Pricing Long-Dated Variable Annuities

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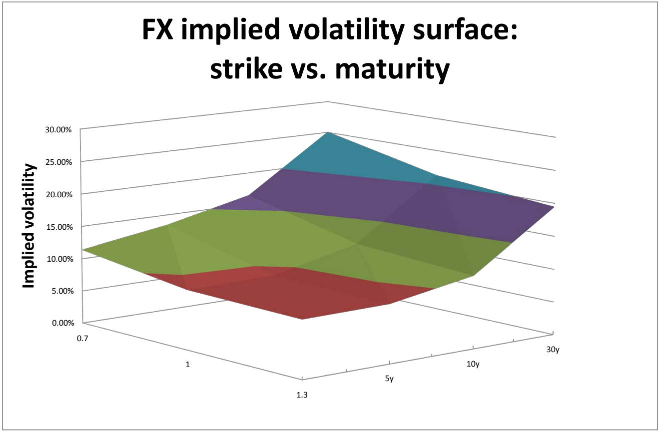
Long-Dated Variable Annuities - Agenda

- Long-term option pricing.
- Need for stochastic volatility, rates and correlation.
- Modelling framework.
- Example I: Equity-Interest Rate Annuities.
- Example II: Inflation-Linked Annuities.
- Conclusion.

- A 'typical' long-maturity implied volatility smile:



- Increasing term structure of implied volatility.
- Pronounced implied volatility smile/skew.
- Smile/skew does not die out for long maturities.

- For the pricing/risk management of long dated exotic structures we need dynamic model capturing these observations, see Piterbarg (2005), Brigo and Mercurio (2006), Andreasen (2007) and many others.
- The increasing term structure of volatility can be explained by the addition of stochastic interest rates. For instance, with FX, the instantaneous forward volatility is given by:

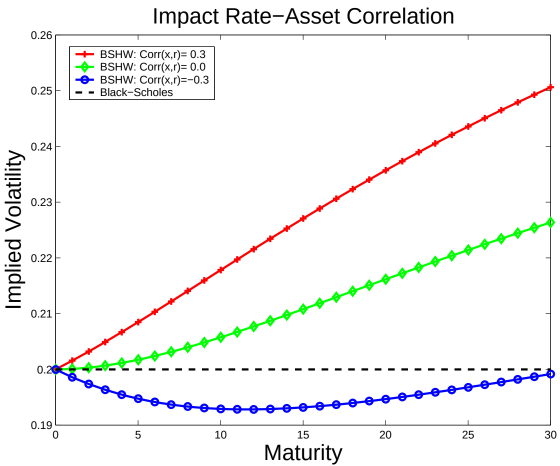
$$\begin{aligned}\sigma_F(t,T) &= ||\sigma(t,T) + \sigma_d(t,T) - \sigma_f(t,T)|| \\ &= ||O(1) + O(T-t) - O(T-t)|| \\ &= O(T-t).\end{aligned}$$

- A risk-neutral explanation for the pronounced smile/skew can be given by stochastic volatility which is correlated with the underlying index.

- Variable Annuities are typically long-dated structures, depending on multiple underlying assets.
- For pricing and hedging we consider a stochastic volatility model with stochastic interest rates under a full correlation structure and with closed-form pricing formulas for vanilla options.
- Having a realistic correlation structure is of practical importance for the pricing and hedging of 'exotic' options with a long-term exposure.
- Closed-form pricing formulas are a big advantage for model calibration and annuity sensitivity analysis.

Impact of correlation

- Varying the correlation between the driving index and the interest rates gives additional flexibility for the at-the-money implied volatility structure:



Correlation is an important factor in long-dated annuity pricing:

- Many variable annuities and other long-dated options heavily depend on the correlation between the Equity/FX/Inflation index and the underlying interest rates.
- TARN (Equity), PRDC (FX), LPI (Inflation) options, Variable Annuities, Unit-Linked and Pension contracts and many other hybrid products.
- Crucial to get a realistic assessment of the correlation risks involved.

- For instance, annuity protection for a pension fund explicitly hinges on the dependency structure between the equity and the underlying interest rates.

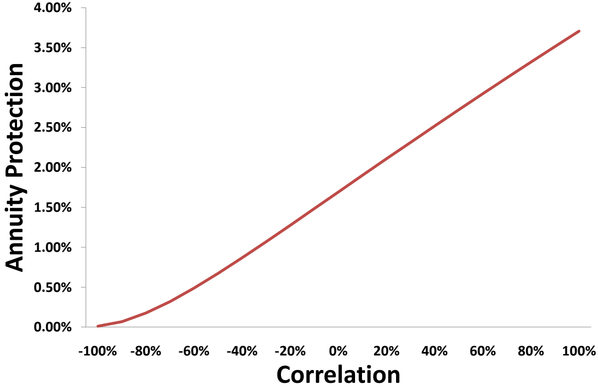


Figure: Put option on the funding ratio of a pension fund, with assets invested for 40% in stocks and 60% in fixed income.

Modelling framework

- The 'standard' model for FX and Inflation derivatives:

$$
\begin{aligned}
dn(t) &= \left[\vartheta_n(t) - a_n n(t)\right] dt + \sigma_n dW_n(t) \\
dr(t) &= \left[\vartheta_r(t) - \rho_{r,I}\sigma_I\sigma_r - a_r r(t)\right] dt + \sigma_r dW_r(t) \\
\frac{dI(t)}{I(t)} &= \left[n(t) - r(t)\right] dt + \sigma_I dW_I(t)
\end{aligned}
$$

- Single-factor Gaussian rates for the both currencies/economies and log-normal index (i.e. FX/CPI/Equity) $I(t)$ in between (e.g. see Jarrow and Yildirim (2003)).
- Ease in calibration as **European Index options** and **forward-starting options** can be handled by variance averaging and Black's formula.

- Consider the following stochastic volatility extension:

$$\begin{aligned} dn(t) &= \left[\vartheta_n(t) - a_n n(t)\right] dt + \sigma_n dW_n(t) \\ dr(t) &= \left[\vartheta_r(t) - \rho_{r,I}\nu(t)\sigma_r - a_r r(t)\right] dt + \sigma_r dW_r(t) \\ \frac{dI(t)}{I(t)} &= \left[n(t) - r(t)\right] dt + \nu(t) dW_I(t) \end{aligned}$$

- Heston (1993) or Schöbel and Zhu (1999) stochastic volatility:

$$\begin{aligned} \text{with: } d\nu(t) &= \kappa\left[\psi - \nu(t)\right] dt + \tau dW_\nu(t) \\ \text{or: } d\nu^2(t) &= \kappa\left[\theta - \nu^2(t)\right] dt + \xi\nu(t) dW_\nu(t) \end{aligned}$$

- **General correlation structure** between the drivers of the volatility, the index, the domestic (nominal) and the foreign (real) rates.

Option Pricing and Characteristic Functions

- European Index Options can be priced by Fourier transforming the characteristic function $\phi_T(u)$ of the log-index price $\log I(T)$:

$$\Big[I(T) - K \Big]^+$$

- Closed-form call price, e.g. see Lewis (2001), obtained by

$$P(t,T)\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[e^{-(\alpha+iv)k} \frac{\phi\Big(v-(\alpha+1)i\Big)}{(\alpha+iv)(\alpha+1+iv)} \right] dv + R\Big(F(t), K, \alpha\Big)$$

- For an optimal choice of the dampening parameter α, minimizing sampling and truncation error, see Lord and Kahl (2007) and van Haastrecht et al. (2009).

- Cliquet-like Equity/FX option payoff:

$$\left[\frac{I(T_2)}{I(T_1)} - K\right]^+$$

- Year-on-Year Inflation option payoff:

$$\left[\frac{CPI(T_2)}{CPI(T_1)} - (1+K)\right]^+$$

- Similarly, for the pricing of forward-starting and inflation options, it suffices to know the characteristic function $\phi_{T_1,T_2}(u)$ of the log index return of the 'different' assets $I(T_1)$ and $I(T_2)$.

Schöbel-Zhu stochastic volatility leads to closed-form expressions for the characteristic functions:

- Applying a change of measure and 'Feynman-Kac', gives us the characteristic function of the log index price

$$\exp\left[A + B \log F(t) + C\,\nu(t) + \frac{1}{2} D\,\nu^2(t)\right]$$

- By the tower-law of conditional expectations, one can find the following expression for the forward-starting characteristic function of the log-index return

$$\mathbb{E}_n^{T_i}\left\{\exp\left[a_0 + a' X + X' B X\right]\right\}, \qquad X \sim N(\bar{0}, \bar{\Sigma}).$$

- Explicit solutions and implementation details in van Haastrecht and Pelsser (2011a) and van Haastrecht and Pelsser (2011b).

Heston stochastic volatility with general correlation structure (to date) does not lead to exact closed-form pricing solutions.

- Under independence, the characteristic functions decompose into independent parts, e.g.

$$\phi_T(u) = \phi_R(u) \cdot \phi_{HE}(u)$$

- Approximations:
  - Project full model onto a degenerated Example, e.g. see Antonov et al. (2008) and Grzelak and Oosterlee (2009).
  - Use degenerate result as control variate for the full model:

Rate-Stock Corr.	Full model Corr.	**Var. Red.**
0.5	99.974%	**1 950**
0.3	99.992%	**5 902**
0.1	99.999%	**55 252**

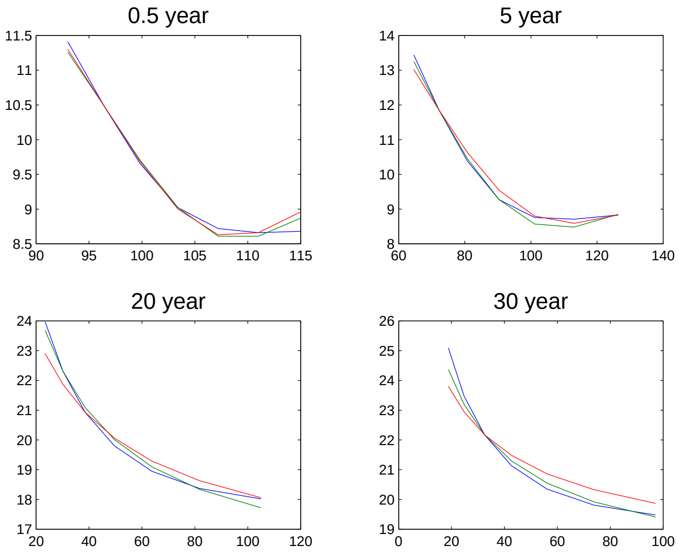


Figure: Market in blue, SZHW-FX in red and Heston-FX in green.

- Multi-factor Gaussian rates can be incorporated, closed-form formulas still hold, see van Haastrecht and Pelsser (2011b).
- Jumps in the price index can be trivially incorporated as we already work in the Fourier domain, i.e. modify the characteristic function in:

$$\psi_{SV}(u) \cdot \psi_J(u)$$

- The model can be used as an approximation for models with other rate processes such as CIR or Cheyette rates, e.g. see Andreasen (2007)

Example I: Equity-Interest Rate Annuities

- A Guaranteed Annuity Option (GAO) with guaranteed rate g has payoff at time T:

$$\begin{aligned} H(T) &= \left(gS(T)\sum_{i=0}^{n} c_i P(T,t_i) - S(T)\right)^+ \\ &= gS(T)\left(\sum_{i=0}^{n} c_i P(T,t_i) - K\right)^+ \end{aligned}$$

  $P(T,t_i)$: discount factor,
  c_i: probability of survival till time t_i, independent of $S(T)$.

- A vast literature on the pricing and risk management of deferred annuity products exists, e.g. see Pelsser (2003), Boyle and Hardy (2003), Bauer et al. (2008) and Marshall et al. (2009).

For a realistic pricing and hedging of Guaranteed Annuities it is evidently important to:

- Incorporate a realistic dependency structure between the equity and the rates.
- Use an equity model which is able to incorporate the volatility skew.
- Efficient formulas for prices and sensitivities.

These characteristics apply for most long-term equity-interest rate hybrids, hence the pricing of GAOs provides a good variable annuity pricing benchmark.

- Special case of general stochastic volatility framework for equity-interest rate hybrids:

$$\frac{dS(t)}{S(t)} = r(t)dt + \nu(t)dW_S^Q(t)$$

- (Multi-factor) Gaussian rates and Heston or Schöbel-Zhu stochastic volatility.
- Full correlation structure between stock returns, interest rates and the stochastic volatility.

Within the Schöbel-Zhu-Hull-White model, the following closed-form formulas can be obtained:

- For **1-factor interest rates**, the GAO price is given by a sum of Black and Scholes (1973) formulas:

$$_x p_R \, g S(0) \sum_{i=0}^{n} c_i \left[F_i N\left(d_1^i\right) - K_i N\left(d_2^i\right) \right]$$

- For **multi-factor interest rates**, the GAO price is given by integrating over a sum of Black and Scholes (1973) formulas multiplied by a Gaussian distribution:

$$_x p_R \, g S(0) \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}{\sigma_x \sqrt{2\pi}} \sum_{i=0}^{n} \left[F_i(x) N\left(h_2(x)\right) - K N\left(h_1(x)\right) \right] dx$$

Impact of stochastic volatility

Stochastic and Deterministic Volatility model imply the following Guaranteed Annuity Prices:

strike g	St. Vol	Det. Vol	Rel. Diff	Corr. Down	Corr. Up
8.23% (ATM)	3.82	3.07	+ 24.5%	3.92	3.76
7%	0.59	0.39	+ 50.7%	0.59	0.59
8%	2.89	2.26	+ 28.0%	2.94	2.85
9%	8.40	7.25	+ 15.8%	8.57	8.24
10%	17.02	15.53	+ 9.6%	17.33	16.73
11%	27.37	25.69	+ 6.5%	27.77	26.98
12%	38.30	36.47	+ 5.0%	38.75	37.86
13%	49.35	47.37	+ 4.2%	49.84	48.87

Table: GAO prices for 55 year old male and positive correlation of 35% between stock returns and long-term rates.

- For a positive correlation, the prices for the guaranteed annuities, using a stochastic volatility model for equity prices are considerably higher in comparison to the constant volatility model.

- Mathematically, the pricing difference is induced by a stochastic quanto correction for the process driving the interest rates:

$$dx(t) = -ax(t)dt + \rho_{xS}\sigma\nu(t)dt + \sigma dW_x^{\mathcal{Q}^S}(t)$$

- For pricing long-dated variable annuities depending on combinations of equity (or: FX) and interest rates it is important to model skew and correlation risks appropriately.
- The stochastic nature of the volatility in combination with a positive correlation, creates a more extreme and skewed dependency structure for variable annuity payoffs than a deterministic volatility model.
- Important to assess remaining model risk by mixing other copulas, stochastic volatility specifications and jump extensions.

Inflation-Linked Annuities

- Market's for inflation-linked annuities grew enormously, partially caused due to pension fund regulations.
- Following Wilkie (1998), we define the following LPI annuities:
  a. $\text{LPI}(t) = \text{CPI}(t)$
  b. $\text{LPI}(t) = \max\left[\text{CPI}(0) \cdot (1 + \text{floor}\ \%)^t, \text{CPI}(t)\right]$
  c. $\text{LPI}(t) = \max\left[\text{CPI}(0), \text{CPI}(1), \ldots, \text{CPI}(t)\right] = \max\left[\text{LPI}(t-1), \text{CPI}(t)\right]$
  d. $\text{LPI}(t) = \text{LPI}(t-1) \cdot \min\left[\max\left[1 + \text{floor}\ \%, \frac{\text{CPI}_{(t)}}{\text{CPI}_{(t-1)}}\right], 1 + \text{cap}\ \%\right]$

  For 0% floors, we have 'Type a' < 'Type b'< 'Type c'< 'Type d'.
- Common contract in the Netherlands is 'Type c' LPIs (or: ratcheted annuities) based on HICP ex. Tobacco, whilst 'Type d' LPI based on RPI became popular in the UK, e.g. see **?**

- Stochastic volatility foreign exchange is also applicable for pricing inflation-linked products:

$$
\begin{aligned}
dn(t) &= \left[\vartheta_n(t) - a_n n(t)\right] dt + \sigma_n dW_n(t) \\
dr(t) &= \left[\vartheta_r(t) - \rho_{r,I}\nu(t)\sigma_r - a_r r(t)\right] dt + \sigma_r dW_r(t) \\
\frac{dI(t)}{I(t)} &= \left[n(t) - r(t)\right] dt + \nu(t) dW_I(t)
\end{aligned}
$$

- Marginal distribution of inflation rates can calibrated using Year-on-Year options.

- Multiple calibrations of the auto-correlation structure are feasible, which could be undesirable for the pricing and hedging of more exotic contracts.

Calibration

Year-on-years options serve as input to imply marginal distributions of inflation rates:

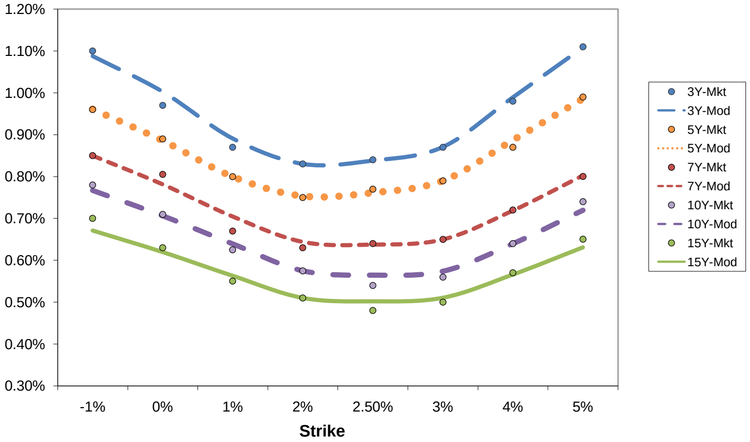


Figure: Market and model implied volatilities for caplets/floorlets maturing in 3, 5, 7, 10, 15 years. Calibration results using Schöbel-Zhu volatility and data corresponding with Mercurio and Moreni (2009).

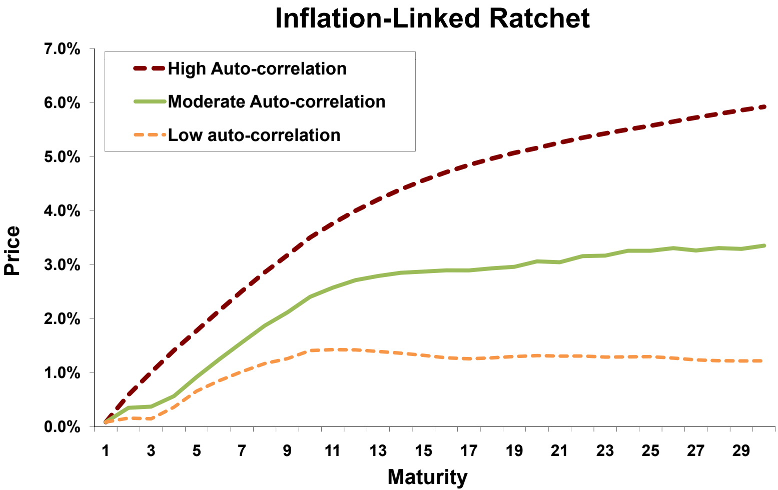


Figure: Impact of varying auto-correlation structure whilst preserving the marginal inflation rate distribution.

- Drawback of foreign exchange framework is that it lacks of explicit control over the auto-correlation structure.
- Real rate volatilities and correlations implicitly determine the auto-correlation structure, unfortunately these unobservable quantities are very hard to estimate, e.g. see Belgrade et al. (2004).
- Luckily, the inflation rate can also directly be parameterized as mean reverting process, implicitly incorporating the real-rate dynamics, e.g. see Dodgson and Kainth (2006) and Jäckel and Bonneton (2010).
- These alternative parameterizations do offer explicit control over the required inflation curve's auto-correlation structures.

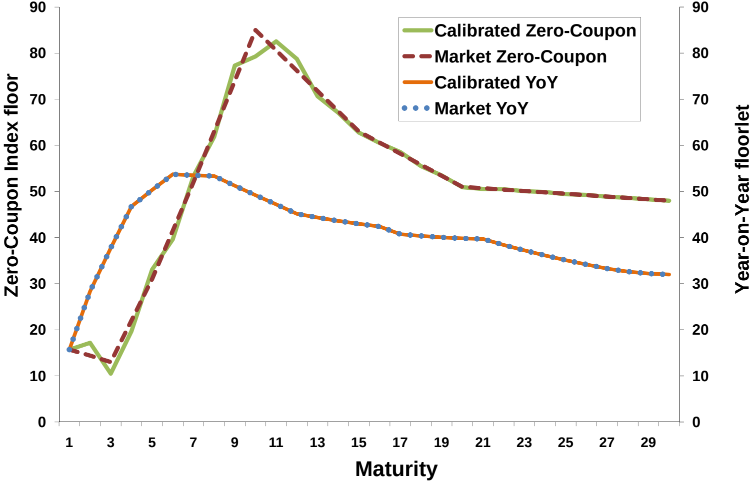


Figure: Time-dependent auto-correlation structure can be implied from a joint calibration to Year-On-Year (marginal distribution) and Zero-Coupon Index Options (terminal distribution).

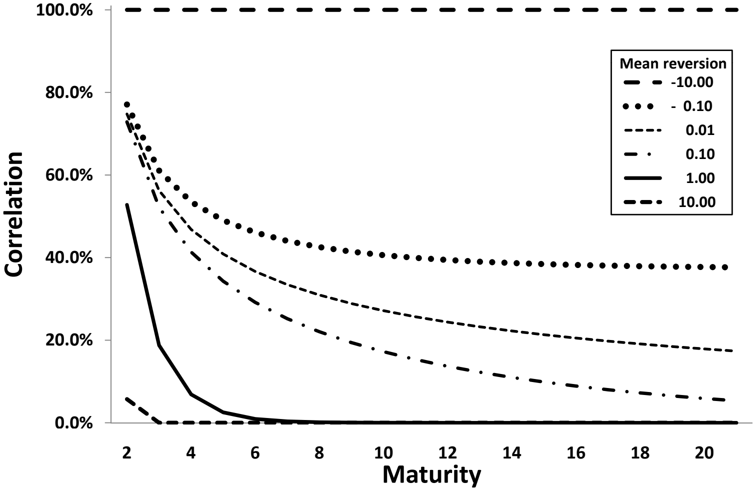


Figure: Time-homogeneous auto-correlation structure between inflation rates for different mean reversion strengths.

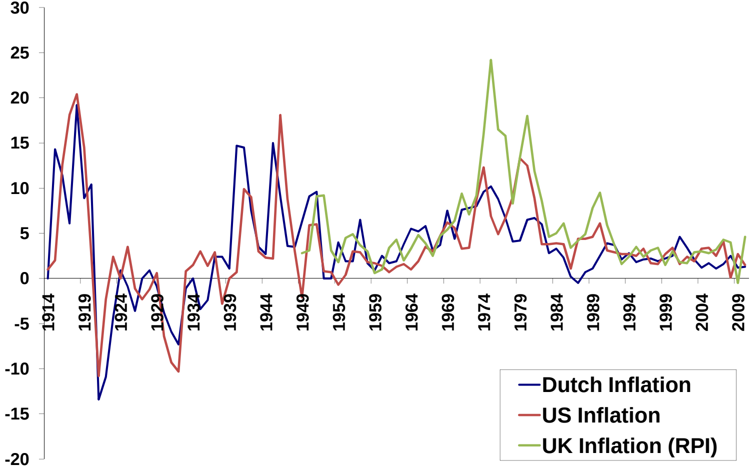


Figure: Dutch, US and UK inflation between 1914-2010 show a relatively high degree of year-on-year auto-correlation 58% (NL), 65% (US) and 78% (UK).

- Calibration to Year-on-Year options is always needed to match the marginal distribution of Inflation Rates.
- Always want a mechanism to understand the relative value between various LPI types and a different calibration of the auto-correlation structure:
  - A time-dependent auto-correlation structure from zero-coupon index option prices (using a piece-wise constant mean reversion).
  - A time-homogenous auto-correlation structure benchmarked against to historical data (using a fixed mean reversion).
  - Mixture of above approaches.
- Ultimately, the choice of a suitable auto-correlation calibration strongly depends on specific the annuity under consideration, market liquidity and the desired level of conservatism.

Conclusion

For the pricing and hedging of Long-Dated Variable Annuities:

- Important to take long-term skew, stochastic interest rates and a full correlation structure jointly into account.
- Models with closed-form pricing formulas for calibration and sensitivity analysis are a big advantage.
- Always want mechanisms for addressing (auto-)correlation risks.

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