Double No-Touches

Market Consistent Pricing with LSV Models

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Global Derivatives - Paris
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Double No-Touches

- First generation exotics – continuous barriers
- Candidate models – LV, SV, LSV
- Sensitivity to skew and smile
- Tuning between local and stochastic volatility

First Generation Exotics – Continuous Barriers

- Conventional European options often perceived as expensive
- Common cheapening feature – introduction of continuously monitored KI or KO barriers
- FX especially, currency rates often seen as rangebound
- Preference not to buy option protection too far away from current level of spot.

Visual Notation for Options

- Several kinds of binary and barrier options
- Let's standardise on a visual notation

	Up and In barrier		Up and Out barrier
	Down and In barrier		Down and Out barrier
✓	Knocks in	✓$	Expires in the money
✕	Knocks out	$	Expires worthless

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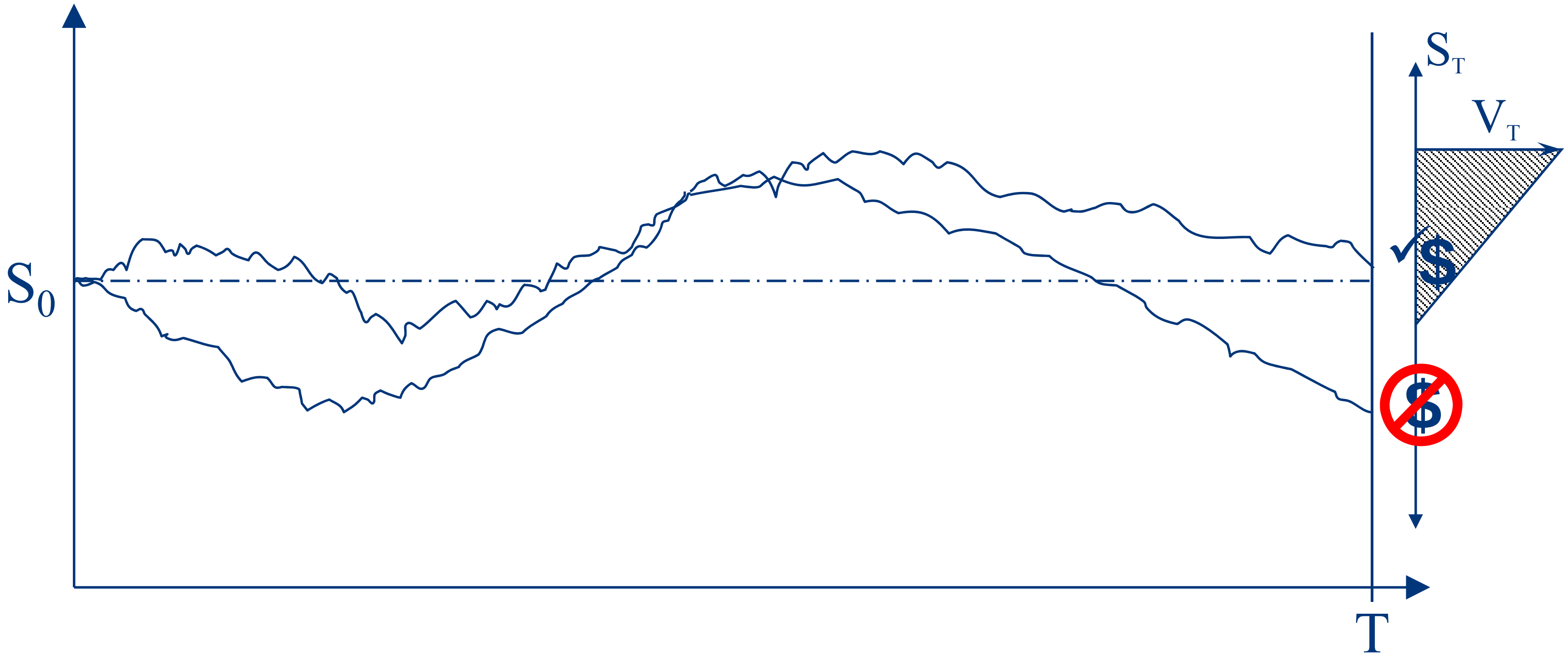
Vanilla Options

- For each trajectory, payoff is simply given by $(S_T - K)^+$



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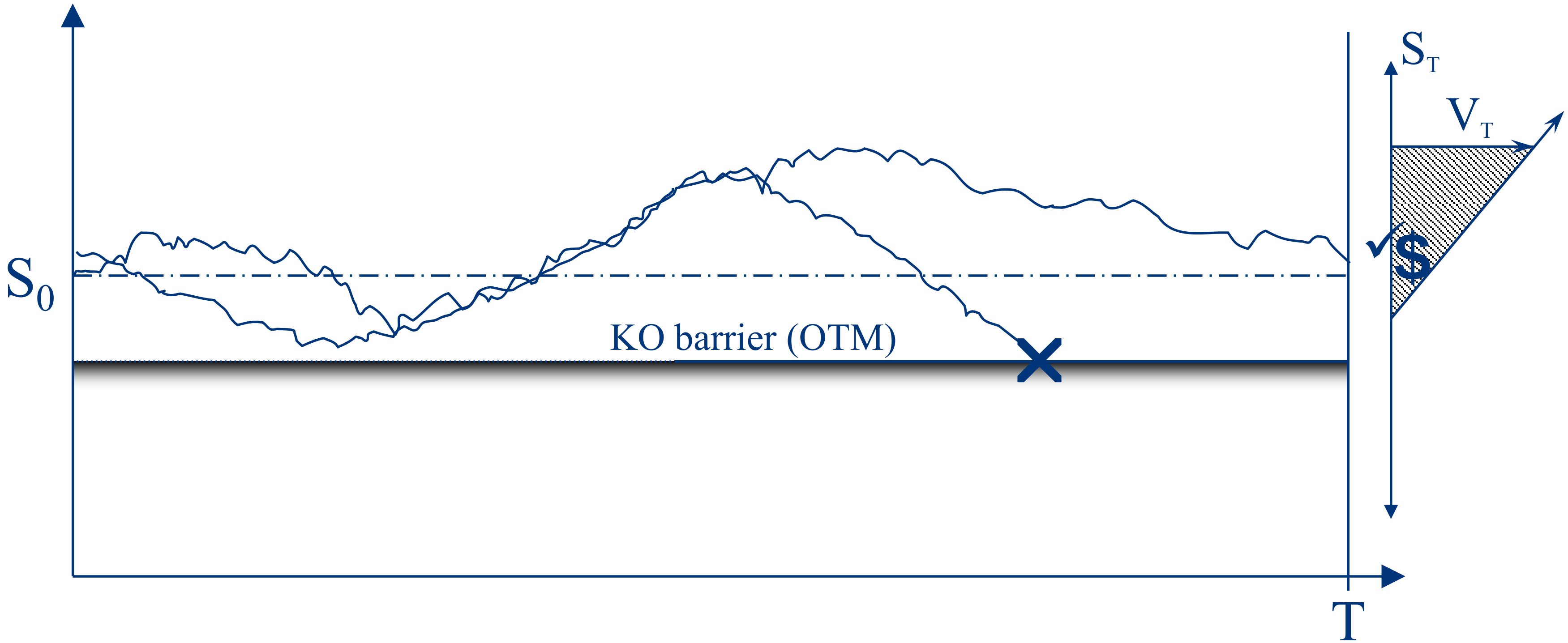
Knock-Out Options

- (Regular) Knock-Out options: barrier in the OTM region



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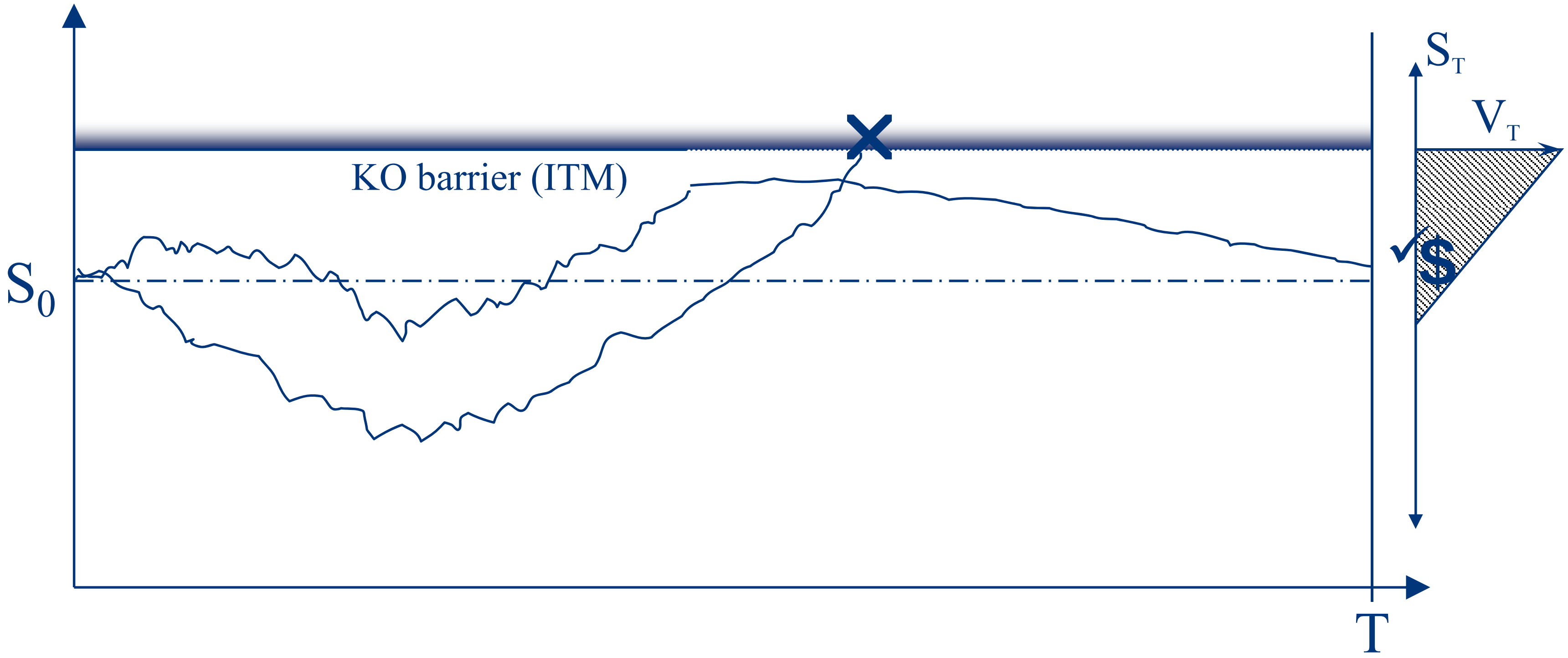
Reverse Knock-Outs

- Reverse Knock-Out options have barrier **in** the money



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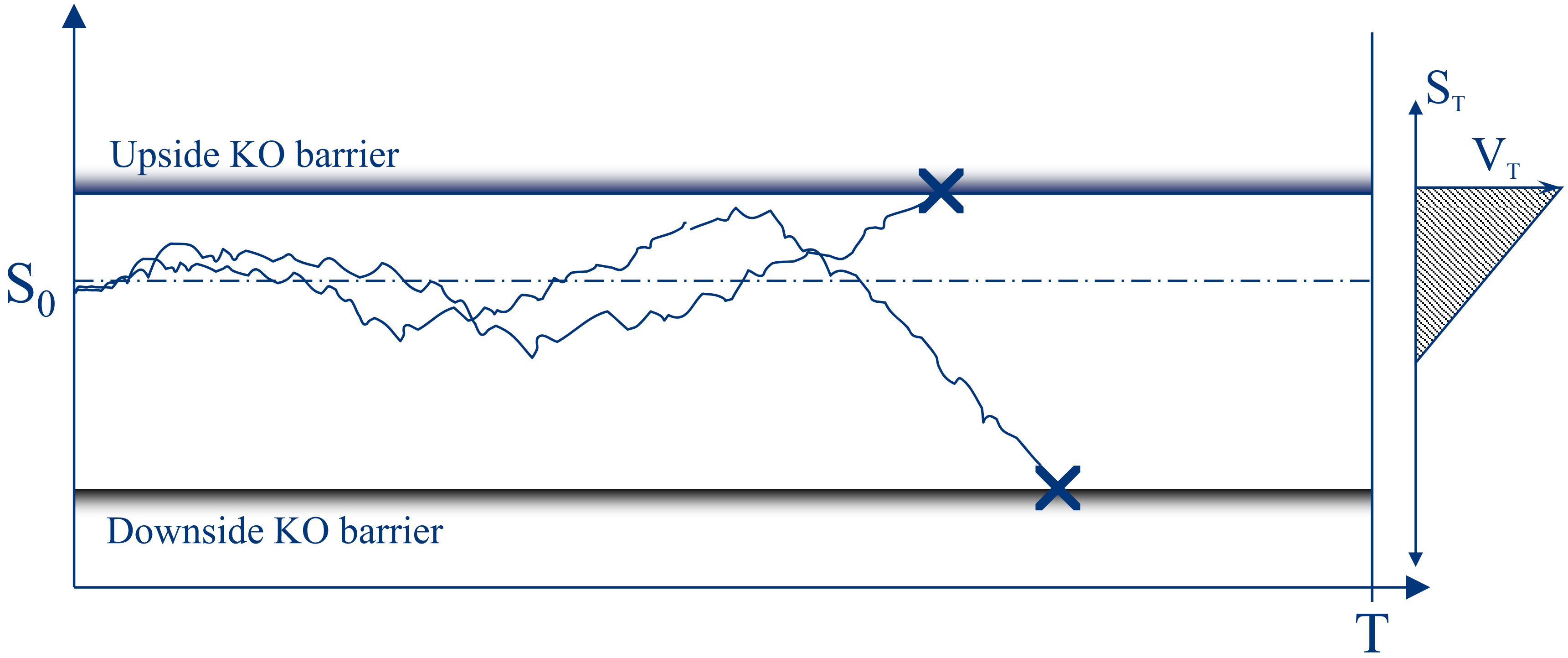
Double Knock-Outs

- Have extinguishing barriers on **both** sides



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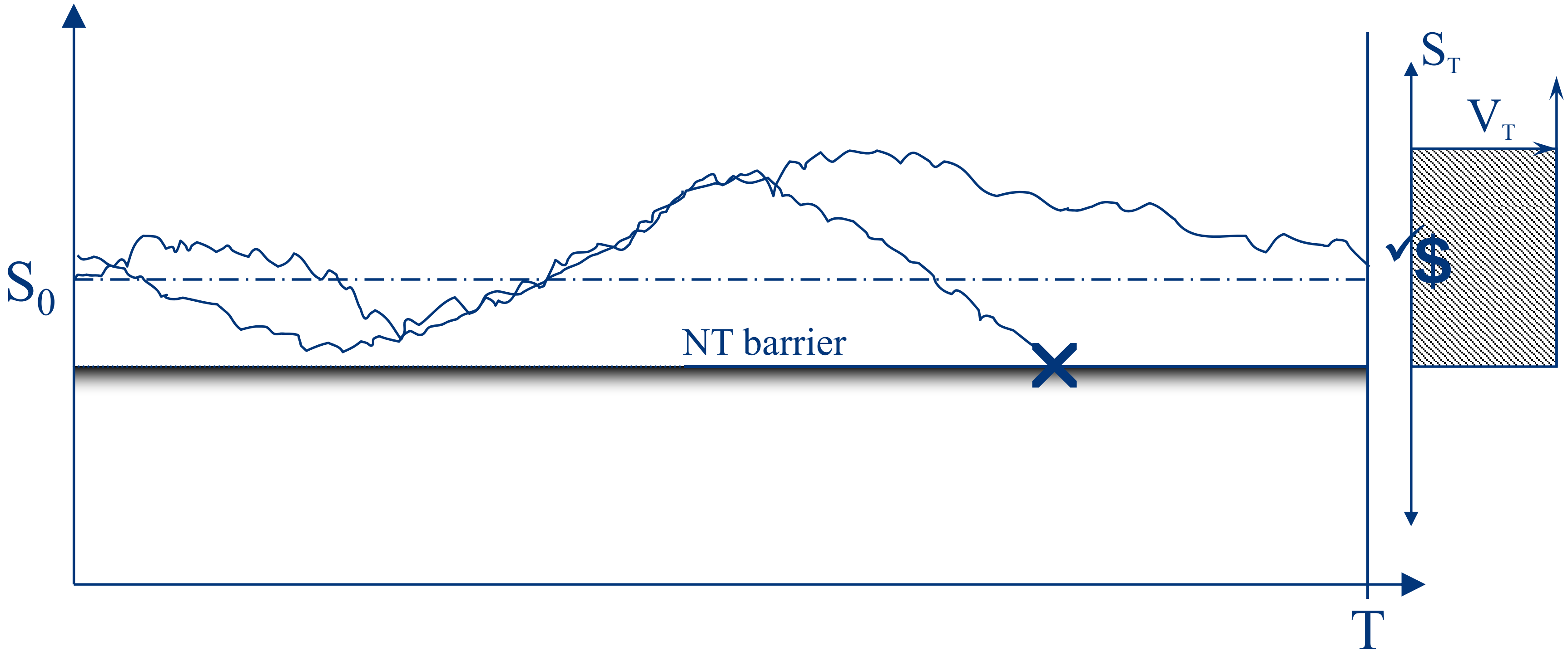
No-Touch Options (single barrier)

- Just like KOs except that if exercised, receive 1 unit of cash



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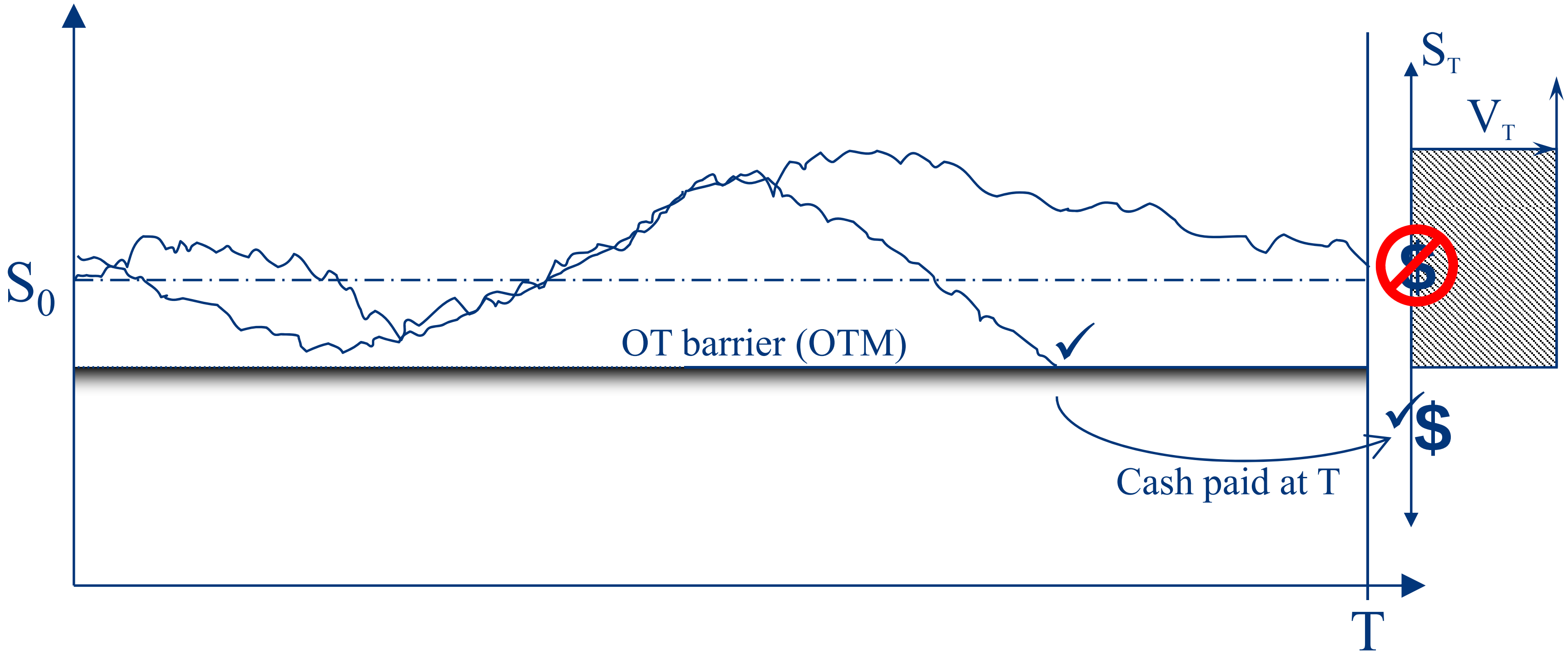
One-Touch Options (single barrier)

- Just like NTs except they pay at expiry if the barrier is touched



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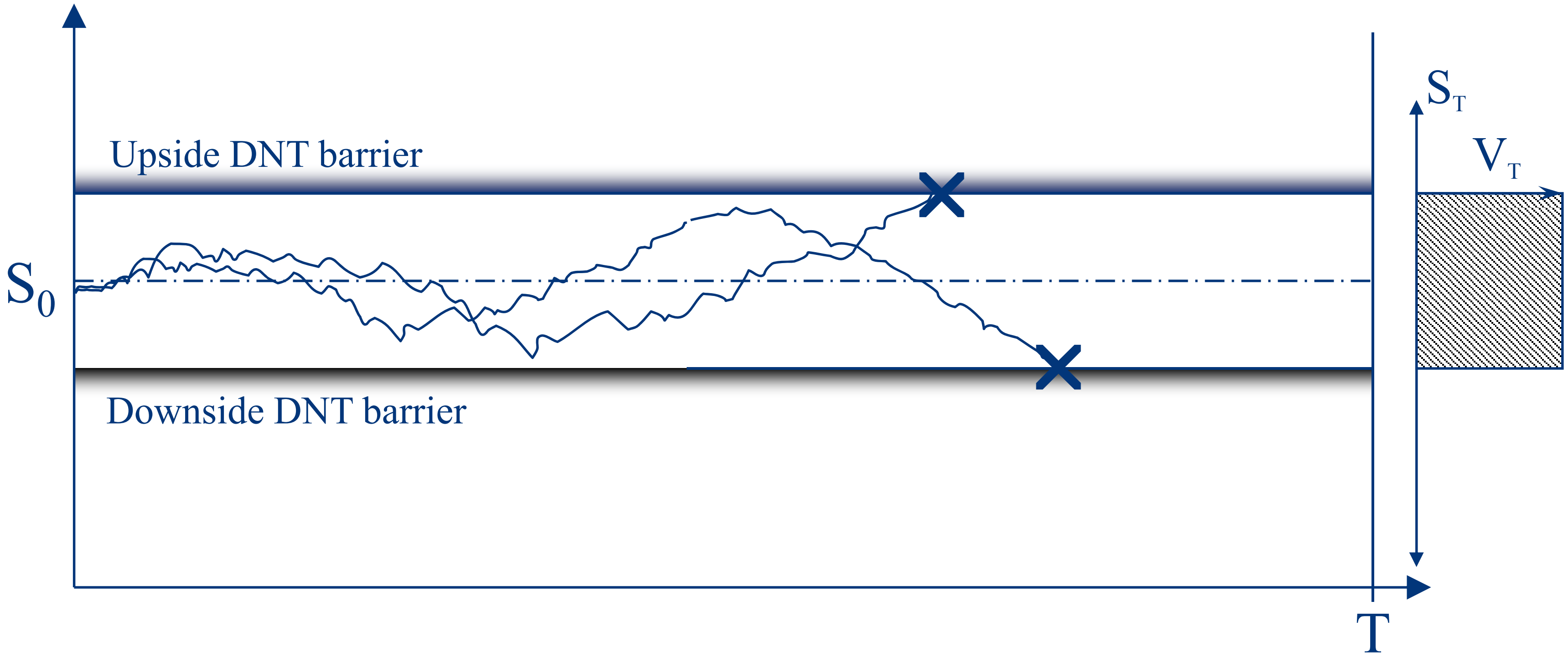
Double No-Touch (double barrier)

- Extinguishing barriers on <u>both</u> sides



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Binary Prices – candidate models

- Black-Scholes [TV] $$dS_t = \mu S_t dt + \sigma S_t dW_t$$
- Term structure [TS] $$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t$$
- Local volatility [LV] $$dS_t = \mu_t S_t + \sigma_{\text{loc}}(S_t, t) S_t dW_t$$
- Stochastic volatility [SV] $$dS_t = \mu_t S_t dt + \sqrt{v_t} S_t dW_t^{(1)}$$

$$\langle W_t^{(1)}, W_t^{(2)} \rangle = \rho dt \qquad dv_t = \kappa(m - v_t)dt + \alpha\sqrt{v_t}\, dW_t^{(2)}$$

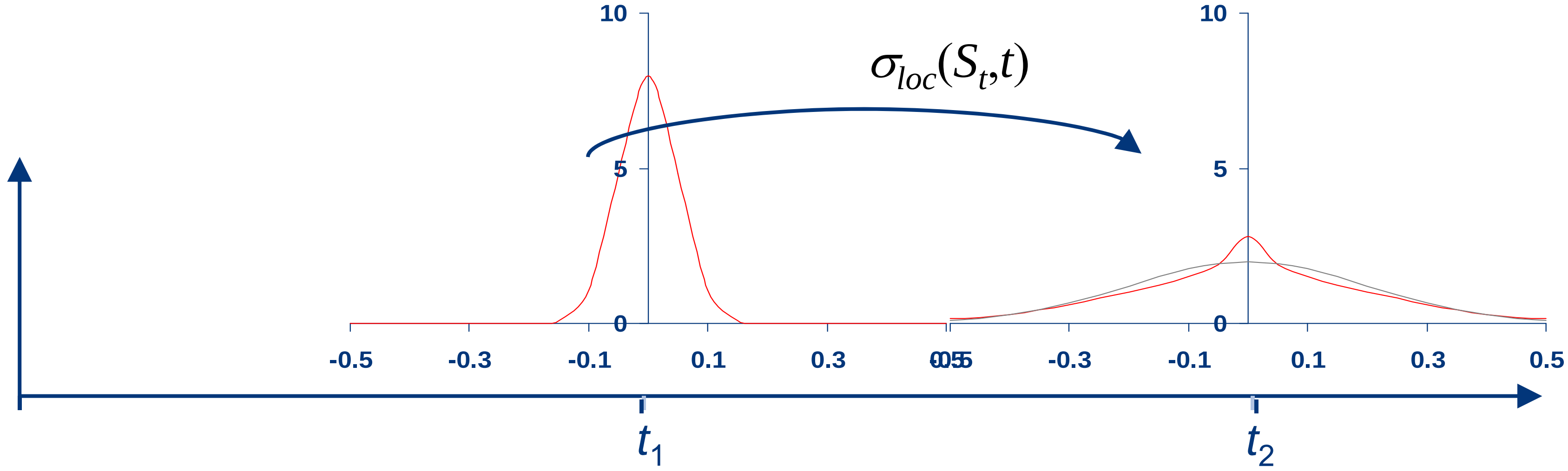
- Local-stochastic volatility [LSV] $$dS_t = \mu_t S_t dt + \sqrt{v_t} A(S_t, t) S_t dW_t^{(1)}$$
$$dv_t = \kappa(m - v_t)dt + \alpha\sqrt{v_t}\, dW_t^{(2)}$$

Binary prices – candidate models

- All of these models can be separately calibrated to the FX volatility surface
  - LV Dupire analysis
  - SV least-squares minimisation of error function
  - LSV forward induction on 2D Fokker-Planck eqn

Local volatility

- What if the marginal distribution at t_1 was lognormal, but at t_2 was strongly leptokurtic?



Local volatility

- Imagine a volatility surface such as

	σ_{ATM}	$\sigma_{25\text{-d-MS}}$	$\sigma_{25\text{-d-RR}}$
1Y	10.00	0.00	0.00
2Y	10.00	0.50	0.00

- Local volatility makes $\sigma_{loc}(S_t,t)$ a function of S_t too

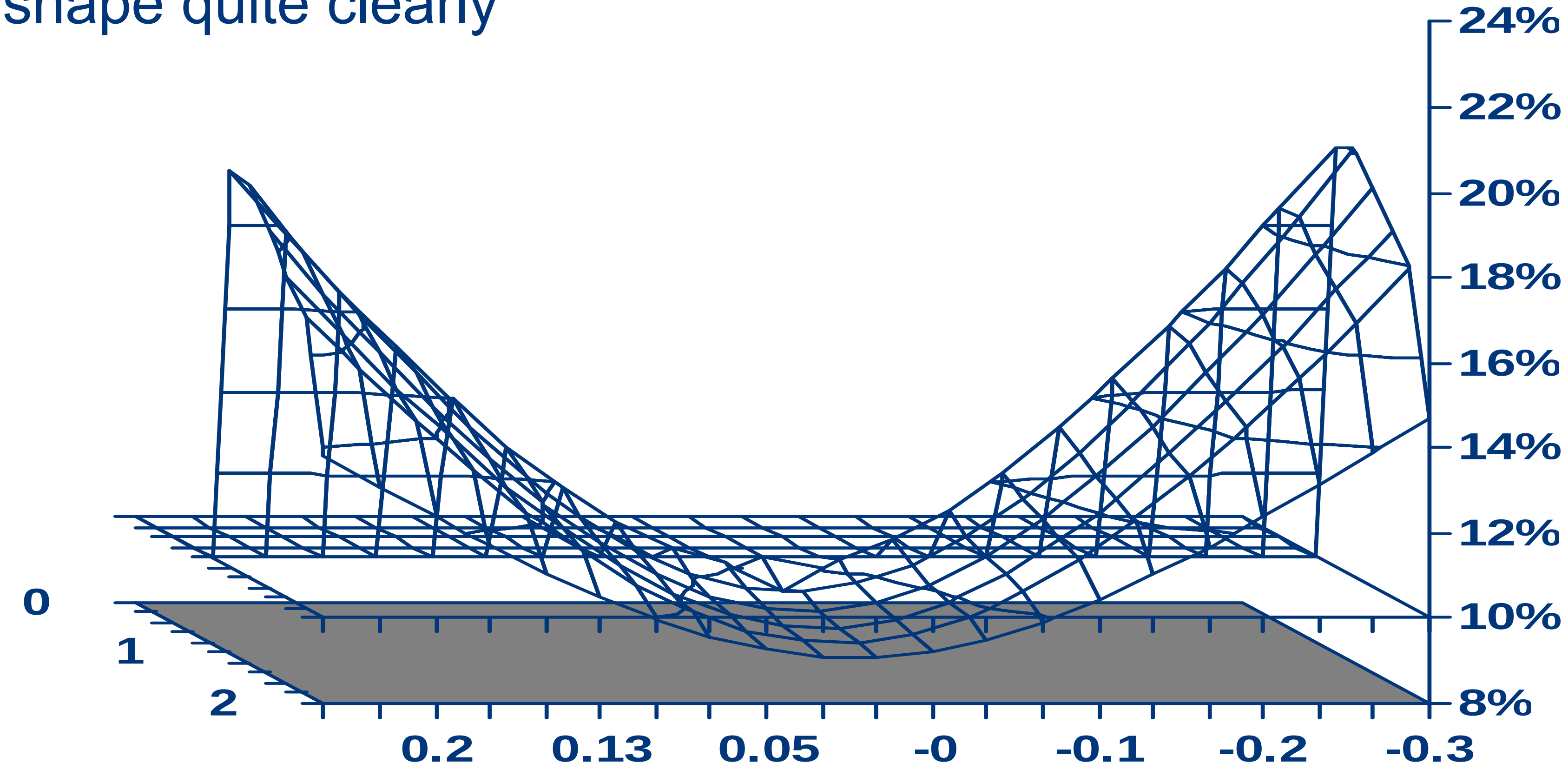
S_t

10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	19.88%	20.07%	19.36%	18.64%	17.99%	14.67%
10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	18.43%	18.46%	17.85%	17.23%	16.68%	13.88%
10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	17.03%	16.93%	16.40%	15.87%	15.41%	13.11%
10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	15.69%	15.48%	15.03%	14.59%	14.21%	12.36%
10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	14.43%	14.14%	13.75%	13.39%	13.08%	11.66%
10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	13.26%	12.93%	12.60%	12.30%	12.05%	11.00%
10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	12.22%	11.87%	11.58%	11.34%	11.14%	10.41%
10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	11.34%	10.98%	10.73%	10.53%	10.37%	9.90%
10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.65%	10.30%	10.07%	9.90%	9.77%	9.49%
10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.19%	9.85%	9.63%	9.48%	9.36%	9.22%
10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	9.42%	9.66%	9.44%	9.29%	9.17%	9.08%
10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.08%	9.72%	9.49%	9.32%	9.20%	9.09%
10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	10.43%	10.05%	9.79%	9.59%	9.44%	9.25%
10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	11.04%	10.62%	10.31%	10.08%	9.89%	9.55%
10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	11.85%	11.41%	11.05%	10.76%	10.53%	9.96%
10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	12.83%	12.38%	11.97%	11.61%	11.32%	10.46%
10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	13.95%	13.52%	13.03%	12.60%	12.24%	11.04%
10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	15.17%	14.79%	14.23%	13.71%	13.26%	11.68%
10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	16.48%	16.18%	15.53%	14.91%	14.38%	12.36%
10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	17.86%	17.66%	16.91%	16.20%	15.57%	13.07%
10.00%	10.00%	10.00%	10.00%	10.00%	10.00%	19.29%	19.22%	18.37%	17.54%	16.81%	13.81%

t_1 t_2

Local volatility

- We can see the functional shape of the local volatility shape quite clearly



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Local volatility

- This means when local volatility depends on S_t we have an inhomogeneous pricing problem

$$dS_t = \mu_t S_t dt + \sigma_{loc}(S_t, t) S_t dW_t$$

- **Ok... But what local volatility should we use?**
- Dupire (1993) asked this – can we construct a state dependent instantaneous volatility $\sigma_{loc}(S_t, t)$ which when fed into the 1D diffusion above, reprices Europeans consistently with $\sigma_{imp}(K, T)$?

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Local volatility

- In fact, earlier work by Gyöngy answered exactly this question – not in context of finance.
- Approach: express marginal pdf in terms of the second partial derivative of $C(K,T)$ and note that marginal pdfs can be thought of as time-slices of these forward transition probabilities.
- The 1D forward Fokker-Planck equation governs the time evolution of these transition probability densities.

Forward Fokker-Planck equation

- Suppose a 1D diffusion is given by

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t$$

- Let $g(x)$ be an arbitrary function of x which tends to 0 as $x \to \infty$. By taking $G_t = g(X_t)$ and applying Itô, we obtain

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial[a(x,t)p(x,t)]}{\partial x} + \frac{1}{2}\frac{\partial^2[b^2(x,t)p(x,t)]}{\partial x^2}$$

Ref: Grigoriu, M. (2002), *Stochastic Calculus: Applications in Science and Engineering*, Birkhauser: Boston.

Forward Fokker-Planck equation

- Now the source solution at $t{=}0$ is known – Dirac delta function – as we know the initial spot level.

$$p(x,t_0) \equiv p(x,t_0,x_0,t_0) = \delta_{x_0}(x)$$

- What do we expect the marginal distribution to look like for future times t ?
- Assume domestic/foreign rates are zero.
- Breeden-Litzenberger: marginal distributions expressible as second derivatives of call prices

$$\varphi_T(K) \equiv f_{S_T}(K) = \frac{\partial^2 C(K,T)}{\partial K^2}$$

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Local volatility

- As interest rates vanish (for ease of exposition), FPE becomes

with $$\frac{\partial p(K,T)}{\partial T} = \frac{1}{2}\frac{\partial^2[b^2(K,T)\,p(K,T)]}{\partial K^2}$$

and $$p(K,T) = \varphi_T(K) = \frac{\partial^2 C}{\partial K^2}$$

- So after a little bit of algebra we obtain

$$b^2(K,T) = \frac{2\dfrac{\partial C}{\partial T}}{\dfrac{\partial^2 C}{\partial K^2}} \qquad \sigma_{loc}(K,T) = \sqrt{\frac{2\dfrac{\partial C}{\partial T}}{K^2\dfrac{\partial^2 C}{\partial K^2}}}$$

Local volatility

- If rates are nonzero the algebra is slightly more involved, and we have

$$\sigma_{loc}(K,T) = \sqrt{2\frac{\frac{\partial C}{\partial T} + (r^d - r^f)K\frac{\partial C}{\partial K} + r^f C}{K^2 \frac{\partial^2 C}{\partial K^2}}}$$

- Problem here is numerics. We have a volatility surface $\sigma_{imp}(K_i,T_j)$ specified by only a handful of strikes K_i and times T_j – and we need to infer a dense set of infinitesimal derivatives
  - 2nd order in moneyness, 1st order in time.

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Forward Fokker-Planck equation

- So we've used the 1D FPE to infer local volatility from the marginals at each future time slice T.
- Local volatility can be interpreted as the domestic risk-neutral expectation of the instantaneous variance V_T at time T, conditional on the asset price S_T being equal to K.

$$\sigma_{loc}^2(K,T) = \mathbf{E}^d\left[V_T \mid S_T = K\right]$$

- And this still holds if variance V_t is stochastic…

Ref: Gatheral, J. (2006), *The Volatility Surface: A Practitioner's Guide*, Wiley: Hoboken, NJ.

Stochastic volatility

- Heston model is a model for stochastic *variance*

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t^{(1)}, \quad \mu = r^d - r^f$$

$$dV_t = \kappa(m - V_t)dt + \alpha\sqrt{V_t}\, dW_t^{(2)}, \qquad \sigma_t = \sqrt{V_t}$$

$$\left\langle dW_t^{(1)}, dW_t^{(2)} \right\rangle = \rho dt$$

- Has characteristic function based "semi-analytic" prices available for Europeans.
- Can be used for fast SV calibration.

Ref: Heston, S.L. (1993), A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *Rev. Financ. Stud.*, **6** (2), 327-343.

Heston model parameters

- Five parameters – quite different effects on the shape of implied volatility surface generated

Parameter	**Effect**
Initial variance V_0	Fixes overall level of implied ATM vol
Vovariance α	Generates volatility smile as α increases
Spot/Variance correlation ρ	Generates volatility skew for nonzero ρ
Mean reversion rate κ	Combined effect: increasing κ, term structure of implied ATM vol shifts in direction of $m^{1/2}$ & smile flattens
Mean reversion level m	

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SV only calibration of the model

- Heston model has no problem generating smiles and skews
- SV calibration is a fairly simple optimisation exercise
- **Terminal calibration:** take as inputs the volatilities at three strikes(25-d-P, ATM, 25-d-C), at one expiry time T. Lock down κ and m. Attempt to minimise objective function which measures the sum of squares of the errors in the vol by varying V_0, ρ, α.
- **Term structure calibration:** With suitably chosen mean reversion parameters κ and m, possible to generate upward sloping or downward sloping ATM volatility surfaces and to tune the butterflies (to some extent).

Stochastic volatility

- One supposed advantage of Heston is that if the Feller condition is satisfied, V_t remains positive with probability 1

$$V_0 > 0 \ \cap \ \Phi \equiv \frac{\alpha^2}{2m\kappa} < 1 \Rightarrow V_t > 0 \ \forall t > 0$$

- One can prove this using speed/scale measure
- For FX, Feller condition on Φ rarely holds.
- Correct handling of $V_t=0$ boundary condition is most important.

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Stochastic volatility

- Even with pure Heston model, we can still construct the 2D forward Fokker-Planck equation.

$$\frac{\partial p}{\partial t} = \kappa \frac{\partial}{\partial V}\left[(V-m)p\right] + \frac{1}{2}\frac{\partial}{\partial x}\left[Vp\right] + \rho\alpha \frac{\partial^2}{\partial x \partial V}\left[Vp\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[Vp\right] + \frac{\alpha^2}{2}\frac{\partial^2}{\partial V^2}\left[Vp\right]$$

$$p(x,V,t_0) \equiv p(x,V,t_0,x_0,V_0,t_0) = \delta_{x_0}(x)\delta_{V_0}(V)$$

- We apply the same initial source solution

Ref: Dragulescu, A.A. and Yakovenko, V. M. (2002), Probability distribution of returns in the Heston model with stochastic volatility, *Quantitative Finance*, **2** 443-453, erratum: C15 (2003)

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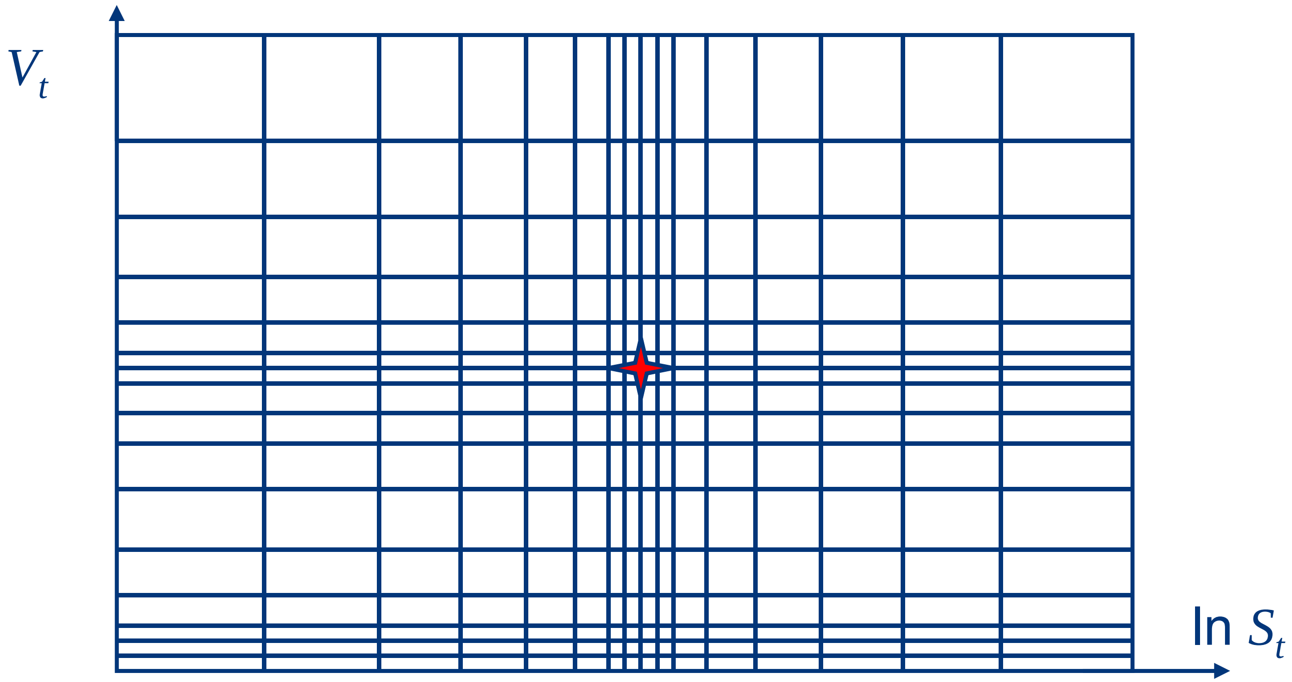
Stochastic volatility

- Note that the source solution is aggressively singular around (x_0,V_0) and a nonuniform mesh is advantageous.



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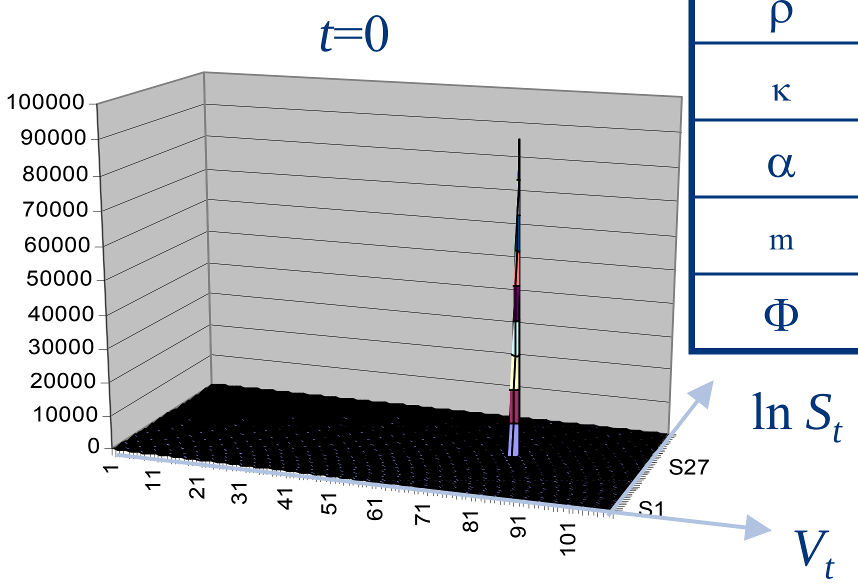
Stochastic volatility

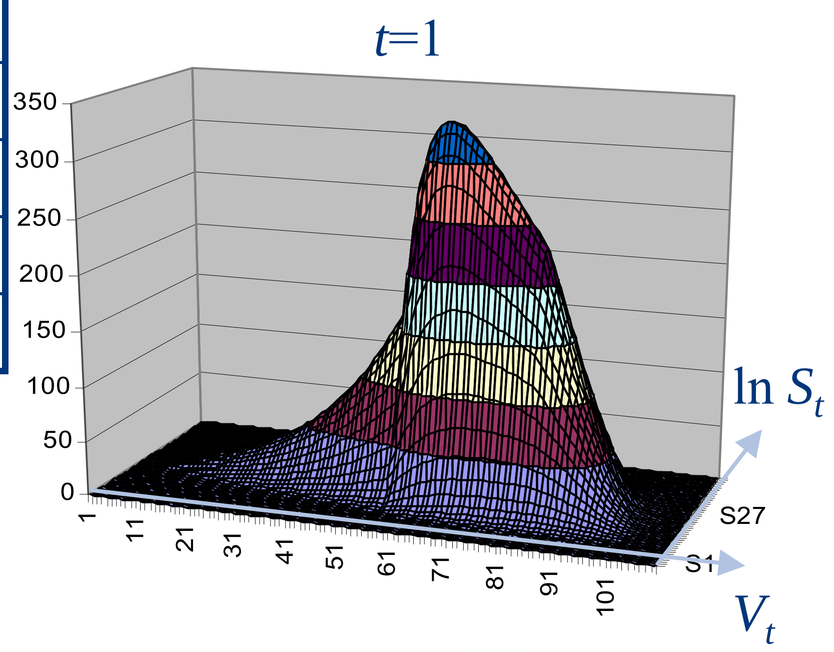
- Numerical solution of Heston FPE with $\Phi < 1$.

	σ_{ATM}	$\sigma_{25\text{-d-MS}}$	$\sigma_{25\text{-d-RR}}$
1Y	10.00	0.20	0.00

V_0	0.01004
ρ	-0.0117
κ	1.5000
α	0.1207
m	0.0104
Φ	0.4655

Mesh looks odd due to nonuniform variance mesh





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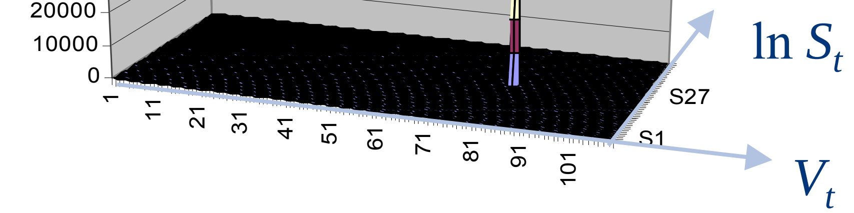
Stochastic volatility

- Numerical solution of Heston FPE with $\Phi > 1$.

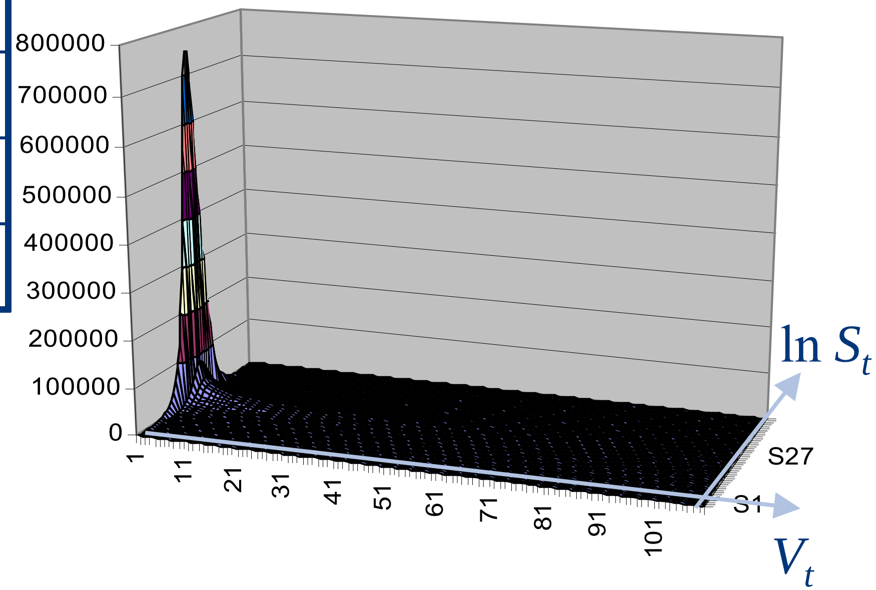
	σ_{ATM}	$\sigma_{25\text{-d-MS}}$	$\sigma_{25\text{-d-RR}}$
1Y	10.00	0.80	0.00

V_0	0.0138
ρ	-0.0385
κ	1.5000
α	0.4540
m	0.0138
Φ	4.9772

t=0



t=1



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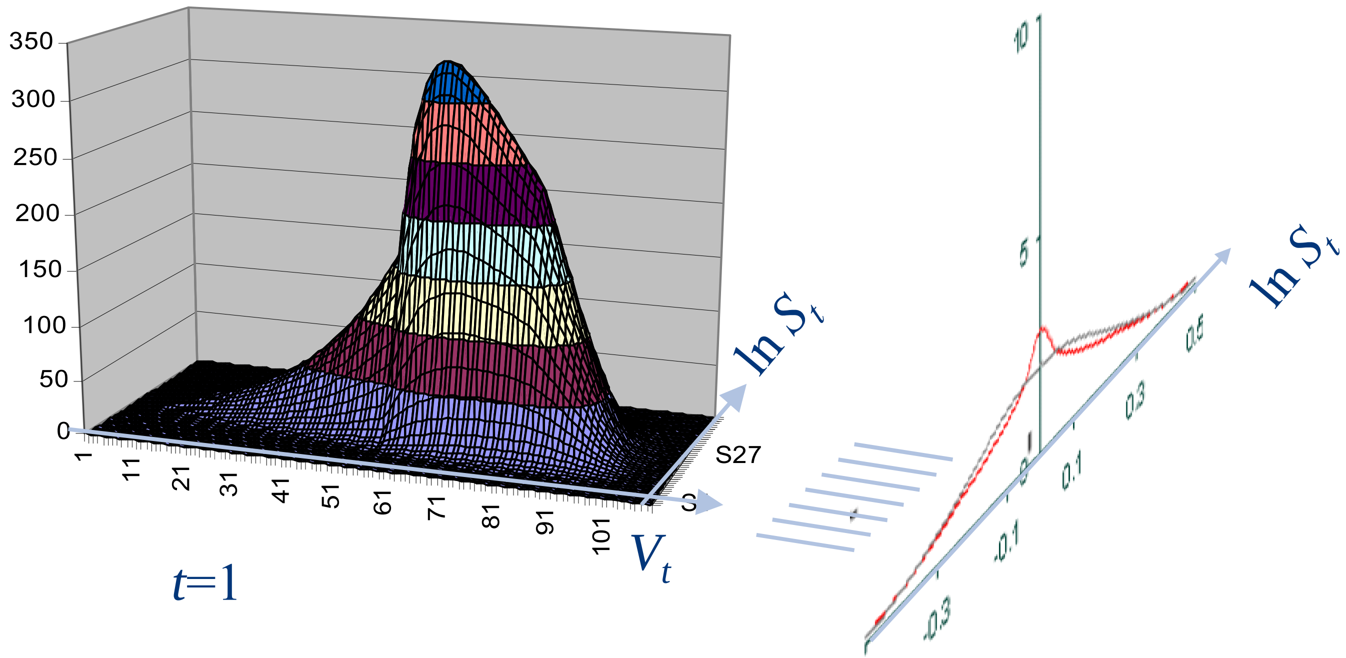
Numerically checking 2D FPE for Heston

- As we have the joint pdf of S_t and V_t we can integrate along the variance direction, and check that the marginal pdf for S_t recovers the prices of Europeans.



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Local Stochastic Volatility

- LSV -- model combining stochastic volatility, such as Heston, *with* a local volatility term $A(S_t, t)$

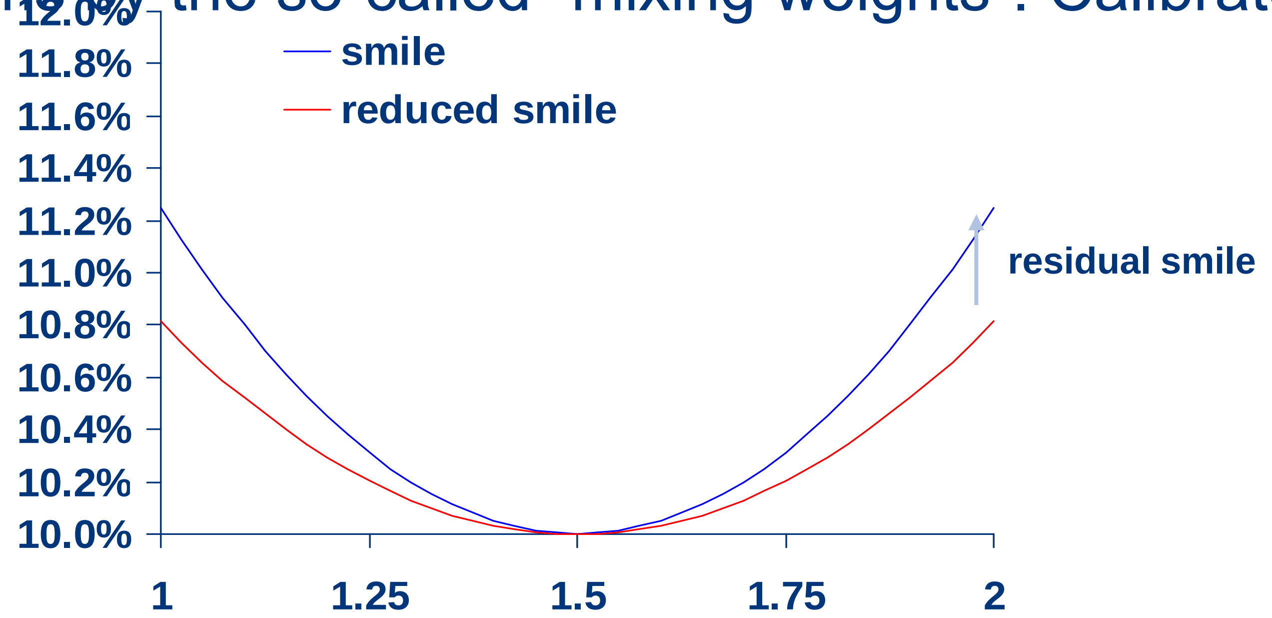
$$dS_t = \mu S_t dt + \sqrt{V_t} A(S_t, t) S_t dW_t^{(1)}, \quad \mu = r^d - r^f$$

$$dV_t = \kappa(m - V_t)dt + \alpha\sqrt{V_t} dW_t^{(2)}$$

$$\left\langle dW_t^{(1)}, dW_t^{(2)} \right\rangle = \rho dt$$

Local Stochastic Volatility - Calibration

- **Phase I:** calibrate pure SV model to reduced smile
  - Mark down convexity either in market terms or models terms by the so-called “mixing weights”. Calibrate SV.



- **Phase II:** calibrate $A(S_t, t)$ to capture residual smile

Finding expected instantaneous variance

- As we have the joint pdf of S_t and V_t, we have

$$\sigma_{loc}^2(K,T) = \mathbf{E}^d\left[V_T A^2(K,T) \mid S_T = K\right]$$

where V_t denotes the stochastic variance factor

- Once this is written, we have

$$A^2(K,T) = \frac{\sigma_{loc}^2(K,T)}{\mathbf{E}^d\left[\sigma_T^2 \mid S_T = K\right]}$$

- Denominator can be numerically computed by

$$\mathbf{E}^d\left[\sigma_T^2 \mid S_T = K\right] = \frac{\mathbf{E}^d\left[\sigma_T^2 \delta_{\{S_T - K\}}\right]}{\mathbf{E}^d\left[\delta_{\{S_T - K\}}\right]} = \frac{\int_V p(K,V,T)\,dV}{\int_V V \cdot p(K,V,T)\,dV}$$

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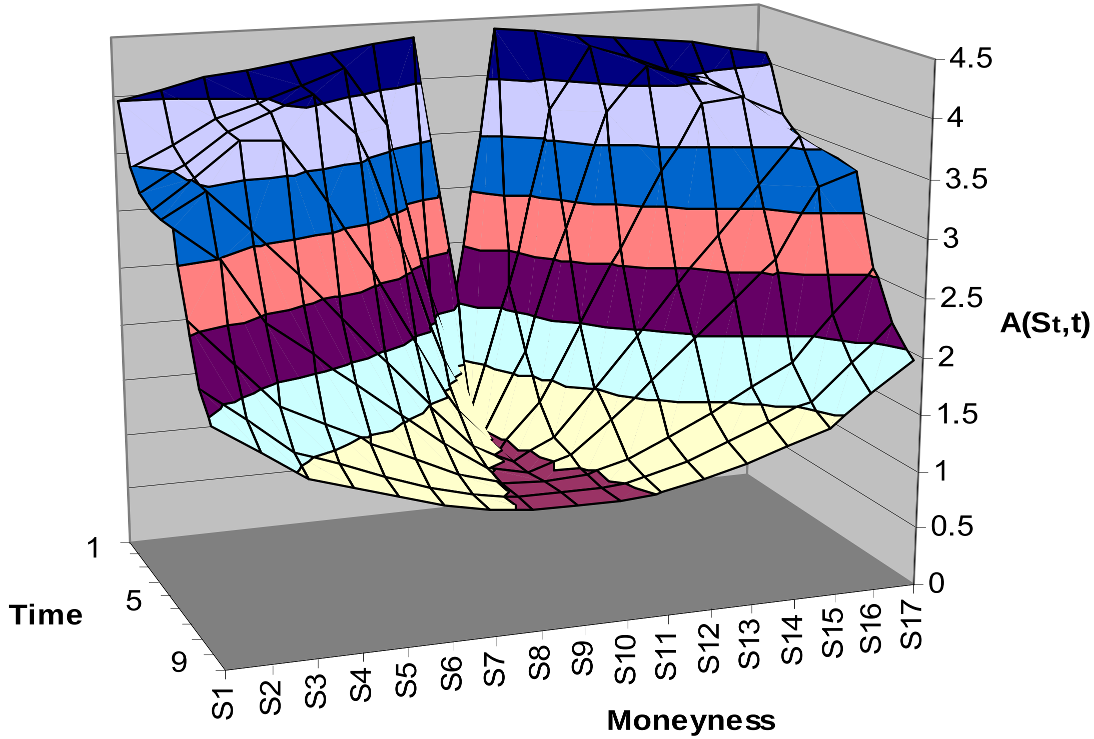
LSV – local volatility contributions

- One expects to find shapes similar to these

	σ_{ATM}	$\sigma_{25\text{-}d\text{-}MS}$	$\sigma_{25\text{-}d\text{-}RR}$
1Y	10.00	0.50	0.00



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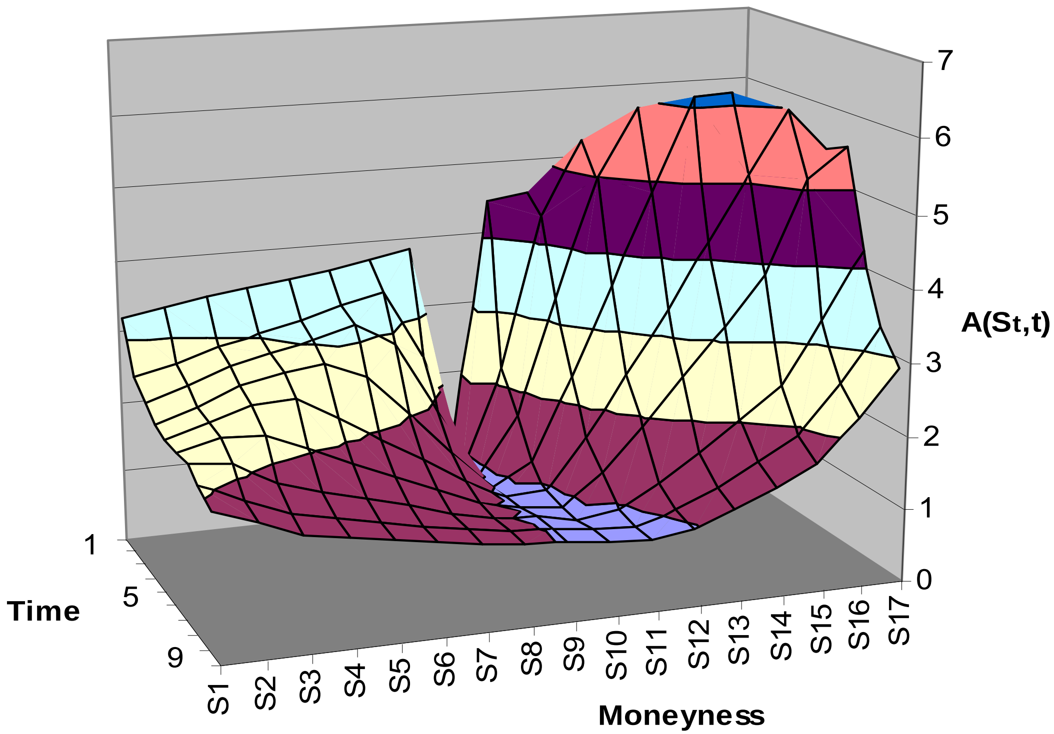
LSV – local volatility contributions

- One expects to find shapes similar to these

	σ_{ATM}	$\sigma_{25\text{-}d\text{-}MS}$	$\sigma_{25\text{-}d\text{-}RR}$
1Y	10.00	0.50	-4.00



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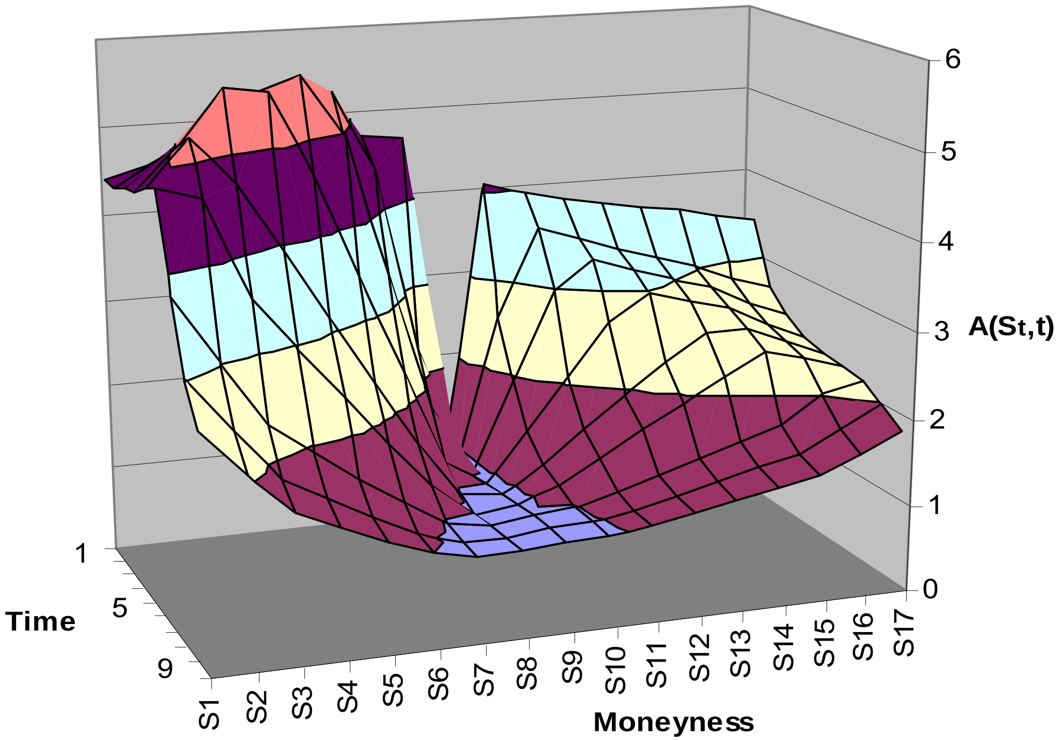
LSV – local volatility contributions

- One expects to find shapes similar to these

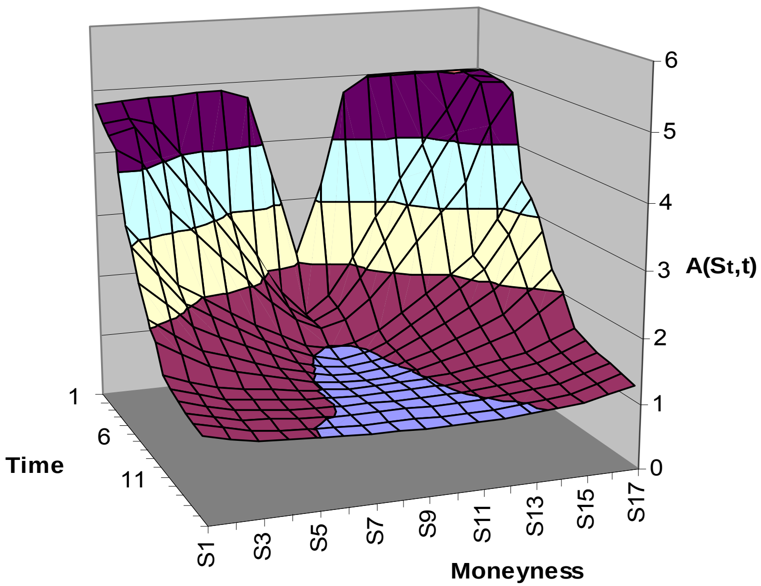
	σ_{ATM}	$\sigma_{25\text{-}d\text{-}MS}$	$\sigma_{25\text{-}d\text{-}RR}$
1Y	10.00	0.50	+4.00



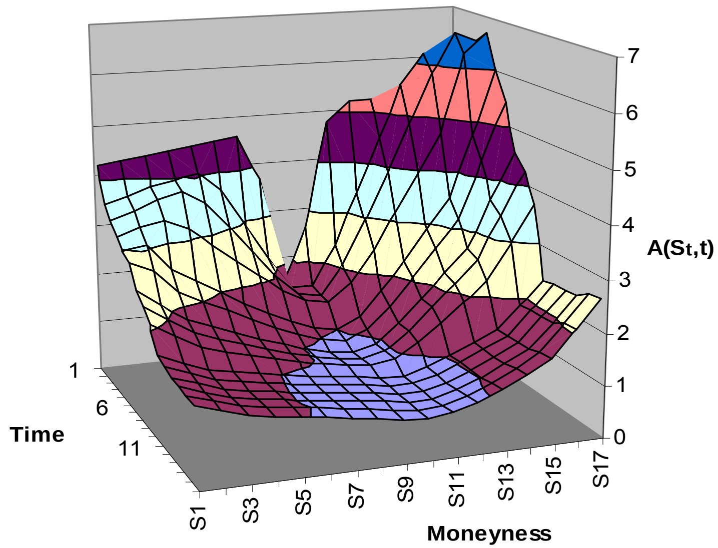
LSV – local volatility contributions

- Real world markets obviously require more structure in $A(S_t,t)$

3Y EURUSD, 16SEP08



3Y USDJPY, 16SEP08



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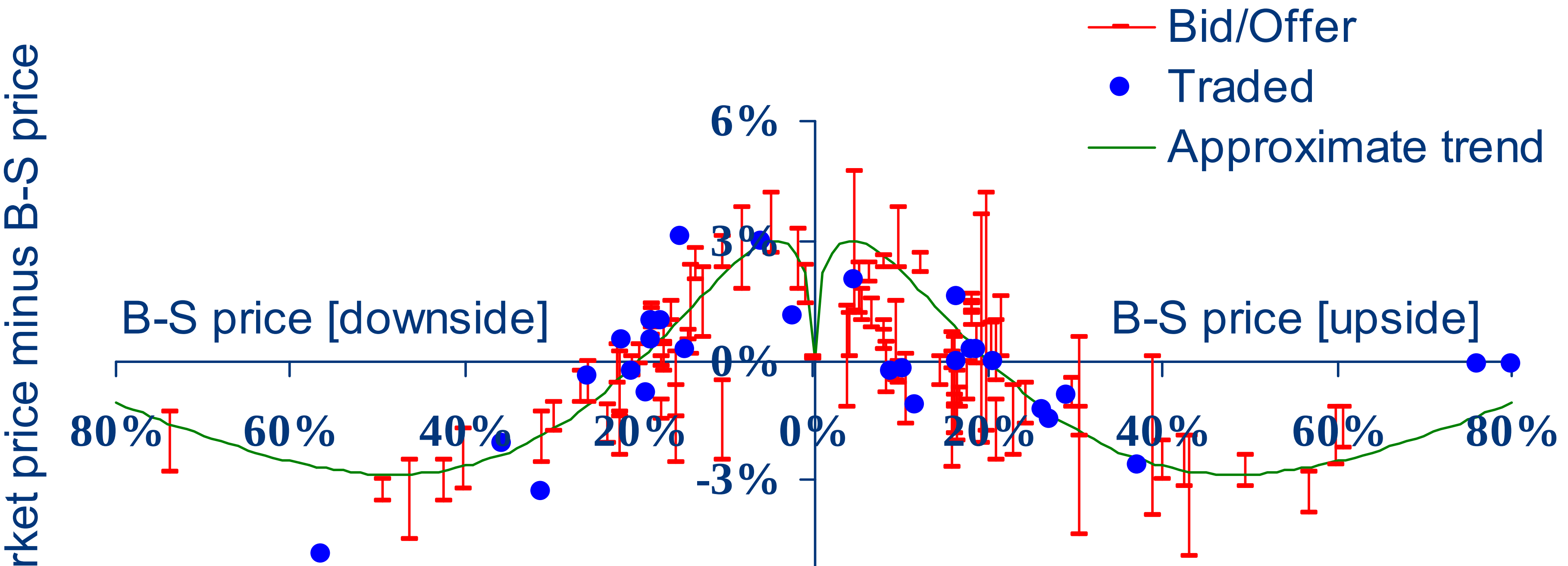
Barriers/touches in FX

- Distant OTs (TV < 20%) typically trade above TV
- Nearer OTs typically trade below TV
  - Structural deviation from B-S prices: **binary moustache**



$$V_T = \mathbf{1}_{\{m_T \le L\}} \qquad V_T = \mathbf{1}_{\{M_T \ge U\}}$$

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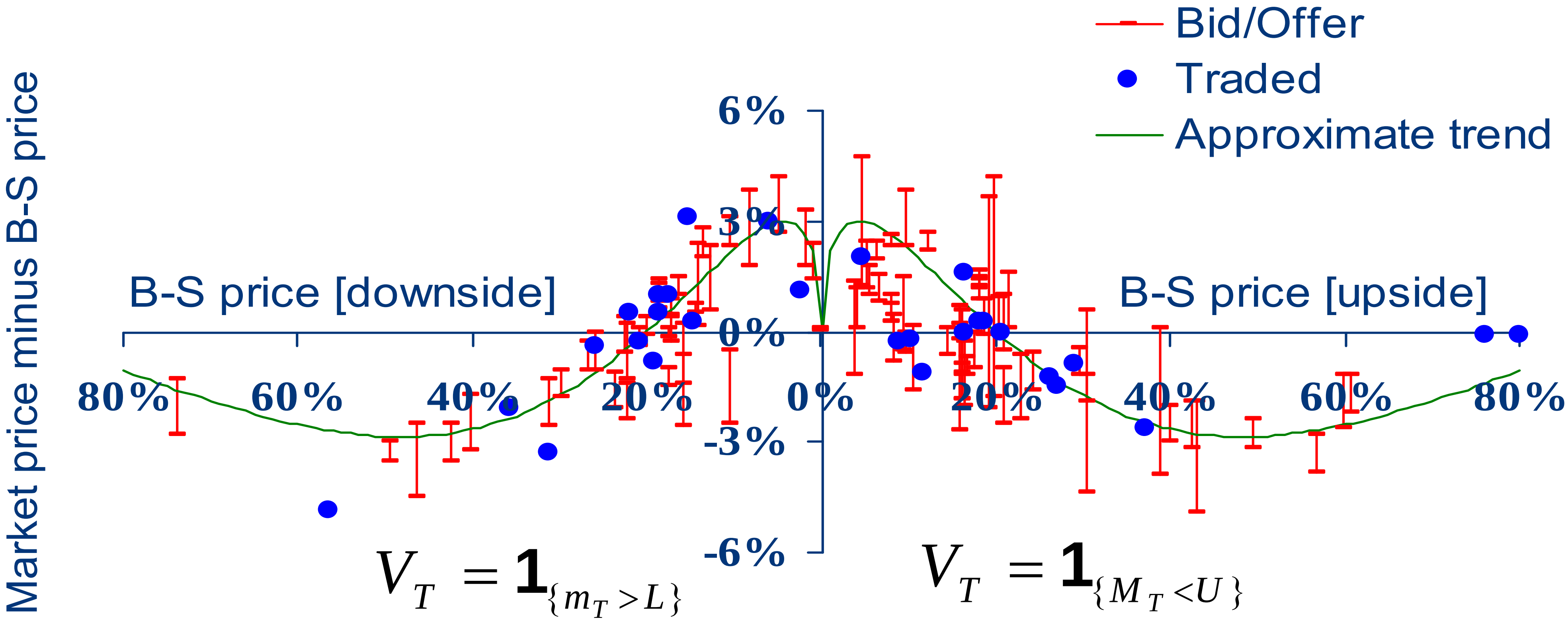
OT binaries in FX

- Distant OTs (TV < 20%) typically trade above TV
- Nearer OTs typically trade below TV



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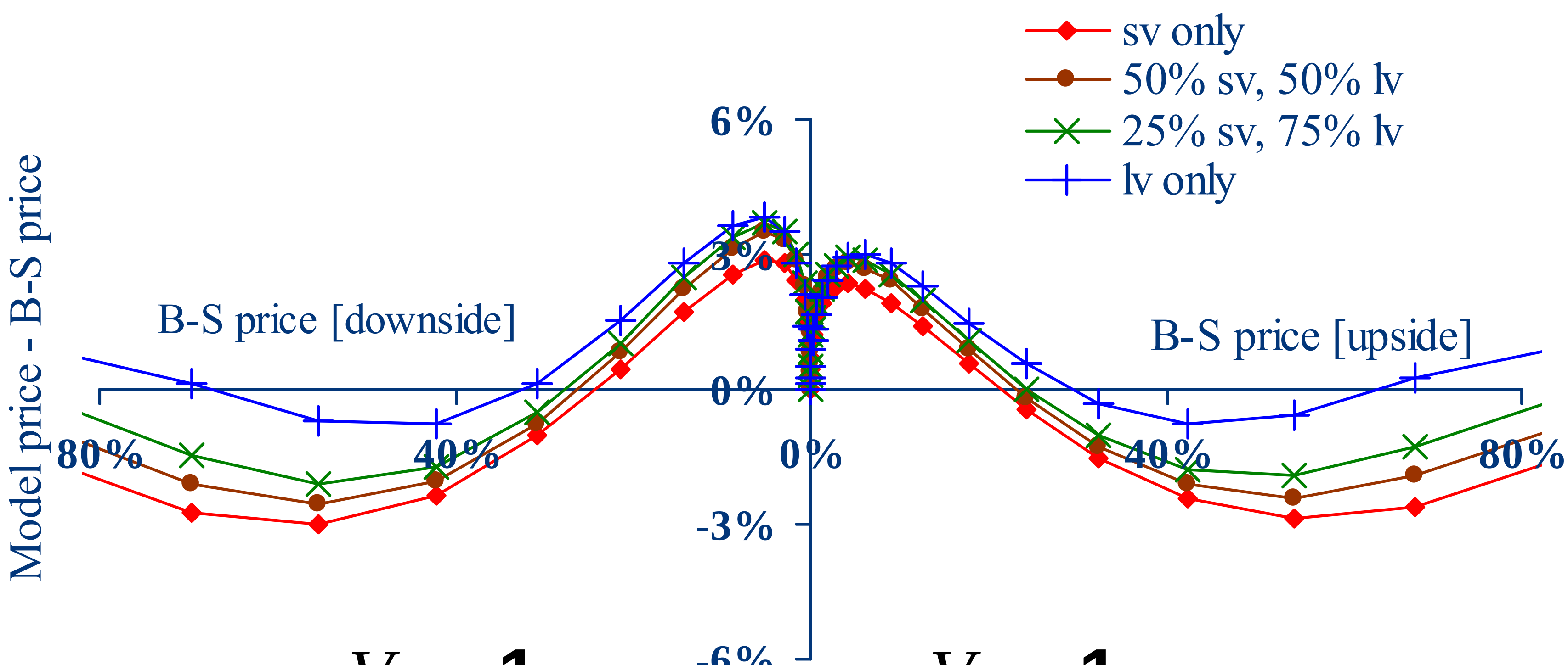
Barriers in FX

- Theoretical "moustache" graph (1Y, S_0=1, r^d=r^f=0, σ=10%, bf=0.5%, minimal risk reversal)



$$V_T = \mathbf{1}_{\{m_T \le L\}} \qquad V_T = \mathbf{1}_{\{M_T \ge U\}}$$

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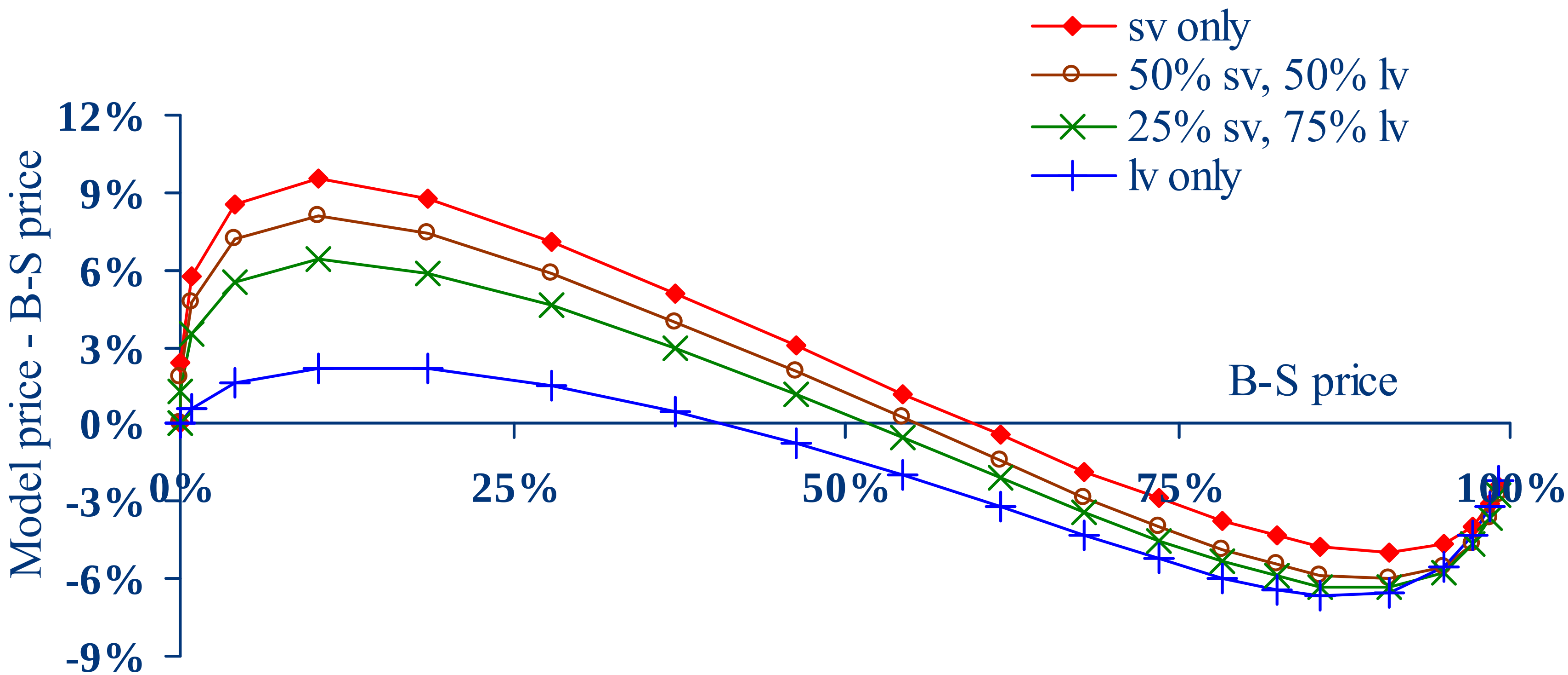
Double barriers in FX

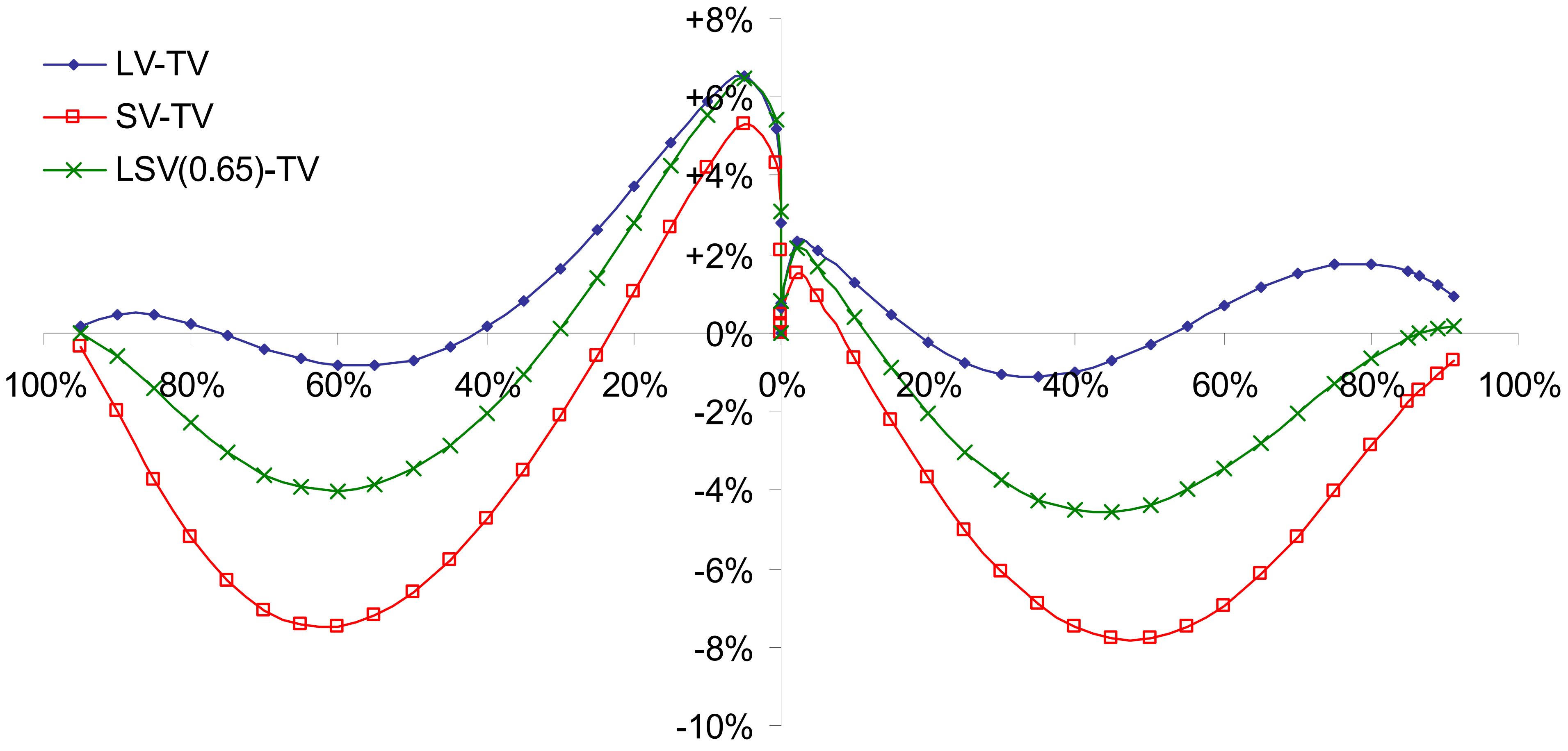
- Theoretical graph of symmetric DNT prices (1Y, S_0=1, r^d=r^f=0, σ=10%, bf=0.5%)



$$V_T = \mathbf{1}_{\{m_T > L\}} \mathbf{1}_{\{M_T < U\}}$$

EURUSD one-touch binaries – 1Y

- Market data as of 3 Dec 2008



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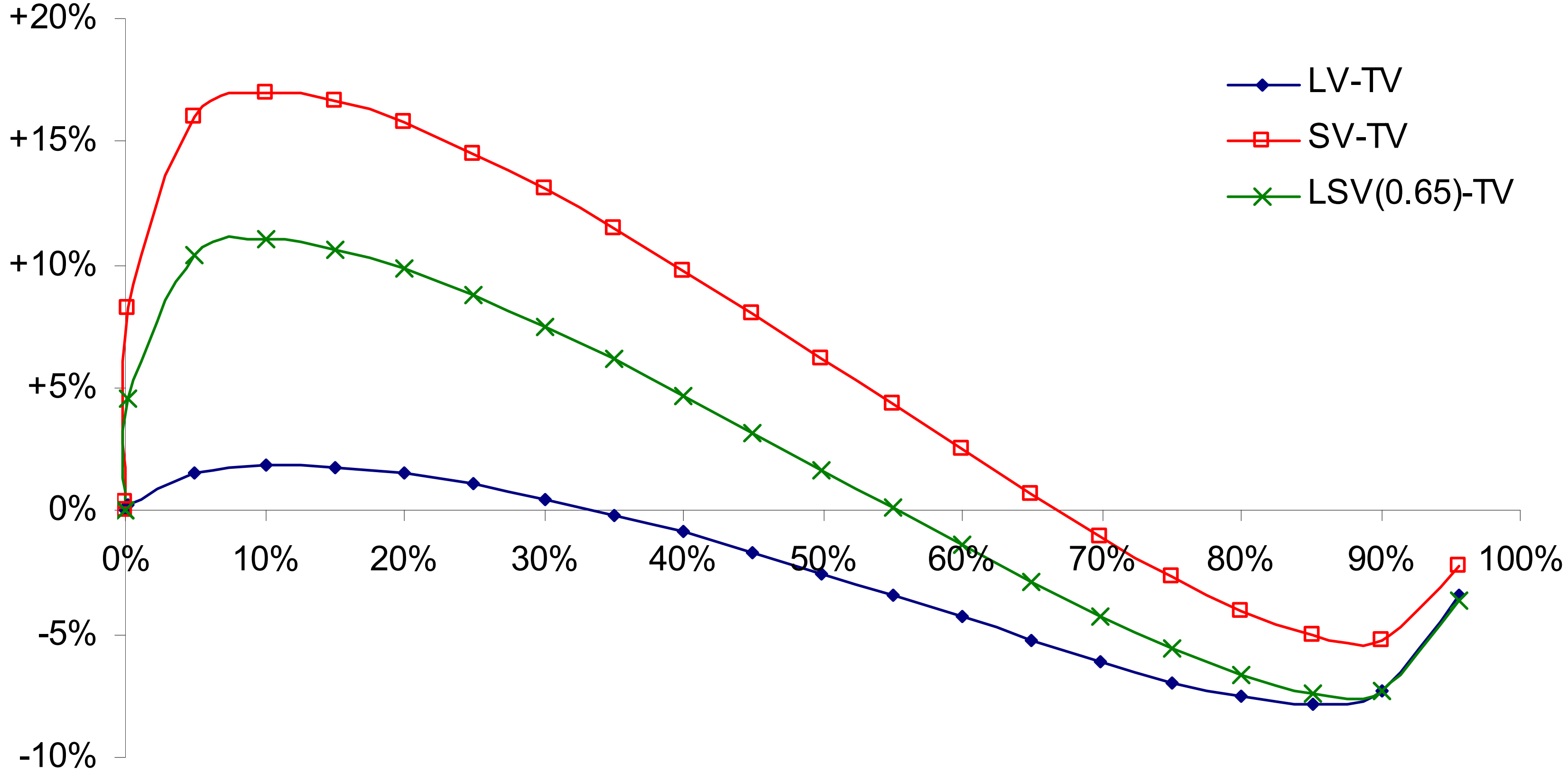
EURUSD double-no-touch binaries – 1Y

- Market data as of 3 Dec 2008



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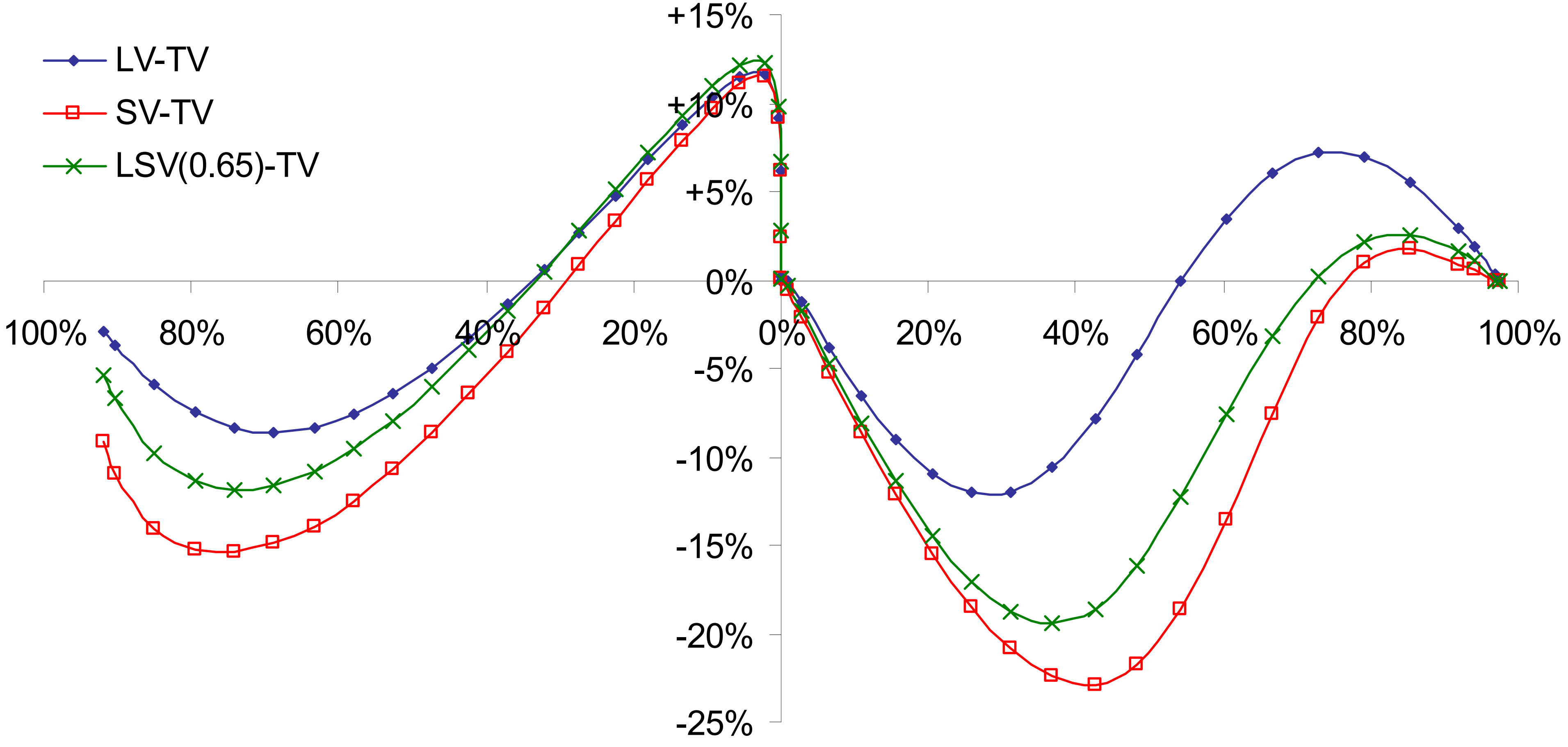
USDJPY one-touch binaries – 1Y

- Market data as of 3 Dec 2008



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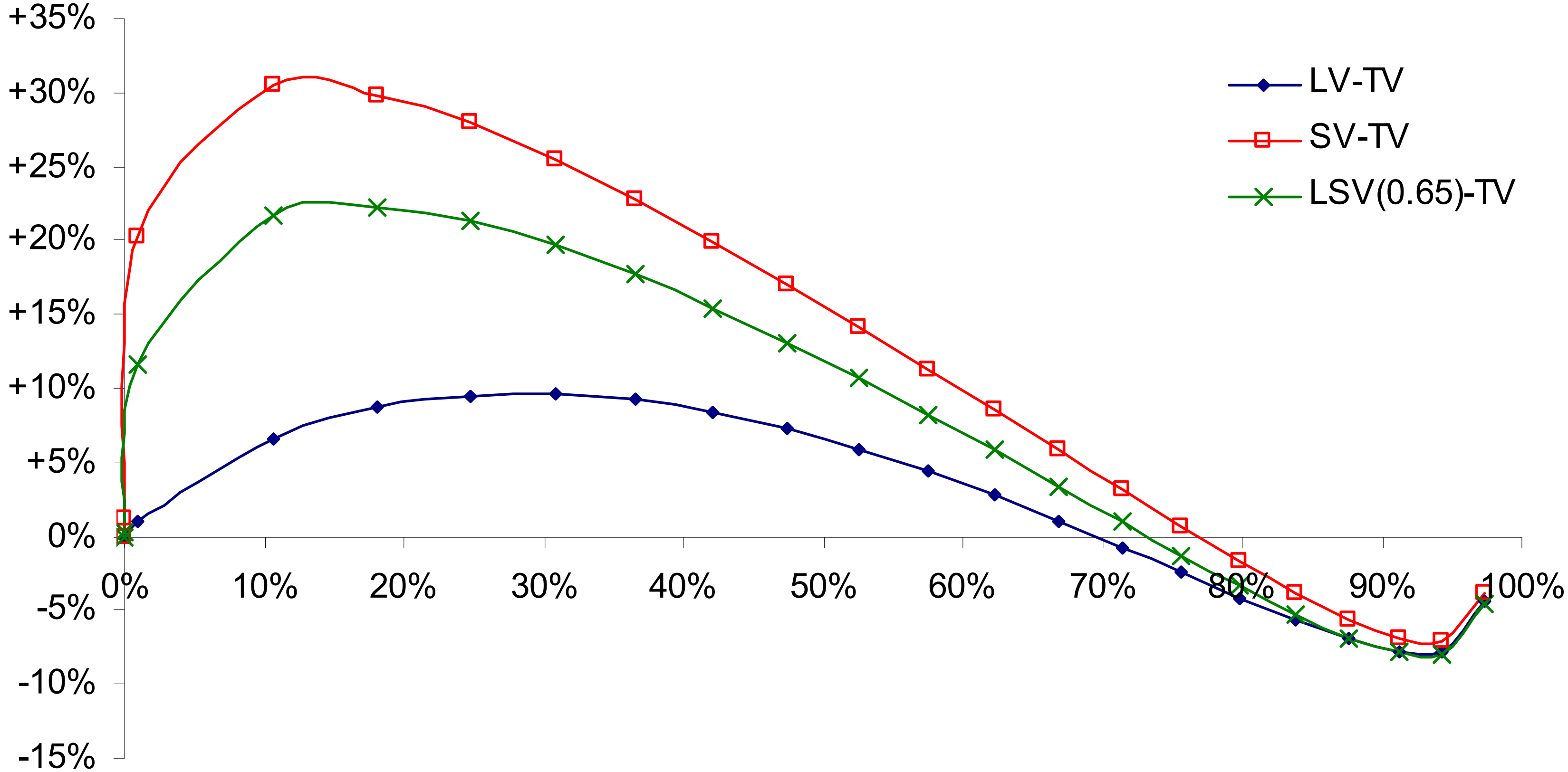
USDJPY double-no-touch binaries – 1Y

- Market data as of 3 Dec 2008



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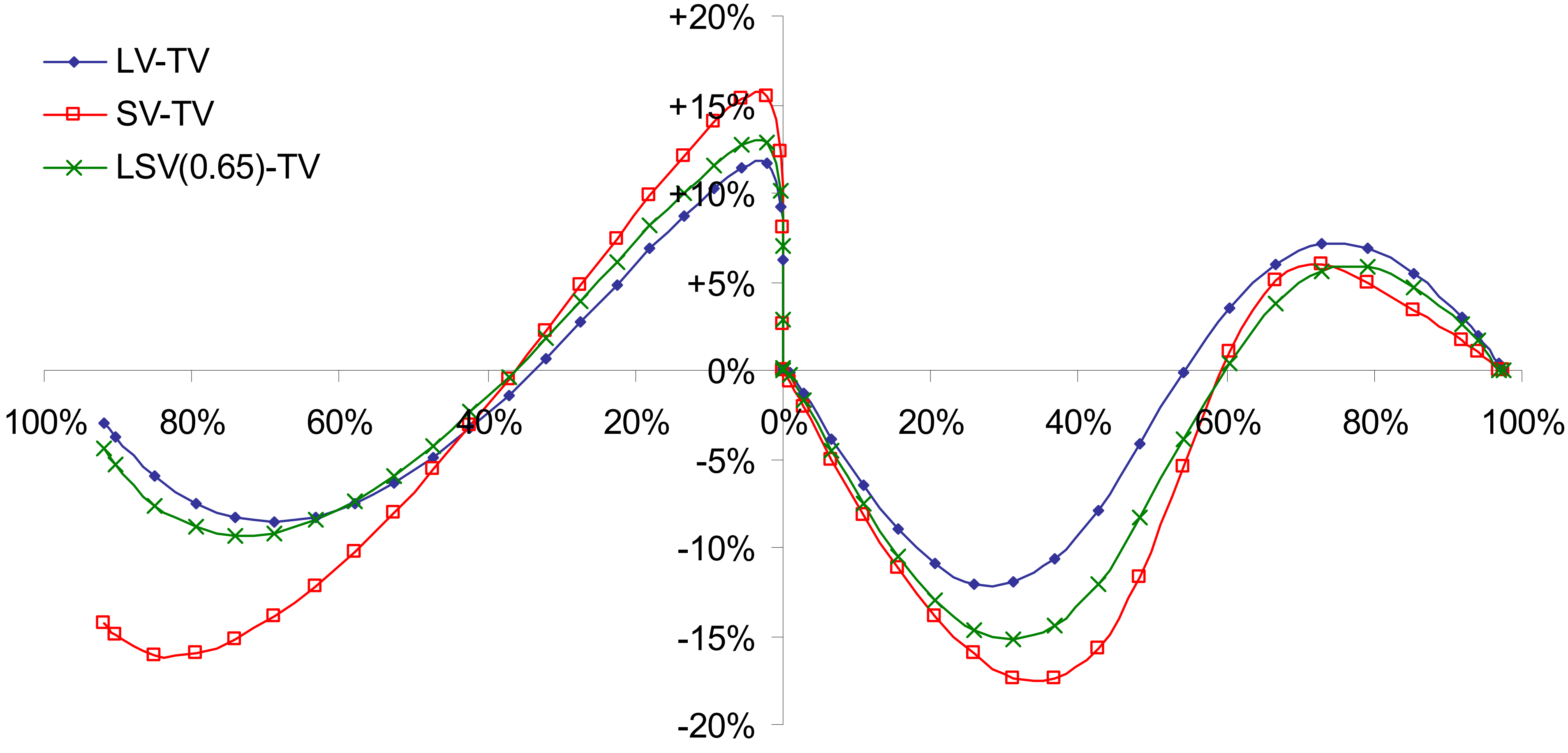
USDJPY one-touch binaries – 5Y

- Market data as of 3 Dec 2008



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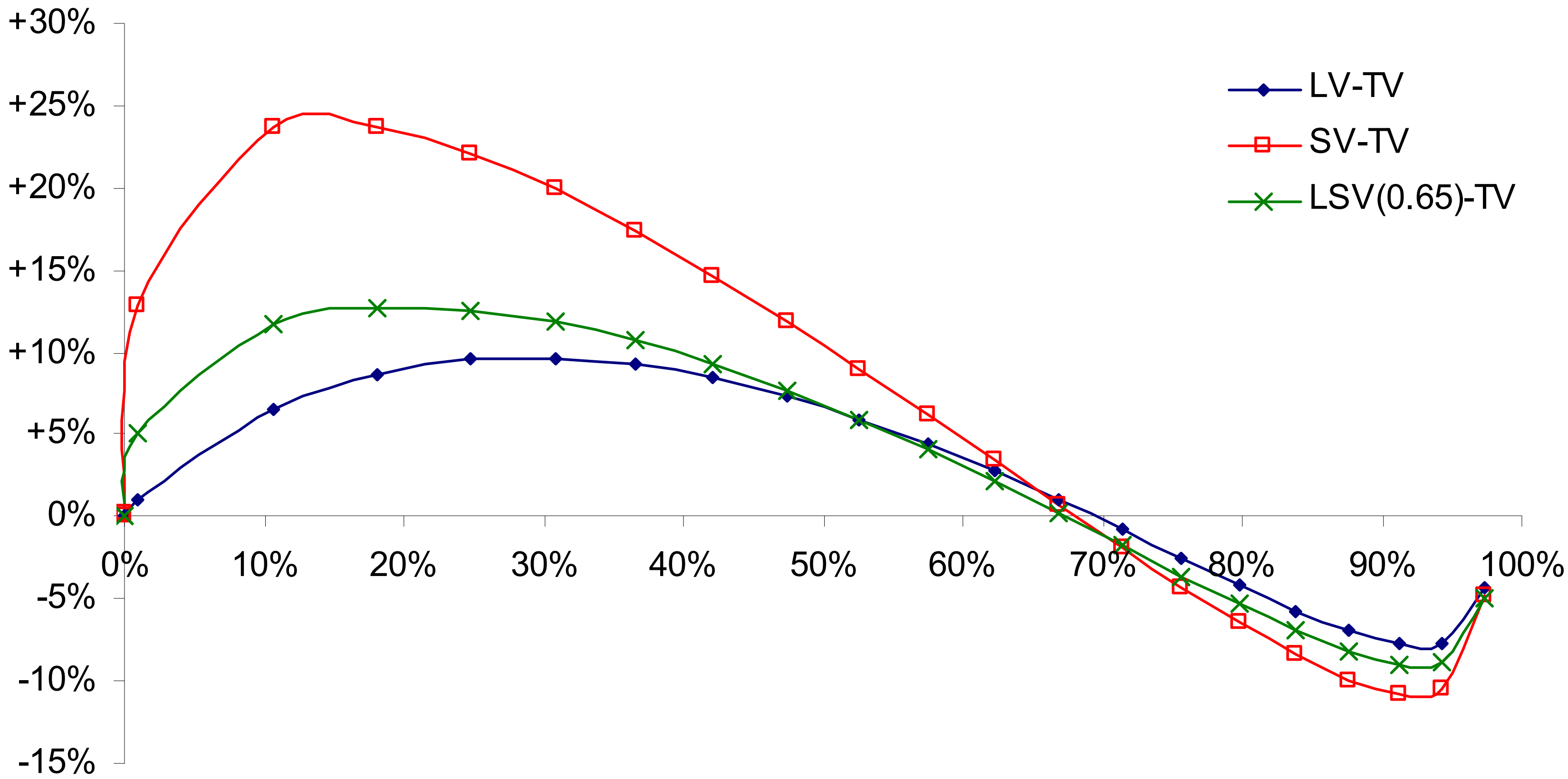
USDJPY double-no-touch binaries – 5Y

- Market data as of 3 Dec 2008



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Summary

- Candidate models for OTs/NTs/DNTs need to capture features from both local and stochastic volatility
- Tuning between these two impacts prices of double no-touches primarily
- Pricing straightforward on PDEs with Dirichlet boundary conditions
- 2-stage calibration process relatively standard (if less straightforward with the numerics)

Thanks for your interest. Questions?

Offline Q’s welcome: iain.clark@standardbank.com
Based on material in www.fxoptionpricing.com

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