Expansion techniques in interest rate modeling : the short rates

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Outline

- Review of the short rates models
- Generalized short rate models with gaussian underlying
- Swaption price approximation
  - Regular perturbation technique for zero-coupon bonds
  - Singular perturbation technique for AD price
  - Numerical integration
- Generalized multi-factor BK model
  (definition, numerical results)
- Bounded rate model
  (definition, numerical results)
- Conclusion

Short rate models

- *HW and its multi-factor generalizations*:
  Vasicek (1977), Hull and White (1990-1994)

- *BK and generalized BK*:
  Black and Karasinski (1991), Tourrucôo, Hagan and Schleiniger (2007)

- *Quadratic short rate model*:
  Piterbarg (2008)

- *CIR model*:
  Cox, Ingersoll and Ross (1985)

Properties

Simplicity and transparency $\rightarrow$ important property for the post-crisis market.

- Analytical tractability
  - Transition probabilities
  - Zero-bond
  - European swaptions
- Rates correlations
- Skew control
- Rates bounds (for example, positive rates for the BK model)
  This property can be important for scenario generation

Gaussian underling models and its properties

Short rate

$$r(t) = f(t, x)$$

is a function of underlying vector mean-reverting Gaussian process x
$\rightarrow$ transition probabilities are available (efficient lattice)

- *HW and its multi-factor generalizations*
  Fully analytically tractable, rates decorrelation for MF case, no skew control, no bounds
- *BK and generalized BK*
  Limited analytical tractability, rates decorrelation for MF case, skew control for GBK, positive rates for BK
- *Quadratic short rate model*
  Analytical tractability, rates decorrelation for MF case, skew control, rates bounded from below

Analytical tractability

- Quadratic model

$$r(t) = x^T\, a(t)\, x + b(t)^T x + c(t)$$

Special case for the exact zero bond price via Riccati eq; European swaption approximation

- GBK model

$$r(t) = r_0(1 + \nu\, x)^{\frac{1}{\nu}}$$

Available approximation for zero bonds (Tourrucôo, Hagan and Schleiniger) however limited for 1F case with non-clear yield curve fit. European swaption is not available

Our results

Calculation of *both* zero-bond and European swaptions for the Generalized short rate models with gaussian underlying (GSRG)

$$r(t) = f(t, x)$$

in the multi-dimensional case.

Technical details can be found in Antonov-Spector.

GSRG

Underlying F-factor mean-reverting process

$$dx_i(t) = (\phi_i(t) - a_i(t)\, x_i(t))\, dt + \lambda_i(t) \cdot dW(t) \quad \text{for } i = 1, \cdots, F$$

for the vector volatilities $\lambda_i(t)$, vector uncorrelated Brownian motions $dW(t)$, mean-reversions $a_i(t)$ and drifts $\phi_i(t)$ serving to link the short rate process to a given yield curve[a].

The short rate process as an arbitrary function of the underlyings

$$r(t) = f(t, x) \quad \text{for } x_i(0) = 0$$

[a] An alternative form of the underlying can contain correlated scalar brownian motions $dW_i(t)$ and scalar volatilities $\gamma_i(t)$, i.e. $dx_i(t) = (\phi_i(t) - a_i(t)\, x_i(t))\, dt + \gamma_i(t)\, dW_i(t)$

Define the numeraire process, a savings account,

$$N_t = e^{\int_0^t dsr(s)}$$

$\rightarrow$ zero-coupon bond with maturity T

$$P(t,T) = N_t\, E\left[\frac{1}{N_T} \mid F_t\right] = E\left[e^{-\int_t^T ds\, r(s)} \mid F_t\right]$$

where $E[\cdots]$ is the expectation operator in the risk-neutral measure.

A *discount factor* $D(T) \equiv P(0,T)$ is a zero bond maturing at T as seen from the origin $\rightarrow$ today's *forward rate*

$$D(T) = e^{-\int_0^T ds\, R(s)}$$

Without loss of generality represent[a] the short rate as "moving" around the today's forward rate

$$f(t, 0) = R(t)$$

The discount factor is considered as the *model input*
$\rightarrow$ the drift parameters $\phi_i(t)$ should be chosen to fit it

$$D(T) = E\left[e^{-\int_0^T ds\, f(s, x(s))}\right]$$

Remark. The drifts will have the second order in volatilities

$$\phi = O(\gamma^2)$$

[a] One can use a proper shift $x_i(t) \rightarrow x_i(t) + \int_0^t ds\, \mu_i(s)$ for some functions $\mu_i(s)$.

It is convenient to remove the mean-reverting term from the underlying x's

$$y_i(t) = x_i(t)\,A_i(t) \quad \text{for} \quad A_i(t) = e^{\int_0^t ds\, a_i(s)}$$

Normal processes y's satisfy an SDE

$$dy_i(t) = \theta_i(t)\,dt + \sigma_i(t)\cdot dW(t) \quad \text{with } y_i(0) = 0$$

where

$$\theta_i(t) = \phi_i(t)\,A_i(t) \quad \text{and} \quad \sigma_i(t) = \lambda_i(t)\,A_i(t).$$

The short rate in terms of the Gaussian y's

$$r(t) = r(t, y) \equiv f\left(t, \left\{A_i(t)^{-1}\,y_i(t)\right\}\right)$$

Swaption price

To calculate a generalized option price with pay-off at time T

$$\left(\sum_n A_n\, P(T, T_n)\right)^+$$

one should be able to evaluate the following expectation

$$V = E\left[\frac{\left(\sum_n A_n\, P(T, T_n)\right)^+}{N_T}\right]$$

Remark. After some experiments with the measure choices we have concluded that the most efficient way is to proceed in the *risk-neutra*l measure.

Arrow-Debreu price

$$
\begin{aligned}
q(t,u) \quad &\equiv \quad E\left[\delta(y(t)-u)\, e^{-\int_0^t ds\, r(s)}\right] \\
&= \quad E\left[e^{-\int_0^t ds\, r(s)} \,|\, y(t)=u\right] E\left[\delta(y(t)-u)\right]
\end{aligned}
$$

where u is a vector and the multi-dimensional delta-function is defined by product,

$$
\delta(y(t)-u) = \prod_{i=1}^{F} \delta(y_i(t)-u_i)
$$

$\rightarrow$ present the option price as a multi-dimensional integral

$$
V = \int du \left(\sum_n A_n\, P(T, T_n; u) \right)^+ q(T,u)
$$

Goal

Approximate

- the zero-bond price $P(t, T; u)$
- the Arrow-Debreu price $q(t, u)$

as function of u.

Remark. The integration should be performed numerically. Given low dimensionality of the problem (two in most of cases) the proposed solution satisfies speed requirements for the calibration.

Zero bond PDE

$$\left(\partial_t - r(t,u) + \sum_i \theta_i(t)\,\partial_i + \frac{1}{2}\sum_{i,j}\sigma_i(t)\cdot\sigma_j(t)\;\partial_i\,\partial_j\right)\,P(t,T;u) = 0$$

with the final condition $P(T,T;u) = 1$ and $\partial_i \equiv \partial_{u_i}$.

Remark. The PDE can be easily derived by the Ito's formula requiring zero drift of the martingale $e^{-\int_0^t ds\, r(s)}\,P(t,T;y(t))$.

AD price PDE

Conjugate PDE

$$\partial_t q(t,u) = \left(-r(t,u) - \sum_i \theta_i(t)\,\partial_i + \frac{1}{2}\sum_{i,j} \sigma_i(t)\cdot\sigma_j(t)\;\partial_i\,\partial_j \right)\; q(t,u)$$

with the initial condition $q(0,u) = \prod_i \delta(u_i - y_i(0))$.

Remark. The integral $P(0,T) = \int du\, P(t,T;u)\, q(t,u)$ is independent of time t. After its differentiation over time we obtain the desired PDE.

Perturbation technique

Apply the perturbation technique to the PDE's considering

$$\theta \to \varepsilon\theta \quad \text{and} \quad \sigma^2 \to \varepsilon\sigma^2$$

- regular perturbations for the zero bond
- singular perturbations for the AD price

Small parameter ε is a *variance* scale

Zero bond

Evolution operator

$$L(t,u) = -r(t,u) + \varepsilon \sum_i \theta_i(t)\,\partial_i + \varepsilon \frac{1}{2} \sum_{i,j} C_{ij}(t)\ \partial_i\,\partial_j$$

with instantaneous covariance matrix

$$C_{ij}(t) = \sigma_i(t) \cdot \sigma_j(t)$$

Unperturbed operator $L_0(t)$ and perturbation $L_1(t)$,

$$\begin{aligned} L_0(t) &= -r(t,u) \\ L_1(t) &= \varepsilon \sum_i \theta_i(t)\,\partial_i + \varepsilon \frac{1}{2} \sum_{i,j} C_{ij}(t)\ \partial_i\,\partial_j \end{aligned}$$

The zero bond leading term

The solution of the *unperturbed* equation

$$(\partial_t + L_0(t))\, P_0(t,T,u) = 0$$

for the unit final value $P_0(T,T,u) = 1$ is very simple

$$P_0(t,T,u) = \exp\left(-\int_t^T d\tau\, r(\tau,u)\right)$$

Rewrite in Dyson terms → unperturbed evolution operator

$$U_0(t_1,t_2) =: e^{\int_{t_1}^{t_2} d\tau L_0(\tau)} := e^{-\int_{t_1}^{t_2} d\tau r(\tau,u)}$$

The bond price $P_0(t,T,u)$ is a result of action of the evolution operator on 1,

$$P_0(t,T,u) = U_0(t,T)\, 1$$

The first correction for the zero bond

$$P_1(t,T,u) = \int_t^T ds \hat{L}_1(t,s) \;\; P_0(t,T,u)$$

where $\hat{L}_1(t,s)$ is the *dressed* perturbation operator,

$$\hat{L}_1(t,s) = U_0(t,s)\, L_1(s)\, U_0^{-1}(t,s)$$

Rewrite the first correction in more convenient way,

$$\begin{aligned} P_1(t,T,u) &= \int_t^T ds\, U_0(t,s)\, L_1(s)\, U_0(s,T)\, 1 \\ &= \int_t^T ds\, P_0(t,s,u)\, L_1(s)\, P_0(s,T,u) \end{aligned}$$

Solution →

$$
\begin{aligned}
P_1(t,T,u) &= \varepsilon\, P_0(t,T,u) \\
&\times \int_t^T ds \left\{ -\sum_i \theta_i(s)\, g_i(s,T,u) \right. \\
&+ \left. \frac{1}{2} \sum_{i,j} C_{ij}(s)(g_i(s,T,u)\, g_j(s,T,u) - g_{ij}(s,T,u)) \right\}
\end{aligned}
$$

with

$$
g_i(s,T,u) = \int_s^T d\tau\, \partial_i\, r(\tau,u) \quad \text{and} \quad g_{ij}(s,T,u) = \int_s^T d\tau\, \partial_i\, \partial_j\, r(\tau,u)
$$

Remark 1. Note that Tourrucoo-Hagan-Schleiniger have presented the zero bond in *exponential* form

$$P(t,T;u) = P_0(t,T,u)\,\exp\left\{\varepsilon\,\Phi_1(t,T;u) + \varepsilon^2\,\Phi_2(t,T;u) + O(\varepsilon^3)\right\}$$

which can sometimes deliver explosive values esp. for large maturities and vols.

Remark 2. The second order correction and further details can be found in Antonov-Spector.

Yield curve match

Fix the drifts to reproduce the yield cure or discount factor $D(T)$.

Equivalently, we can calculate the y's averages

$$E[y_i(t)] = m_i(t) = \int_0^t ds\,\theta_s$$

requiring the fit

$$D(T) = P_0(0,T,0) + P_1(0,T,0)$$

to the leading order in small ε.

Notice that

$$P_0(0,T,0) = \exp\left(-\int_0^T d\tau\, r(\tau,0)\right) = \exp\left(-\int_0^T d\tau\, R_\tau\right) = P(0,T)$$

$\rightarrow$ in order to fit the yield curve we should choose the averages $m_i(t)$ s.t. the second order correction equals to zero, i.e.

$$P_1(0,T,0) = 0$$

or

$$\begin{aligned} & \int_0^T ds \sum_i \theta_i(s)\, g_i(s,T,0) \\ = \quad & \frac{1}{2}\int_0^T ds \sum_{i,j} C_{ij}(s)(g_i(s,T,0)\, g_j(s,T,0) - g_{ij}(s,T,0)) \end{aligned}$$

for all maturities T

Differentiating over T, we obtain

$$\begin{aligned}
&\sum_i \partial_i r(T,0)\ m_i(T) \\
= \quad &\sum_{i,j} \partial_i r(T,0) \int_0^T d\tau \partial_i r(\tau,0)\, V_{ij}(\tau) - \frac{1}{2} S_T \sum_{i,j} \partial_i \partial_j r(T,0)\, V_{ij}(T)
\end{aligned}$$

where $V_{ij}(T)$ is the y's covariance

$$E[(y_i(t) - m_i(t))\,(y_j(t) - m_j(t))] = V_{ij}(t) = \int_0^t ds\, C_{ij}(s)$$

Remark 1. Either of the solutions for individual averages $m_i(T)$ is suitable for the approximation.

Remark 2. Note that Tourrucoo-Hagan-Schleiniger have matched the yield curve by a rough multiplicative adjustment which does not fix the underlying drifts.

AD price

Singular perturbation technique for the forward PDE

$$\partial_t\, q(t,u) = \left(-r(t,u) - \varepsilon \sum_i \theta_i(t)\, \partial_i + \frac{1}{2}\varepsilon \sum_{i,j} C_{ij}(t)\ \partial_i\, \partial_j \right) q(t,u)$$

with the initial condition $q(0,u) = \prod_i \delta(u_i - y_i(0))$.

Applying Dyson technique will require careful calculation of *all* terms in ε due to *singularity* of the initial value.

Instead $\rightarrow$ look for the solution as product of the leading singular part

$$\exp\left(-\frac{1}{2}\varepsilon^{-1}\, u^T\, V(t) u \right)$$

and regular part (in ε) following the *Heat-Kernel expansion technique.*

The Heat-Kernel expansion

Gaussian density (in vector/matrix notations)

$$q_0(t,u) = \frac{\exp\left(-\frac{1}{2}\varepsilon^{-1}\left(u-\varepsilon\, m(t)\right)^T V^{-1}(t)(u-\varepsilon\, m(t))\right)}{(2\pi\,\varepsilon)^{\frac{n}{2}}\left|V(t)\right|^{\frac{1}{2}}}$$

satisfies the PDE

$$\partial_t\, q_0(t,u) = -\varepsilon \sum_i \theta_i(t)\,\partial_i\, q_0(t,u) + \frac{1}{2}\varepsilon \sum_{i,j} C_{ij}(t)\ \partial_i\,\partial_j\, q_0(t,u)$$

with the initial condition $q_0(0,u) = \prod_i \delta(u_i - y_i(0))$. Here $m(t)$ is the $y(t)$ centers and $V(t)$ is the covariance.

We look for the solution of the initial forward PDE as

$$q(t,u) = q_0(t,u)\,\Omega(t,u)$$

where $\Omega(t,u)$ is a *regular* function of ε with the unit initial condition $\Omega(0,u) = 1$.

After substitution $\rightarrow$

$$\begin{aligned}\partial_t\,\Omega(t,u) &= -r(t,u)\,\Omega(t,u) - \varepsilon\sum_i \theta_i(t)\,\partial_i\,\Omega(t,u) + \frac{1}{2}\varepsilon\sum_{i,j} C_{ij}(t)\;\partial_i\,\partial_j\,\Omega(t,u) \\ &+ \varepsilon\sum_{i,j} C_{ij}(t)\,\partial_i \ln q_0(t,u)\,\partial_i\,\Omega(t,u)\end{aligned}$$

The solution Ω as a *regular* expansion in ε

$$\Omega(t,u) = b_0(t,u) + \varepsilon\, b_1(t,u) + \cdots$$

with the initial conditions $b_0(t,u) = 1$ and $b_1(t,u) = 0$.

The leading order part will satisfy the unperturbed PDE

$$\partial_t \, b_0(t,u) + u^T \, V^{-1}(t) \, C(t) \, \partial \, b_0(t,u) + r(t,u) \, b_0(t,u) = 0$$

where we have used vector notations for the first derivatives
$\partial = \{\partial_1, \partial_2, \cdots\}$

This is a linear first order PDE which can be solved by the method of characteristics

$$b_0(t,u) = \exp\left(- \int_0^t ds \, r\left(s, V(s) \, V^{-1}(t) \, u\right)\right)$$

It contributes in a *parallel transport* (in the Heat-Kernel terminology).

Expression for the first correction is more complicated

$$\begin{aligned}\frac{b_1(t,u)}{b_0(t,u)} &= \int_0^t d\tau\,\left(\theta^T(\tau) - m^T(\tau)\,V^{-1}(\tau)\,C(\tau)\right)\,V^{-1}(\tau)\,\hat{h}^{(1)}(\tau;t,u)\\ &+ \frac{1}{2}\int_0^t d\tau\ \mathrm{tr}\ V^{-1}(\tau)\,C(\tau)\,V^{-1}(\tau)\\ &\times \left[\hat{h}^{(1)}(\tau;t,u)\otimes\hat{h}^{(1)}(\tau;t,u) - \hat{h}^{(2)}(\tau;t,u)\right]\end{aligned}$$

where

$$\begin{aligned}\hat{h}_n^{(1)}(\tau;t,u) &= \sum_j \int_0^\tau ds\,V_{nk}(s)\,\tilde{\partial}_k r\left(s, V(s)\,V^{-1}(t)\,u\right)\\ \hat{h}_{nl}^{(2)}(\tau;t,u) &= \sum_{km}\int_0^\tau ds\,V_{nk}(s)\,\tilde{\partial}_k\,\tilde{\partial}_m r\left(s, V(s)\,V^{-1}(t)\,u\right)\,V_{ml}(s)\end{aligned}$$

where the derivative $\tilde{\partial}_k$ denotes the short rate derivative over the *space argument*, $\tilde{\partial}_i\, r(s,z(u)) = \frac{\partial r(s,z(u))}{\partial z_i(u)}$.

Final formula for the AD price

$$
\begin{aligned}
q(t,u) \quad &= \quad \frac{\exp\left(-\frac{1}{2}\varepsilon^{-1}\,(u-\varepsilon\, m(t))^T\, V^{-1}(t)(u-\varepsilon\, m(t))\right)}{(2\pi\,\varepsilon)^{\frac{n}{2}}\,|V(t)|^{\frac{1}{2}}} \\
&\times \quad \exp\left(-\int_0^t ds\, r\left(s, V(s)\, V^{-1}(t)\, u\right)\right)\left\{1+\frac{b_1(t,u)}{b_0(t,u)}\right\}
\end{aligned}
$$

GSRG model examples

Symmetric form of the GSHG models

$$r(t) = f\left(t, \sum_i x_i\right)$$

with normalization properties

- Value at zero $\rightarrow f(t,0) = R(t)$
- Derivative at zero $\rightarrow f'(t,0) = 1$
  (underlying x parameters correspond to *normal* (HW) case)

Remark. We consider *monotone* functions $f(t, X)$ in X when the short rate completely determines a state of the system (in 1F case).

We cover two examples

1. Our version of generalized multi-factor BK model with time-dependent shift parameter
2. Bounded rate model (suitable for scenario generation where the rates evolute inside given boundaries)

Generalized multi-factor BK model

Our version of generalized multi-factor BK model having multiple attractive features including

- transparency
- time-dependent swaption skew control
- multiple factors permitting decorrelations of the rates
- availability of the zero bond approximation and swaption price

The short rate

$$r(t) = f(t, x) = R(t) + \frac{e^{S(t)\sum_i x_i} - 1}{S(t)}$$

where $R(t)$ is the today's forward rate and positive function $S(t)$ is a shift parameter controlling the swaptions skew.

Special cases:

- The standard multi-factor BK model

$$S(t) = R^{-1}(t) \quad \rightarrow \quad r(t) = R(t)\, e^{R^{-1}(t)\sum_i x_i}$$

- The standard multi-factor HW model

$$S(t) = 0 \quad \rightarrow \quad r(t) = R(t) + \sum_i x_i$$

Remark. Typical underlying x parameters correspond to *normal* case, i.e. volatilities have 1-1.5% order of magnitude for any shift.

The analytical expressions can be calculated using the general formulas for the zero-coupon bond and the AD-price.

Explicit expressions for the main derivatives

$$g_i(s,T,u) = \int_s^T d\tau\, \partial_i\, r(\tau,u) = \int_s^T d\tau\, A_i(\tau)^{-1}\, \mathcal{E}(\tau,u)$$

and

$$g_{ij}(s,T,u) = \int_s^T d\tau\, \partial_i\, \partial_i\, r(\tau,u) = \int_s^T d\tau\, A_i(\tau)^{-1}\, A_j(\tau)^{-1}\, S_\tau\, \mathcal{E}(\tau,u)$$

where

$$\mathcal{E}(\tau,u) = e^{S_\tau \sum_i A_i(\tau)^{-1}\, u}$$

The conditions for the drifts simplify to

$$\begin{aligned}\sum_i A_i^{-1}(T)\; m_i(T) \quad &= \quad \sum_{i,j} A_i^{-1}(T) \int_0^T d\tau A_j^{-1}(\tau)\, V_{ij}(\tau) \\ &- \quad \frac{1}{2} S_T \sum_{i,j} A_i^{-1}(T)\, A_j^{-1}(T)\, V_{ij}(T)\end{aligned}$$

In this case one can chose a "symmetric" way

$$m_i(T) = \sum_j \int_0^T d\tau\, V_{ij}(\tau)\, A_j^{-1}(\tau) - \frac{1}{2} S_T \sum_j V_{ij}(T)\, A_j^{-1}(T)$$

Bounded rates

Scenario generation by short rate models (insurance industry)
$\rightarrow$ requirement for a *bounded evolution* of the rates.

The current short rate models do not meet this requirement: there exists a probability s.t. rates can exceed a priori given (positive) value.

We need a short rate model

$$r(t) = f(t, x)$$

with a bounded function $f(t, x)$

Bounded short rates model

$$r(t) = f(t,x) = \frac{U(t)e^{\beta(t)\sum_i x_i} + L(t)\alpha(t)}{e^{\beta(t)\sum_i x_i} + \alpha(t)}$$

where
$L(t)$ is the lower (time-dependent) boundary and $U(t)$ is the upper one.

Comfortable scaling properties

$$f(t,0) = R(t) \quad \text{and} \quad f'(t,0) = 1$$

are satisfied if

$$\alpha(t) = \frac{U(t) - R(t)}{R(t) - L(t)} \quad \text{and} \quad \beta(t) = \frac{U(t) - L(t)}{(R(t) - L(t))(U(t) - R(t))}$$

The short rate

- It is a *monotone* function of $X = \sum_i x_i$
- Its inverse has a simple form,

$$X = \ln \left(\frac{U(t) - R(t)}{U(t) - F} \frac{F - L(t)}{R(t) - L(t)} \right) \beta^{-1}(t)$$

  for $F = f(t, X)$.

- It has two desired limits

$$\lim_{X \to -\infty} f(t, X) = L(t) \quad \text{and} \quad \lim_{X \to +\infty} f(t, X) = U(t)$$

A special case of the bounded model

$$U(t) \to +\infty$$

represents the generalized BK model

$$r(t) = R(t) + \frac{e^{S(t)\,\sum_i x_i} - 1}{S(t)}$$

for the skew parameter

$$S(t) = (R(t) - L(t))^{-1}$$

$$\Downarrow$$

The skewness of the implied volatility depends on the bounds. This property holds also in the general bounded model setup.

The short rate SDE

$$dr(t) = \cdots\, dt + \frac{(r(t) - L(t))(U(t) - r(t))}{(R(t) - L(t))(U(t) - R(t))} \sum_i dx_i(t)$$

The diffusion coefficient, or local volatility,

$$D(t, r) \sim \frac{(r(t) - L(t))(U(t) - r(t))}{(R(t) - L(t))(U(t) - R(t))}$$

can give an approximate form of the implied volatility smile as function of a strike K,

$$\sigma_I(K) = \frac{D\left(t, \frac{R(t)+K}{2}\right)}{\frac{R(t)+K}{2}}$$

Properties

- The implied volatility form depends on the boundaries
- Fixed boundaries $\Leftrightarrow$ absence of the skew control (the model can be calibrated only to one strike (say ATM))
- Fixing one boundary leaves the skew control (for example, with a zero lower boundary there is a skew freedom related with the upper boundary)

Numerical example: generalized BK

Consider 2D case of the generalized BK model

$$r(t) = R(t) + \frac{e^{S(t)\,(x_1+x_2)} - 1}{S(t)}$$

with the underlying processes

$$dx_i(t) = (\phi_i(t) - a_i(t)\,x_i(t))\,dt + \gamma_i(t)\,dW_i(t) \quad \text{for} \quad i = 1, 2$$

starting from zero, $x_i(0) = 0$, with a certain correlation between Brownian motions $\rho(t) = E[dW_1(t)\,dW_2(t)]$.

Use the following standard parameters

Rate	$R(t)$	5.00%
Correlation	$\rho(t)$	-90.00%
Shift	$S(t)$	10
Volatility 1	$\gamma_1(t)$	1.00%
Volatility 2	$\gamma_2(t)$	1.75%
Reversion 1	$a_1(t)$	50.00%
Reversion 2	$a_2(t)$	5.00%

Remark. The shift parameter corresponds to a skew situated between normal and log-normal ones:
– the normal model $\rightarrow S(t) = 0$
– the log-normal model $\rightarrow S(t) = R^{-1}(t) = 20$

We have tested European swaptions with:

- Annual period
- 10Y length
- Maturities 5Y, 10Y and 20Y
- Wide strikes range

The results for Black implied volatilities are presented in the table below. The ATM swaptions are marked in bold.

We compare:

- Approximate analytical (the first order in ε) value (**Approx**)
- Exact value obtained by the Least-Square MC method with large number of paths (**Exact**)

The approximation is performed in the *first* order in variances

$$P(t,T;u) = P_0(t,T;u) + P_1(t,T;u)$$

The 2D integration

$$V = \int du \left(\sum_n A_n \, P(T,T_n;u) \right)^+ q(T,u)$$

is taken numerically, for example, using adaptive grid.

Remark. According to our experiments the *second* order approximation (precision $O(\varepsilon^2)$) does not deliver a reasonable quality increase but can substantially slow down the procedure.

5Y10Y swaption results

Strike (%)	Implied vol (%)		Error (bps)
	Approx	Exact	
3.28	25.77	25.71	5
3.67	25.04	24.99	6
4.10	24.34	24.31	3
4.59	23.69	23.67	2
5.13	**23.09**	**23.09**	**0**
5.73	22.53	22.54	-1
6.41	22.01	22.01	0
7.17	21.52	21.50	2
8.02	21.10	21.06	4

10Y10Y swaption results

Strike (%)	**Implied vol (%)**		**Error (bps)**
	Approx	**Exact**	
2.72	24.39	24.26	13
3.19	23.36	23.27	9
3.74	22.42	22.34	8
4.38	21.54	21.47	7
5.13	**20.75**	**20.67**	**8**
6.01	20.03	19.96	7
7.03	19.39	19.32	7
8.24	18.82	18.75	6
9.65	18.30	18.25	5

20Y10Y swaption results

Strike (%)	Implied vol (%)		Error (bps)
	Approx	Exact	
2.10	21.59	21.24	35
2.62	20.20	19.89	31
3.28	18.94	18.68	26
4.10	17.87	17.62	26
5.13	**16.91**	**16.68**	**23**
6.41	16.05	15.86	19
8.02	15.35	15.15	20
10.03	14.72	14.57	15
12.54	14.19	14.09	11

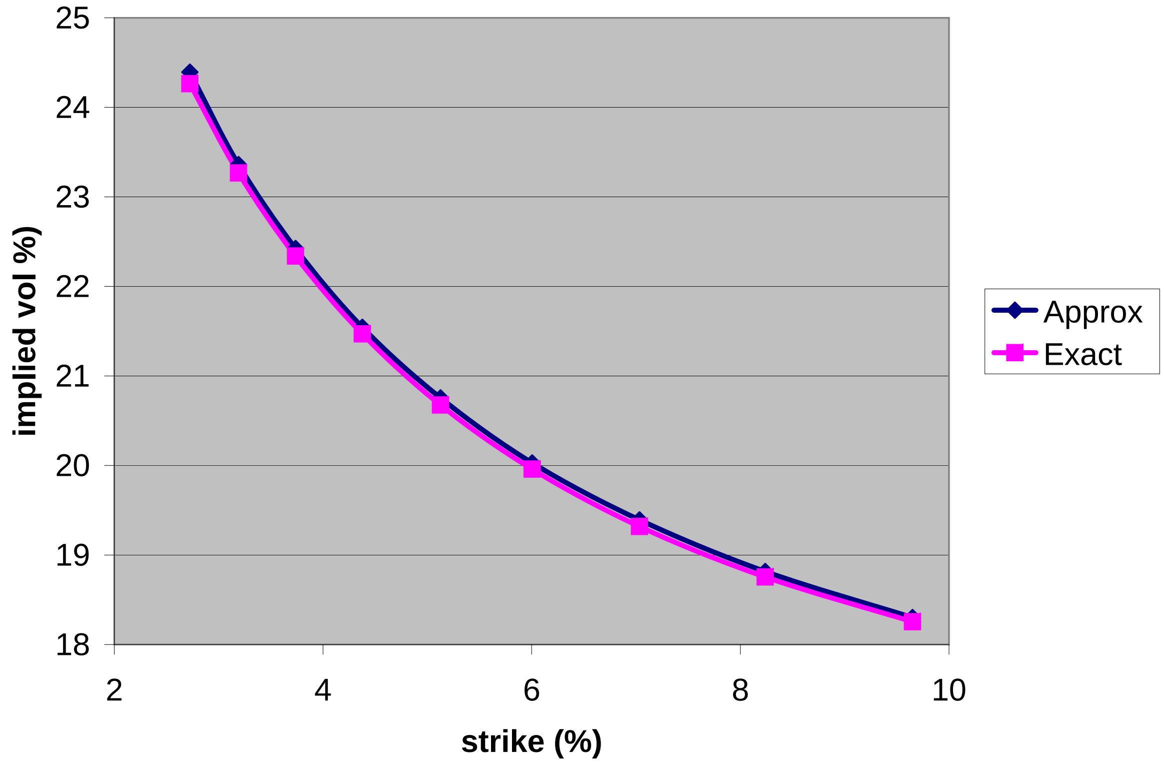

Figure 1: Comparison of the implied vols for 10Y10Y European swaption (generalized BK model).

Numerical example: Bounded short rate model

Consider 1D case of the Bounded short rate model

$$r(t) = f(t, x) = \frac{U(t)e^{\beta(t)\,x} + L(t)\alpha(t)}{e^{\beta(t)\,x} + \alpha(t)}$$

with the underlying process

$$dx(t) = (\phi(t) - a(t)\,x(t))\,dt + \gamma(t)\,dW(t)$$

starting from zero, $x(0) = 0$.

Use the following parameters

Rate	$R(t)$	5.00%
Volatility	$\gamma(t)$	1.5%
Reversion	$a(t)$	5.00%
Lower bound	$L(t)$	0%
Upper bound	$U(t)$	20.00%

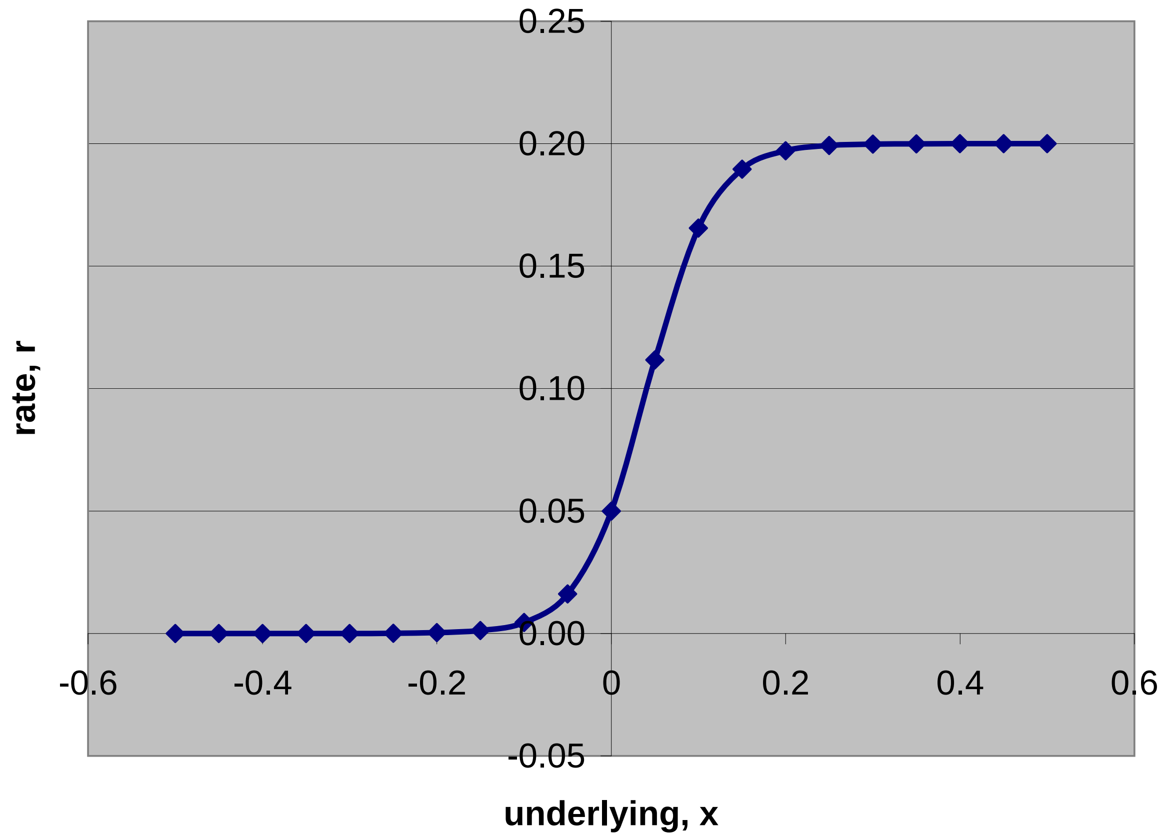

Figure 2: Rate as function of underlying, $r = f(x)$.

We have tested European swaptions with:

- Annual period
- 10Y length
- 10Y maturity
- Wide strikes range

The results for Black implied volatilities are presented in the table below.

We compare:

- Approximate analytical (the first order in ε) value (**Approx**)
- Exact value obtained by the Least-Square MC method with large number of paths (**Exact**)

Strike (%)	**Implied vol (%)**		**Error (bps)**
	Approx	**Exact**	
0.01	29.72	32.52	-280
0.5	19.98	20.54	-56
1	19.01	18.34	67
1.5	18.39	17.79	60
2	17.92	17.49	43
2.5	17.52	17.19	33
3	17.17	16.92	25
3.5	16.85	16.69	16
4	16.56	16.48	8
4.5	16.29	16.27	2
5	16.04	16.07	-3
5.5	15.8	15.87	-7
6	15.57	15.67	-10
6.5	15.35	15.49	-14
7	15.14	15.3	-16
7.5	14.93	15.11	-18
8	14.74	14.93	-19
8.5	14.54	14.74	-20
9	14.34	14.55	-21
9.5	14.15	14.36	-21
10	13.96	14.17	-21
10.5	13.77	13.98	-21

Strike (%)	Implied vol (%)		Error (bps)
	Approx	Exact	
11	13.58	13.79	-21
11.5	13.38	13.6	-22
12	13.19	13.39	-20
12.5	12.99	13.18	-19
13	12.79	12.96	-17
13.5	12.58	12.74	-16
14	12.37	12.52	-15
14.5	12.14	12.31	-17
15	11.92	12.11	-19
15.5	11.69	11.91	-22
16	11.44	11.72	-28
16.5	11.19	11.52	-33
17	10.92	11.32	-40
17.5	10.63	11.12	-49
18	10.33	10.93	-60
18.5	10.01	10.73	-72
19	9.654	9.972	-31.8
19.5	9.263	9.724	-46.1
20	8.824	8.254	57
20.2	8.632	5.852	278
20.4	8.425	4.662	376.3
20.5	8.315	4.63	368.5
21	0	3.99	-399

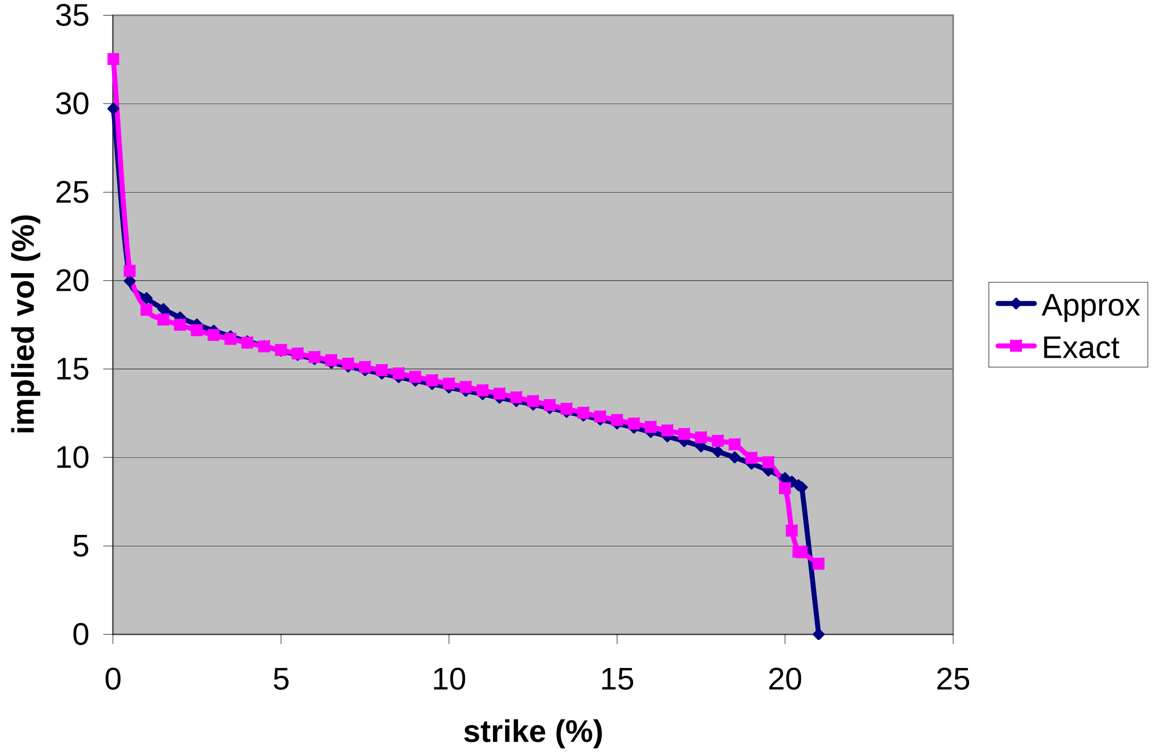

Figure 3: Comparison of the implied vols for 10Y10Y European swaption (bounded short rate model).

Observations

- Very good approximation quality for the first order approximation for short and medium maturities
- Good approximation quality for the first order approximation for large maturities
- Slight quality degeneration for extreme strikes
- Convex unbounded skew for the generalized BK model
- Bounded skew for the bounded short rate model: convex skew for small strikes, concave skew for bigger strikes

Conclusion

We presented:

- Generalized short rate models with gaussian underlying
- Efficient swaption price approximation in a semi-analytical form by small volatility perturbations
- Important special cases of the Generalized short rate model
  - Generalized multi-factor BK model
  - Bounded rate model (especially attractive for scenario generation)

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