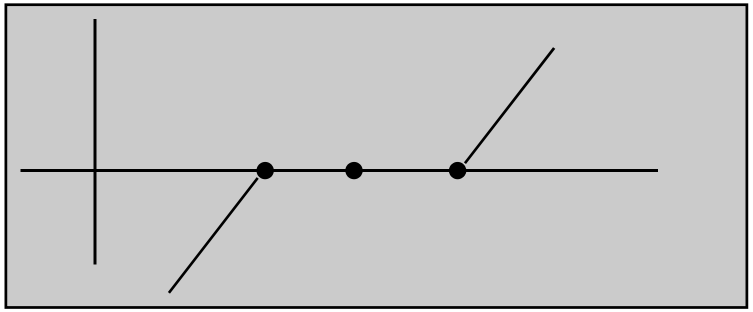




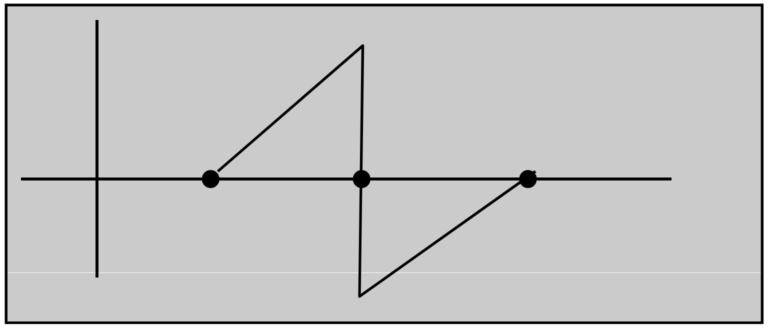
Skew Adjusted Barrier Hedges

$$dS = (aS + b)dW$$

$$DOC_{K,L} \leftrightarrow C_K - \frac{aK+b}{aL+b} P_{\frac{aL^2 + b(2L-K)}{aK+b}}$$



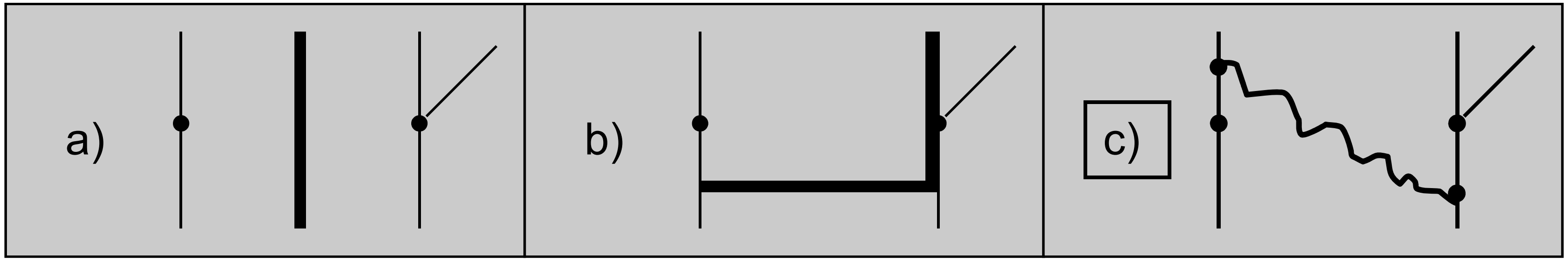
$$UOC_{K,L} \leftrightarrow C_K - (L-K)\left(2Dig_L + \frac{a}{aL+b} C_L\right) - \frac{aK+b}{aL+b} C_{\frac{aL^2 + b(2L-K)}{aK+b}}$$



Boundary Condition Matching

Other Boundary Condition matching:

- If same PF has the same as $DOC_{K,L}$ on some line separating the initial value from the final payoff, it is a static hedge within the model



- For instance, in b) a PF ($C_{K,T} - \int_0^T \alpha(t) P_{K(t),t} dt$) can be computed by time induction to match the boundary condition

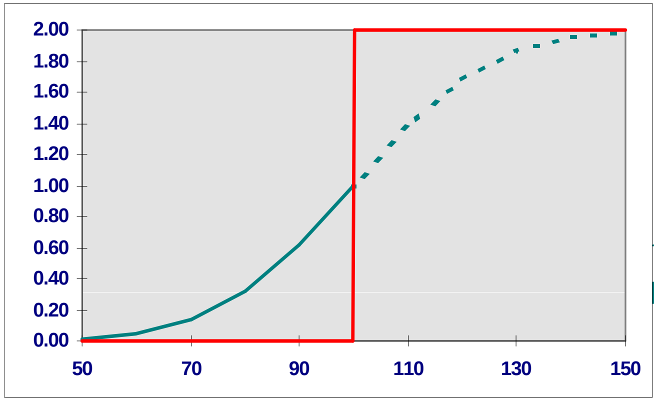
Static hedge: Digitals

Normal Model, r=0

1 American Digital = 2 European Digitals

From reflection principle,

Proba ($Max_{0\text{-}T} > K$) = 2 Proba ($S_T > K$)



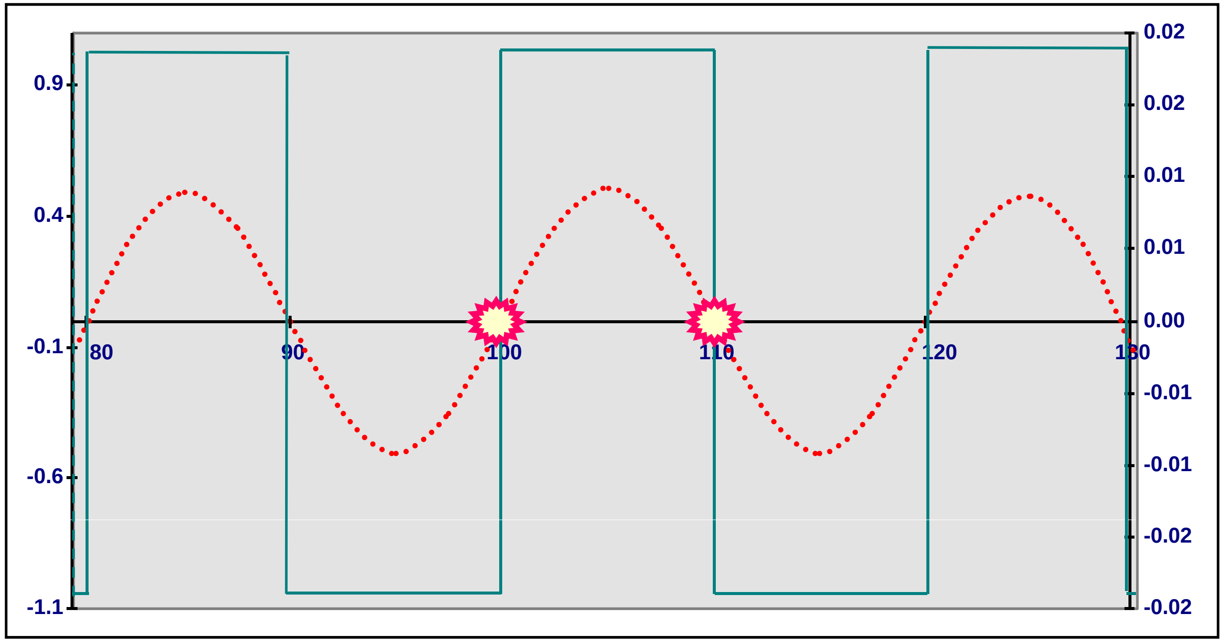
As a hedge, 2 European Digitals meet boundary conditions for the American Digital.

If S reaches K,
the European digital is worth 0.50.

Static hedge: Double knock-out digital

Normal Model, r=0

2 symmetry points: infinite reflections

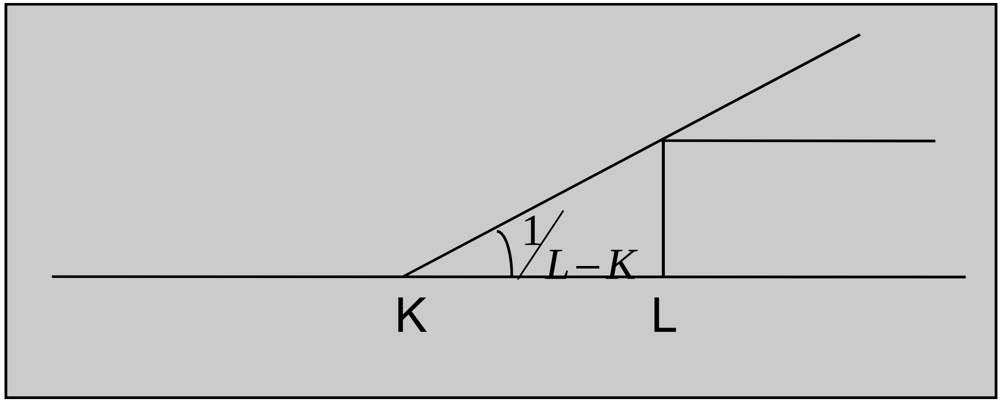


Price & Hedge: infinite series of digitals

Static Hedging 4: Profile Dominance

$OT_{L,T} \equiv One\,Touch$ (give $1 if L touched before maturity T)

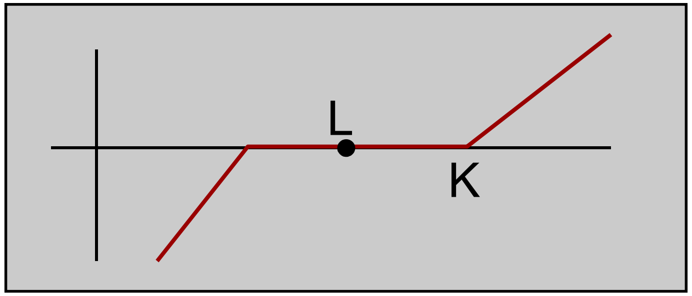
$$\forall K \quad OT_{L,T} \le \frac{1}{L-K} C_{K,T}$$



If L reached, sell $\frac{1}{L-K} C_{K,T} \Rightarrow \quad OT_{L,T}(S_0, t_0) \le \underset{K}{Inf} \frac{1}{L-K} C_{K,T}(S_0, t_0)$

Static Hedging 5: Model Dominance

- Back to $DOC_{K,L}$



- An assumption as the skew at L corresponds to an affine model

  $dS = (aS + b)dW$ (displaced LN)

- $DOC_{K,L}$ priced as in BS with shifted K and L gives new hedging PF which is >0 when L is touched if Skew assumption is conservative

Gamma Projection

- Cancel future $\Gamma \Rightarrow$ no volatility exposure
- Tracking Error of Δ hedging in discrete time
  $= E\left[\int \sigma^2 \Gamma S^2\right]$ (L² norm)

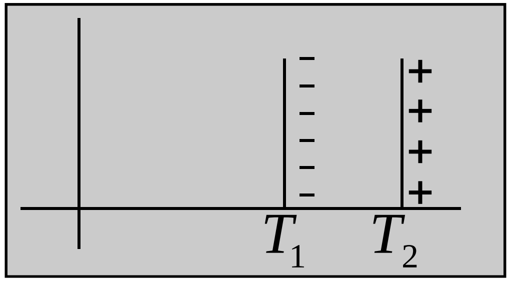
- Minimise future $\Gamma \Leftrightarrow \underset{\alpha_i}{Min}\left\| X - \sum \alpha_i Y_i \right\|$ where Y_i are hedging instruments
- Superbucket hedge cancels future Γ in <u>average</u>

PCA Hedging

- Identify principal moves of implied volatility surface
- Compute sensivities of the PF to the first n factors
- Select a set of hedging instruments and compute their sensitivities
- Find a hedging PF with same sensitivity profile

Cliquet hedge

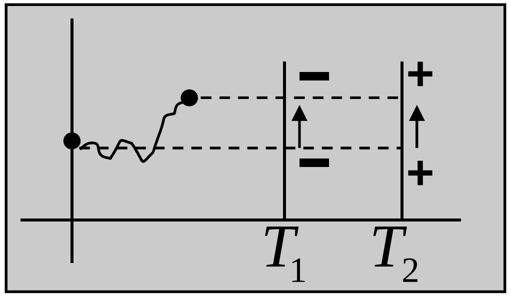
Static



depends on the Skew maturity T_2 observed at T_1

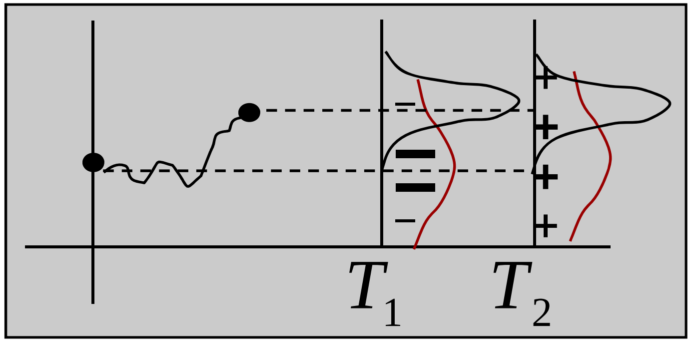
Dynamic

roll ATM straddle



depends on (unlockable) rolling cost

Intermediate density weighted



also subject to the "rolling coaster of rolling costs"

Conclusion

- Pricing assumes a certain model
- Hedging can be performed
  - within the model
  - outside the model, to hedge against the model
- Hedging is more complex than pricing because depends on
  - the events against which to hedge
  - the instruments with which to hedge

Volatility Replication

Volatility Replication

$$\frac{dS}{S} = \sigma_t dW$$ Apply Ito to f(S,t).

$$df = f_S dS + f_t dt + \tfrac{1}{2} f_{SS} \sigma_t^2 S^2 dt$$

$$\Rightarrow \int_0^T f_{SS}(S_t,t)\sigma_t^2 S^2 dt = 2\left[\underbrace{f(S_T,t) - \int_0^T f_t(S_t,t)dt}_{\text{European PF}} - \underbrace{\int_0^T f_S(S_t,t)dS_t}_{\Delta\text{-hedge}} \right]$$

To replicate $\int_0^T g(S,t)\sigma_t^2 dt$,find f / $g(S,t) = f_{SS}(S,t)S^2$: $f = \iint \frac{g}{S^2}$

Examples

Variance Swap	$g(S,t)=1$	$f(S,t)=-\ln(\frac{S}{S_0})$
Corridor Variance Swap	$g(S,t)=1_{[a,b]}(S_t)$	$f(S,t)=-\ln(\frac{S}{S_0})$ on [a,b] + linear extrapolation
FWD Variance Swap	$g(S,t)=1_{[T_1,T_2]}(t)$	$f(S,t)=-\ln(\frac{S}{S_0})\times 1_{[T_1,T_2]}(t)$
Absolute Variance Swap	$g(S,t)=S^2$	$f(S,t)=\frac{(S-S_0)^2}{2}$
Local Time at level K	$g(S,t)=\delta_K(S)$	$f(S,t)=\frac{(S-K)^+}{K^2}$

Conditional Instantaneous FWD Variance

From local time:

$$E\left[\int_0^T \sigma_t^2 \delta_K(S)dt\right] = 2\times\frac{C(K,T)}{K^2}$$

Differentiating wrt T:

$$E\left[\sigma_T^2 \delta_K(S_T)\right] = E\left[\sigma_T^2 \middle| S_T = K\right]\cdot E\left[\delta_K(S_T)\right] = \frac{2}{K^2}\times\frac{\partial C}{\partial T}(K,T)$$

And, as:

$$E\left[\delta_K(S_T)\right] = \frac{\partial^2 C}{\partial K^2}(K,T)$$

$$E\left[\sigma_T^2 \middle| S_T = K\right] = \frac{2}{K^2}\times\frac{\frac{\partial C}{\partial T}(K,T)}{\frac{\partial^2 C}{\partial K^2}(K,T)} = \sigma_{loc}^2(K,T)$$

Quadratic Variation Related Quantities

- If M is a martingale

$X \equiv M^2 - \langle M \rangle$ is a martingale and $X_T = M_0^2 + 2\int_0^T M_t \, dM_t$

$Y \equiv e^{M - \frac{1}{2}\langle M \rangle}$ is a martingale and $Y_T = e^{M_0} + \int_0^T Y_t \, dM_t$

- Both X and Y replicable by Δ hedging.

QV Related Quantities (2)

- <M> appears
  - Additively in X and can be replicated because M^2 is replicated by Europeans
  - Multiplicatively in Y: requires correlation assumption

    Then $e^{-\frac{1}{2}\langle M \rangle}$ becomes replicable. Applying it to λM for various constant λ's and combining them make contingent claims on <M> (e.g. vol swaps or options on vol) replicable

Forward Skew

Forward Skews

In the absence of jump :

model fits market $\Leftrightarrow \forall K,T \quad E[\sigma_T^2 | S_T = K] = \sigma_{loc}^2(K,T)$

This constrains

a) the sensitivity of the ATM short term volatility wrt S;

b) the average level of the volatility conditioned to S_T=K.

a) tells that the sensitivity and the hedge ratio of vanillas depend on the calibration to the vanilla, not on local volatility/ stochastic volatility.

To change them, jumps are needed.

<u>But</u> b) does not say anything on the conditional forward skews.

Sensitivity of ATM volatility / S

At t, short term ATM implied volatility ~ σ_t.

As σ_t is random, the sensitivity $\dfrac{\partial \sigma^2}{\partial S}$ is defined only in average:

$$E_t[\sigma_{t+\delta t}^2 - \sigma_t^2 \big| S_{\delta t} = S_t + \delta S] = \sigma_{loc}^2(S_t + \delta S, t + \delta t) - \sigma_{loc}^2(S_t - t) \approx \frac{\partial \sigma_{loc}^2(S,t)}{\partial S} \cdot dS$$

In average, σ_{ATM}^2 follows σ_{loc}^2 .

Optimal hedge of vanilla under calibrated stochastic volatility corresponds to perfect hedge ratio under LVM.

Options on Realized Variance

Delta Hedging

- We assume no interest rates, no dividends, and absolute (as opposed to proportional) definition of volatility
- Extend f(x) to f(x,v) as the Bachelier (normal BS) price of f for start price x and variance v:

$$f(x,v) \equiv E^{x,v}[f(X)] \equiv \frac{1}{\sqrt{2\pi v}} \int f(y) e^{-\frac{(y-x)^2}{2v}} dy$$

with f(x,0) = f(x)

- Then, $f_v(x,v) = \frac{1}{2} f_{xx}(x,v)$
- We explore various delta hedging strategies

Calendar Time Delta Hedging

- Delta hedging with constant vol: P&L depends on the path of the volatility and on the path of the spot price.
- Calendar time delta hedge: replication cost of $f(X_t, \sigma^2.(T-t))$

$$f(X_0, \sigma^2.T) + \frac{1}{2}\int_0^t f_{xx}(dQV_{0,u} - \sigma^2 du)$$

- In particular, for sigma = 0, replication cost of $f(X_t)$

$$f(X_0) + \frac{1}{2}\int_0^t f_{xx} dQV_{0,u}$$

Business Time Delta Hedging

- Delta hedging according to the quadratic variation: P&L that depends only on quadratic variation and spot price

$$df(X_t, L - QV_{0,t}) = f_x dX_t - f_v dQV_{0,t} + \frac{1}{2} f_{xx} dQV_{0,t} = f_x dX_t$$

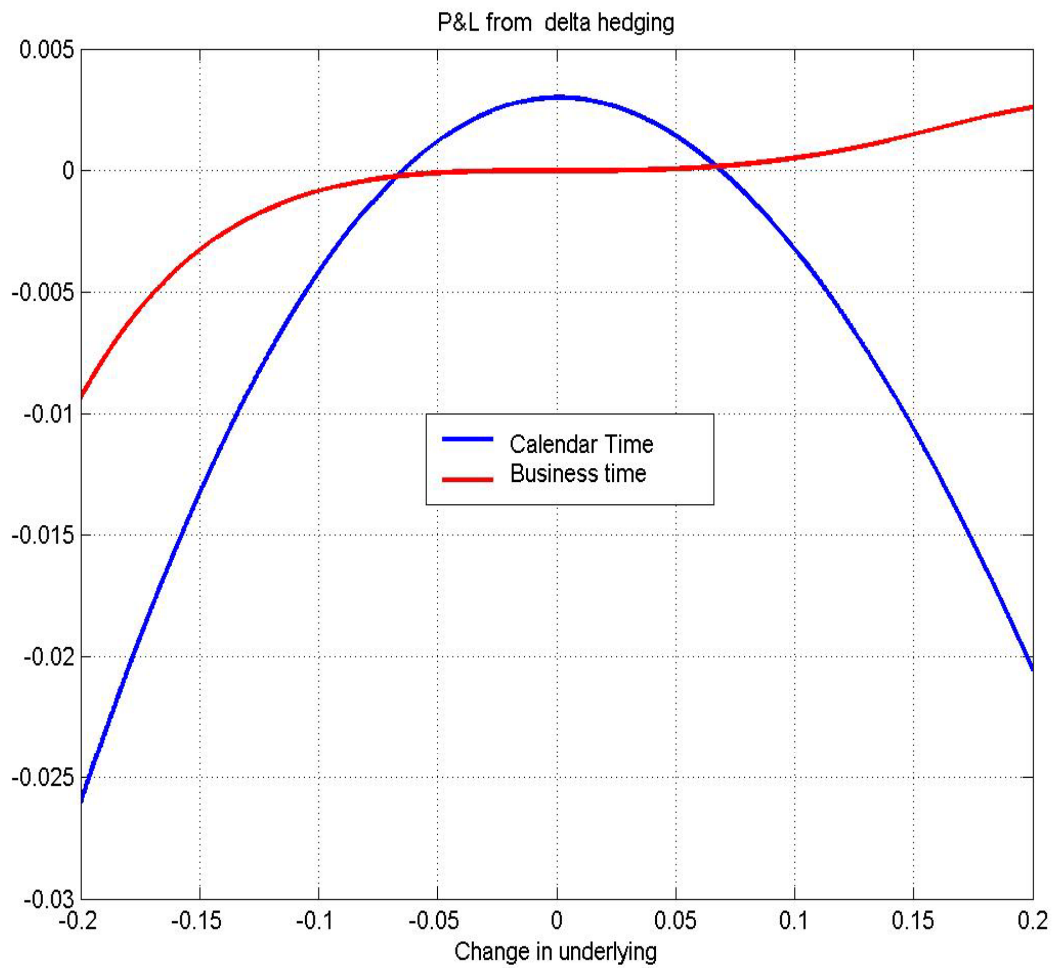
- Hence, for $QV_{0,T} \leq L,$

$$f(X_t, L - QV_{0,t}) = f(X_0, L) + \int_0^t f_x(X_u, L - QV_{0,u})\, dX_t$$

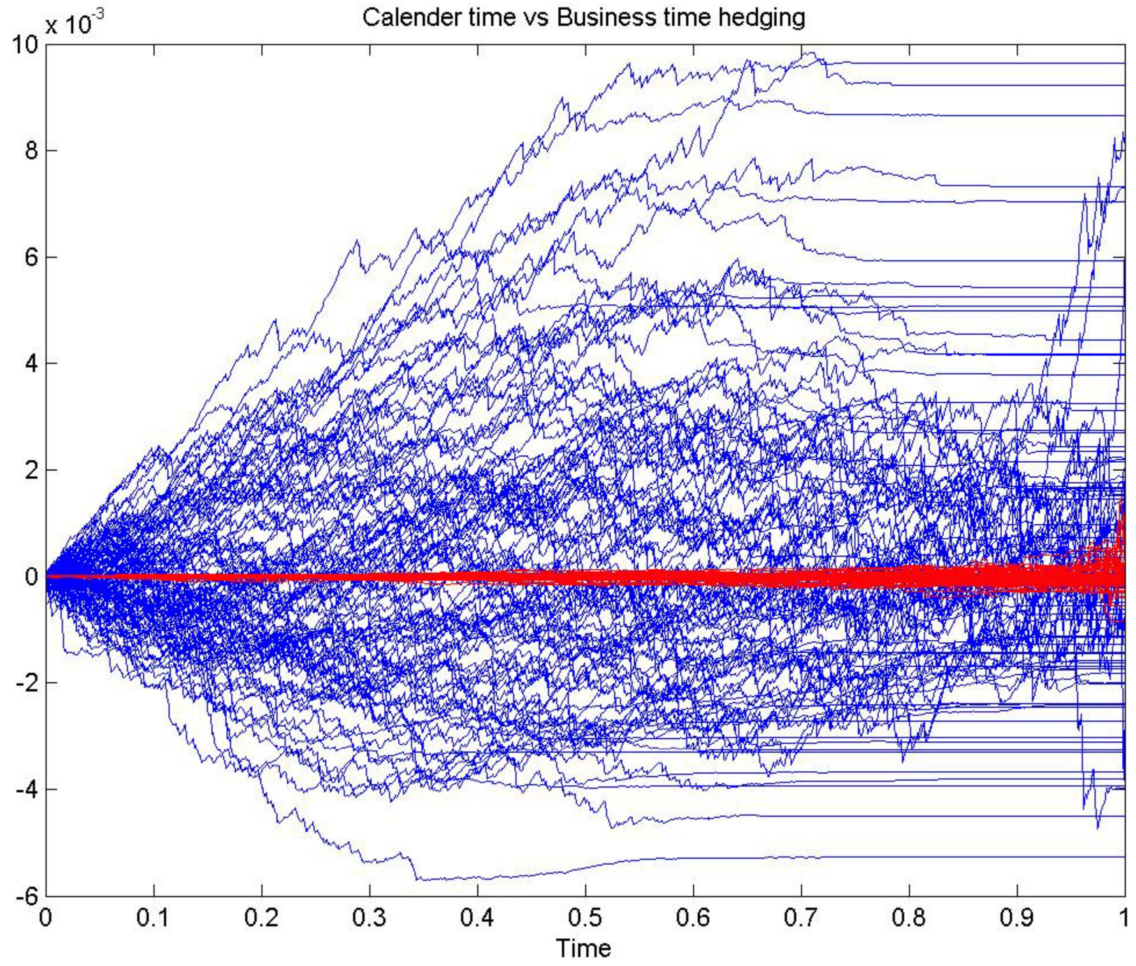
And the replicating cost of $f(X_t, L - QV_{0,t})$is $f(X_0, L)$

$f(X_0, L)$ finances exactly the replication of f until $\tau : QV_{0,\tau} = L$

Daily P&L Variation



Tracking Error Comparison



Hedge with Variance Call

- Start from $f(X_0, L)$ and delta hedge f in "business time"
- If V < L, you have been able to conduct the replication until T and your wealth is $f(X_T, L-V) \geq f(X_T)$
- If V > L, you "run out of quadratic variation" at τ < T. If you then replicate f with 0 vol until T, extra cost:

$$\frac{1}{2}\int_\tau^T f''(X_t)dQV_t \leq \frac{M_f}{2}\int_\tau^T dQV_t = \frac{M_f}{2}(V-L)$$

where $M_f \equiv \sup\{f''(x)\}$

- After appropriate delta hedge, $f(X_0, L) + \frac{M}{2} Call_L^V$

dominates $f(X_T)$which has a market price $f(X_0, L^f)$

Lower Bound for Variance Call

- C_L^V : price of a variance call of strike L. For all f,

$$C_L^V \geq \frac{2}{M_f}\left(f(X_0, L^f) - f(X_0, L)\right)$$

- We maximize the RHS for, say, $M_f \leq 2$
- We decompose f as

$$f(x) = f(X_0) + (x - X_0) f'(X_0) + \int f''(K)\, Vanilla_K(x)\, dK$$

Where $Vanilla_K(x) \equiv K - x$ if $K \leq X_0$ and $x - K$ otherwise

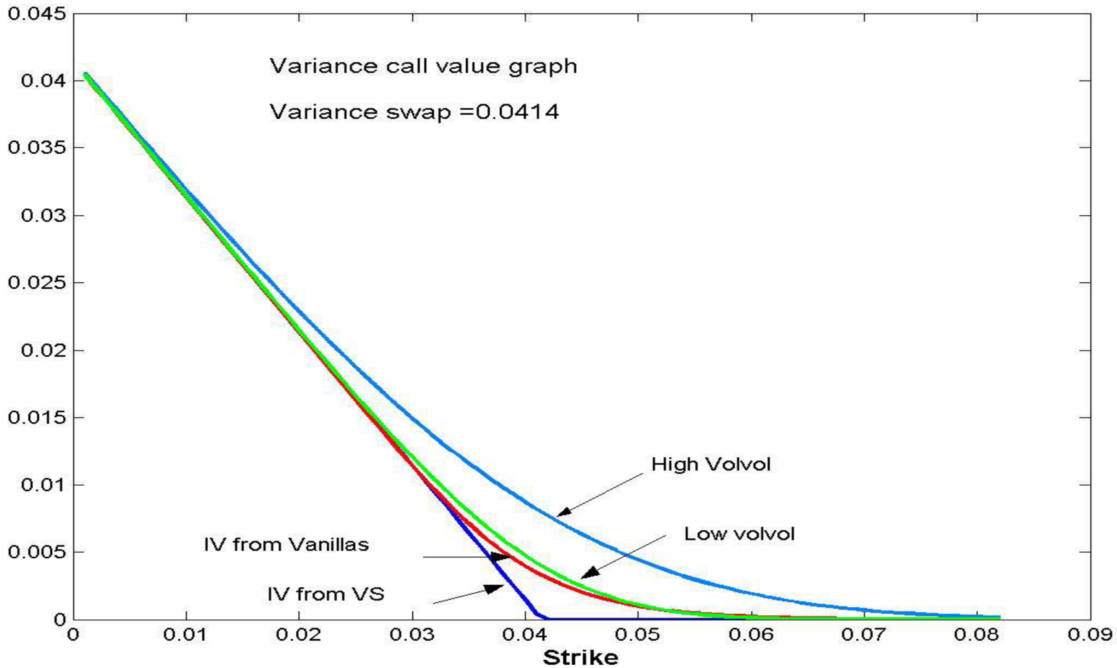
Then, $C_L^V \geq \int f''(K)(Van_K(L^K) - Van_K(L))\, dK$

Where $Van_K(v)$ is the price of $Vanilla_K(x)$ for variance v and L^K is the market implied variance for strike K

Lower Bound Strategy

- Maximum when f'' = 2 on $A \equiv \{K : L^K \geq L\}$, 0 elsewhere
- Then $f(x) = 2\int_A Vanilla_K(x)\, dK$ (truncated parabola)

and $C_L^V \geq 2\int_A (Van_K(L^K) - Van_K(L))\, dK$



Arbitrage Summary

- If a Variance Call of strike L and maturity T is below its lower bound:
- 1) at t = 0,
  - Buy the variance call
  - Sell all options with implied vol $\leq \sqrt{\frac{L}{T}}$
- 2) between 0 and T,
  - Delta hedge the options in business time
  - If $\tau < T$, then carry on the hedge with 0 vol
- 3) at T, sure gain