Valuing Basket Options:
On Smile and Correlation Skew

Peter Austing
Barclays Capital

ICBI Global Derivatives

April 2011

Outline

Overview

- We would like to value European contracts on multiple assets: in particular, basket options
- How can we mark correlation?
- How can we price consistently with smiles and correlation skew?
- And (in principle) hedge?

Foreign Exchange

- In FX correlation is provided via cross-smiles.
- Consider FX assets EURUSD and GBPUSD with spot rates S_1 and S_2.
- EURGBP exchange rate $S_3 = S_1/S_2$ is also liquidly traded.
- Any model we use must correctly revalue EURGBP vanilla options to avoid arbitrage.
- In Black-Scholes this gives the triangle rule:

$$\rho = \frac{\sigma_1^2 + \sigma_2^2 - \sigma_3^2}{2\sigma_1\sigma_2}$$

FX Models

We need a model that correctly prices vanillas on assets and crosses. Three main types of model available:

1. Make instantaneous correlations depend on spot levels: Local Correlation
   - Works but heavy.
2. Copulas
   - For $N > 2$, hard to find a copula that can be both calibrated and simulated.
3. Analytic joint densities
   - For $N > 2$, hard to simulate
   - But certain contracts (BOFs, WOFs, dual digitals) can be valued analytically.

Aim: Construct an analytic density on which baskets can be easily valued.

Review Analytic Density from Best-Of Options

For details, see last year's talk [1]

- Best-of pays

$$\max\{(S_1 - K_1)_+/K_1, (S_2 - K_2)_+/K_2\}$$

- Given triangle of smiles $\sigma_1(K_1)$, $\sigma_2(K_2)$, $\sigma_3(K_3)$ where $K_3 = K_1/K_2$
- A joint cumulative distribution is given by

$$C(K_1, K_2) = \left[1 + \sum_{i=1}^{2} K_i \frac{\partial}{\partial K_i}\right] B[K_1, K_2; \sigma_1(K_1), \sigma_2(K_2), \sigma_3(K_1/K_2)]$$

- Correctly re-prices asset- and cross-smiles

Here $B[K_1, K_2; \sigma_1, \sigma_2, \sigma_3]$ is the Black-Scholes formula for the value of a best-of option.

- Works because
  - Knowledge of best-of prices equivalent to knowledge of pdf
  - Best-ofs become vanillas on assets and cross in simple limits
- Simple to value best-ofs and worst-ofs and dual digitals in the model
- Can be extended to N-assets
- But hard to value an N-asset basket
  - Hard to do all the differentiations to calculate the pdf for $N > 2$
  - Needed to simulate in Monte Carlo

Can we construct a joint density with the property that basket options are easy to value?

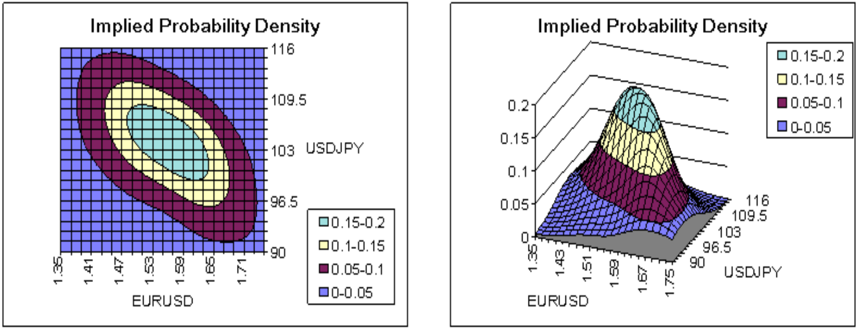

Figure: 1Y implied pdf on 14th April 2008.

Correlation skew in other asset classes

In other asset classes (say equities or precious metals) cross vanillas do not exist.

- Two-asset baskets are more useful
- For simplicity, lets look at equally weighted 2-asset basket options

$$B = (0.5S_1 + 0.5S_2 - K)_+$$

- Maybe some prices for B can be seen in the market
- Or maybe trader takes a view on correlation smile $\rho(K)$ by marking some prices for B.

Can we construct an N-asset density consistent with the asset- and correlation-smiles?

Lipton: knowledge of basket prices for all weights equivalent to knowledge of joint density [2]. Also Carr and Laurence [3].

- The Radon transform of a function f of N variables x is

$$\mathcal{R}f(w,k) = \int dx\, f(x)\delta(w \cdot x - k)$$

- Radon transform can be inverted (see [3] for details)

$$f(x) = \mathcal{R}^{-1}[\mathcal{R}f].$$

- Second derivative of basket price is RT of pdf

$$\begin{aligned} C(w,K) &= \int dS_1 \cdots dS_N\, f_T(S)\,(S \cdot w - K)_+ \\ \frac{\partial^2}{\partial K^2} C &= \mathcal{R}f(w,k). \\ f_T(S) &= \mathcal{R}^{-1}\left[\frac{\partial^2}{\partial K^2} C\right]. \end{aligned}$$

- Inverse transform when n is odd:

$$f(x) = \frac{(i)^{n-1}}{2(2\pi)^{n-1}} \int_{|w|=1} dw \, \frac{\partial^{n-1}}{\partial k^{n-1}} \mathcal{R}f(w,k)|_{k=w\cdot x}$$

- When n is even:

$$f(x) = \frac{(i)^{n}}{(2\pi)^{n}} \int_{|w|=1} dw \int_{-\infty}^{\infty} dk \, \frac{1}{k - w\cdot x} \frac{\partial^{n-1}}{\partial k^{n-1}} \mathcal{R}f(w,k)$$

- Special case (n odd):

$$\delta(x_1 - k_1)\cdots\delta(x_n - k_n) = \frac{(i)^{n-1}}{2(2\pi)^{n-1}} \int_{|w|=1} dw \, \delta^{(n-1)}(w\cdot(x-k))$$

shows how n-variate delta function obtained from derivative of basket payoff.

Task: postulate values for the set of basket prices for all weights and strikes such that

- In the limit the basket becomes a vanilla on any asset it is priced correctly
- In the limit the basket becomes one of the correlation instruments (2-asset baskets) it is priced correctly
- In the limit the asset- and correlation-smiles are flat we get the Black-Scholes basket price

Then we have implicitly constructed a density with the property that asset- and correlation-smiles are correctly repriced.

- We cannot guarantee the density is a probability density, ie positive everywhere
- If smiles are a small deformation from Black-Scholes we should be OK
- Many possible choices for the postulated basket prices.
- Aim for a choice that does not introduce kinks.

One choice:

- Re-write basket payout as

$$\left(\sum w_i \frac{S_i}{F_i} - K\right)_+ \tag{1}$$

  with $\sum w_i^2 = 1$.
- Choose strikes $K_i = KF_i$ and individual implied vols $\sigma_i(K_i)$.
- Choose correlations $\rho_{ij}(K)$ (the correlation you would plug into the Black-Scholes formula to get the 2-asset equally weighted basket price right).
- Plug these into the (best available approximation to) the Black-Scholes formula for the basket option.
- Then in the limit of weights that the basket becomes a vanilla or one of the 2-asset sub-baskets, the price is correct.

FX Baskets

- At first sight, harder: need to match cross-FX smiles
- But we can use the cumulative distribution from BOF prices [1] to price 2-asset baskets with a single numerical integration

$$
\begin{aligned}
P &= \int (K - w_1 S_1 + w_2 S_2)_+ \frac{\partial^2}{\partial S_1 \partial S_2} C(S_1, S_2) \\
&= \int_0^K dU\, C\left(\frac{U}{w_1}, \frac{K-U}{w_2}\right)
\end{aligned}
$$

- Choose correlation $\rho_{ij}(K, w_i, w_j)$ so 2-asset sub-basket with weights w_i, w_j is correct
- Then in the limit that the basket turns into a 2-asset sub-basket with any weights, it is priced correctly.

- But the 2-asset basket prices imply the 2-asset marginal density, and are consistent with cross-smile
- So implicit joint density is consistent with asset- and cross-smiles

In the plots that follow, I am using a slightly different prescription for the asset strikes and correlations:

- Choose strikes $\mathrm{K_i}$ to have equal probabilities that $\mathrm{S_i} > \mathrm{K_i}$ subject to $\sum_\mathrm{i} \mathrm{w_i K_i} = \mathrm{K}$
- Choose correlation ρ_ij so that the 2-asset sub-basket with weights $\mathrm{w_i}$, $\mathrm{w_j}$ and strike $\mathrm{w_i K_i} + \mathrm{w_j K_j}$ is priced correctly.
- This choice also has the property that in the limit of weights that the contract becomes a vanilla or a 2-asset sub-basket, it is priced correctly.

Summary of method:

- Choose asset volatilities $\sigma_i(w, K)$ depending on basket weights w and strike K
- Choose correlations $\rho_{ij}(w, K)$ depending on basket weights w and strike K
- Plug them into (an approximation to) the Black-Scholes formula for a basket
- Vols chosen with the property that when all weights but one go to zero, it is the correct implied vol for the asset vanilla
- Correlations chosen with the property that when all weights but two go to zero, the resulting 2-asset basket price matches the price calculated in the best-of implied density

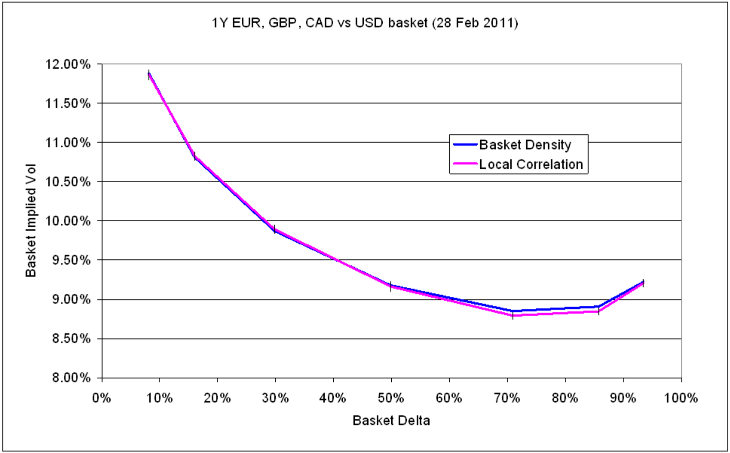

Figure: Equally weighted 3-asset basket in this model versus FX local correlation.

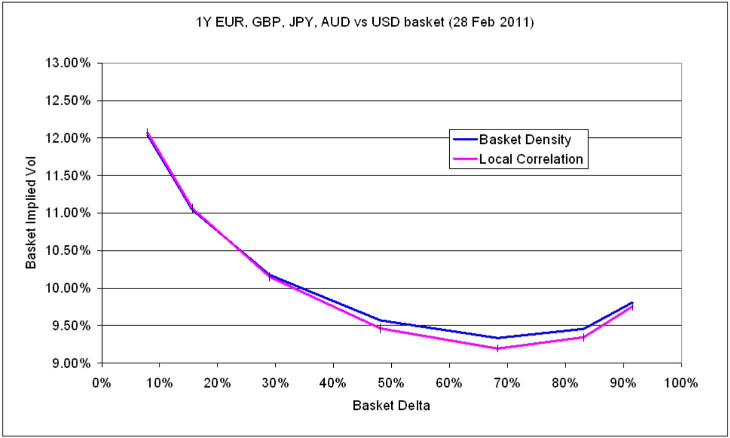

Figure: Equally weighted 4-asset basket in this model versus FX local correlation.

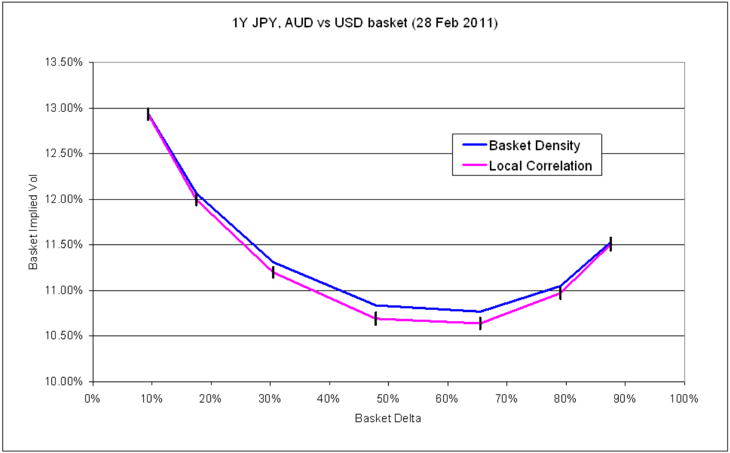

Figure: Two asset basket AUD, JPY vs USD demonstrates model dependence of basket price.

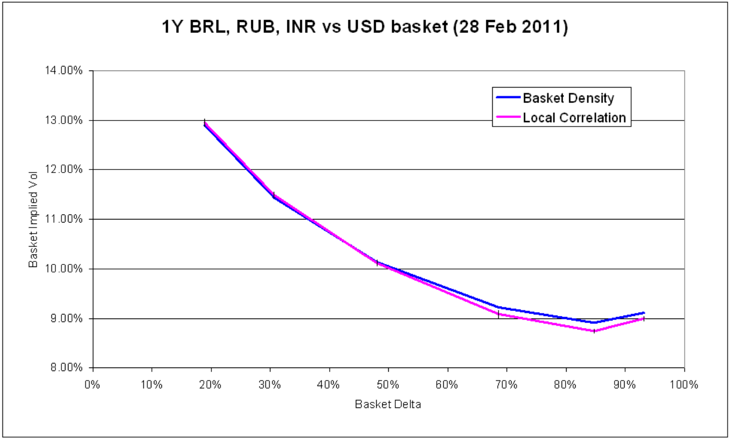

Figure: Equally weighted 3-asset EM basket in this model versus FX local correlation.

Discussion

- We have constructed a density function with the property that asset smiles and correlation smiles are repriced
- We cannot guarantee it is a probability density (ie positive everywhere)
- There is some freedom in setting up the density and it is hard to check the density does not go negative
- But it matches Black-Scholes in the limit of flat smiles, so should be good for small smile effects

For FX baskets:

- Method works by joining 2-dimensional marginal distributions
- Can value FX basket with $N(N-1)/2$ one-dimensional numerical integrations

- Good results have been obtained (comparing to FX LC)

For non-FX baskets

- Correlation skew can be marked with reference to 2-asset baskets. N-asset baskets can then be priced analytically
- Results should be compared to a true model (eg index LC [4])
- There is additional freedom from quadrants with negative weights
- Indicates correlation skew supplied via basket smiles may not be enough to value contracts like spreads

References

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- A. Lipton, Mathematical Methods for Foreign Exchange. World Scientific, 2001.
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