



# A Local Correlation Model : Motivation and Practical Implementation

Exotic Pricing and Hedging

Adil Reghai

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# 1 Motivation

# Literature I – on local correlation

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- 1.A-M. Avenalleda, D. Boyer-Olson, J. Busca and P.Friz, « Reconstructing Volatility », October 2002**
- 2.B-V. Durrleman + N. El-Karoui, « Basket Skew », April 2007**
- 3.C-Bruno Dupire « Basket Skew Asymptotics » working paper 2004**
- 4.D-X. Burtschell, J. Gregory and J-P. Laurent, « Beyond the Gaussian Copula: Stochastic and Local Correlation » , Working Paper, 2005**
- 5.E-A. Langnau, « Introduction Into Local Correlation Modelling » , September 2009.**
- 6.F-B. Jourdain, Mohamed Sbai “Coupling Index and stocks” 2009**

# Literature II – some comments

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**1.A-It gives the framework for calibrating baskets and numerical algorithms for short term asymptotics for pricing**

**2.B-C-It provides a good grasp of the phenomenology with model free approach**

**3.Good for the phenomenology**

**4.D-Simple idea to expand the dimension and obtain stochastic correlation at a cheap cost (is used for the local correlation model)**

**5.E-Simplest (sufficient conditions) local volatility extension plus direct calibration formulae and model risk illustration through the chewing gum effect**

**6.F- Nice numerical method – particle method, specific to baskets**

## Literature II

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**It provides a good grasp of the phenomenology (model-wise + chewing-gum effect)**

**However,**

**Only a short-term asymptotic formula for basket options is available.**

**It assumes that basket options are traded.**

**Numerical pricing with such models is extremely cumbersome. It requires to perform one Cholesky decomposition at each time step of a Monte-Carlo Path.**

# Remaining Issues

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- 1. How to deal with long-term maturity contracts?**
- 2. How to deal with worst-of options?**
- 3. How to handle cases where basket implied dynamics are not observable?**
- 4. How to price, risk-manage and stress-test correlation in a trading environment?**

# Motivation I

$$\frac{d\sigma_t}{\sigma_t} = 2 \sum_{i=1}^n \beta_t^i dq_t^i + 2 \sum_{i=1}^n \beta_t^i q_t^i \frac{d\sigma_t^i}{\sigma_t^i} + \frac{2}{\sigma_t^2} \sum_{1 \leq i < j \leq n} q_t^i q_t^j \sigma_t^i \sigma_t^j d\rho_t^{ij} + \{...\}dt$$

**With**  $B_t = \sum_{i=1}^n w_i S_t^i, \quad q_t^i = \frac{w_i S_t^i}{B_t} \quad \text{and} \quad \beta_t^i = \frac{1}{\sigma_t^2 dt} \text{Cov}\left(\frac{dS_t^i}{S_t^i}, \frac{dB_t}{B_t}\right)$

**Three factors contribute to the basket skew generation:**

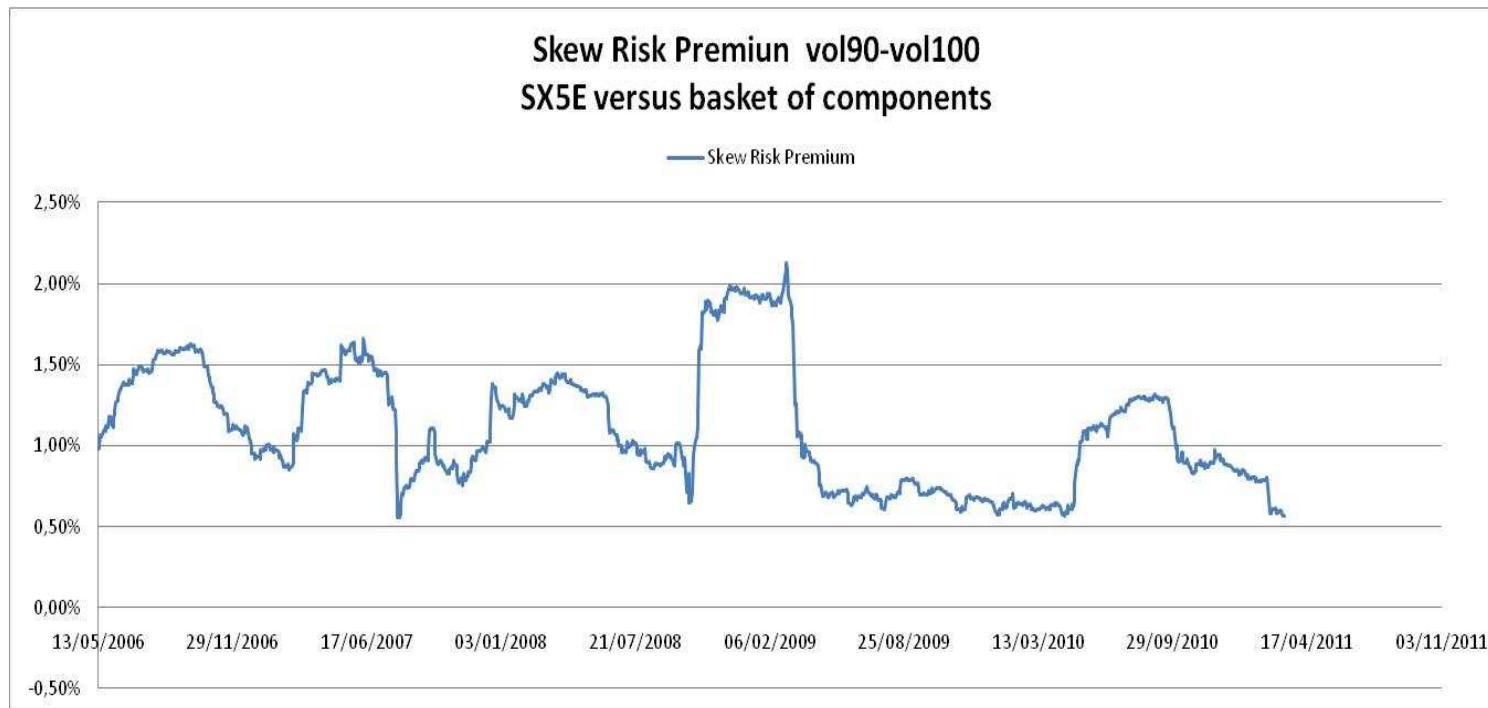
- **S<sup>i</sup>'s relative weight q<sup>i</sup> in the basket;**
- **S<sup>i</sup>'s volatility σ<sup>i</sup> (individual stock skew and their interactions);**
- **S<sup>i</sup> vs. S<sup>j</sup>'s stock correlation ρ<sup>ij</sup>.**

$$\begin{aligned} u_{SV} &= u_{BS} \\ &+ (T-t) \left\{ \sum_{i,j} \frac{1}{12} \alpha_i \alpha_j \sigma_i(t) \sigma_j(t) \rho_{i,j}^{\sigma,\sigma} \frac{\partial u_0}{\partial (\sigma_i \sigma_j)} \right\} \\ &+ (T-t) \left\{ \sum_{i,j} \frac{1}{6} \alpha_i \alpha_j \sigma_i(t) \sigma_j(t) \rho_{i,j}^{\sigma,\sigma} \frac{\partial^2 u_0}{\partial \sigma_i \partial \sigma_j} \right\} + (T-t) \left\{ \sum_{i,j} \frac{1}{2} \sigma_i(t) \alpha_j \sigma_j(t) \rho_{i,j}^{s,\sigma} S_i \frac{\partial^2 u_0}{\partial S_i \partial \sigma_j} \right\} \end{aligned}$$



# Motivation I – illustration on SX5E

**Without stochastic correlation, we can't fit empirical index skews, much steeper than the reconstructed one.**



## Motivation II

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**In the local volatility framework, carry drives daily P&L on a delta-neutral portfolio  $\Pi$  :**

$$\Delta \text{PL}_t \approx \frac{1}{2} \sum_{1 \leq i, j \leq n} \frac{\partial^2 \Pi}{\partial S_t^i \partial S_t^j} S_t^i S_t^j \left( \frac{\Delta S_t^i}{S_t^i} \frac{\Delta S_t^j}{S_t^j} - \rho_t^{ij} \sigma_i(t, S_t^i) \sigma_j(t, S_t^j) \Delta t \right)$$

**Under stress market conditions, realized covariance might exceed expectation...**

$$\rho_t^{ij} \sigma_i(t, S_t^i) \sigma_j(t, S_t^j) \Delta t \ll \frac{\Delta S_t^i}{S_t^i} \frac{\Delta S_t^j}{S_t^j}$$

**... Damaging the risk-management of trading books embedding short-correlation exposure.**

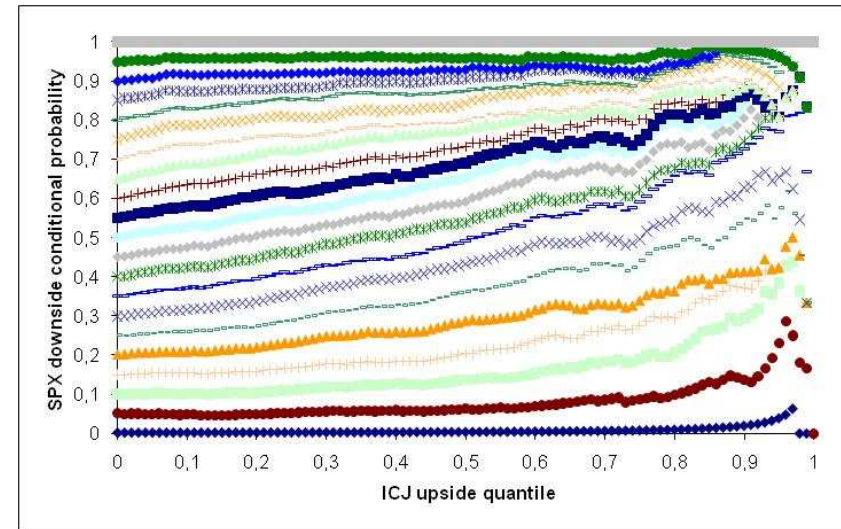
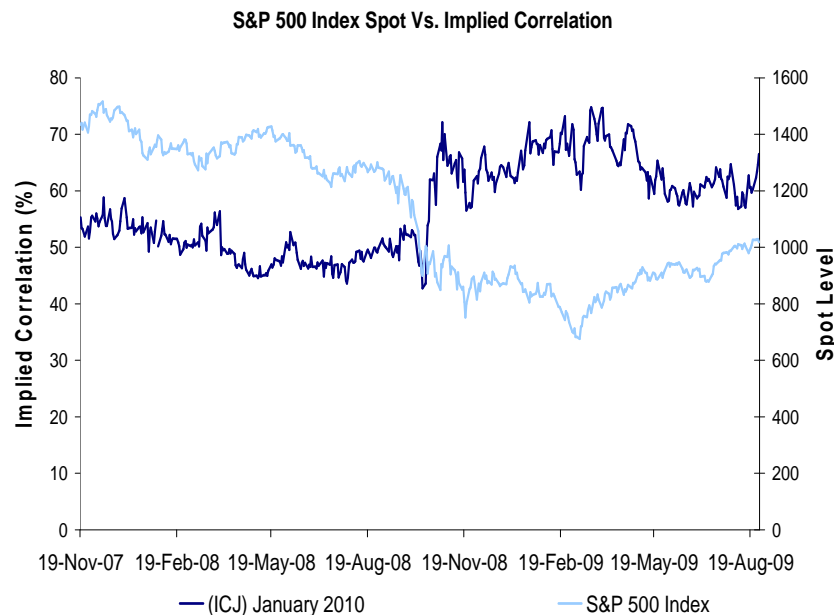
**Assuming well-marked volatilities, correlation need to be shifted by  $\Delta \rho$  so that**

$$(\rho + \Delta \rho)_t^{ij} \sigma_i(t, S_t^i) \sigma_j(t, S_t^j) \Delta t = E \left[ \frac{\Delta S_t^i}{S_t^i} \frac{\Delta S_t^j}{S_t^j} \right]$$

**Incurring a brutal  $(\partial \Pi / \partial \rho) \cdot \Delta \rho$  to stop the bleeding.**

# Motivation III

Historical correlation measures (ICJ) implied from past prices of S&P500 index option expiring on January 2010

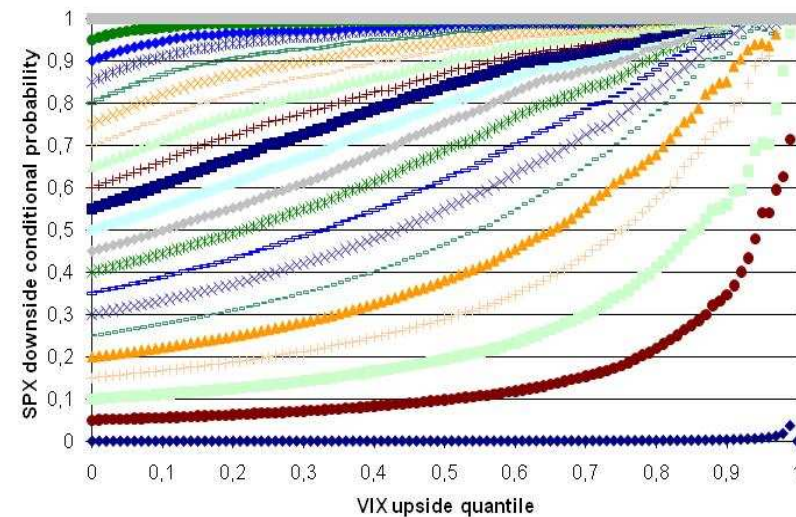
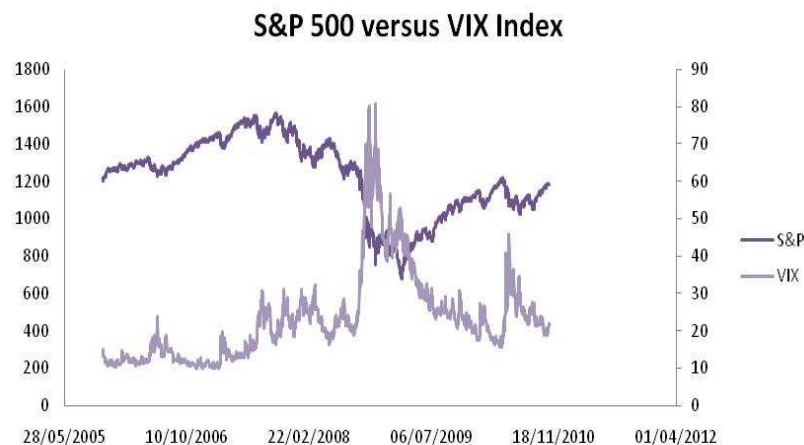


$$P\left(r_{SPX} \leq F_{SPX}^{-1}(u_0) \mid r_{JCJ} \geq F_{JCJ}^{-1}(x)\right)$$

# Motivation IV

Measures of Risk are correlated So it is good to model correlation linked to volatility

**BUT what do you do when markets are dislocated**



$$P\left(r_{SPX} \leq F_{SPX}^{-1}(u_0) \mid r_{VIX} \geq F_{VIX}^{-1}(x)\right)$$

# Motivation V

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**Incorporating correlation risk in the risk-management of multi-underlying options is highly advised.**

**But implementing stochastic correlation is complicated in practice:**

- **Ill-posed calibration problem with very few market information compared to the number of parameters to determine;**
- **Time-consuming diffusion scheme given that a matrix factorization is required at each time step of the discretization grid.**

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# 2 Local Correlation Model

# Models Up and Running In The Banking Industry

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Bank/ Model	Black Pricing	Black Risk Mgt	Black Stress Test	Local Vol Pricing	Local Vol Risk Mgt	Local Vol Stress Test
Top Tier*	✓	✓	✓	✓	✓	✓
Second Tier	✓	✓	✓	✓	X	X
Third Tier	✓	✓	✓	X	X	X

\* "Top Tier" at the technical level.

# Incremental progress

Spot dynamics mostly used by banks are as follows:

$$\left\{ \begin{array}{l} \frac{dS_t^1}{S_t^1} = \sigma_1(t, S_t^1) dW_t^{\rho,1} \\ \vdots \\ \frac{dS_t^n}{S_t^n} = \sigma_n(t, S_t^n) dW_t^{\rho,n} \end{array} \right. \quad < dW_t^{\rho,i}, dW_t^{\rho,j} > = \rho_{i,j}(t) dt$$

A significant enhancement is to consider a local correlation depending on two variables : Time  $t$  + an aggregator  $L$ .

More precisely,

$$\begin{aligned} \rho_{ij}^- &\leftarrow \rho_{ij} * (1 - \lambda) & \lambda(t, S_t^1, \dots, S_t^n) &= f(t, L_t) \\ \rho_{ij}^+ &\leftarrow \rho_{ij} * (1 - \lambda) + \lambda & L_t &= \sum_{i=1}^n w^{(i)} \frac{S_t^{(i)}}{S_0^{(i)}} \end{aligned}$$



# Local Correlation Model

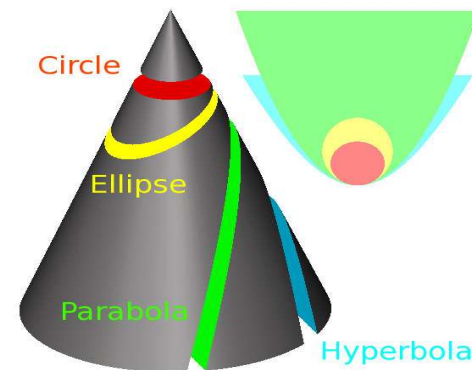
**Instantaneous correlation becomes**

$$\rho^{+, \lambda}(t; S_t^1, \dots, S_t^k) = (1 - \lambda(t; L_t)) \cdot \mathbf{\rho} + \lambda(t; L_t) \cdot \begin{bmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{bmatrix}$$
$$\rho^{-, \lambda}(t; S_t^1, \dots, S_t^k) = (1 - \lambda(t; L_t)) \cdot \mathbf{\rho}$$

**Related Brownian is easily simulated with the following dimension extension:**

$$W^{\rho^{\lambda,+}}(t) = \sqrt{(1 - \lambda(t; L_t))} \cdot W^{\rho}(t) + \sqrt{\lambda(t; L_t)} \cdot W^{\perp}(t)$$
$$W^{\rho^{\lambda,-}}(t) = \sqrt{(1 - \lambda(t; L_t))} \cdot W^{\rho}(t) + \sqrt{\lambda(t; L_t)} \cdot W^{Id}(t)$$

**Similar idea to Pascal's classification of conics:**



## Local Correlation Model : Calibration à la Dupire – Langnau Method

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**Instantaneous moment matching of variance of basket with constituents gives:**

$$\sum_{i=1}^N \rho_{ij} c_{ij} = \sigma_B^2(t, B) B^2$$

**With**

$$c_{ij} := \omega_i \omega_j S_i S_j \sigma_i(t, S_i) \sigma_j(t, S_j)$$

**So we have the following local correlation model**

$$\lambda(t, \vec{S}) = \frac{\sigma_B^2 B^2 - \sum_{ij} \rho_{ij}^{\text{fixed}} c_{ij}}{\sum_{ij} (1 - \rho_{ij}^{\text{fixed}}) c_{ij}} \quad \text{(to increase correlation)}$$

$$\lambda(t, \vec{S}) = \frac{\sigma_B^2 B^2 - \sum_{ij} \rho_{ij}^{\text{fixed}} c_{ij}}{\sum_{ij} (\frac{1}{N-1} + \rho_{ij}^{\text{fixed}}) c_{ij}} \quad \text{(to decrease correlation)}$$

**Provided that:**

$$\sigma_B^2(t, B) B^2 \leq \sum_{ij} c_{ij}$$

# Bad-Press On Local Volatility Model. Yet...

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**Widely used by investment banks to manage their trading books.**

**Main drawbacks are**

- **Model-Wise**  
Its dynamics underestimate
  - forward skew level
  - vovol convexity
- **Numerical**  
Unstable implementation

**But :**

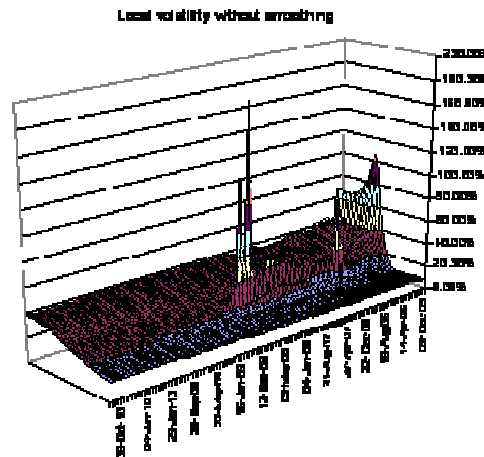
- Its robustness can be improved with a Fixed-Point Approach.
- The “Super Vega Bucket” allows the projection of exotic volatility risk in the vanilla world.

# Fixed Point Algorithm

Calibrate with the Fixed-Point method to generate a smoother surface avoiding numerical issues

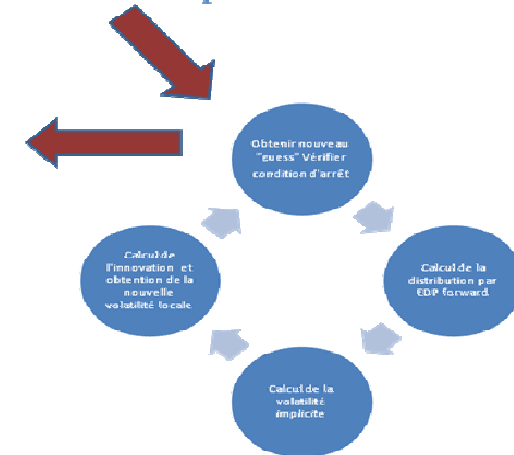
$$\{\sigma_{i+1}\} = \{\sigma_i\} \left\{ \frac{\Sigma_{market, fwd}}{\Phi(\sigma_i)} \right\}$$

## Standard Dupire's Formula

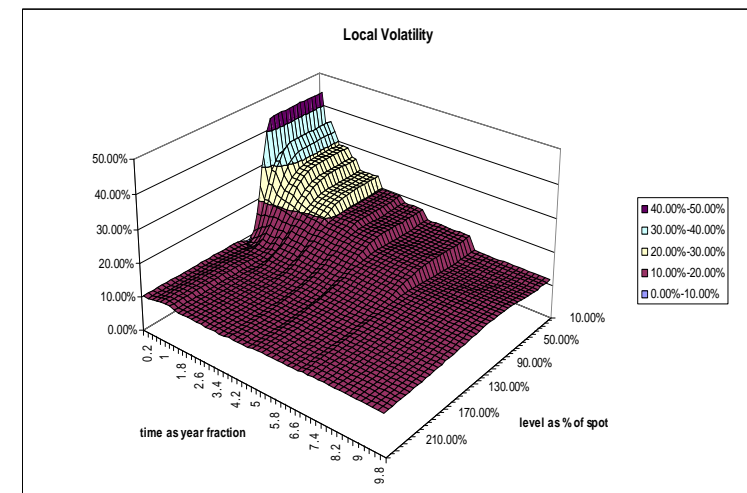


Initialiser la Volatilité locale avec la volatilité implicite

Volatilité locale a convergé: **STOP**



## Fixed Point Algorithm



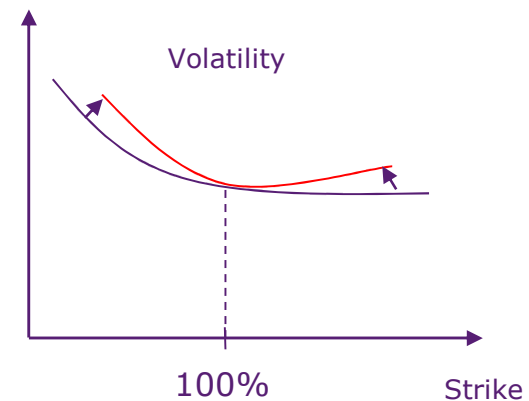
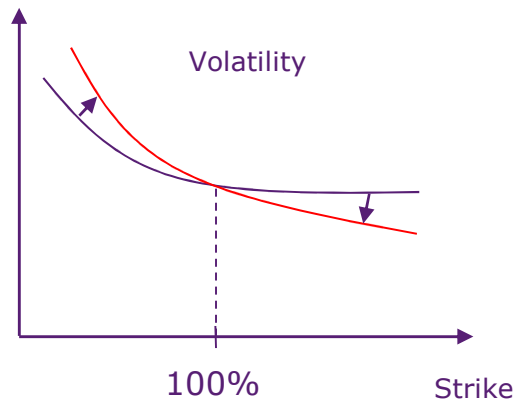
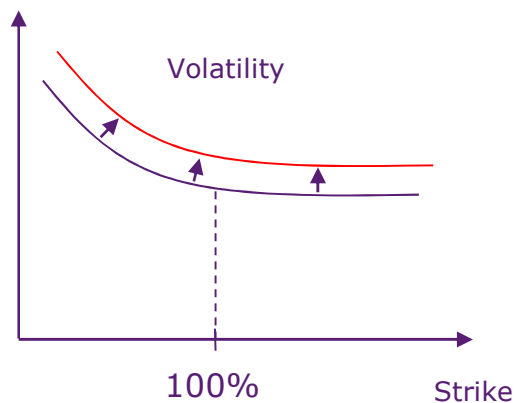
# Vega Scenario Hedging

## Risk Management

- On Exotics, it is essential to hedge the distortions of the implied volatility surface.
- They are mainly due to :
  - Parallel shift
  - Rotation of the smile
  - Change in convexity



*Vega*  
*Vega<sub>Smile</sub>*  
*Vega<sub>curve</sub>*



# But It Is Not Enough...

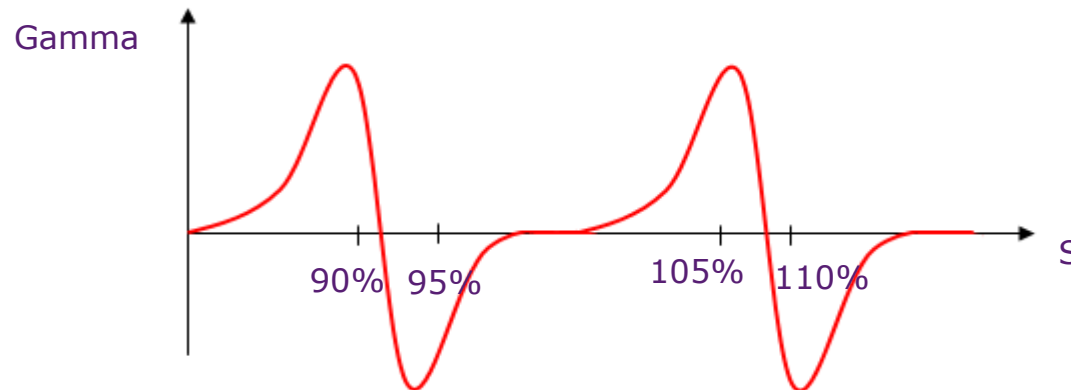
## Example

Consider a call spread with a 90%-down strike and a 110%-up strike to hedge

$$CS(90\%, 110\%) = \text{Call}(90\%) - \text{Call}(110\%)$$

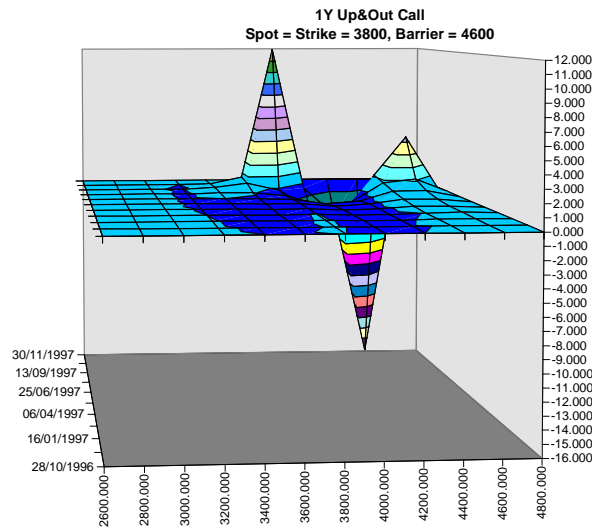
With the parametric volatility approach (*Vega*, *Vega<sub>Smile</sub>*, *Vega<sub>curve</sub>*), we could sell a 95%-105% call spread which features roughly the same sensitivity to volatility deformations.

Gamma of the new portfolio  $CS(90\%, 110\%) - CS(95\%, 105\%)$  would be:

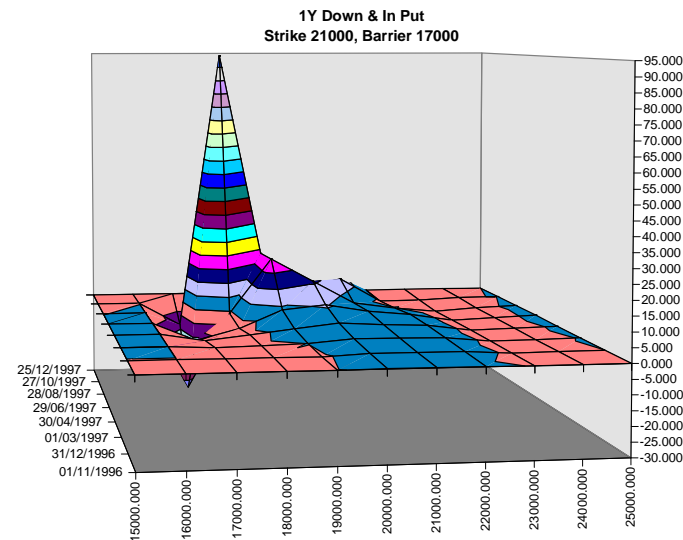


# Local Volatility Allows Super Vega Bucket

**Super Vega Bucket enables to Gamma hedge everywhere on the (K,T)-Grid**



Call Up & Out



Put Down & In

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# 3 Parameters Estimation



# Parameter Estimation

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**The local lambda functional is fine-tuned according to**

- 1. Payoff**
- 2. Available implied information**

**We propose two approaches:**

- The implied approach (world indices, liquid stocks) infers  $\lambda$  from basket option implied volatilities quotes in the market;**
- The statistical Estimation approach analyzes the statistical distributions of the correlation with respect to the basket components as a whole or in some directions.**

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# 3.1 Implied Approach

# Calibration Using Fixed-Point Algorithm

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The algorithm works exactly as in the one factor local volatility model described above:

## 1. Initialization

**Naïve**  $\lambda_0 = 0$   
**Based on Asymptotic approaches**

## 2. Lambda update

$$\lambda_{i+1} = \lambda_i + \ln \left( \frac{\Sigma_{\text{basket, Market}}}{\Phi(\lambda_i)} \right)$$

**$\Phi(\lambda)$  :** Application that transforms the local lambda function into the implied volatility surface of the underlying basket.

## Initial guess : Ito and Short Term Asymptotic

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### Result

$$X_t = B_T \frac{1}{\sigma_B}$$
$$dX_t = \frac{\sum_{i=1}^n \sigma_i \omega_i dW_{i,t}}{\sigma_B} - \frac{1}{2} \ln X_t \frac{d\sigma_B^2}{\sigma_B^2} + \theta_t dt$$
$$\frac{d\sigma_B^2}{\sigma_B^2} = \sum_{i,j=1}^n \underbrace{\beta_{i,j} \frac{d\rho_{i,j}}{\rho_{i,j}}}_{\text{correlation dynamic}} + 2 \sum_{i=1}^n \underbrace{\beta_i \frac{d\sigma_i}{\sigma_i}}_{\text{volatility dynamic}} + 2 \sum_{i=1}^n \underbrace{\beta_i \frac{d\omega_i}{\omega_i}}_{\text{weight's variability}}$$

**We recover MEK and BD results:**

**Three terms contributing to the distortion from a log normal**

- **(1) Weights variability**
- **(2) Each underlying own distortion**
- **(3) Correlation skew**

# Initial Guess : Pat Hagan formula recovered

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**We look at a one stoch vol model - keep one underlying :**

$$X_t = B_T \frac{1}{\sigma_B}$$

$$\frac{dX_t}{X_t} = \sigma(X_t) dZ_t$$

$$\sigma^2(X_t) = 1 + \alpha^2 \ln^2(X_t) - 2\rho_{S,\sigma}\alpha \ln(X_t)$$

**This becomes a local volatility model for which the implied volatility is given by the classical BBF formula.**

**Recover easily the Pat Hagan formula.**

# Initial Guess : Multi Stoch vol with local correlation

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**We keep contribution from each underlying smile**

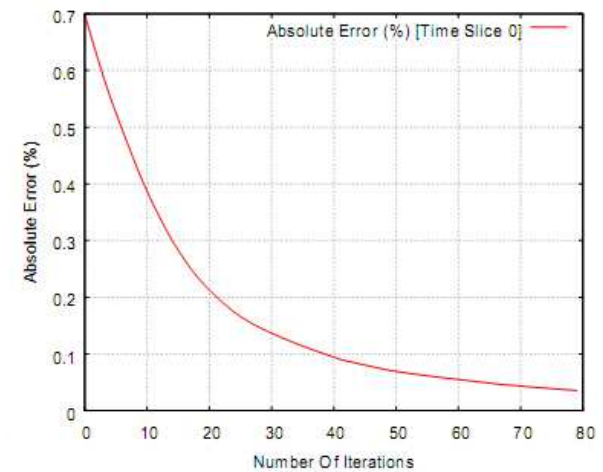
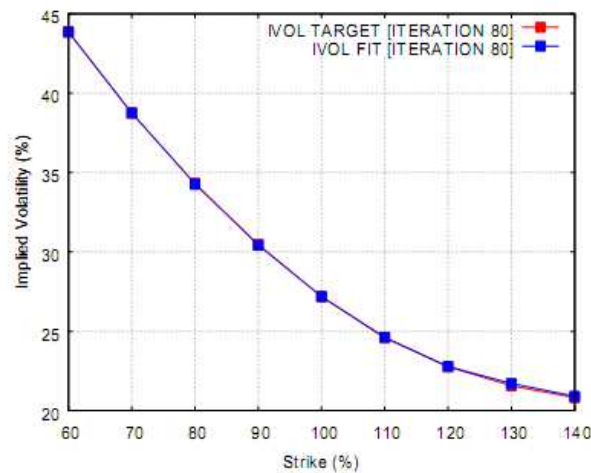
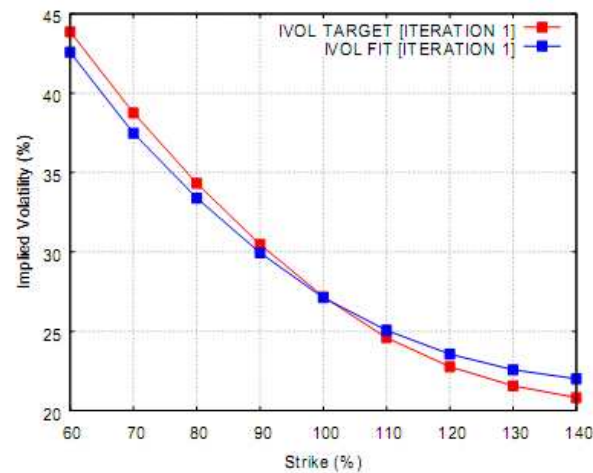
**We neglect the variability of the weights (in practice it is negligible)**

$$\begin{aligned}
 \sigma^2(X_t) = & 1 \\
 & - 2\ln(X_t) \left\{ \sum_{i,j=1}^n \frac{\sigma_i}{\sigma_B} w_i \beta_j \alpha_j \rho_{i,j}^{S,\sigma} \right. \\
 & + \ln^2(X_t) \left\{ \sum_{i,j=1}^n \beta_i \alpha_i \beta_j \alpha_j \rho_{i,j}^{\sigma,\sigma} \right. \\
 & \quad \text{No Correl} \qquad \qquad \qquad \text{Extra term with Correl} \\
 & \quad \left. - \frac{1}{2} \sigma_B \frac{d\lambda}{dB} \sum_{i_1,j_1}^n \beta_{i_1 j_1} \frac{1 - \rho_{i_1 j_1}}{\rho_{i_1 j_1} (1 - \lambda) + \lambda} \right\} \\
 & \quad \left. + \dots \right\}
 \end{aligned}$$

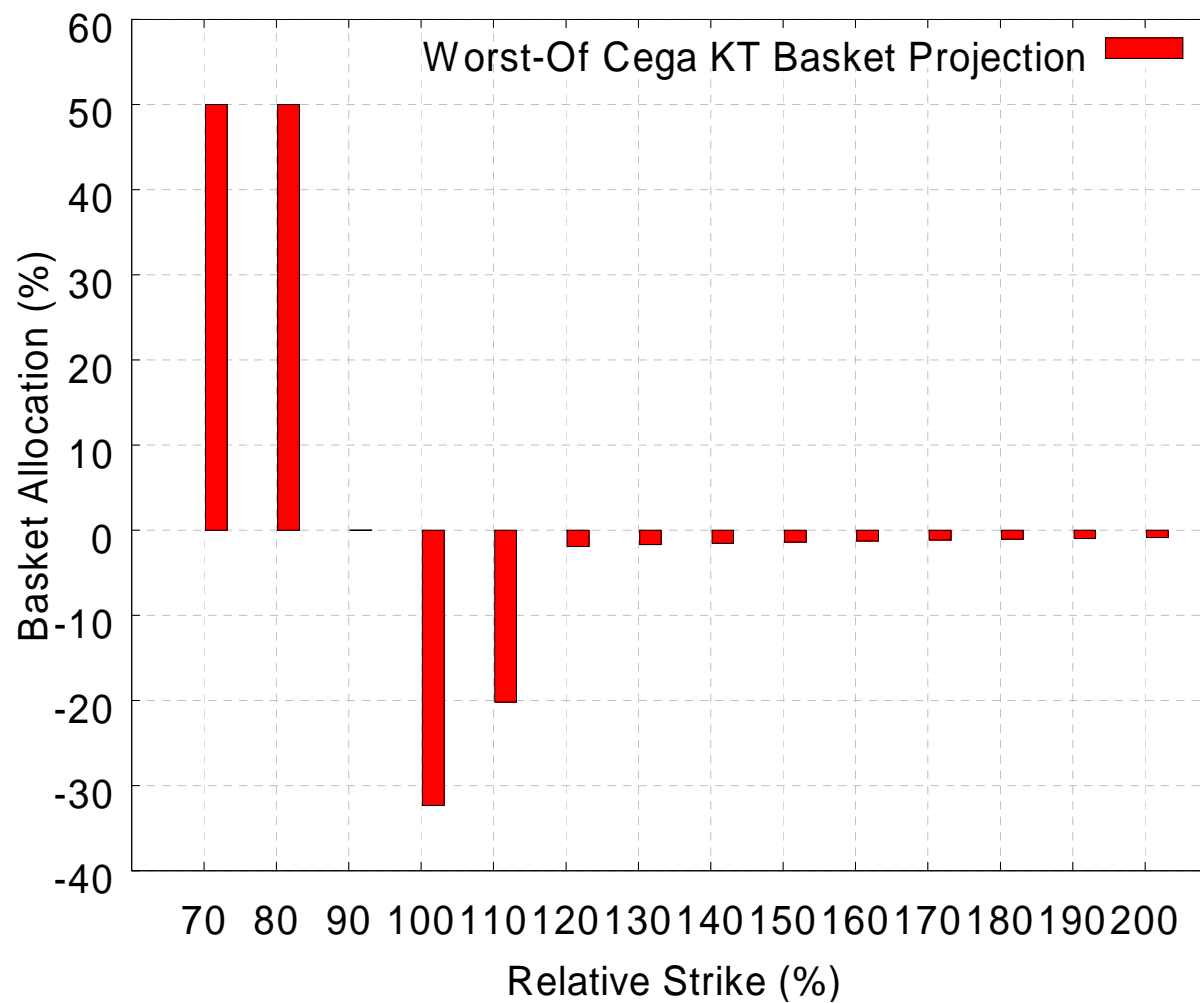
# Calibration Using Fixed-Point Algorithm

Enable to price and manage multi-underlying options with a correlation model fitted on a basket implied volatility surface quoted in the market.

Instantaneous correlation controlled by current {Time + Basket Level}.



# Cega KT





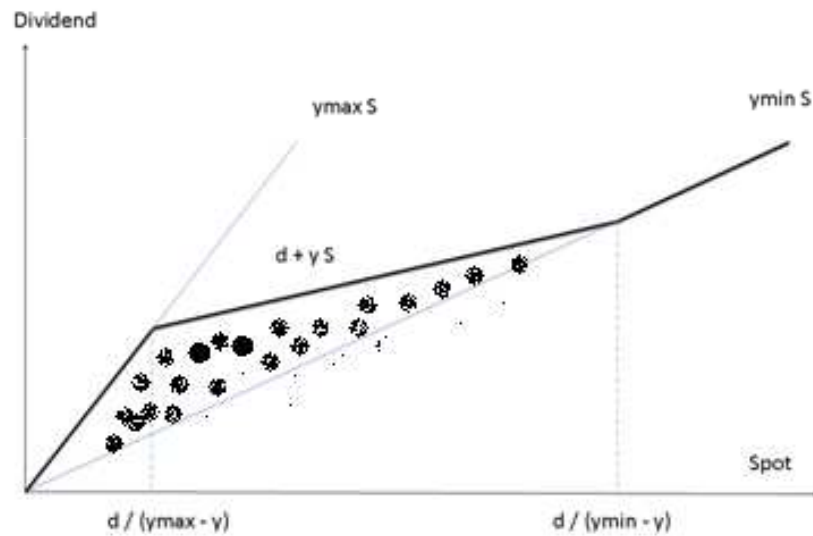
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# 3.2 Statistical Approach

# Model Estimation Using Envelop Approach I

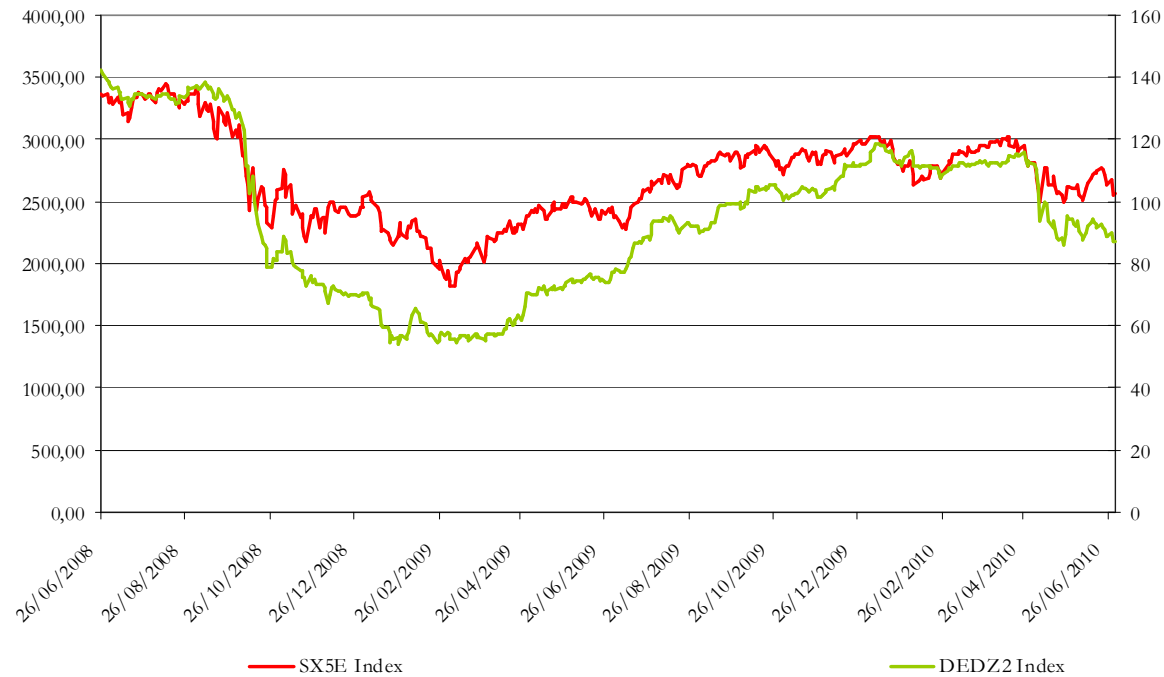
## Description

- Models the upper/lower bound of a random quantity  $Y$  with a hyperplane  $X$ .
- Successfully applied to model dividends with respect to spot:



# Future on Dividend example

- For large movements Spot is a good hedge for dividends



# Model Estimation Using Envelop Approach II

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**Empirical estimation of a correlation envelop with respect to spot levels to define the local lambda function  $\lambda$ .**

**Depends on the product under consideration.**

- **On Best-Of puts, we look for a parametrization which singles out the best performance:**

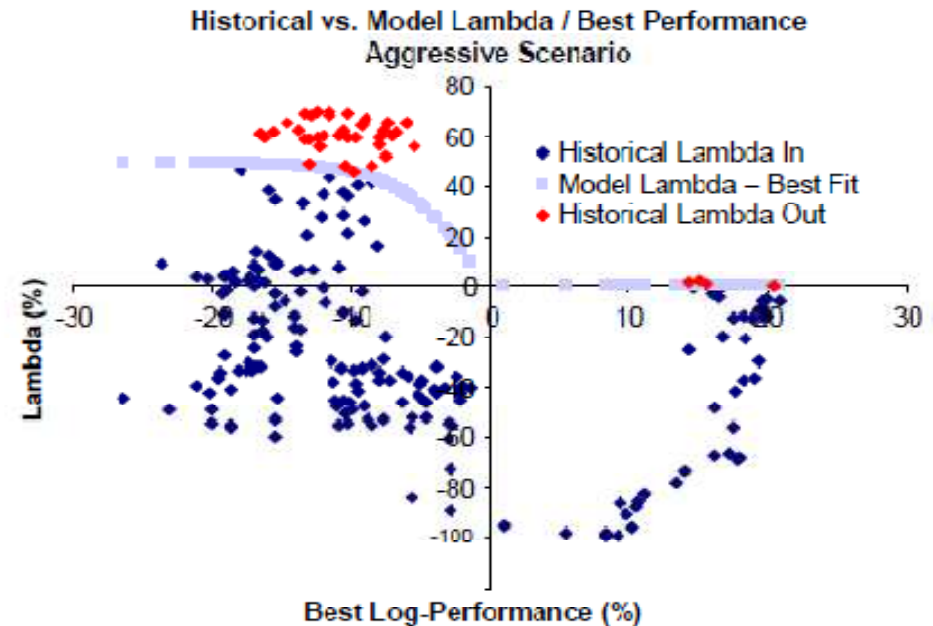
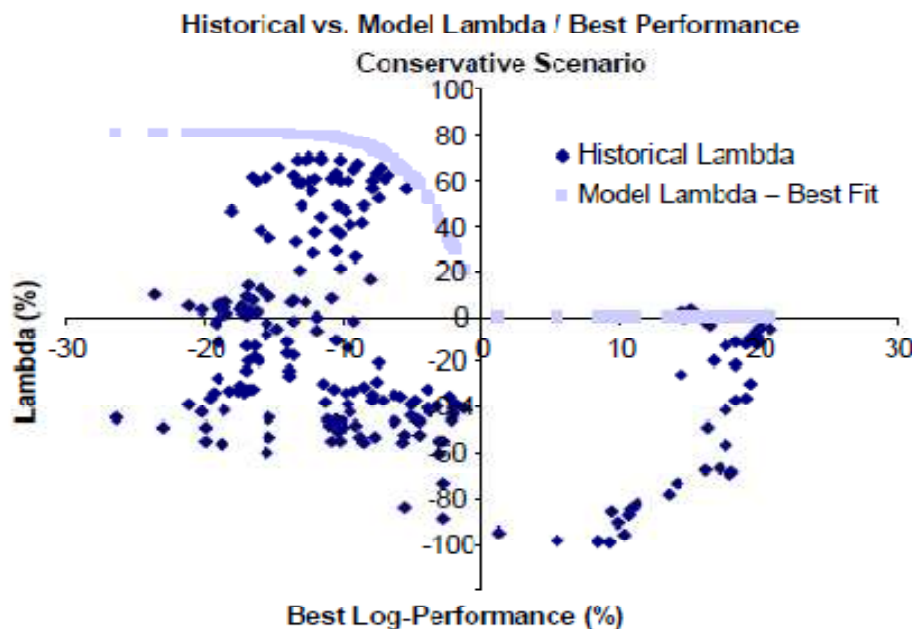
$$\lambda(S_t^1, \dots, S_t^k) = \max \left( -\lambda_0 \times \tanh \left( \frac{s}{\lambda_0} \times \max_{i=1, \dots, n} \ln \left( \frac{S_t^i}{S_0^i} \right) \right), \lambda_{\min} \right)$$

- **$\lambda_0$ ,  $\lambda_{\min}$  and  $s$  are estimated from historical worst-case scenarios.**

# Model Estimation Using Envelop Approach III

Find  $\{ \lambda_0, \lambda_{\min}, s \} : P( \lambda_{\text{histo}} < \lambda_{\text{model}} \mid \text{Best Perf.} < 1 ) = p$

- Conservative scenario :  $p = 100\%$
- Aggressive scenario :  $p = 80\%$



# Delta Hedging Improvement

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In the Local Correlation model, Delta becomes

$$\underbrace{\Delta_{i,LVLC}}_{\text{LocalVolLocalCorrelDelta}} = \underbrace{\Delta_{i,LV}}_{\text{LocalVolatilityDelta}} + \frac{\partial \pi}{\partial \lambda} \frac{\partial \lambda}{\partial S_i}$$

- The delta hedges the change of correlation
- No PnL impact with correlation remarking

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# 4 Conclusion

# Conclusion

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- 1. We proposed a local correlation Model which is an incremental enhancement of the local volatility model**
- 2. We presented pricing and hedging methodologies suited to available information**
- 3. We showed robust numerical techniques to calibrate or estimate this model**
- 4. We discussed a smart implementation that enables to run the model into the pricing libraries of an investment bank**



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# Questions

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**Thank You For Your Attention**

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