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Counterparty Risk Valuation and Credit Models in Crisis

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Unit 1: CREDIT DEFAULT SWAPS AND BIG BANG

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Credit Models and Counterparty Risk Valuation in Crisis

UNIT 1

CREDIT DEFAULT SWAPS

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UNIT 1: SINGLE NAMES CREDIT DERIVATIVES

- Defaultable (corporate) zero coupon bonds;
- Credit Default Swaps (CDS);
- CDS rates;

Defaultable (corporate) zero coupon bonds

Similarly to the zero coupon bond $P(t,T)$ being one of the possible fundamental quantities for describing the interest-rate curve, we now consider a defaultable bond $\bar{P}(t,T)$ as a possible fundamental variable for describing the defaultable market.

DEFAULT FREE		
time t		time T
:	$\longleftarrow$	:
$P(t,T)$		1

with DEFAULT		
time t		time T :
:	$\longleftarrow$	NO DEFAULT: 1
$\bar{P}(t,T)$		DEFAULT: 0

When considering default, we have a random time τ representing the time at which a given company defaults.

$$\boxed{\tau \text{ : Default time of the company}}$$

Defaultable (corporate) zero coupon bonds

The value of a bond issued by the company and promising the payment of 1 at time T, as seen from time t, is the risk neutral expectation of the discounted payoff

BondPrice = Expectation[Discount x Payoff]

$$P(t,T) = \mathbb{E}\{D(t,T)\ 1\ |\mathcal{G}_t\}, \qquad \mathbf{1}_{\{\tau>t\}}\bar{P}(t,T) := \mathbb{E}\{D(t,T)\mathbf{1}_{\{\tau>T\}}|\mathcal{G}_t\}$$

where $\mathcal{G}_t$ represents the flow of information on whether default occurred before t and if so at what time exactly, and on the default free market variables up to t. D is the stochastic discount factor between two dates, depending on interest rates, and represents discounting.

The "indicator" function $\mathbf{1}_{\text{condition}}$ is 1 if "condition" is satisfied and 0 otherwise. In particular, $\mathbf{1}_{\{\tau>T\}}$ reads 1 if default τ did not occur before T, and 0 in the other case.

We understand then that (ignoring recovery) $\mathbf{1}_{\{\tau>T\}}$ is the correct payoff for a corporate bond at time T: the contract pays 1 if the company has not defaulted, and 0 if it defaulted before T, according to our earlier stylized description.

Defaultable (corporate) zero coupon bonds

If we include a recovery amount REC to be paid at default τ in case of early default, we have as discounted payoff at time t

$$D(t,T)\mathbf{1}_{\{\tau>T\}} + \text{REC}\, D(t,\tau)\mathbf{1}_{\{\tau\leq T\}}$$

If we include a recovery amount REC paid at maturity T, we have as discounted payoff

$$D(t,T)\mathbf{1}_{\{\tau>T\}} + \text{REC}\, D(t,T)\mathbf{1}_{\{\tau\leq T\}}$$

Fundamental Credit Derivatives: Credit Default Swaps

"It is a capital mistake to theorize before one has data. Insensibly one begins to twist facts to suit theories, instead of theories to suit facts."
Sherlock Holmes, *A Scandal in Bohemia*, quoted by KMV's J.R. Bohn on a credit risk survey paper.

Credit Default Swaps are basic protection contracts that became quite liquid in the last few years. CDS's are now actively traded and have become a sort of basic product of the credit derivatives area, analogously to interest-rate swaps and FRA's being basic products in the interest-rate derivatives world.

As a consequence, the need is no longer to have a model to be used to value CDS's, but rather to consider a model that can be *calibrated* to CDS's, i.e. to take CDS's as model inputs (rather than outputs), in order to price more complex credit derivatives.

Single name CDS options have never been liquid, as there is more liquidity in the CDS index options. We may expect models will have to incorporate CDS index options quotes rather than price them, similarly to what happened to CDS themselves.

Fundamental Credit Derivatives: CDS's

A CDS contract ensures protection against default. Two companies "A" (Protection buyer) and "B" (Protection seller) agree on the following.

If a third company "C" (Reference Credit) defaults at time τ, with $T_a < \tau < T_b$, "B" pays to "A" a certain (deterministic) cash amount L_{GD}. In turn, "A" pays to "B" a rate R at times $T_{a+1}, \ldots, T_b$ or until default. Set $\alpha_i = T_i - T_{i-1}$ and $T_0 = 0$.

Protection Seller B	$\rightarrow$	protection L_{GD} at default τ_C if $T_a < \tau_C \leq T_b$	$\rightarrow$	Protection Buyer A
	$\leftarrow$	rate R at $T_{a+1}, \ldots, T_b$ or until default τ_C	$\leftarrow$	

(protection leg and premium leg respectively). The cash amount L_{GD} is a *protection* for "A" in case "C" defaults. Typically L_{GD} = notional, or "notional - recovery" $= 1 - \text{R}_{\text{EC}}$.

Fundamental Credit Derivatives: CDS's

A typical stylized case occurs when "A" has bought a corporate bond issued by "C" and is waiting for the coupons and final notional payment from "C": If "C" defaults before the corporate bond maturity, "A" does not receive such payments. "A" then goes to "B" and buys some protection against this risk, asking "B" a payment that roughly amounts to the loss on the bond (e.g. notional minus deterministic recovery) that A would face in case "C" defaults.

Or again "A" has a portfolio of several instruments with a large exposure to counterparty "C". To partly hedge such exposure, "A" enters into a CDS where it buys protection from a bank "B" against the default of "C".

Fundamental Credit Derivatives: CDS's

Protection Seller B	$\rightarrow$	protection LGD at default τ_C if $T_a < \tau_C \leq T_b$	$\rightarrow$	Protection Buyer A
	$\leftarrow$	rate R at $T_{a+1}, \ldots, T_b$ or until default τ_C	$\leftarrow$	

Usually, at evaluation time (t) the amount $R = R_{a,b}(t)$ is set at a value that makes the contract fair, i.e. such that the present value of the two exchanged flows is zero. This is how the market quotes CDS's: CDS are quoted via their fair R's (Bid and Ask). Formally we may write the (Running) CDS discounted value to "B" at time $t < T_a$ as $\Pi\mathrm{RCDS}_{a,b}(t) :=$

$$D(t,\tau)(\tau - T_{\beta(\tau)-1})R\mathbf{1}_{\{T_a<\tau<T_b\}} + \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R\mathbf{1}_{\{\tau>T_i\}} - \mathbf{1}_{\{T_a<\tau\leq T_b\}}D(t,\tau)\ \mathrm{LGD}$$

Accrued rate at default + CDS Rate payments if no default - Protection paym at default

where $T_{\beta(\tau)}$ is the first of the T_i's following τ.

Fundamental Credit Derivatives. CDS's: Risk Neutral Valuation

Denote by $\text{CDS}_{a,b}(t, R, \text{L}_{\text{GD}})$ the time t *price* of the above Running standard CDS's *payoffs*.

As usual, the price associated to a discounted payoff is its *risk neutral expectation*.

The resulting pricing formula depends on the assumptions on interest-rate dynamics and on the default time τ (reduced form models, structural models...).

Fundamental Credit Derivatives. CDS's: Risk Neutral Valuation

In general by risk-neutral valuation we can compute the CDS price at time 0 (or at any other time similarly):

$$\mathsf{CDS}_{a,b}(0, R, \mathsf{L}_{\mathsf{GD}}) = \mathbb{E}\{\Pi\mathsf{RCDS}_{a,b}(0)\},$$

with the CDS discounted payoffs defined earlier. $\mathbb{E}$ denotes the risk-neutral expectation, the related measure being denoted by $\mathbb{Q}$.

However, we will not use the formulas resulting from this approach to price CDS that are already quoted in the market. Rather, we will invert these formulas in correspondence of market CDS quotes to calibrate our models to the CDS quotes themselves. We will give examples of this later.

Now let us have a look at some particular formulas resulting from the general risk neutral approach through some simplifying assumptions.

Default independent of interest rates: CDS Model-independent formulas

Assume the stochastic discount factors $D(s,t)$ to be independent of the default time τ for all possible $0 < s < t$. The premium leg of the CDS at time 0 is:

$$
\begin{aligned}
\mathsf{PremiumLeg}_{a,b}(R) &= \mathbb{E}[D(0,\tau)(\tau - T_{\beta(\tau)-1})R\mathbf{1}_{\{T_a<\tau<T_b\}}] + \\
&\quad + \sum_{i=a+1}^{b} \mathbb{E}[D(0,T_i)\alpha_i R\mathbf{1}_{\{\tau \geq T_i\}}] \\
&= \mathbb{E}\left[\int_{t=0}^{\infty} D(0,t)(t - T_{\beta(t)-1})R\mathbf{1}_{\{T_a<t<T_b\}}\mathbf{1}_{\{\tau\in[t,t+dt]\}}\right] + \\
&\quad + \sum_{i=a+1}^{b} \mathbb{E}[D(0,T_i)]\alpha_i R\,\mathbb{E}[\mathbf{1}_{\{\tau \geq T_i\}}] =
\end{aligned}
$$

$$\text{PremiumLeg}_{a,b}(R) = \int_{t=T_a}^{T_b} \mathbb{E}[D(0,t)(t - T_{\beta(t)-1}) R\, \mathbf{1}_{\{\tau \in [t, t+dt)\}}] +$$

$$+ \sum_{i=a+1}^{b} P(0,T_i)\alpha_i R\, \mathbb{Q}(\tau \geq T_i)$$

$$= \int_{t=T_a}^{T_b} \mathbb{E}[D(0,t)](t - T_{\beta(t)-1}) R\, \mathbb{E}[\mathbf{1}_{\{\tau \in [t, t+dt)\}}] +$$

$$+ \sum_{i=a+1}^{b} P(0,T_i)\alpha_i R\, \mathbb{Q}(\tau \geq T_i)$$

$$= R \int_{t=T_a}^{T_b} P(0,t)(t - T_{\beta(t)-1})\mathbb{Q}(\tau \in [t, t+dt)) +$$

$$+R \sum_{i=a+1}^{b} P(0,T_i)\alpha_i \mathbb{Q}(\tau \geq T_i),$$

where we have used independence in factoring the above expectations.

Default independent of interest rates: CDS Model-independent formulas

We have thus, by rearranging terms and introducing a "unit-premium" premium leg (sometimes called "DV01", "Risky duration" or "annuity"):

$$\text{PremiumLeg}_{a,b}(R; P(0,\cdot), \mathbb{Q}(\tau > \cdot)) = R \;\text{PremiumLeg1}_{a,b}(P(0,\cdot), \mathbb{Q}(\tau > \cdot)),$$

$$\text{PremiumLeg1}_{a,b}(P(0,\cdot), \mathbb{Q}(\tau > \cdot)) := -\int_{T_a}^{T_b} P(0,t)(t - T_{\beta(t)-1})d_t \boxed{\mathbb{Q}(\tau \geq t)}$$

$$+ \sum_{i=a+1}^{b} P(0,T_i)\alpha_i \boxed{\mathbb{Q}(\tau \geq T_i)} \qquad (1)$$

This model independent formula uses the initial market zero coupon curve (bonds) at time 0 (i.e. $P(0,\cdot)$) and the survival probabilities $\mathbb{Q}(\tau \geq \cdot)$ at time 0 (terms in the boxes).

A similar formula holds for the protection leg, again under independence between default τ and interest rates.

Default independent of interest rates: CDS Model-independent formulas

$$\text{ProtecLeg}_{a,b}(\text{L}_{\text{GD}}) = \mathbb{E}[\mathbf{1}_{\{T_a<\tau\le T_b\}}D(0,\tau)\ \text{L}_{\text{GD}}]$$

$$= \text{L}_{\text{GD}}\ \mathbb{E}\left[\int_{t=0}^{\infty}\mathbf{1}_{\{T_a<t\le T_b\}}D(0,t)\mathbf{1}_{\{\tau\in[t,t+dt)\}}\right] = \text{L}_{\text{GD}}\left[\int_{t=T_a}^{T_b}\mathbb{E}[D(0,t)\mathbf{1}_{\{\tau\in[t,t+dt)\}}]\right]$$

$$= \text{L}_{\text{GD}}\int_{t=T_a}^{T_b}\mathbb{E}[D(0,t)]\mathbb{E}[\mathbf{1}_{\{\tau\in[t,t+dt)\}}] = \text{L}_{\text{GD}}\int_{t=T_a}^{T_b}P(0,t)\mathbb{Q}(\tau\in[t,t+dt))$$

so that we have, by introducing a "unit-notional" protection leg:

$$\text{ProtecLeg}_{a,b}(\text{L}_{\text{GD}}; P(0,\cdot),\mathbb{Q}(\tau>\cdot)) = \text{L}_{\text{GD}}\ \text{ProtecLeg1}_{a,b}(P(0,\cdot),\mathbb{Q}(\tau>\cdot)),$$

$$\text{ProtecLeg1}_{a,b}(P(0,\cdot),\mathbb{Q}(\tau>\cdot)) := -\int_{T_a}^{T_b}P(0,t)\ d_t\boxed{\ \mathbb{Q}(\tau\ge t)\ } \qquad (2)$$

Default independent of interest rates: CDS Model-independent formulas

$$\text{ProtecLeg}_{a,b}(\text{L}\text{GD}; P(0,\cdot), \mathbb{Q}(\tau > \cdot)) = \text{L}\text{GD}\ \text{ProtecLeg1}_{a,b}(P(0,\cdot), \mathbb{Q}(\tau > \cdot)),$$

$$\text{ProtecLeg1}_{a,b}(P(0,\cdot), \mathbb{Q}(\tau > \cdot)) := -\int_{T_a}^{T_b} P(0,t)\ d_t \boxed{\mathbb{Q}(\tau \geq t)}$$

This formula too is model independent given the initial zero coupon curve (bonds) at time 0 observed in the market and given the survival probabilities at time 0 (term in the box).

The total (Receiver) CDS price can be written as

$$\text{CDS}_{a,b}(t, R, \text{L}\text{GD}; \mathbb{Q}(\tau > \cdot)) = R\,\text{PremiumLeg1}_{a,b}(\mathbb{Q}(\tau > \cdot)) - \text{L}\text{GD}\ \text{ProtecLeg1}_{a,b}(\mathbb{Q}(\tau > \cdot))$$

$$= R\left[-\int_{T_a}^{T_b} P(0,t)(t - T_{\beta(t)-1})d_t \boxed{\mathbb{Q}(\tau \geq t)} + \sum_{i=a+1}^{b} P(0,T_i)\alpha_i \boxed{\mathbb{Q}(\tau \geq T_i)}\right] +$$

$$+\text{L}\text{GD}\left[\int_{T_a}^{T_b} P(0,t)\ d_t \boxed{\mathbb{Q}(\tau \geq t)}\right]$$

Default independent of interest rates: CDS Model-independent formulas

$$\mathsf{CDS}_{a,b}(t, R, \mathrm{L}\text{GD}; \mathbb{Q}(\tau > \cdot)) = \mathrm{L}\text{GD} \left[\int_{T_a}^{T_b} P(0,t)\ d_t \boxed{\ \mathbb{Q}(\tau \geq t)\ } \right] +$$

$$R \left[-\int_{T_a}^{T_b} P(0,t)(t - T_{\beta(t)-1}) d_t \boxed{\ \mathbb{Q}(\tau \geq t)\ } + \sum_{i=a+1}^{b} P(0,T_i)\alpha_i \boxed{\ \mathbb{Q}(\tau \geq T_i)\ } \right]$$

The integrals in the survival probabilities given in the above formulas can be valued as Stieltjes integrals in the survival probabilities themselves, and can easily be approximated numerically by summations through Riemann-Stieltjes sums, considering a low enough discretization time step.

Default independent of interest rates: CDS Model-independent formulas

Approximation by summations through Riemann-Stieltjes sums

$$\mathrm{CDS}_{a,b}(t, R, \mathrm{L}_{\mathrm{GD}}; \mathbb{Q}(\tau > \cdot)) = \mathrm{L}_{\mathrm{GD}} \left[\sum_{t_j = T_a}^{T_b -} P\left(0, \frac{t_{j+1} + t_j}{2}\right) \boxed{(\mathbb{Q}(\tau \geq t_{j+1}) - \mathbb{Q}(\tau \geq t_j))} \right] +$$

$$R \left[\sum_{t_j = T_a}^{T_b -} P\left(0, \frac{t_{j+1} + t_j}{2}\right) \frac{t_{j+1} - t_j}{2} \boxed{(\mathbb{Q}(\tau \geq t_j) - \mathbb{Q}(\tau \geq t_{j+1}))} + \sum_{i=a+1}^{b} P(0, T_i)\alpha_i \boxed{\mathbb{Q}(\tau \geq T_i)} \right]$$

This formula can be implemented in excel

Default independent of interest rates: CDS Model-independent formulas

The market quotes, at time 0, the fair $R = R_{0,b}^{\text{mkt MID}}(0)$ (actually bid and ask quotes are available for this fair R) equating the two legs for a set of CDS with initial protection time $T_a = 0$ and final protection time $T_b \in \{1y, 2y, 3y, 4y, 5y, 6y.7y, 8y, 9y, 10y\}$, although often only a subset of the maturities $\{1y, 3y, 5y, 7y, 10y\}$ is available. Solve then

$$\text{CDS}_{0,b}(t, R_{0,b}^{\text{mktMID}}(0), \text{L}\textsc{gd}; \mathbb{Q}(\tau > \cdot)) = 0$$

in portions of $\mathbb{Q}(\tau > \cdot)$ starting from $T_b = 1y$, finding the market implied survival $\{\mathbb{Q}(\tau \geq t), t \leq 1y\}$; plugging this into the $T_b = 2y$ CDS legs formulas, and then solving the same equation with $T_b = 2y$, we find the market implied survival $\{\mathbb{Q}(\tau \geq t), t \in (1y, 2y]\}$, and so on up to $T_b = 10y$.

This is a way to strip survival probabilities from CDS quotes in a model independent way. No need to assume an intensity or a structural model for default here.

Default independent of interest rates: CDS Model-independent formulas

However, the market in doing the above stripping typically resorts to hazard functions, assuming existence of hazard functions associated with the default time. We will assume existence of a deterministic intensity, as in deterministic intensity models, and briefly consider the notion of implied deterministic cumulated intensity (hazard function) when introducing intensity models below.

CDS extensions

In the intensity models context we will also explore variants of the basic CDS contract, including

CDS Options.

We will also discuss the CDS big bang launched by ISDA in 2009.

Equity/Interest Rate/commodities/credit Payoffs with Counterparty Risk

When we include counterparty risk in the valuation of an otherwise default-free derivative, or even of a credit derivative, in a way we obtain a hybrid credit derivative.

This holds for example for interest-rate swaps, for equity options and for equity return swaps, for oil swaps and credit default swaps, among other products.

Counterparty risk pricing (CVA) involves the need for an option model for the underlying portfolio and of a credit model for the default of the counterparty, correlated with each other.

We have explored this problem in the initial part of the course.

UNIT 2

REDUCED FORM (INTENSITY) MODELS

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UNIT 2. SINGLE NAME MODELS: REDUCED FORM

- **Modeling Tools: Poisson Processes**;
- Time homogeneous Poisson Processes: deterministic and constant in time credit spread (intensity);
- Time inhomogeneous Poisson Processes: deterministic credit spread (intensity);
- Stochastic intensity Poisson Processes (Cox processes, credit spread volatility).
- CDS Market implied credit spread (deterministic intensity);
- The CDS options market model (embedded stochastic intensity);
- Examples of implied CDS rates volatilities;
- Explicit stochastic intensity modeling: The SSRD Model;
- SSRD Analytic and Automatic calibration to CDS market data and interest rate data;
- CDS options with the SSRD model (with CIR++ stochastic intensity)
- Relationship between CIR++ parameters and implied CDS volatilites;

The Basic Idea of Reduced Form Models

In reduced form or intensity models, the default time τ obeys roughly the following:

Having not defaulted before t, Probability of defaulting in the next dt instants is

$$\text{Prob}(\tau \in [t, t + dt)|\tau > t, \text{market info up to } t) = \lambda(t)dt$$

where the "probability" dt factor λ is called **intensity** or **hazard rate**. It is also an **instantaneous credit spread** (more on this later). Intensity can be

- Constant (τ is first jump of time homogeneous Poisson process);
- Time varying (τ is first jump of time inhomogeneous Poisson process); Can model the term structure of credit spreads; Does not model credit spread volatility; Implied hazard functions;
- Stochastic (τ is first jump of Cox Process); Can model term structure of credit spreads; Can model credit spread volatility;

Modeling Tools: Poisson processes

A "Time homogeneous Poisson process" is a **unit-jump increasing, right continuous process** with **stationary independent increments** and $M_0 = 0$.

Key fact: M time homogeneous Poisson process. Let $\tau^1, \tau^2, \ldots, \tau^m, \ldots$ be the first, second etc. jump times of M. **Then** for some positive contant $\bar{\gamma}$ the random variables $\tau^1, \tau^2 - \tau^1, \tau^3 - \tau^2, \ldots$,
i.e. the times between one jump and the next one, are i.i.d. $\sim$ exponentialRandVar($\bar{\gamma}$)

(EQUIVALENTLY, the R.V's $\bar{\gamma}\tau^1, \bar{\gamma}(\tau^2 - \tau^1), \bar{\gamma}(\tau^3 - \tau^2)$.. are i.i.d. $\sim$ expRandVar(1))

Important Consequence: $\mathbb{Q}\{\bar{\tau} \in [t, t + dt) | \bar{\tau} \geq t\} = \bar{\gamma}\, dt$

In the simplest intensity model **the default time is modeled as** τ^1.

Poisson processes. Default intensity as credit spread

If $\bar{\tau} := \tau^1$ is the first jump time of a Poisson process M_t with intensity $\bar{\gamma}$, then

$$\mathbb{Q}\{\bar{\tau} > t\} = e^{-\bar{\gamma}t}, \quad \mathbb{Q}\{\bar{\tau} \in [t, t+dt) | \bar{\tau} \geq t\} = \bar{\gamma}\, dt.$$

$$\mathbb{Q}\{s < \bar{\tau} \leq t\} = \exp(-\bar{\gamma}s) - \exp(-\bar{\gamma}t) \approx \bar{\gamma}(t-s)$$

The first formula is very important. It tells us that **survival probabilities have the same structure as discount factors, with the default intensity playing the role of interest rates**. This is an extremely important consequence of the exponential distribution for times between jumps.

It is this fundamental property of jumps of poisson processes that allows us to see survival probabilities as discount factors, and thus default intensities as credit spreads. This allows us to use much of the interest-rate technology in default modeling under these kind of reduced form models

Time homogeneous Poisson processes: CDS

Assume as an approximation that the CDS premium leg pays continuosly.

If the intensity γ is taken CONSTANT as in the time homogeneous Poisson process, implying

$$\mathbb{Q}(\tau > t) = e^{-\gamma t}, \quad \mathbb{Q}(\tau \in [t, t+dt) | \tau \geq t) = \gamma \, dt,$$

the CDS price becomes

$$\mathrm{CDS}_{a,b}(t, R, \mathrm{L}_{\mathrm{GD}}; \mathbb{Q}(\tau > \cdot)) = -\mathrm{L}_{\mathrm{GD}} \left[\int_{T_a}^{T_b} P(0,t) \gamma \mathbb{Q}(\tau \geq t) dt \right] + R \left[\int_{T_a}^{T_b} P(0,t) \mathbb{Q}(\tau \geq t) dt \right]$$

If we take $a = 0$ and insert the market CDS rate $R = R_{0,b}^{\text{mkt MID}}(0)$ in the premium leg the CDS present value should be zero. Solve $\mathrm{CDS}_{a,b}(t, R, \mathrm{L}_{\mathrm{GD}}; \mathbb{Q}(\tau > \cdot)) = 0$,

$$\boxed{\gamma = \gamma_{0,b} = \frac{R_{0,b}^{\text{mkt MID}}(0)}{\mathrm{L}_{\mathrm{GD}}}}$$

Time homogeneous Poisson processes: CDS

from which we see that **the CDS rate is indeed a sort of CREDIT SPREAD, or INTENSITY.**

We can play with this formula with a few examples.

CDS of FIAT trades at 300bps for 5y, with recovery 0.3

What is a quick rough calcul for the risk neutral probability that FIAT survives 10 years?

$$\boxed{\gamma = \frac{R_{0,b}^{\text{mkt FIAT}}(0)}{\text{L}_{\text{GD}}}} = \frac{300/10000}{1-0.3} = 4.29\%$$

Time homogeneous Poisson processes: CDS

Survive 10 years:

$$\mathbb{Q}(\tau > 10y) = \exp(-\gamma 10) = \exp(-0.0429 * 10) = 65.1\%$$

Default between 3 and 5 years:

$$\mathbb{Q}(\tau > 3y) - \mathbb{Q}(\tau > 5y) = \exp(-\gamma 3) - \exp(-\gamma 5) = \exp(-0.0429 * 3) - \exp(-0.0429 * 5) = 7.2\%$$

If R_{CDS} goes up and REC remains the same, γ goes up and survival probabilities go down (default probs go up)

If REC goes up and R_{CDS} remains the same, LGD goes down and γ goes up - default probabilities go up

Modeling Tools: time inhomogeneous Poisson Processes

We consider now **deterministic time-varying** intensity $\gamma(t)$, which we assume to be a positive and piecewise continuous function. We define

$$\Gamma(t) := \int_0^t \gamma(u)du,$$

the **cumulated intensity, cumulated hazard rate,** or also **Hazard function**.

If M_t is a Standard Poisson Process, i.e. a Poisson Process with intensity one, than a **time-inhomogeneous Poisson Process** N_t with intensity γ is defined as

$$N_t = M_{\Gamma(t)}.$$

So a time inhomogeneous PP is just a time-changed PP.

N_t is still increasing by jumps of size 1, its increments are still independent, but they are **no longer identically distributed** due to the "time distortion" introduced by Γ.

Modeling Tools: time inhomogeneous Poisson Processes

From $N_t = M_{\Gamma(t)}$ We have obviously

N jumps the first time at $\tau \iff M$ jumps the first time at $\Gamma(\tau)$.

But since we know that M is standard Poisson Process for which first jump time is exponentialRV(1), then

$$\Gamma(\tau) =: \xi \sim \text{exponentialRandVar}(1)$$

By inverting this last equation we have that

$$\tau = \Gamma^{-1}(\xi),$$

with ξ standard exponential random variable.

Time inhomogeneous Poisson Pr: Time varying credit spreads

Also, we have easily

$$\mathbb{Q}\{s < \tau \le t\} = \mathbb{Q}\{\Gamma(s) < \Gamma(\tau) \le \Gamma(t)\} = \mathbb{Q}\{\Gamma(s) < \xi \le \Gamma(t)\} =$$

$$= \mathbb{Q}\{\xi > \Gamma(s)\} - \mathbb{Q}\{\xi > \Gamma(t)\} = \exp(-\Gamma(s)) - \exp(-\Gamma(t)) \text{ i.e.}$$

"prob of default between s and t is "$\mathbf{e^{-\int_0^s \gamma(u)du} - e^{-\int_0^t \gamma(u)du} \approx \int_s^t \gamma(u)du}$"
(where the final approximation is good for small exponents). It is easy to show, along the same lines, that

$$\mathbb{Q}\{\tau \in [t, t+dt) | \tau \ge t\} = \gamma(t)\ dt.$$

"probability that company defaults in (arbitrarily small) "dt" **years given that it has not defaulted so far is $\gamma(t)\ dt$."**

Time inhomogeneous Poisson Pr: Time varying credit spreads

$$\Gamma(t) := \int_0^t \gamma(u)du, \quad \tau = \Gamma^{-1}(\xi) \Rightarrow$$

$$\Rightarrow \quad \mathbb{Q}\{\tau > t\} = e^{-\int_0^t \gamma(u)du}, \quad \mathbb{Q}\{\tau \in [t, t+dt)|\tau \geq t\} = \gamma(t)\ dt.$$

Again, the Poisson Process core structure with exponentially distributed in-between-jump times is allowing us to see survival probabilities as discount factors, and thus default intensities as credit spreads. This allows us to use much of the interest-rate technology in default modeling

But...

ξ **is independent of all default free market quantities and represents an external source of randomness that makes reduced form model incomplete. One cannot hedge** ξ **using equity, interest rate or FX instruments driven by browian gaussian shocks, as these would be independent of** ξ

Time inhomogeneous Poisson Pr: Time varying credit spreads

Here the time-varying nature of γ allows us to take into account the TERM STRUCTURE OF CREDIT SPREADS. It is actually this model that is used to strip default probabilities from CDS quotes, for example.

This formulation does not take into account CREDIT SPREAD VOLATILITY, since γ is deterministic and we have

$$d\gamma(t) = (\ldots)dt + \boxed{0}\, dW_t.$$

We will add credit spread volatility later (particularly relevant in counterparty risk calculations and in CDS options). For the time being we focus on CDS.

CDS Calibration and Implied Hazard Rates/Intensities

Reduced form models are the models that are most commonly used in the market to infer implied default probabilities from market quotes.

Market instruments from which these probabilities are drawn are especially CDS and Bonds.

We will see in some detail the procedure concerning CDS.

The reduced-from model used for this is the time-inhomogeneous Poisson Process, with time varying intensity $\gamma(t)$ and cumulated intensity/hazard function $\Gamma(t) = \int_0^t \gamma(u)du$.

Time inhomogeneous Poisson Process: Examples of Hazard rates stripping from CDS quotes

Recall: Having not defaulted by t, Probability of defaulting in the next dt instants is

$$\text{Prob}(\tau \in [t, t+dt) | \tau > t, \text{market info up to } t) = \gamma(t)dt$$

where the "probability" dt factor γ is called **intensity** or **hazard rate**. It is also an **instantaneous credit spread**. Probability of surviving t is

$$\text{Prob}(\tau > t) = \exp\left(-\int_0^t \gamma(s)ds\right), \ \ \text{Prob}(\tau \in dt) = \gamma(t)\exp\left(-\int_0^t \gamma(s)ds\right)dt$$

Here we take the hazard rate γ to be deterministic and **piecewise constant**:

$$\gamma(t) = \gamma_i \ \text{ for } \ t \in [T_{i-1}, T_i). \quad (\gamma_1, \gamma_2, \ldots, \gamma_i, \ldots)$$

Notice that $\Gamma(t) = \int_0^t \gamma(s)ds = \sum_{i=1}^{\beta(t)-1}(T_i - T_{i-1})\gamma_i + (t - T_{\beta(t)-1})\gamma_{\beta(t)}$

Examples of Hazard rates stripping from CDS quotes

Set $\Gamma_j := \int_0^{T_j} \gamma(s)ds = \sum_{i=1}^{j}(T_i - T_{i-1})\gamma_i$

In this context, compute for example the protection leg term of a CDS:

$$\text{L}_{\text{GD}}\, \mathbb{E}[D(0,\tau)\mathbf{1}_{\{T_a<\tau<T_b\}}] = \text{L}_{\text{GD}} \int_0^{\infty} \mathbb{E}[D(0,u)\mathbf{1}_{\{T_a<u<T_b\}}]\mathbb{Q}(\tau \in du)$$

$$= \text{L}_{\text{GD}} \int_{T_a}^{T_b} \mathbb{E}[D(0,u)]\mathbb{Q}(\tau \in du) = \text{L}_{\text{GD}} \int_{T_a}^{T_b} P(0,u)\gamma(u) \exp\left(-\int_0^u \gamma(s)ds\right) du$$

$$= \text{L}_{\text{GD}} \sum_{i=a+1}^{b} \gamma_i \int_{T_{i-1}}^{T_i} \exp(-\Gamma_{i-1} - \gamma_i(u - T_{i-1}))P(0,u)du$$

Examples of Hazard rates stripping from CDS quotes

With similar computations for the other CDS terms, it can be shown that under this formulation we have

$$\mathrm{CDS}_{a,b}(t, R, \mathrm{L}_{\mathrm{GD}}; \Gamma(\cdot)) = \mathbf{1}_{\{\tau>t\}} \Bigg[R \int_{T_a}^{T_b} P(t,u)(T_{\beta(u)-1} - u) d(e^{-(\Gamma(u)-\Gamma(t))})$$

$$+ \sum_{i=a+1}^{b} P(t,T_i) R \alpha_i e^{-(\Gamma(T_i)-\Gamma(t))} + \mathrm{L}_{\mathrm{GD}} \int_{T_a}^{T_b} P(t,u) d(e^{-(\Gamma(u)-\Gamma(t))}) \Bigg]$$

and in particular

$$\mathrm{CDS}_{a,b}(0, R, \mathrm{L}_{\mathrm{GD}}; \Gamma(\cdot)) = \Bigg[R \int_{T_a}^{T_b} P(0,u)(T_{\beta(u)-1} - u) d(e^{-\Gamma(u)})$$

$$+ \sum_{i=a+1}^{b} P(0,T_i) R \alpha_i e^{-\Gamma(T_i)} + \mathrm{L}_{\mathrm{GD}} \int_{T_a}^{T_b} P(0,u) d(e^{-\Gamma(u)}) \Bigg]$$

Examples of Hazard rates stripping from CDS quotes

So that $\mathsf{CDS}_{a,b}(0, R, \mathsf{L}\textsf{GD}; \Gamma(\cdot)) =$

$$= \left[R \sum_{i=a+1}^{b} \gamma_i \int_{T_{i-1}}^{T_i} \exp(-\Gamma_{i-1} - \gamma_i(u - T_{i-1})) P(0,u)(u - T_{i-1}) du \right.$$

$$+ R \sum_{i=a+1}^{b} P(0,T_i) \alpha_i e^{-\Gamma(T_i)} - \mathsf{L}\textsf{GD} \sum_{i=a+1}^{b} \gamma_i \int_{T_{i-1}}^{T_i} \exp(-\Gamma_{i-1} - \gamma_i(u - T_{i-1})) P(0,u) du]$$

Examples of Hazard rates stripping from CDS quotes

Now in the market $T_a = 0$ and we have fair R quotes for $T_b = 1y, 2y, 3y, \ldots, 10y$, with T_i's resetting quarterly.

$\mathrm{CDS}_{0,b}(0, R, \mathrm{L{\scriptstyle GD}}; \Gamma(\cdot)) =$

$$= \Big[R \sum_{i=1}^{b} \gamma_i \int_{T_{i-1}}^{T_i} \exp(-\Gamma_{i-1} - \gamma_i(u - T_{i-1})) P(0,u)(u - T_{i-1}) du$$

$$+ R \sum_{i=1}^{b} P(0,T_i)\alpha_i e^{-\Gamma(T_i)} - \mathrm{L{\scriptstyle GD}} \sum_{i=1}^{b} \gamma_i \int_{T_{i-1}}^{T_i} \exp(-\Gamma_{i-1} - \gamma_i(u - T_{i-1})) P(0,u) du]$$

We solve

$$\mathrm{CDS}_{0,1y}(0, R^{MKT}_{0,1y}, \mathrm{L{\scriptstyle GD}}; \gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 =: \gamma^1) = 0;$$
$$\mathrm{CDS}_{0,2y}(0, R^{MKT}_{0,2y}, \mathrm{L{\scriptstyle GD}}; \gamma^1; \gamma_5 = \gamma_6 = \gamma_7 = \gamma_8 =: \gamma^2) = 0; \ldots.$$

A Case Study: Parmalat Default Story

After a period of uncertainty on Parmalat's real financial situation, due to deceptive accounting, the real depth of the financial crisis came to light at the end of 2003 and rapidly led to bankruptcy.

September 12, 2003: Parmalat drops plan for a EUR 300 million debt sale.

November 14, 2003: the chief financial officer resigns after questions have been raised on Parmalat financial transactions.

December 9, 2003: Parmalat misses a EUR 150 million bond payment, while the management claims this is due to a customer not paying its bills.

December 19, 2003: a claimed USD 3.9 billion liquidity is revealed not to exist.

December 24, 2003: Parmalat goes into administration.

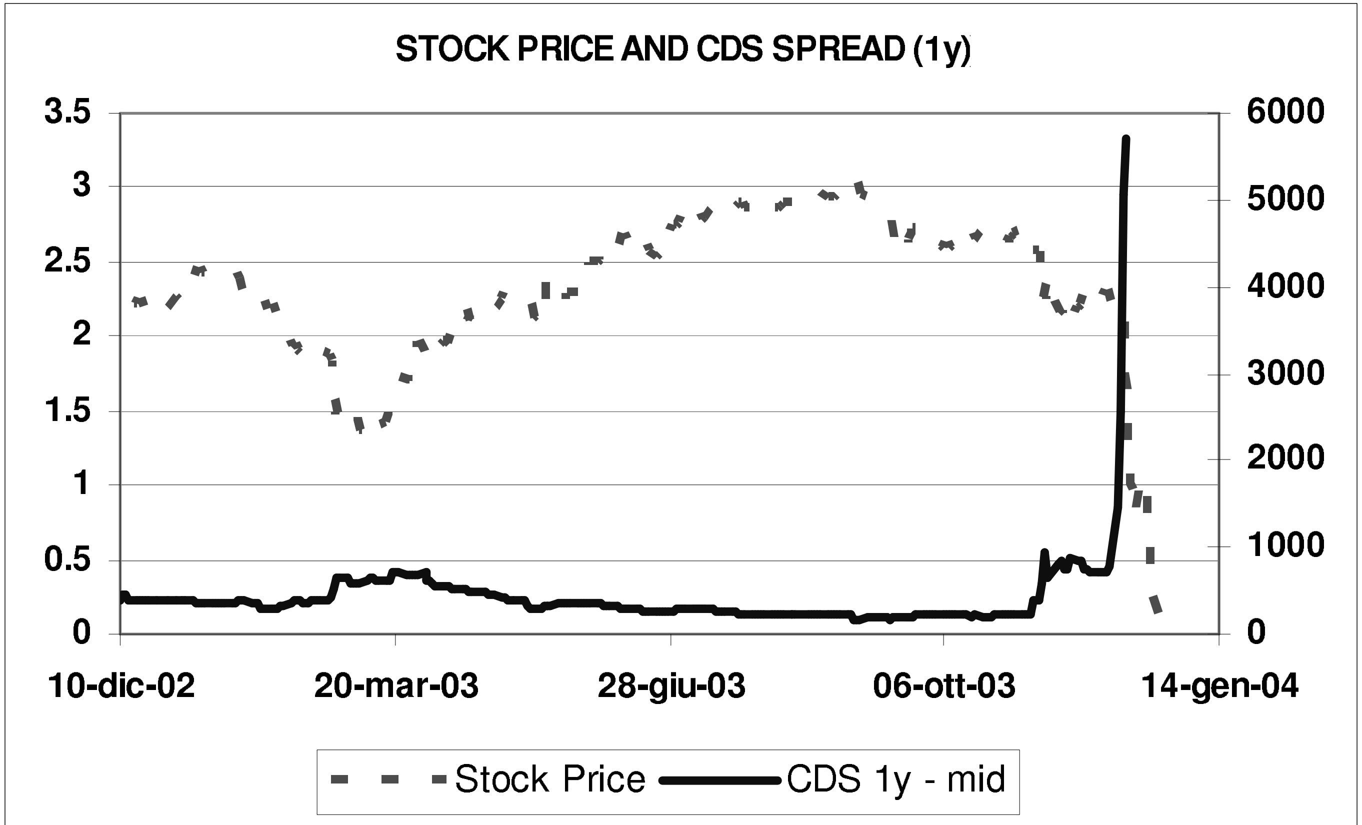
Parmalat CDS data

We consider data from three different days in 2003:

1. **September 10**, just before the beginning of the final Parmalat default story recalled above.
2. **November 28**, after the story of the Parmalat crisis began to unfold but before the pitch of the crisis.
3. **December 10**, when the fraud was not clear yet but the company was openly suspected to be on the verge of bankruptcy.

CDS Maturity T_b	1y	3y	5y	7y	10y
September 10: Premium rate $R_{0,b}$	192.5	215	225	235	235
November 28: Premium rate $R_{0,b}$	725	630	570	570	570
December 8: Premium rate $R_{0,b}$	1450	1200	940	850	850
December 10: Premium rate $R_{0,b}$	5050	2100	1500	1250	1100

Parmalat: CDS quotes and stock prices behavior



Behavior of the stock price (in euros, left scale) and of the 1y-CDS spread (in bps., right scale).

Examples of Hazard rates stripping from CDS quotes

September 10th, 2003 Recovery Rate = 40%

Maturity T_b (yr)	Maturity (dates)	$R_{0,b}$
1	20-Sep-04	192.5
3	20-Sep-06	215
5	20-Sep-08	225
7	20-Sep-10	235
10	20-Sep-13	235

Table 1: Maturity dates and corresponding CDS quotes in bps for September 10th, 2003.

date	intensity γ	survival pr $\exp(-\Gamma)$
10-Sep-03	3.199%	100.000%
20-Sep-04	3.199%	96.714%
20-Sep-06	4.388%	89.552%
22-Sep-08	3.659%	82.508%
20-Sep-10	5.308%	75.357%
20-Sep-13	2.338%	67.078%

Table 2: Calibration with piecewise linear intensity on September 10th, 2003.

date	intensity γ	survival pr $\exp(-\Gamma)$
10-Sep-03	3.199%	100.000%
20-Sep-04	3.199%	96.714%
20-Sep-06	3.780%	89.578%
22-Sep-08	4.033%	82.516%
20-Sep-10	4.458%	75.402%
20-Sep-13	3.891%	66.978%

Table 3: Calibration with piecewise constant intensity on September 10th, 2003.

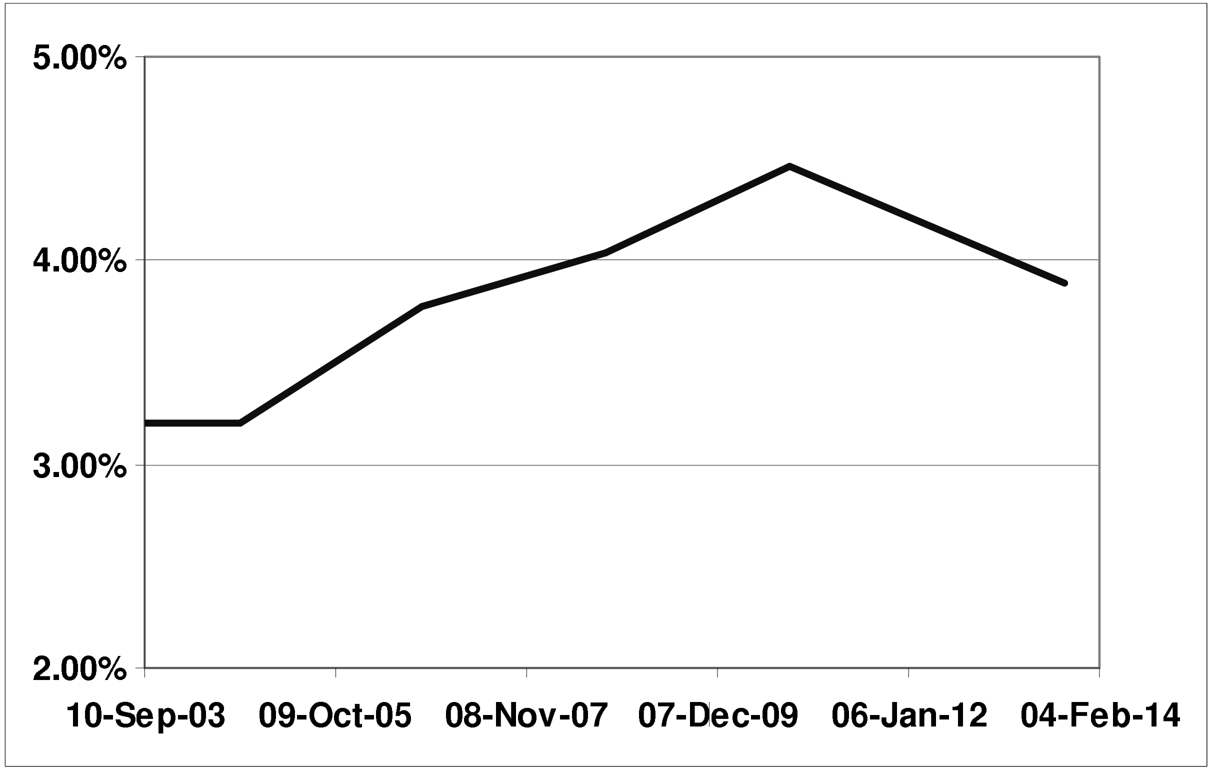


Figure 1: Piecewise linear intensity γ calibrated on CDS quotes on September 10th, 2003.

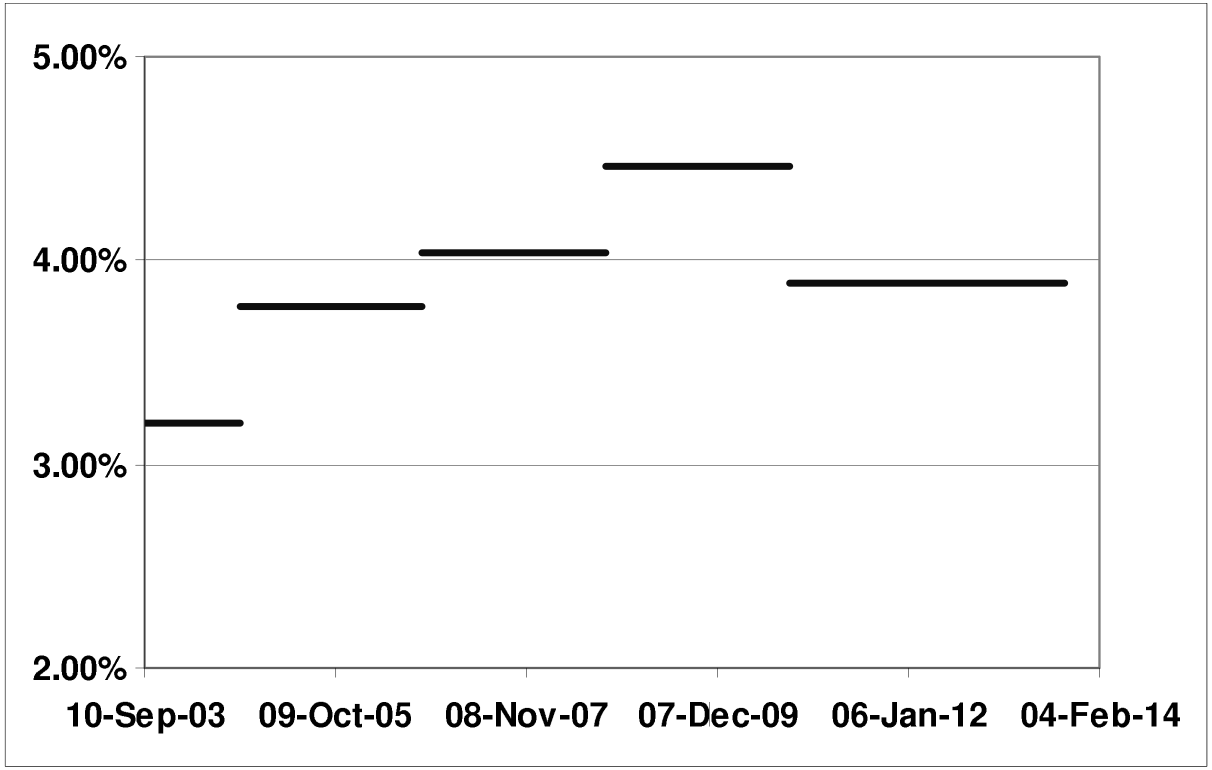


Figure 2: Piecewise constant intensity γ calibrated on CDS quotes on September 10th, 2003.

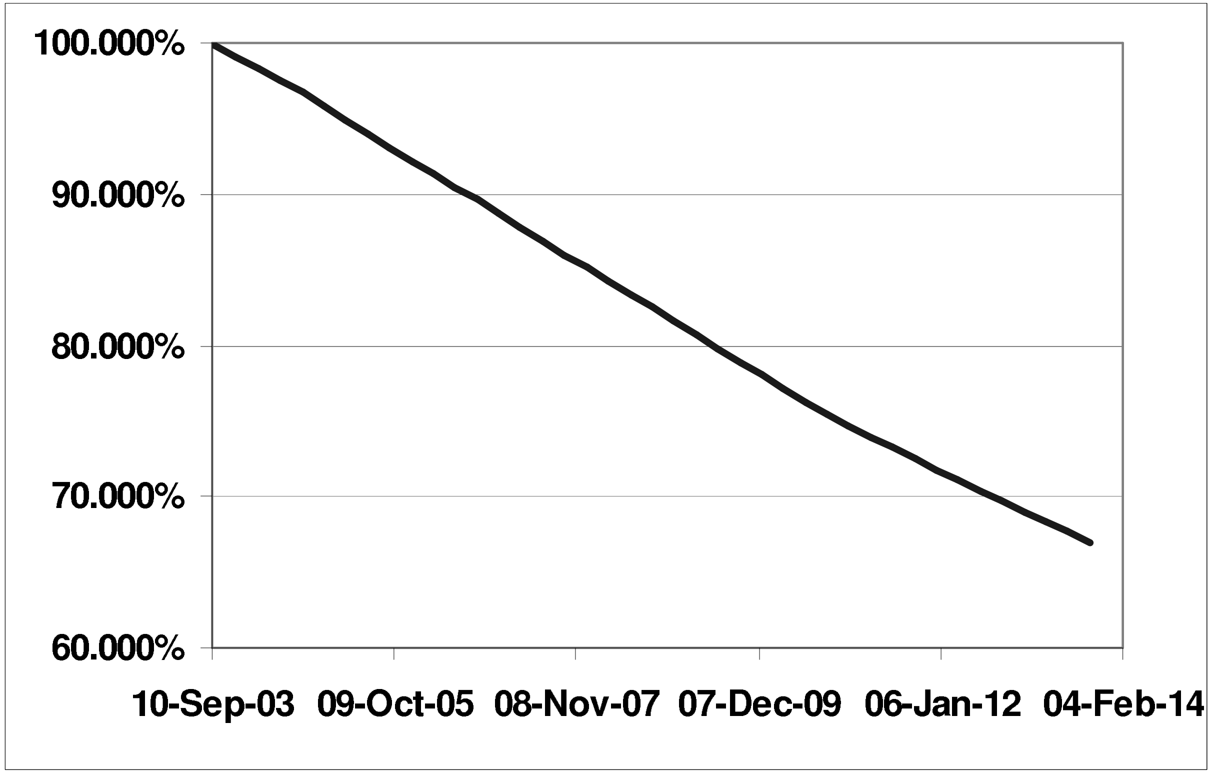


Figure 3: survival probability $\exp(-\Gamma)$ resulting from calibration on CDS quotes on September 10th, 2003.

November 28th, 2003

Recovery Rate $= 40\%$

Maturity T_b (yr)	Maturity (dates)	$R_{0,b}$
1	20-Dec-04	725
3	20-Dec-06	630
5	20-Dec-08	570
7	20-Dec-10	570
10	20-Dec-13	570

Table 4: Maturity Dates and corresponding CDS quotes in bps relative to November 28th, 2003.

date	intensity γ	survival pr $\exp(-\Gamma)$
28-Nov-03	12.047%	100.000%
20-Dec-04	12.047%	87.824%
20-Dec-06	6.545%	72.736%
22-Dec-08	8.226%	62.581%
20-Dec-10	10.779%	51.640%
20-Dec-13	7.880%	38.872%

Table 5: Calibration with piecewise linear intensity on November 28th, 2003.

date	intensity γ	survival pr $\exp(-\Gamma)$
28-Nov-03	12.047%	100.000%
20-Dec-04	12.047%	87.824%
20-Dec-06	9.426%	72.545%
22-Dec-08	7.331%	62.486%
20-Dec-10	9.441%	51.626%
20-Dec-13	9.437%	38.734%

Table 6: Calibration with piecewise constant intensity on November 28th, 2003.

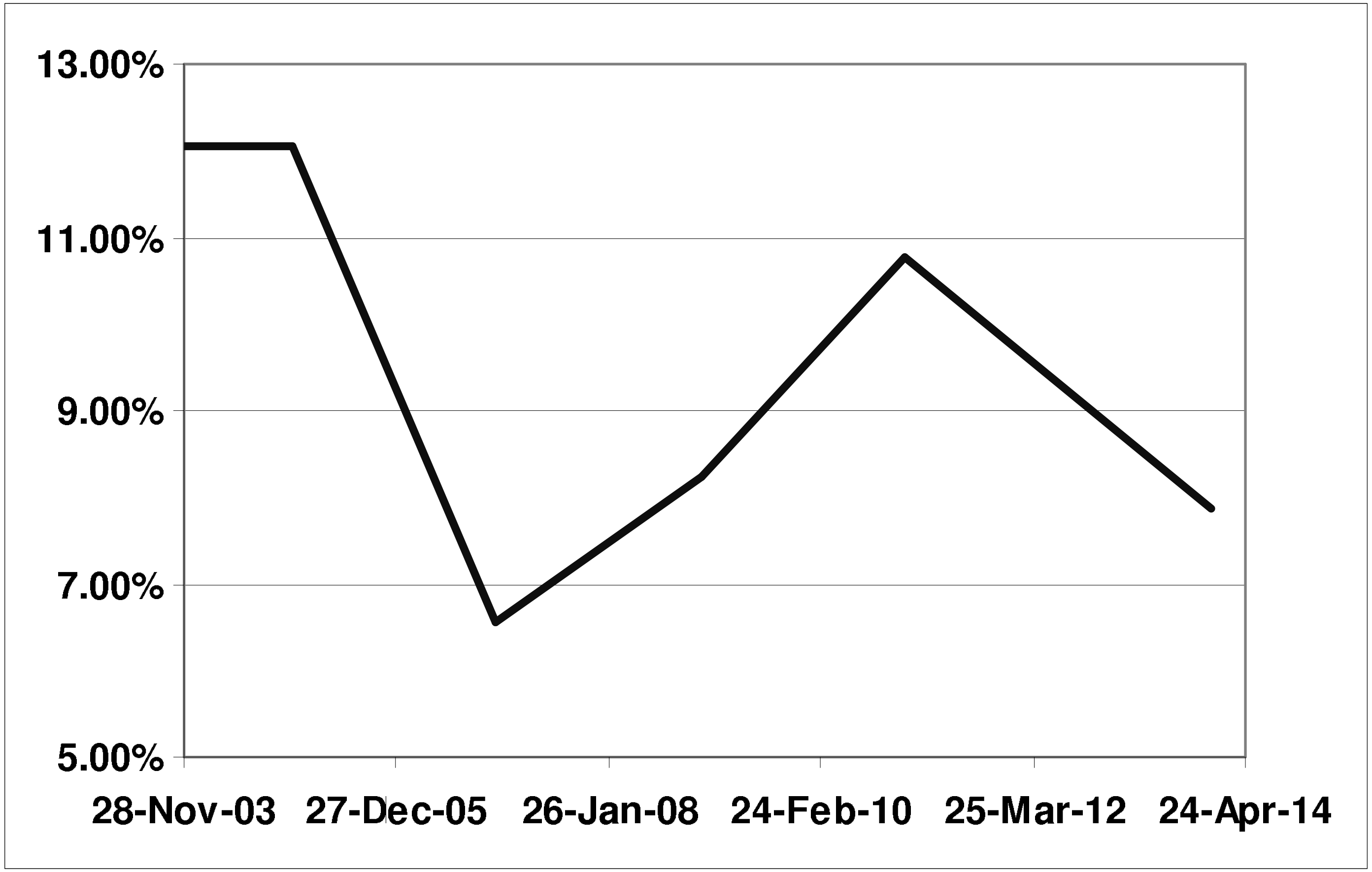


Figure 4: Piecewise linear intensity γ calibrated on CDS quotes on November 28th, 2003.

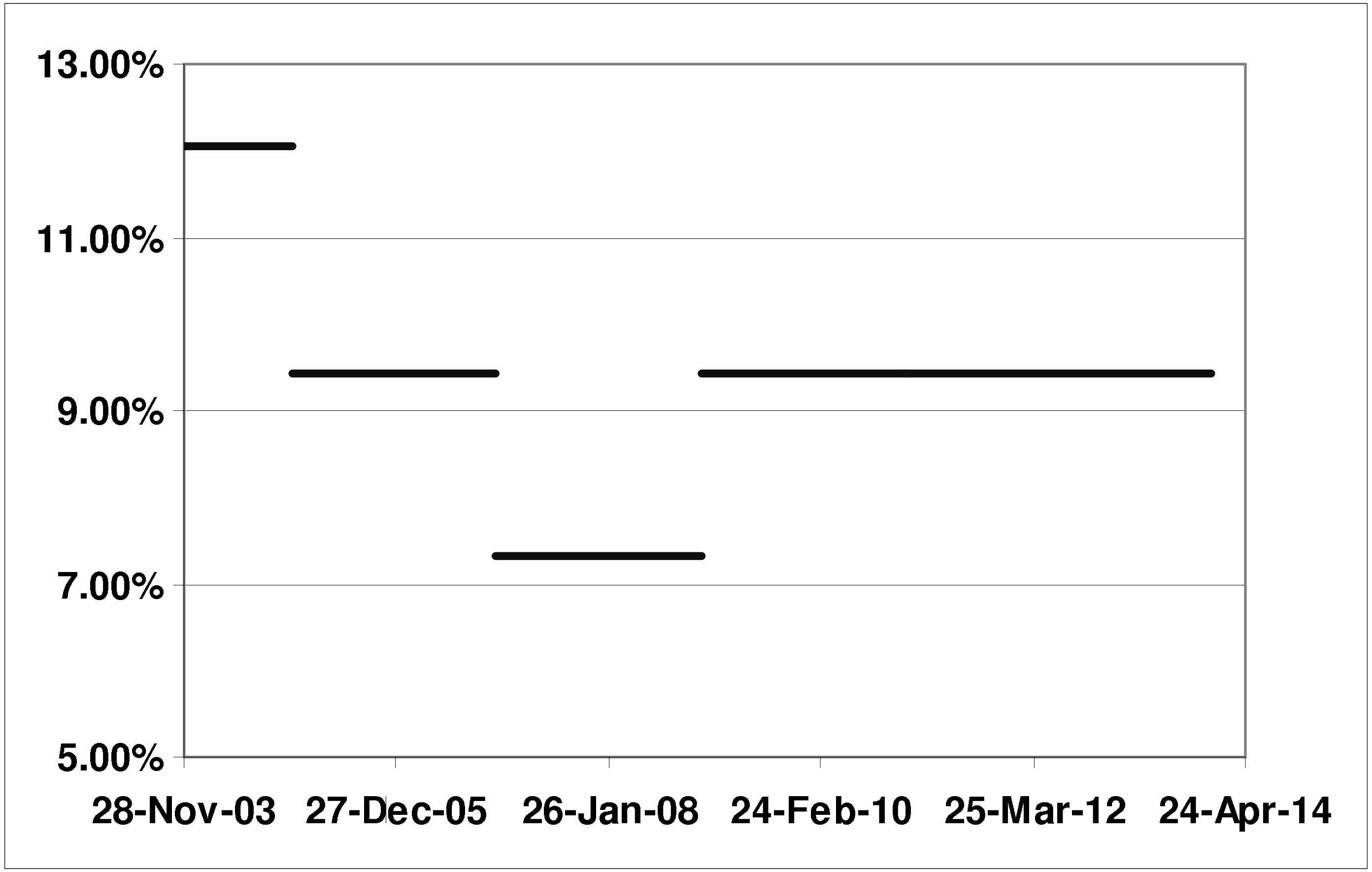


Figure 5: Piecewise constant intensity γ calibrated on CDS quotes on November 28th, 2003.

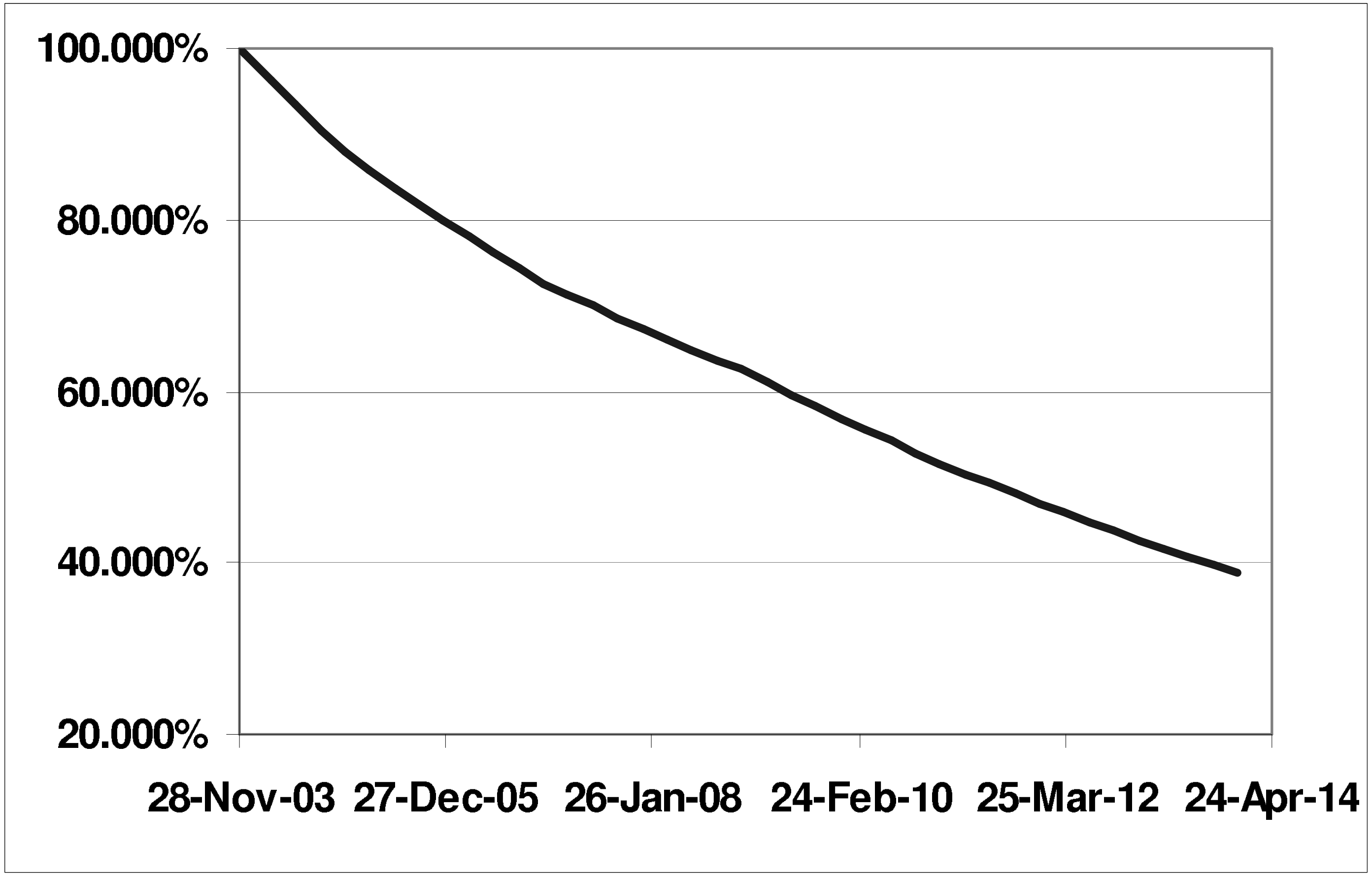


Figure 6: survival probability $\exp(-\Gamma)$ resulting from calibration on CDS quotes on November 28th, 2003.

December 8th, 2003

Recovery Rate $= 25\%$

Maturity T_b (yr)	Maturity (dates)	$R_{0,b}$
1	20-Dec-04	1450
3	20-Dec-06	1200
5	20-Dec-08	940
7	20-Dec-10	850
10	20-Dec-13	850

Table 7: Maturity Dates and corresponding CDS quotes in bps relative to December 8th, 2003.

date	intensity γ	survival pr $\exp(-\Gamma)$
08-Dec-03	19.272%	100.000%
20-Dec-04	19.272%	81.680%
20-Dec-06	7.263%	62.413%
22-Dec-08	2.393%	56.570%
20-Dec-10	11.205%	49.303%
20-Dec-13	11.318%	34.993%

Table 8: Calibration with piecewise linear intensity on December 8th, 2003.

date	intensity γ	survival pr $\exp(-\Gamma)$
08-Dec-03	19.272%	100.000%
20-Dec-04	19.272%	81.680%
20-Dec-06	13.650%	61.931%
22-Dec-08	4.834%	56.126%
20-Dec-10	6.500%	49.213%
20-Dec-13	11.256%	34.934%

Table 9: Calibration with piecewise constant intensity on December 8th, 2003.

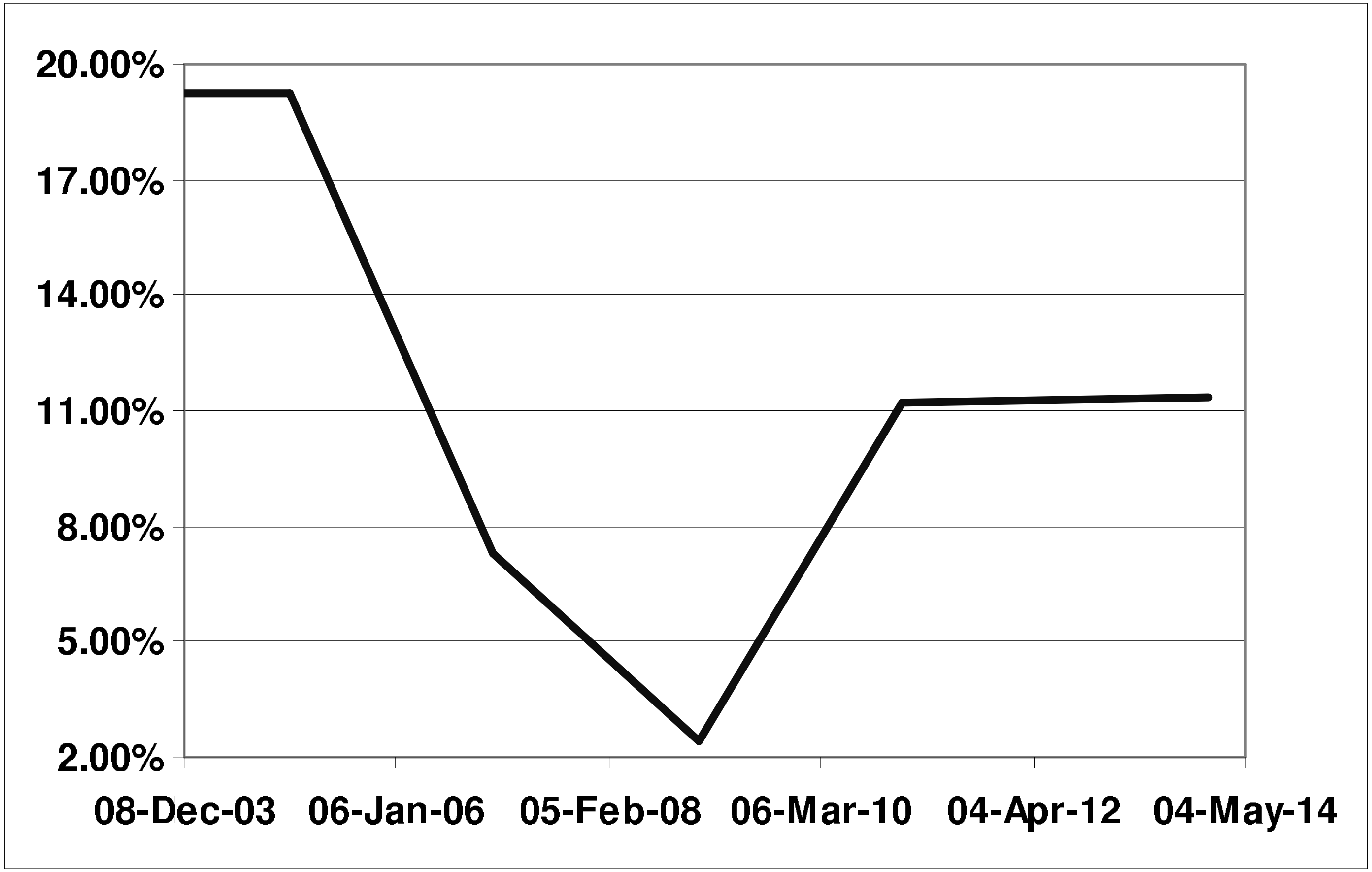


Figure 7: Piecewise linear intensity γ calibrated on CDS quotes on December 8th, 2003.

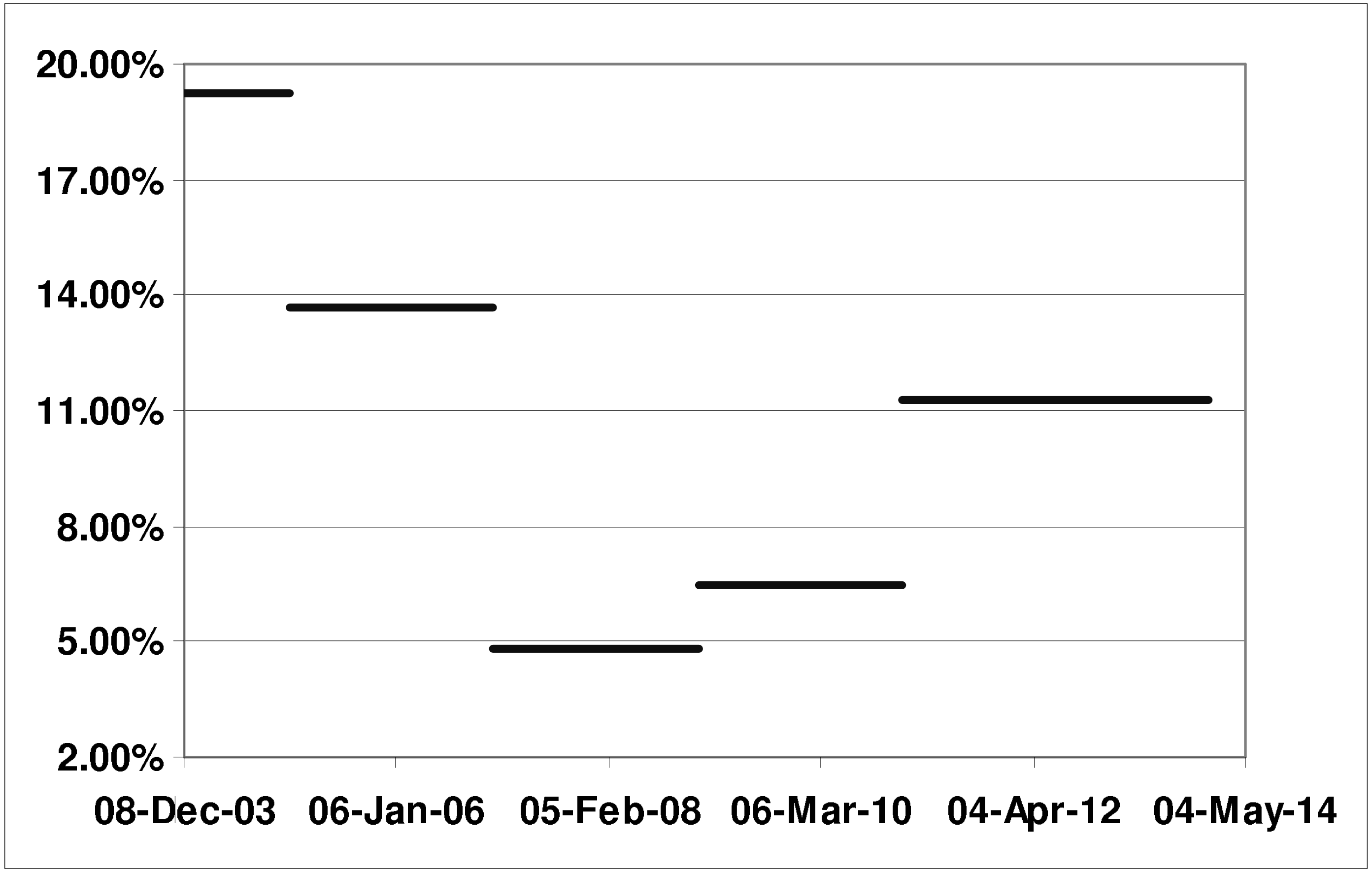


Figure 8: Piecewise constant intensity γ calibrated on CDS quotes on December 8th, 2003.

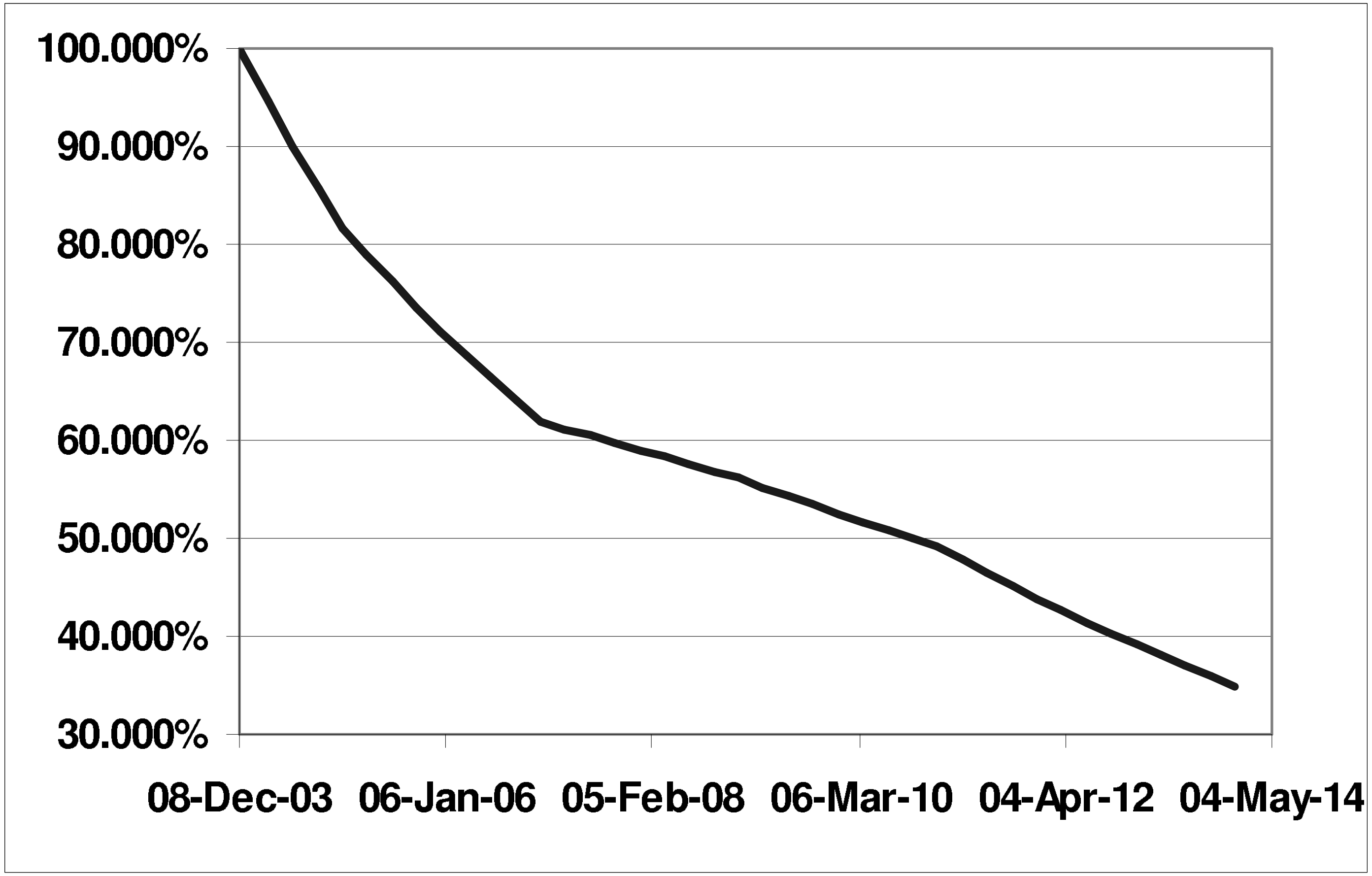


Figure 9: survival probability $\exp(-\Gamma)$ resulting from calibration on CDS quotes on December 8th, 2003.

December 10th, 2003

Recovery Rate $= 15\%$

Maturity T_b (yr)	Maturity (dates)	$R_{0,b}$
1	20-Dec-04	5050
3	20-Dec-06	2100
5	20-Dec-08	1500
7	20-Dec-10	1250
10	20-Dec-13	1100

Table 10: Maturity Dates and corresponding CDS quotes in bps relative to December 10th, 2003.

Payoff Type: Posponed 1

date	intensity γ	survival pr $\exp(-\Gamma)$
10-Dec-03	55.483%	100.000%
20-Dec-04	55.483%	56.018%
20-Dec-06	-61.665%	59.642%
22-Dec-08	84.397%	47.321%
20-Dec-10	-86.408%	48.293%
20-Dec-13	123.208%	27.581%

Table 11: Calibration with pwise linear intensity and postponed payoff 1 on Dec 10, 2003.

date	intensity γ	survival pr $\exp(-\Gamma)$
10-Dec-03	55.483%	100.000%
20-Dec-04	55.483%	56.018%
20-Dec-06	0.807%	55.109%
22-Dec-08	4.017%	50.780%
20-Dec-10	4.292%	46.559%
20-Dec-13	5.980%	38.809%

Table 12: Calibration with pwise const intensity and postponed payoff 1 on Dec 10, 2003.

date	intensity γ	survival pr $\exp(-\Gamma)$
10-Dec-03	64.188%	100.000%
20-Dec-04	64.188%	51.150%
20-Dec-06	-80.163%	60.144%
22-Dec-08	108.007%	45.299%
20-Dec-10	-113.620%	47.944%
20-Dec-13	162.645%	22.732%

Table 13: Calibration with pwise linear intensity and postponed payoff 2.

date	intensity γ	survival pr $\exp(-\Gamma)$
10-Dec-03	64.188%	100.000%
20-Dec-04	64.188%	51.150%
20-Dec-06	-3.270%	54.657%
22-Dec-08	3.900%	50.484%
20-Dec-10	4.282%	46.297%
20-Dec-13	6.065%	38.491%

Table 14: Calibration with piecewise constant intensity and postponed payoff of kind 2.

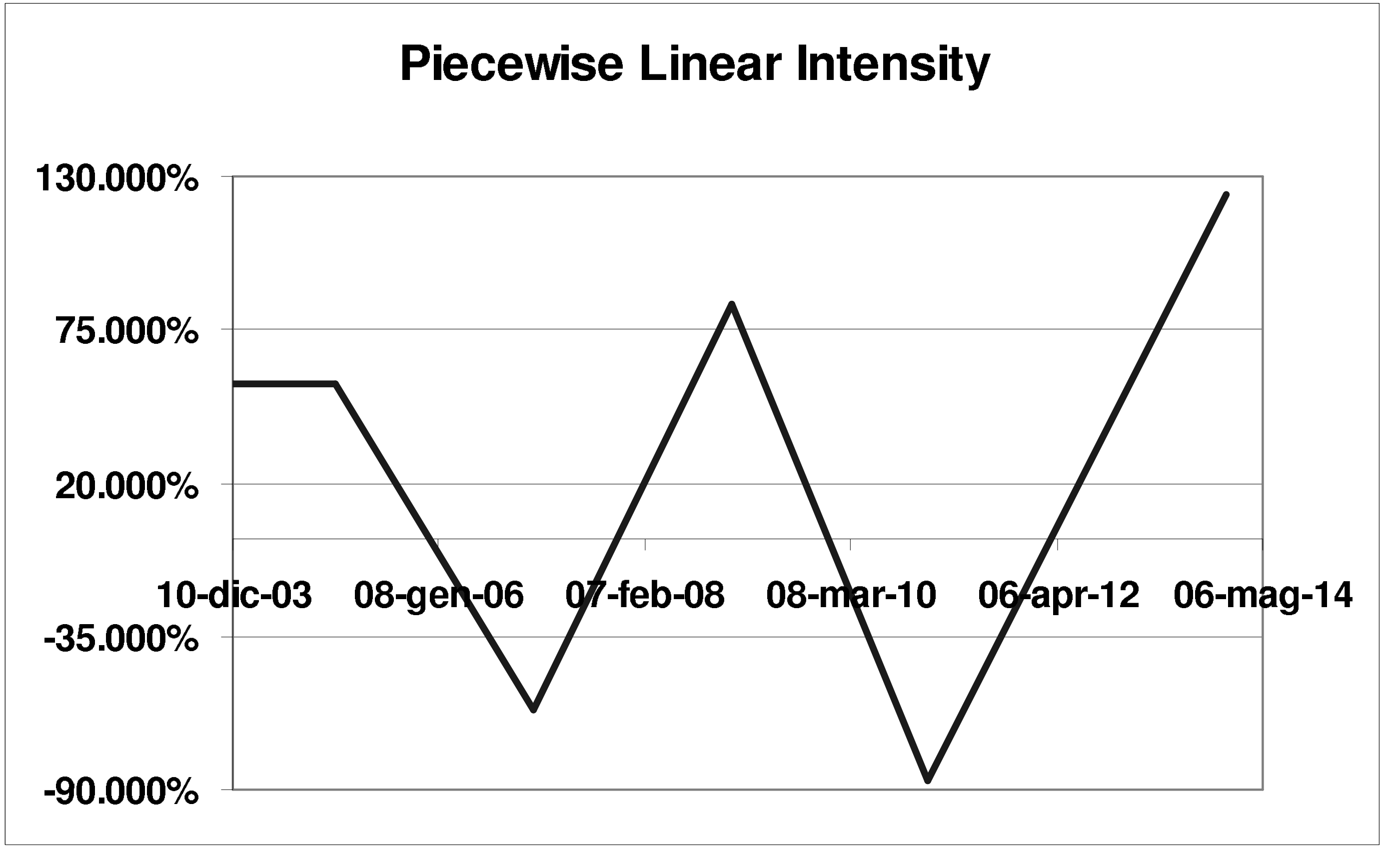


Figure 10: Piecewise linear intensity γ calibrated on CDS quotes on December 10th, 2003, with postponed payoff of kind 1.

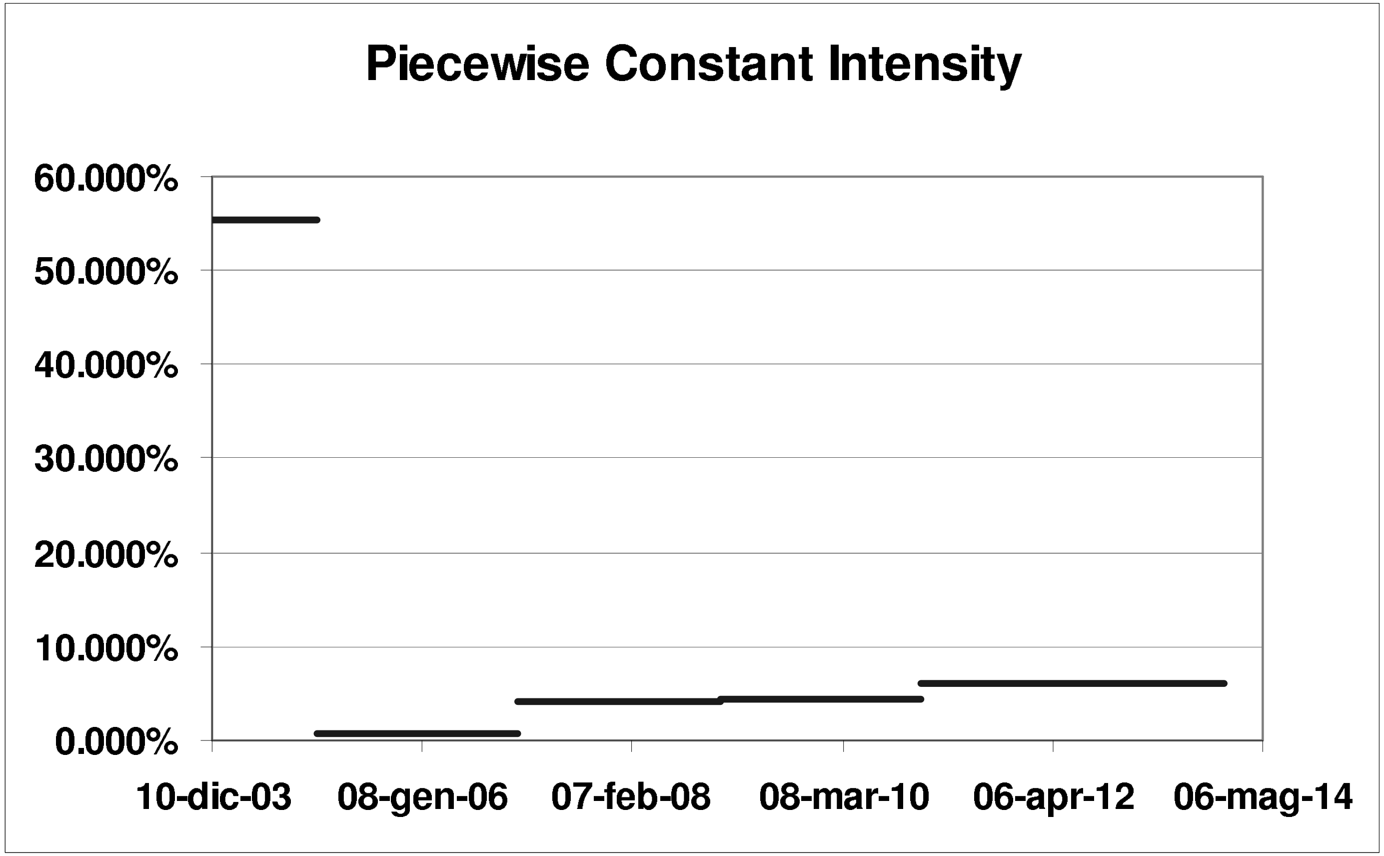


Figure 11: Piecewise constant intensity γ calibrated on CDS quotes on December 10th, 2003, with postponed payoff of kind 1.

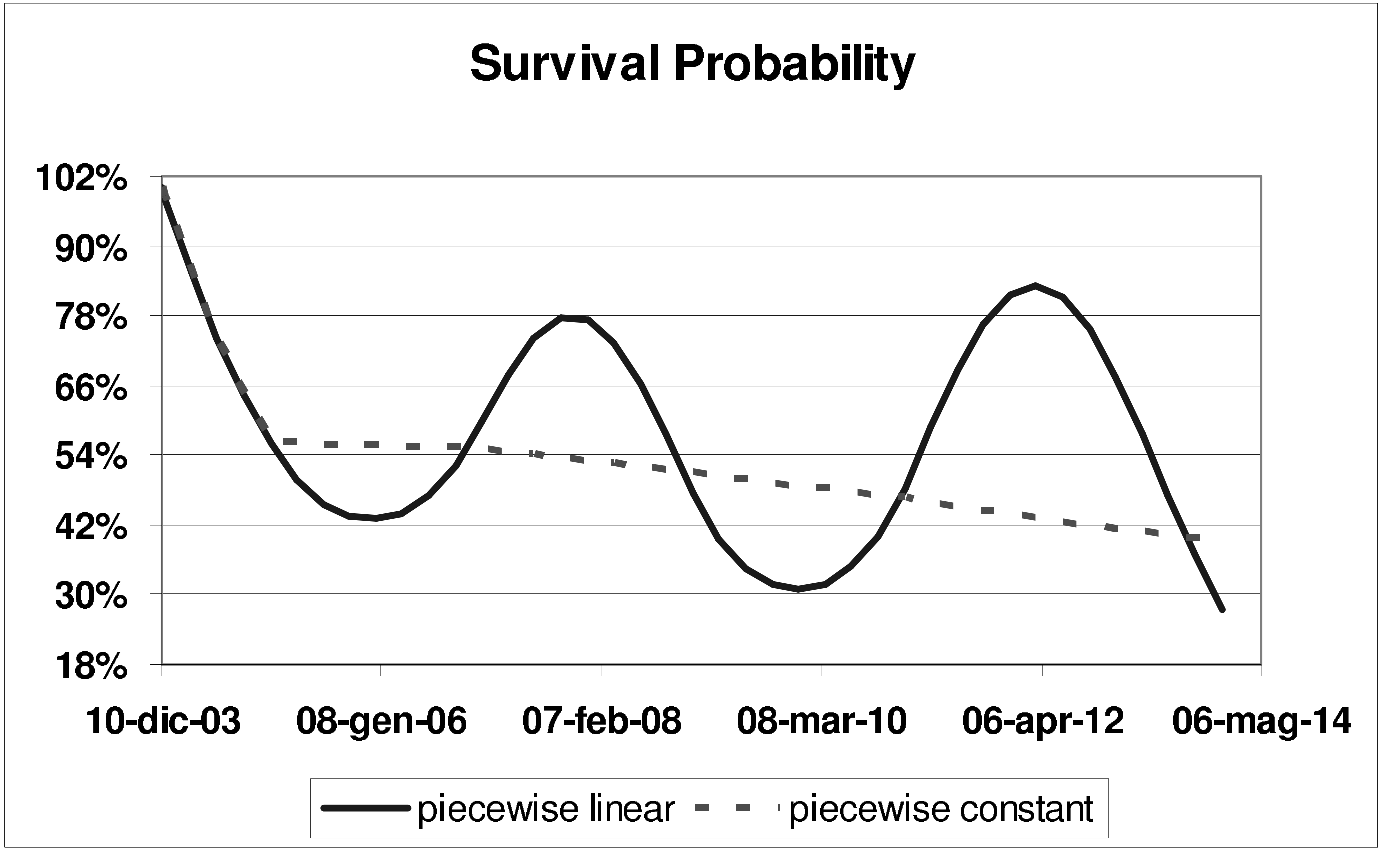


Figure 12: survival probability $\exp(-\Gamma)$ resulting from calibration on CDS quotes on December 10th, 2003, with postponed payoff of kind 1.

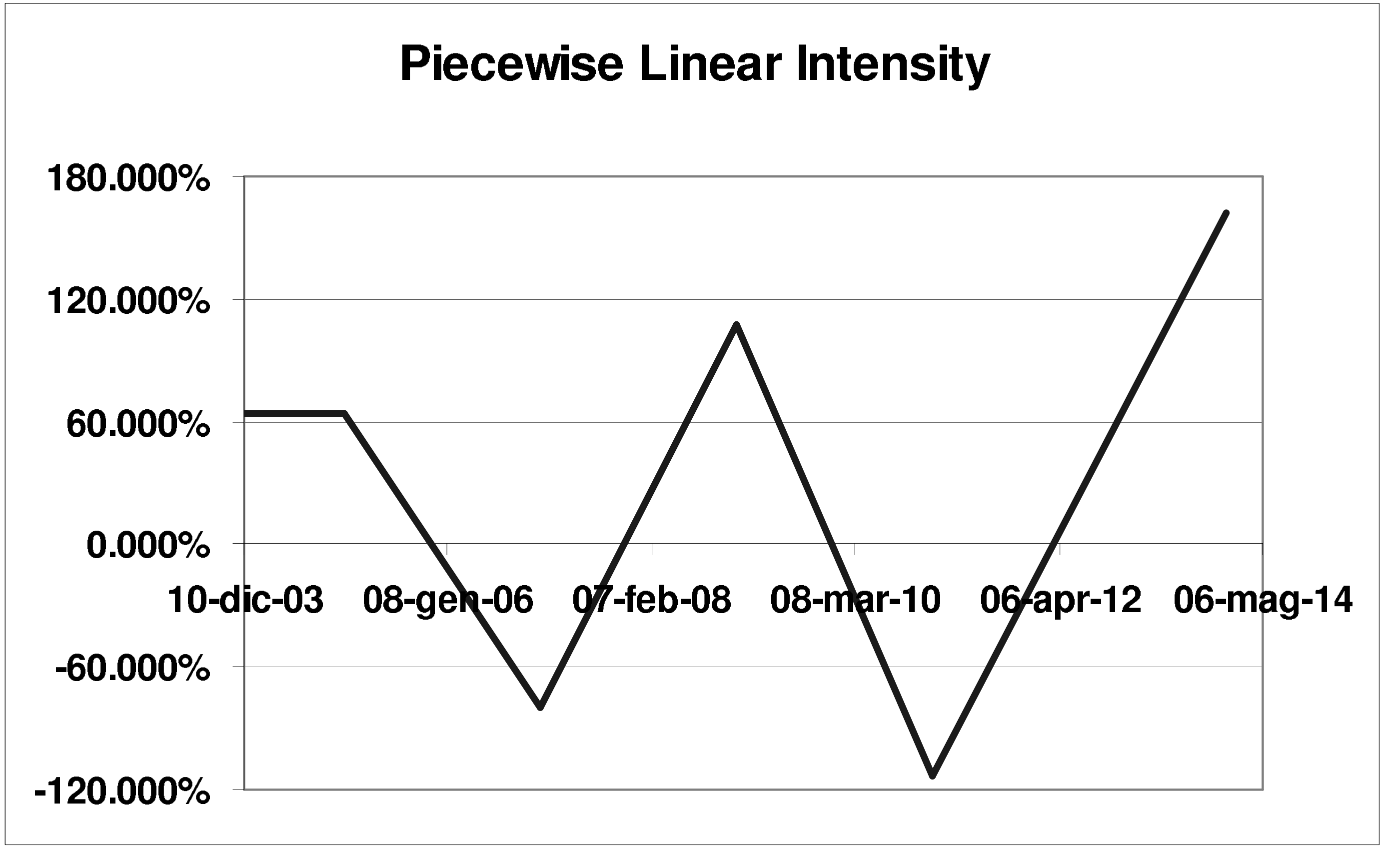


Figure 13: Piecewise linear intensity γ calibrated on CDS quotes on December 10th, 2003, with postponed payoff of kind 2.

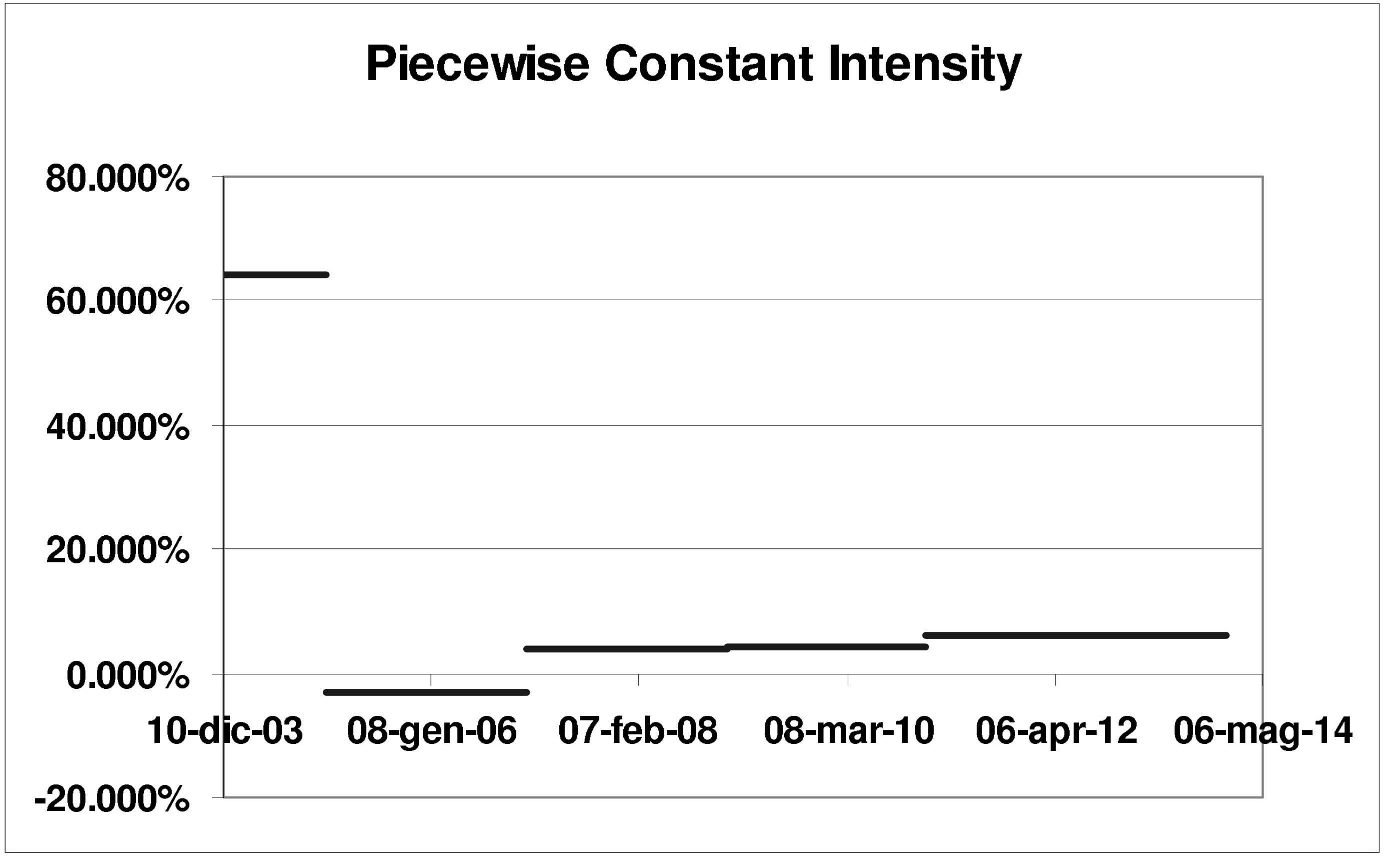


Figure 14: Piecewise constant intensity γ calibrated on CDS quotes on December 10th, 2003, with postponed payoff of kind 2.

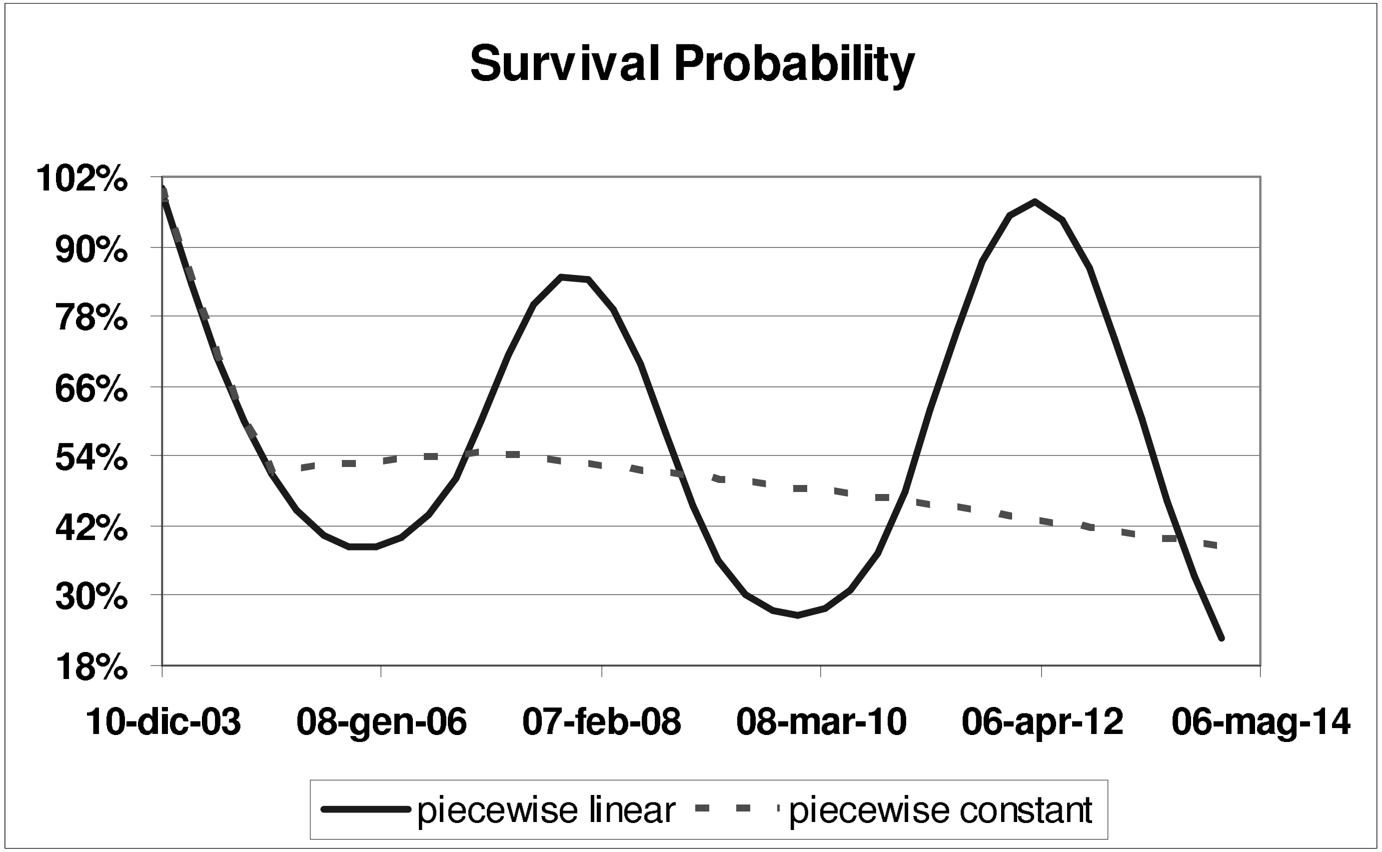


Figure 15: survival probability $\exp(-\Gamma)$ resulting from calibration on CDS quotes on December 10th, 2003, with postponed payoff of kind 2.

A Case Study with AT1P: Lehman Brothers Story

Here we show an intensity model but also the AT1P structural model by Brigo et al (2004-2010), for details on AT1P see

`http://arxiv.org/abs/0912.3028`

`http://arxiv.org/abs/0912.3031`

`http://arxiv.org/abs/0912.4404`

Lehman Brothers CDS Calibration: July 10th, 2007

On the left part of Table 15 we report the values of the quoted CDS spreads before the beginning of the crisis. We see that the spreads are very low. In the middle of Table 15 we have the results of the exact calibration obtained using an intensity model, while on the right part of the Table we have the results of the (exact!) calibration obtained using the AT1P model.

T_i	R_i (bps)	λ_i (bps)	Surv (Int)	σ_i	Surv (AT1P)
10 Jul 2007			100.0%		100.0%
1y	16	0.267%	99.7%	29.2%	99.7%
3y	29	0.601%	98.5%	14.0%	98.5%
5y	45	1.217%	96.2%	14.5%	96.1%
7y	50	1.096%	94.1%	12.0%	94.1%
10y	58	1.407%	90.2%	12.7%	90.2%

Table 15: Results of calibration for July 10th, 2007.

Lehman Brothers CDS Calibration: June 12th, 2008

In Table 16 we report the results of the calibration on June 12th, 2008, in the middle of the crisis. We see that the CDS spreads R_i have increased with respect to the previous case, but are not very high, indicating the fact that the market is aware of the difficulties suffered by Lehman but thinks that it can come out of the crisis. The survival probability resulting from calibration is lower than in the previous case; since the barrier parameter H has not changed, this translates into higher volatilities.

T_i	R_i (bps)	λ_i (bps)	Surv (Int)	σ_i	Surv (AT1P)
12 Jun 2008			100.0%		100.0%
1y	397	6.563%	93.6%	45.0%	93.5%
3y	315	4.440%	85.7%	21.9%	85.6%
5y	277	3.411%	80.0%	18.6%	79.9%
7y	258	3.207%	75.1%	18.1%	75.0%
10y	240	2.907%	68.8%	17.5%	68.7%

Table 16: Results of calibration for June 12th, 2008.

Lehman Brothers CDS Calibration: September 12th, 2008

In Table 17 we report the results of the calibration on September 12th, 2008, just before Lehman's default. We see that the spreads are now very high, corresponding to lower survival probability and higher volatilities than before.

T_i	R_i (bps)	λ_i (bps)	Surv (Int)	σ_i	Surv (AT1P)
12 Sep 2008			100.0%		100.0%
1y	1437	23.260%	79.2%	62.2%	78.4%
3y	902	9.248%	65.9%	30.8%	65.5%
5y	710	5.245%	59.3%	24.3%	59.1%
7y	636	5.947%	52.7%	26.9%	52.5%
10y	588	6.422%	43.4%	29.5%	43.4%

Table 17: Results of calibration for September 12th, 2008.

The CDS Big Bang

In the north American CDS market we have seen in spring 2009 the introduction of new forms of Credit Default Swap (CDS) contracts where protection is paid as an upfront payment plus a fixed coupon, either 100bps or 500bps, depending on the credit quality of the name. Recovery is also fixed to either 20% or 40%. Simplifications doing away with restructuring credit events and other issues have been included and favor standardization. There is aim at having CDS contracts going through exchanges and being guaranteed by clearing houses.

This would reduce counterparty risk

The CDS Big Bang

Given the transition from running to upfront, ISDA proposed a methodology for conversion of CDS quotes between upfront and running (www.cdsmodel.com). The proposed flat hazard rate (FHR) conversion method is to be understood as a rule-of-thumb single-contract quoting mechanism rather than as a modeling device.

They propose to do the conversion on a single deal by deal basis, with a (different) constant hazard rate for each deal.

This implies some problems. First, inconsistency of the model used for converting single CDS on the same name with different maturities. For example, an hypothetical investor who would put the FHR converted running spreads into her old running CDS library would strip wrong hazard rates, inconsistent with those coming directly from the quoted term structure of upfronts.

The CDS Big Bang

This new methodology appears mostly as a device to transit the market towards adoption of the new upfront CDS as direct trading products while maintaining a semblance of running quotes for investors who may be suffering the transition. We caution though that

(i) the conversion done with proper hazard rates consistent across term would produce different results;

(ii) the quantities involved in the conversion should not be used as modeling tools anywhere; and

(iii) For highly distressed names with a high upfront paid by the protection buyer, the conversion to running spreads fails unless, as we propose, a third recovery scenario of 0% is added to the suggested 20% and 40%.

The CDS Big Bang: from upfront to running

Name	Upfronts					Recovery Rate
	20-Jun-10	20-Jun-12	20-Jun-14	20-Jun-16	20-Jun-19	
ArcelorMittal Finance SCA	-8.66%	-14.79%	-17.38%	-18.38%	-18.06%	40%
Continental AG	-16.85%	-23.15%	-25.51%	-26.03%	-25.94%	40%
American International Group Inc.	-25.29%	-32.58%	-34.92%	-35.56%	-36.44%	35%
Hitachi, Ltd.	-0.72%	-3.00%	-5.75%	-8.10%	-11.80%	35%

Table 18: Term structure of upfronts for the four reference entities used in the examples

Proper mechanism

Name	Rec	Fair Spread	Rec	Conventional	Difference
ArcelorMittal Finance SCA	40%	852.57	40%	827.17	25.40
Continental AG	40%	1,112.90	40%	1,037.78	75.12
American International Group Inc.	35%	1,523.00	40%	1,467.23	55.77
Hitachi, Ltd.	35%	234.80	40%	238.72	-3.92

Table 19: Fair and conventional spreads for maturity 20-Jun-19.

The CDS Big Bang: from running to upfront

Name	Spreads					Recovery Rate
	20-Jun-10	20-Jun-12	20-Jun-14	20-Jun-16	20-Jun-19	
ArcelorMittal Finance SCA	1287	1109.86	1009.57	938.57	852.57	40%
Continental AG	2168.64	1607.98	1388.67	1245.89	1112.9	40%
American International Group Inc.	3197.28	2274.91	1913.48	1695.9	1523	35%
Hitachi, Ltd.	157.92	195.57	217.21	223.87	234.8	35%

Table 20: Term structure of spreads for the four reference entities used in the examples

Proper mechanism

Name	Fixed Spread	Rec	PV	Premium PV	Protection PV
ArcelorMittal Finance SCA	500	40%	1,806,384.70	2,561,739.08	4,368,123.78
Continental AG	500	40%	2,593,569.97	2,115,818.22	4,709,388.18
American International Group Inc.	500	35%	3,643,573.65	1,780,827.79	5,424,401.43
Hitachi, Ltd.	100	35%	1,180,156.83	875,487.26	2,055,644.09

Conversion mechanism

Name	Fixed Spread	Rec	PV	Premium PV	Protection PV
ArcelorMittal Finance SCA	500	40%	1,914,127.72	2,714,535.72	4,628,663.44
Continental AG	500	40%	2,823,185.85	2,303,137.42	5,126,323.27
American International Group Inc.	500	40%	3,741,810.30	1,828,841.79	5,570,652.09
Hitachi, Ltd.	100	40%	1,150,463.77	853,459.77	2,003,923.55

Proper - Conversion (as %)

Name	Fixed Spread	Rec	PV	Premium PV	Protection PV
ArcelorMittal Finance SCA	500	40%	-1.08%	-1.53%	-2.61%
Continental AG	500	40%	-2.30%	-1.87%	-4.17%
American International Group Inc.	500	40%	-0.98%	-0.48%	-1.46%
Hitachi, Ltd.	100	40%	0.30%	0.22%	0.52%

Table 21: Present values for a maturity of 20-Jun-19 using the proper mechanism and the proposed conversion mechanism for a notional of 10,000,000 and a zero upfront payment.

CDS Big Bang

Notice also that following december 2008 GM traded above 80 upfront.

It is therefore impossible to convert it into running with the current methodology at CDSMODEL.com

FEATURE ARTICLE

Charting a course through the CDS Big Bang

Stochastic Intensity. The SSRJD model

We have seen in detail CDS calibration and MC simulation in presence of **deterministic** and **time varying** intensity or hazard rates, $\gamma(t)dt = \mathbb{Q}\{\tau \in dt | \tau > t, \mathcal{F}_t\}$

As explained, this accounts for credit spread structure but not for **volatility**.

The latter is obtained moving to stochastic intensity (Cox process). The deterministic function $t \mapsto \gamma(t)$ is replaced by a stochastic process $t \mapsto \lambda(t) = \lambda_t$. The Hazard function $\Gamma(t) = \int_0^t \gamma(u)du$ is replaced by the Hazard process (or cumulated intensity) $\Lambda(t) = \int_0^t \lambda(u)du$.

Recall that $\Lambda(\tau) = \xi$, std exponential RV independent of $\mathcal{F}$, and thus $\tau = \Lambda^{-1}(\xi)$.

Recall that the intensity and the hazard process are $\mathcal{F}_t$-measurable, i.e. they are known at a given time based on default-free market information at that time.

It is the default component ξ that is independent of anything else and thus "impossible to predict".

The SSRJD Model: CIR++ stochastic intensity λ

We model the stochastic intensity as follows: consider

$$\lambda_t = y_t + \psi(t;\beta)\ , \quad t \geq 0,$$

where the intensity has a random component y and a deterministic component ψ to fit the CDS term structure. For y we take a Jump-CIR model

$$dy_t = \kappa(\mu - y_t)dt + \nu\sqrt{y_t}dZ_t + dJ_t,\ \beta = (\kappa, \mu, \nu, y_0),\ 2\kappa\mu > \nu^2.$$

Jumps are taken themselves independent of anything else, with exponential arrival times with intensity η and exponential jump size with a given parameter.

With no jumps J, $y \sim$ nonc. chi-square; Very important: $y > 0$.

Here we focus on the case without jumps, see B and El-Bachir (2006) or B and M (2006) for the case with jumps.

The SSRD Model: CIR++ stochastic intensity λ. Calibrating Implied Default Probabilities

About the parameters of CIR:

$$dy_t = \kappa(\mu - y_t)dt + \nu\sqrt{y_t}dZ_t$$

κ: speed of mean reversion

μ: long term mean reversion level

ν: volatility.

The SSRD Model: CIR++ stochastic intensity λ. Calibrating Implied Default Probabilities

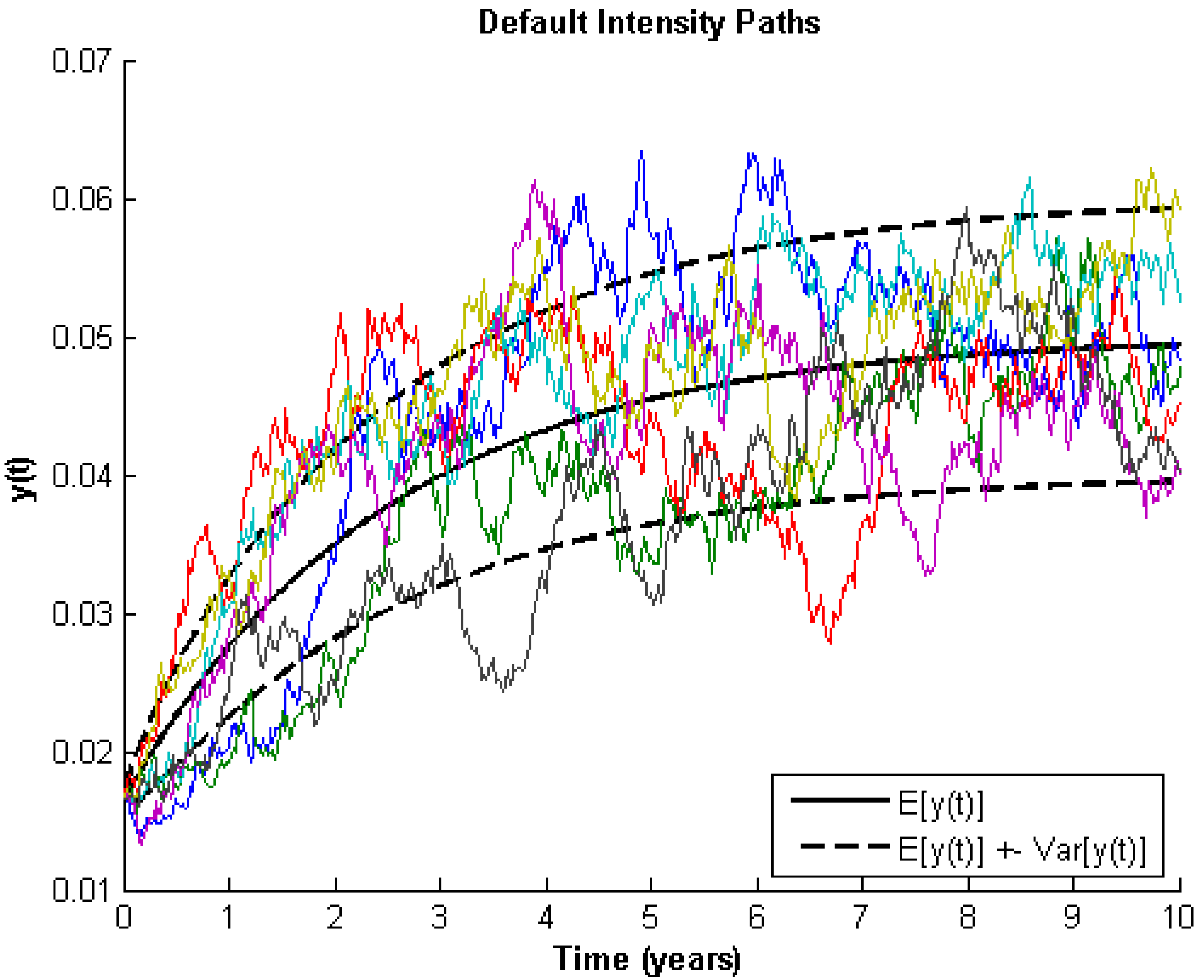
$$E[\lambda_t] = \lambda_0 e^{-\kappa t} + \mu(1 - e^{-\kappa t})$$
$$\mathsf{VAR}(\lambda_t) = \lambda_0 \frac{\nu^2}{\kappa}(e^{-\kappa t} - e^{-2\kappa t}) + \mu \frac{\nu^2}{2\kappa}(1 - e^{-\kappa t})^2$$

After a long time the process reaches (asymptotically) a stationary distribution around the mean μ and with a corridor of variance $\mu\nu^2/2\kappa$.

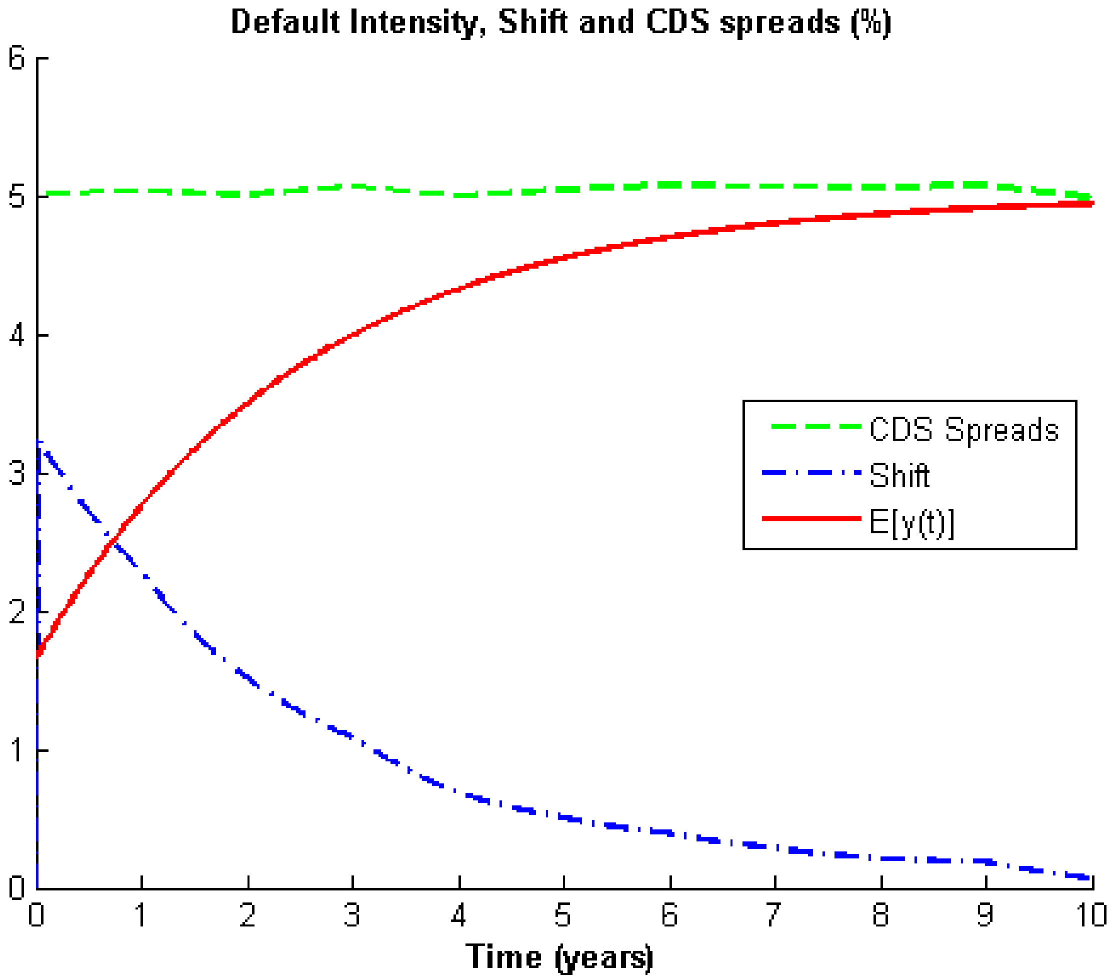
The largest κ, the fastest the process converges to the stationary state. So, ceteris paribus, increasing κ kills the volatility of the credit spread.

The largest μ, the highest the long term mean, so the model will tend to higher spreads in the future in average.

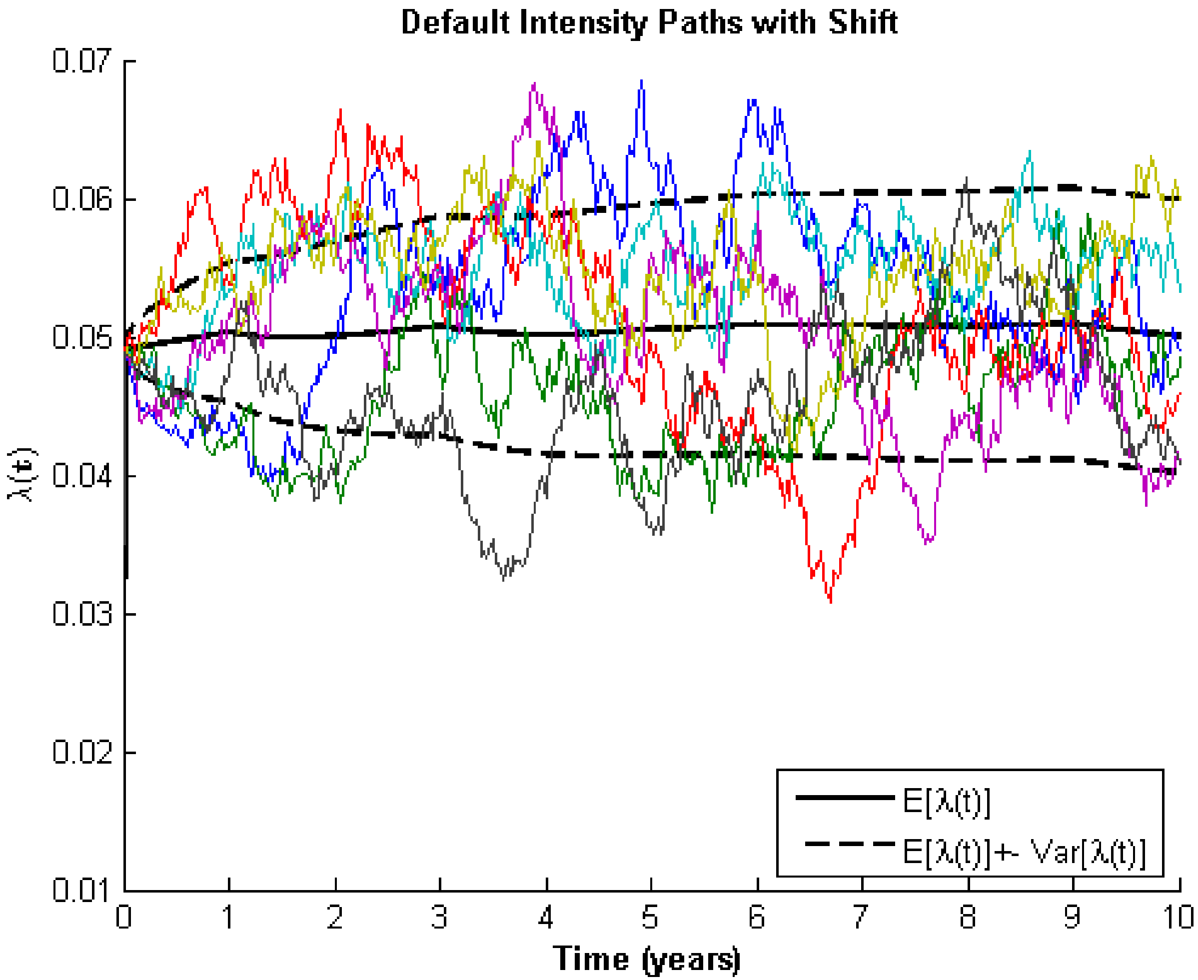
The largest ν, the largest the volatility. Notice however that κ and ν fight each other as far as the influence on volatility is concerned. We see some plots



$$y0 = 0.0165, \kappa = 0.4, \mu = 0.05, \nu = 0.04$$



$y0 = 0.0165, \kappa = 0.4, \mu = 0.05, \nu = 0.04$ shift, mean and mkt hazard rates



$y0 = 0.0165, \kappa = 0.4, \mu = 0.05, \nu = 0.04$ + shift: λ_t.

EXERCISE: The CIR model

EXERCISE. Assume we are given a stochastic intensity process of CIR type,

$$dy_t = \kappa(\mu - y_t)dt + \nu\sqrt{y_t}dW(t)$$

where y_0, κ, μ, ν are positive constants. W is a brownian motion under the risk neutral measure.

a) Increasing κ increases or decreases randomness in the intensity? And ν?

b) The mean of the intensity at future times is affected by k? And by ν?

c) What happens to mean of the intensity when time grows to infinity?

d) Is it true that, because of mean reversion, the variance of the intensity goes to zero (no randomness left) when time grows to infinity?

e) Can you compute a rough approximation of the percentage volatility in the intensity?

f) Suppose that $y_0 = 400bps = 0.04$, $\kappa = 0.3$, $\nu = 0.001$ and $\mu = 400bps$. Can you guess the behaviour of the future random trajectories of the stochastic intensity after time 0?

g) Can you guess the spread of a CDS with 10y maturity with the above stochastic intensity when the recovery is 0.35?

EXERCISE Solutions.

a) We can refer to the formulas for the mean and variance of y_T in a CIR model as seen from time 0, at a given T. The formula for the variance is known to be (see for Example Brigo and Mercurio (2006))

$$\mathsf{VAR}(y_T) = y_0 \frac{\nu^2}{\kappa}(e^{-\kappa T} - e^{-2\kappa T}) + \mu \frac{\nu^2}{2\kappa}(1 - e^{-\kappa T})^2$$

whereas the mean is

$$E[y_T] = y_0 e^{-\kappa T} + \mu(1 - e^{-\kappa T})$$

We can see that for k becoming large the variance becomes small, since the exponentials decrease in k and the division by k gives a small value for large k. In the limit

$$\lim_{\kappa \to +\infty} \mathsf{VAR}(y_T) = 0$$

so that for very large κ there is no randomness left.

We can instead see that VAR(y_T) is proportional to ν^2, so that if ν increases randomness increases, as is obvious from $\nu\sqrt{y_t}$ being the instantaneous volatility in the process y.

b) As the mean is

$$E[y_T] = y_0 e^{-\kappa T} + \mu(1 - e^{-\kappa T})$$

we clearly see that this is impacted by κ (indeed, "speed of mean reversion") and by μ clearly ("long term mean") but not by the instantaneous volatility parameter ν.

c) As T goes to infinity, we get for the mean

$$\lim_{T\to+\infty} y_0 e^{-\kappa T} + \mu(1 - e^{-\kappa T}) = \mu$$

so that the mean tends to μ (this is why μ is called "long term mean").

d) In the limit where time goes to infinity we get, for the variance

$$\lim_{T\to+\infty}[y_0\frac{\nu^2}{\kappa}(e^{-\kappa T}-e^{-2\kappa T})+\mu\frac{\nu^2}{2\kappa}(1-e^{-\kappa T})^2]=\mu\frac{\nu^2}{2\kappa}$$

So this does not go to zero. Indeed, mean reversion here implies that as time goes to infinite the mean tends to μ and the variance to the constant value $\mu\frac{\nu^2}{2\kappa}$, but not to zero.

e) Rough approximations of the percentage volatilities in the intensity would be as follows. The instantaneous variance in dy_t, conditional on the information up to t, is (remember that $VAR(dW(t))=dt$)

$$VAR(dy_t)=\nu^2 y_t dt$$

The percentage variance is

$$VAR\left(\frac{dy_t}{y_t}\right)=\frac{\nu^2 y_t}{y_t^2}dt=\frac{\nu^2}{y_t}dt$$

and is state dependent, as it depends on y_t. We may replace y_t with either its initial value y_0 or with the long term mean μ, both known. The two rough percentage volatilities estimates will then be, for $dt = 1$,

$$\sqrt{\frac{\nu^2}{y_0}} = \frac{\nu}{\sqrt{y_0}}, \quad \sqrt{\frac{\nu^2}{\mu}} = \frac{\nu}{\sqrt{\mu}}$$

These however do not take into account the important impact of κ in the overall volatility of finite (as opposed to instantaneous) credit spreads and are therefore relatively useless.

f) First we check if the positivity condition is met.

$$2\kappa\mu = 2 \cdot 0.3 \cdot 0.04 = 0.024; \quad \nu^2 = 0.001^2 = 0.000001$$

hence $2\kappa\mu > \nu^2$ and trajectories are positive. Then we observe that the variance is very small: Take $T = 5y$,

$$\mathsf{VAR}(y_T) = y_0 \frac{\nu^2}{\kappa}(e^{-\kappa T} - e^{-2\kappa T}) + \theta \frac{\nu^2}{2\kappa}(1 - e^{-\kappa T})^2 \approx 0.0000006.$$

Take the standard deviation, given by the square root of the variance:

$$\mathsf{STDEV}(y_T) \approx \sqrt{0.0000006} = 0.00077.$$

which is much smaller of the level 0.04 at which the intensity refers both in terms of initial value and long term mean. Therefore there is almost no randomness in the system as the variance is very small compared to the initial point and the long term mean.

Hence there is almost no randomness, and since the initial condition y_0 is the same as the long term mean $\mu_0 = 0.04$, the intensity will behave as if it had the value 0.04 all the time. All future trajectories will be very close to the constant value 0.04.

g) In a constant intensity model the CDS spread can be approximated by

$$y = \frac{R_{CDS}}{1 - REC} \Rightarrow R_{CDS} = y(1 - REC) = 0.04(1 - 0.35) = 260bps$$

The SSRD Model: CIR++ stochastic intensity λ. Calibrating Implied Default Probabilities

For restrictions on the β's that keep ψ and hence λ positive, **as is required in intensity models**, we may use the results in B. and M. (2001) or (2006). We will often use the hazard process $\Lambda(t) = \int_0^t \lambda_s ds$, and also $Y(t) = \int_0^t y_s ds$ and $\Psi(t, \beta) = \int_0^t \psi(s, \beta) ds$.

If we can read from the market some implied risk-neutral default probabilities, and associate to them implied hazard functions Γ^{Mkt}, we may wish our stochastic intensity model to agree with them. This happens if

$$\exp(-\Gamma^{\mathsf{Mkt}}(t)) = \exp\left(-\Psi(t, \beta)\right) \mathbb{E}[e^{-\int_0^t y_s ds}]$$

IMPORTANT 1: This is possible only if λ is strictly positive;

IMPORTANT 2: It is fundamental, if we aim at calibrating default probabilities, that the last expected value can be computed analytically.

The only known diffusion model used in interest rates satisfying both constraints is CIR++

The SSRD Model: CIR++ stochastic intensity λ. Calibrating Implied Default Probabilities

$$\exp(-\Gamma^{\mathsf{Mkt}}(t)) = \mathbb{Q}\{\tau > t\} = \exp\left(-\Psi(t,\beta)\right)\mathbb{E}[e^{-\int_0^t y_s ds}]$$

Now notice that $\mathbb{E}[e^{-\int_0^t y_s ds}]$ is simply the bond price for a CIR interest rate model with short rate given by y, so that it is known analytically. We denote it by $P^y(0,t,y_0;\beta)$.

Similarly to the interest-rate case, λ is calibrated to the market implied hazard function Γ^{Mkt} if we set

$$\Psi(t,\beta) := \Gamma^{\mathsf{Mkt}}(t) + \ln(P^y(0,t,y_0;\beta))$$

where we choose the parameters β in order to have a positive function ψ, by resorting to the condition seen earlier.

Market Models for CDS Options

Recall the definition of CDS forward rate $R_{a,b}(t)$ for protection in $[T_a, T_b]$ as that rate in the premium leg of a (forward start) CDS protecting in $[T_a, T_b]$ that makes the forward CDS value equal to zero at the valuation time t. Since
CDS = R AccruedTerm (Accrued Premium) + R Annuity(premium leg) - ProtectionLeg,
we have R = ProtectionLeg/(AccruedTerm + Annuity)

$$0 = \boxed{R}\,\mathbb{E}[D(t,\tau)(\tau - T_{\beta(\tau)-1})\mathbf{1}_{\{T_a<\tau<T_b\}}|\mathcal{F}_t]+$$

$$+\boxed{R}\sum_{i=a+1}^{b}\alpha_i\mathbb{E}[D(t,T_i)\mathbf{1}_{\{\tau>T_i\}}|\mathcal{F}_t] - \mathrm{L{\scriptstyle GD}}\,\mathbb{E}[\mathbf{1}_{\{T_a<\tau\leq T_b\}}D(t,\tau)\;|\mathcal{F}_t]$$

where we have used a technicality (the filtration switching formula to value the CDS given $\mathcal{F}_t$ rather than $\mathcal{G}_t$).

Market Models for CDS Options

Solving in R we get

$$R_{a,b}(t) = \frac{\text{LGD} \sum_{i=a+1}^{b} \mathbb{E}[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}}|\mathcal{F}_t]}{\sum_{i=a+1}^{b} \alpha_i \mathbb{E}[D(t,T_i)\mathbf{1}_{\{\tau>T_i\}}|\mathcal{F}_t] + \mathbb{E}[D(t,\tau)(\tau - T_{\beta(\tau)-1})\mathbf{1}_{\{T_a<\tau<T_b\}}|\mathcal{F}_t]}$$

Consider again the **option** to enter a CDS at the future time T_a paying a fixed rate K at times $T_{a+1}, \ldots, T_b$ or until default, in exchange for a protection payment LGD against possible default in $[T_a, T_b]$ (payer CDS option).

One may wish to introduce implied volatility for CDS options. This would be a volatility associated to the relevant underlying CDS rate R.

Market Models for CDS Options

In order to introduce implied volatility rigorously, one has to come up with an appropriate dynamics for $R_{a,b}$ directly, rather than modeling instantaneous default intensities λ explicitly. This parallels the default-free interest rate market when we resort to the swap market model as opposed for example to a one-factor short-rate model for pricing swaptions.

$$
\begin{array}{ccccc}
\text{Fixed Income} & & \text{inst. interest rate } r & \text{vs} & \text{swap rate } S_{a,b} \\
\updownarrow & & \updownarrow & & \updownarrow \\
\text{Credit} & & \text{inst. intensity } \lambda & \text{vs} & \text{CDS rate } R_{a,b}
\end{array}
$$

CDS Options: Market Models (embedded Stochastic Intensity)

The full derivation of the model and formula is quite technical and involves measure changes with singularity removal. We do not consider it here, interested readers are referred to B. (2005) or B and M (2006).

Assuming a lognormal spread dynamics for $R_{a,b}$ with volatility $\sigma_{a,b}$ (but we could also price with smile and jumps, the framework is completely general) and a few other approximations we get the market formula

CDS Options: Market Models (embedded Stochastic Intensity)

$$\begin{aligned}\text{PayerCDSOpt}_{a,b}(t, K; \text{L}_{\text{GD}}) &= \mathbb{E}\{1_{\{\tau>T_a\}} D(t,T_a)\bar{C}_{a,b}(T_a)(R_{a,b}(T_a)-K)^{+}|\mathcal{G}_t\} \\ &= 1_{\{\tau>t\}}\bar{C}_{a,b}(t)[R_{a,b}(t)N(d_1(t)) - KN(d_2(t))] \\ d_{1,2} &= \left(\ln(R_{a,b}(t)/K) \pm (T_a - t)\sigma^2_{a,b}/2\right)/(\sigma_{a,b}\sqrt{T_a-t}).\end{aligned}$$

$$\bar{C}_{a,b}(t) := \sum_{i=a+1}^{b} \alpha_i \bar{P}(t,T_i).$$

As happens in most markets, this Black-like formula could be used as a implied volatility quoting mechanism rather than as a real model formula.

CDS Options

Now some examples of CDS implied volatilities (Model implementation by Marco Tarenghi).

C1 = Deutsche Telecom; C2 = Daimler Chrysler; C3 = France Telecom

Euro market; CDS options quotes as of March 26, 2004; $\text{R}{\small\text{EC}} = 0.4$; $\text{L}{\small\text{GD}} = 1 - 0.4 = 0.6$; $T_0 =$ March 26 2004 (0); We consider two possible maturities $T_a =$June 20 2004 (86d≈3m) and $T'_a =$Dec 20 2004 (269d≈9m); $T_b =$ june 20 2009 (5y87d); we consider receiver option quotes (puts on R) in basis points (i.e. 1E-4 units on a notional of 1). We obtain

CDS Options

	Option: bid	mid	ask	$R_{0,b}(0)$	$R_{a,b}^{PR}(0)$	$R_{a,b}^{PR2}(0)$	K	$\sigma_{a,b}^{PR}$	$\sigma_{a,b}^{PR(2)}$
C1(T_a)	14	24	34	60	61.497	61.495	60	50.31	50.18
C2	32	39	46	94.5	97.326	97.319	94	54.68	54.48
C3	18	25	32	61	62.697	62.694	61	52.01	51.88
C1(T'_a)	28	35	42	60	65.352	65.344	61	51.45	51.32

Implied volatilities are rather high when compared with typical interest-rate default free swaption volatilities. However, the values we find have the same order of magnitude as some of the values found by Hull and White (2003) via historical estimation.

Further, we see that while the option prices differ considerably, the related implied volatilities are rather similar. This shows the usefulness of a rigorous model for implied volatilities. The mere price quotes could have left one uncertain on whether the credit spread variabilities implicit in the different companies were quite different from each other or similar.

CDS Options

We analyze also the implied volatilities and CDS forward rates under different payoff formulations and under stress. The above Table shows the impact of changing postponement from PR to PR2 (PR2 is a second postponement possibility we did not consider in detail here). When we change formulation we maintain the same market $R_{0,b}(0)$'s and from them we re-strip the hazard functions needed to compute the forwards $R_{a,b}(0)$'s and the numeraire at time 0. The change of postponement leaves both CDS forward rates and implied volatilities almost unchanged.

CDS Options

In the next Table we check the impact of the recovery rate on implied volatilities and CDS forward rates. Every time we change recovery we re-strip the hazard functions from the same market $R_{0,b}(0)$'s. We re-strip hazard functions because $R_{0,b}(0)$'s are given by the market and we cannot change them, whereas our uncertainty is on the recovery rate, that might change. As we can see from the table the impact of the recovery rate is rather small, but we have to keep in mind that the CDS option payoff is built in such a way that the recovery direct flow in LGD cancels and the recovery remains only implicitly inside the initial condition $R_{a,b}(0)$ for the dynamics of $R_{a,b}$. Notice indeed that REC does not appear explicitly in the payoff

$$1_{\{\tau>T_a\}}D(t,T_a)\bar{C}_{a,b}(T_a)(R_{a,b}(T_a)-K)^+$$

CDS Options

	REC = 20%	REC = 30%	REC = 40%	REC = 50%	REC = 60%
$\sigma^{PR}_{a,b}$:					
C1(T_a)	50.02	50.14	50.31	50.54	50.90
C2	54.22	54.42	54.68	55.05	55.62
C3	51.71	51.83	52.01	52.25	52.61
C1(T'_a)	51.13	51.27	51.45	51.71	52.10
$R^{PR}_{a,b}$:					
C1(T_a)	61.488	61.492	61.497	61.504	61.514
C2	97.303	97.313	97.326	97.346	97.374
C3	62.687	62.691	62.697	62.704	62.716
C1(T'_a)	65.320	65.334	65.352	65.377	65.415

Table 22: Impact of recovery rates on the implied volatility and on the CDS forward rates for the PR payoff. Vols are expressed as percentages and rates as basis points

CDS Options

In the next Table we check the impact of a shift in the simply compounded rates of the zero coupon interest rate curve on CDS forward rates and implied volatilities. Every time we shift the curve we recalibrate the hazard functions, while maintaining the same $R_{0,b}(0)$'s. We see that the shift has a more relevant impact than the recovery rate, an impact that remains small.

	shift -0.5%	0	$+0.5\%$
C1(T_a)	49.68	50.31	50.93
C2	54.02	54.68	55.34
C3	51.36	52.01	52.65

shift -0.5%	0	$+0.5\%$
61.480	61.497	61.514
97.294	97.326	97.358
62.677	62.697	62.716

Table 23: Implied volatilities $\sigma_{a,b}$ (left, as percentages) and forward CDS rates $R^{PR}_{a,b}$ (right, as basis points) as the simply compounded rates are shifted uniformly for all maturities.

Constant Maturity CDS with Market Models

Market models can be used for CMCDS as well.

For a description of CMCDS and for a closed form solutions with market models see

`http://arxiv.org/abs/0812.4159`

or

`http://www.damianobrigo.it/cmcdsweb.pdf`

Single Name Models: Reduced Form. Summary

We have seen:

- Explicit Reduced Form (intensity) Models;
  - Constant intensity, standard Poisson process
  - Time varying deterministic intensity, time inhomogeneous Poisson Process, Credit Spread modeling; Piecewise constant and piecewise linear intensities, CDS calibration;
  - Time varying Stochastic intensity, Cox processes, Credit spreads and their volatilities; SSRD CIR++ model; CDS calibration, interest rate calibration. CDS options formula;
- Market Models; CDS options implied volatilty; CDS options smile.

Credit Models and Counterparty Risk Valuation in Crisis

UNIT 3

DEFAULT BASKETS AND CDO's

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UNIT 3. Multi Name Credit Derivatives

In this unit we discuss financial payoffs depending on more than one underlying reference credit. The products we consider are

- First to default baskets;
- n-th to default, Last to default;
- CDO tranches;
- CDO squared tranches.
- Standard Indices: DJ-i-Traxx and related tranches.

First to default

τ_i: default time of reference credit i; τ^i: default time of the i-th name that defaults.

τ^i depends on the trajectories. If in a trajectory ω first defaults name 3, second name 5 and third name 2, we have

$$\tau^1(\omega) = \tau_3(\omega), \quad \tau^2(\omega) = \tau_5(\omega), \quad \tau^3(\omega) = \tau_2(\omega).$$

In a different trajectory the order can be different.

A first to default (FtD) is similar to a CDS on a single name but this time the default that calls for the payment of protection and for the end of the premium leg is the first default in a basket of reference credits $1, 2, 3, \ldots, N$.

First to default

Two companies "A" (Protection buyer) and "B" (Prot seller) agree on the following.

If the first defaulting company among 1,2,...,N (Reference Credits) defaults at time $\tau^1(\omega) = \tau_{i_1(\omega)}(\omega)$, with $T_a < \tau^1 < T_b$, "B" pays to "A" $\text{L}{\rm GD}_{i_1}$. In turn, "A" pays to "B" a rate R at $T_{a+1}, \ldots, T_b$ or until default τ^1. Set $\alpha_i = T_i - T_{i-1}$ and $T_0 = 0$.

Protection Seller B	$\rightarrow$	$\text{L}{\rm GD}_{i_1}$ at 1st default $\tau^1 = \tau_{i_1}$ if in $[T_a, T_b]$	$\rightarrow$	Protection Buyer A
	$\leftarrow$	rate R at $T_{a+1}, \ldots, T_b$ or until 1st default τ^1	$\leftarrow$	

(protection leg and premium leg respectively).

The cash amount $\text{L}{\rm GD}_{i_1}$ is a *protection* for "A" in case the first name defaults before T_b and is name i_1. Typically $\text{L}{\rm GD} = 1 - \text{R}{\rm EC}$. Notice that i_1 is a random variable.

In standard FtD, written by sectors, all names have the same REC (0.5 for financial and 0.4 for telecoms for example).

First to default: Basic concepts

Usually the FtD's are composed by a small number of entities: $N = 5$ or 10.

Again, a FtD works similarly to the standard CDS, but the protection is paid against the first reference entity in the basket experiencing default: the protection seller assumes more risk with respect to selling protection on any single name in the basket.

This fact leads in turn to higher premium rates R paid from the protection buyer.

The protection seller is attracted by the leverage. In case of a default, the protection seller is due to make a single payment relative to a single reference entity, as in single name CDS, but it receives a larger rate before default due to the higher likelihood of default.

From the protection buyer viewpoint, a FtD is seen as a lower cost method of partially hedging multiple credits. However the buyer keeps the risk of multiple defaults.

First to default: Basic concepts

The premium paid for protection is essentially the premium paid for a single name protection **plus** a part due to the likelihood of multiple defaults, that depends on correlation.

Intuitively, in case of perfect dependence, i.e. all reference credits defaulting together, the premium paid for the FtD basket would be equal to the premium paid for the riskiest name. This can be seen also formally (later).

At the same time the premium paid for a FtD has to be lower than the sum of the single different premia of the basket components.

First to default, k-th to default

Formally we may write the (Running) FtD discounted value to "B" at time $t < T_a$ as

$$\Pi\mathrm{RFtD}_{a,b}(t) := D(t,\tau^1)(\tau^1 - T_{\beta(\tau^1)-1})R\mathbf{1}_{\{T_a<\tau^1<T_b\}} +$$

$$+\sum_{i=a+1}^{b} D(t,T_i)\alpha_i R\mathbf{1}_{\{\tau^1>T_i\}} - \mathbf{1}_{\{T_a<\tau^1\leq T_b\}} D(t,\tau^1)\ \mathrm{L}_{\mathrm{GD}_{i_1}}$$

The price of this product can be computed by risk neutral expectation. $D(t,T)$ is the discount factor between t and T (most of times assumed deterministic).

Similarly to the 1st to default, we have contracts such as 2nd to default, k-th to default, last to default. The protection in the generic k-th to default is paid in correspondence of the $k-th$ default τ^k among the names in the basket if this is in $[T_a, T_b]$.

Just substitute τ^1 with τ^k above and i_1 with i_k.

First to default, k-th to default

The initial fair R of the k-th to default, denoted R^k, can be computed as follows

$$R^k_{a,b} = \frac{\mathbb{E}_t[\text{LGD}_{i_k} D(t,\tau^k)\mathbf{1}_{\{T_a<\tau^k\leq T_b\}}]}{\mathbb{E}_t[D(t,\tau^k)(\tau^k - T_{\beta(\tau^k)-1})\mathbf{1}_{\{T_a<\tau^k<T_b\}}] + \sum_{i=a+1}^b \mathbb{E}_t[D(t,T_i)\alpha_i \mathbf{1}_{\{\tau^k>T_i\}}]}$$

We need the Monte Carlo simulation in general, except possibly in case $k = 1$ or $k = N$, where some tricks are available under some particular models;

In general we may be able to avoid simulation also for generic k-th to default under some approximations and homogeneity assumptions on recovery rates. More on this later.

Index CDS and CDO: Intro

First a quick colloquial intro on CDS indices and CDOs and the crisis.

CDO Tranches in particular have become a key point of the critics to Quantitative Analysts and Modeling.

Popular accounts on the crisis resort to quite colorful expressions such as "the formula that killed Wall Street"[1], or "Of couples and copulas: the formula that felled Wall St"[2] just to make two examples.

[1] Recipe for disaster: the Formula that killed Wall Street. Wired Magazine, 17.03.

[2] The Financial Times, Jones, S. (2009). April 24 2009.

Index CDS and CDO: Intro

Another article that brings mathematics and mathematicians (provided that is what one means by "math wizards") into the picture for the blaming is Lohr (2009), in "Wall Street's Math Wizards Forgot a Few Variables", appeared in the New York Times of September 12. Also, Turner[3] (2009) has a section entitled "Misplaced reliance on sophisticated maths".

[3]Turner, J.A. (2009). The Turner Review. March 2009. Financial Services Authority, UK. www.fsa.gov.uk/pubs/other/turner_review.pdf.

Tough contest

Mathematics and Quantitative Models

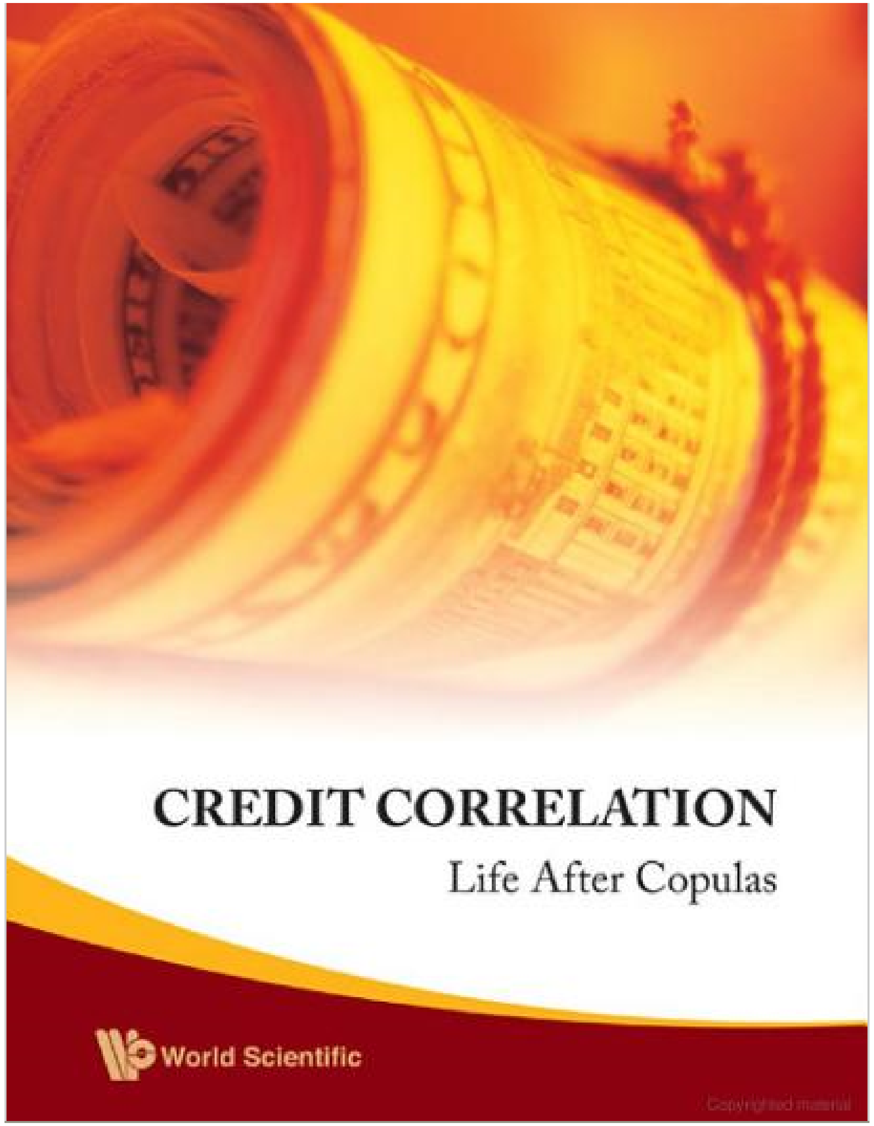
VS

US Home Polices, New Bank - Originate to Distribute system fragility, Volatile Monetary Policies, Myopic Compensation System, Regulatory oversight, Liquidity risk underestimation, NINJAs, Lack of Data, Madoff... (Szegö, 2009-2010).

Mathematical models Reloaded

This overall hostility and blaming attitude towards mathematics and mathematicians, whether in the industry or in academia, is the reason why we feel it is important to point out the following:

The notion that even more mathematically oriented quants have not been aware of the Gaussian Copula model limitations is simply false, as we are going to show, and you may quote us on this.



Proceedings of a Practitioners Conference held in London, 2006, organized by Lipton and Rennie, Merrill Lynch. I was there (as a speaker).

Rebuttal

And what about the earlier 2005 mini credit-correlation crisis when implied correlation went crazy?

September 12, 2005. How a Formula [Base correlation + Gaussian Copula] Ignited Market That Burned Some Big Investors. Wall Street Journal Online

There are several publications that appeared pre-crisis (also stimulated by the 2005 mini-crisis) and that questioned the Gaussian Copula and implied correlation. For example

“Implied Correlation: A paradigm to be handled with care”, SSRN, 2006, again well before the crisis.

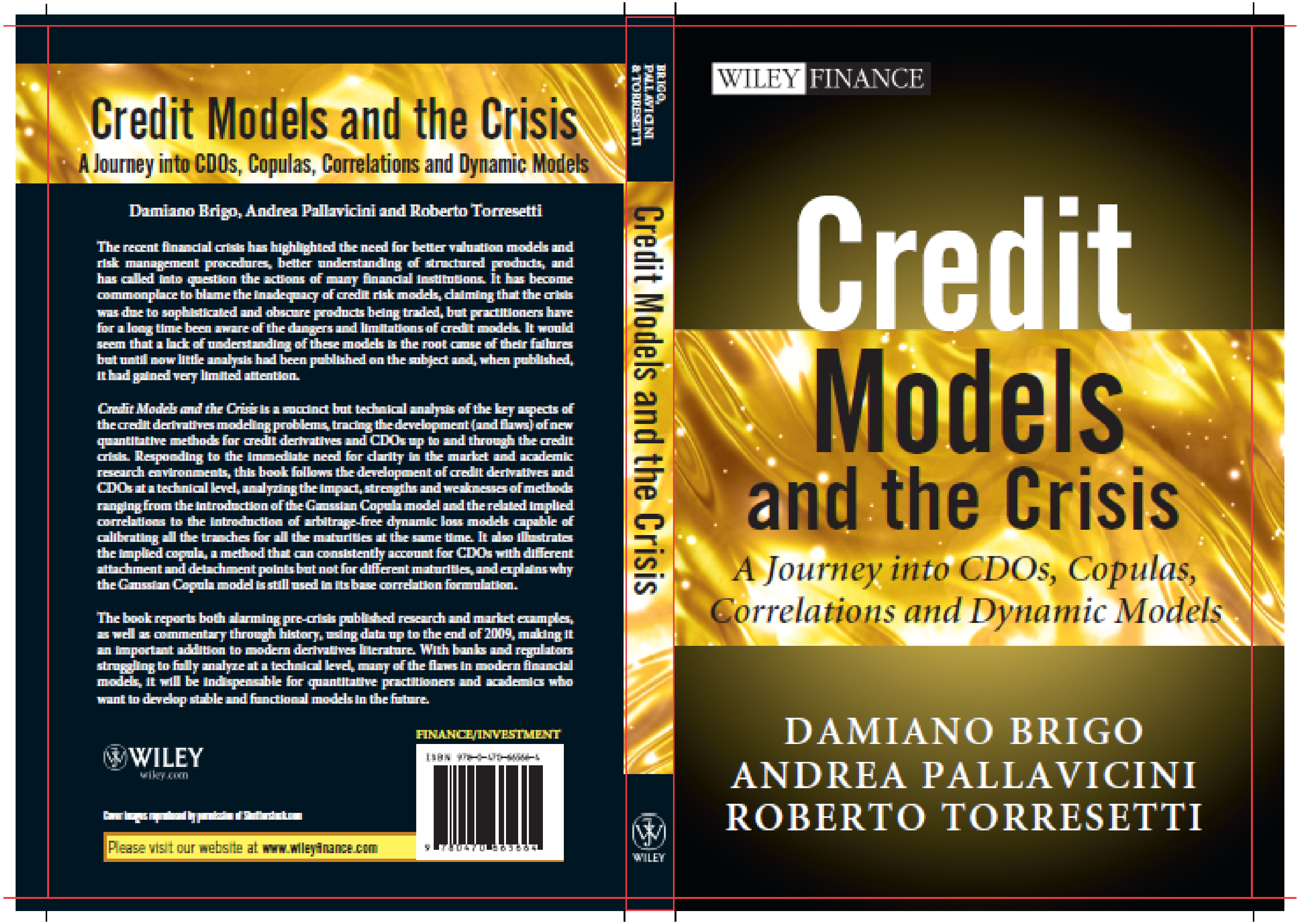
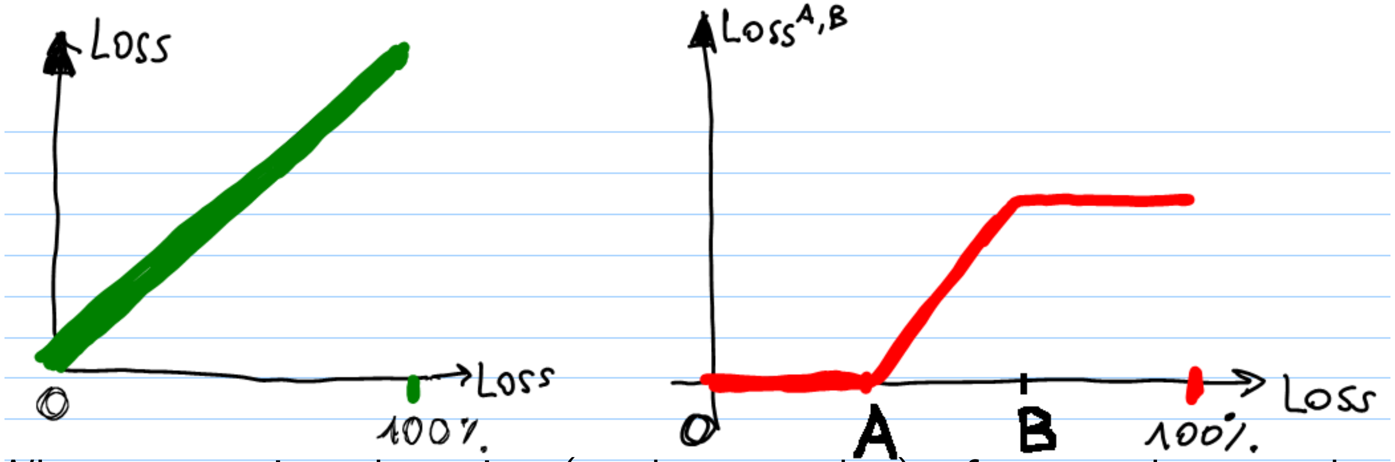


Figure 16: See also "Credit Models and the crisis or: How I learned to stop worrying and love the CDOs". Available at arXiv.org, ssrn.com, defaultrisk.com. Related papers in Mathematical Finance, Risk Magazine, IJTAF



- When computing the price (mark to market) of a tranche, one has to take the expectation of the future tranche losses under the pricing measure.
- From nonlinearity, the tranche expectation will depend on the loss distribution. This is characterized by the marginal distributions of the single names defaults and by the dependency among different names' defaults. Dependency is commonly called "correlation".
- Abuse of language: correlation is a complete description of dependence for jointly Gaussian random variables, but more generally it is not.

Copulas

The complete description is either the whole multivariate distribution or the so-called "copula function", that is the multivariate distribution once the marginal distributions have been standardized to uniform distributions.

THE MONSTER: One-factor Gaussian copula

$$\boxed{\int_{-\infty}^{+\infty} \prod_{i=1}^{125} \Phi\left(\frac{\Phi^{-1}(1-\exp(-\Lambda_i(T))) - \sqrt{\rho_i}m}{\sqrt{1-\rho_i}}\right) dm.}$$

"MEA COPULA!"

David X. Li: Commentators went from suggesting a Nobel award (for introducing a multivariate Gaussian distribution!!!) to blaming him for the whole Crisis.

The scapegoats

Figure 17: David X. Li

Li himself in 2005, two years before the crisis, Wall Street Journal:

[...] "The most dangerous part," Mr. Li himself says of the model, "is when people believe everything coming out of it." Investors who put too much trust in it or don't understand all its subtleties may think they've eliminated their risks when they haven't.

(E.g. These models are static. they ignore Credit Spread Volatilities, that in Credit can be 100%; this has further paradoxical consequences in copula models for wrong way risk, eg Brigo & Chourdakis (2009)).

Tranches and Correlations

The dependence of the tranche on "correlation" is crucial. What the market does is assuming a Gaussian Copula connecting the defaults of the 125 names, parametrized by a correlation matrix with 125*124/2 = 7750 entries. However, when looking at a tranche:

$$7750 \;\; \text{parameters} \longrightarrow \;\; 1 \;\; \text{parameter.}$$

The unique correlation parameter is reverse-engineered to reproduce the price of the liquid tranche under examination. This is called implied correlation, and once obtained it is used to value related products.

Problem: if at a given time the $3\% - 6\%$ tranche for a five year maturity has a given implied correlation, the $6\% - 9\%$ tranche for the same maturity will have a different one. The two tranches on the *same pool* are priced (and hedged!!!) with two inconsistent loss distributions

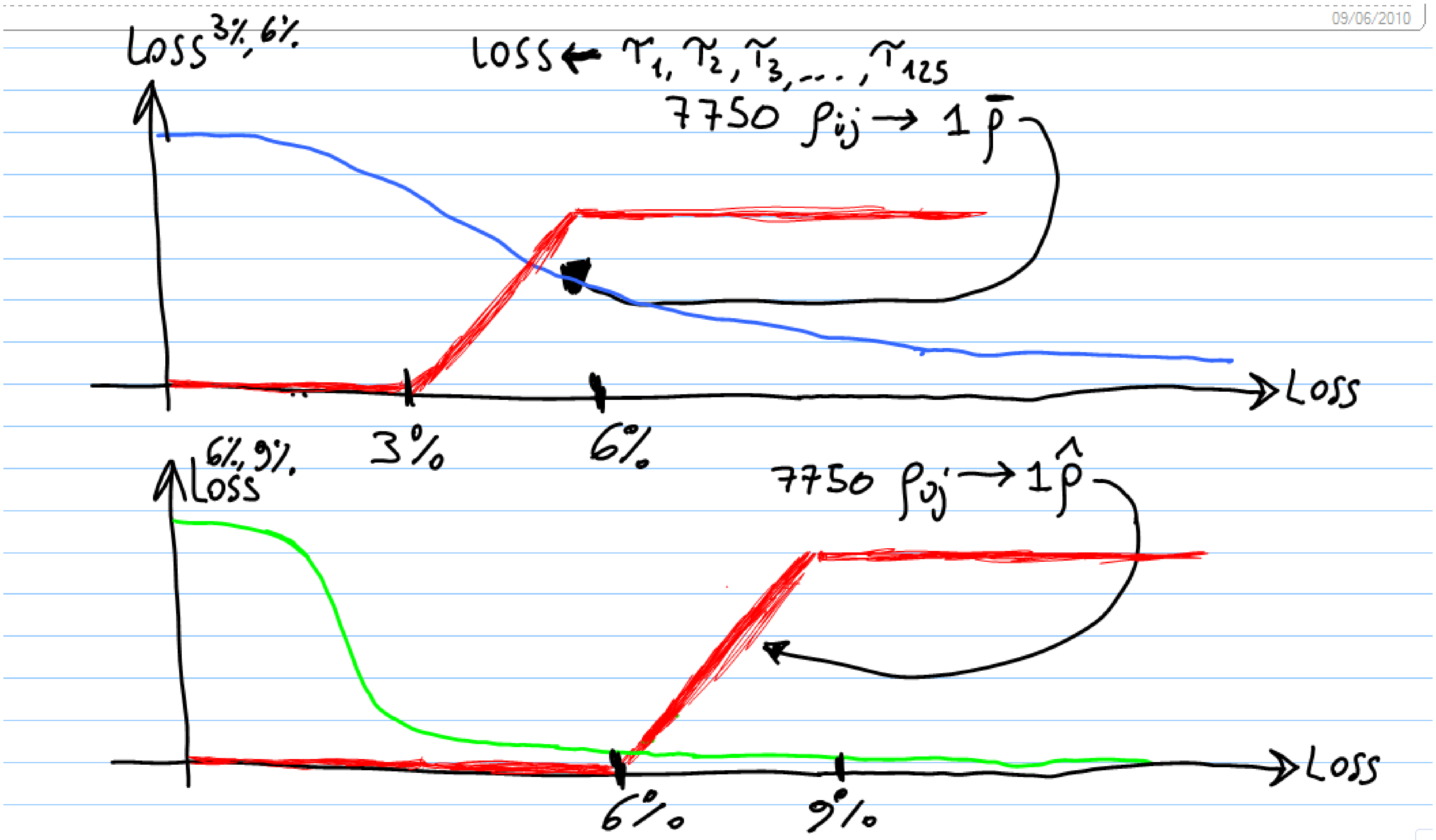


Figure 18: Compound correlation inconsistency

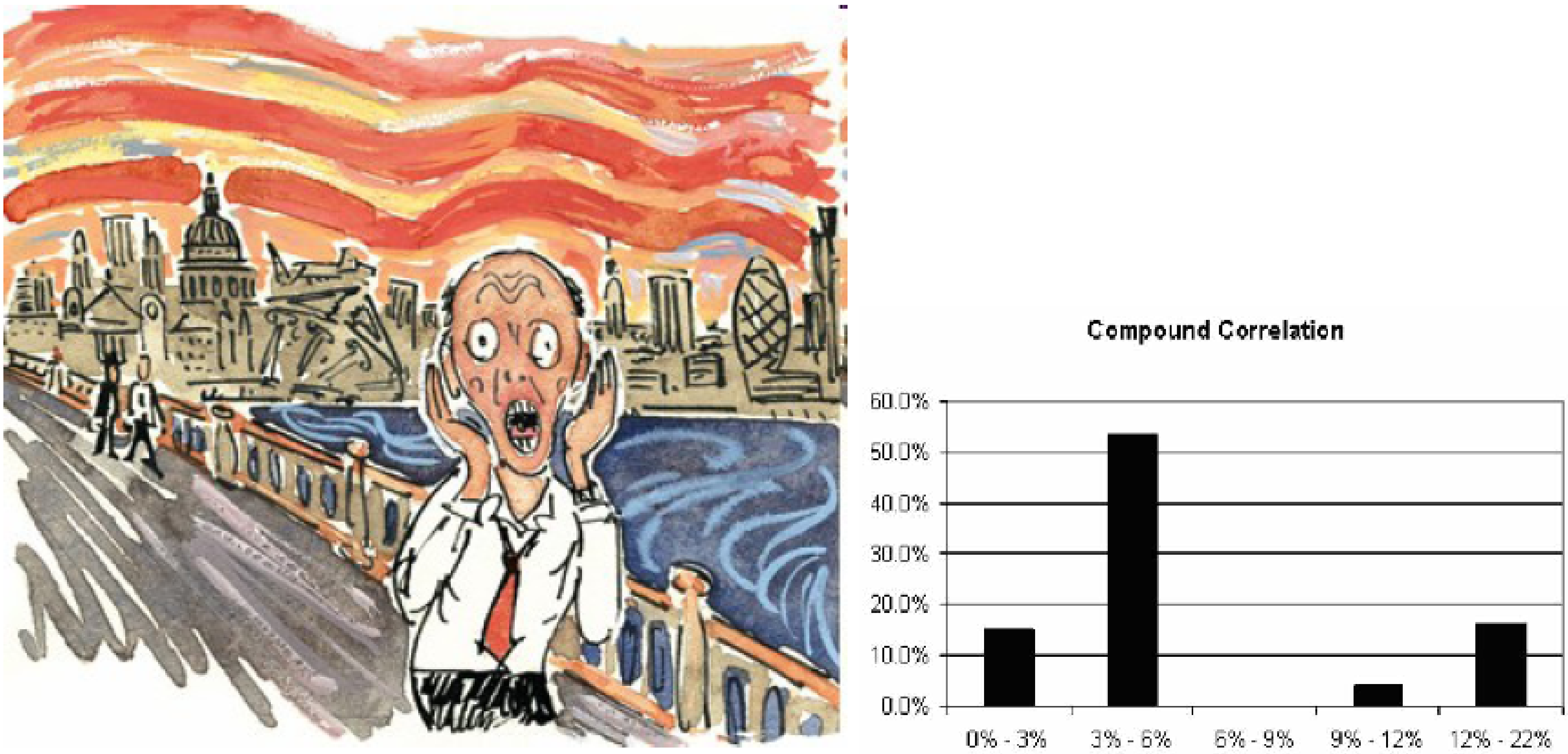


Figure 19: (After Edvard Munch's The Scream;
Compound correlation DJ-iTraxx S5, 10y on 3 Aug 2005)

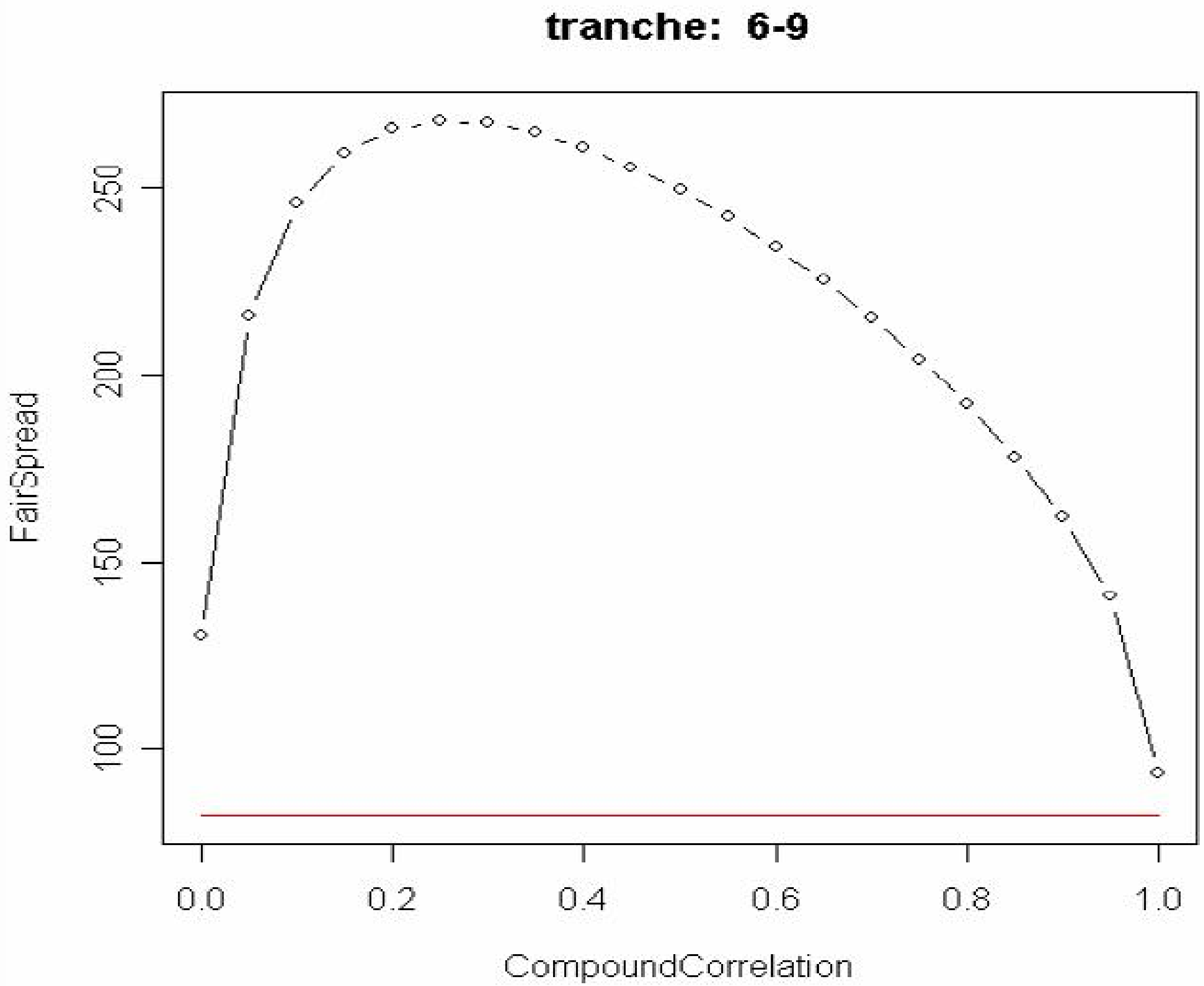


Figure 20: Non-invertibility compound correlation DJ-iTraxx S5, 10y on 3 Aug 2005

Base correlation

The situation is even worse. There are two possible implied correlation paradigms: compound correlation and base correlation. The second one is the one that is prevailing in the market.

Base correlation is easier to interpolate but is inconsistent even at single tranche level, in that it prices the $3\% - 6\%$ tranche by decomposing it into the $0\% - 3\%$ tranche and $0\% - 6\%$ tranche and using two different correlations (and hence distributions) for those. This inconsistency shows up occasionally in negative losses (i.e. in defaulted names resurrecting).

[in the graph we use put-call parity to simplify]

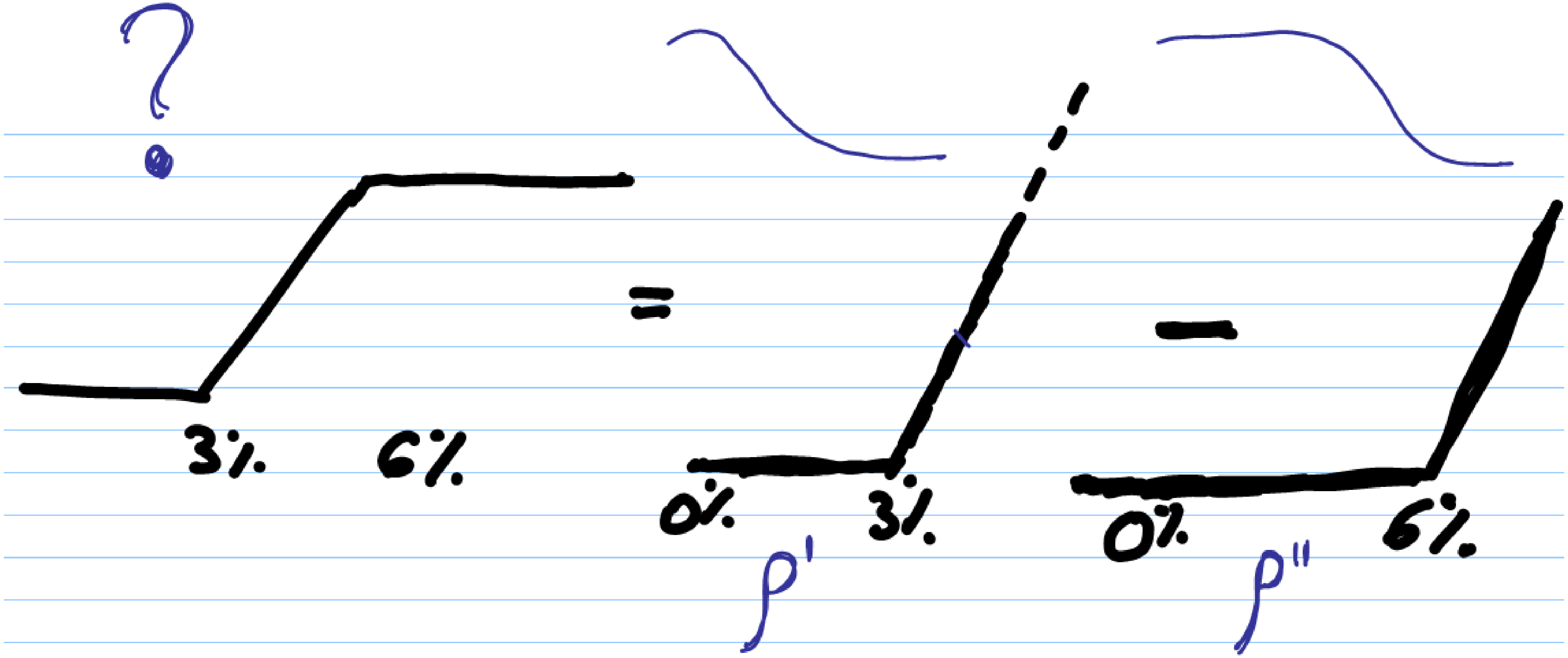
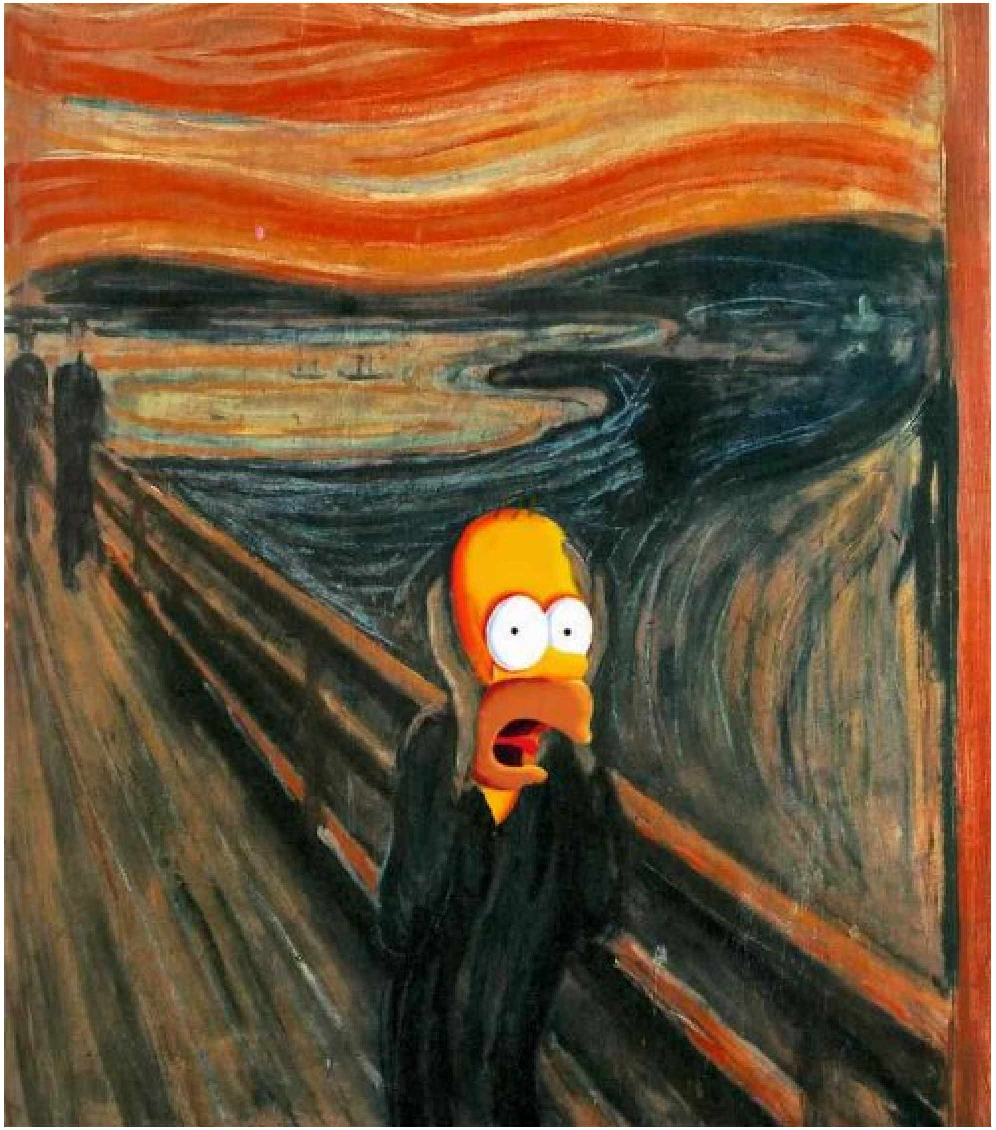


Figure 21: Base correlation inconsistency



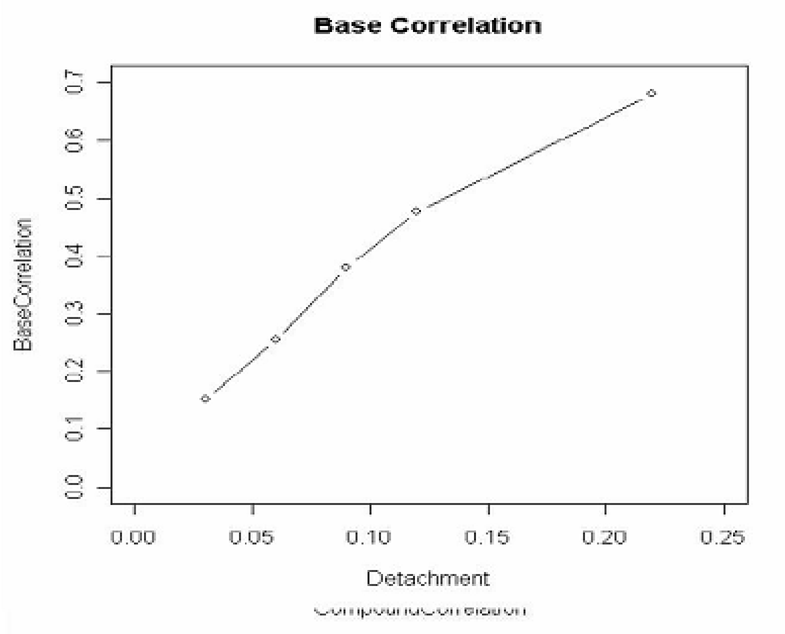


Figure 22: (After Edvard Munch's The Scream; Base correlation DJ-iTraxx S5, 10y on 3 Aug 2005)

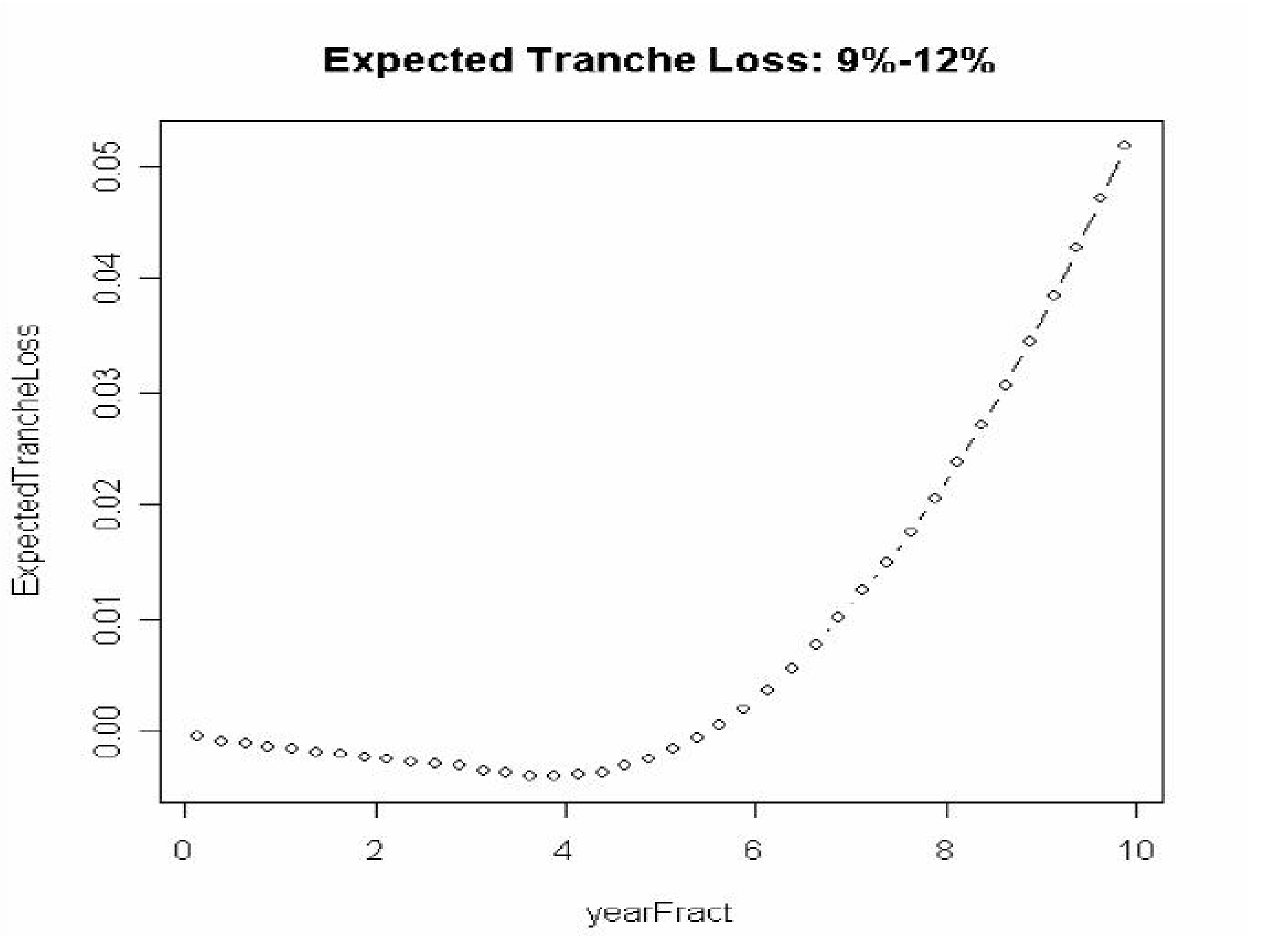


Figure 23: Expected tranche loss coming from Base correlation calibration, 3d August 2005, First published in 2006.

Index CDS's (iTraxx, CDX...)

Given a pool of names $1, 2, \ldots, M$, typically $M = 125$, each with notional $1/M$, so that the pool has total notional 1, the index default leg pays to the protection buyer the loss increment occurring each time one or more names default, until final maturity $T = T_b$ arrives or until all the names in the pool have defaulted.

$$\textsc{Loss}(T) = \frac{1}{M}\sum_{i=1}^{M}(1-\textsc{Rec}_i)1_{\{\tau_i \le T\}}, \quad \textsc{Outno}(T) = \frac{1}{M}\sum_{i=1}^{M} 1_{\{\tau_i > T\}} = 1 - \frac{\text{n. of defaults by } T}{M}$$

In exchange a periodic premium rate (or "spread") R is paid from the protection buyer to the protection seller, until final maturity T_b. This is computed on a notional that decreases each time a name in the pool defaults, and decreases of an amount corresponding to the notional of that name. Important: The whole notional, **irrespective of the recovery**.

$$\boxed{\begin{array}{c}\text{Prot}\\ \text{Seller}\end{array}} \begin{array}{ccc} \rightarrow & \text{Loss increment } d\textsc{Loss}(t) \text{ at all } t \in [T_0, T_b] & \rightarrow \\ \leftarrow & \text{rate } R \text{ at } T_1, \ldots, T_b \text{ on OutstandingNotional} & \leftarrow \end{array} \boxed{\begin{array}{c}\text{Prot}\\ \text{Buyer}\end{array}}$$

Index CDS's (iTraxx, CDX...)

The price of the two legs of the index at time 0 is given as follows:

$$\text{PriceDefaultLeg}(0) = \mathbb{E}\left[\int_0^T D(0,u)d\text{Loss}(u)\right]$$

$$\text{PricePremiumLeg}(0) = R_{\text{index}}\mathbb{E}\left[\sum_{i=1}^{n}\alpha_i D(0,T_i)\,\text{OutNo}_{\text{index}}(T_i)\right].$$

where $\alpha_i = T_i - T_{i-1}$ is the year fraction.

One should not be confused by the integral, the loss $\text{Loss}(t)$ changes with discrete jumps each time one or more names in the pool default.

Index CDS's (iTraxx, CDX...)

$$\text{Price}\textsc{PremiumLeg}(0) = R_{\text{index}}\mathbb{E}\left[\sum_{i=1}^{n}\alpha_i D(0,T_i)\,\textsc{OutNO}_{\text{index}}(T_i)\right].$$

The quantity

$$\text{DV01}_{\text{index}} = \text{Annuity}_{\text{index}} = \mathbb{E}\left[\sum_{i=1}^{n}\alpha_i D(0,T_i)\,\textsc{OutNO}_{\text{index}}(T_i)\right]$$

represents the value of the premium leg corresponding to a unit basis point spread R_{index}, or also the change in the premium leg when the index spread changes by one basis point (keeping everything else fixed).

It is often called the (defaultable) annuity of the index

Index CDS's (iTraxx, CDX...)

Prot Seller	$\rightarrow$	Loss increment $d\text{Loss}(t)$ at all $t \in [T_0, T_b]$	$\rightarrow$	Prot Buyer
	$\leftarrow$	rate R at $T_1, \ldots, T_b$ on OutstandingNotional	$\leftarrow$	

$$\text{OutNo}_{\text{index}}(t) = 1 - (\text{number of defaults by } t)/\text{number of names in the pool}$$

Differently from what will happen with the tranches, **the recovery is not considered when computing the outstanding notional**. Thus the index contains information both on the actual loss (including recoveries) and on the number of defaults alone.

The forward index spread is the R_{index} that sets the 2 legs to the same value at time 0

$$R_{\text{index}}(0) = \frac{\mathbb{E}\left[\int_0^T D(0,u)d\text{Loss}(u)\right]}{\mathbb{E}\left[\sum_{i=1}^n \alpha_i D(0,T_i)\,\text{OutNo}_{\text{index}}(T_i)\right]}$$

The market quotes the value of $R_{\text{index}}(0)$ at specific dates $t = 0$ that, for different maturities T, balances the two legs.

CDO tranches

Synthetic CDO tranches with maturity T are obtained by "tranching" the loss "$\text{Loss}(t)$" of the index pool. The tranched loss at points A and B is

$$\text{Loss}^{tr}_{A,B}(t) := \frac{1}{B-A}\left[(\text{Loss}(t) - A)1_{\{A<\text{Loss}(t)\leq B\}} + (B-A)1_{\{\text{Loss}(t)>B\}}\right].$$

The contract consists of two legs, the **default leg** and the **premium leg**.

Default Leg. Once enough names have defaulted and the loss has reached A, the count starts. Each time the loss increases the corresponding loss change re-scaled by the tranche thickness $B-A$ is paid to the protection buyer, until maturity arrives or until the total pool loss exceeds B, in which case the payments stop.

The discounted **default leg** can then be written as

$$\text{DefLeg}_{A,B}(0) = \int_0^T D(0,t)d\text{Loss}^{tr}_{A,B}(t)$$

CDO's tranches

Premium leg. As usual, in exchange for the loss payments, a premium rate $R_{0,T}^{A,B}(0)$, fixed at time $T_0 = 0$, is paid periodically, say at times $T_1, T_2, \ldots, T_b = T$, from the protection buyer to the protection seller. Part of the premium can be paid at time $T_0 = 0$ as an upfront $U_{0,T}^{A,B}(0)$. This time the rate is paid on the "survived" positive notional at the relevant payment time and not on the whole notional. **This notional decreases of the same amounts as the tranched loss increases, taking into account the recovery, differently from the index**. This notional is $\textsc{Outno}_{A,B}^{tr}(t) = 1 - \textsc{Loss}_{A,B}^{tr}(t)$.

The **premium leg** payoff is

$$\textsc{PrLeg}(0) = U_{0,T}^{A,B}(0) + \sum_{i=1}^{b} D(0, T_i) R_{0,T}^{A,B}(0) \int_{T_{i-1}}^{T_i} \textsc{Outno}_{A,B}^{tr}(t) dt.$$

If we assume that the payments are made on the notional remaining at each payment date T_i, the premium leg can be written by replacing the integral with $\alpha_i(1 - \textsc{Loss}_{A,B}^{tr}(T_i))$.

CDO's tranches

Prot Seller	$\rightarrow$	Tranched loss increment $d\mathsf{Loss}^{tr}_{A,B}(t)$ at all $t \in [T_0, T_b]$	$\rightarrow$	Prot Buyer
	$\leftarrow$	(upfront U at T_0, and) rate R at $T_1, \ldots, T_b$ on OutNo	$\leftarrow$	

The premium rate $R^{A,B}_{0,T}(0)$ that makes the contract fair (i.e. zero valued) at inception can be derived by equating the risk neutral expectation of the two legs and solving in R:

$$R^{A,B}_{0,T}(0) = \frac{\mathbb{E}_0\left[\int_0^T D(0,t)d\mathsf{Loss}^{tr}_{A,B}(t)\right] - U^{A,B}_{0,T}(0)}{\mathbb{E}_0\left[\sum_{i=1}^{b} \alpha_i D(0,T_i)(1 - \mathsf{Loss}^{tr}_{A,B}(T_i))\right]}$$

and the above expression can be easily recast in terms of the upfront premium U for tranches that are quoted in terms of upfront fees. Again, **differently from the index, the tranche has the same loss in numerator and denominator**. Thus tranche data alone (without index) do not allow for splitting the information on number of defaulted names and actual loss, or, they do not allow to single out the **recovery** from the loss.

CDO's tranches

The market quotes either the periodic premiums rate $R_{0,T}^{A,B}(0)$ of these tranches or their upfront premium rate $U_{0,T}^{A,B}(0)$ for maturities $T = 3y, 5y, 7y, 10y$ and standard attachment points A, B, on standardized pools of names.

Tranches with low detachment points $(B \leq 3\%)$ are usually quoted in terms of the upfront premium, while tranches with higher detachment points are quoted in terms of the periodic premium.

Information contained in CDO quotes

Recall the market quoted spreads for indices and tranches on standardized pools:

$$R_{\text{index}}(0) = \frac{\mathbb{E}_0\left[\int_0^T D(0,u)d\textsf{Loss}(u)\right]}{\mathbb{E}_0\left[\sum_{i=1}^n \alpha_i D(0,T_i)\,\textsf{Outno}_{\text{index}}(T_i)\right]}$$

$$\textsf{Outno}_{\text{index}}(t) = 1 - (\text{number of defaults by } t)/M$$

$$R_{0,T}^{A,B}(0) = \frac{\mathbb{E}_0\left[\int_0^T D(0,t)d\textsf{Loss}_{A,B}^{tr}(t)\right] - U_{0,T}^{A,B}(0)}{\mathbb{E}_0\left[\sum_{i=1}^b \alpha_i D(0,T_i)(1-\textsf{Loss}_{A,B}^{tr}(T_i))\right]}$$

If these spreads R are the only implied correlation we have in the market, **we see that the only information we can infer are "expected losses", "expected tranche losses" and "expected number of defaults".**

CDS Indices: DJ-iTRAXX in detail

DJ-iTRAXX is a family of CDS indices, which spans the main credit market in Europe.

This family was created with the purpose to standardize market quotes, and also to create a reference liquid multi-name credit derivative. These indices constitute now the most liquid quotations in the credit-derivatives market.

Since the quotation paradigm is standardized (we see the details below) these reference quotes are practically the only safe source of market cross-sectional default correlation.

There are also indices for different areas. We can find indices relative to Europe, the US (CDX), Japan, Asia, Australia, high yields and emerging markets.

We will concentrate on the European index: The DJ iTraxx Europe.

CDS Indices: DJ-iTRAXX Europe

These credit derivatives indices provide exposure to the high grade credit markets, and in particular to the most liquid reference entities.

In Europe (and also in the US), these indices also provide exposure to the main sub-sectors of the high grade corporate bond markets, and also to high yield.

The credit indices are constructed in order to provide exposure to the most liquid segments of the credit markets. This is achieved by **selecting the most liquid CDS** in the market and **equally weighting them in the index**

Each index is subject to regular rebalancing every 6 months in March and September. Rebalancing follows the same rules as the initial composition of the indices.

CDS Indices: DJ-iTRAXX Europe

DJ iTraxx Europe comprises a variety of indices. The main four are:

- **DJ iTraxx Benchmark ("main index")** - Top 125 European names in terms of the CDS volumes traded by the market makers in the past six months. Dealer liquidity poll every six months. Sectorial diversification; 3,5,7 and 10y maturities;
- **DJ iTraxx HiVol Index** - This index is a subset of DJ iTraxx Benchmark (30 names), but the reference entities are supposed to be the ones with widest spreads among non-financial entities; 3,5,7 and 10y maturities;
- **DJ iTraxx Crossover** - 45 names as selected by a dealers' poll, 45 most liquid sub-investment grade entities. Non-financial European names with rating no better than Baa3 or BBB- and on Negative Watch or worse.

All indices roll every six months, in **March** and in **September**.

CDS Indices: DJ-iTRAXX Europe

We concentrate on the most important index, i.e. the DJ iTraxx main index.

We have seen before that when a **credit event** happens in one of the names in the credit index, the seller of protection pays an amount equal to the Loss Given Default (LGD) of that particular name, proportionally to the weight of the name in the pool (the names are equally weighted).

On the other side the protection buyer still pays the index premium rate, but on the decreased notional corresponding to the names still "alive".

CDS Indices: DJ-iTRAXX Europe

DJ iTraxx Europe can easily be used as a simple and cheap instrument to trade the general direction of credit spreads. It has a number of advantages:

- Immediate diversification. DJ iTraxx Europe enables the investor to gain immediate diversification in a single liquid transaction.
- Accurate market tracking. The inclusion of only the most liquid names and the fact that these are updated every six months, ensures that DJ iTraxx Europe accurately reflects the composition of the European credit market.
- Low bid/offer spread compared to single names.
- High liquidity. The large number of market makers ensures that investors can trade large sizes without affecting the market.

CDS Indices: Weighted- vs uniform- average DJ-iTRAXX spread

Consider all the names in the i-TRAXX portfolios. Take the 5y maturity. Consider an index replicating portfolio of CDS on each of the 125 names of i-traxx and assume premium rates (or spreads) R^i for each name to be paid at times $T_1, T_2, \ldots, T_b$.

The protection leg of this portfolio is the sum of the protection legs of the 125 individual CDS. Each protection leg is equal to the premium leg of the same CDS, given that at inception the CDS price is null. Then the protection leg for each name i is

$$\text{ProtLeg}_i = R^i_{0,5y} \sum_{j=1}^{b} \alpha_j \bar{P}_i(0, T_j) = R^i_{0,5y} A_i$$

where $\bar{P}_i(0, T_j)$ is the corporate zero coupon bond for name i and maturity T_j and A_i is the related annuity (or DV01). As usual α_j is the year fract between T_{j-1} and T_j.

CDS Indices: Theoretical vs actual DJ-iTRAXX spread

If we look for a single premium rate $\bar{R}$ to be paid at periods $1, 2, \ldots, b$, in all the premium legs of the replicating portfolio, that balances all the protection legs, we have to solve the equation between the related premium leg and the total protection leg

$$\bar{R}\sum_{i=1}^{125}\sum_{j=1}^{b}\alpha_j \bar{P}_i(0,T_j) = \sum_{i=1}^{125}\mathrm{ProtLeg}_i$$

or, given the above equality

$$\bar{R}\sum_{i=1}^{125}A_i = \sum_{i=1}^{125}R^i_{0,5y}A_i \qquad \text{so that}$$

$$\boxed{\bar{R} = \sum_{i=1}^{125} w_i R^i_{0,5y}}\;, \quad w_i = \frac{A_i}{\sum_{k=1}^{125} A_k}$$

CDS Indices: Theoretical vs actual DJ-iTRAXX spread

$$\boxed{\bar{R} = \sum_{i=1}^{125} w_i R^i_{0,5y}}\ , \quad w_i = \frac{A_i}{\sum_{k=1}^{125} A_k}$$

This would be the correct index premium rate (or spread) obtained through the replicating portfolio; If we computed instead, as one is tempted to do for simplicity,

$$\boxed{\hat{R} = \sum_{i=1}^{125} \frac{1}{125} R^i_{0,5y}}$$

this would be the correct spread only if defaults never occured. The **dispersion bias** $\bar{R} - \hat{R}$ depends on the single name CDS survival probabilities and it is larger when single name CDS spreads are more dispersed. The default probabilities embedded in the $\bar{P}_i$'s are of course stripped from CDS; They can be stripped not only from the 5y CDS premium rates $R^i_{0,5y}$ of the different names but also from $R^i_{0,1y}$, $R^i_{0,3y}$ etc.

CDS Indices: DJ-iTRAXX tranches

As we have seen before for CDO's the index can be traded also in terms of tranches.

The difference with earlier general CDO's is that now tranches are standardized.

For the DJ-iTraxx Europe, the traded tranches are: An **Equity tranche**, responsible for all losses between $A = 0\%$ and $B = 3\%$, then other **mezzanine** and **senior** tranches covering

$$A - B : \quad 3\% - 6\%,\ 6\% - 9\%,\ 9\% - 12\%,\ 12\% - 22\%,\ 22\% - 100\%.$$

For the main US index, the DJ CDX NA the tranche sizes are different:

$$0\% - 3\%,\ 3\% - 7\%,\ 7\% - 10\%,\ 10\% - 15\%\ 15\% - 30\%,\ 30\% - 100\%.$$

CDS Indices: DJ-iTRAXX tranches

Example: Investor sells EUR 10mn protection on the 3%-6% tranche. We assume a credit spread of 135bp. Therefore, the market maker pays the investor 135bp per annum quarterly on a notional amount of EUR 10mn. We assume $\text{R}_{\text{EC}} = 40\%$ for all names.

- Each single name in the portfolio has a credit position in the index of $1/125 = 0.8\%$ and participates to the aggregate loss in terms of 0.8% x $\text{L}_{\text{GD}} =$ 0.8% x 0.6 $=$ 0.48%.
- This means that each default corresponds to a loss of 0.48% in the global portfolio.
- After 6 defaults, the total loss in the portfolio is EUR 0.48% x 6 $=$ 2.88%, and the tranche buyer is still protected.
- When the 7th name in the pool defaults the total loss amounts to 3.36% and the lower attachment point of the tranche is reached.
- To compute the loss of the tranche we have to normalize the total loss with respect to the tranche size: The net loss in the tranche is then (3.36%-3%)/3% x 10mn $=$ EUR 1.2mn which is immediately paid by the protection seller.

CDS Indices: DJ-iTRAXX tranches

Example (Continued)

- The notional amount on which the premium is paid reduces to 10mn - 1.2mn = EUR 8.8mn, and the investor receives every month a premium of 135bp on EUR 8.8mn until maturity or until the next default.
- Each following default leads to change in the tranche loss (paid by the protection seller) of 0.48%/3% x 10mn = EUR 1.6mn, and the tranche notional decreases correspondingly.
- After the 13th default the total loss exceeds 6% (13 x 0.48% = 6.24%) and the tranche is completely wiped out.
- In this case one last payment is made of (6%-5.76%)/3% x 10mn = EUR 0.8mn to the protection buyer, which in turn stops paying the premium since the outstanding notional has reduced to zero.

CDS Indices: DJ-iTRAXX tranches

An equity tranche buyer suffers of every default in the portfolio, which leads to a decrease of tranche notional on which the periodic premium is paid and conversely to contingent protection payments.

On the other side, the buyer of more senior tranches is more protected against few defaults: Indeed these tranches are affected only in case of a large number of defaults.

This difference leads to different premia paid to buy protection. It is natural to see that the periodic premium paid to buy protection is inversely proportional to the tranche seniority: the larger A and B, the smaller $R^{A,B}$.

A technical note: The premium for the equity tranche is usually very large, so it is market practice to pay it as a fixed running premium $R = 500$bps plus an upfront payment U (still computed so that the total value is zero at inception).

CDS Indices: DJ-iTRAXX tranches. Quotes examples

d1: 03-Oct-05; d2 : 23-May-06; quotes in bps;
0-3% equity tranche is quoted upfront + 500bps running

	Rindex3y	**Upfront0-3(3Y)**	**R3-6(3Y)**	**R6-9(3Y)**
d1	21.25	0.0475	20	7
d2	17.5	0.005	3.5	1.125

	Rindex5y	**Up0-3(5y)**	**R3-6(5y)**	**R6-9**	**R9-12**	**R12-22**	**R22-100**
d1	36	0.2775	91	27.5	11.5	6.5	2.42301
d2	32	0.24175	69.25	21	9.25	4.25	1.55

	Ridx10y	**Up0-3(10y)**	**R3-6(10y)**	**R6-9**	**R9-12**	**R12-22**	**R22-100**
d1	55.75	0.565	482.5	101.5	48.5	20.5	9.83936
d2	53.5	0.515	552.5	131	62	23.5	4.5

CDO Squared

As CDOs have become more popular, issuers have started looking for new assets against which to collateralize them, including other CDOs. This explains the birth of CDOs of ABS, CDOs of CDOs and also CDOs of a mixture of ABS and CDOs. These products are usually called CDOs squared. Due to leverage effects, these products create different risk profiles from the regular synthetic CDO, resulting in higher spreads for the investors.

CDO Squared

The attachment points of the mezzanine tranches in the "baby CDO's" can be different and the underlying portfolios are also different for each baby CDO. Obviously, given the limited number of credit entities present in the market, a certain overlapping of names across basic CDO's to be retranched is inevitable. The whole portfolio is then re-tranched and sold to the investors.

The basic idea is quite simple and it is similar to the regular CDO case: The fundamental variable is always the loss. But here we have a sort of double protection: In the regular CDO each default impacts on some tranche (for example the first default attacks the equity tranche). Here before impacting the squared tranche, the underlying tranches must be impacted.

CDO Squared

For example if only one name defaults, we could have no effect on the underlying mezzanines and hence no effect on the squared tranches. On the other side, when the mezzanine tranches are impacted, every successive default impacts deeply on the mezzanine tranche and hence on the squared tranches. This is why we have higher spreads for the same risk profile when compared to regular CDO tranches.

Furthermore it is clear that the overlapping effect is very relevant, since the default of one single name which is present in many baby CDOs could lead to great portfolio losses.

CDO Squared

Summarizing, CDO squared are CDO of CDO's (hence the name) in the following sense.

Now suppose for a moment that we take a set of $1, 2, \ldots, M$ tranches of different "basic" baby CDO's as underlyings of a global CDO. In principle the underlying baby CDO's can be written on different portfolios and can have different attachment points.

Here, for simplicity, we assume the basic CDO's to have the same maturity T_b, the same attachment points A, B and different underlying portfolios built using an equal number of components.

CDO Squared

Now add the tranched losses $(\text{Loss}^{tr}_{A,B})^{(1)}, (\text{Loss}^{tr}_{A,B})^{(2)}, \ldots, (\text{Loss}^{tr}_{A,B})^{(M)}$ of all these basic CDO's $1, 2, \ldots, M$...

...and tranche again the obtained total tranched losses $\mathcal{L}$ along two points X, Y!!

Roughly speaking, a CDO squared is a contract offering protection against this re-tranched loss of a pool of CDO tranches.

CDO Squared

Formally, define the total Loss of the basic CDO tranches

$$\mathcal{L}(t) := (\mathsf{Loss}^{tr}_{A,B})^{(1)} + (\mathsf{Loss}^{tr}_{A,B})^{(2)}(t) + \ldots + (\mathsf{Loss}^{tr}_{A,B})^{(M)}(t);$$

Notice that since each basic tranche loss $\mathsf{Loss}^{tr}_{A,B}(t)$ is always a number between 0 and 1, we have easily

$$\mathcal{L}(t) \in [0 \quad M].$$

Now tranche again at $X, Y \in [0 \quad M]$:

$$\mathcal{L}_{X,Y}(t) := \frac{1}{Y - X}\left[(\mathcal{L}(t) - X)1_{\{X < \mathcal{L}(t) \leq Y\}} + (Y - X)1_{\{\mathcal{L}(t) > Y\}}\right]$$

As for the basic CDO tranches, protection is offered against this loss in exchange for a flow of interest rate payments R occurring at given times $T_1, T_2, \ldots, T_b$ on the total notional minus the losses incurred up to that point.

CDO Squared

Also in CDO squared tranches one is interested in the premium rate $(R_{0,T}^{X,Y})^{(2)}(0)$ that makes the contract fair (i.e. zero valued) at inception.

We may write the CDO squared discounted payoff to the protection buyer as

$$\int_0^T D(0,t)d\mathcal{L}_{X,Y}(t) - \sum_{i=1}^{n}(R_{0,T}^{X,Y})^{(2)}(0)\alpha_i D(0,T_i)(1-\mathcal{L}_{X,Y}(T_i))$$

As usual, the fair premium rate at inception is defined as the one that sets to zero the risk neutral price of the CDO squared:

$$(R_{0,T}^{X,Y})^{(2)}(0) = \frac{\mathbb{E}_0\left[\int_0^T D(0,t)d\mathcal{L}_{X,Y}(t)\right]}{\mathbb{E}\left[\sum_{i=1}^n \alpha_i D(0,T_i)(1-\mathcal{L}_{X,Y}(T_i))\right]}$$

CDO Squared

The difference with basic CDO's is not in the shape of the payoff in terms of a given loss and premium rates (they are identical) but is in the the underlying loss to be tranched.

$\mathcal{L}_{X,Y}(t)$ here contains much more structure than the loss tranched in the basic CDO tranches.

For example, the expected loss $\mathbb{E}[\mathcal{L}]$ depends on the correlation structure while the expected loss $\mathbb{E}[\text{Loss}]$ of the basic pool of names (CDO's) does not.

Given the double tranching, this product is highly nonlinear and sensitivities are quite difficult to calculate and can behave unstably.

Conclusions

We have seen that in general in multi name products the interdependence among the different components in the portfolio is quite relevant.

For example in k-th to default products the correlation among default times is relevant to evaluate the distribution of the τ's.

Analogously the correlation has a deep impact on the distribution of the portfolio loss, which is the main variable to take into account when pricing CDO's and iTraxx tranches.

This topic will be addressed later in the discussion, when we will introduce the concepts of default correlation and copula functions.

UNIT 4

MULTI NAME REDUCED FORM MODELS AND COPULAS

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UNIT 4. Multi Name Reduced Form Models and Copulas

- Bottom up approach: from single defaults to the loss;
- Introducing dependence in defaults;
- Correlating intensities: Why doesn't it work;
- Correlating jump components: is linear correlation adequate?
- Copula functions.
- Factor Copulas.
- Pricing CDO's etc with factor copulas: Building the Loss distribution
  - Monte Carlo; Recurrence Approach ; Fast Fourier Transform
  - Probability Shifting ; Large Pool
- Implied correlation: Base and Compound.
- Consistency with the whole correlation skew: the Implied copula of Hull and White

BOTTOM UP: Introducing dependence across single Defaults

Especially in the CDO payoffs, we have seen that the real underlying of the credit market is often the LOSS of the pool.

Models that consider directly the aggregate loss and worry later (if at all) about single name defaults are called TOP DOWN. We see these models later.

Models that consider single defaults, introduce dependence across them and then consider the aggregate loss are called BOTTOM UP.

These are the models we consider now. Mostly, they are based on copula functions.

BOTTOM UP. Credit Correlation

A first idea of correlation is given by the correlation between default indicators default correlation, given by (see for example Hull and White, 2000)

$$\rho = \frac{E[\mathbf{1}_{\{\tau_1<T\}}\mathbf{1}_{\{\tau_2<T\}}] - E[\mathbf{1}_{\{\tau_1<T\}}]E[\mathbf{1}_{\{\tau_2<T\}}]}{\mathsf{STD}[\mathbf{1}_{\{\tau_1<T\}}]\mathsf{STD}[\mathbf{1}_{\{\tau_2<T\}}]}$$

where τ_i is the default time for name i.

This is not a good definition for credit correlation: In fact **it is not possible to estimate it historically**. We would need the default data for identical companies, but it is obvious to see that this is not an easy task. Also, being the indicators non elliptically distributed in general, correlation would not be a good measure of dependence between them (more on this below).

Another possibility is to use **equity correlation as a proxy for credit correlation**, but this has **no formal justification**, except in particular structural models and under special conditions; also, typically **credit correlations are smaller than equity ones**.

Introducing dependence in Defaults

Given names $1, 2, ..., n$, we may define dependency between the default times in a reduced form model setting

$$\tau_1 = \Lambda_1^{-1}(\xi_1), \ldots, \tau_n = \Lambda_n^{-1}(\xi_n)$$

essentially in three ways.

1. Put dependency in (stochastic) intensities of the different names and keep the ξ of the different names independent;
2. Put dependency among the ξ of the different names and keep the (stochastic or trivially deterministic)intensities independent;
3. Put dependency both among (stochastic) intensities of the different names and among the ξ of the different names;

Introducing dependence in Defaults

1) Put dependency in (stochastic) intensities of the different names and keep the ξ of the different names independent;

In this case, one may induce dependence among the $\lambda_i(t)$ by taking diffusion dynamics for each of them and correlating the brownian motions.

$$d\lambda_i(t) = \mu_i(t, \lambda_i(t))dt + \sigma_i(t, \lambda_i(t))dW_i(t), \quad d\lambda_j(t) = \mu_j(t, \lambda_j(t))dt + \sigma_j(t, \lambda_j(t))dW_j(t),$$

$$dW_i dW_j = \rho_{i,j} dt, \quad \xi_i, \xi_j \text{independent}$$

Advantages: possible tractability; ease of implementation; default of one name does not affect the intensity of other names; The correlation can be estimated historically from time series of credit spreads;

Disadvantages: unrealistically low dependence across defaults $1_{\{\tau_i < T\}}, 1_{\{\tau_j < T\}}$. See e.g. Roncalli et al `http://gro.creditlyonnais.fr/content/wp/copula-intensity.pdf`

Introducing dependence in Defaults

2) Put dependency among the ξ of the different names and keep the (stochastic or trivially deterministic)intensities independent;

This is the framework that is currently used for correlation products in the market, especially for defining implied correlation.

Advantages: can take deterministic intensities, which makes life easier for the stripping of single name default probabilities; can reproduce sufficient levels of dependence across default times by putting dependence structures (called "copula functions") on the ξ's.

Disadvantages: no natural historical source for estimating the copula, often calibrated by means of dubious considerations; Default of one name affects the intensity of other names (realistic but untractable); Ignores credit spreads volatilities (large) and correlations.

Introducing dependence in Defaults

3) Put dependency among the ξ of the different names and among the stochastic intensities of different names (combine 1 and 2 above, very rarely used);

This is the most complicated framework.

Advantages: takes into account possible credit spread volatility, and can produce a sufficient amount of dependence among default times by putting dependence structures (called "copula functions") on the ξ's.

Disadvantages: no natural historical source for estimating the copula, often calibrated by means of dubious considerations; Default of one name affects the intensity of other names (realistic but untractable); Calculations are quite complicated, due to the presence of stochasticity both in the intensities and in the ξ's.

Introduction to Copula Functions

It is well known that linear correlation is not enough to express the dependence between two random variables in an efficient way in general.

Example: take X standard Gaussian and take $Y = X^3$. Y is a deterministic one-to-one transformation of X, so that the two variables give exactly the same information and should have maximum dependence. However, if we take the linear correlation between X and Y we easily get

$$(E(X^4) - E(X^3)E(X))/(\text{Std}(X^3)\text{Std}(X)) = 3/\sqrt{15} = \sqrt{3}/\sqrt{5} < 1$$

We get a dependence measure that is smaller than 1 (1 means maximum dependence).

So **correlation is not a good measure of dependence in this case**.

Introduction to Copula Functions

In standard financial models this problem with correlation as a dependence measure is usually absent because we are concerned with dependence between instantaneous Brownian shocks, which are Gaussian. **Correlation works well for Gaussian variables**.

In credit derivatives with intensity models we may find ourselves in the situation where we need to introduce dependence between the *exponential* components $\xi = \Lambda(\tau)$ of Poisson processes for different names.

This is usually done by means of Copula functions.

Introduction to Copula Functions

Given a random variable X, we may transform it in several ways through a deterministic function: $2X$, X^5, $\exp(X)$,.. A particularly interesting transformation function is the cumulative distribution function F_X of X. Let $U = F_X(X)$.

$$F_U(u) = \mathbb{Q}(U \leq u) = \mathbb{Q}(F_X(X) \leq u) = \mathbb{Q}(X \leq F_X^{-1}(u))$$

$$= \mathbb{Q}(X \leq z) = F_X(Z) = F_X(F_X^{-1}(u)) = u$$

However, the identity distribution function $F_U(u) = u$ is characteristic of a uniform random variable in $[0\ 1]$. This means that $U = F_X(X)$ is a **uniform distribution function**. Notice that since F_X is one to one, U contains the same information as X.

The idea then is to transform all random variables X by their F_X obtaining all uniform variables that contain the same information as the starting X. This way we rid ourselves of marginal distributions, obtaining only uniform rv's, *and can concentrate on introducing dependence directly for these standardized uniforms*.

Introduction to Copula Functions

Let $(U_1, ..., U_n)$ be a random vector with uniform margins and joint distribution $C(u_1, ..., u_n)$. $C(u_1, ..., u_n)$ is the *copula* of the random vector. It can be characterized by a number of properties that we do not repeat here.

An important result is given by **Sklar's theorem**: Let H be an n-dimensional distribution function with margins $F_1, ..., F_n$. Then there exists an n-copula C such that for all $\mathbf{x}$ in $\bar{\mathbb{R}}^n$,

$$H(x_1, ..., x_n) = C(F_1(x_1), ..., F_n(x_n)). \tag{3}$$

This result tells us that what we hinted at in the previous slide works. Indeed, one would intuitively write

$$\begin{aligned} H(x_1, ..., x_n) &= \mathbb{Q}(X_1 \leq x_1, \ldots, X_n \leq x_n) \\ &= \mathbb{Q}(F_1(X_1) \leq F_1(x_1), ..., F_n(X_n) \leq F_n(x_n)) \\ &= \mathbb{Q}(U_1 \leq F_1(x_1), ..., U_n \leq F_n(x_n)) = C(F_1(x_1), ..., F_n(x_n)) \end{aligned}$$

where C is the joint distribution function of uniforms $U_1, ..., U_n$.

Introduction to Copula Functions

Sklar's theorem: For any joint distribution function $H(x_1, \ldots, x_n)$ with margins $F_1, \ldots, F_n$ there exists a copula function $C(u_1, \ldots, u_n)$ (i.e. a joint distribution function on n uniforms) such that

$$H(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$$

Notice that C contains the pure dependence information. Notice the important point: Correlation between two variables is just a number, whereas a copula function between two variables is a two dimensional function.

We may also write

$$C(u_1, \ldots, u_n) = H(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)). \tag{4}$$

from which we see that we may use any known joint distribution function to define a copula

Introduction to Copula Functions

INVARIANCE: if $g_1, ..., g_n$ are (say strictly increasing) one-to-one transformations, then the copula of some given $X_1, ..., X_n$ is the same as the copula for $g_1(X_1), ..., g_n(X_n)$ (not so for correlation). So **the copula is invariant for deterministic transformations that preserve the information**.

This tells us again that Copulas are really expressing the core of dependence.

In particular we can find some quantities (beside the standard linear correlation) expressing the level of *concordance* between two continuous random variables whose copula is C.

We now give two examples.

Introduction to Copula Functions

We present here the two most important measures of concordance. They provide the perhaps best alternatives to the linear correlation coefficient as a measure of dependence for pairs of non-gaussian (and non-elliptical) distributions, for which the linear correlation coefficient is inappropriate and often misleading.

- **Kendall's Tau** between two random variables X, Y

$$\tau(X,Y) = \mathbb{Q}\{(X-\tilde{X})(Y-\tilde{Y})>0\} - \mathbb{Q}\{(X-\tilde{X})(Y-\tilde{Y})<0\}$$

  where $(\tilde{X},\tilde{Y})$ is i.i.d. as (X,Y). It can be proved that if (X,Y) is a couple of continuous random variables with copula C, then

$$\tau(X,Y) = 4\int\int_{[0,1]^2} C(u,v)dC(u,v) - 1$$

Kendall's Tau for a pair of random variables (X,Y) is invariant under strictly increasing componentwise transformations.

Introduction to Copula Functions

- **Spearman's Rho** between two random variables X, Y

$$\rho_S(X,Y) = 3\mathbb{Q}\{(X-\tilde{X})(Y-Y')>0\} - \mathbb{Q}\{(X-\tilde{X})(Y-Y')<0\}$$

where (X,Y), (X',Y') and $(\tilde{X},\tilde{Y})$ are i.i.d. pairs. It can be proved that if (X,Y) is a couple of continuous random variables with copula C, then

$$\rho(X,Y) = 12\int\int_{[0,1]^2} C(u,v)du\,dv - 3$$

Spearman's Rho for a pair of random variables (X,Y) is invariant under strictly increasing componentwise transformations.

Also, it can be proved that every copula C is bounded between tho functions C^+ and C^-, which are known as the Fréchet-Hoeffding bounds:

$$C(u_1,u_1,\ldots,u_n)^- \le C(u_1,u_1,\ldots,u_n) \le C(u_1,u_1,\ldots,u_n)^+$$

Introduction to Copula Functions

$$C(u_1, u_1, \ldots, u_n)^- \leq C(u_1, u_1, \ldots, u_n) \leq C(u_1, u_1, \ldots, u_n)^+$$

where

$$C(u_1, u_1, \ldots, u_n)^- = \max(u_1 + u_2 + \ldots + u_n - 1, 0)$$

and

$$C(u_1, u_1, \ldots, u_n)^+ = \min(u_1, u_2, \ldots, u_n)$$

as in our example above.

While C^+ is a copula, C^- is a copula only in dimension 2.

We can define also an orthogonal copula $C^\perp$ corresponding to independent variables:

$$C(u_1, u_1, \ldots, u_n)^\perp = u_1 \cdot u_2 \cdot \ldots \cdot u_n$$

Introduction to Copula Functions

Consider now (X,Y) a pair of random variables with copula C, then

$$\begin{aligned} C = C^{+} \quad &\rightarrow \quad \tau_C = \rho_C = 1 \\ C = C^{\perp} \quad &\rightarrow \quad \tau_C = \rho_C = 0 \\ C = C^{-} \quad &\rightarrow \quad \tau_C = \rho_C = -1 \end{aligned}$$

Then C^{+} is the copula corresponding to the maximum dependence (correspondence) and C^{-} is the copula corresponding to the minimum dependence (correspondence). $C^{\perp}$ corresponds to perfect independence between two variables.

Before starting to introduce the most important families of copulas, let us define the concept of tail dependence.

Introduction to Copula Functions

The concept of tail dependence relates to the amount of dependence in the upper-right quadrant tail or lower-left-quadrant tail of a bivariate distribution. It is a concept that is relevant for the study of dependence between extreme values. Roughly speaking, it is the idea of "fat tails" for the dependence structure.

It turns out that tail dependence between two continuous random variables X and Y is a copula property and hence the amount of tail dependence is invariant under strictly increasing transformations of X and Y .

Definition: Let (X,Y) be a couple of continuous random variables with marginal distribution functions F and G. The coefficient of upper tail dependence of (X,Y) is

$$\lim_{u \uparrow 1} \mathbb{Q}\{Y > G^{-1}(u) | X > F^{-1}(u)\} = \lambda_U$$

provided that the limit $\lambda_U \in [0, 1]$ exists. If $\lambda_U \in (0, 1]$ X and Y are said to be asymptotically dependent in the upper tail; if $\lambda_U = 0$, X and Y are said to be asymptotically independent in the upper tail.

Introduction to Copula Functions

$$\lim_{u \uparrow 1} \mathbb{Q}\{Y > G^{-1}(u) | X > F^{-1}(u)\} = \lambda_U$$

Since $\mathbb{Q}\{Y > G^{-1}(u) | X > F^{-1}(u)\}$ can be rewritten as

$$\frac{1 - \mathbb{Q}\{X \leq F^{-1}(u)\} - \mathbb{Q}\{Y \leq G^{-1}(u)\} + \mathbb{Q}\{X \leq F^{-1}(u), Y \leq G^{-1}(u)\}}{1 - \mathbb{Q}\{X \leq F^{-1}(u)\}}$$

an alternative and equivalent definition (for continuous random variables), from which it is seen that the concept of tail dependence is indeed a copula property, is the following:

$$\lim_{u \uparrow 1} (1 - 2u + C(u,u))/(1-u) = \lambda_U$$

The concept of lower tail dependence can be defined in a similar way. If the limit

$$\lim_{u \downarrow 0} \mathbb{Q}\{Y \leq G^{-1}(u) | X \leq F^{-1}(u)\} = \lim_{u \downarrow 0} C(u,u)/u = \lambda_L$$

exists, then C has lower tail dependence if $\lambda_L \in (0,1]$, and lower tail independence if $\lambda_L = 0$.

Some important copulas: Gaussian copulas

The canonical copula is the Gaussian (or Normal) copula, obtained using a multivariate normal distribution Φ_R^n as H:

$$C(u_1, ..., u_n) = \Phi_R^n(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_n)) \tag{5}$$

where the Gaussian margins have all zero mean and unit variance, R is the correlation matrix and Φ^{-1} is the inverse of the usual standard normal cdf. Unfortunately this copula cannot be expressed in closed form. Indeed, in the 2-dimensional case we have:

$$C_R(u, v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left\{-\frac{s^2 - 2\rho st + t^2}{2(1-\rho^2)}\right\} ds\, dt, \tag{6}$$

ρ being the (only) correlation parameter in the matrix R. Notice that in case we are modelling the dependence among N names, the correlation matrix R in principle has $N(N-1)/2$ free parameters. Some properties:

- neither upper nor lower tail dependence
- $C(u, v) = C(v, u)$ i.e. *exchangeable copula*.

Some important copulas: Archimedean copulas

Archimedean copulas are an important class of copulas with a great quality: they can be expressed in closed form. In general Archimedean copulas arise from a particular function φ called the *generator* of the copula. In particular, if $\varphi : [0,1] \to [0,\infty)$ is a continuous, strictly decreasing function such that $\varphi(1) = 0$, then

$$C(u,v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)) \tag{7}$$

is a copula if and only if φ is convex (in other words it must be $\varphi' < 0$ and $\varphi'' > 0$ under differentiability). We recall that $\varphi^{[-1]}$ is the *pseudo-inverse* of φ defined as:

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t) & 0 \le t \le \varphi(0) \\ 0 & t > \varphi(0) \end{cases}$$

If $\lim_{t\to 0} \varphi(t) = +\infty$ we say that φ is a *strict* generator and the copula is said to be a *strict* copula. According to the particular generator used, we have different families of copula functions. We give examples in dimension 2 but they can be easily generalized.

It happens that even if Archimedean copulas are known in closed form, they are difficult to simulate. On the contrary Gaussian copulas are not known in closed form but are easier to simulate. We will face later this issue of sampling from copulas.

Clayton family

Let us choose $\varphi(t) = (t^{-\theta} - 1)/\theta$ where $\theta \in [-1, \infty)\backslash\{0\}$. Then the Clayton family is:

$$C_\theta(u, v) = \max([u^{-\theta} + v^{-\theta} - 1], 0)^{-1/\theta}. \tag{8}$$

If $\theta > 0$ the copulas are strict and the copula expression simplifies to

$$C_\theta(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}. \tag{9}$$

The Clayton copula has lower tail dependence for $\theta > 0$, and $C_{-1} = C^-$, $\lim_{\theta\to 0} C_\theta = C^\perp$ and $\lim_{\theta\to\infty} C_\theta = C^+$.

Frank family

Let us choose $\varphi(t) = -\ln \frac{e^{-\theta t}-1}{e^{-\theta}-1}$, where $\theta \in \mathbb{R}\backslash\{0\}$. This originates the Frank family

$$C_\theta(u,v) = -\frac{1}{\theta}\ln\left(1 + \frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1}\right). \tag{10}$$

The Frank copulas are strict Archimedean copulas. Furthermore $\lim_{\theta\to-\infty} C_\theta = C^-$, $\lim_{\theta\to\infty} C_\theta = C^+$ and $\lim_{\theta\to 0} C_\theta = C^\perp$. The members of the Frank family are the only Archimedean copulas which satisfy the equation $C(u,v) = \breve{C}(u,v)$ (remark: only in two dimensions!!!). The members of the Frank family have no tail dependence (neither upper nor lower).

$\breve{C}(u,v)$ is the *survival copula* defined as

$$\mathbb{P}\big[X_1 > x_1,\ \ldots,\ X_n > x_n\big] = \breve{C}(\bar{F}_1(x_1),\ \ldots,\ \bar{F}_n(x_n))$$

where the $\bar{F}$'s are the margins survival functions. The survival copula is not linked to the copula in a simple way: It can be proved that in two dimensions the following relation holds:

$$\breve{C}(u,v) = u + v - 1 + C(1-u, 1-v).$$

Gumbel family

Let us choose $\varphi(t) = (-\ln t)^\theta$, where $\theta \geq 1$. This gives the Gumbel family

$$C_\theta(u, v) = \exp(-[(-\ln u)^\theta + (-\ln v)^\theta]^{1/\theta}). \tag{11}$$

The Gumbel copulas are strict Archimedean copulas. Furthermore $\lim_{\theta \to \infty} C_\theta = C^+$, $C_1 = C^\perp$. Gumbel copulas describe **only positive dependence** between random variables; moreover they feature upper tail dependence.

Marshall Olkin Copula

This copula is outside the Archimedean family and is given by

$$C_{\alpha_1,\alpha_2}(u_1, u_2) = \min(u_1^{1-\alpha_1} u_2, u_1 u_2^{1-\alpha_2})$$

for two numbers $0 \leq \alpha_1, \alpha_2 \leq 1$. This copula has both an absolutely continuous and a singular component.

Its survival copula

$$\check{C}(u_1, u_2) = u_1 u_2 \min(u_1^{-\alpha_1}, u_2^{-\alpha_2})$$

links the first jump times of correlated Poisson processes coming from a Common Poisson Shock framework (more on this later with the GPCL loss model).

If $\alpha_1 = \alpha_2 = 0$ we have independence, otherwise if it is 1 we have C^+.

Some important copulas: t-Copulas

If the vector $\mathbf{X}$ of random variables has the stochastic representation $\mathbf{X} \sim \mu + \frac{\sqrt{\nu}}{\sqrt{S}}\mathbf{Z}$ where $\mu \in \mathbb{R}^n$, ν is a positive integer, $S \simeq \chi^2_\nu$ and $\mathbf{Z} \simeq \mathcal{N}_n(\mathbf{0}, \Sigma)$ are independent, then $\mathbf{X}$ has an n-variate t_ν-distribution with mean μ (for $\nu > 1$) and covariance matrix $\frac{\nu}{\nu-2}\Sigma$ (for $\nu > 2$). If $\nu \leq 2$ then Cov$(\mathbf{X})$ is not defined. In this case we just interpret Σ as being the shape parameter of the distribution of $\mathbf{X}$.

The copula of $\mathbf{X}$ defined above is can be written as

$$C^t_{\nu,R}(\mathbf{u}) = t^n_{\nu,R}(t^{-1}_\nu(u_1), \ldots, t^{-1}_\nu(u_n))$$

where $R_{ij} = \Sigma_{ij}/\sqrt{\Sigma_{ii}\Sigma_{jj}}$ for $i, j \in \{1, \ldots, n\}$ and where $t^n_{\nu,R}$ denotes the distribution function of $\sqrt{\nu}\mathbf{Y}/\sqrt{S}$ where $S \simeq \chi^2_\nu$ and $\mathbf{Y} \simeq \mathcal{N}_n(\mathbf{0}, R)$ are independent. Here t_ν denotes the (equal) margins of $t^n_{\nu,R}$, i.e. the distribution function of $\sqrt{\nu}Y_1/\sqrt{S}$.

Some important copulas: t-Copulas

In the bivariate case the copula expression can be written as

$$C^t_{\nu,R}(u,v) = \int_{-\infty}^{t_\nu^{-1}(u)} \int_{-\infty}^{t_\nu^{-1}(v)} \frac{1}{2\pi(1-R_{12}^2)^{1/2}} \left\{1+\frac{s^2 - 2R_{12}st + t^2}{\nu(1-R_{12}^2)}\right\}^{-(\nu+2)/2} ds\, dt$$

Note that R_{12} is simply the usual linear correlation coefficient of the corresponding bivariate t_ν-distribution if $\nu > 2$. If (X_1,X_2) has a standard bivariate t-distribution with ν degrees of freedom and linear correlation matrix R, then $X_2|X_1 = x$ is t-distributed with $\nu+1$ degrees of freedom and $\mathbb{E}(X2|X1 = x) = R_{12}x$, $\text{Var}(X_2|X1 = x) = \left(\frac{\nu+x^2}{\nu+1}\right)(1-R_{12}^2)$.

Summary on copula properties

In the following table we collect the properties of the different copulas considered so far.

Copula	*Positive Dependence*	*Independence*	*Negative Dependence*	*Upper Tail Dependence*	*Lower Tail Dependence*
Clayton $\theta \in [-1, +\infty)$ $\theta \neq 0$	$C \to C^+$ $\theta \to +\infty$	$C \to C^\perp$ $\theta \to 0$	$C = C^-$ $\theta = -1$	no	only for $\theta > 0$
Frank $\theta \in \mathbb{R}\backslash\{0\}$	$C \to C^+$ $\theta \to +\infty$	$C \to C^\perp$ $\theta \to 0$	$C \to C^-$ $\theta \to -\infty$	no	no
Gumbel $\theta \in [1, +\infty)$	$C \to C^+$ $\theta \to +\infty$	$C = C^\perp$ $\theta = 1$	no negative dependence	yes	no
Gaussian $\rho \in (-1, 1)$	$C \to C^+$ $\rho \to +1$	$C = C^\perp$ $\rho = 0$	$C \to C^-$ $\rho \to -1$	no	no
t-Copula $R_{12} \in (-1, 1)$	$C \to C^+$ $R_{12} \to +1$ $\nu \to \infty$	$C = C^\perp$ $R_{12} = 0$ $\nu \to \infty$	$C \to C^-$ $R_{12} \to -1$ $\nu \to \infty$	yes	yes

Sampling from Copulas

The Gaussian copula has no closed form but is simple to simulate. Instead the Archimedean copulas have closed form but are difficult to sample, especially in high dim.

Here we present a general algorithm to sample out from copulas. The algorithm is based on general considerations. Let

$$C_k(u_1, \ldots, u_k) = C(u_1, \ldots, u_k, 1, \ldots, 1), \quad k = 2, \ldots, n-1$$

denote the $k-$dimensional margins of C, with $C_1(u_1) = u_1$ and $C_n(u_1, \ldots, u_n) = C(u_1, \ldots, u_n)$. Let $U_1, \ldots, U_n$ have a joint distribution function C. Then the conditional distribution of U_k given the values of $U_1, \ldots, U_{k-1}$, is given by

$$\begin{aligned} C_k(u_k|u_1, \ldots, u_{k-1}) &= \mathbb{P}\{U_k \le u_k | U_1 = u_1, \ldots, U_{k-1} = u_{k-1}\} \\ &= \frac{\partial^{k-1} C_k(u_1, \ldots, u_k)}{\partial u_1 \ldots \partial u_{k-1}} \Big/ \frac{\partial^{k-1} C_{k-1}(u_1, \ldots, u_{k-1})}{\partial u_1 \ldots \partial u_{k-1}} \end{aligned}$$

given that the numerator and the denominator exist and that the latter is $\neq 0$.

Sampling from Copulas

ARCHIMEDEAN COPULAS

- Simulate a random variable u_1 from $U(0,1)$.
- Simulate a random variable u_2 from $C_2(\cdot|u_1)$.

$\vdots$

- Simulate a random variable u_n from $C_n(\cdot|u_1,\ \ldots,\ u_{n-1})$.

GAUSSIAN COPULAS

- Find the Cholesky decomposition A of R where R is the correlation matrix
- Simulate n independent random variables $z_1,\ \ldots,\ z_n$ from $\mathcal{N}(0,1)$
- Set $\mathbf{x} = A\mathbf{z}$
- Set $u_i = \Phi(x_i)$, $i = 1, \ldots, n$
- $(u_1,\ \ldots,\ u_n)^T \sim C_R$.

Sampling from Copulas

t-COPULAS

- Find the Cholesky decomposition A of R
- Simulate n independent random variables $z_1, \ldots, z_n$ from $\mathcal{N}(0,1)$
- Simulate a random variate s from χ^2_ν independent of $z_1, \ldots, z_n$
- Set $\mathbf{y} = A\mathbf{z}$
- Set $x_i = \frac{\sqrt{\nu}}{\sqrt{s}} y_i$
- Set $u_i = t_\nu(x_i)$, $i = 1, \ldots, n$
- $(u_1, \ldots, u_n)^T \sim C^t_{\nu,R}$.

Sampling from Copulas

If we need to simulate a single default time τ we need to simulate first its intensity and then an independent uniform random variable; at this point we take minus the log of the uniform and apply the inverse of the cumulated intensity.

What in case of multiple dependent defaults? Recall that if U is uniform, $-\ln(1-U)$ is exponential.

$$\tau_1 = \Lambda_1^{-1}(-\ln(1-U_1)), \tau_2 = \Lambda_2^{-1}(-\ln(1-U_2)), ..., \tau_N = \Lambda_N^{-1}(-\ln(1-U_N)).$$

Usually λ's are taken deterministic or independent of each other, and the dependency between the τ's is loaded into a copula on the U's.

One Factor Copula

In general practitioners use the Gaussian copula as a standard tool in pricing credit derivatives. But, as mentioned above, the number of free parameters in this case is $N(N-1)/2$ and it could be quite a problem when dealing with many names. Moreover the Gaussian copula has no analytical tractability, and this is an undesirable feature.

Still, the Gaussian (or t-) copula has the advantage to break complete dependence into sets of pairwise dependencies.

(One-)Factor copulas are a way to simplify the pricing process, since they reduce the dimensionality of the problem and also present more analytical tractability. This is the reason why in the past they attracted interest and became a standard when pricing CDOs and CDS index tranches.

One Factor Copula

One Factor copulas in the intensity/reduced form framework are particular Gaussian copulas on the $U = -\ln(1-\xi)$ coming from a factor structure.

However, it is possible to give a derivation of this copula inspired by the other (structural) framework, and by Merton's model in particular.

Recall that in Merton's model a company incurs in a default if the value of the firm falls below the debt value at the debt maturity.

Let us call V the gaussian log-firm value and K the log debt value: there is a default when $V < K$. Let us assume to have standardized V (zero mean and unit variance).

Prob to have a default is $\mathbb{Q}\{V < K\} = \Phi(K)$

Now let us imagine to deal with $i = 1, 2, \ldots, N$ different companies: Then each company has value V_i and debt K_i.

One Factor Copula

Following the work of Vasicek (Vasicek, 1987), we assume that each process V_i is driven by a common (systematic) factor and a specific (hydiosyncratic) factor.

$$V_i = \sqrt{\rho_i}\, M + \sqrt{1-\rho_i}\, Y_i$$

where Y_i, M are i.i.d as $N(0,1)$ so that corr$(V_i, V_j) = \sqrt{\rho_i \rho_j}$.

Now the N parameters ρ_i are sufficient to construct the whole correlation structure, and this is an improvement with respect to the standard Gaussian copula (N (N-1)/2). In particular if $\rho_i = \rho$ then corr$(V_i, V_j) = \rho$, unique correlation parameter. Important: conditional on the common M defaults are **independent** and default prob is:

$$\mathbb{Q}\{V_i < K_i | M\} \quad = \quad \mathbb{Q}\left(\sqrt{\rho_i}\, M + \sqrt{1-\rho_i}\, Y_i < K_i \middle| M\right) = \Phi\left(\frac{K_i - \sqrt{\rho_i}\, M}{\sqrt{1-\rho_i}}\right).$$

The value of the debt K_i is inversely computed as $K_i = \Phi^{-1}(p_i)$ where p_i is the default probability of each company i and is usually computed apart.

One Factor Copula

This is our inspiration, but we define now the copula directly on the U_i's leading to the default time in the intensity framework,

$$\tau_1 = \Lambda_1^{-1}(-\ln(1 - U_1)), \tau_2 = \Lambda_2^{-1}(-\ln(1 - U_2)), ..., \tau_N = \Lambda_N^{-1}(-\ln(1 - U_N)).$$

We take $U_i = \Phi(X_i)$ where the X_i's are Gaussian variables defined as

$$X_i = \sqrt{\rho_i}\, M + \sqrt{1 - \rho_i}\, Y_i, \tag{12}$$

Y_i, M as before. Let us consider the Gaussian copula (5) in this framework:

$$C(u_1, \ldots, u_n) = \mathbb{Q}(X_1 < \Phi^{-1}(u_1), \ldots, X_n < \Phi^{-1}(u_n)). \tag{13}$$

By iterated expectations, we can write the previous term as:

$$\mathbb{E}\left(\mathbb{Q}(X_1 < \Phi^{-1}(u_1), \ldots, X_n < \Phi^{-1}(u_n)|M)\right), \tag{14}$$

One Factor Copula

$$\mathbb{E}\left(\mathbb{Q}(X_1 < \Phi^{-1}(u_1), \ldots, X_n < \Phi^{-1}(u_n)|M)\right), \tag{15}$$

By independence of X's, conditional on M, the probability is the product of probabilities

$$\mathbb{Q}_{|M}(X_i < \Phi^{-1}(u_i)) = \mathbb{Q}_{|M}(\sqrt{\rho_i}\, M + \sqrt{1-\rho_i}\, Y_i, < \Phi^{-1}(u_i)) = \Phi\left(\frac{\Phi^{-1}(u_i) - \sqrt{\rho_i} m}{\sqrt{1-\rho_i}}\right),$$

given that Y's are independent standard Gaussians, of which we take the external expectation:

$$C(u_1, \ldots, u_n) = \int \left(\prod_{i=1}^{n} \Phi\left(\frac{\Phi^{-1}(u_i) - \sqrt{\rho_i} m}{\sqrt{1-\rho_i}}\right)\right) \varphi(m)\, dm, \tag{16}$$

where $\varphi(v)$ is the usual standard Gaussian density (of M). **Thus, the n-dimensional copula is computed through a one dimensional integral.**

One Factor Copula

Notice also that, with deterministic intensities $\Lambda = \Gamma$:

$$\boxed{\mathbb{Q}(\tau_1 \le t_1, \dots, \tau_n \le t_n) = \int \prod_{i=1}^{n} \Phi\left(\frac{\Phi^{-1}(1-\exp(-\Gamma_i(t_i))) - \sqrt{\rho_i}m}{\sqrt{1-\rho_i}}\right)\varphi(m)\,dm}$$

This allows knowledge of the joint distribution of default times (and hence pricing of a credit derivative) from a one-dimensional integration against the gaussian density.

In general also multi factor decompositions are possible, but this increases the complexity (see e.g. Laurent and Gregory, 2004 and Hull and White, 2004). We will see applications of this copula below.

Monte Carlo pricing with Copulas

To price multi-name credit products we need i) a pricing model and ii) an estimate of the value of the default dependence (copula) parameters to put in the model.

Let us skip for a moment the second issue (later) and let us suppose to have somehow estimated the copula parameters and concentrate on the pricing.

If we can write the payoff of the contract with respect to the series of default times, than it is natural to evaluate the contract by means of Monte Carlo simulations.

More precisely we can simulate a collection of default times $\tau_1, \ldots, \tau_N$ for the N components of the portfolio by means of copula and intensity, as seen before, and evaluate the realization of the payoff under each scenario.

Then we repeat it many times and average over scenarios, getting the price.

Monte Carlo pricing of a First to Default

For example we can think to price a First to Default (FtD) using this procedure.

As we have seen earlier, the (Running) FtD discounted payoff to the protection seller at time $t < T_a$ is $\Pi\text{RFtD}_{a,b}(t) :=$

$$D(t,\tau^1)(\tau^1 - T_{\beta(\tau^1)-1})R\mathbf{1}_{\{T_a<\tau^1<T_b\}} + \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R\mathbf{1}_{\{\tau^1>T_i\}} - \mathbf{1}_{\{T_a<\tau^1\leq T_b\}}D(t,\tau^1)\ \text{LGD}_{i_1}$$

So we just simulate a series of default times and compute the payoff for that collection $\tau_1,\ldots,\tau_n$. Then we repeat this procedure simulating other sets of default times and have an estimate of $\text{FtD}_{a,b}(t) = \mathbb{E}_t\{\Pi\text{RFtD}_{a,b}(t)\}$.

First to default: Special cases

TOTAL DEPENDENCE. Assume positive deterministic intensities γ_i, cumulated intensities Γ_i and defaults of each name i, τ_i, as first jump of a related Poisson process:

$$\tau_1 = \Gamma_1^{-1}(\xi_1), \ldots, \tau_N = \Gamma_N^{-1}(\xi_N)$$

If the ξ are all equal from total dependence, the smallest τ_i (i.e. the first default) is the one corresponding to the largest Γ_i, i.e. to the riskiest single name. Then the related premium rate R will be the rate of the corresponding CDS

At the same time the premium paid for a FtD has to be lower than the sum of the single different premia of the basket components:

The sum of the CDS rates R_i corresponds to protection against all of the names up to time τ. As such, this includes in particular protection against the first name, and then the premium rate of the sum of CDS has to be larger than the premium rate in the FtD.

First to default: Special cases

Usually in the case of k-th to default baskets, the number of components is around 5 or 10, so the computational effort of a Monte carlo approach is not dramatic. The situation changes when considering CDO's and CDS's indices, where the number is around 100.

Monte Carlo pricing of a First to Default

If one wants to speed up the computation one can use the control variate variance reduction technique. Possible "analytical" payoffs (see earlier treatment of control variate on single name intensity models) to be used as control variate are

$$1_{\{\tau_1<T,\tau_2<T\}} + 1_{\{\tau_1<T,\tau_3<T\}} + \ldots + 1_{\{\tau_{N-1}<T,\tau_N<T\}}$$

where the expectation of each term, in case of Gaussian copulas, depends on pairwise correlations and can be computed analytically given approximations of the bivariate normal distribution function. Another possible payoff is the indicator

$$1_{\{\tau^1<T\}} = 1 - 1_{\{\tau_1\geq T,\tau_2\geq T,\ldots,\tau_N\geq T\}}$$

(without discount $D(0,\tau^1)$) whose expectation is the survival copula associated with the given copula model underlying default dependence. If this is known in closed form, we have an analytical value for this payoff.

CDO pricing: One-Factor approach

$$\mathbb{Q}(\tau_i < T | M = m) = \Phi\left(\frac{\Phi^{-1}(1 - \exp(-\Gamma_i(T))) - \sqrt{\rho_i} m}{\sqrt{1 - \rho_i}}\right)$$

As noticed before, conditional on M defaults are independent, and joint default probabilities are just products of these objects.

The final step is to determine the expected loss of the underlying portfolio in terms of these default times. Though factor models provide an appealing framework to perform this task the mathematical complexity and analytical tractability of the model itself will vary according to the specific properties of the underlying portfolio of credits.

CDO pricing: One-Factor approach

In particular the effectiveness of Factor models will depend on two main measures to assess the complexity of a portfolio of obligors:

- Portfolio Size: The number N of credits in the underlying portfolio
- Homogeneity: The uniformity of the underlying credits with respect to notional, recovery REC_i, default probability (see Γ_i) and correlation to common factors (ρ_i).

We are going to illustrate some methods to evaluate the Loss Distribution of the portfolio in the Factor model according to different specifics of the underlying portfolio itself.

CDO: One-Factor approach: Finite Size, Homogeneous Portfolio

We assume homogeneity in the portfolio in the sense that we consider **equal recovery rates**, **equal default probabilities** and **equal correlation** for the names in the basket.

Then, the probability of a single default in the portfolio, conditional on M, is

$$\mathbb{Q}(\tau_i < T|M = m) = \Phi\left(\frac{\Phi^{-1}(1-\exp(-\Gamma(T))) - \sqrt{\rho}m}{\sqrt{1-\rho}}\right)$$

and it is independent on the defaulting name i.

The main feature of the factor approach is that **conditionally to the common factor** M **the names are independent**. So we can easily compute the conditional probability to have n defaults among the N obligors with the Binomial distribution, given that now all names have the same statistics:

$$\mathbb{Q}\{n\,\text{defaults}|M = m\} = \binom{N}{n}\mathbb{Q}(\tau < T|M = m)^n\left(1 - \mathbb{Q}(\tau < T|M = m)\right)^{N-n}$$

CDO: One-Factor approach: Finite Size, Homogeneous Portfolio

$$\mathbb{Q}\{n\,\text{defaults}|M=m\} = \binom{N}{n} \mathbb{Q}(\tau < T|M=m)^n \left(1 - \mathbb{Q}(\tau < T|M=m)\right)^{N-n}$$

In order to compute the unconditional probability to have n defaults among the N obligors we just integrate the previous expression w.r. to the common factor M

$$\mathbb{Q}\{n\,\text{defaults}\} = \int_{-\infty}^{+\infty} \mathbb{Q}\{n\,\text{defaults}|M=m\}\varphi(m)dm$$

In general this integral has to be computed numerically: This is the reason why the factor approach is called **semi-analytical**. Indeed we have closed formulas for the number of defaults but we need to solve integrals numerically.

In the homogeneous portfolio framework, the computation of the associated loss distribution is straightforward. Given a constant recovery rate Rec, for the underlying credits in the portfolio, the probability of having a percentage portfolio loss $L = n \cdot (1 - \text{Rec})$ given by n names defaulting in the portfolio is given by

$$\mathbb{Q}\{\text{Loss} = L\} = \mathbb{Q}\{n\,\text{defaults}\}$$

CDO: One-Factor approach: Infinite (Large) Size, Homogeneous Portfolio

If the numbers of reference entities in the portfolio becomes reasonably high, the formulation of the loss probability simplifies further.

If the portfolio is relatively large, conditional on M the fraction of credits of the underlying portfolio defaulting over a specific time horizon should be roughly equal to the common individual default probability of the underlying credits for that horizon (assumed equal across names due to homogeneity).

This is a consequence of a probabilistic result known as the Law of Large Numbers (which in turn can be derived as a consequence of the Central Limit Theorem). Let us see this in more detail.

CDO: JPMorgan Approach: The Large Pool Model

Let us introduce the definition of Clean Spread γ_{clean} for the pool, taking the itraxx as fundamental example:

$$\gamma_{clean} = \frac{R_T^{\text{i-Traxx}}}{1 - \text{Rec}} = \frac{R_T^{\text{i-Traxx}}}{\text{Lgd}}$$

($R_{0,T}^{\text{i-Traxx}}$ (bps) is the quoted premium rate (spread) of the CDS index for maturity T).

Justification of this formula has been given in the single name case.

If we denote by T the maturity of the contract, then the average cumulative default probability of the portfolio can be approximated by:

$$\text{PD}(T) = 1 - \exp\left(-\frac{\gamma_{clean}}{10000}T\right)$$

as in standard constant intensity model. The factor 10000 is due to expressing γ in bps.

CDO. JPMorgan Approach: The Large Pool Model

Conditional on the systemic factor M defaults are independent and the probability of one default is, as we have seen many times, the same for all names and given by:

$$\mathsf{PD}_1(M;\rho) = \Phi\left(\frac{\Phi^{-1}(\mathsf{PD}(T)) - \sqrt{\rho}M}{\sqrt{1-\rho}}\right)$$

Here the Large Pool assumption comes into play: we can assume the number of names going to infinity and check what happens to the default fraction in the pool.

CDO. JPMorgan Approach: The Large Pool Model

The default fraction or rate (DR) of the pool at a given time T is the number of defaulted names over the total number of names in the pool, and is what matters for a CDO payoff. We may write, conditional on M:

$$DR_T^N(M) = \frac{1}{N}\sum_{i=1}^{N} 1_{\{\tau_i \le T|M\}}$$

But given that, conditional on M, $1_{\{\tau_i \le T|M\}}$ are i.i.d. with mean $\mathrm{PD}_1(M;\rho)$, the law of large numbers tells us that

$$DR_T^N(M) \to \mathrm{PD}_1(M;\rho) \text{ as } N \to \infty$$

Since for equal recovery across names the loss is simply $(1 - \textsc{Rec})$ times the default fraction, the loss under infinite pool in our case is simply

$$\textsc{Loss}_T^{\infty}(M;\rho) = (1 - \textsc{Rec})\mathrm{PD}_1(M;\rho)$$

CDO. JPMorgan Approach: The Large Pool Model

If we consider a $[0, B]$ tranche we have that the **tranched loss** (conditional on M) is $\min(\mathsf{Loss}_T(M), B)$. JPMorgan computes the **expected tranched loss** by using the above result on the loss under large pool, so that, conditional on M, there is no need to take expectation, due to the simplification obtained through the law of large numbers:

$$\mathbb{E}[\mathsf{Loss}_{0,B}^{tr}](M, \rho) = \mathsf{Loss}_{0,B}^{tr,\infty}(M, \rho) := \frac{1}{B} \min\left(\mathsf{Loss}_T^{\infty}(M; \rho), B\right)$$

(notice that no expectation is acting really), and, unconditionally,

$$\mathbb{E}[\mathsf{Loss}_{0,B}^{tr}](\rho) = \mathsf{Loss}_{0,B}^{tr,\infty}(\rho) := \int \mathsf{Loss}_{0,B}^{tr,\infty}(m, \rho)\varphi(m)dm$$

CDO. JPMorgan Approach: The Large Pool Model

In particular, substituting all expressions and summing up, we find $\text{Loss}_{0,B}^{tr,\infty}(\rho) =$

$$= \boxed{\int \frac{1}{B} \min \left(\Phi \left(\frac{\Phi^{-1}\left(1 - \exp\left(\frac{-R_T^{\text{i-Traxx}}\ T}{\text{Lgd}\ 10000}\right)\right) - \sqrt{\rho} m}{\sqrt{1-\rho}} \right) (1 - \text{Rec}), B \right) \varphi(m) dm}$$

We can value this in closed form, obtaining

$$= \Phi(A_1) + \frac{\text{Lgd}}{B} \Phi_2 \left(-A_1, \Phi^{-1}\left(1 - \exp\left(\frac{-R_T^{\text{i-Traxx}}\ T}{\text{Lgd}\ 10000}\right)\right), -\sqrt{\rho} \right)$$

$$A_1 = \frac{1}{\sqrt{\rho}} \left[\Phi^{-1}\left(1 - \exp\left(\frac{-R_T^{\text{i-Traxx}}\ T}{\text{Lgd}\ 10000}\right)\right) - \sqrt{1-\rho}\ \Phi^{-1}(B/\text{Lgd}) \right]$$

and where $\Phi_2(\cdot, \cdot, c)$ is the bivariate standard normal cumulative distribution function with correlation c.

CDO. JPMorgan Approach: The Large Pool Model

For a generic tranche A, B, we have through the above formula

$$\mathsf{Loss}_{A,B}^{tr,\infty}(\rho_A, \rho_B) = \frac{1}{B - A}\left[B\ \mathsf{Loss}_{0,B}^{tr,\infty}(\rho_B) - A\ \mathsf{Loss}_{0,A}^{tr,\infty}(\rho_A)\right]$$

where we remember that we are computing everything at maturity T.

If we removed the infinite pool assumption we would need to put EXPECTED tranche losses in the payout.

Consistency of the model would impose $\rho_A = \rho_B$. We keep the possible inconsistency in view of implied correlation.

CDO. JPMorgan Approach: The Large Pool Model

Now, let us compute the outstanding tranche notional

$$\mathsf{ON}^{tr,\infty}_{A,B}(T,\rho_A,\rho_B) = \Big(1 - \mathsf{Loss}^{tr,\infty}_{A,B}(\rho_A,\rho_B)\Big)$$

Here too we do not need to take expectations since randomness has been ruled out by the law of large numbers in the basic loss. Since the payments are quarterly made, we can define a survival rate in the following way (T is expressed in years, annual compounding):

$$\mathsf{ON}^{tr,\infty}_{A,B}(T,\rho_A,\rho_B) = \left(1 + \frac{\mathsf{SurvivalRate}(\rho_A,\rho_B)}{4}\right)^{-4T}$$

CDO. JPMorgan Approach: The Large Pool Model

We can then extend, using this rate as pivot, the outstanding notional (as if it were a bond price) at all times t by

$$\mathsf{ON}_{A,B}^{tr,\infty}(t,\rho_A,\rho_B) = \left(1 + \frac{\mathsf{SurvivalRate}(\rho_A,\rho_B)}{4}\right)^{-4t}$$

Now assume, as JPMorgan does, that interest rates are null, or, equivalently, $P(0,t) = D(0,t) = 1$ for all t. Then we can write the tranche default leg price as

$$\mathsf{DefaultLeg}_{a,b}^{A,B}(0)(\rho_A,\rho_B) = \sum_{i=a+1}^{b} \left(\mathsf{ON}_{A,B}^{tr,\infty}(T_{i-1},\rho_A,\rho_B) - \mathsf{ON}_{A,B}^{tr,\infty}(T_i,\rho_A,\rho_B)\right)$$

and the tranche premium leg for unit spread as

$$\mathsf{PremiumLeg1}_{a,b}^{A,B}(0)(\rho_A,\rho_B) = \sum_{i=a+1}^{b} \alpha_i \mathsf{ON}_{A,B}^{tr,\infty}(T_i,\rho_A,\rho_B)$$

CDO. JPMorgan Approach: The Large Pool Model

In this setup the fair tranche spread $R^{A,B}_{0,T_b}$ paid in the premium leg that balances the default leg solves

$$R^{A,B}_{0,T_b}\ \mathsf{PremiumLeg1}^{A,B}_{0,b}(0)(\rho_A,\rho_B) = \mathsf{DefaultLeg}^{A,B}_{0,b}(0)(\rho_A,\rho_B)$$

from which

$$R^{A,B}_{0,T_b} = \frac{\mathsf{DefaultLeg}^{A,B}_{0,b}(0)(\rho_A,\rho_B)}{\mathsf{PremiumLeg1}^{A,B}_{0,b}(0)(\rho_A,\rho_B)}$$

This approach will be fundamental in defining **implied correlation**.

CDO. One-Factor approach: Large Homogeneous Portfolio

This approach has some limits:

- It can be used only for very large portfolios (typically basket of 100 names or more). This is the case for CDO's but not for k-th to default swaps
- The model cannot produce consistent probabilities for low number of defaults (zero or one). This is due to the fact that the model does not consider the absolute number of defaults but only the (continuous) fraction of the defaulted portfolio

CDO. One-Factor approach: Finite, Not Homogeneous Portfolio

This is the most typical case in the CDO world, but also the most difficult. As usual we need to compute the portfolio loss distribution. But here we have different recoveries REC_i (leading to different additive losses $\text{LGD}_i = 1 - \text{REC}_i$) and different default probabilities for the underlying names.

Find a way to compute the **conditional** on M portfolio loss distribution, so that we can obtain the **unconditional** distribution just integrating on the common factor.

There are three main approaches to compute the conditional distribution:

- Fast Fourier Transform Approach
- Recurrence Relation Approach
- Probability Shifting Approach

Loss Calculation: Factor Copula without MC. Fast Fourier Transform

We need to compute the $\text{Loss}(t)$ distribution at each time t, where

$$\text{Loss}(t) = \sum_{i=1}^{n} \text{Loss}_i \mathbf{1}_{\{\tau_i < t\}}$$

If the names are equally weighted in the portfolio we can substitute Loss_i with $\text{LGD}_i = 1 - \text{Rec}_i$. Computing the conditional version

$$\text{Loss}_m(t) = \sum_{i=1}^{n} \text{Loss}_i \mathbf{1}_{\{\tau_i < t | M = m\}} = \sum_{i=1}^{n} L_{i,m}(t)$$

where $L_{i,m} = \text{Loss}_i \mathbf{1}_{\{\tau_i < t | M = m\}}$ are independent variables. We know that the characteristic function φ of a sum of independent variables is given by the product of single characteristic functions (convolution product in the space of densities).

$$\varphi_{\text{Loss}_m}(u) = \mathbb{E}[e^{iu\text{Loss}_m}] = \mathbb{E}[e^{iu\sum_j L_{j,m}}] = \mathbb{E}[\prod_j e^{iuL_{j,m}}] = \prod_j \mathbb{E}[e^{iuL_{j,m}}] = \prod_j \varphi_{L_{j,m}}(u)$$

Loss Calculation: Factor Copula without MC. Fast Fourier Transform

$$\varphi_{\mathsf{Loss}_m}(u) = \prod_j \varphi_{L_{j,m}}(u) \iff \mathsf{DFT}(p_{\mathsf{Loss}_m}) = \mathsf{DFT}(p_{L_{1,m}}) \cdots \mathsf{DFT}(p_{L_{n,m}}),$$

since in general the characteristic function $\varphi_{\mathsf{Loss}^m}(u)$ of a (discrete) random variable is a (discrete) Fourier transform DFT of the density of the random variable Loss_m, and the same holds for single loss terms $\varphi_{L_{j,m}}(u)$. With the inverse discrete Fourier transform IDFT we can compute the loss density as

$$p_{\mathsf{Loss}_m} = \mathsf{IDFT}(\mathsf{DFT}(p_{L_{1,m}}) \cdots \mathsf{DFT}(p_{L_{n,m}}))$$

The Fast Fourier Transform is a particular method to efficiently compute the DFT.

The method is explained in great detail in Robertson, J. P., (1992). The Computation of Aggregate Loss Distributions

`http://www.defaultrisk.com/pp_related_01.htm`

Loss Calculation: Factor Copula without MC. Recurrence Relation

This approach is quite simple but it can be applied only when the recovery rates and the notionals are the same for each underlying name (see Hull and White, 2004).

The different realizations of the loss are just LGD times the number of defaults and we need the probability to have that specific number of defaults.

Here we do not assume default prob or correlation across pairs to be equal.

Since conditional on M the defaults are independent, it is possible to find a recurrence formula for the corresponding probabilities. The prob to have zero defaults at t is:

$$\pi_t(0|M) = \prod_{i=1}^{N} \mathbb{Q}(\tau_i > t|M)$$

We have seen earlier how to compute the conditional survival probability in a factor model

$$\mathbb{Q}(\tau_i > T|M = m) = 1 - \Phi\left(\frac{\Phi^{-1}(1 - \exp(-\Gamma_i(T))) - \sqrt{\rho_i}m}{\sqrt{1 - \rho_i}}\right)$$

Loss Calculation: Factor Copula without MC. Recurrence Relation

Then it is easy to prove that the probability to have exactly one default is (either defaults name 1 and names $2, \ldots, N$ survive, or defaults name 2 and names $1, 3, \ldots, N$ survive, and these two events are disjoint, and so on):

$$\pi_t(1|M) = \pi_t(0|M) \sum_{i=1}^{N} \frac{1 - \mathbb{Q}(\tau_i > t|M)}{\mathbb{Q}(\tau_i > t|M)}$$

Applying the same reasoning and defining

$$w_i = \frac{1 - \mathbb{Q}(\tau_i > t|M)}{\mathbb{Q}(\tau_i > t|M)}$$

we have:

$$\pi_t(k|M) = \pi_t(0|M) \sum w_{p(1)} w_{p(2)} \ldots w_{p(k)}$$

where $\{p(1), p(2), \ldots, p(k)\}$ is a set of k numbers among $\{1, 2, \ldots, N\}$ and the summation is taken over all the possible $\frac{N!}{(N-k)!k!}$ combinations.

Loss Calculation: Factor Copula without MC. Recurrence Relation

Let us define $P_k = \sum w_{p(1)} w_{p(2)} \dots w_{p(k)}$ and $A_k = \sum_{i=1}^{N} w_i^k$; then, there is a Recurrence Formula saying

$$
\begin{aligned}
P_1 &= A_1 \\
2P_2 &= A_1 P_1 - A_2 \\
3P_3 &= A_1 P_2 - A_2 P_1 + A_3 \\
&\vdots \\
kP_k &= A_1 P_{k-1} - A_2 P_{k-2} + A_3 P_{k-3} - \dots + (-1)^{k+1} A_k
\end{aligned}
$$

the A's are easily computed based on conditional probabilities on M given earlier. The scheme then give the P's that, in turn, give the probability $\pi(k|M)$ of having k defaults and hence the loss if all the LGD and notionals are equal.

This method results to be computationally very fast, but has the limitation that it works only under strict requirements on the recoveries and the notionals.

Loss Calculation: Factor Copula without MC. Probability Shifting

A detailed description of this method can be found in Hull and White (2004) or in Andersen et al. (2003).

The idea is to divide the spectrum of possible losses into buckets of small size and to compute, conditional on M, the probability to have the loss inside one of these buckets.

The probability distribution is computed by recurrence: if we have the probability distribution of $n-1$ names, we can update the distribution when adding the n-th name.

In detail: Let us divide the possible loss range $[0, \text{Loss}^{MAX}]$ into buckets $\{0, b^0\}, \{b^0, b^1\}, \ldots, \{b^{K-1}, b^K\}$. We denote by $b_j = \{b^{j-1}, b^j\}$ the j-th bucket.

Then we indicate with p_j^n the probability that the loss is in the bucket b_j when n names are present. We assume that the loss is concentrated in the middle of the interval, so that if the loss is in the bucket j, we set $L_j = \frac{b^{j-1}+b^j}{2}$.

Loss Calculation: Factor Copula without MC. Probability Shifting

As usual we compute conditional on M (independence) and then integrate.

At the beginning we imagine to have a portfolio of zero names, so $p_j^0 = 0$ for all j.

Now add a particular credit reference with default probability d_1, survival $1 - d_1$ and recovery REC_1. Then we update the probability distribution of the loss for a portfolio with one name. Obviously $p_0^1 = 1 - d_1$ and then there is a particular bucket containing $L_0 + \text{LGD}_1$ having probability d_1.

Loss Calculation: Factor Copula without MC. Probability Shifting

In general if we have all the series of the p_j^{n-1}'s with $n-1$ names considered, we can compute the updated p_j^n's when adding the $n-th$ name in the following way

$$p_{u(j)}^n \quad = \quad p_j^{n-1} \cdot d_n + p_{u(j)}^{n-1} \cdot (1 - d_n)$$

where $u(j)$ indicates the bucket containing the loss of n names if the loss of $n-1$ names was in the bucket j.

This method has two main advantages: (i) Works well also for different notionals and recoveries, (ii) the width of the buckets is arbitrary.

Single Tranche CDO Pricing

Once we have the distribution of the portfolio loss, obtained by one of the methods above, we can compute the price of each tranche of a CDO.

As before (with the large pool model) we can extend tranched loss knowledge at time T_b to earlier loss knowledge by defining a survival rate for the expected outstanding tranche notional: Solve

$$1 - \mathbb{E}[\mathsf{Loss}^{tr}_{A,B}(T)] = \left(1 + \frac{\mathsf{SurvivalRate}}{4}\right)^{-4T}$$

in SurvivalRate and assume

$$1 - \mathbb{E}[\mathsf{Loss}^{tr}_{A,B}(t)] = \left(1 + \frac{\mathsf{SurvivalRate}}{4}\right)^{-4t}$$

Or else, to be more precise, we should apply the above methods for different maturities, to get the expected loss at different t's.

Single Tranche CDO Pricing

As we have already seen, a CDO tranche contract is made up by two legs of payments, the default leg

$$\text{PriceDefLeg}_{A,B}(0) = \int_0^T P(0,t)\mathbb{E}[d\text{Loss}^{tr}_{A,B}(t)]$$

and the premium leg

$$\text{PricePrLeg}(0) = R^{A,B}_{0,T}(0)\ \text{PricePrLeg1}(0),$$

$$\text{PricePrLeg1}(0) := \sum_{i=1}^{b} P(0,T_i)\alpha_i(1 - \mathbb{E}[\text{Loss}^{tr}_{A,B}(T_i)])$$

Still using the Factor copula, instead of resorting to one of the above approximations (large pool, probability shifting, recurrence relation) one can go for a Monte Carlo simulation, that makes no approximation on the loss evolution in time but computes it exactly. This can be numerically more intensive though.

Single Tranche CDO Pricing

We can compute the Tranche fair spread as

$$R_{0,T_b}^{A,B}(0) = \frac{\mathbb{E}[\int_0^{T_b} P(0,t)d\mathsf{Loss}_{A,B}^{tr}(t)]}{\mathbb{E}[\sum_{i=1}^{b} P(0,T_i)\alpha_i(1-\mathsf{Loss}_{A,B}^{tr}(T_i))]}$$

If we use Monte Carlo, typically the numerator term inside the expectation has larger variance than the corresponding term in the denominator. The standard error can be computed only for the numerator assuming the denominator expectation to be exact. Otherwise, a conservative window built on standard errors for the numerator and the denominator can be built.

We typically use 200.000 paths without variance reduction techniques. In C++ it is a matter of few seconds.

The approximated methods can be helpful here. Particularly quick and precise is the probability shifting approach, although it gets slow if we apply it for different maturities to have a more precise evolution of the loss.

Implied Correlation

Up to now we have discussed how to compute the loss distribution starting from a given value for the Gaussian copula correlation, always neglecting how we can find this value from market quotes.

In market practice the default correlation is extracted implicitly from quotes of very liquid multi-name products such as the CDO index tranches. The correlation obtained in this way is called **implied correlation**.

Dealers provide quotes of the premium $R_{0,T_b}^{A,B}(0)$ paid for each tranche. Usually a Gaussian copula (standard or One Factor) is used for the pricing.

A single correlation value among all pairs of names in the portfolio is used in the Gaussian copula, that is the market quote **does not diversify across sector, country, etc, but assesses a sort of "average" correlation.**

Implied Correlation

In the tranche prices formulas write the losses as

$$\text{PriceDefLeg}_{A,B}(0) = \int_0^T P(0,t) d[B\ \mathbb{E}[\text{Loss}^{tr}_{0,B}(t)] - A\ \mathbb{E}[\text{Loss}^{tr}_{0,A}(t)]]/(B-A)]$$

$$= \underbrace{\frac{B}{B-A}}_{\beta} \text{PriceDefLeg}_{0,B}(0) - \underbrace{\frac{A}{B-A}}_{\alpha} \text{PriceDefLeg}_{0,A}(0)$$

and similarly the premium leg per unit spread

$$\text{PricePrLeg1}_{A,B}(0) = \sum_{i=1}^{b} P(0,T_i)\alpha_i(1-[B\ \mathbb{E}[\text{Loss}^{tr}_{0,B}(T_i)]-A\ \mathbb{E}[\text{Loss}^{tr}_{0,A}(T_i)]]/(B-A))$$

$$= \beta \text{PricePrLeg1}_{0,B}(0) - \alpha \text{PricePrLeg1}_{0,A}(0)$$

Implied Correlation

Now assume we put a Gaussian Factor copula with the same correlation parameter ρ_A in all loss calculations for the tranche $[0, A]$ and ρ_B for all loss calculations for $[0, B]$.

$$\text{PriceDefLeg}_{A,B}(0, \rho_A, \rho_B) = \beta \text{PriceDefLeg}_{0,B}(0, \rho_B) - \alpha \text{PriceDefLeg}_{0,A}(0, \rho_A)$$

$$\text{PricePrLeg1}_{A,B}(0, \rho_A, \rho_B) = \beta \text{PricePrLeg1}_{0,B}(0, \rho_B) - \alpha \text{PricePrLeg1}_{0,A}(0, \rho_A)$$

To obtain the implied correlation for a given maturity T_b, we insert the market quoted tranche spread $R^{A,B}_{0,T_b}(0)$ in the premium leg and find correlation parameters such that the two legs match:

$$\text{PriceDefLeg}_{A,B}(0, \rho_A, \rho_B) = R^{A,B\ \mathrm{Mkt}}_{0,T_b}\ \text{PricePrLeg1}_{A,B}(0, \rho_A, \rho_B)$$

Implied Correlation

Consider the i-traxx tranches for example, to clarify the procedure. For a given maturity in 3y, 5y, 7y, 10y the market quotes

$$\text{Upfront}^{0,3\% \text{ Mkt}} + 500bps \text{ running},$$

$$R^{3,6\% \text{ Mkt}}, R^{6,9\% \text{ Mkt}}, R^{9,12\% \text{ Mkt}}, R^{12,22\% \text{ Mkt}}$$

To obtain the implied correlation we proceed as follows.

Implied Correlation

First solve in $\rho_{3\%}$ for the equity tranche (this should be done with the upfront but here we assume we have converted it into an equivalent spread)

$$\text{PriceDefLeg}_{0,3\%}(0, \rho_3) = 500bps\ \text{PricePrLeg1}_{0,3}(0, \rho_3) + \text{Upfront}^{0,3\%\ \text{Mkt}}$$

The move on: now you have to choices: solve

$$\text{PriceDefLeg}_{3,6}(0, \rho_3, \rho_6) = R^{3,6\%\ \text{Mkt}}\ \ \text{PricePrLeg1}_{3,6}(0, \rho_3, \rho_6)$$

in ρ_6 (base correlation) or solve

$$\text{PriceDefLeg}_{3,6}(0, \bar{\rho}_{3,6}, \bar{\rho}_{3,6}) = R^{3,6\%\ \text{Mkt}}\ \ \text{PricePrLeg1}_{3,6}(0, \bar{\rho}_{3,6}, \bar{\rho}_{3,6})$$

in $\bar{\rho}_{3,6}$ (compound correlation).

Implied Correlation: Base VS Compound

$$\text{PriceDefLeg}_{3,6}(0, \rho_3, \rho_6) = R^{3,6\%\ \text{Mkt}}\ \ \text{PricePrLeg1}_{3,6}(0, \rho_3, \rho_6)$$

solve in ρ_6 (base correlation), or solve

$$\text{PriceDefLeg}_{3,6}(0, \bar{\rho}_{3,6}, \bar{\rho}_{3,6}) = R^{3,6\%\ \text{Mkt}}\ \ \text{PricePrLeg1}_{3,6}(0, \bar{\rho}_{3,6}, \bar{\rho}_{3,6})$$

in $\bar{\rho}_{3,6}$(compound correlation).

Compound correlation is more consistent at the level of single tranche: We value the whole payoff of the tranche premium and default legs with one single copula (model) with parameter $\bar{\rho}_{3,6}$.

Base correlation is inconsistent at the level of single tranche: we value different parts of the same payoff with different models, i.e. part of the payoff (involving $\text{Loss}_{0,3}$) is valued with a copula in ρ_3, while a different part (involving $\text{Loss}_{0,6}$) of **the same** payoff is valued with a copula in ρ_6.

Implied Correlation: Base VS Compound

The procedure goes on similarly:

$$\text{PriceDefLeg}_{6,9}(0, \rho_6, \rho_9) = R^{6,9\%\ \text{Mkt}}\ \text{PricePrLeg1}_{6,9}(0, \rho_6, \rho_9)$$

solve in ρ_9 (base correlation), or solve

$$\text{PriceDefLeg}_{6,9}(0, \bar{\rho}_{6,9}, \bar{\rho}_{6,9}) = R^{6,9\%\ \text{Mkt}}\ \text{PricePrLeg1}_{6,9}(0, \bar{\rho}_{6,9}, \bar{\rho}_{6,9})$$

in $\bar{\rho}_{6,9}$ (compound correlation).

And so on.

Implied Correlation: Model or simply Quoting Mechanism?

If the Gaussian copula assumptions were consistent with market tranche prices, there should be a unique Gaussian copula model consistent with the market.

In other terms all values $\rho_3, \rho_6, \rho_9, \ldots$ (base correlation) should be equal.

Or is we resort to compound correlation all values $\bar{\rho}_{0,3} = \rho_3, \bar{\rho}_{3,6}, \bar{\rho}_{6,9} \ldots$ should be equal to each other.

As we are going to see this does not happen: plotting base correlations

$$\rho_3, \rho_6, \rho_9, \ldots$$

we obtain a **correlation skew**, while when plotting compound correlations

$$\bar{\rho}_{0,3}, \bar{\rho}_{3,6}, \bar{\rho}_{6,9} \cdots$$

we obtain a **correlation smile**.

Implied (Compound) Correlation

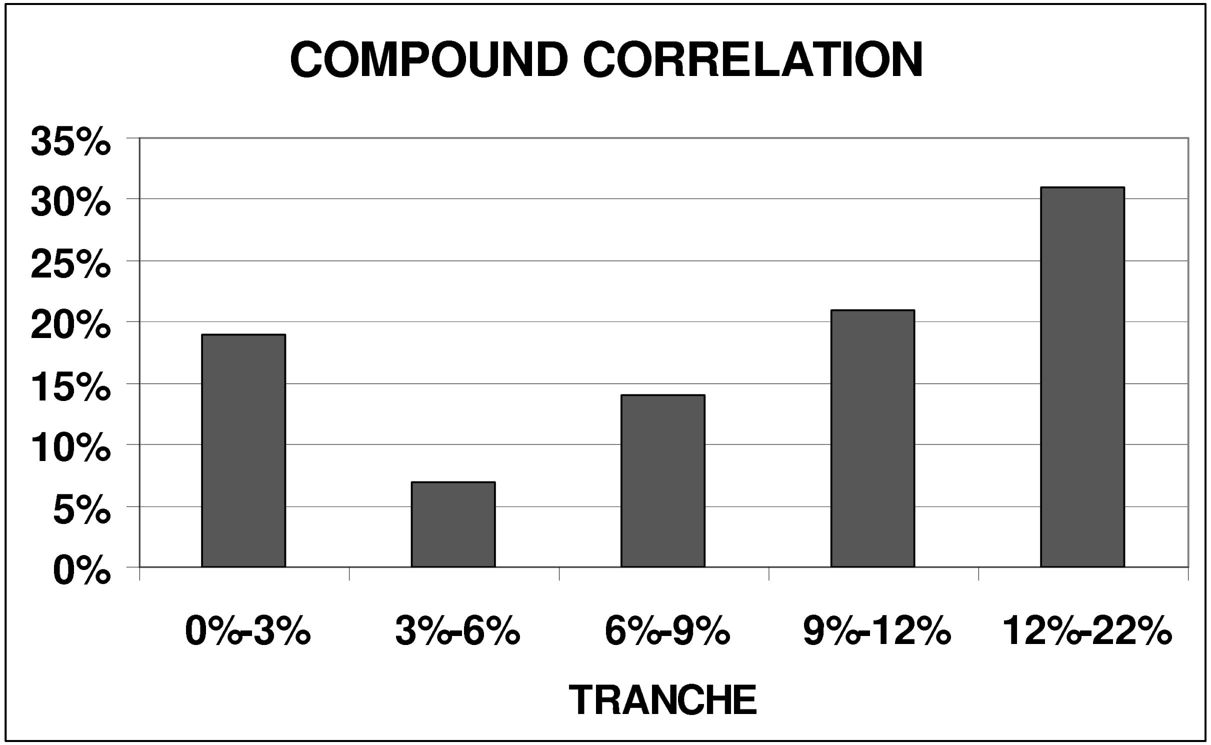


Figure 24: Example of compound correlation structure $\bar{\rho}$ for the DJ iTraxx.

Implied (Compound) Correlation

The value of correlation corresponding to the 1st mezzanine tranche (3%-6%) is lower than the one corresponding to the equity tranche, and after that the correlation grows.

The reason of this behavior must be sought for in the shape of the loss distribution leading to different investment considerations.

Consider the senior tranches: To reach the attachment points we need high losses, but this means that many defaults have to occur and this results in high correlations.

To obtain significant spreads (justifying the investment) we need high correlations.

On the contrary, if we think of the equity tranche $0 - 3\%$ it is clear that to obtain significant spreads we need low correlations: this tranche is impacted by every default, while a large correlation would imply a low probability of a single default (otherwise we would have a large probability to have many defaults with high correlation and in general this is not the case).

Implied (Base) Correlation

Now, let us suppose that we need to price a bespoke tranche (i.e. a non standard tranche, for example having different attachment points, e.g. 4%-7%).

Which value of correlation should we use? Is there a way to extract the correct value of correlation from the compound correlation smile?

The answer is negative: The **values of the implied compound correlations are peculiar** of the **PAIRS of standard attachment points**.

To overcome the problem of a correlation value linked to a pair of points a notion of correlation depending on a single point is better. We use then *base correlation*.

The idea is the following: We use market prices for quoted tranches and then we consider only equity tranches with the same lower attachment point. For example the DJ iTraxx tranches are transformed in 0%-3%, 0%-6%, 0%-9%, 0%-12%, 0%-22%.

Implied (Base) Correlation

Now, suppose we are interested in pricing the 4%-7% tranche: First we compute the base correlation curve, then we simply interpolate the curve to obtain the values ρ_4 and ρ_7 at 4% and 7%. Now we can price the considered tranche as a difference between the 0%-7% and the 0%-4% equity tranches.

Market contributors usually provide implied values both for the base and the compound correlation. Notice that by construction the compound correlation and the base correlation relative to the equity tranches are the same.

The term compound correlation derives from the fact that the loss for a mezzanine tranche is

$$\mathsf{Loss}^{tr}_{A,B} = \frac{1}{B-A}\left[(\mathsf{Loss} - A)^+ - (\mathsf{Loss} - B)^+\right],$$

i.e. it is given by the composition (sum) of two (call) options on the loss.

Implied (Base) Correlation

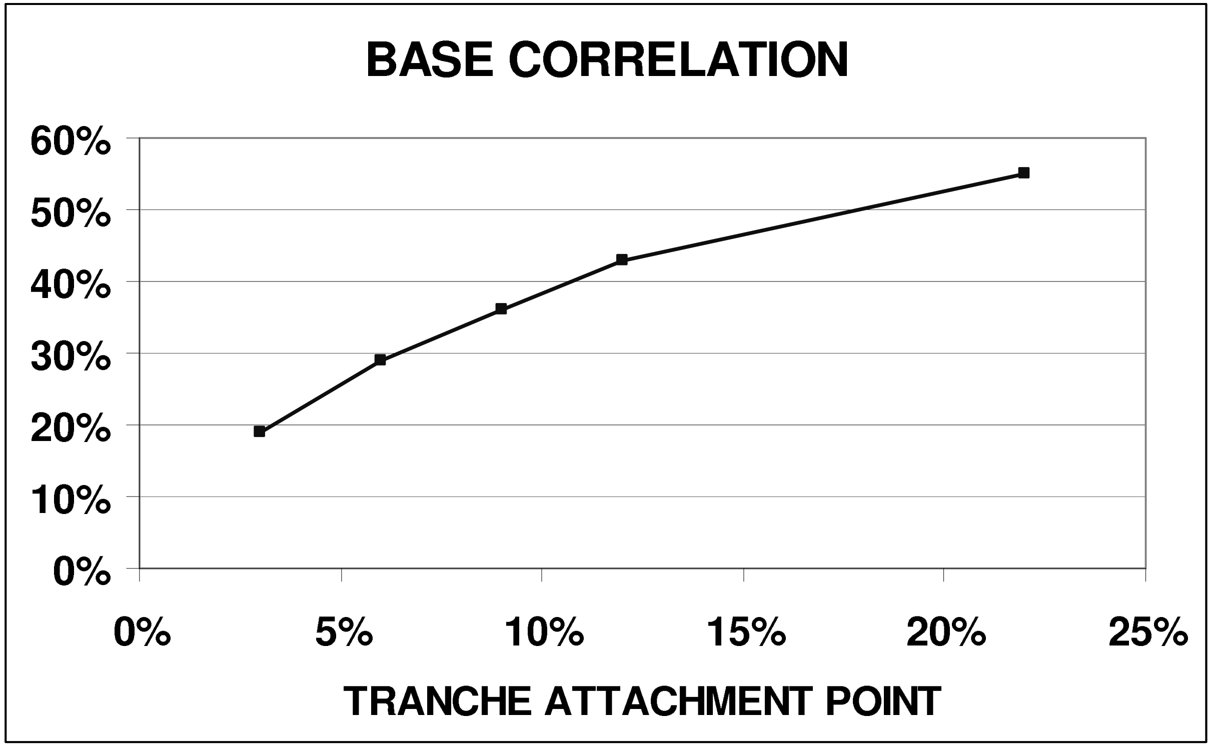


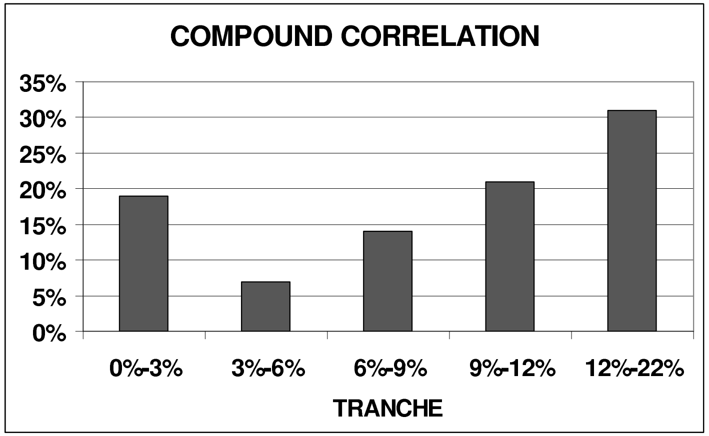
Figure 25: Example of base correlation structure for the DJ iTraxx.

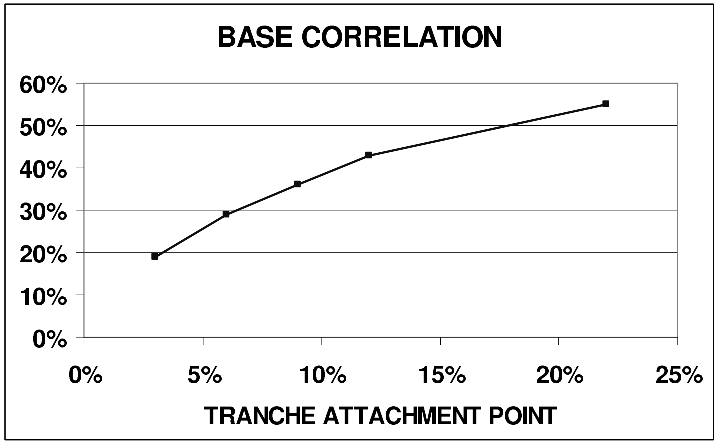
Implied (Base) Correlation

Notice two main differences:

First:The graph for the base correlation is a line while the graph for the compound is a collection of columns. This because the base correlation can be associated to a single (high) attachment point while the compound is associated to an entire tranche, so it would be incorrect to plot it as a line.

Second The values of base correlation are always increasing as a function on the attachment point, and are always greater than corresponding values of compound correlation.





Implied Correlation: Base VS Compound

$$\text{PriceDefLeg}_{3,6}(0, \rho_3, \rho_6) = R^{3,6\%\ \text{Mkt}}\ \ \text{PricePrLeg1}_{3,6}(0, \rho_3, \rho_6)$$

solve in ρ_6 (base correlation), or solve

$$\text{PriceDefLeg}_{3,6}(0, \bar{\rho}_{3,6}, \bar{\rho}_{3,6}) = R^{3,6\%\ \text{Mkt}}\ \ \text{PricePrLeg1}_{3,6}(0, \bar{\rho}_{3,6}, \bar{\rho}_{3,6})$$

in $\bar{\rho}_{3,6}$(compound correlation)

Compound correlation at times does not exist: the second equation may have no solution in some market conditions.

In these cases base correlation still exists: the fact that we (inconsistently) have two different ρ's in different parts of the payoff gives more freedom to the possible solutions of the first equation.

In these cases, however, the base correlation implies **negative expected tranche LOSS realizations** and is thus arbitrageable in principle. This happens when the base corr skew is very steep.

Implied Correlation: Base VS Compound

FEATURE ARTICLE

IMPLIED CORRELATION IN CDO TRANCHES:

A Paradigm to be Handled With Care

Stripping Correlations: MC vs Large Pool

We have seen how one strips implied correlations. However, in solving e.g.

$$\text{PriceD}\textsc{ef}\text{L}\textsc{eg}_{3,6}(0, \rho_3, \rho_6) = R^{3,6\%\ \mathsf{Mkt}}\ \ \text{PriceP}\textsc{r}\text{L}\textsc{eg}1_{3,6}(0, \rho_3, \rho_6)$$

solve in ρ_6 (base correlation), or solve

$$\text{PriceD}\textsc{ef}\text{L}\textsc{eg}_{3,6}(0, \bar{\rho}_{3,6}, \bar{\rho}_{3,6}) = R^{3,6\%\ \mathsf{Mkt}}\ \ \text{PriceP}\textsc{r}\text{L}\textsc{eg}1_{3,6}(0, \bar{\rho}_{3,6}, \bar{\rho}_{3,6})$$

in $\bar{\rho}_{3,6}$(compound correlation)

One can value the legs either by the large pool model assumption, by the Probability Shifting or Recurrence Relation method, or by full monte carlo valuation.

Stripping Correlations: MC vs Large Pool

Recall large pool model: all names have same default probabilities, recoveries, expected tranche loss is computed as tranched expected loss, zero interest rates etc.

The value of implied correlation changes considerably in the two cases

i) Monte carlo or Probability Shifting vs ii) large pool model (JPMorgan)

In the following we present two examples

Monte Carlo Approach

The simulation is performed using a standard Gaussian copula with a unique value of correlation. We simulate many a collection of default times and compute the expected values of the payment legs.

In this case we used the names of the DJ-iTraxx index: In particular we considered all the different spreads of the portfolio components (but a common recovery of 40%).

This method has a great disadvantage: It is quite heavy from a numerical viewpoint. In fact the stripping of correlations involves an iterative procedure for evaluating the tranche changing the correlation value: This means that at each round of the stripping (that is for each trial ρ) we need to run the Monte Carlo simulation, and this could be quite cumbersome.

Monte Carlo Approach

In the following Table we report an example of stripping of implied correlation starting from quoted tranches. In brackets we report the quoted correlation values.

Tranche	Spread	Base Correlation	Compound Correlation
0%-3%	24.125%	18.94%(19%)	18.94%(19%)
3%-6%	134	28.50%(29%)	4.83%(5%)
6%-9%	45	36.17%(37%)	13.32%(13%)
9%-12%	31	41.92%(43%)	21.98%(22%)
12%-22%	15.25	57.29%(55%)	30.44%(31%)

Table 24: Quoted DJ iTraxx tranche spreads on November 11th, 2004 by BNP Paribas. The maturity is March 20th, 2010 and the reference index spread is 37 bps. Both base correlation (third column) and compound correlation (forth column) are reported. The spread of the equity tranche is composed by a running spread of 500 bps plus an upfront premium: Here we report only this upfront part.

Monte Carlo Approach

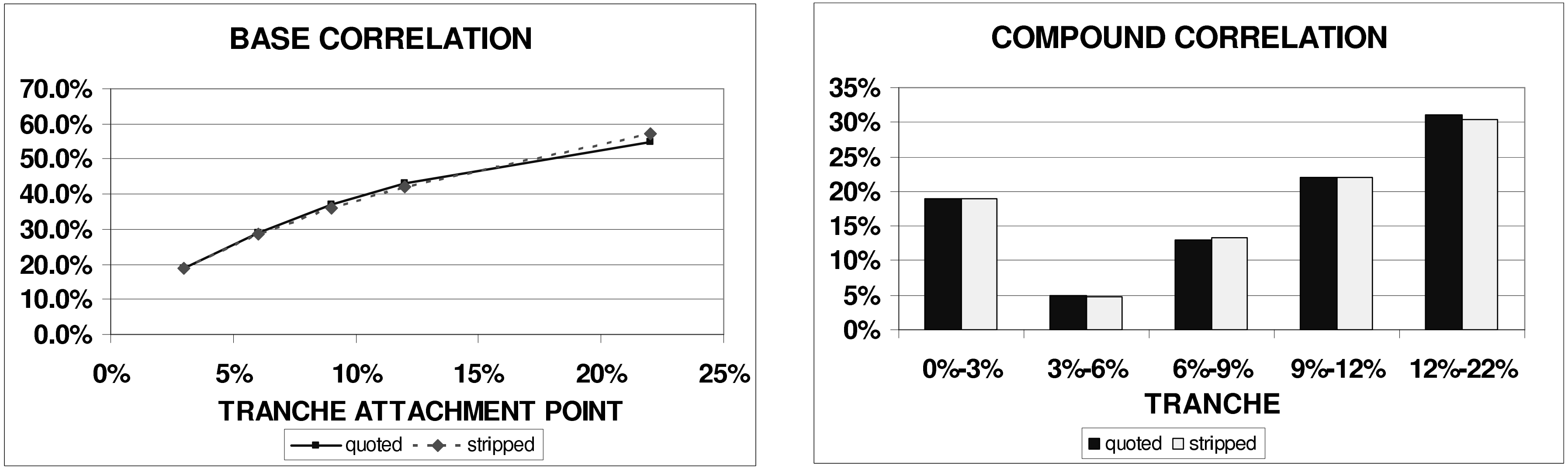


Figure 26: Comparison between quoted correlation curves and curves stripped with Monte Carlo. On the left we have the base correlation, on the right the compound correlation.

We used 100000 paths with no variance reduction technique, but 50.000 can often be sufficient.

Large Pool: JPMorgan Approach

The one-factor/large pool approach we have seen earlier is an alternative, and in particular JPMorgan has implemented and freely distributed a simplified version of it. For a detailed description of how JPM model works see also McGinty and Ahluwalia (2004). Initially to make implied correlation more objective JPMorgan used to assume interest rates all equal to 0, $P(t,T) = D(t,T) = 1$ for any t, T in premium and default legs.

Finally we remark the fact that they quote only base correlation: This is in line with the current practice to give importance only to base correlation (even if sometimes other dealers quote also the compound correlation).

One Factor Copula: JPMorgan Approach

We report the values of the base correlation stripped by JPM quotes of iTraxx tranches obtained implementing the Large Pool Model (in brackets we report the quoted values of correlation).

Tranche	Spread	Base Correlation Large Pool	Base Correlation MC
0%-3%	24.05%	25.9% (25.7%)	18.94%
3%-6%	134	35.5% (35.3%)	28.5 %
6%-9%	47	43.4% (43.2%)	36.17%
9%-12%	31.5	49.1% (48.7%)	41.92%
12%-22%	155	64.3% (63.9%)	57.29%

Table 25: Results of the stripping with JPMorgan approach.

Notice the difference with respect to the implied base correlations we have derived earlier and compared with Paribas quotes for example (and by most other entities), obtained with a more refined method (e.g. Monte Carlo) and without large pool assumptions.

One Factor Copula: JPMorgan Approach

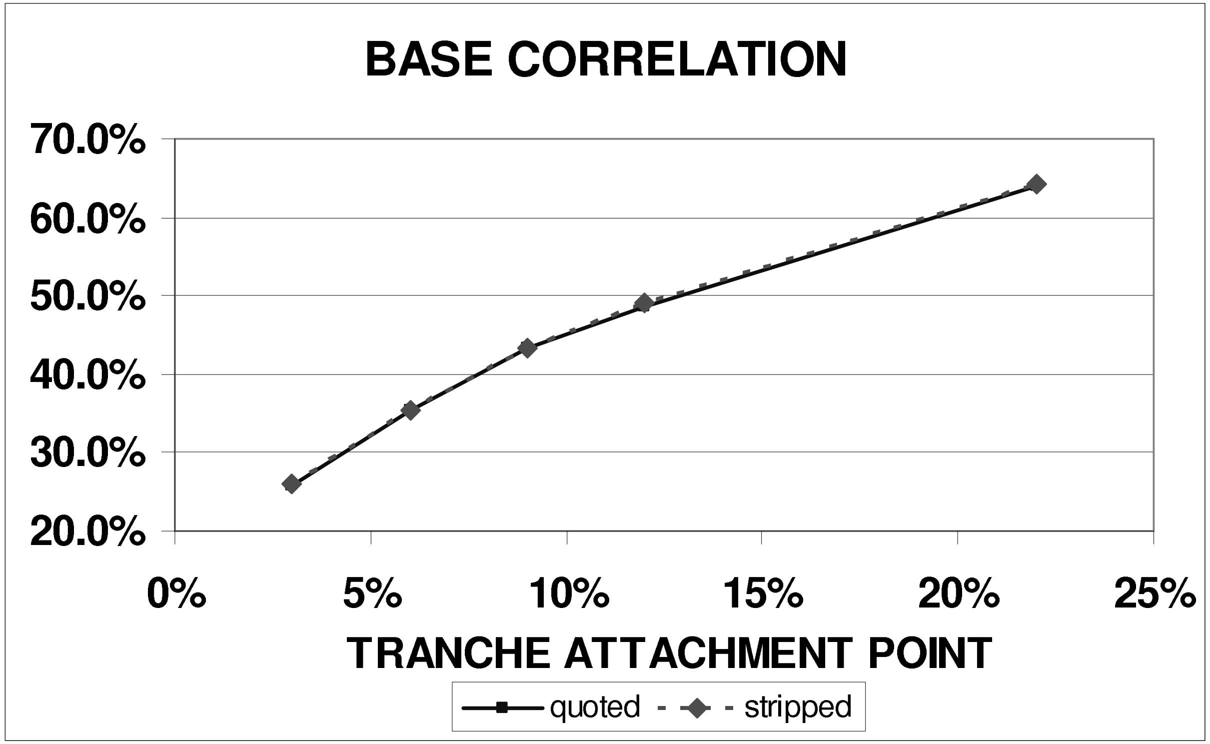


Figure 27: Comparison between quoted correlation curves and curves stripped with the JPMorgan approach.

One Factor Copula: JPMorgan Approach

JPMorgan used to say that they are not really providing the market with an estimation technique but merely with a quoting mechanism. And in order for the quoting to be uniquely determined, they do away with interest rates, different credit spreads etc. This way even two entities that disagree say on the zero curve for interest rates will agree on base correlation. Thus JPMorgan implied correlation is only based on inputs from the liquid credit market indices on which all parties agree. This quoting method however has been abandoned now even by JPMorgan, since base correlation has a number of problems even under more precise implementations.

Calibrating all tranches consistently: The Implied Factor Copula (IFC) approach (Hull & White)

We give a summary of the method, trying to highlight its advantages and drawbacks.

A factor copula structure is assumed, similarly to the Gaussian 1 factor copula approach seen earlier, which we recall here:

$$\tau_1 = \Lambda_1^{-1}(-\ln(1-U_1)), \tau_2 = \Lambda_2^{-1}(-\ln(1-U_2)), \ldots, \tau_N = \Lambda_N^{-1}(-\ln(1-U_N)).$$

where we took $U_i = \Phi(X_i)$, with X_i's Gaussian variables defined as

$$X_i = \sqrt{\rho_i}\, M + \sqrt{1-\rho_i}\, Y_i,$$

with Y_i, M standard independent gaussian variables. We had

$$\mathbb{Q}(\tau_i < T|M) = \Phi\left(\frac{\Phi^{-1}(1-\exp(-\Gamma(T))) - \sqrt{\rho_i}M}{\sqrt{1-\rho_i}}\right)$$

with Γ the hazard function, assumed common for all single names (homogeneous pool)

Calibrating all tranches consistently: The Implied Factor Copula (IFC) approach

This time, instead:

- We do not model the copula explicitly, but
- we model default probabilities conditional on the systemic factor M of the copula.
- The copula will then be "hidden" inside these conditional probabilities.
- We assume a homogeneous model, in that the default probabilities of single names are all equal to each other.

Let us consider, for simplicity, survival (or equivalently default) probabilities that are associated to a constant-in-time hazard rate. We know that if we have a constant-in-time (possibly random) hazard rate λ for name i then the survival probability is

$$\mathbb{Q}(\tau_i > t) = \mathbb{E}[\exp(-\lambda t)]$$

Calibrating all tranches consistently: The Implied Factor Copula (IFC) approach

The implied copula approach postulates the following "scenario" distribution for the hazard rate λ conditional on the systemic factor M:

$$\lambda | M \sim \left\{ \begin{array}{ccc} \text{conditional hazard rate} & \text{Systemic scenario} & \text{Scenario probability} \\ \lambda_1 & M = m_1 & p_1 \\ \lambda_2 & M = m_2 & p_2 \\ \vdots & \vdots & \vdots \\ \lambda_s & M = m_s & p_s \end{array} \right.$$

Calibrating all tranches consistently: The Implied Factor Copula (IFC) approach

This way the default probability for a single name is, conditional on the systemic factor M,

$$\mathbb{Q}(\tau_i < t | M = m_j) = \ 1 - \exp(-\lambda_j t).$$

Compare with the Gaussian factor copula case:

$$\mathbb{Q}(\tau_i < T | M = m_j) = \Phi\left(\frac{\Phi^{-1}(1 - \exp(-\Gamma(T))) - \sqrt{\rho_i} m_j}{\sqrt{1 - \rho_i}}\right)$$

Unconditionally:

$$\mathbb{Q}(\tau_i < t) = \mathbb{E}[\mathbb{Q}(\tau_i < t|M)] = \sum_{j=1}^{s} p_j \mathbb{Q}(\tau_i < t|M = m_j) = \sum_{j=1}^{s} p_j (1 - \exp(-\lambda_j t))$$

Calibrating all tranches consistently: The Implied Factor Copula (IFC) approach

Conditional on M, all default times are independent, have the same hazard rate and their hazard rates are given by the above scenarios. Then to integrate against M we simply sum over all possible hazard rate scenarios multiplying by the scenario probability:

$$\textsc{Tranche}_{A,B}(0,R) = \sum_{j=1}^{s} p_j \textsc{Tranche}_{A,B}(0,R; \text{indep defaults across names with const haz rate } \lambda_j)$$

Each term on the right hand side can be easily computed according to the methods we have seen earlier (large pool model approximation, FFT, recurrence approach, Probability shifting approach, Monte Carlo...).

Calibrating all tranches consistently: The Implied Factor Copula (IFC) approach by Hull and White

Now we move to calibration: How can we calibrate say all the 5y DJ-i-Traxx tranches?

When the market gives us the tranche spread quote $R_{0,5y}^{A_i,B_i,\text{MKT}}$ for two canonical attachments (e.g. $A = 3\%, B = 6\%$) we know that this should make the tranche net present value equal to zero. We might then try to solve

$$\sum_{j=1}^{s} p_j \text{T}\textsf{RANCHE}_{A_i,B_i}(0, R_{0,5y}^{A_i,B_i,\text{MKT}}; \text{indep defaults across names with const haz rate } \lambda_j) = 0$$

$$\text{for all } \; A_i, B_i.$$

In practice, we try and minimize the sum of squares of the above expression across the five canonical tranches:

$$\mathbf{p}^\star = \text{argmin}_{p_1,\ldots,p_s} \sum_{i=1}^{5} \left[\sum_{j=1}^{s} p_j \text{T}\textsf{RANCHE}_{A_i,B_i}(0, R_{0,5y}^{A_i,B_i,\text{MKT}}; \text{ind defaults with const haz rate} \lambda_j) \right]^2$$

Calibrating all tranches consistently: The Implied Factor Copula (IFC) approach by Hull and White

$$\mathbf{p}^\star = \mathrm{argmin}_{p_1,\ldots,p_s} \sum_{i=1}^{5} \left[\sum_{j=1}^{s} p_j \textsc{Tranche}_{A_i,B_i}(0, R_{0,5y}^{A_i,B_i,\mathsf{MKT}}; \text{ind defaults with const haz rate } \lambda_j) \right]^2$$

Indeed, this is the approach suggested by Hull and White: calibrate the scenario probabilities p while pre-assigning the haz rate scenarios λ_j exogenously.

One may also calibrate the index itself besides the tranches, by adding the squared difference between the model implied index and the market index in the target function. The model implied index is easily computed by recalling that

$$\mathbb{Q}(\tau_i > t) = \sum_{j=1}^{s} p_j \exp(-\lambda_j t).$$

Calibrating all tranches consistently: The Implied Factor Copula (IFC) approach by Hull and White

One may naively think that since we have 6 quotes (5 tranches plus the index) then six parameters p could be enough to fit the market quotes, getting a low squared error out of the minimization. So one could set the number of scenarios $s = 6$.

Actually, this does not work. One may have to go up to $s = 30$ in order to be able to fit market data with a good precision.

We tried the model and checked that indeed, even with 27 points the error remains large, no matter how smartly one pre-selects the $\lambda_1, \ldots, \lambda_{27}$.

However, even with $s = 30$ one more problem surfaces. If we do not impose some further constraint or smoothing on the minimization problem, we are finding multiple solutions each time we change the initial guess.

Calibrating all tranches consistently: The Implied Factor Copula (IFC) approach by Hull and White

Hull and White propose to add to the target function a quantity that penalizes changes in convexity in the patterns of the scenario probabilities plotted against the default probabilities associated to each scenario:

$$\text{add} \;\; c\sum_{j=2}^{s-1} \frac{(p_{j+1}+p_{j-1}-2p_j)^2}{0.5[\exp(-\lambda_{j-1}5y)-\exp(-\lambda_{j+1}5y)]} \;\; \text{to the target function}$$

This term's numerators are second order differentials of the p's, and they penalize departures from convexity. Therefore, when we plot the graph $\{(i,p_i), i=1,\ldots,s\}$ all kinds of humps are penalized. The constant c controls the degree of smoothing: too large a c privileges the smoothing over the fit, and we cannot recover exactly the market quotes. Too small a c fails to guarantee substantial uniqueness in the calibration solution. We found good results with $c = 1$.

Calibrating all tranches consistently: The Implied Factor Copula (IFC) approach by Hull and White

As a final remark, we mention a relationship between default probabilities and recovery rates that may be necessary to fit the market correlation skew in periods of turmoil (e.g July 2005).

Following results of an empirical study by Hamilton et al (2005) HW suggest to change recovery in each scenario by linking it to the conditional probability of default in that scenario:

$$\mathrm{Rec}_j = 0.52 - 6.9(1 - \exp(-\lambda_j 5y)).$$

Then in computing the target function to be minimized we set recovery rates (and thus the related $\mathrm{Lgd} = 1 - \mathrm{Rec}$ contributing to the loss function) in each scenario according to the above relationship. The final minimization problem reads:

Calibrating all tranches consistently: The Implied Factor Copula (IFC) approach by Hull and White

$$\mathbf{p}^\star = \mathsf{argmin}_{p_1,\ldots,p_s} \sum_{i=1}^{5} \left[\sum_{j=1}^{s} p_j \mathsf{TRANCHE}_{A_i,B_i}(0, R_{0,5y}^{A_i,B_i,\mathsf{MKT}}; \lambda = \lambda_j; \mathsf{REC} = \mathsf{REC}_j) \right]^2$$

$$+ \left[125 \left(\sum_{j=1}^{s} p_j R_{0,5y}^{\mathsf{MKT}} \sum_{k=1}^{b} \alpha_k P(0,T_k) e^{-\lambda_j T_k} - \sum_{j=1}^{s} p_j \mathsf{LGD}_j \sum_{k=1}^{b} P(0,T_k)(e^{-\lambda_j T_{k-1}} - e^{-\lambda_j T_k}) \right) \right]^2$$

$$+ c \sum_{j=2}^{s-1} \frac{(p_{j+1} + p_{j-1} - 2p_j)^2}{0.5[\exp(-\lambda_{j-1} 5y) - \exp(-\lambda_{j+1} 5y)]}$$

under the constraints $p_i > 0,\ p_1 + \ldots + p_{30} = 1.$

This problem can be effectively solved with the excel solver.

Calibrating all tranches consistently: The Implied Factor Copula (IFC) approach by Hull and White

Now we present a calibration example based on the same 5y i-Traxx data of November 11, 2004 we have seen before for the base/compound correlation examples.

This is a plot of the calibrated probabilities p^* against the default probabilities:

$$(1 - e^{-\lambda_1 5y}, p_1), (1 - e^{-\lambda_2 5y}, p_2), \ldots, (1 - e^{-\lambda_{30} 5y}, p_{30})$$

We take $c = 1$ in the smoothing part. The calibration error is negligible.

$1 - e^{-\lambda_i 5y}$	p_i
0.09%	19.78%
1.16%	17.97%
2.19%	16.04%
3.15%	13.82%
4.07%	11.26%
4.97%	8.46%
5.83%	5.64%
6.66%	3.14%
7.48%	1.29%
8.28%	0.28%
9.05%	0.00%
9.81%	0.00%
10.54%	0.00%
11.27%	0.00%
11.98%	0.01%

$1-e^{-\lambda_i 5y}$	p_i
12.68%	0.03%
13.39%	0.12%
14.10%	0.25%
14.84%	0.34%
15.64%	0.37%
16.49%	0.30%
17.47%	0.17%
18.59%	0.05%
19.90%	0.00%
21.38%	0.00%
23.03%	0.00%
25.05%	0.06%
28.89%	0.16%
43.54%	0.24%
62.90%	0.23%

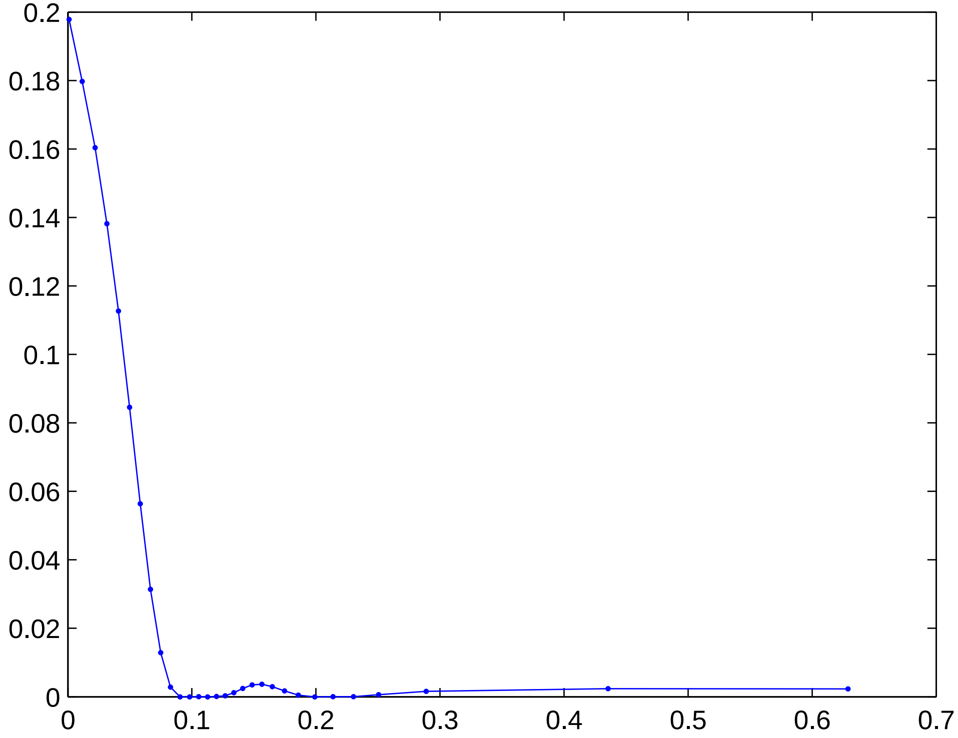


Figure 28: Implied copula calibrated parameters consistent with the whole correlation skew, plot of $(1 - e^{-\lambda_i 5y}, p_i)$.

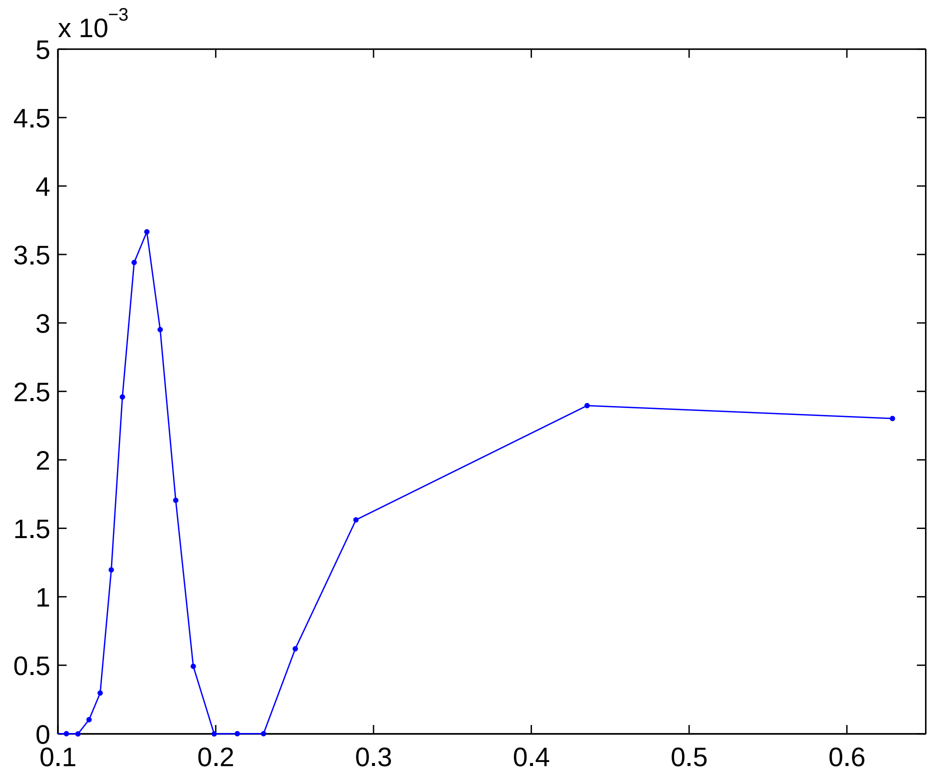


Figure 29: Implied copula calibrated parameters consistent with the whole correlation skew, plot of $(1 - e^{-\lambda_i 5y}, p_i)$ (zoom)

Calibrating all tranches consistently: The Implied Factor Copula (IFC) approach by Hull and White

Once the model is calibrated, any payoff can be priced with a weighted sum of prices each under a model assuming independent defaults and time-constant hazard rates homogeneous across all names.

$$\text{PriceIFC} = \sum_{j=1}^{s} p_j \ \text{Price(indep defaults,} \ \lambda = \lambda_j \ \text{across all names} \ i = 1 \ldots N)$$

This is powerful. A drawback of this approach, like all other copula approaches, is that the core dependence across defaults is modelled as static, and there is no dependence dynamics. Pricing a tranche option would require in principle a dynamical model.

Also, there is no differentiation for single name probabilities, which are the same (although scenario-based) across names.

Calibrating all tranches consistently: The Implied Factor Copula (IFC) approach. Infinite pool.

Also with the implied factor copula one can resort to the infinite pool simplification, as we did for the gaussian factor copula. Conditional default indicators $1_{\{\tau_i<T|M=m_j\}}$ are i.i.d., so that their sample average as their number tends to infinity tends to the single common true mean:

$$DR^{j,N}(T) := \frac{1}{N}\sum_{i=1}^{N} 1_{\{\tau_i<T|M=m_j\}} \rightarrow \mathbb{E}1_{\{\tau_i<T|M=m_j\}} = \mathbb{Q}\{\tau_i < T|M = m_j\} = 1-e^{-\lambda_j T}$$

when N tends to ∞.

Again, this way we avoid taking expectations, except the final one with respect to M. But conditional on $M = m_j$ all randomness has been ruled out by the law of large numbers and both the default rate and the loss are completely determined.

Calibrating all tranches consistently: The Implied Factor Copula (IFC) approach. Infinite pool.

$$DR(T)|M \quad \sim \quad \left\{ \begin{array}{ccc} \text{conditional default rate} & \text{Systemic scenario} & \text{Scenario probability} \\ DR^1(T) = 1 - \exp(-\lambda_1 T) & M = m_1 & p_1 \\ DR^2(T) = 1 - \exp(-\lambda_2 T) & M = m_2 & p_2 \\ \vdots & \vdots & \vdots \\ DR^s(T) = 1 - \exp(-\lambda_s T) & M = m_s & p_s \end{array} \right. \tag{17}$$

The model is then used to calibrate the quoted index and tranches. For a preferred maturity T (typically 5 years) one sets the default rates DR to

$$DR^1(T) = 0,\ DR^2(T) = 1/125,\ DR^3(T) = 2/125,\ \ldots,\ DR^{125}(T) = 124/125$$

and inverts in the λ_j, solving

$$\lambda_j = -\ln(1 - DR^j(T))/T.$$

Calibrating all tranches consistently: The Implied Factor Copula (IFC) approach. Infinite pool.

Then one proceeds using

$$\text{Loss}_t|\{M = m_j\} = (1-\text{Rec}_j)(1-\exp(-\lambda_j t)), \;\; \text{Rec}_j = -0.1056289\ln(1-\exp(-\lambda_j)).$$

The recovery specification follows results of an empirical study we performed, see the first feature article by Torresetti, Brigo and Pallavicini (2006a) for more details and further references.

From the conditional loss $\text{Loss}_t|\{M = m_j\}$ one builds the payoff of tranches conditional on M, and then averages over M with a simple linear combination of the prices under each M scenario. See Torresetti, Brigo and Pallavicini (2006a) for more details and examples.

Calibrating all tranches consistently: The Implied Factor Copula (IFC) approach. Infinite pool.

FEATURE ARTICLE 2.

RISK NEUTRAL vs OBJECTIVE LOSS DISTRIBUTION AND CDO VALUATION.

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UNIT 5

BEYOND COPULAS: DYNAMIC LOSS MODELS

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UNIT 5. Dynamic Loss Models

- Bottom Up (Copula) vs Top Down models;
- Dynamic Loss Models: the Generalized Poisson Loss (GPL) model;
- Calibration examples;
- Consistency with single names: Common Poisson Shock framework;
- Avoiding infinite defaults: the Cluster Adjusted GPL model (GPCL);
- Calibration examples;
- Calibration in-crisis

Towards Dynamics Loss Models: Information in CDO quotes

Recall again the market quoted fair spreads for indices and tranches:

$$R_0 = \frac{\mathbb{E}_0\Big[\int_0^T D(0,u)d\text{Loss}_u\Big]}{\mathbb{E}_0\Big[\sum_{i=1}^b \delta_i D(0,T_i)(1-\bar{C}_{T_i})\Big]}$$

$$R_0^{A,B} = \frac{\mathbb{E}_0\Big[\int_0^T D(0,u)d\text{Loss}_u^{A,B}\Big] - U_0^{A,B}}{\mathbb{E}_0\Big[\sum_{i=1}^b \delta_i D(0,T_i)(1-\text{Loss}_{T_i}^{A,B})\Big]}$$

where $\text{Loss}_{T_i}^{A,B}$ is the tranched loss at points A, B divided by the tranche thickness $B - A$. C_T is the number of defaults up to T, whereas $\bar{C}_T$ is the same quantity divided by the pool size (default fraction of the pool). Typically $\text{Loss}_T = (1 - \text{Rec})\bar{C}_T$.

If R_0 and $R_0^{A,B}$ are the only data on default correlation in the market, **we see that the only information are "expected losses", "expected tranche losses" and "expected number of defaults".**

Loss models: The "TOP DOWN" approach

TOP DOWN APPROACH: Model loss-related quantities directly rather than patching single defaults models through a copula.

- a "Market Model" appeal: Focuses on more direct market objects, avoiding arbitrary assumptions on single name default dependencies;
- Possibility to have an authentically dynamic model;
- Calibrate indices and tranches consistently across attachments/maturities;
- Possibility to infer synthetic recovery information on a pool; **BUT...**
- How do losses of different pools "talk" to each other? (CDO squared);
- Consistency with single names: Random Thinning?

Dynamical dependence models: Top Down Approach

Earlier we modeled single default probabilities with intensity models and patched single default together with a copula, building the loss afterwards (bottom up).

As an alternative, we have then focused on **expected tranche losses (ETL)** as model independent quantities but did not postulate any dynamics underlying them.

Now we go one step further. Rather than contenting ourselves with the ETL, we focus on the Loss as a process to be modeled directly and realistically, and whose dynamics has to be made consistent with market index and tranche data for a start (top down).

Dynamical dependence models: Top Down Approach

Recently, some TOP DOWN models for the loss distribution dynamics and/or loss rates have been proposed. These models would be suited to price tranche options, forward starting CDO's and other loss dynamics dependent payoffs.

Some of these models are only "TOP", in that one never goes down to single name level. We will see examples of both TOP (GPL model) and rigorously "TOP DOWN" (GPCL model) approaches.

If the model remains only "TOP", it can be OK when pricing payoffs that are direct functions of the loss, avoiding the cumbersome modeling of the whole detailed dependence structure of default times, but may lead to problems when facing for example CDO squared payoffs or credit payoffs depending on the fine structure of defaults. Further, partial hedges and sensitivities with respect to single names are difficult without the "DOWN" feature.

Top Down Approach: The GPL Model

The basic Generalized Poisson Loss (GPL) model is an example of the TOP (down?) approach and can be formulated as follows.

Consider a number n of independent Poisson processes $N_1, \ldots, N_n$ with intensities $\lambda_1, \ldots, \lambda_n$. Define the stochastic process

$$Z_t = \sum_{j=1}^{n} \alpha_j N_j(t),$$

for increasing integers $\alpha_1, \ldots, \alpha_n$, and model the number of defaults as Z_t.

Top Down Approach: The GPL Model

$$\text{Example :} \quad M = 125, \quad Z_t = 1\ N_1(t) + 2\ N_2(t) + \ldots + 125\ N_{125}(t).$$

If N_1 jumps there has been just one default (idiosyncratic default), if N_{125} jumps there are 125 defaults and the whole pool defaults one shot (systemic risk), otherwise for other N_i's we have intermediate situations.

Some N's may have zero intensity, which is equivalent to say that the corresponding multiplier is set to zero.

This model explicitly contemplates the possibility of multiple defaults in small time intervals, contrary for example to Schönbucher (2005) and Errais, Giesecke and Goldberg (2006).

Top Down Approach: The GPL Model

A drawback of the model is that the number of defaults in time may increase without limit. If our pool contains M names, we may then consider

$$C_t := \min(Z_t, M) = Z_t 1_{\{Z_t < M\}} + M 1_{\{L_t \geq M\}}$$

as actual number of defaults. If Z has a known distribution, the distribution of C_t can be easily derived as a byproduct:

$$\mathbb{Q}(C_t \leq x) = 1_{\{x<M\}} \mathbb{Q}(Z_t < x) + 1_{\{x \geq M\}}$$

The GPL Model

The law of Z_t (and thus of C_t) is directly known through its characteristic function. We have easily, thanks to independence of N_i's,

$$\varphi_{Z_t}(u) = \prod_{j=1}^{n} \mathbb{E}_0[\,\exp(-iu\alpha_j N_j(t))\,] = \prod_{j=1}^{n} \varphi_{N_j(t)}(\alpha_j u),$$

where $\varphi_{N_j(t)}$ is the characteristic function of the Poisson process N_j. Since we know the Poisson char function, we obtain easily

$$\varphi_{Z_t}(u) = \prod_{j=1}^{n} \exp\left[\left(\int_0^t \lambda_j(v)dv\right)\left(e^{i\alpha_j u} - 1\right)\right] = \exp\left[\sum_{j=1}^{n} \Lambda_j(t)\left(e^{i\alpha_j u} - 1\right)\right]$$

The density of Z_t can be obtained as the inverse Fourier transform

Recovery assumptions

In order to ensure an arbitrage-free dynamics, the portfolio cumulated loss ($\textsc{Loss}_t$) and the re-scaled number of defaults ($\bar{C}_t$) must be non-decreasing processes taking values in the $[0, 1]$ interval, the former with increments always smaller or equal than the increment of the latter.

$$d\textsc{Loss}_t \leq d\bar{C}_t.$$

When we write expression like dX_t for jump processes X (that we assume to be right continuous with left limit) we mean $X_t - X_{t-}$, where X_{t-} is the left limit of X at t.

Recovery assumptions

The portfolio cumulated loss and the number of defaults cannot be independetly modelled, since they are coupled by the forward realization of the recovery rate ($\textsc{Rec}_t$) at default

$$d\textsc{Loss}_t = [1 - \textsc{Rec}_t]d\bar{C}_t,$$

where we define $\textsc{Rec}_t$ as the "recovery rate at default", assuming it is a $\mathcal{G}_t$-adapted and left continuous (and hence predictable) process taking values in the interval $[0, 1]$.

Calibration

The GPL model is calibrated to the market quotes observed on March 1 and 6, 2006. Deterministic discount rates are listed in Brigo, Pallavicini and Torresetti (2006). Tranche data and DJi-TRAXX fixings, along with bid-ask spreads, are

	Att-Det	**March, 1 2006**		**March, 6 2006**		
		5y	7y	3y	5y	7y
Index		35(1)	48(1)	20(1)	35(1)	48(1)
Tranche	0-3	2600(50)	4788(50)	500(20)	2655(25)	4825(25)
	3-6	71.00(2.00)	210.00(5.00)	7.50(2.50)	67.50(1.00)	225.50(2.50)
	6-9	22.00(2.00)	49.00(2.00)	1.25(0.75)	22.00(1.00)	51.00(1.00)
	9-12	10.00(2.00)	29.00(2.00)	0.50(0.25)	10.50(1.00)	28.50(1.00)
	12-22	4.25(1.00)	11.00(1.00)	0.15(0.05)	4.50(0.50)	10.25(0.50)
Tranchlet	0-1	6100(200)	7400(300)			
	1-2	1085(70)	5025(300)			
	2-3	393(45)	850(60)			

Calibration

The objective function f to be minimized in the calibration is the squared sum of the errors shown by the model to recover the tranche and index market quotes weighted by market bid-ask spreads:

$$f(\alpha, \Lambda) = \sum_i \epsilon_i^2, \quad \epsilon_i = \frac{x_i(\alpha, \Lambda) - x_i^{\mathrm{Mid}}}{x_i^{\mathrm{Bid}} - x_i^{\mathrm{Ask}}}$$

where the x_i, with i running over the market quote set, are the index values R_0 for DJi-TRAXX index quotes, and either the index periodic premium rates $R_0^{A,B}$ or the upfront premium rates $U_0^{A,B}$ for the DJi-TRAXX tranche quotes.

Calibration: All standard tranches up to seven years

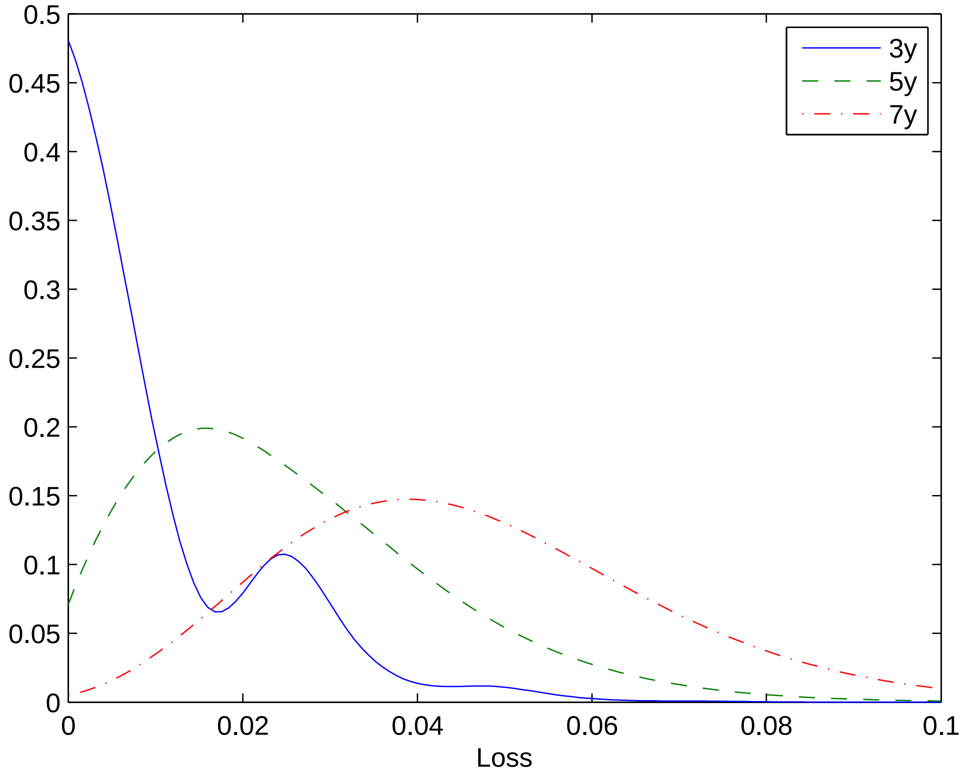
As a first calibration example we consider standard DJi-TRAXX tranches up to a maturity of 7y with constant recovery rate of 40%.

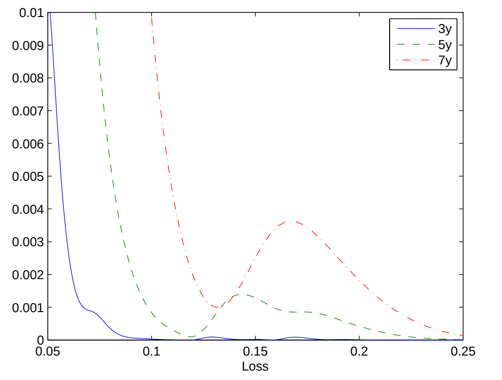
The calibration procedure selects five Poisson processes. The 18 market quotes used by the calibration procedure are almost perfectly recovered. In particular all instruments are calibrated within the bid-ask spread (we show the ratio calibration error / bid ask spread).

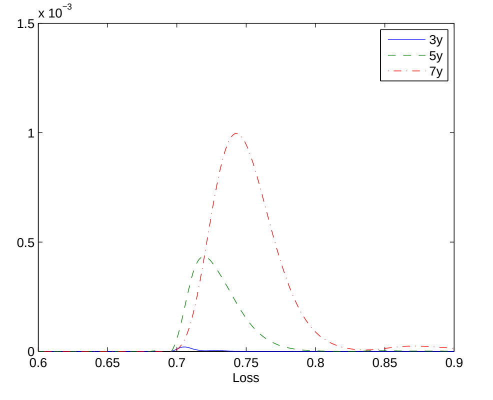
	Att-Det	**Maturities**		
		3y	5y	7y
Index		-0.4	-0.2	-0.9
Tranche	0-3	0.1	0.0	-0.7
	3-6	0.0	0.0	0.7
	6-9	0.0	0.0	-0.2
	9-12	0.0	0.0	0.0
	12-22	0.0	0.0	0.2

$\boldsymbol{\alpha}$	$\boldsymbol{\Lambda(T)}$		
	3y	5y	7y
1	0.535	2.366	4.930
3	0.197	0.266	0.267
16	0.000	0.007	0.024
21	0.000	0.003	0.003
88	0.000	0.002	0.007

Calibration: All standard tranches up to seven years







Calibration: All standard tranches up to seven years

One possible comparison of our implied loss distribution according to the GPL model is with the implied loss distribution according to the (static) "implied copula" approach seen earlier.

If we compare the implied loss distribution resulting from the calibration of the five year index and tranche quotes with the implied copula approach as reformulated in Torresetti et al. (2006) (Implied Default Rate approach, seen earlier in Feature article 1), we find a qualitative pattern similar to the pattern we have above.

Calibration: All standard tranches up to seven years

Notice in particular the large portion of mass concentrated near the origin, the subsequent modes when moving along the loss distribution for increasing values, and the bumps in the far tail.

These features are common to both approaches. In our GPL models the bumps in the tails of the loss distributions, which seem to be necessary in order to be able to recover the market quotes, are obtained thanks to the multiple jumps components contributing to the loss distribution. In particular, the components with higher α's are giving rise to the little bumps in the far tail of the loss distribution and help with senior tranches.

Calibration: More recent results (2006). i-Traxx

	Att-Det	Maturities			
		3y	5y	7y	10y
Index		18(0.5)	30(0.5)	40(0.5)	51(0.5)
Tranche	0-3	350(150)	1975(25)	3712(25)	4975(25)
	3-6	5.50(4.0)	75.00(1.0)	189.00(2.0)	474.00(4.0)
	6-9	2.25(3.0)	22.25(1.0)	54.25(1.5)	125.50(3.0)
	9-12		10.50(1.0)	26.75(1.5)	56.50(2.0)
	12-22		4.00(0.5)	9.00(1.0)	19.50(1.0)
	22-100		1.50(0.5)	2.85(0.5)	3.95(0.5)

Table 26: DJi-TRAXX index and tranche quotes in basis points on October 2, 2006, along with the bid-ask spreads. Index and tranches are quoted through the periodic premium, whereas the equity tranche is quoted as an upfront premium.

α_j	$\Lambda_j(T)$			
	3y	5y	7y	10y
1	0.778	1.318	3.320	4.261
3	0.128	0.536	0.581	1.566
15	0.000	0.004	0.024	0.024
19	0.000	0.007	0.011	0.028
32	0.000	0.000	0.000	0.007
79	0.000	0.000	0.003	0.003
120	0.000	0.002	0.003	0.008

Table 27: DJi-TRAXX pool. Cumulated intensities, integrated up to tranche maturities, of the basic GPL model. Each row j corresponds to a different Poisson component with jump amplitude α_j. The recovery rate is 40%. All tranches are calibrated within one bid-ask

Calibration: More recent results (2006). CDX

	Att-Det	**Maturities**			
		3y	5y	7y	10y
Index		24(0.5)	40(0.5)	49(0.5)	61(0.5)
Tranche	0-3	975(200)	3050(100)	4563(200)	5500(100)
	3-7	7.90(1.6)	102.00(6.1)	240.00(48.0)	535.00(21.4)
	7-10	1.20(0.2)	22.50(1.4)	53.00(10.6)	123.00(7.4)
	10-15	0.50(0.1)	10.25(0.6)	23.00(4.6)	59.00(3.5)
	15-30	0.20(0.1)	5.00(0.3)	7.20(1.4)	15.50(0.9)

Table 28: CDX index and tranche quotes in basis points on October 2, 2006, along with the bid-ask spreads. Index and tranches are quoted through the periodic premium, whereas the equity tranche is quoted as an upfront premium.

α_j	$\Lambda_j(T)$			
	3y	5y	7y	10y
1	1.132	3.043	4.247	7.166
2	0.189	0.189	0.812	1.625
6	0.011	0.091	0.091	0.091
18	0.000	0.006	0.028	0.028
23	0.000	0.004	0.005	0.032
32	0.000	0.000	0.000	0.009
124	0.000	0.003	0.005	0.010

Table 29: CDX pool. Left side: cumulated intensities, integrated up to tranche maturities, of the basic GPL model. Each row j corresponds to a different Poisson component with jump amplitude α_j. The recovery rate is 40%. All tranches are calibrated within one bid-ask

GPL in-Crisis: Fix the α's

Fix the independent Poisson jump amplitudes to the levels just above each tranche detachment, when considering a 40% recovery.

For the DJi-Traxx, for example, this would be realized through jump amplitudes $a_i = \alpha_i/125$ where

$$\alpha_5 = \text{roundup}\left(\frac{125 \cdot 0.03}{(1 - \text{Rec})}\right),\ \alpha_6 = \text{roundup}\left(\frac{125 \cdot 0.06}{(1 - \text{Rec})}\right),\ \alpha_7 = \text{roundup}\left(\frac{125 \cdot 0.09}{(1 - \text{Rec})}\right),$$

$$\alpha_8 = \text{roundup}\left(\frac{125 \cdot 0.12}{(1 - \text{Rec})}\right),\ \alpha_9 = \text{roundup}\left(\frac{125 \cdot 0.22}{(1 - \text{Rec})}\right),$$

$$\alpha_{10} = 125$$

and, in order to have more granularity, we add the sizes 1,2,3,4:

$$\alpha_1 = 1,\ \alpha_2 = 2,\ \alpha_3 = 3,\ \alpha_4 = 4.$$

GPL in-Crisis

In total we have $n = 10$ jump amplitudes. We then modify slightly the obtained sizes in order to account also for CDX attachments that are slightly different.

$$\alpha_i \equiv 125 \cdot a_i \in \{1, 2, 3, 4, 7, 13, 19, 25, 46, 125\}$$

Given these amplitudes, we obtain the default counting process fraction as

$$\bar{C}_t = 1_{\{N_n(t)=0\}}\bar{c}_t + 1_{\{N_n(t)>0\}} \ , \quad \bar{c}_t := \min\left(\sum_{i=1}^{n-1} a_i N_i(t), 1\right) \ .$$

GPL in-Crisis

Now let the random time $\hat{\tau}$ be defined as the first time where $\sum_{i=1}^{n} a_i N_i(t)$ reaches or exceeds the relative pool size of 1.

$$\hat{\tau} = \inf\{t : \sum_{i=1}^{n} a_i N_i(t) \geq 1\} \ .$$

We define the loss fraction as

$$\begin{aligned} \bar{L}_t := 1_{\{\hat{\tau}>t\}}(1 - \text{Rec})1_{\{N_n(t)=0\}}\bar{c}_t + 1_{\{\hat{\tau}\leq t\}} \left[(1 - \text{Rec})1_{\{N_n(\hat{\tau})=0\}} + 1_{\{N_n(\hat{\tau})>0\}} (1 - \text{Rec}\,\bar{c}_{\hat{\tau}})\right] \\ = 1_{\{\hat{\tau}>t\}}(1 - \text{Rec})1_{\{N_n(t)=0\}}\bar{c}_t + 1_{\{\hat{\tau}\leq t\}} (1 - \text{Rec}\,\bar{c}_{\hat{\tau}}) \quad . \end{aligned} \tag{18}$$

GPL in-Crisis

- Whenever the armageddon component N_n jumps the first time, the default counting process $\bar{C}_t$ jumps to the entire pool size and no more defaults are possible.
- Whenever armageddon component N_n jumps the first time we will assume that the recovery rate associated to the remaining names defaulting in that instant will be zero.
- The pool loss however will not always jump to 1 as there is the possibility that one or more names already defaulted before N_n jumped, with recovery $\textsc{Rec}$.

GPL in-Crisis

This way whenever N_n jumps at a time when the pool has not been wiped out yet, we can rest assured that the pool loss will be above $1 - \text{Rec}$.

We do this because the market in 2008 has been quoting CDOs with prices assuming that the super-senior tranche would be impacted to a level impossible to reach with fixed recoveries at 40%.

For example there was a market for the DJi-Traxx 5 year $60 - 100\%$ tranche on 25-March-2008 quoting a running spread of 24bps bid.

GPL in-Crisis

We know how to calculate the distribution of both $\bar{C}_t$ and $\bar{L}_t$ given that:

- the distribution of $\bar{c}_t = \min\left(\sum_{i=1}^{n-1} a_i N_i(t), 1\right)$ is obtained running a 'reduced' GPL, i.e. a GPL where the jump N_n is excluded.
- N_n is independent from all other processes N_i so that we can factor expectations when calculating the risk neutral discounted payoffs for tranches and indices.

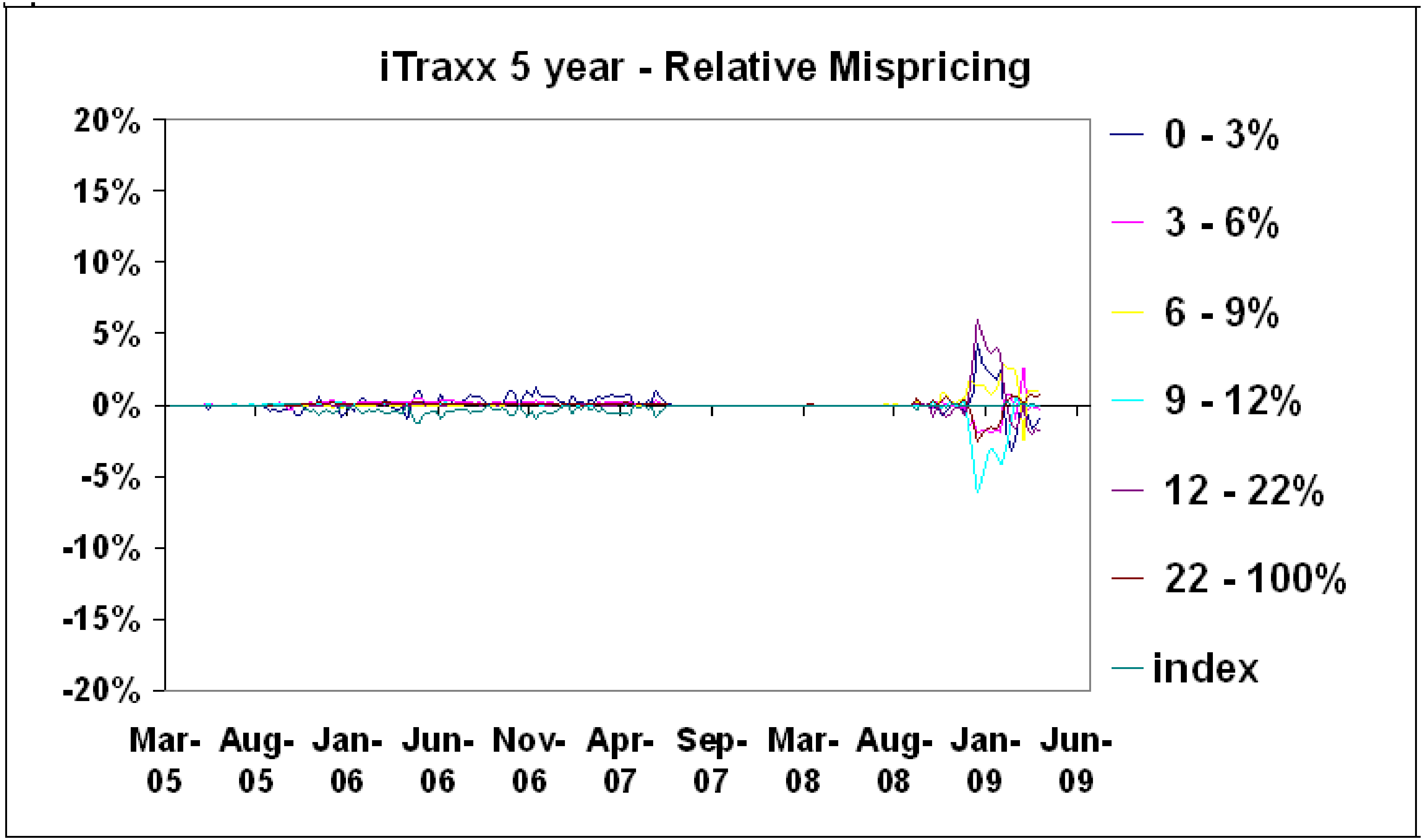
GPL in-Crisis

Concerning recovery issues, in the dynamic loss model recovery can be made a function of the default rate $\bar{C}$ or other solutions are possible, see Brigo Pallavicini and Torresetti (2007) for more discussion.

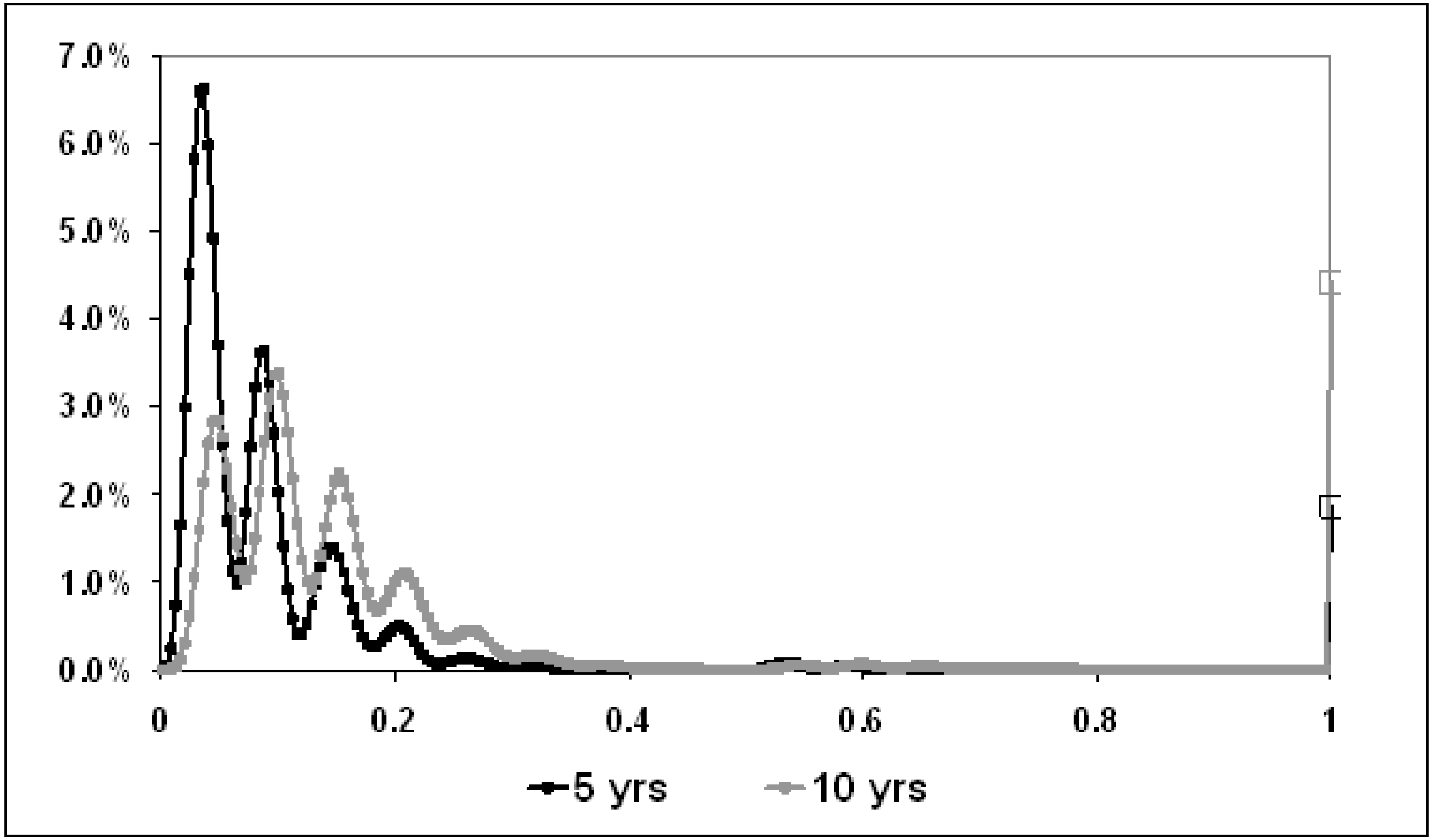
Here we use the above simple methodology to allow losses of the pool to penetrate beyond $(1 - \text{Rec})$ and thus affect severely even the most senior tranches, in line with market quotations.

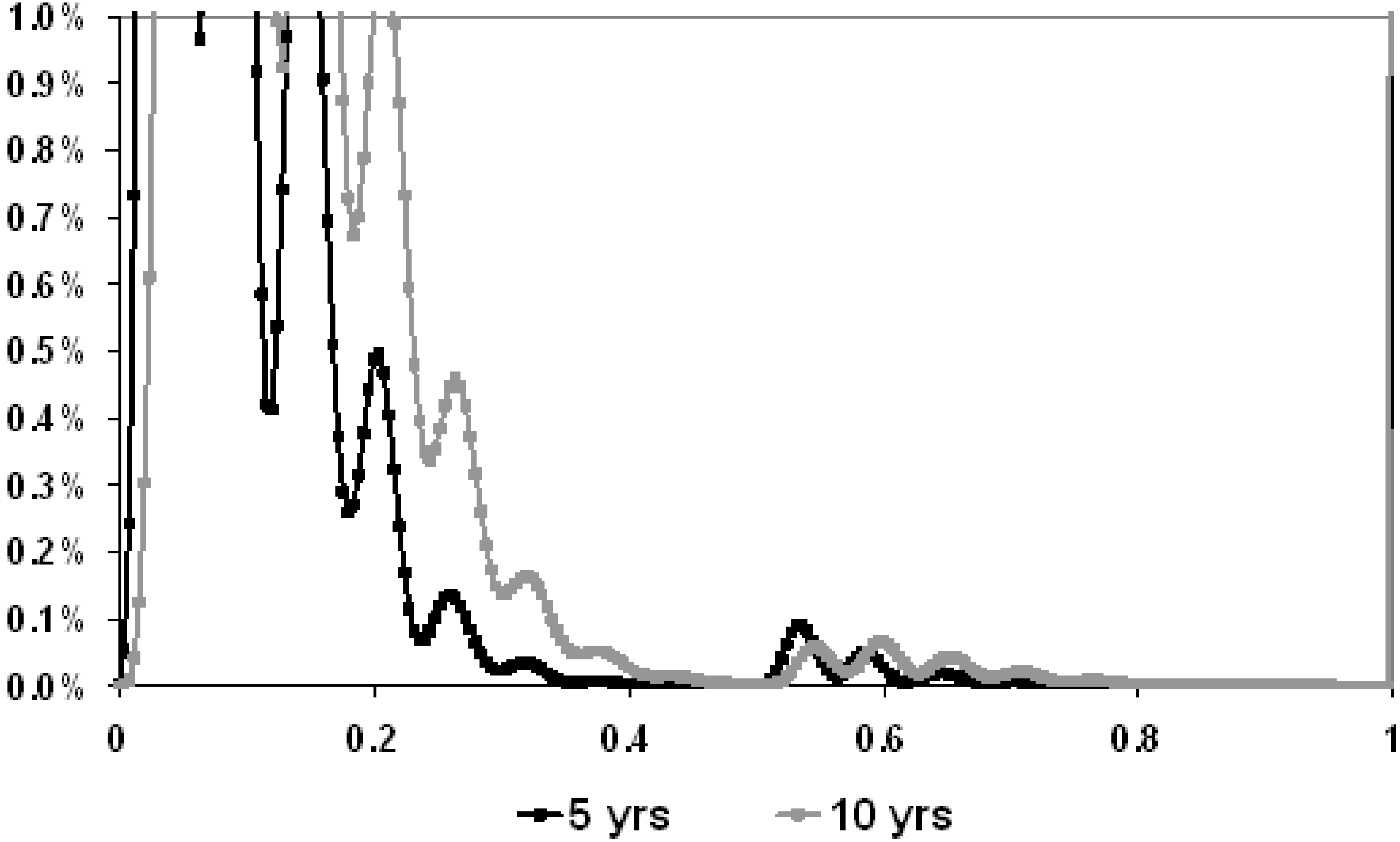
GPL in-Crisis

History of calibration in-crisis with a different parametrization (α's fixed a priori):


iTraxx 5 year - Relative Mispricing
20%
15%
10%
5%
0%
-5%
-10%
-15%
-20%
Mar-05
Aug-05
Jan-06
Jun-06
Nov-06
Apr-07
Sep-07
Mar-08
Aug-08
Jan-09
Jun-09
0 - 3%
3 - 6%
6 - 9%
9 - 12%
12 - 22%
22 - 100%
index

Default Rate Distribution (CDX_3-oct-08)




1.0%
0.9%
0.8%
0.7%
0.6%
0.5%
0.4%
0.3%
0.2%
0.1%
0.0%
0
0.2
0.4
0.6
0.8
1
5 yrs
10 yrs