Calibration of Stochastic and Local Stochastic Volatility Models

Global Derivatives Trading & Risk Management 2011

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AGENDA

- INTRODUCTION
- STOCH VOL CLOSED FORM CALIBRATION
- ROBUSTIFICATION TECHNIQUES
- MONTE CARLO CALIBRATION
- LOCAL STOCH VOL CALIBRATION
- CONCLUSIONS

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Motivation

- Model calibration is a key step before using any financial market model for pricing and hedging purposes
- Large derivative houses frequently need to calibrate hundreds of underlyings to market data
- If the model does not allow an analytic calibration, usually a least squares fit is computed with suitable optimization methods
- Algorithm speed and robustness are necessary to obtain accurate and stable PnL and Greeks in a front office environment

→ Modern algorithms are needed for the solution of calibration problems

Selected approaches in the literature:

Andersen/Andreasen (2000), Hamida/Cont (2005), Mikhailov/Noegel (2003), Turinici (2009)

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Example: Calibration of the Time-Dependent Heston Model

Goal: Choose the model parameters x such that the model matches the market prices $C^{\text{market}}(T_i, K_j)$ of n given standard calls with maturities T_i and strikes K_j

$$\min_{x=(v_0,\kappa_1,\theta_1,\sigma_1,\rho_1,\ldots,\kappa_m,\theta_m,\sigma_m,\rho_m)^T} \sum_{i=1}^{N_T}\sum_{j=1}^{N_K} w_{i,j}\left[C^{\text{market}}(T_i,K_j) - C^{\text{model}}(T_i,K_j;x)\right]^2$$

s.t.

$$C^{\text{model}}(T_i,K_j;x) = e^{-rT_i}E_Q(S_{T_i}(x) - K_j)^+$$

$$\begin{aligned} dS_t &= (r-d)S_t dt + \sqrt{v_t}S_t dW_t^{(1)}, \quad S_0 > 0 \\ dv_t &= \kappa(t)[\theta(t) - v_t]dt + \sigma(t)\sqrt{v_t}dW_t^{(2)}, \quad dW_t^{(1)}dW_t^{(2)} = \rho(t)dt \end{aligned}$$

$$v_0, \kappa_i, \theta_i, \sigma_i \geq 0, \ |\rho_i| \leq 1, \quad \frac{\sigma_i^2}{2} - \kappa_i\theta_i \leq 0, \quad i = 1, \ldots, m$$

with piecewise constant model parameters $\kappa_i, \theta_i, \sigma_i, \rho_i$ on $[t_{i-1}, t_i), \ i = 1, \ldots, m.$

→ The model admits a semi-closed form solution for plain vanilla call options C (see e.g. Mikhailov & Noegel, 2003)

Nonlinear Least Squares Problem and Gauss-Newton Approach

Hence the calibration problem can be rephrased as a (deterministic) nonlinear least squares problem

$$\min_x \ \tfrac{1}{2} \parallel R(x) \parallel_2^2 =: f(x) \qquad \textbf{s.t.} \quad \boldsymbol{c(x)} \le \mathbf{0}$$

with residual function
$$R(x) = \begin{pmatrix} R_1(x) \\ \vdots \\ R_n(x) \end{pmatrix} := \begin{pmatrix} \sqrt{w_{1,1}} \times \left[C_{1,1}^{\text{market}} - C_{1,1}^{\text{model}}(x) \right] \\ \vdots \\ \sqrt{w_{N_T,N_K}} \times \left[C_{N_T,N_K}^{\text{market}} - C_{N_T,N_K}^{\text{model}}(x) \right] \end{pmatrix}$$

The Hessian of f can be approximated based on the Jacobian $J_R(x)$ of R(x)

$$\nabla^2 f(x) \ = \ J_R(x)^T J_R(x) + \textstyle\sum_{j=1}^n R_j(x) \nabla^2 R_j(x) \approx J_R(x)^T J_R(x)$$

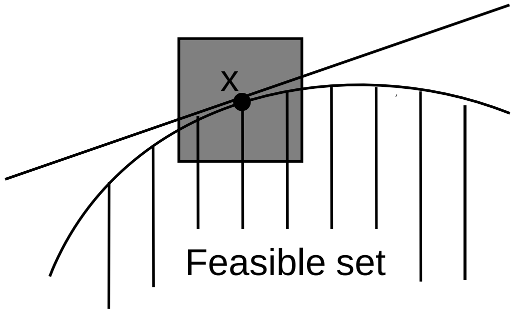
since the residuals $R_j(x)$ in the optimal point are usually quite small.

→ We get a very good approximation of the second derivative by solely making use of first order information (Gauss Newton approximation)

Sketch of the Algorithm

Idea: Combine Gauss-Newton approximation of the Hessian with a feasible point trust region SQP algorithm developed by Wright and Tenny

To compute a stationary point the SQP algorithm successively solves



$$(\text{QP}) \; \min_{d} \; \nabla f(x)^T d + \tfrac{1}{2} d^T H d$$

$$\text{s.t.} \quad c(x) + J_c(x)^T d \leq 0, \qquad \| d \|_\infty \leq \Delta$$

where H is a Gauss-Newton-approximation of the Hessian of the Lagrangian

$$L(x, \lambda) := f(x) + \lambda^T c(x)$$

To preserve feasibility of the iterates we project the solution of (QP) onto the feasible set after each iteration.

The Feasible Set of the Heston Calibration Problem

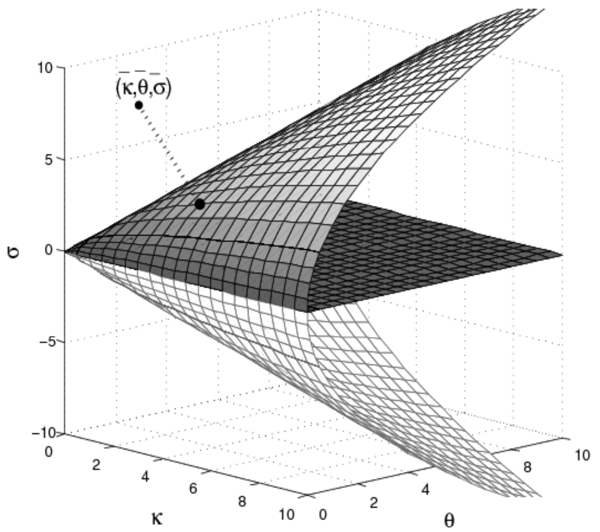
Projection Theorem: (Gerlich, Giese, M., Sachs, 2011)

The set of Heston constraints $v_0, \kappa_i, \theta_i, \sigma_i \geq 0,\ |\rho_i| \leq 1,\ \ \frac{\sigma_i^2}{2} - \kappa_i\theta_i \leq 0,\ i = 1, ..., m$

is equivalent to

$$v_0 \geq 0,\ |\rho_i| \leq 1,\ X_i := \begin{pmatrix} \kappa_i & \frac{\sigma_i}{\sqrt{2}} \\ \frac{\sigma_i}{\sqrt{2}} & \theta_i \end{pmatrix} \in \mathcal{S}_+^2 \text{ and } \sigma_i \geq 0,\ i = 1, \ldots, m.$$

with an explicit solution of the associated projection problem.



Furthermore, if additional lower and upper bounds are imposed on the parameters, we can solve the projection problem via the semidefinite program

$$\min_{Y,Z \in \mathcal{S}^2} Tr(Z) \quad \text{s.t.} \quad \begin{pmatrix} I & (Y - \bar{Y})^T \\ Y - \bar{Y} & Z \end{pmatrix} \succeq 0$$

$$Y \succeq 0, \quad \tilde{l}_i \leq y_i \leq \tilde{u}_i,\ i = 1, 2, 3$$

Sketch of the FPSQP Algorithm (Based on Wright/Tenny, 2004)

While $\|\nabla_x L(x_k, \lambda_k)\| \geq TOL$ do

1. Solve (QP) to obtain search direction d_k
2. Project $x_k + d_k$ onto the feasible set by solving the SDP
3. Pursue a trust region step size strategy to achieve convergence
4. Update the Lagrange multipliers and the Hessian (via Gauss-Newton)

End While

Properties / Advantages:

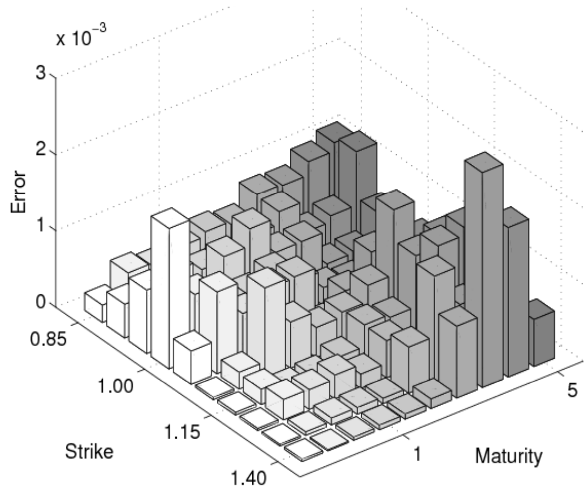
- Algorithm makes full use of the structure of the problem (Gauss Newton approximation of the Hessian / projections via SDPs)
- Closed form is only evaluated for parameters inside the feasible set
- Convergence to a Karush-Kuhn-Tucker point is guaranteed (if TOL=0)
- Trust region strategy implicitely regularizes the ill-posed inverse problem

Sample Iteration Run

Algorithm output for the calibration of the constant parameter Heston model to the dataset taken from Andersen and Brotherton (1997).

k	$\|\nabla_x L\|_2$	$\|R(x^k)\|_2^2$	$\sqrt{v_0^k}$	κ^k	$\sqrt{\theta^k}$	σ^k	ρ^k
0	3.87e+00	9.76e-01	40.00	60.00	40.00	40.00	-70.00
1	3.04e-01	8.82e-03	1.00	85.26	21.78	28.44	-95.00
2	1.66e-01	5.42e-04	13.26	60.95	17.26	19.06	-70.00
3	1.09e-03	9.71e-05	12.52	61.89	16.72	18.60	-66.99
4	3.38e-02	8.44e-05	12.43	86.89	16.32	21.52	-68.13
5	2.52e-02	6.64e-05	12.29	111.90	15.99	23.92	-70.36
6	1.32e-02	5.79e-05	12.17	132.65	15.77	25.69	-72.12
7	1.71e-03	5.58e-05	12.12	140.61	15.68	26.30	-72.80
8	1.28e-04	5.58e-05	12.10	142.99	15.66	26.49	-73.00
9	7.84e-06	5.58e-05	12.10	143.60	15.66	26.54	-73.04
10	1.63e-06	5.58e-05	12.10	143.75	15.66	26.55	-73.05
11	4.61e-07	5.58e-05	12.10	143.78	15.66	26.55	-73.05

Calibration error at optimal solution



→ The calibration takes less than one second on a desktop PC

→ A combination of nonlinear and semidefinite programming leads to a robust and rapidly converging algorithm

Comparison to Derivative-Free Algorithms

Many quants apply derivative-free global optimization algorithms, because

- They are easy to apply
- Ill-conditioning and/or local minima can lead to instable parameters

→But Gauss-Newton methods may actually be more robust in finance applications:

Benchmark: Direct search simul. annealing algorithm of Hedar and Fukushima

Statistics of optimal solutions for 100 randomly chosen start points of the algorithms

	Global	Feasible Point SQP Algorithm			Direct Search Simulated Annealing		
	Min. x^*	Mean	Std.dev.	Max.dev.	Mean	Std.dev.	Max.dev.
$\sqrt{v_0}$	25.0000%	25.0000%	0.0000%	0.0000%	25.0832%	0.2976%	0.8797%
κ	50.0000%	50.0000%	0.0001%	0.0001%	67.7480%	39.4530%	132.0436%
$\sqrt{\theta}$	30.0000%	30.0000%	0.0000%	0.0000%	29.7508%	2.2761%	17.6526%
σ	30.0000%	30.0000%	0.0000%	0.0002%	31.8125%	7.4429%	22.3409%
ρ	-70.0000%	-70.0000%	0.0000%	0.0001%	-72.8945%	5.9187%	25.6014%
$\|R(x)\|^2$	0.00e+00	1.08e-14	1.22e-14	5.10e-14	3.34e-05	5.69e-05	2.28e-04

→The calibration results of the FPSQP code are much more stable, although the DSSA algorithm took 40 times longer (Ø 3100 calls of f) to solve the problem

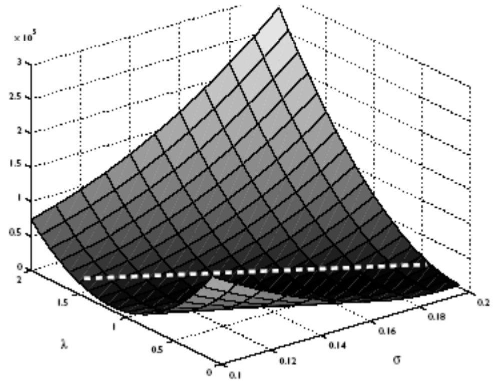
Comparison to General Purpose Quasi-Newton Algorithms

Apart from derivative-free methods, Quasi-Newton algorithms like SNOPT, IPOPT can be applied for the solution of general nonlinear least squares problems.

In our experience Gauss-Newton (GN) methods are superior to Quasi Newton (QN) codes for the calibration of financial market models.

→ Example for time-dependent Heston calibration: Same accuracy reached with 16 Gauss-Newton iterations vs. 378 IPOPT Quasi-Newton iterations!

Sketch of an ill-conditioned function



Reason for superior performance of GN vs. QN:

- The residuals $R_j(x)$ are usually small
- The Gauss-Newton approximation better captures the curvature of ill-conditioned objective functions

→ Hint: Simply pass a finite difference Gauss-Newton Hessian to your favorite Newton optimizer

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Robustification for Time-Dependent Parameters

Feasible Point SQP calibration results for an increasing number m of model parameters:

m	1	2	3	5	10
$\|R(x^*)\|_2^2$	5.58e-05	2.22e-05	9.83e-06	7.97e-06	7.11e-06
Iterations	11	13	14	36	64
runtime [sec]	0.4	0.9	1.5	6.3	22.4
$\text{cond}_2 H(x^*)$	2.79e+06	7.42e+06	2.63e+07	6.22e+08	1.52e+12

→ The time-dependent calibration problem is much more ill-conditioned and requires additional regularization, e.g. $f(x) = \frac{1}{2}\|R(x)\|_2^2 + \gamma p(x)$ with

$$p(x) = \sum_{i=1}^{m-1} \left(\frac{\kappa_{i+1}-\kappa_i}{t_{i+1}-t_i}\right)^2 + \left(\frac{\sqrt{\theta_{i+1}}-\sqrt{\theta_i}}{t_{i+1}-t_i}\right)^2 + \left(\frac{\sigma_{i+1}-\sigma_i}{t_{i+1}-t_i}\right)^2 + \left(\frac{\rho_{i+1}-\rho_i}{t_{i+1}-t_i}\right)^2$$

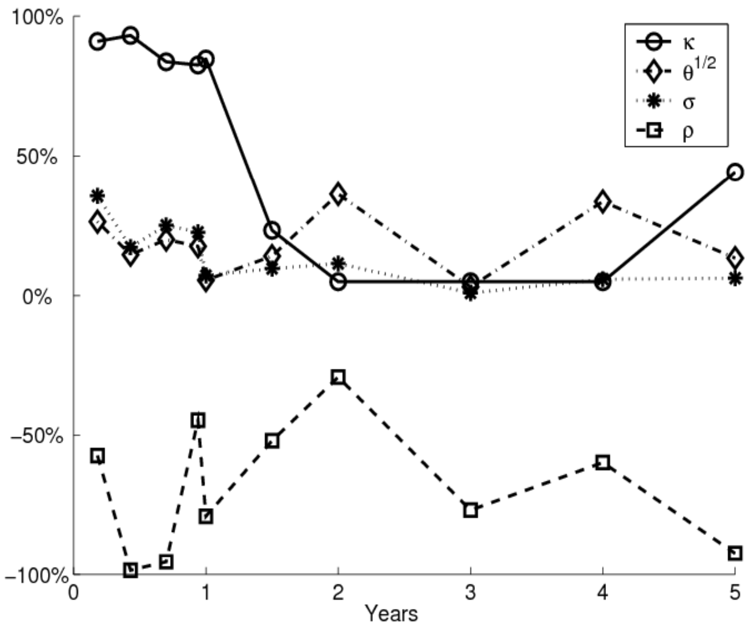
Results for the regularized calibration with an increasing number m of parameters:

m	1	2	3	5	10
$\|R(x^*)\|_2^2$	5.58e-05	2.36e-05	1.05e-05	9.88e-06	9.84e-06
Iterations	11	14	17	17	16
runtime [sec]	0.4	1.0	1.8	3.0	5.6
$\text{cond}_2 H(x^*)$	2.79e+06	3.61e+06	3.76e+07	4.75e+07	7.31e+07

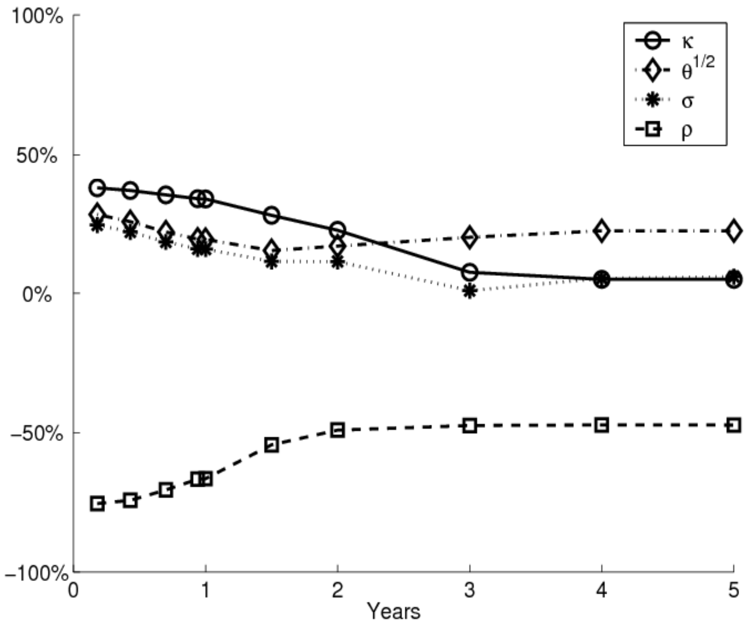
Time-Dependent Parameters: Plot of Optimal Solutions

In addition to improving the condition of the optimization problem the regularization term also leads to smoother optimal solutions:

Solution of unregularized problem



Solution of regularized problem



→ The time-dependent parameters reflect the curvature of the volatility surface

Stabilizing the Calibration Result

Additional fitting criteria like the skew help to stabilize the calibration outcome:

$$\min_x \sum_{i=1}^{N_T} \sum_{j=1}^{N_K} w_{i,j} \left[C_{i,j}^{\text{market}} - C_{i,j}^{\text{model}}(x)\right]^2 + \sum_{i=1}^{N_T} w_i^{\text{skew}} \left[\text{skew}_i^{\text{market}} - \text{skew}_i^{\text{model}}(x)\right]^2$$

Analogous terms can be added for the ATM level or the convexity of the implied vol surface

$$\sum_{i=1}^{N_T} w_i^{\text{atm}} \left[\text{atm}_i^{\text{market}} - \text{atm}_i^{\text{model}}(x)\right]^2, \quad \sum_{i=1}^{N_T} w_i^{\text{conv}} \left[\text{conv}_i^{\text{market}} - \text{conv}_i^{\text{model}}(x)\right]^2.$$

Example:

- Fit A and Fit B have exactly the same sum of squared implied vol errors
- But Fit B exactly fits the skew!

→ Additional fitting criteria help to choose among equal imperfect fits

Pinning Down the Forward-Skew (Weber, M., 2011)

- Just fitting the spot start implied vol surface leaves the forward skew dynamics of the model to a large extent uncontrolled (with risk of undesired fwd skew changes)
- **Idea:** Pin down the forward skew via additional constraints to match a prespecified target skew (e.g. stationary fwd skew):

$$\min_x \sum_{i=1}^{N_T} \sum_{j=1}^{N_K} w_{i,j} \left[C_{i,j}^{\text{market}} - C_{i,j}^{\text{model}}(x) \right]^2$$

$$\text{s.t.} \quad \begin{pmatrix} \text{fwdskew}_1^{\text{model}}(x) - \text{fwdskew}_1^{\text{target}} \\ \vdots \\ \text{fwdskew}_{N_T}^{\text{model}}(x) - \text{fwdskew}_{N_T}^{\text{target}} \end{pmatrix} = \vec{0}$$

.STOXX50E Cliquet Prices (Maturity 3y, Tenor 3m, Local Floor -5%, Local Cap 5%)	
Independent Increment Model:	**4.70%**
Local Volatility Model (spot start calibration)	-0.25%
Stochastic Volatility Model (spot start calibration):	1.50%
Stochastic Volatility Pinned Fwd-Skew Model:	**4.70%**

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Monte Carlo Calibration: Discretization & Convergence

If closed forms are not available, the call prices can be approximated via Monte Carlo based on M samples and stepsize Δt_n in step n:

$$
(P_{M,\Delta t}) \qquad
\begin{aligned}
&\min_{x\in X} \hat{f}_{M,\Delta t} := \frac{1}{2}\sum_{i=1}^{N_T}\sum_{j=1}^{N_K}\left[C^{\text{market}}(T_i,K_j) - \hat{C}^{\text{model}}_{M,\Delta t}(T_i,K_j;x)\right]^2 \\
&\text{where } \hat{C}^{\text{model}}_{M,\Delta t}(T_i,K_j;x) := e^{-rT_i}\frac{1}{M}\sum_{m=1}^{M}\left(s^m_{N_i}(x) - K_j\right)^+ \\
&\text{s.t. } y^m_{n+1}(x) = y^m_n(x) + a(x,y^m_n(x))\Delta t_n + b(x,y^m_n(x))\Delta W^m_n, \; y^m_0 = Y_0.
\end{aligned}
$$

→ Do solutions of problem $(P_{M,\Delta t})$ converge to a solution of the original problem?

Theorem: (Käbe, M., Sachs, 2007)
Suppose that the MC approximations and their derivatives converge uniformly

$$
\lim_{k\to\infty}\sup_{x\in X}\left|\hat{C}^{\text{model}}_{M_k,\Delta t_k}(T_i,K_j;x) - C^{\text{model}}(T_i,K_j;x)\right| = 0 \quad (a.s.)
$$

$$
\lim_{k\to\infty}\sup_{x\in X}\left\|\nabla\hat{C}^{\text{model}}_{M_k,\Delta t_k}(T_i,K_j;x) - \nabla C^{\text{model}}(T_i,K_j;x)\right\|_2 = 0 \quad (a.s.)
$$

for given sequences $M_k \to \infty, \Delta t_k \to 0$. Then every limit point x* of first order critical points x_k of $(P_{M_k,\Delta t_k})$ almost surely satisfies the first order optimality conditions of the original calibration problem.

Adjoint-Equation for the Monte Carlo Objective‘s Gradient

Compute the gradient with the so-called adjoint approach:

- First applied in finance by Giles and Glasserman (2006) to compute Greeks
- For smoothed payoffs/SDEs we may apply adjoints in a calibration setting
- An explicit derivation of the equation can be complicated (→ autom. differentiation)

Theorem: (Käbe, M., Sachs, 2007)

The derivative of $\hat{C}^{\text{model}}_{M,\Delta t}(T_i, K_j; x)$ (neglecting corners) can be computed via

$$\frac{\partial}{\partial x}\hat{C}^{\text{model}}_{M,\Delta t}(T_i, K_j; x) = e^{-rT_i}\frac{1}{M}\sum_{m=1}^{M}\sum_{n=0}^{N_i-1}(\lambda^m_{n+1})^T\left[\frac{\partial}{\partial x}a(x, y^m_n)\Delta t_n + \frac{\partial}{\partial x}\left(b(x, y^m_n)\Delta W^m_n\right)\right]$$

where $\lambda^m_n \in I\!R^L$ is the solution of the adjoint equation

$$\lambda^m_n = \left[I + \frac{\partial}{\partial y}a(x, y^m_n)\Delta t_n + \frac{\partial}{\partial y}\left(b(x, y^m_n)\Delta W^m_n\right)\right]^T \lambda^m_{n+1},$$

$$n = N_i - 1, N_i - 2, ..., 1, \;\; m = 1, ..., M, \;\; \lambda^m_{N_i} = \left[1_{\{s^m_{N_i}(x) > K_j\}}, 0, ..., 0\right] \in \mathbb{R}^L.$$

Advantages:

- We obtain the exact gradient with <u>one</u> forwards/backwards solution
- If dim_SDEs = L << P = num_parameters a significant speedup can be expected

SpeedUp for the Lognormal Variance Model

Calibration of the lognormal variance model with 10 parameter time buckets (P=41):

Calibration with finite diffs. and one MC layer $(M_1, \Delta t_1)$ = (100k, 9e-3)

Iter	$\|\nabla_x L\|_2$
0	3.7985e+00
1	3.5643e+00
2	3.3203e+00
3	3.0844e+00
4	2.8543e+00
5	2.6302e+00
6	2.4152e+00
⋮	⋮
31	1.9312e-03
32	2.4012e-04
33	4.4683e-05
34	9.7093e-06
35	4.9502e-07
Time	182:47 min

Adjoint-based calibration with two Monte Carlo layers $(M_1, \Delta t_1)$ = (10k,3e-2), $(M_2, \Delta t_2)$ = (100k,9e-3)

Iter	$\|\nabla_x L\|_2$
0	3.7374e+00
1	3.5214e+00
⋮	⋮
30	1.6864e-04
31	8.9735e-05
32	4.8856e-05
0	2.3308e-03
1	5.1196e-03
⋮	⋮
5	1.8100e-04
6	4.0502e-05
7	3.7831e-06
8	2.0280e-07
Time	9:45 min

Observations:

- Computation time is reduced from > 3 hours to < 10 minutes
- Multi layer methods reduce the number of expensive iterations
- Adjoints significantly reduce the computation time per iteration:

P	Finite diff.	Adjoint	Speedup
5	100	71	1.4
9	167	72	2.3
13	235	71	3.3
17	301	72	4.2
21	367	73	5.0
25	436	74	5.9
29	501	72	7.0
33	574	73	7.9
37	634	73	8.7
41	702	73	9.6

Monte Carlo calibration is feasible on a desktop PC!

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LSV Calibration by Solving the Forward Kolmogorov PDE

- Local stochastic volatility models combine the best of both the LV and SV worlds:

$$\begin{aligned} df_t^{\mathrm{LSV}} &= \sigma(f_t^{\mathrm{LSV}}, t)\sqrt{v_t} f_t^{\mathrm{LSV}} dW_t^1 \\ dv_t &= \kappa(\theta - v_t)dt + \xi\sqrt{v_t} dW_t^2 \end{aligned} \qquad \text{(Heston+LV)}$$

→ But a least squares calibration is extremely expensive (no closed form solution)!

Alternative: Fixed point approach (Ren, Madan, Qian, 2007)

1. Solve forward Kolmogorov PDE (in x = ln(S/fwd)) with a given estimate of $\sigma(f,t)$

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left[\frac{1}{2}v\sigma^2 p\right] - \frac{\partial}{\partial v}\left[\kappa(\theta - v)p\right] + \frac{\partial^2}{\partial x^2}\left[\frac{1}{2}v\sigma^2 p\right] + \frac{\partial^2}{\partial x \partial v}\left[\rho\sigma\xi v p\right] + \frac{\partial^2}{\partial v^2}\left[\frac{1}{2}v\xi^2 p\right]$$

2. Use the density from 1. to compute the conditional expected value of v_t given f_t^{LSV}

$$E[v_t | f_t^{\mathrm{LSV}} = f] = \frac{\int_0^\infty v p(t,f,v)dv}{\int_0^\infty p(t,f,v)dv}$$

3. Adjust σ according to Gyoengy's identity for the local vols of the LSV model

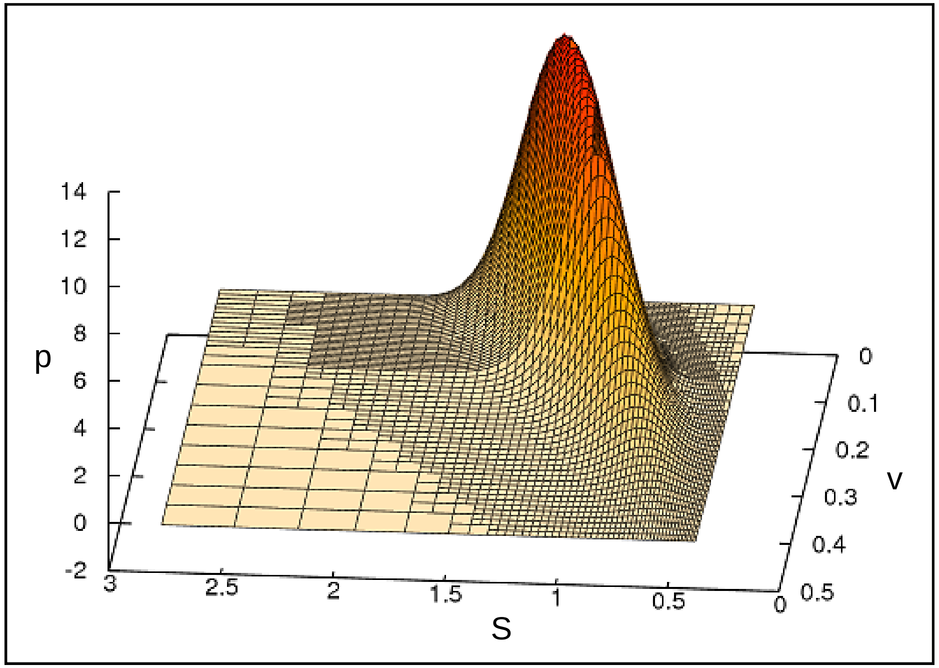
$$(\sigma_{\mathrm{LV}}^{\mathrm{LSV}})^2(f,t) \quad = \quad \sigma^2(f,t)E[v_t | f_t^{\mathrm{LSV}} = f] \stackrel{!}{=} (\sigma_{\mathrm{LV}}^{\mathrm{market}})^2(f,t)$$

→Repeat steps 1-3 until $\sigma(f,t)$ has converged (in most cases 1-2 loops suffice)

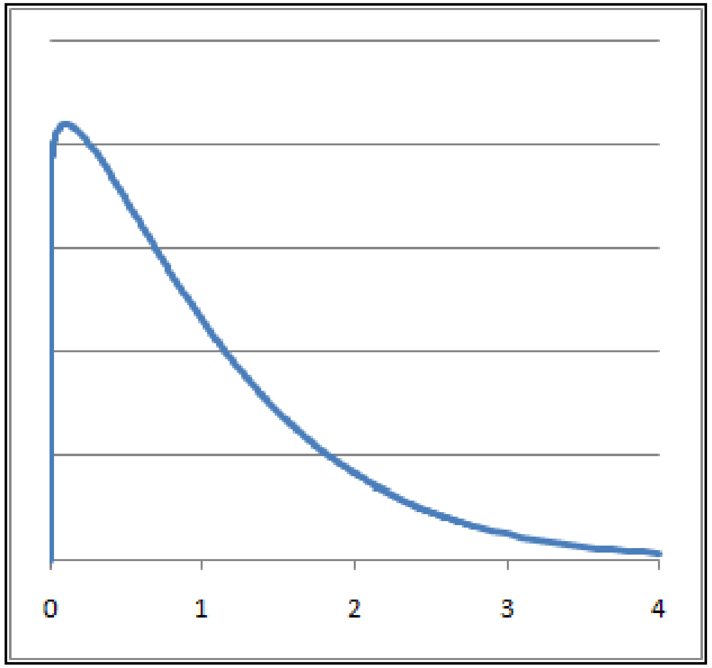
Numerical Solution of the Forward Kolmogorov PDE

Problem: The forward Kolmogorov PDE is costly and difficult to solve in equities

Density of joint distribution of (S,v)



Problematic slope of Heston density of v



→ Most studies switch to lognormal variance, but still robustness in combination with acceptable computation time is very hard to achieve for long maturities

LSV Calibration via LV Ratios (Piterbarg, 2007, Labordère, 2009)

Idea: Applying Gyoengy's theorem twice (for the starting and the target LSV model) avoids the need for conditional expectations

- Compute the local vols of an LSV and an SV model via Gyoengy:

$$\begin{aligned}(\sigma_{\mathrm{LV}}^{\mathrm{LSV}})^2(f,t) &= \sigma^2(f,t)E[v_t|f_t^{\mathrm{LSV}}=f] \stackrel{!}{=} (\sigma_{\mathrm{LV}}^{\mathrm{market}})^2(f,t)\\ (\sigma_{\mathrm{LV}}^{\mathrm{SV}})^2(x,t) &= E[v_t|f_t^{\mathrm{SV}}=x]\end{aligned}$$

- Taking the ratio and solving for the unknown function $\sigma(\cdot,\cdot)$, we obtain

$$\sigma(f,t) = \frac{\sigma_{\mathrm{LV}}^{\mathrm{market}}(f,t)}{\sigma_{\mathrm{LV}}^{\mathrm{SV}}(x,t)}\sqrt{\frac{E[v_t|f_t^{\mathrm{SV}}=x]}{E[v_t|f_t^{\mathrm{LSV}}=f]}} \stackrel{x=H(f,t)}{\approx} \frac{\sigma_{\mathrm{LV}}^{\mathrm{market}}(f,t)}{\sigma_{\mathrm{LV}}^{\mathrm{SV}}(H(f,t),t)}$$

with an <u>approximative</u>, strictly monotonically increasing map $H(f,t)$ satisfying

$$f_t^{\mathrm{SV}} = H(f_t^{\mathrm{LSV}},t),$$

which can be computed via harmonic averages:

$$H(f,t) = \Phi_t^{-1}\left(\int_{f_0}^{f}\frac{1}{y\sigma_{\mathrm{LV}}^{\mathrm{market}}(y,t)}dy\right) \quad \text{with} \quad \Phi_t(x) = \int_{\Lambda(t)}^{x}\frac{1}{y\sigma_{\mathrm{LV}}^{\mathrm{SV}}(y,t)}dy$$

LSV Calibration Process

Step-by-step calibration process:

1. Calibrate the pure stochastic volatility (SV) model
2. Compute the local volatilities of the market
3. Compute the Gyoengy local volatilities of the SV model
4. Determine the mapping H(f,t)
5. Compute the unknown local vol function for the LSV model:

$$\sigma(f,t) = \frac{\sigma_{\text{LV}}^{\text{market}}(f,t)}{\sigma_{\text{LV}}^{\text{SV}}(H(f,t),t)}$$

Observations:

- This process yields an approximative, instantaneous calibration of the LSV model!
- In steps 2 / 3 a presmoothing of input vols help to obtain a smooth output

→ For further steps see Henry-Labordère (2009)

Example: Price Fit of the Plain Vanilla Surface in Basis Points

Price fit for the pure Heston model:

Strikes \ Maturities	0.50	1.00	1.50	2.00	2.50	3.00	3.50	4.00
40%	-0.02%	-0.16%	-0.37%	-0.44%	-0.48%	-0.53%	-0.60%	-0.61%
50%	-0.13%	-0.44%	-0.63%	-0.64%	-0.62%	-0.64%	-0.72%	-0.72%
60%	-0.30%	-0.62%	-0.74%	-0.69%	-0.64%	-0.65%	-0.73%	-0.72%
70%	-0.47%	-0.68%	-0.71%	-0.62%	-0.55%	-0.56%	-0.64%	-0.62%
80%	-0.55%	-0.58%	-0.55%	-0.45%	-0.39%	-0.39%	-0.47%	-0.44%
90%	-0.38%	-0.32%	-0.28%	-0.21%	-0.17%	-0.18%	-0.25%	-0.21%
100%	0.00%	0.00%	-0.02%	0.00%	0.00%	0.00%	-0.06%	0.00%
110%	0.17%	0.14%	0.08%	0.06%	0.02%	0.04%	-0.01%	0.07%
120%	0.05%	0.07%	0.02%	-0.02%	-0.07%	-0.04%	-0.06%	0.01%
130%	0.00%	0.00%	-0.04%	-0.10%	-0.17%	-0.15%	-0.16%	-0.09%
140%	0.00%	-0.01%	-0.04%	-0.12%	-0.21%	-0.22%	-0.24%	-0.20%
150%	0.00%	0.00%	-0.02%	-0.09%	-0.17%	-0.22%	-0.27%	-0.27%
160%	0.00%	0.00%	0.00%	-0.04%	-0.12%	-0.18%	-0.24%	-0.29%
170%	0.00%	0.00%	0.00%	-0.02%	-0.07%	-0.13%	-0.19%	-0.26%
180%	0.00%	0.00%	0.00%	-0.01%	-0.04%	-0.08%	-0.14%	-0.21%
190%	0.00%	0.00%	0.00%	-0.01%	-0.02%	-0.05%	-0.10%	-0.17%
200%	0.00%	0.00%	0.00%	0.00%	-0.01%	-0.03%	-0.07%	-0.13%

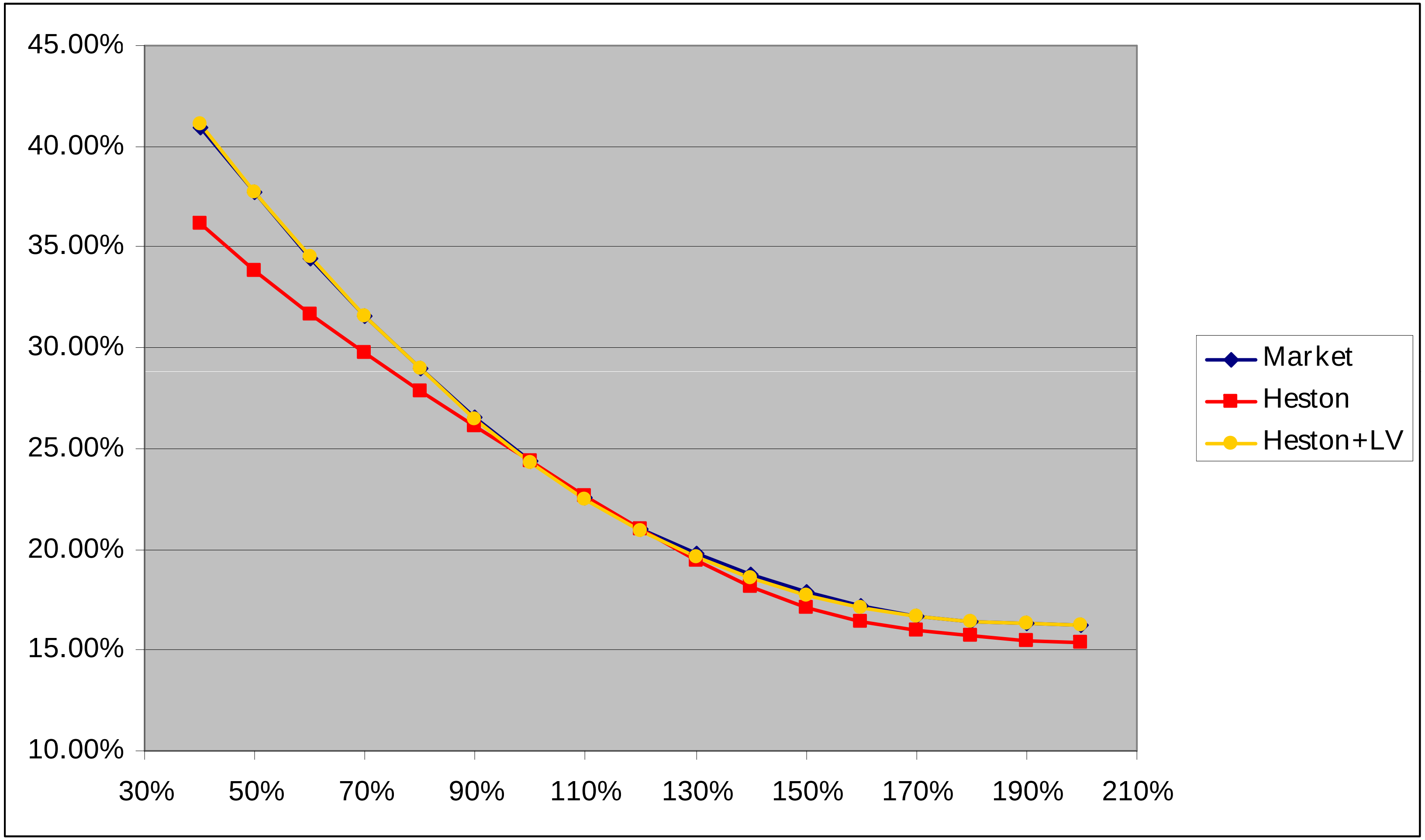
The steep skew of the market cannot be reproduced by the Heston model (Feller enforced).

Price fit for the Heston+LV model:

Strikes \ Maturities	0.50	1.00	1.50	2.00	2.50	3.00	3.50	4.00
40%	0.00%	0.02%	0.02%	0.01%	0.01%	0.01%	0.01%	0.01%
50%	-0.03%	-0.04%	-0.02%	0.00%	0.01%	0.01%	0.01%	0.01%
60%	-0.01%	0.01%	0.02%	0.02%	0.02%	0.01%	0.01%	0.01%
70%	0.03%	0.03%	0.02%	0.02%	0.01%	0.01%	0.01%	0.00%
80%	0.02%	0.00%	0.00%	0.00%	0.00%	-0.01%	-0.01%	-0.02%
90%	-0.02%	-0.02%	-0.02%	-0.01%	-0.01%	-0.02%	-0.02%	-0.03%
100%	-0.01%	-0.01%	-0.01%	-0.01%	-0.02%	-0.02%	-0.03%	-0.04%
110%	0.00%	-0.01%	-0.02%	-0.03%	-0.05%	-0.05%	-0.06%	-0.07%
120%	-0.01%	-0.03%	-0.04%	-0.05%	-0.06%	-0.07%	-0.08%	-0.09%
130%	0.00%	-0.02%	-0.03%	-0.05%	-0.06%	-0.07%	-0.08%	-0.09%
140%	0.00%	0.00%	-0.01%	-0.03%	-0.05%	-0.06%	-0.07%	-0.08%
150%	0.00%	0.00%	0.00%	-0.01%	-0.03%	-0.04%	-0.06%	-0.07%
160%	0.00%	0.00%	0.00%	0.00%	-0.01%	-0.03%	-0.04%	-0.06%
170%	0.00%	0.00%	0.00%	0.00%	-0.01%	-0.01%	-0.03%	-0.05%
180%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.01%	-0.02%	-0.03%
190%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.01%	-0.02%
200%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.01%	-0.02%

The ad-hoc LSV calibration leads to a near-perfect fit of the market!

Example: SVLV Implied Volatilities for the 2y Time Slice



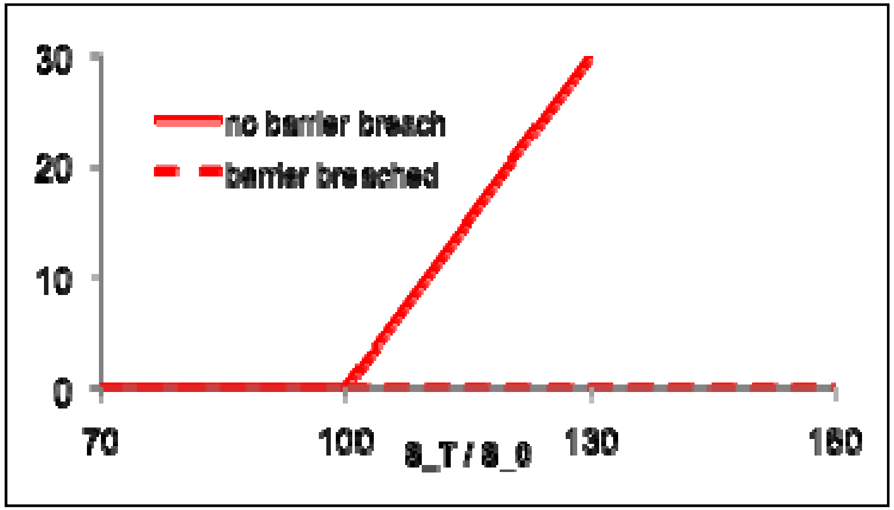
Choosing the Forward Skew Dynamics

As shown in several studies, the SV forward skew dynamics are mainly preserved under the LSV correction.

→ The Stoch Vol calibration is a significant step when building an LSV model

Example: Consider an up-and-out call with maturity T = 3 years and payoff

$$\text{UOC} = \max\left[\frac{S_T}{S_0} - 100\%, 0\%\right] \times 1_{\left\{\max_{0\le t\le T} \frac{S_t}{S_0} < 130\%\right\}}$$



Model prices:

- Local Volatility: 1.95%
- LSV w/o pinned SV fwd-skew: 2.60%
- LSV with pinned SV fwd skew: 2.70% (steep stationary fwd skew)

→ Long fwd skew exposure motivated by static hedging arguments

AGENDA

- INTRODUCTION
- STOCH VOL CLOSED FORM CALIBRATION
- ROBUSTIFICATION TECHNIQUES
- MONTE CARLO CALIBRATION
- LOCAL STOCH VOL CALIBRATION
- CONCLUSIONS

Summary and Conclusions

- Gauss Newton algorithms are the methods of choice for model calibrations in the form of nonlinear least squares problems
- The proposed feasible point trust region Gauss Newton SQP algorithm with semidefinite programming projections outperforms all considered benchmarks
- To stabilize the calibration, suitable penalty terms or constraints can be added to the minimization problem (target ATM levels, skews, fwd skews etc.)
- Adjoint and multi-layer techniques significantly speed up Monte Carlo calibrations
- Approximative on-the-fly LSV corrections offer a feasible alternative to computationally intensive exact PDE calibration methods
- Based on robust and fast SV calibrations, LSV models become feasible for large-scale applications in the front office

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