Successfully implementing Stochastic Intrinsic Currency Volatility Models

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Outline of this presentation

- Overview of the intrinsic currency framework
- Overview of the stochastic intrinsic currency volatility model ("SticVol")
- Techniques for obtaining stable calibrations to the SticVol model
- Extension to multiple less liquid currencies
  - Using maximum entropy to parameterise the correlation matrix
- Derivative pricing: FX basket options
- Using intrinsic currency volatility to trade FX correlation and covariance swaps
- Conclusion

Motivation for intrinsic currency modelling

- Most people in the world who travel between different countries understand the concept of “intrinsic currency values”, because
  - they can hold the notes of each currency
  - they understand how much of each currency is required to buy the different things that they want to buy

- However forex market practitioners typically don’t understand intrinsic currency values!
  - Forex market practitioners usually talk in currency pairs
- The intrinsic currency framework aims to restore the natural intuition about individual currencies

What is the intrinsic currency framework?

- Basic definitions:

  X_i = ***intrinsic value*** of 1 unit of currency i

  so each currency has its own value variable. Observable FX rates are given by $X_{ij} = X_i/X_j$. If X_i follow log-normal processes with volatility σ_i then the instantaneous volatility of X_{ij} is

  $$\sigma_{ij} = \sigma(\sigma_i, \sigma_j) = \sqrt{\sigma_i^2 - 2\rho_{ij}\sigma_i\sigma_j + \sigma_j^2} \quad .$$

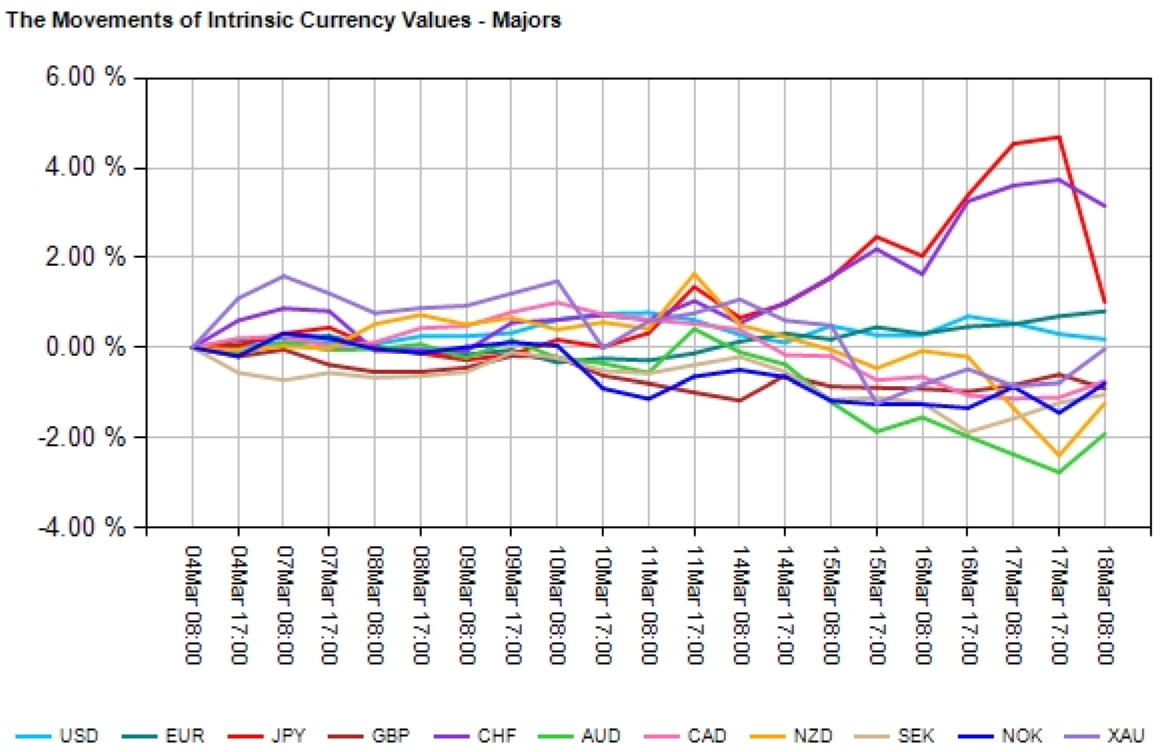
  where ρ_{ij} is the correlation between X_i and X_j
- The units which measure the value of the X_i are not specified because they cancel out in the ratios X_i/X_j. Not necessary to determine the units of the X_i to use the framework.

Dealing with the extra degree of freedom

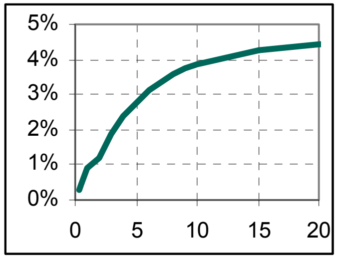
- With N currencies, there are N–1 degrees of freedom which are determined by the usual FX rates. However, the intrinsic currency framework has an extra degree of freedom, because it's got a value variable for each currency.
- Use the concept of ***maximum information entropy*** to handle the extra degree of freedom in the covariance matrix between the variables of the intrinsic currency framework.
- Use ***maximum likelihood estimation*** to handle the extra degree of freedom when estimating changes in the intrinsic currency values themselves.

Graphs of intrinsic currency values

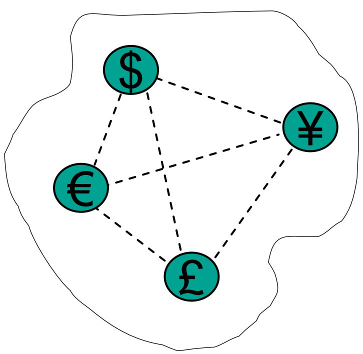
- In the week following the recent Japanese Earthquake, JPY strengthened significantly, but less obvious was that CHF was highly correlated with a lot of the JPY movements:



Motivation for the SticVol pricing model



- With interest rate derivatives pricing, options on individual rates can be priced using the Black formula, without any regard for the behaviour of related rates along the yield curve. Libor Market Models, which deal with the whole yield curve, are more powerful but harder to formulate.



- With FX derivative pricing, the focus is usually to model FX rates individually. Simultaneously modelling all the cross rates between a set of currencies should be more powerful, if such a model can be found.
- The stochastic intrinsic currency volatility model ("**SticVol** model") is capable of doing just that.

Stochastic intrinsic currency volatility model

- A SABR-style model for the FX market.
- Chose currency k as the numéraire. Then the stochastic processes for X_i and σ_i are given by:

$$\frac{dX_i}{X_i} = (\tilde{\lambda} - r_i + \varepsilon^2 \rho_{ik} \sigma_i \sigma_k) dt + \varepsilon \sigma_i dW_i$$

$$\frac{d\sigma_i}{\sigma_i} = \varepsilon^2 \tilde{\rho}_{ik} \upsilon_i \sigma_k dt + \varepsilon \upsilon_i dZ_i$$

where r_i is the interest rate in currency i, $\tilde{\lambda}$ is a variable which is the same for all X_i, and where the correlation matrices are defined by

$$\begin{pmatrix} \mathbf{dW} \\ \mathbf{dZ} \end{pmatrix} \begin{pmatrix} \mathbf{dW}' & \mathbf{dZ}' \end{pmatrix} = \begin{pmatrix} \rho & \tilde{\rho}' \\ \tilde{\rho} & \mathbf{r} \end{pmatrix} dt \quad ,$$

where $'$ denotes vector and matrix transpose.

Variables of the SticVol model for N currencies

- N stochastic intrinsic currency volatilities σ_i.
- An N×N symmetric matrix $\boldsymbol{\rho}$ of correlations between the N intrinsic currency values X_i.
- N volatility of volatility variables ν_i. It turns out that there is significant tenor dependency to the ν_i, so that 1 month ν_i are typically around 200%-230%, with 1 year ν_i typically around 65%-85%.
- An N×N symmetric matrix of correlations $\boldsymbol{r}$ between the N intrinsic currency volatilities σ_i. These are typically all positively correlated, which makes intuitive sense.
- An N×N matrix $\tilde{\rho}$ between all the intrinsic currency values X_i and all the intrinsic currency volatilities σ_i. These tend to reflect the market's risk-reversals.

Features of SticVol

- SABR itself cannot be used consistently with multiple FX rates, because if X_{ij}, X_{jk} are SABR processes then $X_{ik}=X_{ij}\,X_{jk}$ is not a SABR process
- With SticVol, FX rates are given by $X_{ij} = X_i/X_j$, so any intrinsic currency formulation automatically has the required symmetries provided:
  - it produces the correct risk-neutral processes for X_{ij}, and
  - it behaves correctly when numéraire currency changed
- When using SticVol, an approximation formula is available for the log-normal implied volatility σ_B as a function of strike K, as a function of the SticVol variables σ_i, ν_i, $\boldsymbol{\rho}$, **r** and $\tilde{\boldsymbol{\rho}}$.
- Smile shape of SABR and SticVol very similar

New SABR style formula for FX volatility σ_B

$$\sigma_B = \varepsilon \sigma_{ij} \frac{z}{x(z,\sigma_i,\sigma_j)} \left(1 + \varepsilon^2 \left(\frac{1}{4}(a_1 \sigma_{ij} + 2a_3 + 2a_4) - \frac{1}{6} a_5 + \frac{2a_2 - a_1^2}{24} \right) \tau_{ex} \right)$$

$$z = \frac{\ln\left(\frac{X_{ij}^F}{K} \right)}{\varepsilon \sigma_{ij}} \quad , \quad x(z,\sigma_i,\sigma_j) = \frac{\ln\left(\frac{\sqrt{1 - 2\varepsilon a_1 z + \varepsilon^2 (a_2 + a_5) z^2} - \frac{a_1}{\sqrt{a_2 + a_5}} + \varepsilon \sqrt{a_2 + a_5}\, z}{1 - \frac{a_1}{\sqrt{a_2 + a_5}}} \right)}{\varepsilon \sqrt{a_2 + a_5}}$$

$$a_1 = \frac{1}{\sigma_{ij}} D_\sigma \sigma_{ij} \quad , \quad a_3 = \tilde{\rho}_{ij} \frac{\upsilon_i \sigma_i \sigma_j}{\sigma_{ij}} \frac{\partial \sigma_{ij}}{\partial \sigma_i} + \tilde{\rho}_{jj} \frac{\upsilon_j \sigma_j^2}{\sigma_{ij}} \frac{\partial \sigma_{ij}}{\partial \sigma_j}$$

$$a_2 = \frac{\upsilon_i^2 \sigma_i^2}{\sigma_{ij}^2} \left(\frac{\partial \sigma_{ij}}{\partial \sigma_i} \right)^2 + \frac{2 r_{ij} \upsilon_i \upsilon_j \sigma_i \sigma_j}{\sigma_{ij}^2} \frac{\partial \sigma_{ij}}{\partial \sigma_i} \frac{\partial \sigma_{ij}}{\partial \sigma_j} + \frac{\upsilon_j^2 \sigma_j^2}{\sigma_{ij}^2} \left(\frac{\partial \sigma_{ij}}{\partial \sigma_j} \right)^2$$

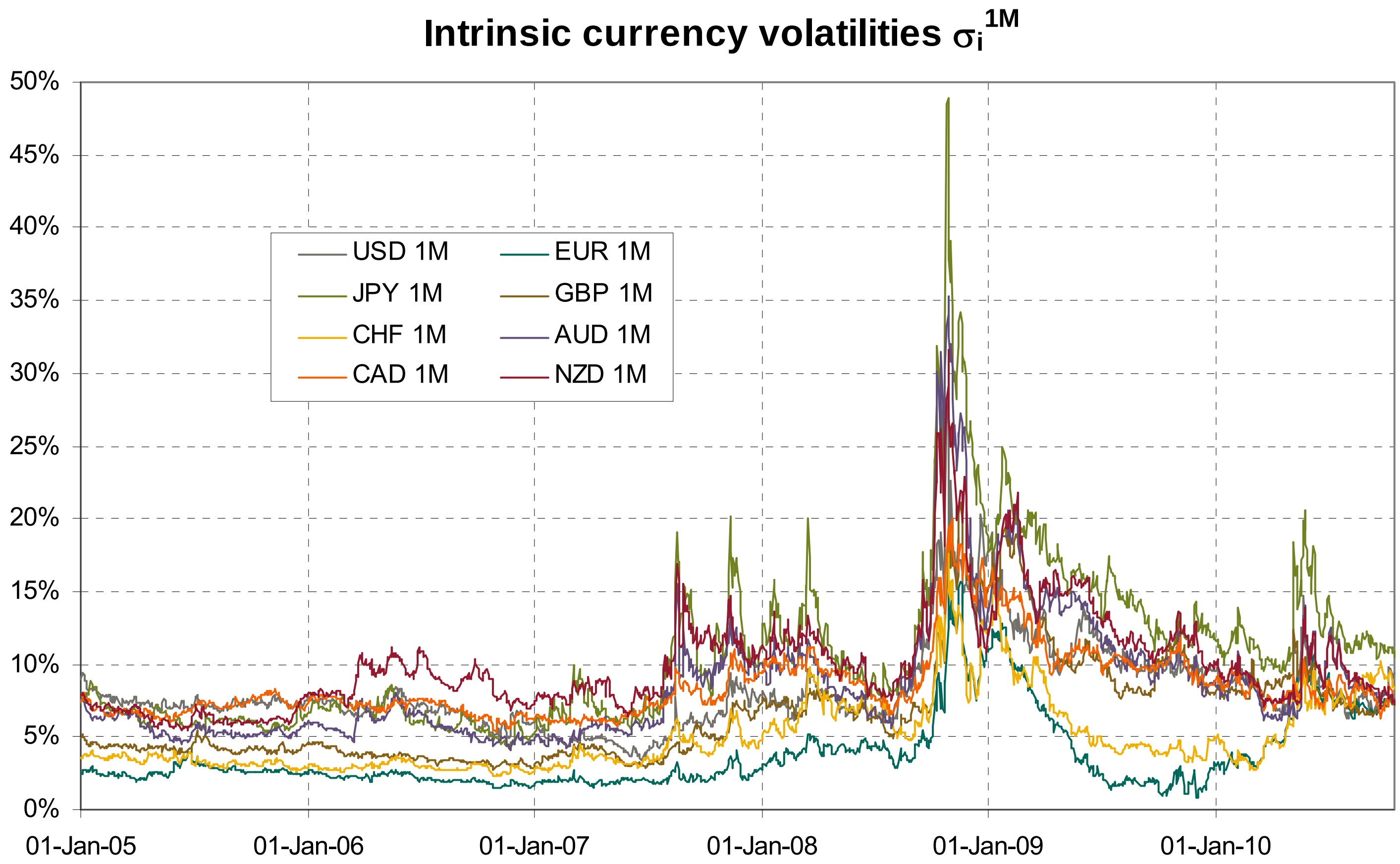
$$a_4 = \frac{\sigma_i^2 \sigma_j^2}{\sigma_{ij}^4} \left(1 - \rho_{ij}^2\right) \left(\frac{1}{2} \upsilon_i^2 - r_{ij} \upsilon_i \upsilon_j + \frac{1}{2} \upsilon_j^2 \right) \quad , \quad a_5 = D_\sigma a_1$$

Advantages of intrinsic currency modelling

- SticVol demonstrates that modelling the stochastic behaviour of intrinsic currencies is a good way to build models which handle more than two currencies at the same time.
  - SticVol is just one possible multi-currency model. Other stochastic processes could be postulated.
  - The intrinsic currency approach guarantees that all natural symmetries of the forex market are respected
- For pricing models which include interest rates as well as forex rates, modelling correlations between interest rate variables and intrinsic currency values is a natural way to build models
  - A reasonable starting point might be to assume that interest rates in a particular currency are independent of intrinsic currency values in all other currencies

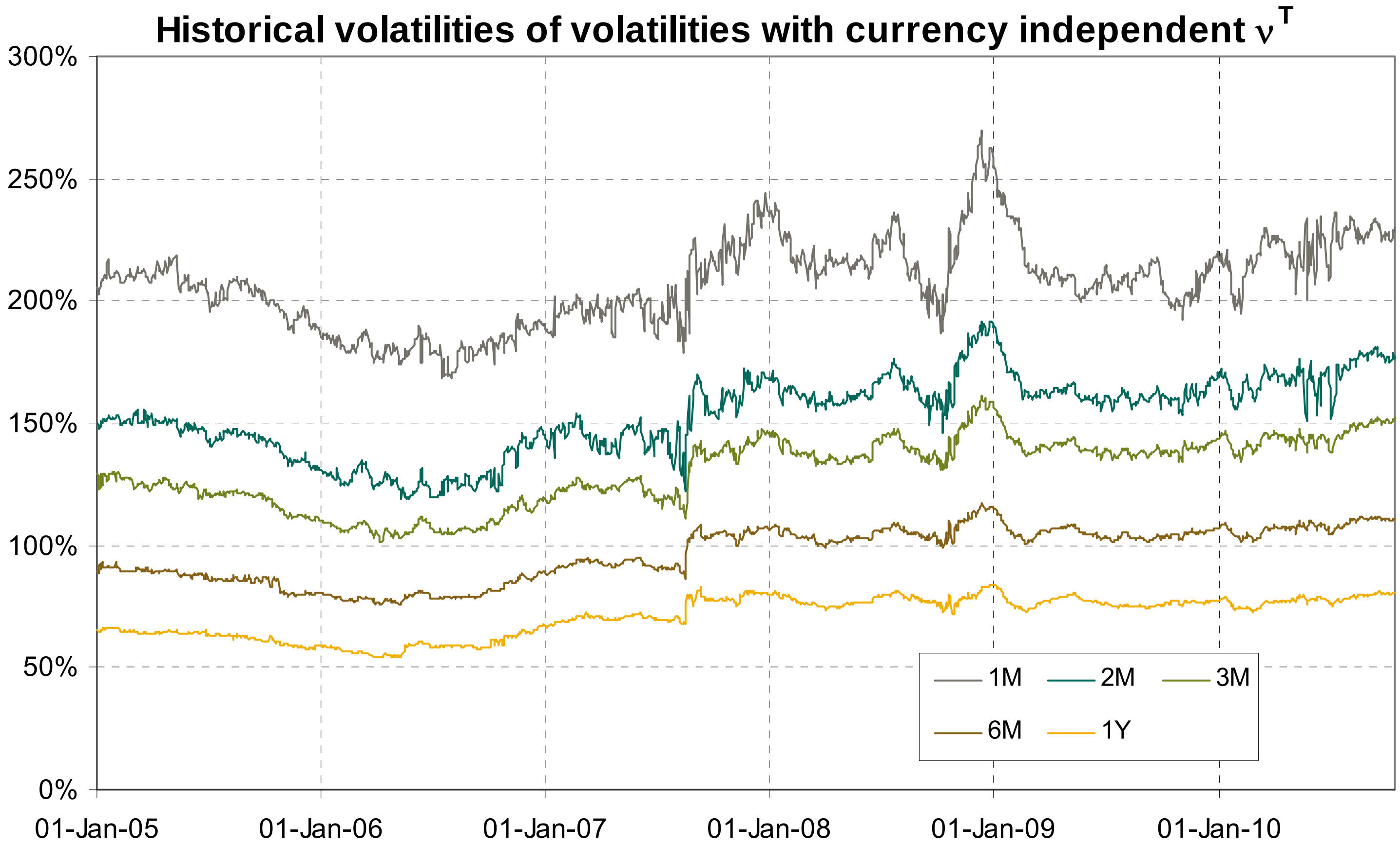
1-month intrinsic currency volatilities

- Up until the Greece crisis, EUR had been the safest currency, having the lowest volatility:



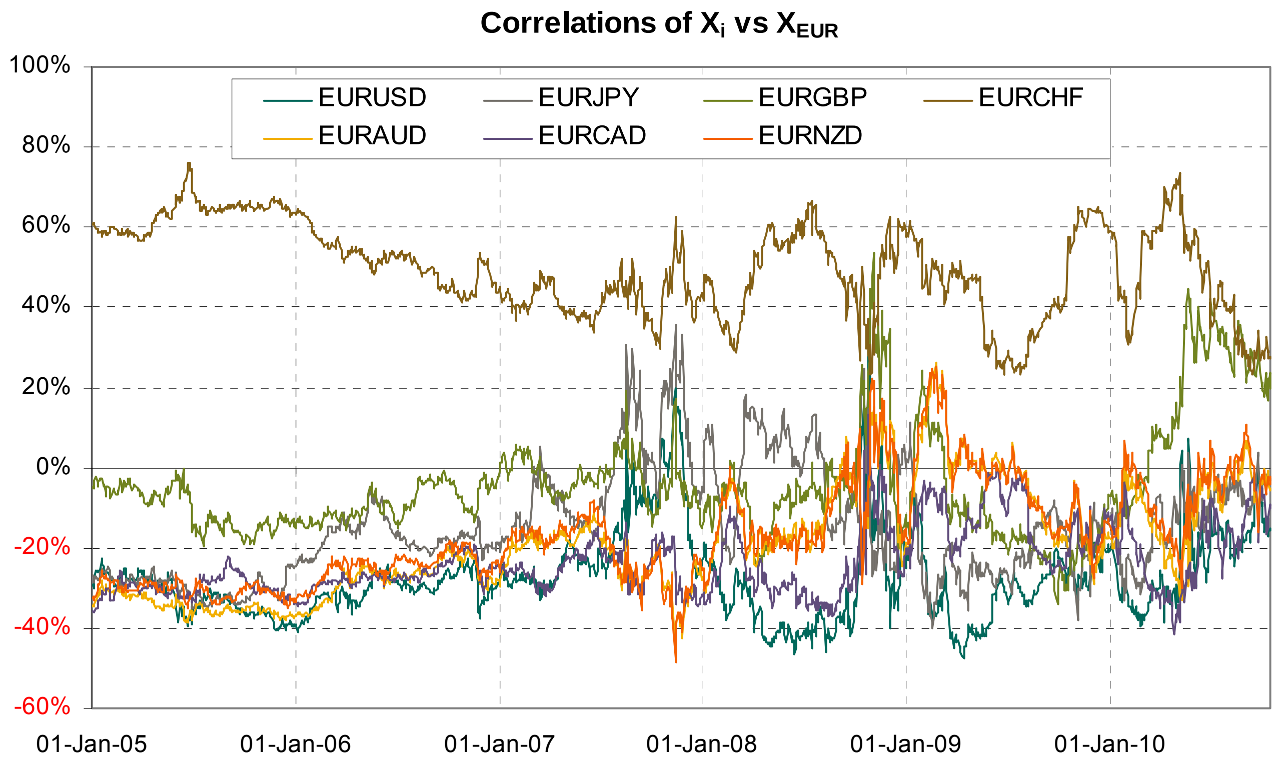
Volatility of volatilities ν^T

- The volatilities of volatilities ν^T exhibit a strong tenor dependence:



Correlations of X_i with X_{EUR}

- As with the previous work, there is a significant correlation between X_{EUR} and X_{CHF}:



Calibrating SticVol involves a lot of market data

Market data on 2-Feb-2010 (all 420 data points):

	At-the-money volatilities					Risk reversals					Market strangles				
	1M	2M	3M	6M	1Y	1M	2M	3M	6M	1Y	1M	2M	3M	6M	1Y
EUR/USD	10.4%	10.9%	11.2%	11.8%	12.2%	1.2%	1.2%	1.3%	1.3%	1.4%	0.2%	0.2%	0.3%	0.4%	0.4%
USD/JPY	12.3%	12.7%	13.0%	13.5%	14.0%	1.0%	1.4%	1.8%	2.1%	2.6%	0.2%	0.3%	0.3%	0.3%	0.2%
GBP/USD	10.8%	11.4%	11.7%	12.5%	13.0%	1.3%	1.3%	1.5%	1.7%	1.8%	0.3%	0.3%	0.3%	0.4%	0.4%
USD/CHF	9.9%	10.3%	10.6%	11.3%	11.8%	-0.5%	-0.5%	-0.6%	-0.6%	-0.6%	0.2%	0.3%	0.3%	0.4%	0.5%
AUD/USD	14.1%	14.5%	14.9%	15.3%	15.6%	2.1%	2.4%	2.7%	2.8%	3.0%	0.2%	0.3%	0.4%	0.5%	0.6%
USD/CAD	11.5%	12.2%	12.6%	13.0%	13.2%	-1.0%	-1.1%	-1.1%	-1.1%	-1.1%	0.3%	0.3%	0.4%	0.5%	0.5%
NZD/USD	14.5%	15.3%	15.6%	16.3%	16.7%	1.7%	2.2%	2.6%	3.0%	3.3%	0.2%	0.3%	0.4%	0.5%	0.6%
EUR/JPY	13.2%	13.7%	14.1%	14.8%	15.4%	1.9%	2.3%	2.7%	3.2%	3.8%	0.2%	0.2%	0.2%	0.2%	0.2%
EUR/GBP	9.2%	9.8%	10.1%	10.7%	11.0%	-0.3%	-0.5%	-0.7%	-0.8%	-0.9%	0.2%	0.3%	0.3%	0.4%	0.4%
EUR/CHF	3.9%	4.0%	4.1%	4.2%	4.4%	0.2%	0.3%	0.5%	0.8%	1.0%	0.2%	0.3%	0.3%	0.3%	0.3%
EUR/AUD	10.2%	10.8%	11.1%	11.6%	12.0%	-0.8%	-1.0%	-1.3%	-1.5%	-1.6%	0.2%	0.3%	0.4%	0.4%	0.5%
EUR/CAD	10.3%	10.7%	11.1%	11.5%	11.8%	0.3%	0.3%	0.3%	0.4%	0.4%	0.3%	0.4%	0.4%	0.5%	0.5%
EUR/NZD	10.3%	10.9%	11.3%	12.1%	12.5%	-1.0%	-1.4%	-1.7%	-2.0%	-2.2%	0.3%	0.3%	0.4%	0.5%	0.6%
GBP/JPY	14.6%	15.4%	16.0%	16.8%	17.4%	2.2%	2.6%	3.0%	3.5%	4.1%	0.2%	0.2%	0.2%	0.2%	0.2%
CHF/JPY	12.8%	13.4%	13.6%	14.1%	14.4%	1.8%	2.3%	2.4%	2.8%	3.2%	0.1%	0.1%	0.1%	0.2%	0.2%
AUD/JPY	18.8%	19.5%	20.0%	20.5%	21.5%	2.4%	3.4%	4.2%	5.1%	6.6%	0.2%	0.1%	0.1%	0.1%	-0.2%
CAD/JPY	15.7%	16.2%	16.5%	17.1%	17.7%	1.9%	2.4%	2.8%	3.3%	3.8%	0.2%	0.3%	0.3%	0.2%	0.2%
NZD/JPY	19.3%	20.0%	20.5%	21.0%	22.0%	2.4%	3.3%	4.1%	5.1%	6.6%	0.2%	0.1%	0.2%	0.1%	-0.2%
GBP/CHF	9.6%	10.3%	10.5%	11.1%	11.4%	0.6%	0.7%	0.9%	1.1%	1.2%	0.2%	0.3%	0.3%	0.3%	0.3%
GBP/AUD	12.7%	13.1%	13.3%	13.5%	13.8%	-0.7%	-0.8%	-1.1%	-1.1%	-1.2%	0.3%	0.3%	0.4%	0.5%	0.6%
GBP/CAD	11.8%	12.3%	12.6%	13.0%	13.3%	0.5%	0.5%	0.5%	0.6%	0.6%	0.3%	0.4%	0.5%	0.5%	0.6%
GBP/NZD	12.8%	13.3%	13.6%	14.2%	14.6%	-0.2%	-0.6%	-0.8%	-1.0%	-1.3%	0.3%	0.3%	0.4%	0.5%	0.6%
AUD/CHF	10.7%	11.3%	11.8%	12.8%	13.3%	0.7%	1.0%	1.2%	1.5%	1.9%	0.3%	0.3%	0.3%	0.4%	0.4%
CAD/CHF	10.5%	11.1%	11.6%	12.1%	12.5%	-0.1%	-0.1%	-0.1%	0.0%	0.1%	0.3%	0.4%	0.5%	0.5%	0.6%
NZD/CHF	10.8%	11.4%	11.8%	13.0%	13.6%	1.0%	1.3%	1.7%	2.6%	3.0%	0.3%	0.3%	0.4%	0.4%	0.4%
AUD/CAD	10.7%	11.0%	11.2%	11.4%	11.5%	1.5%	1.7%	1.9%	1.9%	2.0%	0.2%	0.2%	0.3%	0.4%	0.5%
AUD/NZD	7.6%	7.6%	7.7%	7.9%	8.1%	-0.1%	-0.1%	-0.1%	-0.1%	-0.1%	0.1%	0.1%	0.1%	0.2%	0.2%
NZD/CAD	10.8%	11.5%	11.8%	12.2%	12.5%	1.0%	1.4%	1.7%	2.0%	2.3%	0.2%	0.3%	0.4%	0.5%	0.5%

Calibrating the SticVol model

- To calibrate the model, minimise the target function $f(\sigma_i, \nu_i, \boldsymbol{\rho}, \mathbf{r}, \tilde{\rho}) = Z_1 + Z_2 + Z_3$ where
  - Z_1 is the χ^2 statistic, so that minimising Z_1 minimises the difference between the model and the market

$$\chi^2 = \sum_{\text{data points}} \left[\frac{(\text{market-value}) - (\text{model-value})}{\frac{1}{2}(\text{bid-offer spread in market})} \right]^2$$

  - Z_2 is minus the information entropy in the correlation matrix, so minimising Z_2 maximises the information entropy;
  - Z_3 is a term that encourages desirable properties in the calibration results, as discussed below.
- Use a conjugate gradient technique, defined in terms of variables which ensure that $0<\sigma_i<\sigma_{max}$, $0<\nu_i<\nu_{max}$, and that the correlation matrix (comprised of $\boldsymbol{\rho}, \mathbf{r}, \tilde{\rho}$) remains positive definite.

Imposing constraints on σ_i and ν_i

- Imposing constraints on σ_i and ν_i helps a lot because conjugate gradient schemes test "silly" values as they search for the minimum.
- Allow σ_i to be tenor dependent (i.e. σ_i^T), but for stability impose the condition that ν_i is tenor dependent but currency independent (i.e. ν^T).
- σ_{max} = 80% and ν_{max} = 500% worked well in practice, and produced reasonable fits in the turbulent markets following the Lehman default.
- Use the functions $\sigma_i^T(x_i^T)$ and $\nu^T(y^T)$ where

$$\sigma_i^T(x_i^T) = \frac{\sigma_{\max}}{1+e^{-x_i^T}} \quad , \quad \upsilon^T(y^T) = \frac{\upsilon_{\max}}{1+e^{-y^T}} \quad .$$

Keeping the correlation matrix positive definite

- Parameterise the Cholesky decomposition of the correlation matrix, i.e.

$$\begin{pmatrix} \boldsymbol{\rho} & \tilde{\boldsymbol{\rho}}' \\ \tilde{\boldsymbol{\rho}} & \mathbf{r} \end{pmatrix} = \mathbf{C}\mathbf{C}' = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ \cos\theta_{21} & \sin\theta_{21} & 0 & \cdots \\ \cos\theta_{31} & \sin\theta_{31}\cos\theta_{32} & \sin\theta_{31}\sin\theta_{32} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & \cos\theta_{21} & \cos\theta_{31} & \cdots \\ 0 & \sin\theta_{21} & \sin\theta_{31}\cos\theta_{32} & \cdots \\ 0 & 0 & \sin\theta_{31}\sin\theta_{32} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

  so that the correlation matrix is defined in terms of angles θ_{ij} which are unconstrained.

- Any scheme where the sums of squares of the rows of **C** equal 1 can be used, so that the correlation matrix has 1's down the leading diagonal. Above scheme uses hypersphere angles to satisfy that condition.

Hypersphere angle parameterisation

- If c_{ij} are the elements of the Cholesky decomposition, the hypersphere parameterisation is given by

$$c_{ij} = \begin{cases} \cos\theta_{i1} & j = 1 \\ \sin\theta_{i1}\cos\theta_{i2} & j = 2 \\ \sin\theta_{i1}\sin\theta_{i2}\cos\theta_{i3} & j = 3 \\ \ldots & \\ \sin\theta_{i1}\ \ldots\ \sin\theta_{i,i-2}\cos\theta_{i.i-1} & j = i-1 \\ \sin\theta_{i1}\ \ldots\ \sin\theta_{i,i-2}\sin\theta_{i,i-1} & j = i \\ 0 & j > i \end{cases}$$

- However the pyramid nature of this scheme means that for large i, all elements in row i depend on θ_{i1}, but only the last two depend on $\theta_{i,i-1}$. A more balanced scheme works better.

Alternative angle parameterisation

- Distribute the sin/cos terms more evenly:

$$i=2 \quad \begin{cases} c_{21} = \cos\theta_{21} \\ c_{22} = \sin\theta_{21} \end{cases} \qquad i=3 \quad \begin{cases} c_{31} = \cos\theta_{31}\cos\theta_{32} \\ c_{32} = \sin\theta_{31} \\ c_{33} = \cos\theta_{31}\sin\theta_{32} \end{cases}$$

$$i=4 \quad \begin{cases} c_{41} = \cos\theta_{41}\cos\theta_{42} \\ c_{42} = \sin\theta_{41}\cos\theta_{43} \\ c_{43} = \cos\theta_{41}\sin\theta_{42} \\ c_{44} = \sin\theta_{41}\sin\theta_{43} \end{cases} \qquad i=5 \quad \begin{cases} c_{51} = \cos\theta_{51}\cos\theta_{52}\cos\theta_{54} \\ c_{52} = \sin\theta_{51}\cos\theta_{53} \\ c_{53} = \cos\theta_{51}\sin\theta_{52} \\ c_{54} = \sin\theta_{51}\sin\theta_{53} \\ c_{55} = \cos\theta_{51}\cos\theta_{52}\sin\theta_{54} \end{cases}$$

- This scheme is more balanced because e.g. up to i=8, there are no more than 3 trigonometric functions in each c_{ij}.

Calculate the derivatives analytically

- With 8 currency pairs and 5 tenors, there are 165 variables to calibrate. To make the scheme computationally feasible, use the chain rule to calculate analytically the derivatives required for conjugate gradient, i.e.

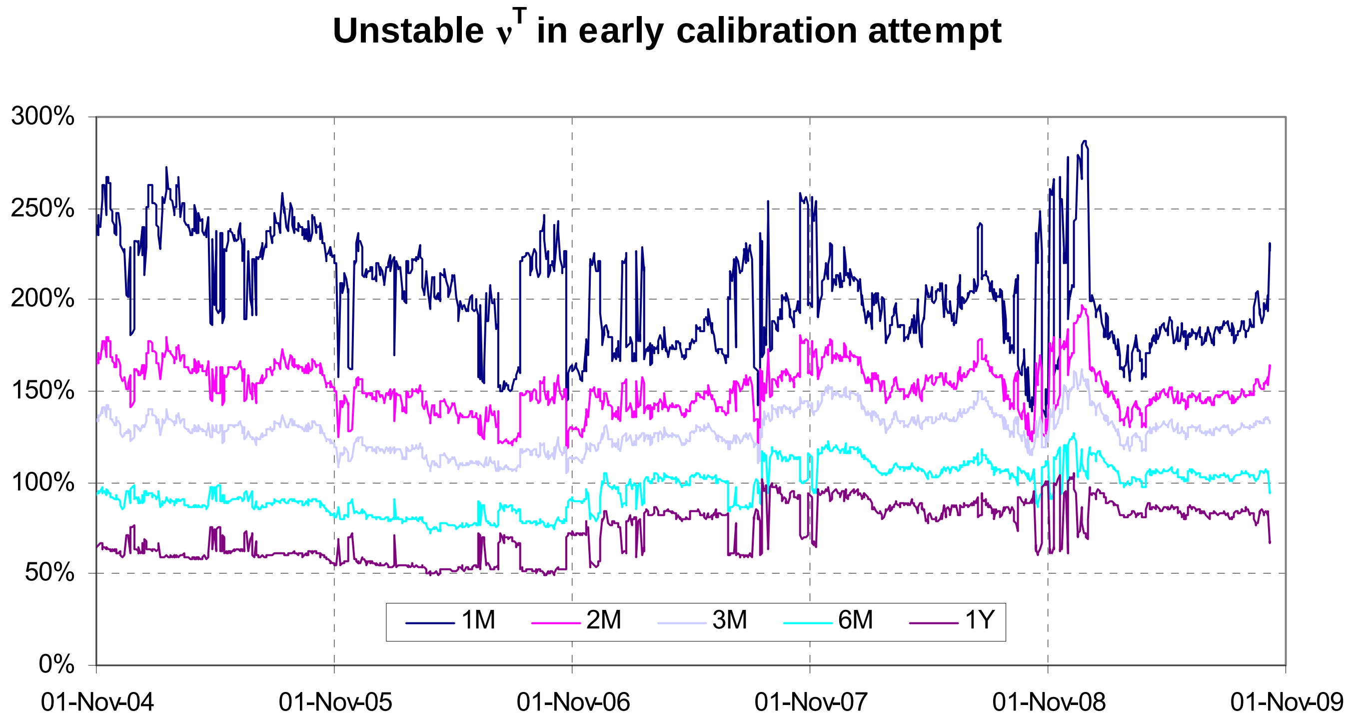
$$\frac{\partial f}{\partial x_i^T} = \frac{\partial f}{\partial \sigma_i^T} \frac{d\sigma_i^T}{dx_i^T} \quad , \quad \frac{\partial f}{\partial y^T} = \frac{\partial f}{\partial \upsilon^T} \frac{d\upsilon^T}{dy^T}$$

$$\frac{\partial f}{\partial \theta_{ij}} = \sum_{kl} \left(\frac{\partial f}{\partial \rho_{kl}} \frac{\partial \rho_{kl}}{\partial \theta_{ij}} + \frac{\partial f}{\partial r_{kl}} \frac{\partial r_{kl}}{\partial \theta_{ij}} + \frac{\partial f}{\partial \tilde{\rho}_{kl}} \frac{\partial \tilde{\rho}_{kl}}{\partial \theta_{ij}} \right) \quad \text{etc}$$

- Use the analytical SABR-style approximation for σ_B, so need to calculate all the derivatives of σ_B with respect to σ_i, ν_i , $\boldsymbol{\rho}$, $\mathbf{r}$, $\tilde{\rho}$.

Define Z_3 to make solutions more stable

- An early calibration attempt produced the following unstable results for ν^T:



Unstable ν^T related to shape of strange errors

- Between 27-Nov-06 and 28-Nov-06, (1M, 1Y) market strangle errors switch sign $(-,+) \rightarrow (+,-)$:

	27-Nov-06 calibration errors ($\nu^{1M} = 170\%$)					28-Nov-06 calibration errors ($\nu^{1M} = 220\%$)				
	1M MS	2M MS	3M MS	6M MS	1Y MS	1M MS	2M MS	3M MS	6M MS	1Y MS
EURUSD	-0.0%	-0.0%	-0.0%	0.0%	0.0%	0.0%	0.0%	-0.0%	-0.0%	-0.1%
USDJPY	-0.0%	-0.0%	0.0%	0.0%	0.0%	0.1%	0.0%	0.0%	-0.0%	-0.1%
GBPUSD	-0.0%	-0.0%	-0.0%	0.0%	0.0%	0.0%	0.0%	-0.0%	-0.0%	-0.1%
USDCHF	-0.0%	-0.0%	-0.0%	-0.0%	0.0%	0.0%	-0.0%	-0.0%	-0.1%	-0.1%
AUDUSD	-0.0%	0.0%	0.0%	0.0%	0.1%	0.1%	0.0%	0.0%	-0.0%	-0.0%
USDCAD	-0.0%	0.0%	0.0%	0.1%	0.2%	0.1%	0.0%	0.0%	-0.0%	-0.0%
NZDUSD	-0.1%	-0.0%	0.0%	0.0%	0.1%	0.0%	0.0%	-0.0%	-0.0%	-0.1%
EURJPY	-0.0%	-0.0%	-0.0%	-0.0%	0.0%	0.0%	0.0%	-0.0%	-0.0%	-0.1%
EURGBP	-0.0%	-0.0%	-0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-0.0%	-0.0%
EURCHF	-0.0%	-0.0%	-0.0%	0.0%	0.1%	0.0%	0.0%	-0.0%	-0.0%	-0.0%
EURAUD	-0.0%	-0.0%	0.0%	0.0%	0.1%	0.0%	0.0%	0.0%	-0.0%	-0.0%
EURCAD	-0.0%	0.0%	0.0%	0.0%	0.1%	0.0%	0.0%	0.0%	0.0%	0.0%
EURNZD	-0.0%	0.0%	0.1%	0.1%	0.2%	0.1%	0.1%	0.1%	0.0%	0.0%
GBPJPY	-0.0%	-0.0%	-0.0%	-0.0%	0.0%	0.0%	0.0%	0.0%	-0.0%	-0.1%
CHFJPY	-0.0%	-0.0%	-0.0%	-0.0%	0.0%	0.0%	0.0%	-0.0%	-0.1%	-0.1%
AUDJPY	-0.0%	-0.0%	-0.0%	0.0%	0.0%	0.0%	0.0%	-0.0%	-0.0%	-0.1%
CADJPY	-0.0%	-0.0%	-0.0%	0.0%	0.0%	0.1%	0.0%	0.0%	-0.0%	-0.1%
NZDJPY	-0.0%	0.0%	0.0%	0.0%	0.1%	0.1%	0.0%	0.0%	-0.0%	-0.1%
GBPCHF	-0.0%	-0.0%	-0.0%	0.0%	0.0%	0.0%	0.0%	-0.0%	-0.0%	-0.0%
GBPAUD	-0.0%	-0.0%	-0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	-0.0%	-0.0%
GBPCAD	-0.0%	-0.0%	0.0%	0.0%	0.1%	0.0%	0.0%	0.0%	0.0%	-0.0%
GBPNZD	-0.0%	0.0%	0.1%	0.1%	0.2%	0.1%	0.1%	0.0%	0.0%	-0.0%
AUDCHF	-0.1%	-0.0%	-0.0%	0.0%	0.0%	0.0%	-0.0%	-0.0%	-0.0%	-0.0%
CADCHF	-0.0%	-0.0%	-0.0%	0.0%	0.1%	0.0%	0.0%	-0.0%	0.0%	-0.0%
NZDCHF	-0.0%	0.0%	0.0%	0.1%	0.1%	0.1%	0.0%	0.0%	0.0%	-0.0%
AUDCAD	-0.0%	-0.0%	0.0%	0.0%	0.1%	0.0%	0.0%	0.0%	-0.0%	-0.0%
AUDNZD	0.0%	0.0%	0.1%	0.1%	0.2%	0.1%	0.1%	0.1%	0.1%	-0.0%
NZDCAD	-0.0%	0.0%	0.0%	0.1%	0.1%	0.1%	0.0%	0.0%	0.0%	-0.0%

Define Z_3 to make solutions more stable

- Include in Z_3 the term Z_{MS} defined by

$$Z_{MS} = \sum_{T} \left(\sum_{ij} \left([MS_{\text{Market}}]_{ij}^{T} - [MS_{\text{Model}}]_{ij}^{T} \right) \right)^2$$

  which penalises calibrations where all the market strangle errors for a particular tenor have the same sign.

- Also include a term which, for each currency pair, for the set of 5 ATM volatilities (1M, 2M, 3M, 6M and 1Y), the set of 5 Risk Reversals and the set of 5 Market Strangles, encourages some of the error terms in each set to be positive and some to be negative.

Sample results for a single day: 2-Feb-2010

Calibration errors: ModelValue – MarketValue

	Errors in at-the-money volatilities					Errors in the risk reversals					Errors in the market strangles				
	1M	2M	3M	6M	1Y	1M	2M	3M	6M	1Y	1M	2M	3M	6M	1Y
EURUSD	-0.0%	0.0%	-0.0%	-0.1%	-0.1%	-0.3%	-0.2%	-0.1%	-0.0%	-0.0%	-0.0%	-0.0%	-0.0%	-0.1%	-0.1%
USDJPY	-0.3%	-0.1%	0.0%	0.2%	0.3%	0.4%	0.4%	0.2%	0.1%	-0.2%	0.0%	0.1%	0.2%	0.3%	0.4%
GBPUSD	-0.0%	0.1%	0.2%	0.1%	0.0%	-0.2%	-0.0%	-0.1%	-0.0%	-0.1%	-0.1%	-0.0%	-0.0%	-0.1%	-0.1%
USDCHF	0.2%	0.1%	0.1%	-0.1%	-0.2%	0.0%	-0.1%	-0.1%	-0.1%	-0.1%	-0.0%	-0.0%	-0.0%	-0.0%	-0.0%
AUDUSD	-0.0%	-0.0%	-0.0%	0.0%	0.1%	-0.1%	-0.0%	-0.1%	-0.0%	-0.2%	-0.0%	-0.1%	-0.1%	-0.2%	-0.4%
USDCAD	0.1%	0.1%	-0.0%	0.1%	0.1%	0.3%	0.1%	0.1%	-0.1%	-0.2%	-0.1%	-0.0%	-0.1%	-0.1%	-0.1%
NZDUSD	0.2%	-0.4%	-0.2%	-0.1%	-0.0%	0.3%	0.1%	-0.1%	-0.2%	-0.4%	-0.0%	-0.1%	-0.1%	-0.2%	-0.4%
EURJPY	0.2%	0.0%	-0.0%	-0.2%	-0.2%	0.4%	0.4%	0.2%	-0.0%	-0.4%	0.0%	0.1%	0.1%	0.2%	0.2%
EURGBP	0.1%	-0.0%	0.0%	0.0%	0.1%	-0.2%	-0.2%	-0.0%	0.0%	0.1%	0.0%	0.0%	-0.0%	0.0%	-0.0%
EURCHF	-0.1%	-0.1%	0.1%	0.2%	0.4%	0.1%	0.1%	-0.0%	-0.2%	-0.3%	-0.1%	-0.1%	-0.1%	-0.1%	-0.1%
EURAUD	0.5%	0.1%	0.1%	-0.1%	-0.3%	-0.3%	-0.3%	-0.1%	-0.0%	0.1%	0.0%	0.0%	0.0%	-0.0%	-0.1%
EURCAD	0.1%	0.3%	0.0%	-0.1%	-0.2%	-0.1%	-0.1%	-0.1%	-0.1%	-0.0%	-0.1%	-0.1%	-0.1%	-0.1%	-0.1%
EURNZD	0.7%	-0.1%	0.1%	-0.2%	-0.1%	-0.3%	-0.1%	0.1%	0.2%	0.4%	-0.0%	-0.0%	-0.0%	-0.1%	-0.1%
GBPJPY	0.3%	0.1%	-0.1%	-0.2%	-0.1%	0.1%	0.1%	-0.1%	-0.3%	-0.7%	-0.0%	0.0%	0.0%	0.1%	0.1%
CHFJPY	0.2%	-0.2%	-0.0%	-0.1%	0.2%	0.4%	0.3%	0.4%	0.3%	0.0%	0.1%	0.2%	0.2%	0.2%	0.2%
AUDJPY	0.0%	-0.3%	-0.4%	-0.4%	-0.8%	0.9%	0.5%	-0.0%	-0.7%	-2.0%	0.0%	0.1%	0.1%	0.1%	0.4%
CADJPY	0.1%	0.3%	0.3%	0.2%	-0.0%	0.2%	0.1%	-0.0%	-0.2%	-0.5%	-0.0%	0.0%	0.1%	0.1%	0.1%
NZDJPY	0.0%	-0.7%	-0.5%	-0.3%	-0.5%	1.0%	0.5%	0.0%	-0.5%	-1.9%	0.0%	0.1%	0.1%	0.1%	0.4%
GBPCHF	0.0%	-0.3%	-0.2%	-0.3%	-0.3%	0.1%	0.1%	-0.1%	-0.1%	-0.2%	0.0%	-0.0%	-0.0%	0.0%	0.0%
GBPAUD	-0.3%	-0.1%	-0.0%	0.4%	0.3%	-0.1%	-0.1%	0.1%	0.1%	0.1%	-0.0%	0.0%	0.0%	-0.0%	-0.0%
GBPCAD	-0.2%	0.1%	-0.1%	0.2%	0.0%	-0.2%	-0.2%	-0.1%	-0.0%	-0.0%	-0.1%	-0.1%	-0.1%	-0.1%	-0.1%
GBPNZD	0.1%	-0.2%	0.1%	0.2%	0.3%	-0.6%	-0.3%	-0.2%	-0.1%	0.1%	-0.0%	0.0%	0.0%	-0.0%	-0.1%
AUDCHF	0.5%	-0.0%	-0.3%	-1.1%	-1.3%	0.5%	0.4%	0.3%	0.1%	-0.3%	-0.0%	-0.0%	-0.0%	-0.1%	-0.0%
CADCHF	0.1%	-0.0%	-0.5%	-0.7%	-1.0%	0.1%	0.1%	0.1%	-0.0%	-0.2%	-0.1%	-0.1%	-0.1%	-0.1%	-0.2%
NZDCHF	0.7%	-0.3%	-0.1%	-0.8%	-0.9%	0.4%	0.2%	-0.0%	-0.7%	-1.1%	-0.1%	-0.1%	-0.1%	-0.1%	-0.0%
AUDCAD	-0.0%	0.3%	0.4%	0.6%	0.7%	-0.2%	-0.2%	-0.2%	-0.0%	-0.1%	0.0%	0.1%	0.0%	-0.0%	-0.1%
AUDNZD	-0.2%	-0.1%	0.3%	0.5%	0.6%	0.0%	0.1%	0.0%	-0.0%	-0.1%	0.1%	0.2%	0.2%	0.3%	0.2%
NZDCAD	0.3%	-0.0%	0.2%	0.3%	0.3%	0.3%	0.1%	-0.1%	-0.1%	-0.2%	0.0%	0.0%	-0.0%	-0.1%	-0.1%

Perfect or imperfect calibration to market data?

- Perfect calibrations to market data are not possible when market data allows arbitrage
  - This happened recently in the interest rate markets, where there were arbitrages between CMS caps and CMS spread options. (NB: for mark to market, each product should be marked to its own market, but for exotics pricing a model can't be consistent with both!)
- SticVol relates implied volatility curves between different currency pairs, so the market data could contain subtle inconsistencies because the market usually doesn't look at things like that
- Forcing perfect calibrations (via e.g. tenor dependent correlation) carries the danger that the dynamics end up distorted

Extension to multiple less liquid currencies

- With the major currencies, ATM volatilities, risk reversals, market strangles are available for all combinations of currencies, but when adding e.g. BRL, only USD/BRL, EUR/BRL reliable
- To handles less liquid currencies, adopt a two stage process:
  - Calibrate the core model to all the major currencies and fix the corresponding σ_i^T, ν^T, $\boldsymbol{\rho}$, $\mathbf{r}$, $\tilde{\rho}$.
  - Extend the model by adding the required currencies using only the market data against USD and EUR
- Calibrate the much bigger correlation matrix by using a functional form which automatically imposes maximum information entropy

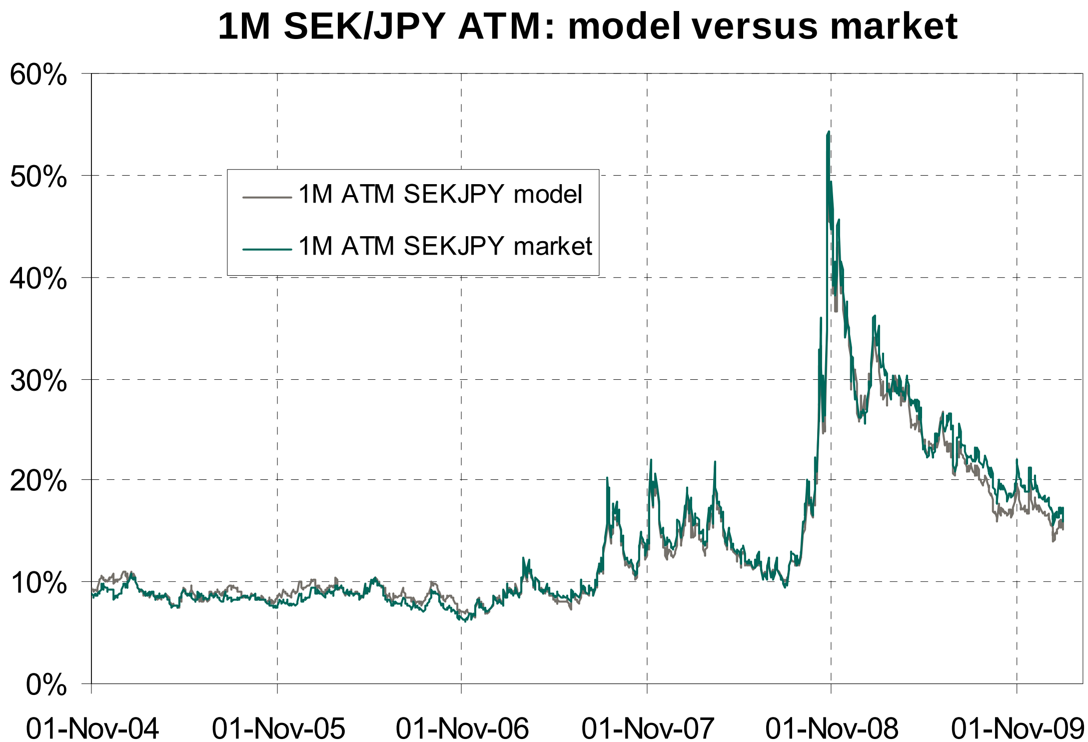
Correlation matrix imposing maximum entropy

- Re-order the correlation matrix so that $\boldsymbol{\rho}$, **r**, $\tilde{\rho}$ for the M<N major currencies are in the top left of **CC′**, with the USD and EUR variables in the 4x4 top left hand corner. Maximum entropy means maximum determinant of **C**, so since det(**C**)=$\prod_i C_{ii}$, **C** must have the form:

$$\mathbf{C}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ C_{21} & C_{22} & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ C_{41} & C_{42} & C_{43} & C_{44} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ C_{(2M)1} & C_{(2M)2} & C_{(2M)3} & C_{(2M)4} & C_{(2M)5} & \cdots & C_{(2M)(2M)} & 0 & \cdots & 0 \\ C_{(2M+1)1} & C_{(2M+1)2} & C_{(2M+1)3} & C_{(2M+1)4} & 0 & \cdots & 0 & C_{(2M+1)(2M+1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ C_{(2N)1} & C_{(2N)2} & C_{(2N)3} & C_{(2N)4} & 0 & \cdots & 0 & 0 & \cdots & C_{(2N)(2N)} \end{pmatrix}$$

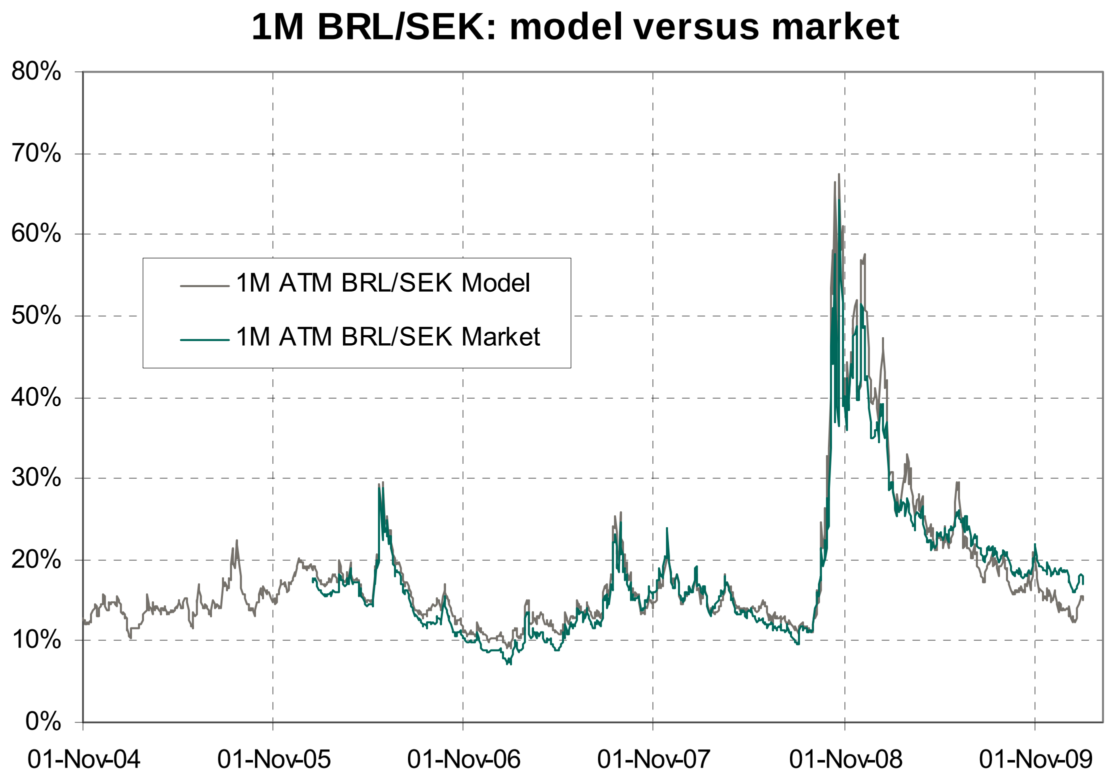
Example: calculation of 1M SEK/JPY ATM

- SticVol first calibrated to 28 currency pairs between USD, EUR, JPY, GBP, CHF, AUD, CAD, NZD, then extended to include SEK, NOK, BRL, calibrated against USD and EUR



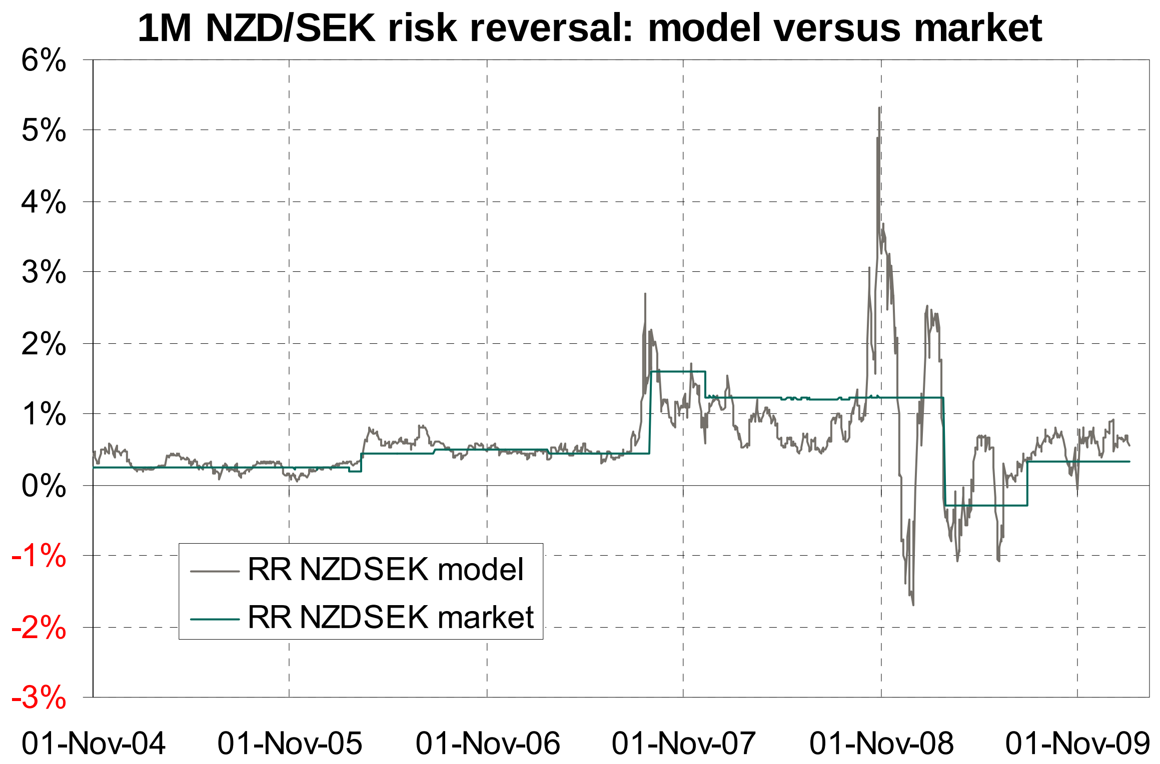
Example: calculation of 1M ATM BRL/SEK

- SticVol first calibrated to 28 currency pairs between USD, EUR, JPY, GBP, CHF, AUD, CAD, NZD, then extended to include SEK, NOK, BRL, calibrated against USD and EUR



Example: calculation of 1M NZD/SEK RR

- Comparing the model to more esoteric market data points, the match is less good, however the market data here is unreliable so the model is *probably* more accurate



Derivative pricing: FX basket options

- The advantage of SticVol is that it's a model that calibrates to the implied volatility curves of ALL currency pairs under consideration.
- Difficult to find other models which can do that reliably when dealing with a lot of currencies
  - For example, with a local vol model, the covariance matrix depends on the spot rates and extreme spot rates often end up breaking the positive definiteness
- To illustrate SticVol derivative pricing, consider the option to exchange an amount $K_\$$ of \$ for a basket of 7 other currencies with \$ value $B_\$$

$$B_\$ = A_{€} X_{€,\$} + \frac{A_{JPY}}{X_{\$,JPY}} + A_{£} X_{£,\$} + \frac{A_{CHF}}{X_{\$,CHF}} + \frac{A_{CAD}}{X_{\$,CAD}} + A_{AUD} X_{AUD,\$} + A_{NZD} X_{NZD,\$}$$

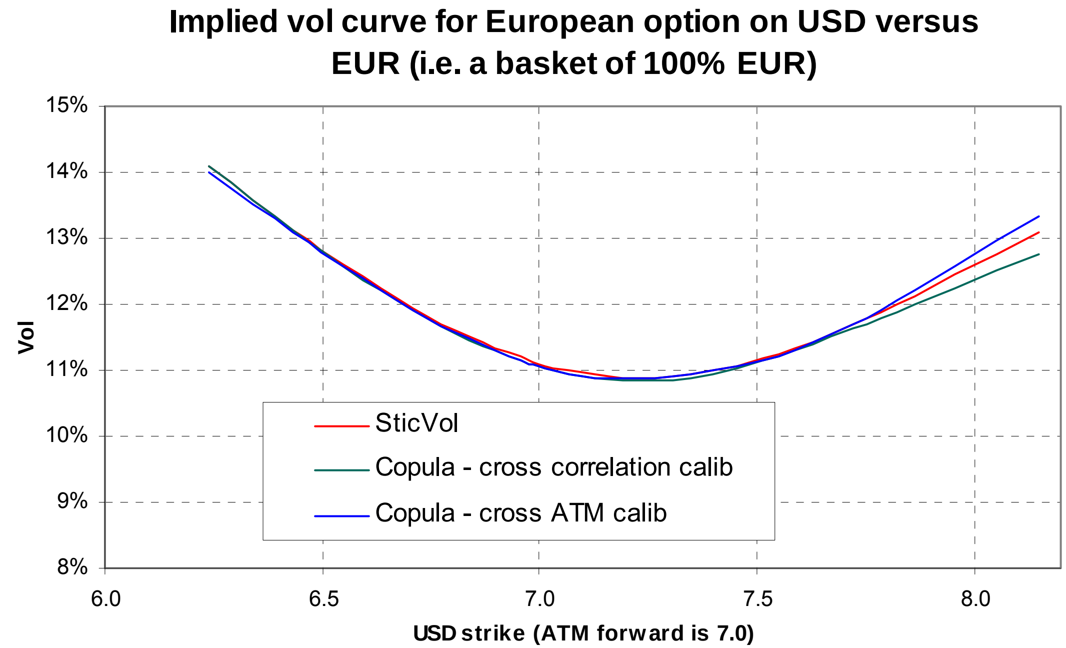
SticVol versus Gaussian copula models

- Gaussian copula models can be calibrated robustly to multiple currency pairs, although they're incapable of a calibrating to full implied volatility curves of all currency pairs.
- For the option on Max($K_\$$–$B_\$$,0), calibrate the Gaussian copula model to the full implied volatility curves for $ against each basket currency, and then determine the correlations
    1. Cross ATM calibration: so that the ATM volatility of each cross is correct, or
    2. Cross correlation calibration: using the formula

$$\rho_{XY} = \frac{\sigma_{X\$}^2 + \sigma_{Y\$}^2 - \sigma_{XY}^2}{2\sigma_{X\$}\sigma_{Y\$}}$$

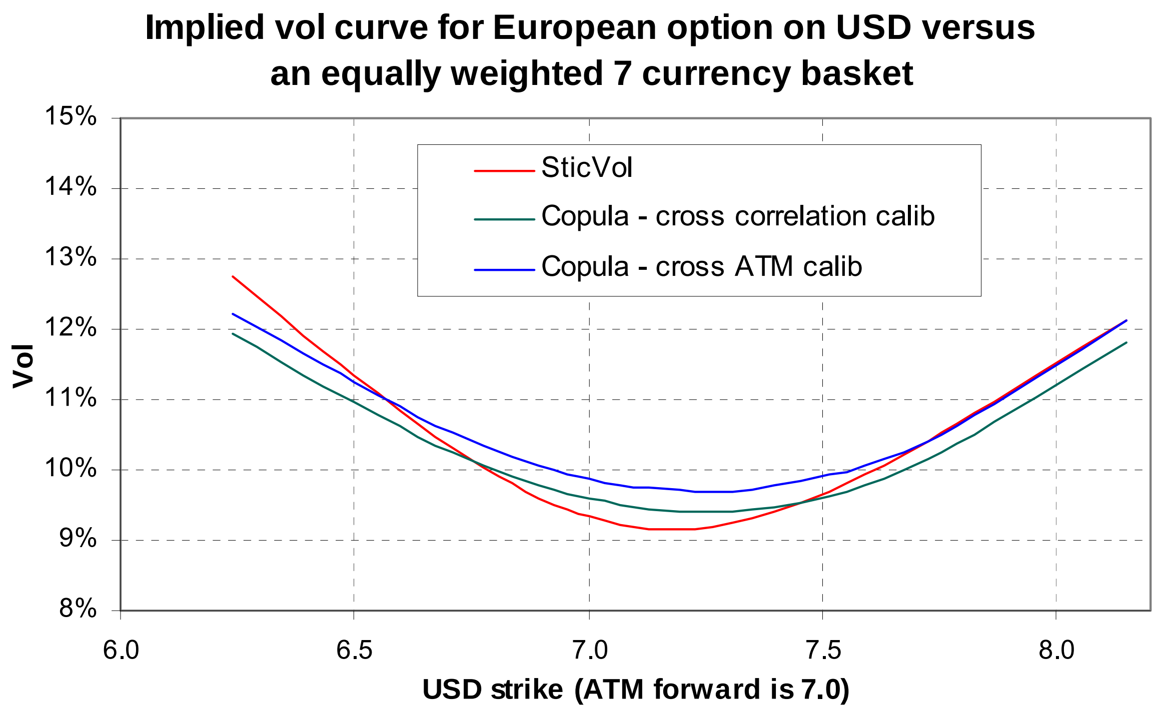
Calibration: SticVol vs Gaussian Copula

- To illustrate SticVol derivative pricing, 3-month options were considered, using Monte-Carlo simulations for pricing. For all currency pairs involving USD, the two model calibrations were very close, for example with EUR/USD:



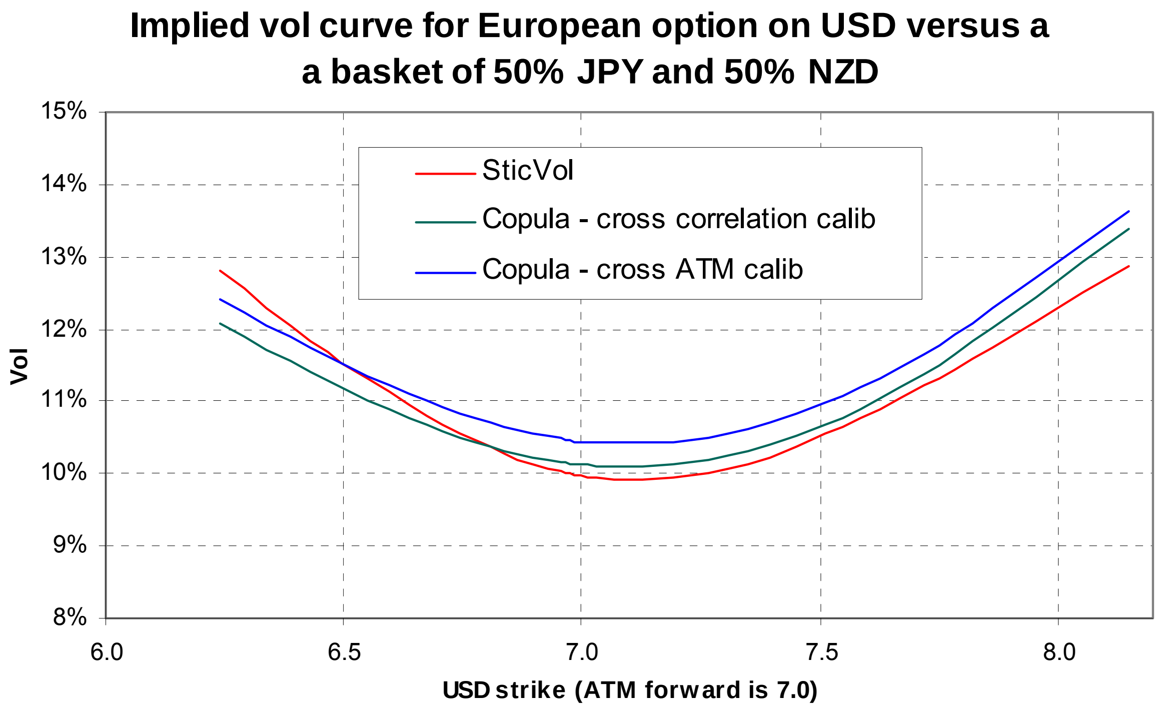
Basket option 1: SticVol vs Gaussian Copula

- All the non-USD currency amounts A_i were set so that at option expiry, the forward value in USD of all A_i are equal to the USD amount $A_\$$. In the volatility graph, strike K is in units of $A_\$$:



Basket option 2: SticVol vs Gaussian Copula

- As another example, a basket of 50% JPY and 50% NZD is chosen, because NZD/JPY has a huge risk-reversal. SticVol pricing is generally lower, except for out-of-the-month USD calls



FX correlation and covariance swaps

- These products have settlements of the form

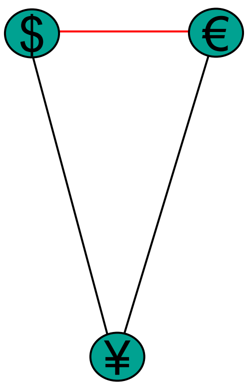
$$\text{Payment} = \text{Notional} \times (\text{Realised - Strike}) \quad ,$$

  where 'Realised' is the realised correlation or covariance over the lifetime of the product, and 'Strike' is a fixed reference correlation or covariance which is set in advance.

- Example: long correlation position in \$/¥ vs €/¥ in \$10,000 per 0.01 of correlation. Typical vega positions in the underlying currency pairs are:

\$/¥	22,355
€/¥	49,284
€/\$	-73,991

FX correlation and covariance swaps



- Taking \$/¥ vs €/¥ as an example
  - A covariance swap between \$/¥ and €/¥ which pays the fixed covariance is mostly a long position in intrinsic ¥ volatility;
  - A correlation swap between \$/¥ and €/¥ which pays the fixed correlation is a long position in intrinsic ¥ volatility coupled with smaller short positions in intrinsic \$ volatility and intrinsic € volatility
- Intuition: as ¥ vol gets bigger and bigger while \$, € vol remain constant, eventually the ¥ vol completely dominates. So for very large ¥ vol compared to \$, € vol, the correlation between \$/¥ and €/¥ will be close to 1. Hence as ¥ vol rises or falls, the covariance and correlation between \$/¥ and €/¥ must rise and fall too.

Formulas for covariance and correlation

- For two spot FX rates X_{ik} and X_{jk}

$$\text{Covariance}(X_{ik}, X_{jk}) = \sigma_k^2 + \rho_{ij}\sigma_i\sigma_j - \rho_{ik}\sigma_i\sigma_k - \rho_{jk}\sigma_j\sigma_k$$

$$\text{Correlation}(X_{ik}, X_{jk}) = \frac{\sigma_k^2 + \rho_{ij}\sigma_i\sigma_j - \rho_{ik}\sigma_i\sigma_k - \rho_{jk}\sigma_j\sigma_k}{\sqrt{\sigma_i^2 - 2\rho_{ik}\sigma_i\sigma_k + \sigma_k^2}\sqrt{\sigma_j^2 - 2\rho_{jk}\sigma_j\sigma_k + \sigma_k^2}}$$

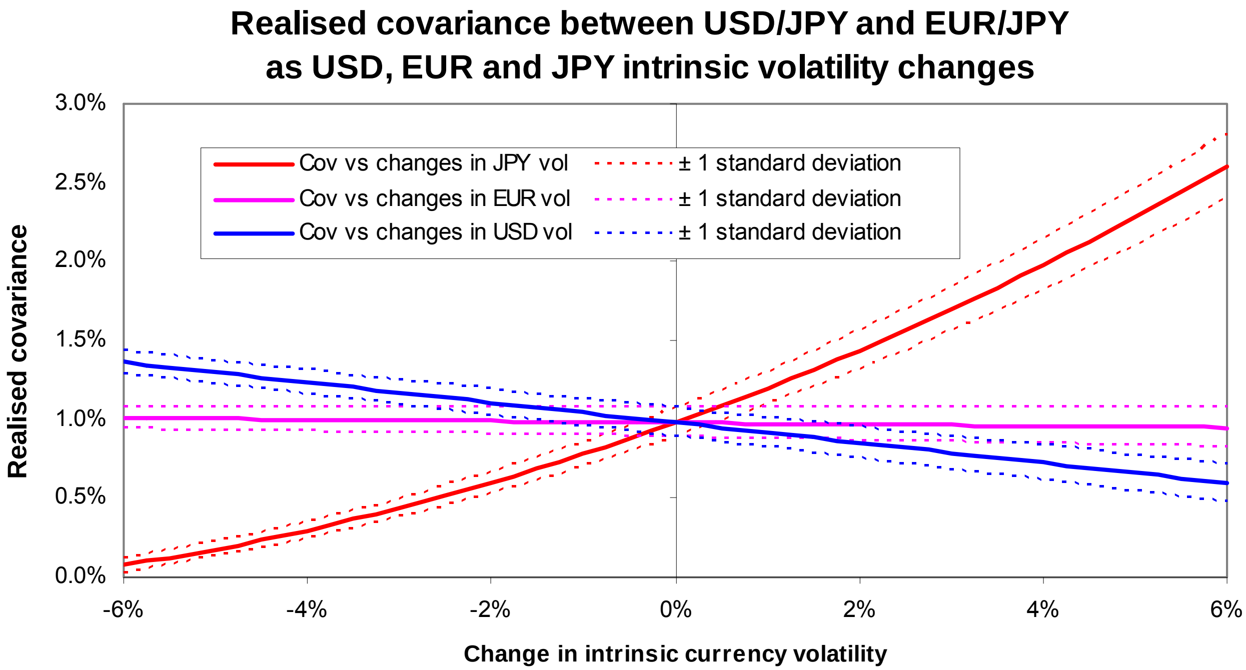
- However, intrinsic currency correlations tend to be small (because maximum possible entropy is zero correlation), so good approximations are

$$\text{Correlation}(X_{ik}, X_{jk}) \approx \sigma_k^2$$

$$\text{Correlation}(X_{ik}, X_{jk}) \approx \frac{\sigma_k^2}{\sqrt{\sigma_i^2 + \sigma_k^2}\sqrt{\sigma_j^2 + \sigma_k^2}}$$

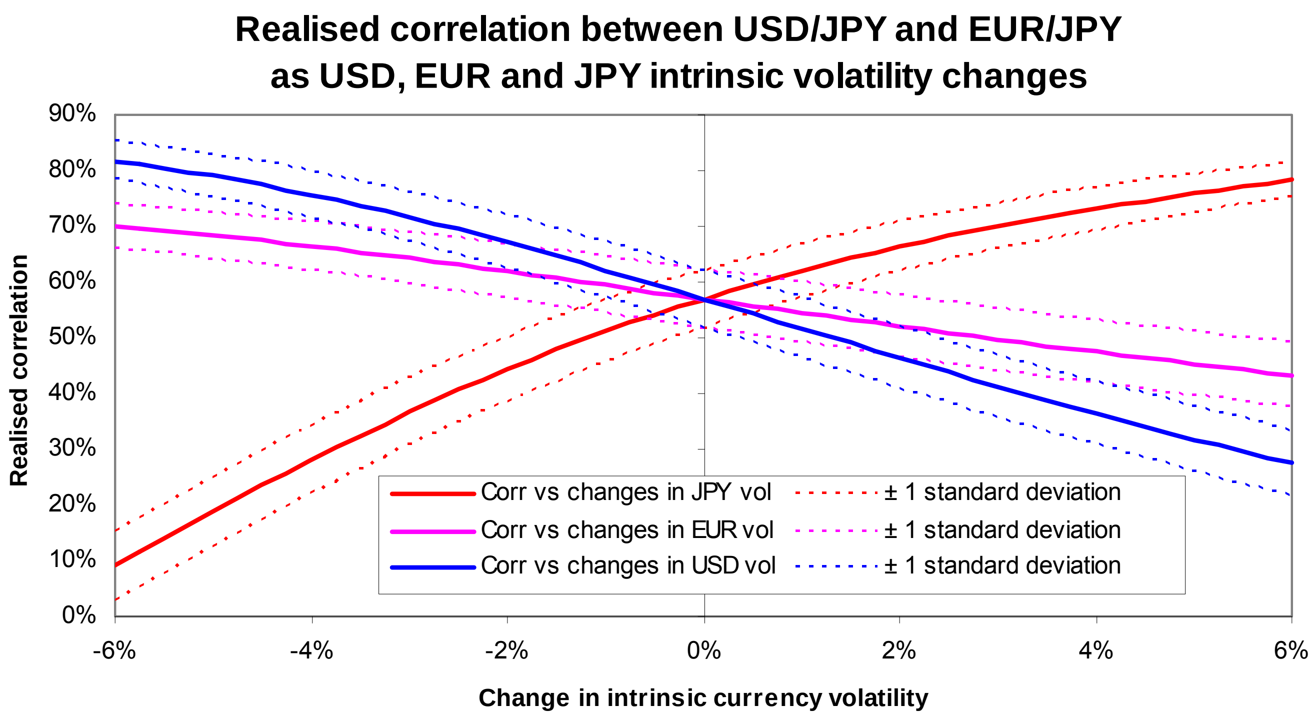
Example $/¥ vs €/¥ covariance swap

- Taking a 1y covariance swap as an example, as advertised this swap is mostly a long position in intrinsic ¥ volatility:



Example $/¥ vs €/¥ correlation swap

- Taking a 1y correlation swap as an example, as advertised this swap is a long position in intrinsic ¥ volatility coupled with smaller short positions in intrinsic $ and intrinsic € volatilities



Conclusion

- It is straightforward to calibrate SticVol using conjugate gradient techniques.
  - Calculate derivatives of the target function analytically
- SticVol can be used to calculate the implied volatility curves of less liquid currency pairs.
  - With incomplete market data, imposing maximum entropy via the Cholesky decomposition is a big help in simplifying the calibration problem
- SticVol provides a robust model for pricing multi currency FX derivatives.
- The intrinsic currency concept provides a new way of looking at correlation and covariance swaps.

Intrinsic currency references (1/5)

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  - For the pure log-normal model, this article introduces the maximum entropy principle, and dynamic bid-offers for at-the-money volatilities.
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  - Shows how to use SticVol to calculate implied volatility curves of less liquid currency pairs
- The stochastic intrinsic currency volatility framework, FX basket option pricing, Paul Doust, 18-Mar-2011
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