Pricing CMS Spread Options

- Copulae and Multi-SABR -

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Disclaimer

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Introduction

- In this presentation we wish to outline methods for approaching CMS spread option valuation and hedging issues.
- We introduce a new method for modifying the SABR density to be usable for copula pricing methods.
- Finally, we consider a spread model where the marginals are modeled by a SABR dynamic and the spread is projected to a displaced diffusion SABR dynamic.

The Market and Products

- Trading the slope of the yield curve
- Trading CMS spread volatility
- Trading CMS rates correlation
- Building blocks of Structured Products (IFRS floors and caps, Range Accruals, TARNs, ...)
- Need for hedging flows in CMS spreads and single rates

The Market Quotes - CMS

Single CMS rates are quoted as a spread over a floating rate, for instance 3M-Euribor

CMS01

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Constant Maturity Swaps

	2Y Index	5Y Index	10Y Index	20Y Index	30Y Index
5Y Swaps	65.5/71.5	120.6/129.6	167.7/177.7	170.3/190.3	144.1/169.1
10Y Swaps	53.6/59.6	92.7/101.7	129.5/139.5	103.1/123.1	75.5/100.5
15Y Swaps	45.1/51.1	74.4/83.4	96.5/106.5	61.7/81.7	40.9/65.9
20Y Swaps	39.5/45.5	63.7/72.7	78.8/98.8	44.9/64.9	26.0/56.0

Prices Quoted Q A/360 vs 3M Euribor
* Please call for swap fixing
Will Ferguson/Eric Morisset/Cristiano Cosso on +44 (0)20 7532 3050
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Figure: Market Quotes for CMS Rates as Spread quoted on Euribor (04.01.2011)

The Market Quotes - CMS Spread Options I

CMS Spread options are quoted as CMS Spread Floors and Caps for 10Y2Y, 30Y2Y or 30Y10Y.

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EUR 10/2 CMS Spread Options

	FWD	ATM	Flr -0.25	Flr -0.10	Flr 0.00	Cap 0.25	Cap 0.50	Cap 0.75	Cap 1.00	Cap 1.50
1y	1.55	26.5	0.2	0.3	0.3	99.6	80.9	62.5	44.8	15.5
2y	1.41	81.1	3.2	3.7	4.1	210.0	168.1	127.5	89.7	33.0
3y	1.26	151.0	10.1	11.6	12.7	293.6	231.0	172.1	119.7	46.7
4y	1.13	234.0	22.4	25.3	27.6	359.5	279.1	206.0	143.6	60.3
5y	1.02	321.2	37.7	42.6	46.6	415.5	319.9	235.3	165.2	73.6
7y	0.87	500.7	75.2	85.6	93.9	520.1	399.0	294.7	210.3	101.5
10y	0.75	788.0	153.6	173.1	188.3	676.2	522.9	393.4	290.1	156.9
15y	0.51	1321.5	359.3	402.2	434.6	850.2	667.9	515.7	394.8	236.8
20y	0.41	1807.2	556.9	619.2	665.5	1044.9	834.1	657.3	515.0	323.0
2x4	0.87	131.4	19.1	21.6	23.5	149.5	111.0	78.4	53.8	27.3
5x10	0.45	427.0	115.9	130.5	141.8	260.7	203.0	158.1	124.9	83.4
5X15	0.20	908.0	321.6	359.6	388.1	434.7	348.0	280.4	229.6	163.2
10x15	-0.10	456.6	205.7	229.1	246.3	174.0	145.0	122.3	104.7	79.8
10X20	-0.06	914.2	403.2	446.1	477.2	368.7	311.2	263.9	224.9	166.0
Digitals										
5y	1.02				36.7		371.4		261.6	
10y	0.75				143.2		587.1		381.8	
15y	0.51				308.8		694.8		447.2	

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EUR 30/2 CMS Spread Options

	FWD	ATM	Flr -0.25	Flr -0.10	Flr 0.00	Cap 0.25	Cap 0.50	Cap 0.75	Cap 1.00	Cap 1.50
1y	1.69	35.1	0.5	0.6	0.7	110.3	91.9	73.8	56.4	26.1
2y	1.45	110.4	5.9	7.0	7.8	222.6	182.4	144.0	108.3	50.9
3y	1.22	205.0	18.2	21.3	23.7	297.1	239.4	186.0	138.5	65.8
4y	1.01	311.5	38.5	45.1	50.2	346.4	275.7	212.2	157.3	75.7
5y	0.83	423.9	67.8	79.1	87.9	383.1	302.8	232.2	172.1	84.2
7y	0.55	658.9	152.3	176.1	194.1	440.6	347.0	266.6	199.4	102.5
10y	0.27	1005.5	320.2	365.5	399.2	510.5	404.0	314.0	239.5	132.4
15y	0.01	1539.3	624.6	704.2	762.2	633.4	511.6	409.8	326.0	205.1
20y	-0.01	1987.8	829.0	928.7	1001.3	813.1	667.1	545.5	445.5	300.3
2x4	0.60	165.0	32.6	38.1	42.4	123.7	93.3	68.3	48.9	24.9
5x10	-0.36	462.0	252.4	286.4	311.2	127.4	101.1	81.8	67.4	48.2
5X15	-0.49	927.8	556.8	625.0	674.3	250.3	208.8	177.6	153.9	120.9
10x15	-0.65	457.7	304.4	338.7	363.0	122.9	107.7	95.9	86.5	72.7
10X20	-0.40	916.9	508.8	563.2	602.1	302.6	263.2	231.6	206.0	167.9
Digitals										
5y	0.83				74.0		313.9		232.0	
10y	0.27				295.9		412.5		286.2	
15y	0.01				520.8		470.2		321.9	

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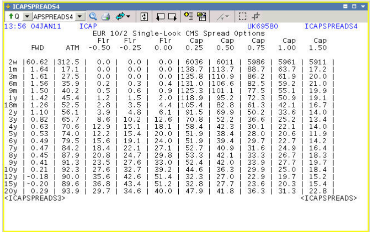
Figure: CMS Spread Floors and Caps for 10Y2Y (left) and 30Y2Y (right)

The Market Quotes - CMS Spread Options II

CMS Spread options are also quoted as Single Look CMS Spread Floors and Caps



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EUR 10/2 Single-Look CMS Spread Options

	FWD	ATM	Flr -0.50	Flr -0.25	Flr 0.00	Cap 0.25	Cap 0.50	Cap 0.75	Cap 1.00	Cap 1.50
2w	60.62	312.5	0.0	0.0	0.0	6036	6011	5986	5961	5911
1m	1.64	17.1	0.0	0.0	0.0	138.7	113.7	88.7	63.7	17.2
3m	1.61	27.5	0.0	0.0	0.0	135.8	110.9	86.2	61.9	20.0
6m	1.56	35.9	0.2	0.3	0.4	131.0	106.6	82.5	59.2	21.0
9m	1.50	40.2	0.5	0.6	0.9	125.3	101.1	77.5	55.1	19.9
1y	1.42	45.4	1.2	1.5	2.0	118.9	95.2	72.3	50.9	19.1
18m	1.26	52.5	2.8	3.5	4.4	105.4	82.8	61.3	42.1	16.7
2y	1.10	56.1	3.9	4.8	6.1	91.5	69.9	50.2	33.6	14.0
3y	0.82	65.7	8.6	10.2	12.6	70.8	52.2	36.6	25.2	13.4
4y	0.63	70.6	12.9	15.1	18.1	58.4	42.3	30.1	22.1	14.0
5y	0.53	74.0	12.2	15.4	20.0	51.9	38.4	28.0	20.6	11.9
6y	0.49	79.5	15.6	19.1	24.0	51.9	39.4	29.7	22.7	14.2
7y	0.47	84.2	18.4	22.1	27.1	52.7	40.9	31.6	24.9	16.4
8y	0.45	87.9	20.8	24.7	29.8	53.3	42.1	33.3	26.7	18.3
9y	0.41	91.3	23.5	27.6	33.0	52.4	42.0	33.9	27.7	19.7
10y	0.21	92.3	27.6	32.7	39.2	44.6	36.3	29.9	25.0	18.4
12y	-0.18	90.0	35.6	42.6	51.4	32.3	27.0	22.9	19.7	15.2
15y	-0.20	89.6	36.8	43.4	51.2	32.8	27.7	23.6	20.3	15.4
20y	0.29	93.9	29.7	34.6	40.0	47.9	41.8	36.3	31.3	22.8

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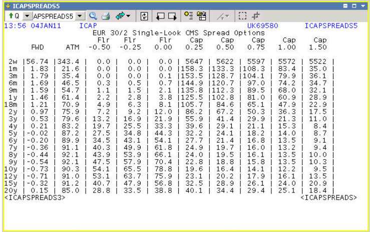

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EUR 30/2 Single-Look CMS Spread Options

	FWD	ATM	Flr -0.50	Flr -0.25	Flr 0.00	Cap 0.25	Cap 0.50	Cap 0.75	Cap 1.00	Cap 1.50
2w	56.74	343.4	0.0	0.0	0.0	5647	5622	5597	5572	5522
1m	1.83	21.6	0.0	0.0	0.0	158.3	133.3	108.3	83.4	35.0
3m	1.79	35.4	0.0	0.0	0.1	153.5	128.7	104.1	79.9	36.1
6m	1.69	46.5	0.3	0.5	0.7	144.9	120.7	97.0	74.2	34.7
9m	1.59	54.7	1.1	1.5	2.1	135.8	112.3	89.5	68.0	32.1
1y	1.46	61.4	2.2	2.8	3.8	125.5	102.8	81.0	60.9	28.9
18m	1.21	70.9	4.9	6.3	8.1	105.7	84.6	65.1	47.9	22.9
2y	0.97	75.9	7.2	9.2	12.0	86.2	67.2	50.3	36.3	17.5
3y	0.53	79.6	13.2	16.9	21.9	55.9	41.4	29.9	21.3	11.0
4y	0.21	83.2	19.7	25.5	33.3	39.6	29.1	21.1	15.3	8.4
5y	-0.02	87.2	27.5	34.8	44.3	32.2	24.1	18.2	14.0	8.7
6y	-0.20	89.9	34.5	43.1	54.1	27.7	21.4	16.8	13.5	9.1
7y	-0.36	91.1	40.3	49.9	61.8	24.9	19.7	16.0	13.2	9.4
8y	-0.44	92.1	43.9	53.9	66.1	24.0	19.5	16.1	13.5	10.0
9y	-0.54	92.1	47.5	57.9	70.4	22.8	18.8	15.8	13.5	10.3
10y	-0.73	90.3	54.1	65.5	78.8	19.6	16.4	14.1	12.2	9.5
12y	-0.71	91.0	53.1	63.7	75.9	23.1	20.2	17.9	16.1	13.5
15y	-0.32	91.2	40.7	47.9	56.8	32.5	28.9	26.1	24.0	20.9
20y	0.15	85.0	28.8	33.5	38.8	40.1	34.4	29.4	25.1	18.4

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Figure: CMS Spread Single Look for 10Y2Y (left) and 30Y2Y (right)

Hedging, Pricing and Calibration to Market Quotes

To calibrate the model we use

- CMS Caplet prices across different strikes
- Swaption prices across different strikes
- CMS prices
- CMS Spread option prices (across different strikes)

Pitfalls

- Market participants have black box solutions
- Some use simple models without fine tuning (interpolation, extrapolation)
- Use off the shelf formulae (for implied vol or copulae)
- Often such methods are the basis for pricing exotic derivatives

Copula - Definition

A copula C is a function

$$C : [0,1]^d \to [0,1]$$

having the following properties:

- C is increasing
- $C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i$ for all $i = 1, \ldots, d$
- For all $\underline{a}$, $\underline{b} \in [0,1]^d$ such that $a_i \leq b_i$ we have

$$\sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_1 + \ldots i_d} C(u_{1 i_1}, \ldots, u_{d i_d}) \geq 0$$

where $u_{j1} = a_j$ and $u_{j2} = b_j$ for all i.

Copulae and Distributions

The joint cumulative probability distribution and the density are given by:

$$F_C(x_1, \ldots, x_d) = C\left(F_{X_1}(x_1), \ldots, F_{X_d}(x_d)\right) \quad (1)$$

$$f_C(x_1, \ldots, x_d) = c\left(F_{X_1}(x_1), \ldots, F_{X_d}(x_d)\right) \prod_{i=1}^{d} f_{X_i}(x_i) \quad (2)$$

For continuous distributions the representation is unique, see Sklar, A. (1959) or Nelsen, R. B. (2006).

Copulae and Option Pricing

Let p be the payoff of an option on d assets where the joint distribution is given by the marginals F_{X_i}, the densities f_{X_i}, $i = 1, \ldots, d$ and a copula C.
The price of a European option with payoff p is given by:

$$\int_D p(x) c(F_X(x)) \prod_{i=1}^{d} f_i(x_i) dx \tag{3}$$

Pricing Formula I - Copula

Using the general pricing formula (3) we compute the price of a spread option by setting $d = 2$ and $p(x_1, x_2) = (x_1 - x_2 - K)^+$.

$$\int_0^\infty \int_0^\infty p(x_1, x_2) c\left(F_{X_1}(x_1), F_{X_2}(x_2)\right) f_{X_1}(x_1) f_{X_2}(x_2) dx_1 dx_2 \quad (4)$$

For the implementation the cumulative distributions F_{X_i} as well as the densities f_{X_i} have to be evaluated numerically. Furthermore, the integral has to be truncated to apply some numerical integration scheme.

Pricing Formula II - Copula Density

Another standard approach is to apply a transform to restrict the domain of integration to a bounded region. In this case $[0,1] \times [0,1]$. The corresponding formula is

$$\int_0^1 \int_0^1 \left(F_{X_1}^{-1}(u_1) - F_{X_2}^{-1}(u_2) - K \right)^+ c(u_1, u_2) du_1 du_2 \qquad (5)$$

For implementation the inverse cumulative distributions $F_{X_i}^{-1}$ and the densities f_{X_i} have to be evaluated numerically. For integration we can apply an adaptive integration method.

Computational Issues

- It is well known that the application (4) and (5) should rely on stable integration methods
- Numerical evaluation of the marginal densities, the cumulative density or the inverse cumulative density has to be performed and should be stable and fast
- The integration can be reduced to one dimensional integrals which is computationally much more efficient

Pricing Formula III - Copula and Marginal

Another method is to use the conditional distribution. The payoff of the spread option can be written in integral form:

$$(S_1(T) - S_2(T) - K)^+ = \int_0^\infty 1_{\{S_1(T) > x+K\}} 1_{\{S_2(T) < x\}} dx \quad (6)$$

Using (6) and taking expectations we find:

$$C(K,T) = DF(0,T) \int_0^\infty \mathbb{P}[S_1(T) > x + K, S_2(T) < x] dx \quad (7)$$

Finally, equation (7) is expressed in terms of the copula C

$$\mathbb{P}[S_1(T) > x+K, S_2(T) < x] = F_{X_2}(x+K) - C\left(F_{X_1}(x+K), F_{X_2}(x)\right) \quad (8)$$

Pricing Formula IV - Super- and Sub-Hedge

In the following we wish to apply a super- and sub-hedge argument for pricing CMS Spread options. Let $S = S_1 - S_2$ and K_1, K_2 such that $K = K_1 + K_2$. We consider the following lower and upper bound on the payoff of a spread option:

$$(S_1 - K_1)^+ - (S_2 - K_2)^+ \leq (S - K)^+ \leq (S_1 - K_1)^+ + (K_2 - S_2)^+ \quad (9)$$

Thus, it is possible to hedge a spread option by trading in the marginal underlyings S_1 and S_2. We have, taking expectations:

$$C(K_1, T) - C(K_2, T) \leq C(S, K) \leq C(K_1, T) + P(K_2, T) \quad (10)$$

Pricing Formula using the Sub-Hedge

Let us consider the mid strike $M = (K_1 - K_2)/2$ and the average strike $A = (K_1 + K_2)/2$. Using $K_1 = A + M$ and $K_2 = A - M$ we have

$$\begin{aligned}
\mathbb{E}[(S-K)^+] &= \mathbb{E}[(S_1 - K_1)^+] - \mathbb{E}[(S_2 - K_2))^+] \\
&+ \int_{-\infty}^{A} \mathbb{P}\left(S_1 > x + M, S_2 < x - M\right) dx \\
&+ \int_{A}^{\infty} \mathbb{P}\left(S_1 < x + M, S_2 > x - M\right) dx
\end{aligned}$$

Pricing Formula using the Super-Hedge

Using the same notation we find another equation for the price of the spread option.

$$
\begin{aligned}
\mathbb{E}[(S-K)^+] &= \mathbb{E}[(S_1-(A+M))^+] + \mathbb{E}[((A-M)-S_2)^+] \\
&\quad - \int_{-\infty}^{A} \mathbb{P}\left(S_1 < x+M, S_2 < x-M\right) dx \\
&\quad - \int_{A}^{\infty} \mathbb{P}\left(S_1 > x+M, S_2 > x-M\right) dx
\end{aligned}
$$

CMS Pricing and Copulae - Tying the Loose Ends

The probabilities from (7), (11) and (11)

$$\mathbb{P}\left(S_1 > x + M, S_2 < x - M\right)$$
$$\mathbb{P}\left(S_1 < x + M, S_2 > x - M\right)$$
$$\mathbb{P}\left(S_1 < x + M, S_2 < x - M\right)$$
$$\mathbb{P}\left(S_1 > x + M, S_2 > x - M\right)$$

can be expressed using the copula C. See Nelsen, R. B. (2006).

Sub-Hedge Pricing and Copulae

For the sub hedge we have, using equation (11):

$$\begin{aligned}\mathbb{E}[(S-K)^+] &= C(K_1,T;S_1) - C(K_2,T;S_2)\\ &+ \int_{-\infty}^{A} D(x-M;S_2) - C(D(x+M;S_1), D(x-M);S_2)dx\\ &+ \int_{A}^{\infty} D(x+M;S_1) - C(D(x+M;S_1), D(x-M;S_2))dx\end{aligned}$$

If we expect the assets to be positively correlated the integrals should be small!

Super-Hedge Pricing and Copulae

For the super hedge there is a corresponding formula, using equation (11):

$$
\begin{aligned}
\mathbb{E}[(S-K)^+] &= C(K_1, T; S_1) - P(K_2, T; S_2) \\
&- \int_{-\infty}^{A} C(D(x+M; S_1), D_2(x-M))dx \\
&- \int_{A}^{\infty} 1 - D(x+M; S_1) - D(x-M; S_2) \\
&\quad -C(D(x+M; S_1), D(x-M; S2))dx
\end{aligned}
$$

If we expect the assets to be negatively correlated the integrals should be small!

Modeling Approaches

- Term Structure Models
  Based on a Displaced Diffusion Stochastic Volatility Libor Market Model

$$
\begin{aligned}
dL_i(t) &= \mu_i(t)dt + \varphi_{\mathrm{DD}}(\beta_i(t), L_i(t))\sigma_i(t)\sqrt{V(t)}dW(t) \\
dV(t) &= \kappa(t)\left(\theta(t) - V(t)\right)dt + \nu(t)\sqrt{V(t)}dZ(t)
\end{aligned}
$$

  a swap rate model can be derived and has been considered by Antonov, A. and Arneguy, M. (2009), Kiesel, R. and Lutz, M. (2010) or Andersen, L. and Piterbarg, V. (2010a).
- Fixed Maturity Models
  Smile fitting using some parametric model such as SABR or SVI considered by Hagan, P.S., Kumar, D., Lesniewski A.S. and Woodward, D.E. (2002) or Gatheral, J. (2006)

We focus on the second kind of models!

Marginal Distributions

We focus (which is common practice) on the case that the marginals S_1 and S_2 are specified by a SABR dynamic given by:

$$\begin{aligned} dS_i(t) &= \alpha S_i(t)^{\beta_i} dW_i(t) \\ d\alpha_i(t) &= \nu_i \alpha_i(t) dZ_i(t) \\ S_i(0) &= S_{i,0} \\ \alpha_i(0) &= \alpha_{i,0} \\ \langle dW_i(t), dZ_i(t) \rangle &= \rho_i dt \end{aligned}$$

For European Call and Put option prices there are no closed form solutions available. However, Hagan, P.S., Kumar, D., Lesniewski A.S. and Woodward, D.E. (2002) propose an approximation formula for the implied Black volatility.

Problems with SABR

- Computing the Convexity Adjustment when applying replication of CMS rates sometimes is an increasing function with respect to strike
- For the application of copula methods the probability distribution is necessary
- Extrapolation for very low and very high strikes is necessary
- Generally the SABR approximation formula

$$\sigma \approx \frac{-\nu \log\left(\frac{K}{S_0}\right)\left(1 + T\left(\frac{(\beta-1)^2}{24}\frac{\alpha_0^2}{(S_0K)^{1-\beta}} + \frac{1}{4}\frac{\rho\nu\alpha_0\beta}{(S_0K)^{1-\beta}} + \frac{2-3\rho^2}{24}\nu^2\right)\right)}{\log\left(\frac{\sqrt{1-2\rho+q^2}+q-\rho}{1-\rho}\right)} \tag{11}$$

$$q = \frac{\nu}{\alpha_0}\frac{S_0^{1-\beta} - K^{1-\beta}}{1-\beta}$$

might imply too high volatilities for high strikes and for low strikes arbitrage might be possible since the density calculated from (11) can get negative.

Possible Solutions

There are some solutions for this problem:

- Constant extrapolation
- Regimes of volatility of volatility
- Price extrapolation
- Density extrapolation

Constant Extrapolation

Some time it has been market practice to cut off the smile for low and high value and use constant extrapolation. This method can be formalized as follows. Let us fix strikes k_l and k_u and we assume that the approximation formula is valid for $k \in (k_l \leq k \leq k_u)$.

$$\tilde{\sigma}(k) := \begin{cases} \sigma(k_l) & , \quad k < k_l \\ \sigma(k) & , \quad k_l \leq k \leq k_u \\ \sigma(k_u) & , \quad k > k_u \end{cases} \tag{12}$$

Regimes of Volatility

Since constant extrapolation is not an appropriate solution other methods have been used by practitioners. One is the choice of general volatility regimes. In the simplest case we might choose:

$$\tilde{\sigma}(k) := \begin{cases} \sigma(k) \cdot f_1(k) & , \quad k < k_l \\ \sigma(k) & , \quad k_l \leq k \leq k_u \\ \sigma(k) \cdot f_2(k) & , \quad k > k_u \end{cases} \tag{13}$$

Possible choice of the functions f_1 and f_2 could be

$$f_n(k) = (1 + a|k - k_l|)^{-1}, n = 1, 2 f_n(k) = (1 + a(k - k_l)^2)^{-1}, n = 1, 2$$

The constant extrapolation is a special case if we choose $f_1(k) = \sigma(k_l)$ and $f_2(k) = \sigma(k_u)$.

Price Extrapolation

Extrapolating the price has been suggested in Benaim, S., Dodgson, M., and Kainth, D. (2010). We consider

$$\tilde{C}(k) := \begin{cases} C_1(k;\alpha) & , \quad k < k_l \\ C(k,\sigma(k)) & , \quad k_l \le k \le k_u \\ C_2(k;\beta) & , \quad k > k_u \end{cases} \tag{14}$$

with $\alpha = \{\alpha_1, \ldots, \alpha_{d_1}\}$ and $\beta = \{\beta_1, \ldots, \beta_{d_2}\}$ some parameter sets and C_1 and C_2 are functions. The parameters can be chosen such that the prices fit smoothly at k_l and k_u and the tails can be controlled via choosing C_1 and C_2 appropriately.
Negative densities can be avoided most of the time and for some parameter sets the admissible range k_l, k_{ATM} is very small.

Density Extrapolation - Proposed Solution

We consider the following approach to extrapolation. With respect to the difficulties of the approaches discussed so far we propose to extrapolate the density. This allows for positive densities and - with some numerical effort - to match the observed prices within the observed bid-ask spread.

$$p(x) = x^{\mu_l} \exp(a_l + b_l x + c_l x^2) \tag{15}$$

$$p(x) = x^{\mu_u} \exp(a_u + b_u x^{-1} + c_u x^{-2}) \tag{16}$$

This representation allows to smoothly fit the modified density at the cut off points k_l and k_u. Furthermore, it is possible to directly control the tail behavior using the constants $\mu_l > 0$ and $\mu_u < 0$.

Density Extrapolation - Parameters and Implementation

- The choice of k_l and k_u can be used to more closely fit option prices
- The parameters μ_l and μ_u can be used to control the tails
- The parameters a_l, b_l and c_l as well as a_u, b_u and c_u can be applied to smoothly fit the density
- Option prices have to be computed numerically
- F_X and F_X^{-1} can be computed numerically and efficiently
- Can be directly applied for copula pricing

Density Extrapolation - Example

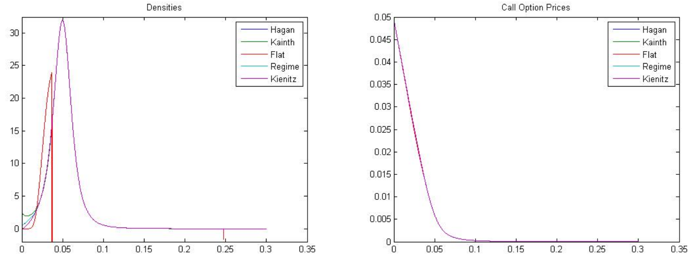


Figure: We have plotted the densities and the call option prices using different methods. The parameters are $\alpha = 0.1339 f^{1-\beta}$, $\beta = 0.5$, $\nu = 0.3843$, $\rho = -0.1595$ and $f = 0.0495$

Density Extrapolation - Example

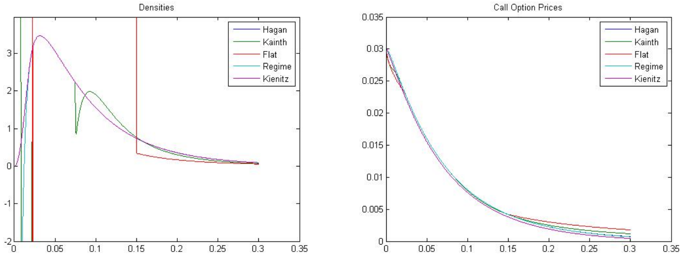


Figure: We have plotted the densities and the call option prices using different methods. The parameters are $\alpha = 0.9f^{1-\beta}$, $\beta = 0.5$, $\nu = 0.2$, $\rho = -0.2$ and $f = 0.03$

Density Extrapolation - Example

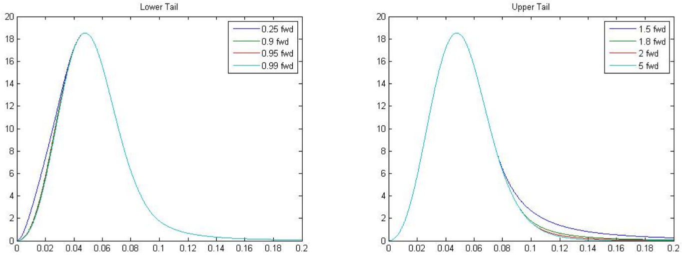


Figure: Using the attachment points as multiple of the forward to construct the distribution for the lower (left) and the upper (right) tail

Density Extrapolation - Example

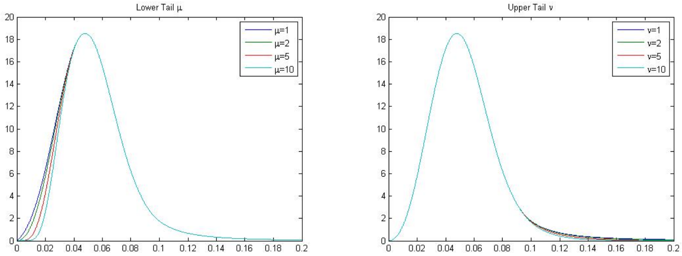


Figure: Using the decay parameters μ_l (left) and μ_u (right) for controlling the tails

Further Questions

- How to calibrate μ_l and μ_u?
- There are SABR parameters which are pathologic that no construction of a nice density is possible

And the Answers

The answers to the questions raised above are given in Dirkmeier, M. and Kienitz, J. (2011)

- There are far out of the money quotes for swaptions which help to determine the right tail. Via Call-Put parity we can handle the left tail
- To find the admissible region we do not use SABR but the Vanna-Volga approach to pricing

Application to Monte Carlo Simulation

- Since the probability density can be easily calculated it is also suitable for Monte Carlo applications
- The inverse of the cumulative density can be evaluated very fast
- We compared the results to Chen, B., Oosterlee, C.W. and van der Weide, H. (2010) and find that our method leads to comparable results but are faster since we can apply one step sampling
- Using this approach it can be applied for LMM-SABR simulation since it enables long time stepping

Lower and Upper Bounds from Copulae

CMS spread option prices are bounded by choosing special 'copulae'

- Frêchet-Hoeffding Lower Bound

$$C_l(u_1, u_2) = (u_1 + u_2 - 1)^+$$

- Frêchet-Hoeffding Upper Bound

$$C_u(u_1, u_2) = \min_{i=1,2} u_i$$

For any copula C we have

$$C_l(u_1, u_2) \leq C(u_1, u_2) \leq C_u(u_1, u_2)$$

Lower and Upper Bounds from Copulae - Example

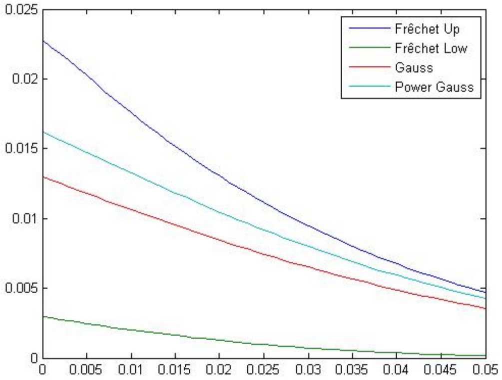


Figure: Pricing the CMS Spread Option using Different Copulae

Optimal Lower Bound

- Lower bounds can be derived by choosing the Frêchet lower bound as the copula
- By considering (8) we can also derive a lower bound involving the marginals
- We get optimal values for strikes if we consider the digitals appearing in the integrals
- McLoud, P. (2011) derives an optimal lower bound by considering (11)
  In general we get admissible domains $I = \cup I_i$ with the endpoints of each Interval $I_i = [x_i, x^i]$ corresponding to a local max and a local min.

Optimal Upper Bound

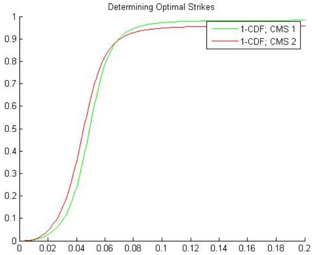
- Upper bounds can be derived by choosing the Frêchet upper bound as the copula
- By considering (8) we can also derive an upper bound involving the marginals
- We get optimal values for strikes if we consider the digitals appearing in the integrals
- McLoud, P. (2011) derives an optimal upper bound by considering (11)
  In general it is possible to identify the strike x_0 leading to a minimum of the upper bound which then is the optimal upper bound

Optimal Bounds from Hedging Approach

- It is possible to compute the optimal values considering intersections of the cumulative distributions of the marginals
- Our approach for the stable calculation of the density can be applied to identify the optimal values
- The numerical integration procedure is stable and fast

Lower and Upper Bounds from Hedge - Example

For the numerical example we use a mid strike of 0.01.



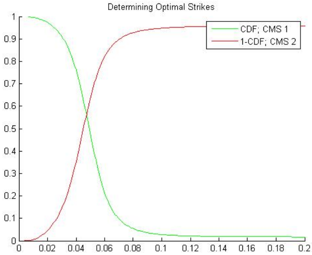


Figure: Computing Optimal Lower and Upper Bounds - $k_{\text{low},1} = 0.0096$, $k_{\text{low},2} = 0.0675$ and $k_{\text{up}} = 0.0479$

Inequalities - Sub and Superhedge

Using the sub and the superhedge we find for the CMS spread option price $V(T)$:

$$
\begin{aligned}
V(T) &\leq C(k_{\text{up}} + M; S_1) + P(k_{\text{up}} - M; S_2) \\
V(T) &\geq \Big(C(k_{\text{low},2} + M; S_1) - C(k_{\text{low},1} - M; S_2)\Big) \\
&\quad + \Big(P(k_{\text{low},1} - M; S_2) - P(k_{\text{low},2} + M; S_1)\Big)
\end{aligned}
$$

Copula Choice

Despite calculating the lower and upper bounds for the spread option price using the Frêchet-Hoeffding bounds there are many choices for the dependency structure in terms of copulae.
Suggested copulae are

- Symmetric Copulae (Gaussian Copula)
- Skewed Copulae (Power Gaussian Copula, Skewed Normal Copula)

For calibrating to market prices the Power Gauss Copula and Skewed copulae are useful.

The Power Gauss Copula

For $\theta_1, \theta_2 \in [0,1]$ and denoting by C_{G} the Gaussian copula with correlation $\rho \in (0,1)$ the Power Gauss copula, see Andersen, L. and Piterbarg, V. (2010b) is given by:

$$C_{\text{PG}}(u_1, u_2) = u_1^{1-\theta_1} u_2^{1-\theta_2} C_{\text{G}}(u_1^{\theta_1}, u_2^{\theta_2})$$

The correlation ρ is used to move the height of the implied volatility smile whereas the parameters θ_1 and θ_2 control the slope and the curvature of the smile. Choosing $\theta_1 = \theta_2 = 1$ leads to the (symmetric) Gaussian copula.

Other Skewed Copulae

- In Kainth, D. (2010) the application of the skewed normal copula and the skewed t-copula has been proposed
- These copula are related to the skewed normal distribution and skewed t-distribution.
- It is argued that the skewed t-copula gives better fit to market observed data than the skewed normal copula
- In general asymmetric copulae give much better fit than symmetric ones.

Conclusions and Summary

- CMS spread pricing using copulae needs the density and the cumulative distribution of the marginals or the inverse cumulative distribution
- We suggested a stable method to compute these which is numerically cheap
- We identified lower and upper bounds for the CMS spread option prices
- We applied sub and super hedges for hedging using simple rates options
- As a by product we derived a method for Monte Carlo simulation of a SABR model

Two-Dimensional SABR Model

Another way to model the spread is to use the marginals and derive some equation for the spread. To this end consider $S_i(t)$, $i = 1, 2$ is modeled by

$$\begin{aligned}
dS_i(t) &= \alpha_i(t) S_i(t)^{\beta_i} dW_i(t) \\
d\alpha_i(t) &= \nu_i \alpha_i(t) dZ_i(t) \\
S_i(0) &= s_i^0 \\
\alpha_i(0) &= \alpha_i^0 \\
\langle dW_1(t), dW_2(t) \rangle &= \rho dt \\
\langle dW_i(t), dZ_j(t) \rangle &= \gamma_{ij} dt \\
\langle dZ_1(t), dZ_2(t) \rangle &= \xi dt. \qquad (17)
\end{aligned}$$

ρ_{ij} is the correlation, γ_{ij} the cross-skew and ξ_{ij} is the decorrelation between the stochastic volatilities.

Definition - Displaced Diffusion SABR Model

To apply the technique of Markovian projection, Piterbarg (2006), Antonov and Audet (2008) or Antonov, A. and Misirpashaev, T. (2009), we follow the approach in Kienitz, J. and Wittke, M. (2010) and propose to project the original model to a displaced SABR diffusion with $\beta = 1$.

$$\begin{aligned} dS(t) &= \alpha(t)F(S(t))dW(t) \qquad (18) \\ d\alpha(t) &= \nu\alpha(t)dZ(t) \\ \langle dW(t), dZ(t)\rangle &= \gamma dt \\ F(S(t)) &= p + q(S(t) - S(0)) \\ p &= F(S(0)) \\ q &= F'(S(0)) \end{aligned}$$

This choice allows negative realizations to some extend and is therefore a reasonable choice.

Pricing Formula for Caps

Suppose we have the SDE (18) and let the implied SABR volatility function be σ_{SABR}. The solution of the projected SDE can be evaluated in closed form using the pricing formulae for displaced diffusions, for instance for a caplet on the CMS spread:

$$\begin{aligned}
\text{CMScaplet} &= B(0,T)E_T[S(T)-K]^+ \\
&= B(0,T)E_T[S(T)+a-(K+a)]^+ \\
&= B(0,T)\{(E_{P^T}[S(T)]+a)N(d1)-(K+a)N(d2)\}
\end{aligned}$$

$$d_{1/2} = \frac{\ln\left(\frac{E_{P^T}[S(T)]+a}{K+a}\right) \pm \frac{1}{2}\sigma_{\text{SABR}}^2 T}{\sigma_{\text{SABR}}\sqrt{T}}$$

The volatility σ_{SABR} is given by an approximation formula for a standard SABR model.

The Projected Diffusion

We find for the parameters in (18):

$$
\begin{aligned}
p &= \sqrt{p_1^2 + p_2^2 - 2p_1 p_2 \rho} \\
q &= \frac{p_1 q_1 \rho_1^2 - p_2 q_2 \rho_2^2}{p} \\
\eta &= \sqrt{(p_1\nu_1\rho_1)^2 + (p_2\nu_2\rho_2)^2 - 2\xi_{12} p_1\nu_1\rho_1 p_2\nu_2\rho_2} / p \\
\gamma &= \frac{1}{\eta p^2}\left(p_1^2\nu_1\rho_1\gamma_{11} + p_2^2\nu_2\rho_2\gamma_{22} - p_1p_2\nu_2\rho_2\gamma_{21} - p_1p_2\nu_1\rho_1\gamma_{12}\right) \\
s_0 &= S_1(0) - S_2(0) \\
p_1 &= \alpha_1 S_1^{\beta_1}; p_2 = \alpha_2 S_2^{\beta_2} \\
q_1 &= \beta_1\alpha_1 S_1^{(\beta_1-1)}; q_2 = \beta_2\alpha_2 S_2^{(\beta_2-1)} \\
\rho_1 &= \frac{p_1 - p_2\rho}{p}; \rho_2 = \frac{p_1\rho - p_2}{p}
\end{aligned}
$$

Performance of the Method - $T = 1$

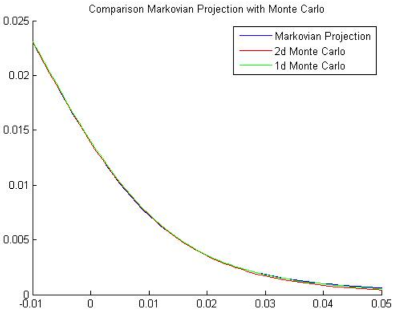


Figure: Comparison of the Approximation with Simulation for $T = 1$

Performance of the Method - $T = 5$

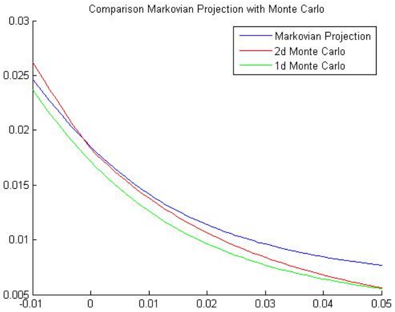


Figure: Comparison of the Approximation with Simulation for $T = 5$

Performance of the Method - $T = 10$

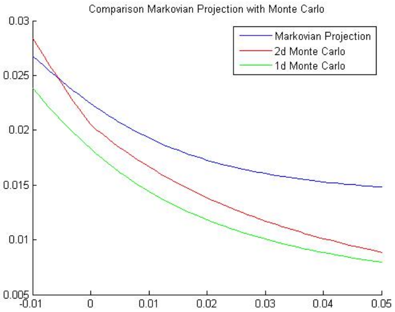


Figure: Comparison of the Approximation with Simulation for $T = 10$

Controlling the Smile

- Kienitz, J. and Wittke, M. (2010) show that the parameters γ_{12}, γ_{21}, ξ and ρ can be chosen to model dependency.
- Ideally these can be fitted to the observed CMS spread smile.
- We show how the parameters effect the prices, resp. the smile

Changing γ_{12} and γ_{21}

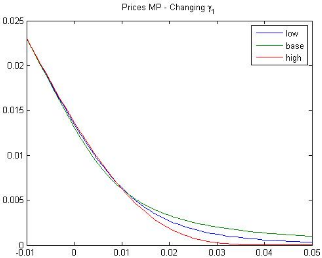


Figure: Changing the parameters γ_{12} and γ_{21} - Effect on Price

Changing γ_{12} and γ_{21}

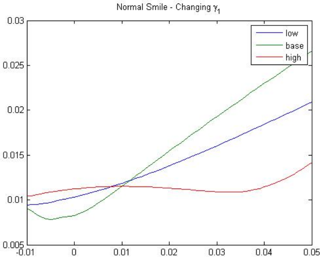


Figure: Changing the parameters γ_{12} and γ_{21} - Effect on Volatility

Changing ξ

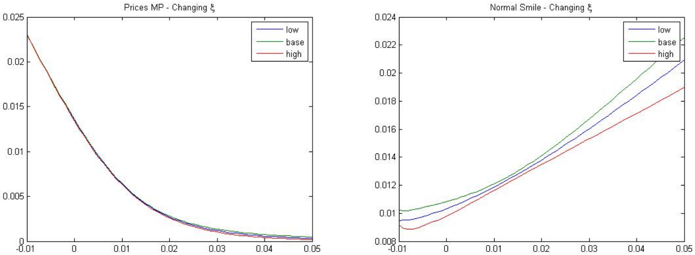


Figure: Changing the parameters γ_{12} and γ_{21} - Effect on Price and Volatility

Changing ρ

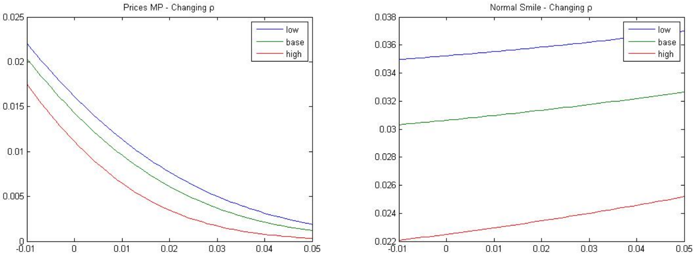


Figure: Changing the parameters γ_{12} and γ_{21} - Effect on Price and Volatility

Conclusions and Summary

- We outlined the use of a 2d SABR model for pricing CMS spread options
- We derived a easy to use pricing formula which is suitable for calibration
- We showed that the approach is applicable for reasonable small times (up to $T = 5$)
- We showed how the additional parameters in a 2d model can be used to control the smile

Overall Conclusions and Summary

- The SABR model has to be handled with care
- CMS Spread option pricing is a complex task (performance, stability, choice of copula)
- Methods from general stochastic volatility modeling is feasible for SABR (Markovian Projection, Parameter Averaging)
- The performance is well for short dated options but other models show more stable results

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