

An Efficient Implementation of Stochastic Volatility by the method of Conditional Integration

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Presentation Overview

The purpose of this presentation is to

- ▶ apply pseudo-analytic techniques for the accurate pricing of options for the stochastic volatility model:

$$\begin{aligned}dF(t, T) &= \sigma(t)F(t, T)dW_F \\dX(t) &= (\theta - \alpha X(t))dt + \xi dW_X\end{aligned}$$

where $X(t) = \log(\sigma(t)^2)$ and $\langle dW_F, dW_X \rangle = \rho dt$.

This model naturally includes that of SABR ($\beta = 1$), with the extension to $\beta < 1$ also being considered.

The presentation will consist of three major sections:

- ▶ The Vanilla Smile
- ▶ The Forward Smile
- ▶ Correlated Stochastic Volatility



The Vanilla Smile

Uncorrelated Stochastic Volatility - Hull and White [1987]

As is familiar, in Hull and White [1987] it was noted that in the presence of uncorrelated stochastic volatility, the value of a vanilla option of maturity T and strike K , denoted $C(K, T)$ can be expressed as

$$C(K, T) = \int_0^\infty C(K, T, \vartheta) \psi(\vartheta) d\vartheta$$

where ϑ is the mean integrated variance, defined as

$$\vartheta = \frac{1}{T} \int_0^T \sigma(t)^2 dt$$

with distribution $\psi(\vartheta)$, and $C(K, T, \vartheta)$ is the associated Black-Scholes option price with implied volatility $\sigma_{imp}(0, T)^2 = \vartheta$.

Uncorrelated Stochastic Volatility - Hull and White [1987]

The Hull-White approach is to perform a Taylor expansion of $C(K, T, \vartheta)$ about $\mathbb{E}[\vartheta]$ resulting in

$$C(K, T) = C(K, T, \mathbb{E}[\vartheta]) + \frac{1}{2} \frac{\partial^2 C}{\partial \vartheta^2} \Big|_{\vartheta=\mathbb{E}[\vartheta]} \mathbb{E}[(\vartheta - \mathbb{E}[\vartheta])^2] + \dots$$

The method relies upon the calculation of the expected powers of $\vartheta - \mathbb{E}[\vartheta]$, $\mathbb{E}[\vartheta^2], \dots$
For their chosen volatility process

$$dV = \mu V dt + \xi V dW$$

where $V = \sigma(t)^2$ then

$$\begin{aligned} \mathbb{E}[\vartheta] &= \frac{e^{\mu T} - 1}{\mu T} V_0 \\ \mathbb{E}[\vartheta^2] &= \left[\frac{2e^{(2\mu+\xi^2)T}}{(\mu+\xi^2)(2\mu+\xi^2)T^2} + \frac{2}{\mu T^2} \left(\frac{1}{2\mu+\xi^2} - e^{\mu T} \mu + \xi^2 \right) \right] V_0^2 \end{aligned}$$

The focus of Hull-White is on $C(K, T, \vartheta)$. Although intuitively useful, practically it is not. The focus of Stein and Stein on $\psi(\vartheta)$ is a more powerful approach.

Uncorrelated Stochastic Volatility - Stein and Stein [1991]

Although Stein and Stein is most often referenced in the context of the progression of stochastic volatility processes considered in the 1990's, the content of their [1991] paper is particularly relevant to this presentation.

- ▶ They consider the mean reverting volatility process

$$d\sigma(t) = (\theta - \alpha\sigma(t))dt + \xi dW$$

- ▶ A closed form solution for the distribution of the asset, $\phi(S, t)$ is presented
- ▶ They present an approximate solution to $\phi(S, t)$ as

$$\phi(S, t) = \int_0^\infty L(\sigma) m_t(\sigma) d\sigma$$

where $L(\sigma)$ is lognormal and $m_t(\sigma)$ a “mixing distribution”.

- ▶ They demonstrate that approximating the volatility distribution accurately captures the prices of options

Uncorrelated Stochastic Volatility - Stein and Stein [1991]

Stein and Stein make (at least) two interesting remarks:

Although our approximation technique appears to work relatively well in pricing options (particularly those away from the money), one might question its usefulness, given that the exact formula can itself be quite easily implemented. However, it should be noted that our basic approximation methodology may be helpful in attacking more general models than the one studied here, where exact solutions prove less tractable.

The approximation of the variance distribution and associated quantities will be the basis of this presentation.

Uncorrelated Stochastic Volatility - Stein and Stein [1991]

Stein and Stein make (at least) two interesting remarks:

Consider, for example, a constant elasticity of volatility (CEV) generalization of Equation (1), that is,

$$dP = rP dt + \sigma P^j dz, \quad (1')$$

where $0 < j < 1$. This extension, when combined with Equation (2), captures other empirically relevant aspects of volatility, including the tendency for percentage returns to be more volatile when prices are low. We do not know whether a tractable exact solution exists for stock price distributions generated by (1') and (2).

However, it would appear that we can apply a variant of our approximation technique. The logic is as follows. Suppose we know the stock price distribution corresponding to just (1') with σ fixed.¹⁵ It can be shown that an analogue to Equation (11) holds in that our desired exact distribution for the stochastic volatility case can be represented as a mixture of fixed σ CEV distributions. The mixing distribution is somewhat more complicated than $m_i(\sigma)$, but has a similar form. This suggests that even if we cannot solve for the exact distribution as readily as above, we may be able to use the same method of approximation. As in Equation (13), we might use a simple substitute for the mixing distribution to generate an approximate stock price distribution.

Although unable to relax the zero correlation assumption, they suggest introducing skew via a CEV process and forming the asset distribution as an integral over CEVs. This has recently been revisited by e.g. Rebonato [2010] in the context of the SABR model.

Uncorrelated Stochastic Volatility - McGhee [2008]

Returning to the expression of the option price as an integral over the mean integrated variance, ϑ ,

$$C(K, T) = \int_0^\infty C(K, T, \vartheta) \psi(\vartheta) d\vartheta$$

then the distribution of $\psi(\vartheta)$ can be approximated using a distribution expansion such as Gram-Charlier

$$\psi(\vartheta) \sim \psi_{GC} = \psi_{LN}(\vartheta) \left[1 + \frac{\kappa_3}{3! \beta^{3/2}} H_3 \left(\frac{\log \vartheta - \alpha}{\sqrt{\beta}} \right) + \frac{\kappa_4}{4! \beta^2} H_4 \left(\frac{\log \vartheta - \alpha}{\sqrt{\beta}} \right) + \dots \right]$$

with κ_3 controlling skewness, κ_4 kurtosis and $H_3(x) = x^3 - 3x$ and $H_4(x) = x^4 - 6x^2 + 3$ Hermite polynomials. The associated moments of ϑ are obtained via

$$\mathbb{E}[\vartheta^\lambda] = \left(1 + \frac{\kappa_3}{6} \lambda^3 + \frac{\kappa_4}{24} \lambda^4 \right) \exp \left(\left(\alpha + \frac{1}{2} \beta \lambda \right) \lambda \right)$$

Provided $\psi(\vartheta)$ is a “small” perturbation of lognormal, the above can be used to calibrate the degree of curvature for many markets (e.g. a local-stochastic volatility model where all correlation dependence is driven through the local volatility function).

Uncorrelated Stochastic Volatility

An alternative to using a distribution expansion is to feed more known information into the distribution calculation. For most processes the distribution of the terminal instantaneous volatility, $\sigma(T)$ is known - $\psi(\sigma(T))$ (throughout $\psi(x)$ will generically denote “the probability density function of x ”) - and $\psi(\vartheta)$ is expressed as

$$\psi(\vartheta) = \int_0^\infty \psi(\vartheta|\sigma(T))\psi(\sigma(T))d\sigma(T)$$

The conditioning on $\sigma(T)$ dampens the higher order moments of $\vartheta|\sigma(T)$ making the conditional distribution well approximated (where this weakness will be discussed later) via a lognormal $\phi(\vartheta|\sigma(T)) \sim \phi_{LN}(\vartheta|\sigma(T))$ The unconditional moments of ϑ are determined via

$$\mathbb{E}[\vartheta^\lambda] = \int_0^\infty \exp\left(\left(\alpha_{\sigma(T)} + \frac{1}{2}\beta_{\sigma(T)}\lambda\right)\lambda\right) \psi(\sigma(T))d\sigma(T)$$

where $\alpha_{\sigma(T)}$ and $\beta_{\sigma(T)}$ are chosen to match $\mathbb{E}[\vartheta|\sigma(T)]$ and $\mathbb{E}[\vartheta^2|\sigma(T)]$. The option prices are then given by

$$C(K, T) = \int_0^\infty \left(\int_0^\infty C(K, T, \vartheta)\psi(\vartheta|\sigma(T))d\vartheta \right) \psi(\sigma(T))d\sigma(T)$$

Correlated Stochastic Volatility on lognormal asset

Using standard notation, the asset forward process will be given by

$$dF(t, T) = \sigma(t)F(t, T)dW$$

with $\sigma(t)$ some stochastic volatility process correlated with $F(t, T)$. Integrating

$$F(T, T) = F(0, T) \exp \left(-\frac{1}{2} \int_0^T \sigma(t)^2 dt + \int_0^T \sigma(t) dW \right)$$

Decomposing $dW = \rho dW_\sigma + \bar{\rho} dW_F$ (where $\bar{\rho} = \sqrt{1 - \rho^2}$) then

$$F(T, T) = F(0, T) \exp \left(-\frac{1}{2} \vartheta T + \rho \int_0^T \sigma(t) dW_\sigma + \bar{\rho} \int_0^T \sigma(t) dW_F \right)$$

Given a volatility path $\sigma(0, T)$ then $F(T, T)|\sigma(0, T)$ is lognormal with

$$\mathbb{E}[F(T, T)|\sigma(0, T)] = F(0, T) \exp \left(-\frac{1}{2} \rho^2 \vartheta T + \rho \int_0^T \sigma(t) dW_\sigma \right)$$

and log-variance

$$(1 - \rho^2) \int_0^T \sigma(t)^2 dt = (1 - \rho^2) \vartheta T$$

This dates back to Willard [1997] in the context of Conditional Monte Carlo for path *independent* options.

Correlated Stochastic Volatility on lognormal asset

Option valuation is therefore driven by the two stochastic quantities

$$\vartheta = \frac{1}{T} \int_0^T \sigma(t)^2 dt \quad \text{and} \quad I_{\sigma dW_\sigma} = \int_0^T \sigma(t) dW_\sigma$$

derived from a volatility path $\sigma(0, T)$ with associated conditional option price denoted

$$C(K, T, \sigma(0, T)) = C(K, T, \vartheta, I_{\sigma dW_\sigma})$$

Integrating over all such volatility paths $\Omega(\sigma(0, T))$ gives

$$\begin{aligned} C(K, T) &= \int_{\Omega(\sigma(0, T))} C(K, T, \sigma(0, T)) d\Omega(\sigma(0, T)) \\ &= \int_0^\infty d\vartheta \int_{-\infty}^\infty C(K, T, \vartheta, I_{\sigma dW_\sigma}) \psi(\vartheta, I_{\sigma dW_\sigma}) dI_{\sigma dW_\sigma} \end{aligned}$$

With $\psi(\vartheta, I_{\sigma dW_\sigma})$ the joint density of ϑ and $I_{\sigma dW_\sigma}$.

Correlated Stochastic Volatility on lognormal asset

As in the zero correlation case it is advantageous to condition further on the terminal instantaneous volatility $\sigma(T)$ yielding

$$C(K, T) = \int_0^\infty d\sigma(T) \psi(\sigma(T)) \int_0^\infty d\vartheta \int_{-\infty}^\infty C(K, T, \vartheta, I_{\sigma dW_\sigma}) \psi(\vartheta, I_{\sigma dW_\sigma} | \sigma(T)) dI_{\sigma dW_\sigma}$$

This unwieldy looking expression simplifies significantly for the case of lognormal volatility present in the SABR model

The special case of lognormal volatility

For the simple case of lognormal volatility

$$d\sigma(t) = \xi\sigma(t)dW_\sigma$$

then the term adjusting the forward rate for nonzero correlation $I_{\sigma dW_\sigma}$ is trivially

$$I_{\sigma dW_\sigma} = \int_0^T \sigma(t)dW_\sigma = \int_0^T \frac{d\sigma(t)}{\xi} = \frac{1}{\xi}(\sigma(T) - \sigma(0))$$

Further

$$\sigma(T) = \sigma(0) \exp\left(-\frac{1}{2}\xi^2 T + \xi W(T)\right)$$

yielding $\psi(\sigma(T))$ therefore the expression for the option price reduces to

$$C(K, T) = \int_0^\infty d\sigma(T) \psi(\sigma(T)) \int_0^\infty d\vartheta C\left(K, T, \vartheta, \frac{1}{\xi}(\sigma(T) - \sigma(0))\right) \psi(\vartheta|\sigma(T))$$

The only unknown here is the distribution of ϑ conditional on the terminal instantaneous volatility $\sigma(T)$, namely $\psi(\vartheta|\sigma(T))$.

Lognormal approximation of $\psi(\vartheta|\sigma(T))$

The lognormal approximation to $\psi(\vartheta|\sigma(T))$ (the conditioning restricts the higher order moments making this a reasonable assumption) requires the calculation of $\mathbb{E}[\vartheta|\sigma(T)]$ and $\mathbb{E}[\vartheta^2|\sigma(T)]$. These can be determined by numerical integration or more directly. Firstly,

$$\mathbb{E} \left[\frac{1}{T} \int_0^T \sigma(t)^2 dt \middle| \sigma(T) \right] = \frac{\sigma(0)^2}{T} \frac{\sqrt{2\pi T}}{2\xi} \exp \left(\frac{1}{2} q^2 \frac{\xi^2}{T^2} \right) [N(\phi(T)) - N(\phi(0))]$$

where

$$\phi(t) = \frac{2\xi}{\sqrt{T}} \left(t - \frac{1}{2} q \right) \quad \text{and} \quad q = T + \frac{1}{\xi^2} \log \left(\frac{\sigma(T)}{\sigma(0)} \right)$$

Secondly,

$$\mathbb{E} \left[\left(\frac{1}{T} \int_0^T \sigma(t)^2 dt \right)^2 \middle| \sigma(T) \right] = \frac{2}{T^2} \int_0^T dt_1 \int_{t_1}^T dt_2 \mathbb{E} \left[\sigma(t_1)^2 \sigma(t_2)^2 \middle| \sigma(T) \right]$$

Here, the inner integral can be determined analytically leaving at most just one time numerical integration.

Using Yor [1992], Islah [2009] provides the density $\psi(\vartheta, \sigma(T))$ in exact form as well as an extensive solution to the SABR problem.

How well is $\psi(\vartheta)$ approximated?

In order to understand how effective the method of integrating over conditional lognormal distributions is at capturing the distribution of ϑ , consider

$$\sigma_k(T) = \left(\mathbb{E} \left[\left(\frac{1}{T} \int_0^T \sigma^2(t) dt \right)^k \right] \right)^{1/2k}$$

This is known analytically for lognormal volatility with the first few moments being

$$\begin{aligned} \mathbb{E} \left[\frac{1}{T} \int_0^T \sigma^2(t) dt \right] &= \frac{\sigma^2(0)}{\xi^2 T} (e^{\xi^2 T} - 1) \\ \mathbb{E} \left[\left(\frac{1}{T} \int_0^T \sigma^2(t) dt \right)^2 \right] &= \frac{2\sigma^4(0)}{\xi^4 T^2} \left(\frac{1}{30} e^{6\xi^2 T} - \frac{1}{5} e^{\xi^2 T} + \frac{1}{6} \right) \\ \mathbb{E} \left[\left(\frac{1}{T} \int_0^T \sigma^2(t) dt \right)^3 \right] &= \frac{4\sigma^6(0)}{\xi^6 T^3} \left(\frac{1}{1890} e^{15\xi^2 T} - \frac{1}{270} e^{6\xi^2 T} + \frac{1}{70} e^{\xi^2 T} - \frac{1}{90} \right) \end{aligned}$$

and so on ...

How well is $\psi(\vartheta)$ approximated?

Here the analytic calculations are compared with those from the conditional integration for AUDJPY parameters. In addition, a lognormal approximation to $\psi(\vartheta)$ is given for comparative purposes:

T	$\sigma(0)$	ξ	k	Analytic $\sigma_k(T)$	Conditional Integration	Lognormal $\psi(\vartheta) \sim \psi_{LN}(\vartheta)$
2Y	20%	60%	1	24.20%	24.20%	24.20%
2Y	20%	60%	2	34.38%	34.38%	34.38%
2Y	20%	60%	3	54.57%	54.34%	48.82%
2Y	20%	60%	4	93.84%	93.01%	69.34%
5Y	20%	50%	1	28.23%	28.23%	28.23%
5Y	20%	50%	2	59.14%	59.14%	59.14%
5Y	20%	50%	3	156.08%	154.92%	123.90%
5Y	20%	50%	4	455.28%	446.85%	259.58%
30Y	20%	23%	1	31.31%	31.31%	31.31%
30Y	20%	23%	2	87.17%	87.17%	87.17%
30Y	20%	23%	3	321.68%	318.80%	242.70%
30Y	20%	23%	4	1314.30%	1239.95%	675.71%

- ▶ Up to moderate maturities, the conditional integration captures the distribution very well.
- ▶ For the longest maturities the conditional integration is beginning to lose accuracy in the higher order moments. This can be corrected for by an intermediate integration step.

Comparing SABR process and the SABR approximation

There are two comparative studies that can be performed:

- ▶ Fixing the SABR parameters upfront, then comparing the SABR integration and SABR approx of Hagan et al [2002] against a reference scheme;
- ▶ Calibrating the SABR integration, SABR approximation and the reference scheme to market volatilities and comparing the resulting volatility smiles.

The reference scheme of choice is a 2F ADI scheme on the pricing equation

$$V_t + \frac{1}{2}\sigma^2 F^2 V_{FF} + \frac{1}{2}\xi^2 V_{\sigma\sigma} + \rho\sigma^2 \xi F V_{F\sigma} - rV = 0$$

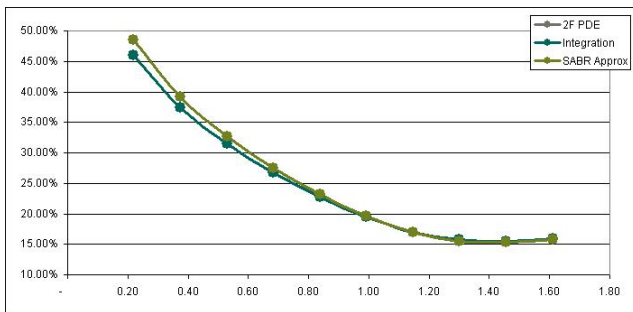
with appropriately chosen boundary conditions.

Fixing the SABR parameters will be considered first, with the calibration to market volatilities considered later in this presentation.

SABR process vs approximation - 2YR AUDJPY

The first example is typical of the smile in the 2YR AUDJPY market:

SABR Implied Volatility: $T=2Y$, $\sigma(0) = 20\%$, $\xi = 60\%$, $\rho = -70\%$, $\beta = 1.0$



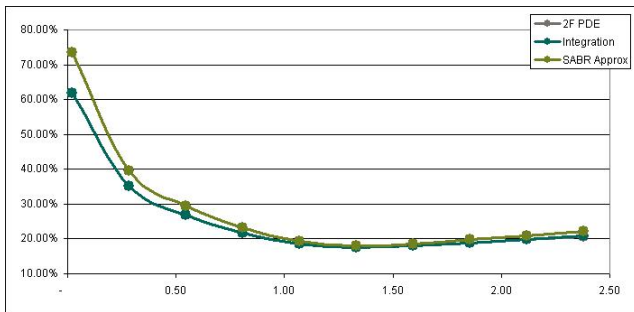
Strike	0.2200	0.3744	0.5289	0.6833	0.8378	0.9922	1.1467	1.3011	1.4556	1.6100
2F PDE	45.97%	37.46%	31.47%	26.72%	22.76%	19.47%	17.01%	15.72%	15.51%	15.90%
Integration	46.02%	37.48%	31.47%	26.72%	22.76%	19.46%	16.99%	15.71%	15.53%	15.93%
SABR Approx	48.51%	39.16%	32.64%	27.50%	23.22%	19.64%	16.93%	15.50%	15.30%	15.78%

The 2F PDE and SABR Integration are so close as to be indistinguishable.

SABR process vs approximation - 5YR AUDJPY

The second example is typical of the smile in the 5YR AUDJPY market:

SABR Implied Volatility: $T=5Y$, $\sigma(0) = 20\%$, $\xi = 50\%$, $\rho = -50\%$, $\beta = 1.0$



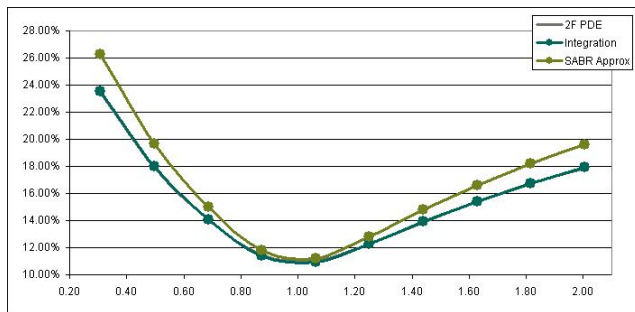
Strike	0.0225	0.2844	0.5464	0.8083	1.0703	1.3322	1.5942	1.8561	2.1181	2.3800
2F PDE	61.85%	35.18%	26.92%	21.75%	18.63%	17.61%	18.00%	18.85%	19.79%	20.71%
Integration	61.92%	35.21%	26.92%	21.71%	18.53%	17.55%	18.02%	18.93%	19.90%	20.83%
SABR Approx	73.42%	39.53%	29.42%	23.12%	19.27%	18.03%	18.59%	19.74%	20.99%	22.19%

The 2F PDE and SABR Integration are so close as to be indistinguishable.

SABR process vs approximation - high vol-of-vol market

The third example is a $\rho = 0$, high x_i relative to $\sigma(0)$ market:

SABR Implied Volatility: $T=5Y$, $\sigma(0) = 10\%$, $\xi = 50\%$, $\rho = 0\%$, $\beta = 1.0$



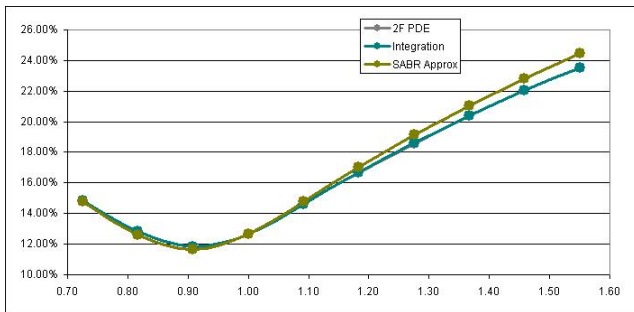
Strike	0.3085	0.4970	0.6855	0.8740	1.0625	1.2510	1.4395	1.6280	1.8165	2.0050
2F PDE	23.45%	17.94%	14.04%	11.44%	10.99%	12.29%	13.88%	15.37%	16.70%	17.89%
Integration	23.51%	17.98%	14.06%	11.41%	10.92%	12.28%	13.90%	15.40%	16.74%	17.93%
SABR Approx	26.27%	19.65%	14.97%	11.79%	11.21%	12.83%	14.78%	16.57%	18.17%	19.60%

The 2F PDE and SABR Integration remain very close with the SABR approximation showing significant deviation.

SABR process vs approximation - EURTRY market

The final example is similar to the EURTRY market:

SABR Implied Volatility: $T=1Y$, $\sigma(0) = 12\%$, $\xi = 85\%$, $\rho = 45\%$, $\beta = 1.0$



Strike	0.7250	0.8167	0.9083	1.0000	1.0917	1.1833	1.2750	1.3667	1.4583	1.5500
2F PDE	14.84%	12.81%	11.86%	12.65%	14.59%	16.65%	18.59%	20.37%	22.02%	23.54%
Integration	14.85%	12.82%	11.83%	12.67%	14.59%	16.65%	18.59%	20.38%	22.03%	23.55%
SABR Approx	14.77%	12.62%	11.66%	12.64%	14.79%	17.04%	19.14%	21.06%	22.82%	24.45%

The 2F PDE and SABR Integration remain very close with the SABR approximation deviating on the upside.

SABR process ($\beta = 1$) sample calculation times

For the 2yr AUDJPY example:

- ▶ Test machine:
 - ▶ Intel Xeon CPU 2.33GHz
 - ▶ RAM 3.25GB
- ▶ 50 integration steps per integral
- ▶ Runtime (standard integration scheme):
 - ▶ $\sim 0.00065s$: 1 strike
 - ▶ $\sim 0.00196s$: 10 strikes
 - ▶ $\sim 0.01592s$: 100 strikes
- ▶ From this the volatility and option calculations can be split:
 - ▶ Initial volatility calculations: $\sim 0.00050s$
 - ▶ Conditional vanilla calculations: $\sim 0.00015s$ per strike
- ▶ If necessary, this can be run many times faster by enhanced integration schemes.

The SABR model for $\beta \neq 1$

Building on this foundation - the integration under the terminal instantaneous volatility and the conditional integrated variance distribution - the other variants of the SABR model can be considered:

- ▶ $\beta = 0$ - replacement of the conditional Black-Scholes lognormal price with normal equivalent;
- ▶ $0 < \beta < 1$ and $\rho = 0$ - replacement of the conditional Black-Scholes lognormal price with CEV price (as originally pointed out by Stein and Stein);
- ▶ $0 < \beta < 1$ replacement of the conditional Black-Scholes lognormal price with a modified CEV price accounting for non-zero correlation.

The CEV process as a squared Bessel process

It is worth reviewing the calculation of the CEV price. Assuming

$$dS = \sigma(t)S^\beta dW(t)$$

it is standard to write $X(t) = S(t)^{2(1-\beta)}/(1-\beta)$ resulting in

$$dX = (1 - 2\beta)\sigma^2(t)dt + 2\sqrt{X}\sigma(t)dW(t)$$

then perform a time-change (see Goldenberg [1991]) by defining

$$\phi(t) = \int_0^t \sigma^2(t)dt$$

so that

$$dW(\phi(t)) = \sigma(t)dW(t) \quad \text{and} \quad d\phi(t) = \sigma^2(t)dt$$

such that

$$dX = (1 - 2\beta)d\phi(t) + 2\sqrt{X}dW(\phi(t))$$

X is a *squared Bessel process of order* $1 - 2\beta$ which can be tackled via Feller [1951]. See further Cox [1975] and Schroder [1989].

Conditional CEV Price

Returning to the CEV equation

$$dF(t, T) = \sigma(t)F(t, T)^\beta dW(t)$$

then under the intermediate change of variables $X(t) = F(t, T)^{1-\beta} / (1-\beta)$ and conditioning on the volatility path

$$dX(t) = -\frac{1}{2} \frac{\beta}{1-\beta} \frac{1}{X(t)} \sigma^2(t) dt + \rho \sigma(t) dW_\sigma(t) + \bar{\rho} \sigma(t) dW_S(t)$$

and so

$$X(T) = X(0) + \frac{\rho}{\xi} (\sigma(T) - \sigma(0)) - \frac{1}{2} \frac{\beta}{1-\beta} \int_0^T \frac{1}{X(t)} \sigma^2(t) dt + \bar{\rho} \int_0^T \sigma(t) dW_S(t)$$

Conditional CEV Price - Islah [2009]

Islah [2009] proposes as an approximation for the non-zero correlation case to use

$$d\bar{X}(t) = -\frac{1}{2} \frac{\beta}{1-\beta} \frac{1}{\bar{X}(t)} \sigma^2(t) dt + \bar{\rho} \sigma(t) dW_S(t)$$

with

$$\bar{X}(0) = X(0) + \frac{\rho}{\xi} (\sigma(T) - \sigma(0))$$

Then with $Y(t) = \bar{X}(t)^2$ and time change

$$\phi(t) = (1 - \rho^2) \int_0^t \sigma^2(s) ds$$

reduces $Y(t)$ to the form

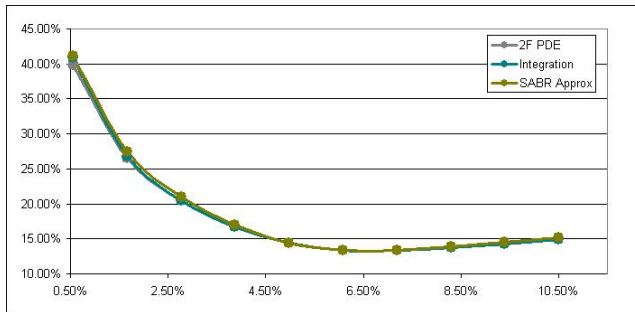
$$dY(\phi(t)) = \left(\frac{1 - 2\beta - \rho^2(1 - \beta)}{(1 - \beta)(1 - \rho^2)} \right) d\phi(t) + 2\sqrt{Y(\phi(t))} dW_{\phi(t)}$$

which is once again a squared Bessel process.

SABR process vs approximation - 10YR AUD CMS

This example shows the 5yr expiry for AUD 10yr CMS:

SABR Implied Volatility: $T=5Y$, $\sigma_{atmf} = 13.75\%$, $\xi = 33\%$, $\rho = -10\%$, $\beta = 0.5$



Strike	0.55%	1.66%	2.76%	3.87%	4.98%	6.08%	7.19%	8.29%	9.40%	10.51%
2F PDE	39.82%	26.59%	20.46%	16.65%	14.33%	13.33%	13.30%	13.72%	14.28%	14.85%
Integration	40.84%	26.76%	20.51%	16.67%	14.33%	13.33%	13.31%	13.73%	14.29%	14.86%
SABR Approx	41.10%	27.46%	20.99%	16.92%	14.44%	13.39%	13.41%	13.91%	14.55%	15.19%

The 2F PDE and SABR Integration remain very close with the SABR approximation showing a small degree of difference.

SABR process ($\beta < 1$) sample calculation times

In this case the conditional vanilla (CEV) valuation is significantly slower requiring application of enhanced numerical integration techniques:

- ▶ Test machine:
 - ▶ Intel Xeon CPU 2.33GHz
 - ▶ RAM 3.25GB
- ▶ 50 integration steps per integral
- ▶ Runtime (standard integration scheme):
 - ▶ $\sim 0.0844s$: 1 strike
 - ▶ $\sim 0.8593s$: 10 strikes
 - ▶ $\sim 8.6891s$: 100 strikes
- ▶ Runtime (enhanced integration scheme):
 - ▶ $\sim 0.0035s$: 1 strike
 - ▶ $\sim 0.0355s$: 10 strikes
 - ▶ $\sim 0.3595s$: 100 strikes

Long dated integration - 30YR USDJPY

For the longest maturity trades, a single integration step is insufficient to achieve the desired accuracy as the conditional lognormal assumption is weakened. It is possible to therefore take an intermediate integration set (Integration II, below) to correct for this.

Strike	Implied Vol 2F PDE	Implied Vol Integration I	Implied Vol Integration II
0.1	28.44%	28.53%	28.51%
0.3	23.34%	23.41%	23.36%
0.5	20.89%	20.87%	20.89%
0.7	19.33%	19.25%	19.33%
0.9	18.26%	18.14%	18.25%
1.1	17.53%	17.36%	17.51%
1.3	17.04%	16.85%	17.02%
1.5	16.72%	16.53%	16.70%
1.7	16.54%	16.36%	16.52%
1.9	16.45%	16.29%	16.43%

SABR Implied Volatility: $T=30Y$, $\sigma_{atmf} = 19.45\%$, $\xi = 21.87\%$, $\rho = -46.75\%$, $\beta = 1.0$

The SABR model in practice

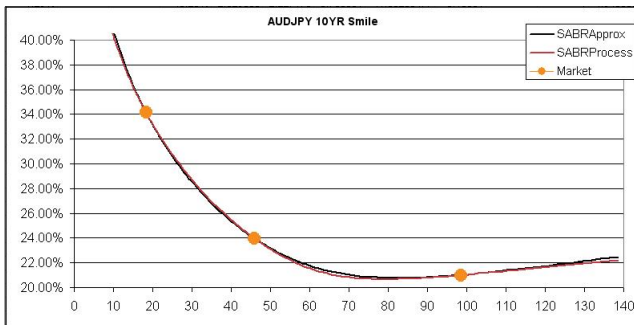
Previously, the parameters of the SABR model were fixed and the resulting SABR Approx, SABR Process Integration and SABR Process PDE methods compared. In practice, the input is market rates to which the SABR model is calibrated. For AUDJPY:

Model Parameters for SABR Process					Model Parameters for SABR Approx				
Tenor	ATMVol	ATM/Sigma	VolVol	Rho	Tenor	ATMVol	ATM/Sigma	VolVol	Rho
1W	14.00%	13.73%	374.29%	-26.82%	1W	14.00%	13.99%	365.83%	-25.79%
1M	15.80%	15.48%	213.30%	-43.24%	1M	15.80%	15.76%	206.91%	-41.42%
2M	15.75%	15.42%	167.87%	-52.58%	2M	15.75%	15.66%	161.50%	-50.12%
3M	16.20%	15.93%	144.28%	-61.44%	3M	16.20%	16.06%	138.27%	-58.89%
6M	17.40%	17.16%	116.04%	-66.69%	6M	17.40%	17.13%	109.94%	-63.83%
1Y	18.90%	18.71%	97.16%	-69.24%	1Y	18.90%	18.37%	89.53%	-65.81%
2Y	19.40%	20.23%	63.03%	-72.66%	2Y	19.40%	19.40%	58.56%	-70.49%
3Y	20.40%	21.80%	56.07%	-72.94%	3Y	20.40%	20.40%	51.02%	-70.82%
4Y	21.40%	23.29%	51.89%	-69.91%	4Y	21.40%	21.40%	46.08%	-67.55%
5Y	22.20%	24.62%	49.48%	-67.00%	5Y	22.20%	22.20%	42.81%	-64.42%
6Y	22.80%	25.91%	49.16%	-63.91%	6Y	22.80%	22.80%	41.05%	-61.02%
7Y	23.15%	27.17%	49.81%	-61.73%	7Y	23.15%	23.15%	39.92%	-58.79%
8Y	23.40%	28.72%	51.41%	-60.05%	8Y	23.40%	23.40%	39.24%	-57.44%
9Y	23.65%	30.46%	52.52%	-58.44%	9Y	23.65%	23.65%	38.31%	-56.56%
10Y	23.90%	32.02%	52.04%	-57.49%	10Y	23.90%	23.90%	36.91%	-56.49%

This is consistent with earlier results where, for common parameters, the SABR approx had excess smile and more negative skew than the SABR process.

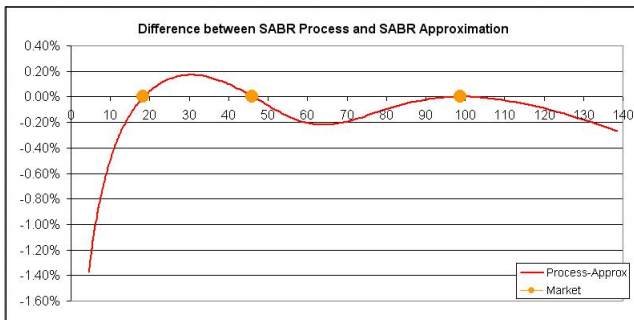
Associated smiles in the calibration region

The calibration region is sufficiently restricted that all models are close to one another between 15P and 15C, with deviations outside this region.



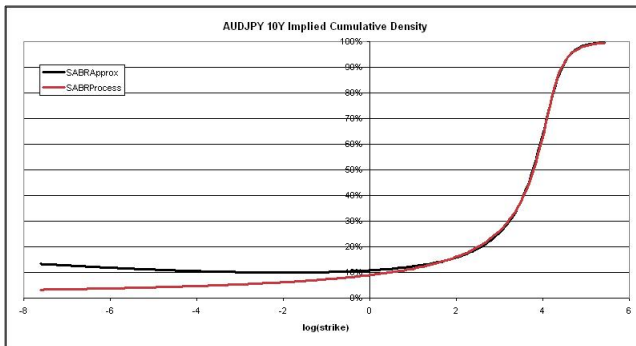
Difference between the SABR Process and SABR Approximation Smiles

Graphing the difference between the SABR Process and the SABR Approx, shows that the SABR process would have significantly lower volatilities in the deep tails:



Difference between the SABR Process and SABR Cumulative Density

The high downside volatilities in the SABR approximation lead to regions of negative density (as shown by the region of *decreasing* cumulative density below).



By construction, as an integral will defined densities, the SABR Conditional Integration will always have positive density.

The general case of mean reverting volatility

Returning to the more general case of mean reverting volatility

$$dX(t) = (\theta - \alpha X)dt + \xi dW_\sigma$$

where $X = \log \sigma^2(t)$. Then this can be integrated to yield

$$X(T) = X(0)e^{-\alpha T} + \frac{\theta}{\alpha}(1 - e^{-\alpha T}) + \frac{\xi}{\sqrt{2\alpha T}}(1 - e^{-2\alpha T})^{1/2}\overline{W}_\sigma(T)$$

and so $\sigma^2(T)$ is lognormally distributed with expectation

$$\mathbb{E}[\sigma^2(T)] = \mathbb{E}[e^{X(T)}] = \exp\left(X(0)e^{-\alpha T} + \frac{\theta}{\alpha}(1 - e^{-\alpha T}) + \frac{\xi^2}{4\alpha}(1 - e^{-2\alpha T})\right)$$

allowing, at least, for the numerical calculation of the unconditional moments $\mathbb{E}[\vartheta]$, $\mathbb{E}[\vartheta^2]$, etc.

The general case of mean reverting volatility

In order to calculate the conditional moments of the mean integrated variance distribution, then it is necessary to calculate $X(t)|X(T)$. This can be written in the form

$$X(t)|X(T) = X(0)e^{-\alpha t} + \frac{\theta}{\alpha}(1 - e^{-\alpha t}) + \bar{\rho}(t, T)(1 - e^{-2\alpha t})^{1/2}\varsigma \\ \frac{\xi}{\sqrt{2\alpha t}}(1 - e^{-2\alpha t})^{1/2}(1 - \bar{\rho}^2(t, T))^{1/2}W(t)$$

where

$$\varsigma = \frac{X(T) - X(0)e^{-\alpha T} - \frac{\theta}{\alpha}(1 - e^{-\alpha T})}{(1 - e^{-2\alpha T})^{1/2}}$$

is related to the number of standard deviations associated with $X(T)$ and

$$\bar{\rho}(t, T) = \frac{(1 - e^{-2\alpha t})^{1/2}}{(1 - e^{-2\alpha T})^{1/2}}e^{-\alpha(T-t)}$$

As before, the conditional moments $\mathbb{E}[\vartheta|\sigma(T)]$ and $\mathbb{E}[\vartheta^2|\sigma(T)]$ can now be calculated.

Correlation adjustment under mean reversion

Unlike the case of lognormal volatility, the adjustment to the forward under mean reverting volatility conditional upon $\sigma(T)$ is path dependent and can be written

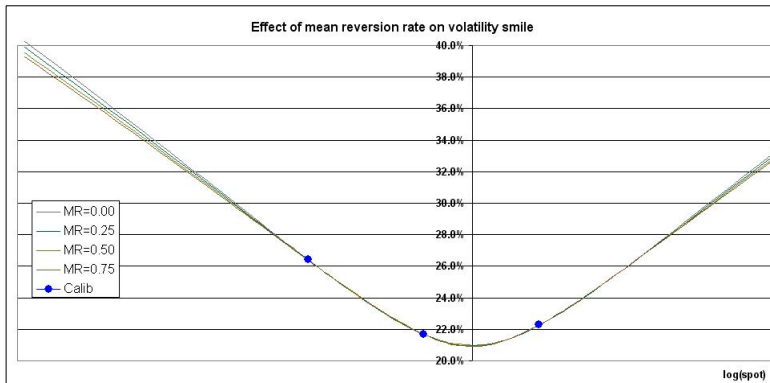
$$\mathbb{E} \left[\int_0^T \sigma dW_\sigma \middle| X(T) \right] \\ = \frac{2}{\xi} (\sigma(T) - \sigma(0)) - \left(\frac{\xi}{4} + \frac{\theta}{\xi} \right) \int_0^T \mathbb{E} \left[e^{X(t)/2} \middle| X(T) \right] + \frac{\alpha}{\xi} \int_0^T \mathbb{E} \left[e^{X(t)/2} X(t) \middle| X(T) \right] dt$$

is closely related to $\vartheta|X(T)$ and can be modelled / approximated in a number of ways.

This term reduces to the lognormal volatility under $\xi = 2\xi_{ln}$ and $\theta = -\xi^2/4 = -\xi_{ln}^2$.

Effect of introducing mean reversion

The effect of mean reverting volatility is clearly seen - providing a way of controlling the wings:



Vanilla Smile Summary

- ▶ The use of the conditional integration technique allows for the accurate calculation of vanilla option prices under a range of stochastic volatility models, including SABR;
- ▶ Conditional integration ensures fast, efficient and robust calculation of the vanilla smile across strikes and maturities;
- ▶ Vanilla calibration is a key component of more complex models e.g. 4-factor local-stochastic volatility with stochastic interest rates used in the long dated FX space (see “Mixed Volatility Dynamics: Pricing & Calibration of Long dated Multi-Asset Products” by Han Lee / Oliver Brochhaus);
- ▶ By construction, the methodology ensures a well defined probability density function (unlike SABR Approx);
- ▶ The methodology can be used to study the mean reverting volatility problem and thereby provides a way of controlling the volatility smile wings;
- ▶ The technique of constructing the joint density of the asset level and instantaneous volatility allows for an investigation of the forward smile ...



The Forward Smile

Joint and Conditional Densities of the Asset and Instantaneous Volatility

The previous methodology to price vanilla options can equally well be used to construct the joint density of the asset and instantaneous volatility, $\psi(F(T, T), \sigma(T))$ as

$$\psi(F(T, T), \sigma(T)) = \int_0^\infty d\vartheta \int_{-\infty}^\infty \psi(F(T, T)|\vartheta, I_{\sigma dW}) \psi(\vartheta, I_{\sigma dW}|\sigma(T)) dI_{\sigma dW} \psi(\sigma(T))$$

Of particular interest are the conditional densities

$$\psi(F(T, T)|\sigma(T)) = \frac{\psi(F(T, T), \sigma(T))}{\psi(\sigma(T))}$$

representing the asset distribution conditional upon the instantaneous volatility $\sigma(T)$ and

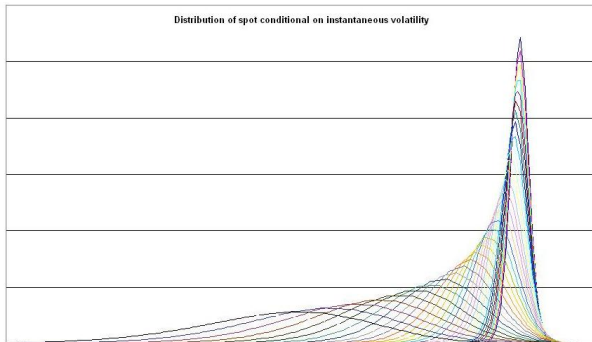
$$\psi(\sigma(T)|F(T, T)) = \frac{\psi(F(T, T), \sigma(T))}{\psi(F(T, T))}$$

the distribution of the instantaneous volatility conditional upon the asset level, $F(T, T)$. The latter is particularly relevant in the analysis of the forward smile distribution.

Asset density conditional on instantaneous volatility $\psi(F(T, T)|\sigma(T))$

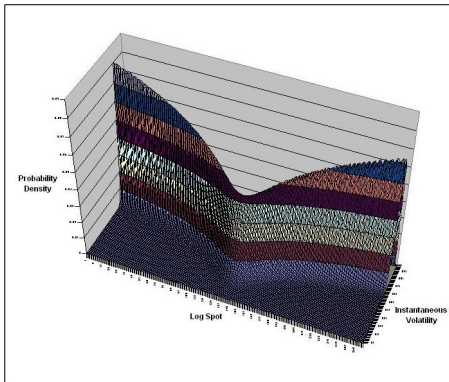
The graph below shows the case of SABR ($\beta = 1$) with $\rho < 0$

This is the distribution of $\psi(F(T, T)|\sigma(T))$ for a range of $\sigma(T)$:



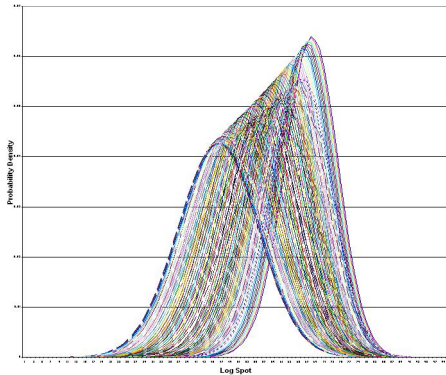
- ▶ for low $\sigma(T)$ the level of volatility will be low and the spot distribution will be narrow and shifted higher.
- ▶ for high $\sigma(T)$ the level of volatility will be high and the spot distribution will be wide and shifted lower.

Instantaneous volatility conditional on spot $\psi(\sigma(T)|F(T, T))$



The characteristic “V” shape originates from large spot movements being the result of high volatility paths; where spot is little moved will be the result of low volatility paths. The asymmetry here is caused by a negative spot-volatility correlation.

A cross-section of $\psi(\sigma(T)|F(T, T))$



- ▶ The densities to the right correspond to large asset moves as they are associated with few high volatility paths - and so narrow.
- ▶ The densities to the left correspond to the asset moving very little - all volatility paths will contribute some probability to the asset being unchanged. As such the distribution of instantaneous volatility here is wide.

Forward Volatility Surfaces

- ▶ In a local volatility model there is a unique implied volatility surface at a future asset level $F(t, t)$ - denote this by $\Pi_{LV}(F(t, t))$.
- ▶ In stochastic volatility models there is a distribution of implied volatility surfaces represented by $\psi(\sigma(t)|F(t, t))$ where for each such $\sigma(t)$ the implied volatility surface will be denoted $\Pi(F(t, t), \sigma(t))$.
- ▶ Given a future asset level there are two expected implied volatility surfaces that can be defined:

- ▶ the expected implied volatility per strike:

$$\int_0^{\infty} d\sigma(t) \psi(\sigma(t)|F(t, t)) IV \left\{ C((K, T - t, \Pi(F(t, t), \sigma(t)))) \right\}$$

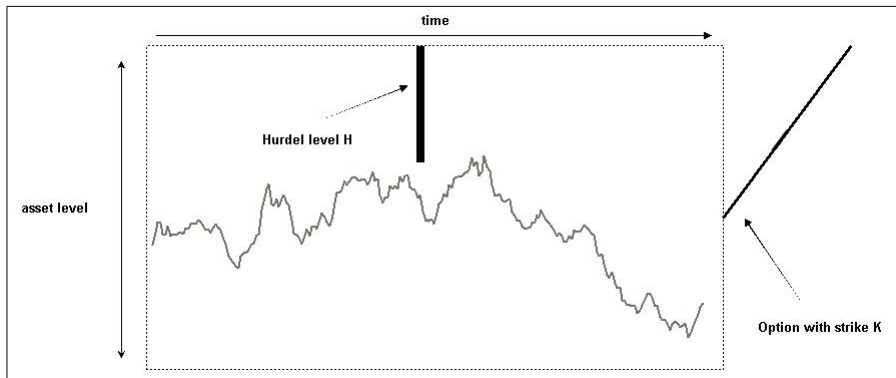
- ▶ the implied volatility of the expected price per strike

$$IV \left\{ \int_0^{\infty} d\sigma(t) \psi(\sigma(t)|F(t, t)) C(K, T - t, \Pi(F(t, t), \sigma(t))) \right\}$$

- ▶ This latter definition will generate the expected implied volatility surface denoted $\Pi_e(F(t, t))$ whose smile curves will be called *expected forward smiles*.

Single Hurdle Options and the associated Expected Forward Smile

Consider a vanilla option expiring at T with strike K which can knockout at time t if the asset is above a level H . This is known as a hurdle option:



Single Hurdle Options and the associated Expected Forward Smile

The hurdle option, denoted V_{H^+} , is given by

$$\begin{aligned}
 V_{H^+} &= \int_0^H dF(t, t) \psi(F(t, t)) \int_0^\infty d\sigma(t) \psi(\sigma(t) | F(t, t)) C(K, T - t, \Pi(F(t, t), \sigma(t))) \\
 &= \int_0^H dF(t, t) \psi(S(t)) C(K, T - t, \Pi_e(F(t, t), \sigma(t)))
 \end{aligned}$$

using the definition of the expected implied volatility surface, $\Pi_e(F(t, t))$.

Differentiating with respect to H and using

$$\psi(H) = \left. \frac{\partial V_{DG^-}}{\partial K} \right|_{K=H}$$

with V_{DG^-} a digital put then

$$\Pi_e(F(t, t) = H) = IV \left\{ \frac{\partial V_{H^+}}{\partial H} \middle/ \left. \frac{\partial V_{DG^-}}{\partial K} \right|_{K=H} \right\}$$

- *From the prices of single period hurdle options, the expected forward smile can be calculated*

The larger the distribution of $\sigma(t)|F(t, t)$ the more the curvature in $\Pi_e(F(t, t))$

Suppose that $\Pi(F(t, t), \sigma(t)) = \sigma(t)$ i.e. the forward smiles are flat. What does the expected forward smile look like?

Writing $V(K, T - t, \sigma(t))$ as $V(\sigma(t))$ to ease notation and expanding $V(\sigma(t))$ about $\mathbb{E}[\sigma(t)]$ i.e.

$$V(\sigma(t)) \sim V(\mathbb{E}[\sigma(t)]) + \left. \frac{\partial V}{\partial \sigma} \right|_{\sigma(t)} (\sigma(t) - \mathbb{E}[\sigma(t)]) + \frac{1}{2} \left. \frac{\partial^2 V}{\partial \sigma^2} \right|_{\sigma(t)} (\sigma(t) - \mathbb{E}[\sigma(t)])^2$$

Then

$$\int_0^\infty d\sigma(t) \psi(\sigma(t)|S(t)) V(\sigma(t)) \sim V(\mathbb{E}[\sigma(t)]) + \frac{1}{2} \left. \frac{\partial^2 V}{\partial \sigma^2} \right|_{\sigma(t)} \mathbb{E}[(\sigma(t) - \mathbb{E}[\sigma(t)])^2]$$

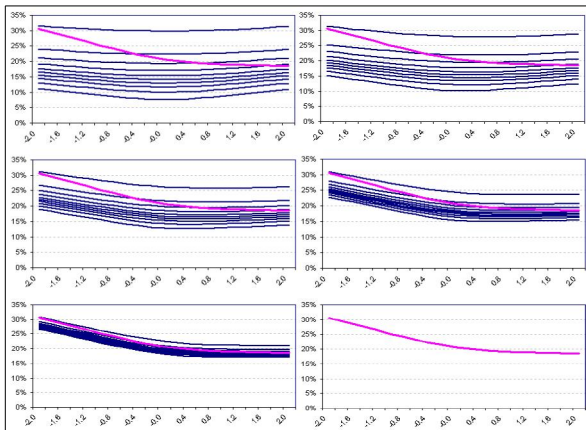
with

$$\frac{\partial^2 V}{\partial \sigma^2} > 0$$

the resulting smile will have positive curvature originating from the *distribution* of flat smiles (this is simply Hull and White [1987] once again).

Local-Stochastic Volatility


Calibrate a LSV model to a zero correlation stochastic volatility smile. Consider the downside forward smile at some future date:



As the volatility of volatility is decreased (blue lines) the distribution of forward smiles will collapse onto that of local volatility (pink lines): $\Pi_e(F(t, t)) \rightarrow \Pi_{LV}(F(t, t))$.

Forward Smile Summary

- ▶ The use of conditional integration techniques allow for the construction of the joint density of the asset and the instantaneous volatility;
- ▶ The two conditional densities of asset conditional on instantaneous volatility and instantaneous volatility conditional on asset level can easily be constructed;
- ▶ The density of the instantaneous volatility conditional on the asset level is particularly relevant as it determines the distribution of forward smiles within the stochastic volatility model;
- ▶ An understanding of the forward smile distribution has application across exotics pricing and provides insight into the evolution of the implied volatility surface itself.



Correlated Stochastic Volatility

Correlated Stochastic Volatility Processes

Consideration will be given to correlated (log)normal asset processes with lognormal volatility:

$$dR_X = \mu_X(t)dt + \sigma_X(t)dW_X^R$$

$$dR_Y = \mu_Y(t)dt + \sigma_Y(t)dW_Y^R$$

$$d\sigma_X = \xi_X(t)\sigma_X(t)dW_X^\sigma$$

$$d\sigma_Y = \xi_Y(t)\sigma_Y(t)dW_Y^\sigma$$

Relevant correlations are

$$\text{asset-asset} \quad \langle dW_X^R, dW_Y^R \rangle = \rho(R_X, R_Y)dt$$

$$\text{asset-volatility} \quad \langle dW_{X,Y}^R, dW_{X,Y}^\sigma \rangle = \rho(R_{X,Y}, \sigma_{X,Y})dt$$

$$\text{volatility-volatility} \quad \langle dW_X^\sigma, dW_Y^\sigma \rangle = \rho(\sigma_X, \sigma_Y)dt$$

Here attention will be paid to asset-asset and volatility-volatility correlations.

Correlation Swap Contract definition

Given $N + 1$ observations on dates $\{T_0, T_1, \dots, T_N\}$ of two assets X and Y , denoted $S_X(i)$ and $S_Y(i)$, then the correlation swap pays

$$\text{Notional} \times (\rho_{\text{swap}}(S_X, S_Y) - K)$$

where K is the strike and

$$\rho_{\text{swap}}(S_X, S_Y) = \frac{\sum_{i=1}^N (R_X(i) - \bar{R}_X)(R_Y(i) - \bar{R}_Y)}{\sqrt{\sum_{i=1}^N (R_X(i) - \bar{R}_X)^2 \sum_{i=1}^N (R_Y(i) - \bar{R}_Y)^2}} \equiv \frac{C_{X,Y}}{\sqrt{V_X V_Y}}$$

with $C_{X,Y}$ the covariance between X and Y , V_X , V_Y their respective variances and

$$R_{X,Y}(i) = S_{X,Y}(i) - S_{X,Y}(i-1)$$

for normal assets,

$$R_{X,Y}(i) = \log(S_{X,Y}(i)/S_{X,Y}(i-1))$$

for lognormal assets. The asset means are given by $\bar{R}_{X,Y} = \frac{1}{N} \sum_{i=1}^N R_{X,Y}(i)$

Empirical observations of the effect of (correlated) stochastic volatility

Building a 4-factor Monte Carlo allows for the effect of (correlated) stochastic volatility on the pricing of correlation swaps to be investigated.

Denoting $\rho_{swap} = \mathbb{E}[\rho_{swap}(S_X, S_Y)]$, ρ_{spot} the input spot correlation and ρ_{vol} the input correlation between the stochastic volatility processes, then the following observations are made:

- ▶ With no stochastic volatility,

$$\rho_{swap}(SV = 0) \sim \rho_{spot}$$

- ▶ With uncorrelated stochastic volatility,

$$\rho_{swap}(SV \neq 0, \rho_{vol} = 0) < \rho_{swap}(SV = 0) \sim \rho_{spot}$$

- ▶ With correlated stochastic volatility,

$$\rho_{swap}(SV \neq 0, \rho_{vol} = 0) < \rho_{swap}(SV \neq 0, \rho_{vol} > 0) < \rho_{swap}(SV = 0) \sim \rho_{spot}$$

Conditional Monte Carlo for Correlation Swaps under Stochastic Volatility

Given a joint space of volatility paths associated with X and Y , denoted $\Omega\{\sigma_X, \sigma_Y\}$, and the expected value of the correlation swap conditional upon these paths, $\rho_{swap}(\sigma_X, \sigma_Y)$, then

$$\rho_{swap} = \int_{\Omega} \rho_{swap}(\sigma_X, \sigma_Y) \psi(\sigma_X, \sigma_Y) d\sigma_X d\sigma_Y$$

where $\psi(\sigma_X, \sigma_Y)$ is joint density of the volatility paths.

Usually this integral will be performed under Monte Carlo by simulating both spot paths and both volatility paths, however, using Lipton and McGhee [1999] then $\rho_{swap}(\sigma_X, \sigma_Y)$ is known in expansion form under a term structure of forward curve and volatility. As a result, only the correlated volatility paths need be generated.

Further, if the distribution of correlated volatility paths could be captured numerically then a direct conditional integration approach can be attempted.

Correlation Swaps under Black-Scholes term structure

Given

$$\begin{aligned}R_X(i) &= \mu_X(i)\Delta T(i) + \sigma_X(i)\Delta W_X(i) \\ R_Y(i) &= \mu_Y(i)\Delta T(i) + \sigma_Y(i)\Delta W_Y(i)\end{aligned}$$

for non-stochastic parameters $\mu_{X,Y}(i)$ and $\sigma_{X,Y}(i)$, time increments $\Delta T(i)$ and $\Delta W_{X,Y}(i) \sim N(0, \sqrt{\Delta T(i)})$ with asset-asset correlation $\rho_{X,Y}(i)$ such that

$$\langle \Delta W_X(i), \Delta W_Y(i) \rangle = \rho_{X,Y}(i)\Delta T(i)$$

then, making the dependency on the volatility term structures σ_X and σ_Y explicit

$$\rho_{swap}(\sigma_X, \sigma_Y) = \mathbb{E} \left[\frac{C_{X,Y}}{V_X^{1/2} V_Y^{1/2}} \middle| \sigma_X, \sigma_Y \right]$$

Correlation Swap Expansion of Lipton and McGhee [1999]

Writing

$$\rho_{swap}(\sigma_X, \sigma_Y) = \mathbb{E} \left[\frac{C_{X,Y}}{V_X^{1/2} V_Y^{1/2}} \middle| \sigma_X, \sigma_Y \right]$$

then the function $f(C_{X,Y}, V_X, V_Y) = C_{X,Y} V_X^{-1/2} V_Y^{-1/2}$ can be expanded about $\mathbb{E}[C_{X,Y}]$, $\mathbb{E}[V_X]$ and $\mathbb{E}[V_Y]$ so that

$$\begin{aligned} \mathbb{E}[f(C_{X,Y}, V_X, V_Y)] &\sim f(\mathbb{E}[C_{X,Y}], \mathbb{E}[V_X], \mathbb{E}[V_Y]) \\ &+ \frac{1}{2} \frac{\partial^2 f}{\partial C_{X,Y}^2} \mathbb{E}[(C_{X,Y} - \mathbb{E}[C_{X,Y}])^2] + \dots \\ &+ \frac{\partial^2 f}{\partial C_{X,Y} \partial V_X} \mathbb{E}[(C_{X,Y} - \mathbb{E}[C_{X,Y}])(V_X - \mathbb{E}[V_X])] + \dots \end{aligned}$$

and so to second order requires the calculation of the terms $\mathbb{E}[C_{XY}]$, $\mathbb{E}[V_X]$, $\mathbb{E}[V_Y]$, $\mathbb{E}[C_{XY}^2]$, $\mathbb{E}[V_X^2]$, $\mathbb{E}[V_Y^2]$, $\mathbb{E}[C_{XY} V_X]$, $\mathbb{E}[C_{XY} V_Y]$ and $\mathbb{E}[V_X V_Y]$ which can be done analytically.

Effect of Volatility Term Structure

Input correlation: $\rho_{spot} = 60\%$

Weekly observation for 1yr

0-6m $\sigma_X = 10\%$; 6m-12m $\sigma_X = \lambda_X \times 10\%$

0-6m $\sigma_Y = 20\%$; 6m-12m $\sigma_Y = \lambda_Y \times 20\%$

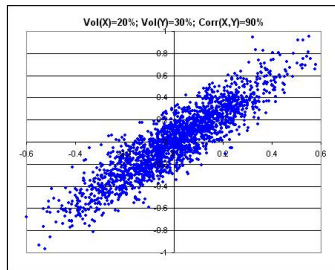
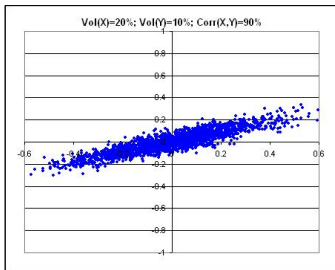
$\lambda_X \backslash \lambda_Y$	0.4	0.6	0.8	1.0	1.2	1.4	1.6
0.4	60.00%	59.23%	57.42%	55.15%	52.78%	50.51%	48.42%
0.6	59.23%	60.00%	59.46%	58.21%	56.65%	55.02%	53.45%
0.8	57.42%	59.46%	60.00%	59.63%	58.79%	57.73%	56.62%
1.0	55.15%	58.21%	59.63%	60.00%	59.75%	59.18%	58.46%
1.2	52.78%	56.65%	58.79%	59.75%	60.00%	59.83%	59.44%
1.4	50.51%	55.02%	57.73%	59.18%	59.83%	60.00%	59.89%
1.6	48.42%	53.45%	56.62%	58.46%	59.44%	59.89%	60.00%

In this simplified case using only the leading term of the expansion

$$\rho_{swap} = \rho_{spot} \left(\frac{1 + \lambda_X \lambda_Y}{\sqrt{1 + \lambda_X^2} \sqrt{1 + \lambda_Y^2}} \right)$$

Intuitive explanation of observations

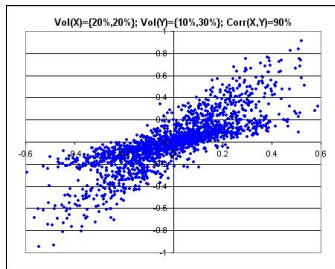
Generate two series of correlated normal variates to represent $R_X(i)$ and $R_Y(i)$.



In the above unnormalised graphs, the correlation can be thought of as measuring the degree of dispersion about the straight line (the slope of that straight line being the ratio of volatilities **not** the correlation). Each of these graphs corresponds to a flat volatility term structure with the same constant correlation.

Intuitive explanation of observations

Generate two series of correlated normal variates to represent $R_X(i)$ and $R_Y(i)$.



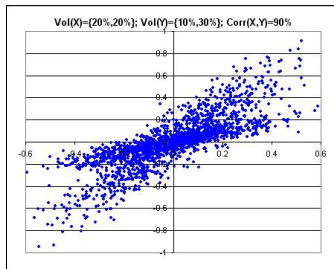
Now a term structure of volatility is introduced by combining the previous two scenarios:

- ▶ $(\sigma_X, \sigma_Y) = (20\%, 10\%)$ for the first half of the data series;
- ▶ $(\sigma_X, \sigma_Y) = (20\%, 30\%)$ for the second half of the data series.

The degree of dispersion has increased and so the correlation has dropped.

Intuitive explanation of observations

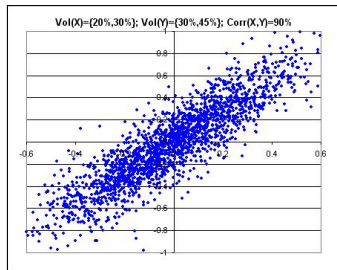
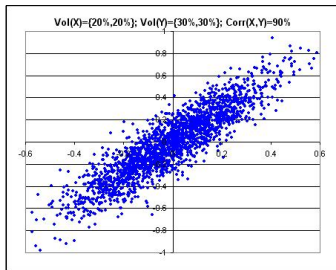
Generate two series of correlated normal variates to represent $R_X(i)$ and $R_Y(i)$.



Considering the zero stochastic volatility case as that of flat term structure, then introducing **uncorrelated** stochastic volatility corresponds to an integral over volatility term structures and results in decorrelation.

Intuitive explanation of observations

Generate two series of correlated normal variates to represent $R_X(i)$ and $R_Y(i)$.



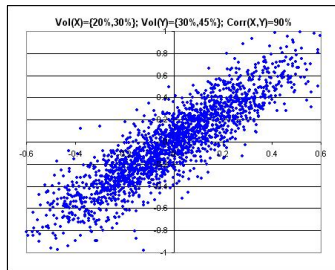
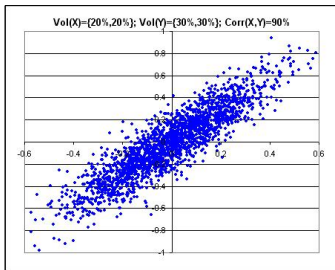
Now compare the case of flat volatility $(\sigma_X, \sigma_Y) = (20\%, 30\%)$ with that of:

- ▶ $(\sigma_X, \sigma_Y) = (20\%, 30\%)$ for the first half of the data series;
- ▶ $(\sigma_X, \sigma_Y) = (30\%, 45\%)$ for the second half of the data series.

As the volatilities move together then the degree of decorrelation is significantly reduced.

Intuitive explanation of observations

Generate two series of correlated normal variates to represent $R_X(i)$ and $R_Y(i)$.



Considering the zero stochastic volatility case as that of flat term structure, then introducing **correlated** stochastic volatility corresponds to an integral over volatility term structures and results in reduced decorrelation.

Effect of Uncorrelated Stochastic Volatility - Conditional MC

Input correlation: $\rho_{spot} = 60\%$; $\rho_{vol} = 0\%$

Weekly observation for 1yr

$\sigma_X(0) = 10\%$;

$\sigma_Y(0) = 20\%$;

Correlation swap price with varying vol-of-vol

$\xi_X \backslash \xi_Y$	0%	20%	40%	60%	80%	100%
0%	60.00%	59.80%	59.21%	58.28%	57.02%	55.53%
20%	59.80%	59.60%	59.03%	58.12%	56.84%	55.36%
40%	59.21%	59.03%	58.46%	57.59%	56.37%	54.97%
60%	58.28%	58.12%	57.59%	56.76%	55.61%	54.29%
80%	57.02%	56.84%	56.37%	55.61%	54.64%	53.41%
100%	55.53%	55.36%	54.97%	54.29%	53.41%	52.23%

For illustration only the leading order term is used here.

Effect of Correlated Stochastic Volatility - Conditional MC

Input correlation: $\rho_{spot} = 60\%$; $\rho_{vol} = 60\%$

Weekly observation for 1yr

$\sigma_X(0) = 10\%$;

$\sigma_Y(0) = 20\%$;


Correlation swap price with varying vol-of-vol

$\xi_X \backslash \xi_Y$	0%	20%	40%	60%	80%	100%
0%	60.00%	59.80%	59.21%	58.28%	57.02%	55.53%
20%	59.80%	59.84%	59.49%	58.79%	57.70%	56.37%
40%	59.21%	59.49%	59.38%	58.91%	58.07%	56.96%
60%	58.28%	58.79%	58.91%	58.68%	58.09%	57.20%
80%	57.02%	57.70%	58.07%	58.09%	57.79%	57.15%
100%	55.53%	55.37%	56.96%	57.20%	57.15%	56.73%

For illustration only the leading order term is used here.

Summary of Correlated Stochastic Volatility

- ▶ Correlation swaps are sensitive to both stochastic volatility and the correlation between stochastic volatility processes;
- ▶ The observed behaviour is conceptually described by pricing of the contract under Black-Scholes term structure;
- ▶ Pricing under stochastic volatility is analogous to integrating over a term structure of markets (as described by conditioning methodology);
- ▶ More widely, the methodology can be applied to value spread options, cross options and quanto/compo options;
- ▶ The method is sufficiently fast, robust and accurate to be used in practical applications.



Concluding Remarks

Concluding remarks

- ▶ There remain a number of analytic and pseudo-analytic techniques which allow for the detailed exploration of the behaviour of stochastic volatility and correlated stochastic volatility processes;
- ▶ Weaknesses in existing approximation and simulation methods can be replaced by robust, efficient, accurate algorithms;
- ▶ Pseudo-analytic techniques in stochastic volatility modelling provide an intuitive understanding of the forward smile dynamics and a link to exotics pricing and the dynamics of the macro volatility surface;
- ▶ In the multi-asset arena the techniques presented here provide for significant efficiency gains as well as an understanding of the effects of correlated stochastic volatility.

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