Option Portfolio Optimization

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ICBI Global Derivatives

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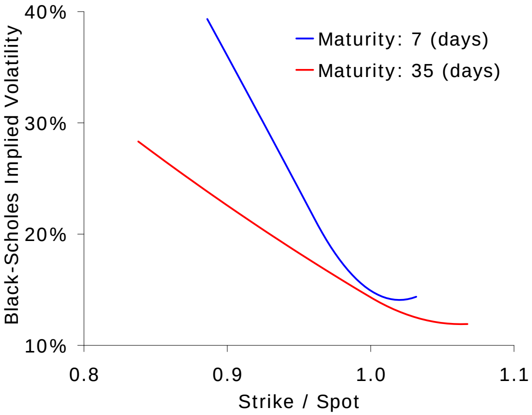
Investing in Options



- As an investor or trader, what option trades to make?
- Can standard portfolio optimization techniques be applied to these securities?

Related Literature

- Carr, P. and D. Madan, 2001, "Optimal Positioning in Derivative Securities," *Quantitative Finance*, Vol.1, No.1.
- Cochrane, J. and J. Saa-Requejo, 2000, "Beyond Arbitrage: Good-Deal Asset Price Bounds in Incomplete Markets," *Journal Political Economy*, Vol.108, 79-119.
- Driessen, J. and P. Maenhout, 2007, "Empirical Portfolio Perspective on Option Pricing Anomalies," *Review of Finance*, Vol.11, No.4, 561-603.
- Meucci, A., 2008, "Fully Flexible Views: Theory and Practice," *Risk*, Vol.21, No.10 (October), 97-102.
- Zymler, S., B. Rustem, D. Kuhn, 2011, "Robust Portfolio Optimization with Derivative Insurance Guarantees," *European Journal of Operations Research*, Vol.210, No.2, 410-424.

Issues

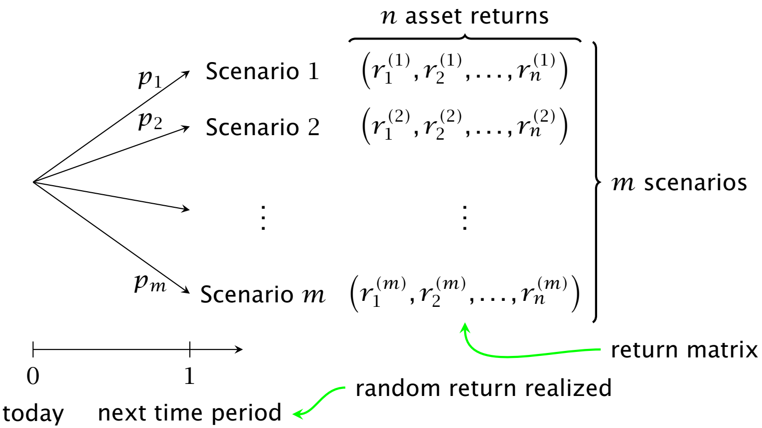
- Prices
  - Risk-neutral distribution
  - Finite number traded strikes and maturities
  - Identify trades to take advantage of mispricing
  - Transaction costs: bid-offer spreads
- Preferences
  - Risk aversion: mean-variance or expected utility
  - Option returns are highly skewed, so third and higher moments are important
- Probabilities
  - Real-world distribution: view, historical data

Standard Mean-Variance Formulation

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \sigma_P^2 \equiv \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \sigma_{ij} x_j \\ \text{subject to} \quad & \mu_P \equiv \sum_{j=1}^{n} \mu_j x_j \geq \mu_{\text{min}} \\ & \sum_{j=1}^{n} x_j = 1, \quad x \geq 0 \end{aligned}$$

- Covariance matrix: $\sigma_{ij} = \mathrm{E}[(R_i - \mu_i)(R_j - \mu_j)]$
- Coskewness tensor: $\mathrm{E}[(R_i - \mu_i)(R_j - \mu_j)(R_k - \mu_k)]$
  - Too many parameters to reliably estimate
  - Even factor models of coskewness are complicated
  - Cokurtosis even more complicated
- How to proceed?

Portfolio Optimization: Scenario Formulation



- Scenarios can be multidimensional (stock prices, volatility and other factors)
- Buy-and-hold investment: markets are incomplete, options are not redundant

Mean-Variance Portfolio Optimization

$$
\begin{array}{lrcll}
\underset{x}{\text{minimize}} & \sigma_P^2(x) & = & \displaystyle\sum_{i=1}^{m} p_i \left(r^{(i)}(x) - \mu_P(x) \right)^2 & \\
\text{subject to} & r^{(i)}(x) & = & \displaystyle\sum_{j=1}^{n} r_j^{(i)} x_j, & 1 \le i \le m \\
 & \mu_P(x) & = & \displaystyle\sum_{i=1}^{m} p_i r^{(i)}(x) \ge \mu_{\text{min}} & \\
 & & & \displaystyle\sum_{j=1}^{n} x_j = 1, \quad x \ge 0 &
\end{array}
$$

Reduced Form

Reduced Form

$$\underset{x}{\text{minimize}} \quad \sum_{j=1}^{n} \sum_{j'=1}^{n} x_j \sigma_{jj'} x_{j'}$$

$$\text{subject to} \quad \sum_{j=1}^{n} \mu_j x_j \geq \mu_{\min}$$

$$\sum_{j=1}^{n} x_j = 1, \quad x \geq 0$$

Scenario Form

$$\underset{x}{\text{minimize}} \quad \sum_{i=1}^{m} p_i \left(r^{(i)}(x) - \mu_P(x) \right)^2$$

$$\text{subject to} \quad r^{(i)}(x) = \sum_{j=1}^{n} r_j^{(i)} x_j, \quad 1 \leq i \leq m$$

$$\mu_P(x) = \sum_{i=1}^{m} p_i r^{(i)}(x) \geq \mu_{\min}$$

$$\sum_{j=1}^{n} x_j = 1, \quad x \geq 0$$

- The reduced form compresses all information about scenarios, probabilities, etc. into a mean vector and a covariance matrix
- The two formulations are equivalent
- However, the scenario form is more flexible
  - Alternative constraints: skewness, kurtosis, etc.
  - Alternative objectives: semi-variance, average downside risk, etc.

Alternative Constraints, Objectives, and Bid-Offer Spreads

Mean-Variance-Skewness Portfolio Optimization

$$\begin{array}{lrcl}
\underset{x}{\text{minimize}} & \sigma_P^2(x) & = & \sum_{i=1}^{m} p_i \left(r^{(i)}(x) - \mu_P(x) \right)^2 \\
\text{subject to} & r^{(i)}(x) & = & \sum_{j=1}^{n} r_j^{(i)} x_j, \qquad 1 \le i \le m \\
 & \mu_P(x) & = & \sum_{i=1}^{m} p_i r^{(i)}(x) \ge \mu_{\text{min}} \\
 & \text{skew}_P(x) & = & \sum_{i=1}^{m} p_i \left(r^{(i)}(x) - \mu_P(x) \right)^3 \ge \text{skew}_{\text{min}} \\
 & & & \sum_{j=1}^{n} x_j = 1, \quad x \ge 0
\end{array}$$

Can handle portfolio kurtosis in the same way

Semi-Variance

$$\text{semi variance}(x) = \mathrm{E}\left[\max\left(\mu_P(x) - r(x), 0\right)^2\right]$$

$$\left.\begin{aligned}
\underset{x,d}{\text{minimize}} \quad & \sum_{i=1}^{m} p_i d_i^2 \\
\text{subject to} \quad & r^{(i)}(x) = \sum_{j=1}^{n} r_j^{(i)} x_j, \quad 1 \le i \le m \\
& \mu_P(x) = \sum_{i=1}^{m} p_i r^{(i)}(x) \ge \mu_{\text{min}} \\
& d_i \ge \mu_P(x) - r^{(i)}, \quad 1 \le i \le m \\
& \sum_{j=1}^{n} x_j = 1, \quad x \ge 0, \quad d \ge 0
\end{aligned}\right\} \quad \text{quadratic program}$$

Average Downside Risk

$$\text{average downside risk}(x) = \mathrm{E}\left[\max\left(\mu_P(x) - r(x), 0\right)\right]$$

$$\left.\begin{array}{ll} \underset{x,d}{\text{minimize}} & \displaystyle\sum_{i=1}^{m} p_i d_i \\ \text{subject to} & r^{(i)}(x) = \displaystyle\sum_{j=1}^{n} r_j^{(i)} x_j, \quad 1 \le i \le m \\ & \mu_P(x) = \displaystyle\sum_{i=1}^{m} p_i r^{(i)}(x) \ge \mu_{\text{min}} \\ & d_i \ge \mu_P(x) - r^{(i)}, \quad 1 \le i \le m \\ & \displaystyle\sum_{j=1}^{n} x_j = 1, \quad x \ge 0, \quad d \ge 0 \end{array}\right\} \quad \text{linear program}$$

Expected Utility Formulation

$$\begin{aligned} \underset{x}{\text{minimize}} \quad \mathrm{E}[U] &= \sum_{i=1}^{m} p_i U\left(W_0(1 + r^{(i)}(x))\right) \\ \text{subject to} \quad r^{(i)}(x) &= \sum_{j=1}^{n} r_j^{(i)} x_j, \qquad 1 \le i \le m \\ &\sum_{j=1}^{n} x_j = 1, \quad x \ge 0 \end{aligned}$$

Common choices for the utility function U

- Power utility, CRRA: $U(W) = \dfrac{W^\gamma}{\gamma} \quad (\gamma < 1, \gamma \neq 0)$
- Log utility: $U(W) = \log(W)$ $(\gamma = 0)$
- Many others ...

Portfolio Optimization with Bid-Offer Spreads

Let x_j be the quantity of the instrument j purchased

Let y_j be the quantity of the instrument j sold

$$
\begin{array}{lll}
\underset{x,y}{\text{minimize}} & \text{Var}\,[W_T] & \Leftarrow \text{minimize variance} \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} x_j P_0^j - \sum_{j=1}^{n} y_j Q_0^j = w_0 & \Leftarrow \text{budget constraint} \\
 & \text{E}\,[W_T] \geq w_{\text{min}} & \Leftarrow \text{expected wealth constraint} \\
 & \displaystyle W_T^{(i)} = \sum_{j=1}^{n} w_0 (1 + r_j^{(i)} x_j), & 1 \leq i \leq m
\end{array}
$$

- Can formulate in terms of wealth or return
- P_0^j are prices to buy; Q_0^j are prices to sell

Portfolio Optimization with the Scenario Formulation

Flexible formulation

- Can include mean, variance, skewness, kurtosis and any other moments of the portfolio return distribution
- Easy to replace mean-variance formulation with expected utility
- Easy to include bid and offer prices

Additional ways to control risk

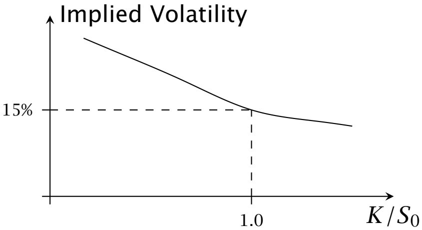
- Bounds on security holdings
- Bounds on return by scenario
- Bounds on conditional value-at-risk (CVAR), expected excess loss and other risk measures

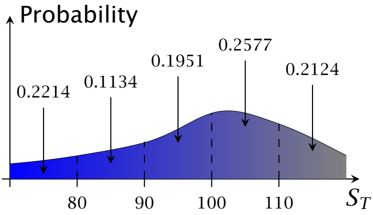
How to specify investor views, i.e., real-world scenarios and probabilities?

- Historical: straightforward
- Parametric: use favorite option pricing model with real-world parameters

Specifying Investor Views:
Real-World Scenarios and Probabilities

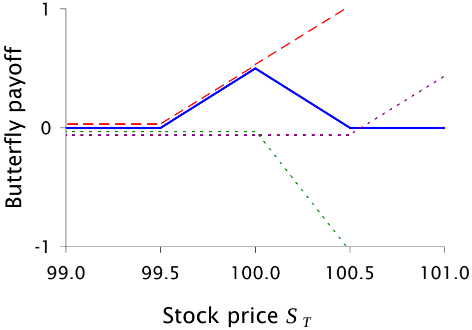
Specifying Investor Views





- Implied volatility curve ⇔ asset price distribution
- View: choose real-world parameters of an option pricing model, then infer scenarios and probabilities
    - Resulting scenarios are smooth
    - Procedure can be used with historical data using standard econometric techniques

Scenario Probabilities From Butterflies

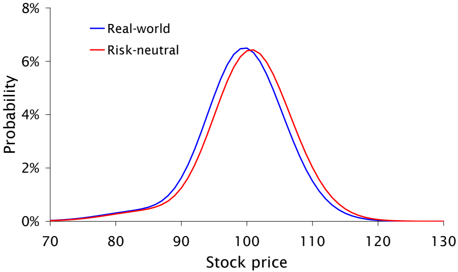


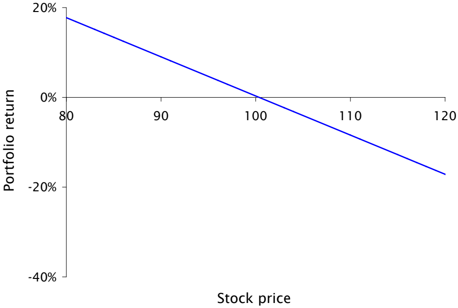
Strike	Trade
99.5	Buy 1 call
100.0	Sell 2 calls
100.5	Buy 1 call

- Positive payoff when $99.5 < S_T < 100.5$
- Butterfly price is related to the probability $S_T = 100$
- Scenario probability:
  $P(K - \frac{1}{2} < S_T < K + \frac{1}{2}) \approx 4e^{rT}(C(K - \frac{1}{2}) - 2C(K) + C(K + \frac{1}{2}))$,
  where $C(K)$ is the Merton call option price with strike K
- Breeden and Litzenberger (1978)

Examples

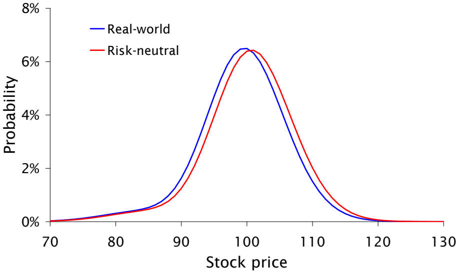
Bearish Investor: Stock and Bond Only

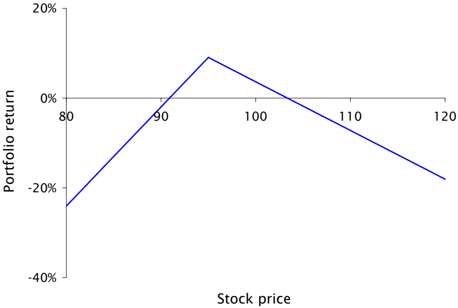




- Time horizon: $T = 1/12$, interest rate $r = 2\%$
- Market prices in red
- Real-world view: $\mu = -10\%$ (stock return parameter, annual)
- Constraint: $\sigma_{\text{RW}} \leq 5\%$ (monthly, i.e., 17% annual)
- Optimal portfolio: stock -90%, bond 190%
- Real-world expected return: 13% (annual)

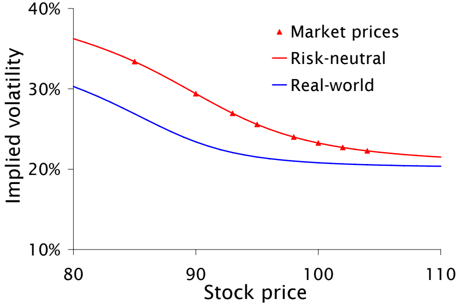
Bearish Investor: Stock, Bond and Put Option

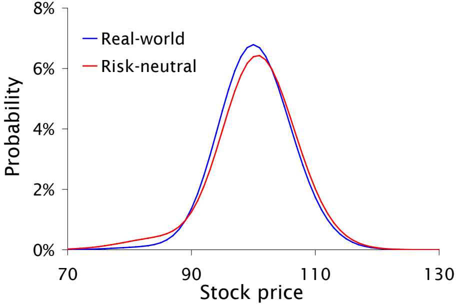




- Time horizon: $T = 1/12$, put strike $K = 95$
- Market prices in red
- Real-world view: $\mu = -10\%$
- Constraint: $\sigma_{\text{RW}} \leq 5\%$ (monthly, i.e., 17% annual)
- Optimal portfolio: stock -110%, bond 210%, put -3%
- Real-world expected return: 25% (annual)
- Including an option increases expected return with the same volatility

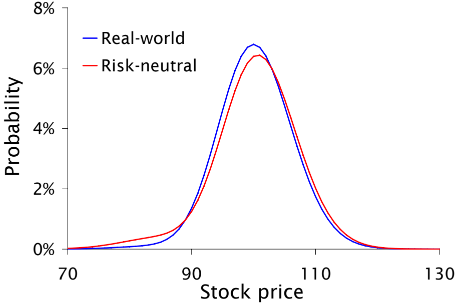
View: Real-World Volatility Less than Risk-Neutral

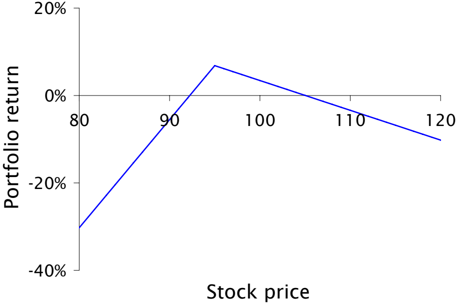




- Time horizon: $T = 1/12$, interest rate $r = 2\%$
- Market prices in red
- Real-world view (Merton, J): $\lambda = 20\%$, $\sigma = 20\%$, $\mu_S = -15\%$, $\sigma_S = 5\%$, $\mu = 2\%$

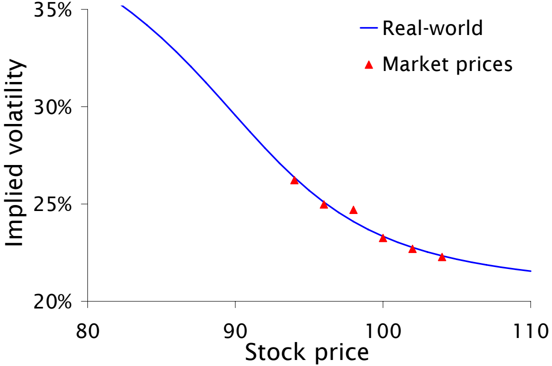
View: Real-World Volatility Less than Risk-Neutral





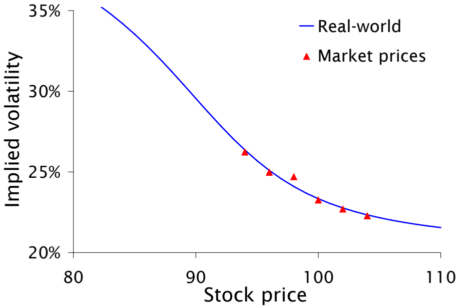
- Time horizon: $T = 1/12$; stock, bond and put with strike $K = 95$
- Market prices in red
- Real-world view (Merton, J): $\lambda = 20\%$, $\sigma = 20\%$, $\mu_S = -15\%$, $\sigma_S = 5\%$
- Constraint: $\sigma_{\text{RW}} \leq 5\%$ (monthly, i.e., 17% annual)
- Optimal portfolio: stock -70%, bond 170%, put -3%
- Optimal portfolio: similar profile to selling a straddle
- Real-world expected return: 16% (annual)

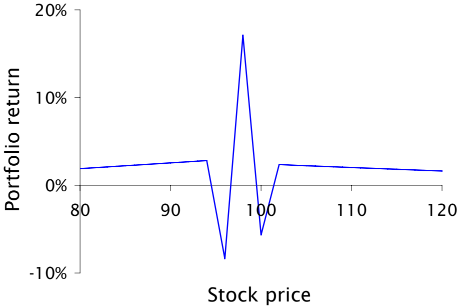
Mispriced Option



- Time horizon: $T = 1/12$, interest rate $r = 2\%$
- Market prices in red
- Real-world view: smooth fit to market prices
- $K = 98$ strike put appears to be mispriced

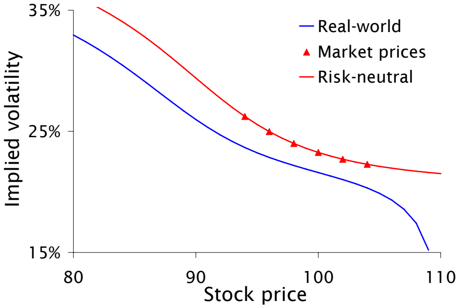
Mispriced Option

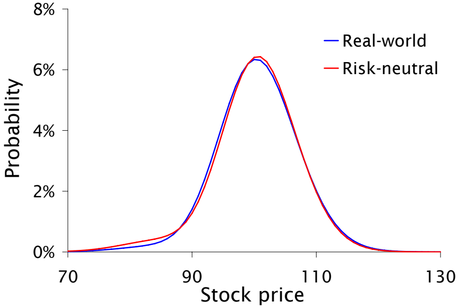




- Time horizon: $T = 1/12$, interest rate $r = 2\%$
- Market prices in red
- Real-world view: smooth fit to market prices
- $K = 98$ strike put appears to be mispriced
- Constraint: $\sigma_{\text{RW}} \leq 5\%$ (monthly, i.e., 17% annual)
- Optimal portfolio includes puts and calls of all available strikes
- Real-world expected return: 24% (annual)

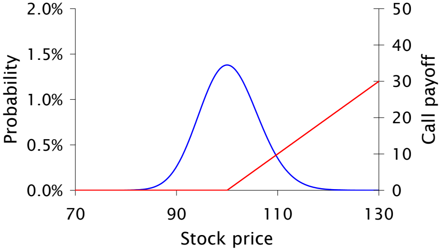
Example: Stock, Bond and Twelve Options

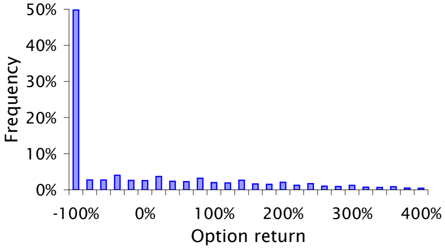




- Time horizon: $T = 1/12$, interest rate $r = 2\%$
- Market prices in red
- Real-world view: $\mu = 5\%$, lower volatility

Expected Option Return



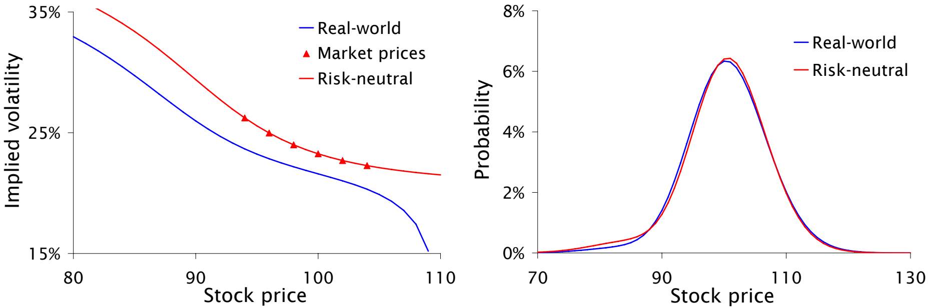


$$\text{RW expected call return} = \frac{\mathsf{E}^P[\max(S_T - K, 0)] - C_0}{C_0}$$

$$= \frac{e^{\mu T}\mathsf{E}^P[e^{-\mu T}\max(S_T - K, 0)] - C_0}{C_0}$$

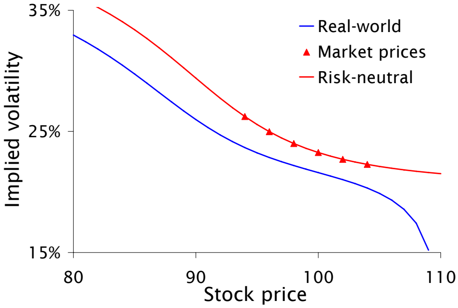
- Option price formula gives a formula for expected option return
- Formulas for variance and other moments as well
- Option investing: real-world (E^P) vs. risk-neutral probabilities (E^Q)

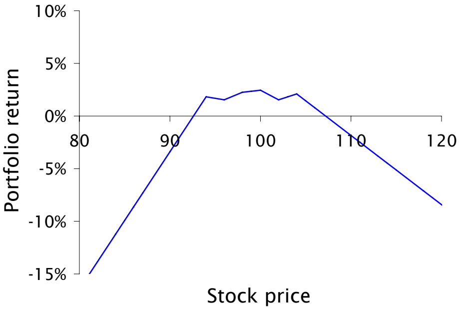
Expected Return and Standard Deviation



	Stock	Bond	Put: 94	96	98	100	102	104
Exp ret	0.4%	0.3%	−36%	−29%	−22%	−16%	−11%	−8%
Std dev	6%	0%	255%	213%	176%	145%	120%	99%
	Stock	**Bond**	**Call: 94**	**96**	**98**	**100**	**102**	**104**
Exp ret	0.4%	0.3%	−1%	−2%	−4%	−6%	−9%	−12%
Std dev	6%	0%	74%	89%	108%	131%	161%	201%

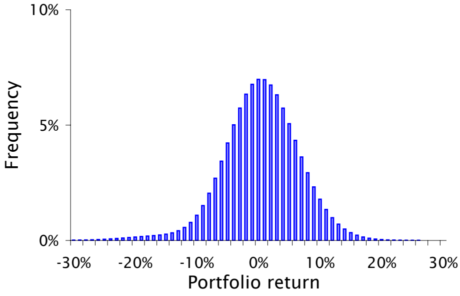
Example: Stock, Bond and Twelve Options

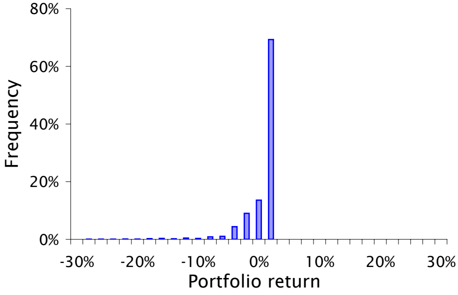




- Time horizon: $T = 1/12$, interest rate $r = 2\%$
- Market prices in red
- Real-world view: $\mu = 5\%$, lower volatility
- Constraint: $\sigma_{\text{RW}} \leq 3\%$ (monthly, i.e., 10% annual)
- Optimal portfolio includes puts and calls of all available strikes
- Solution has bond-like payoff, with real-world expected return: 8% (annual)

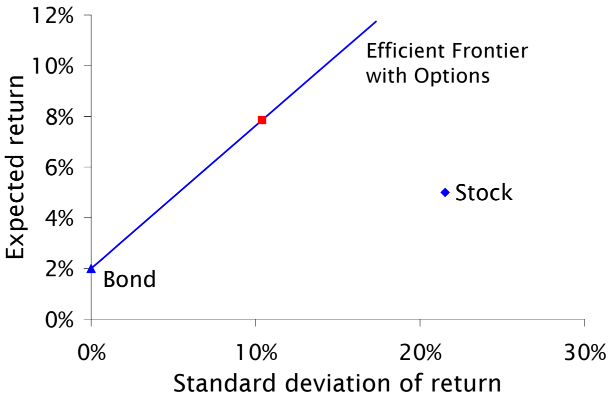
Example: Stock, Bond and Twelve Options





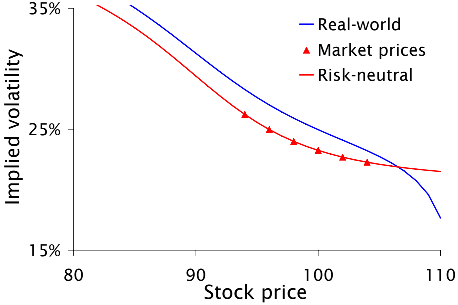
- Real-world view: $\mu = 5\%$, lower volatility
- Constraint: $\sigma_{\text{RW}} \leq 3\%$ (monthly, i.e., 10% annual)
- Optimal portfolio includes puts and calls of all available strikes
- Solution has bond-like payoff, with real-world expected return: 8% (annual)

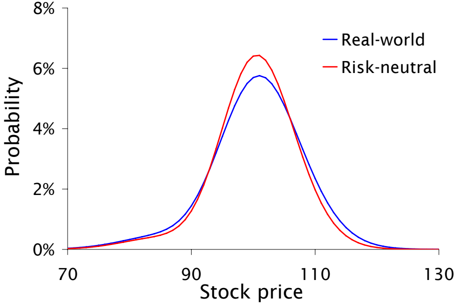
Efficient Frontier: Stock, Bond and Twelve Options



- Real-world view: $\mu = 5\%$, lower volatility
- Efficient frontier dominates stock-bond combinations

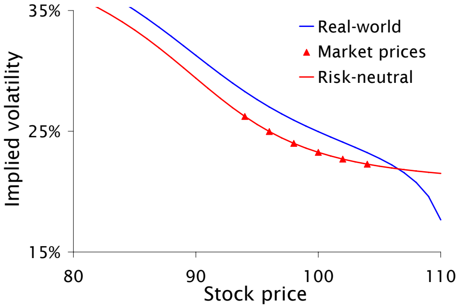
Example 2: Stock, Bond and Twelve Options

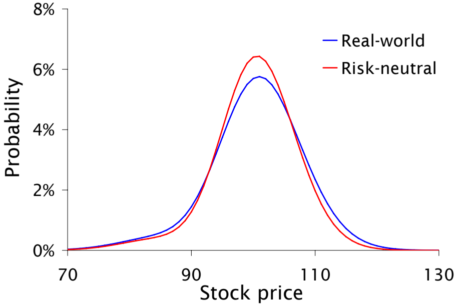




- Time horizon: $T = 1/12$, interest rate $r = 2\%$
- Market prices in red
- Real-world view: $\mu = 5\%$, higher volatility

Expected Return and Standard Deviation

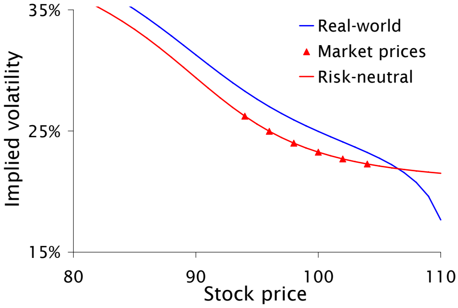


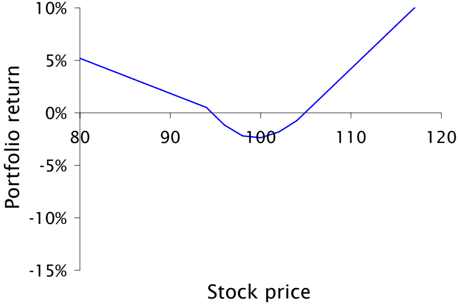


	Stock	Bond	Put: 94	96	98	100	102	104
Exp ret	0.4%	0.3%	20%	16%	12%	8%	4%	2%
Std dev	8%	0%	383%	305%	240%	189%	151%	123%

	Stock	Bond	Call: 94	96	98	100	102	104
Exp ret	0.4%	0.3%	6%	8%	12%	16%	22%	30%
Std dev	8%	0%	87%	104%	127%	159%	204%	268%

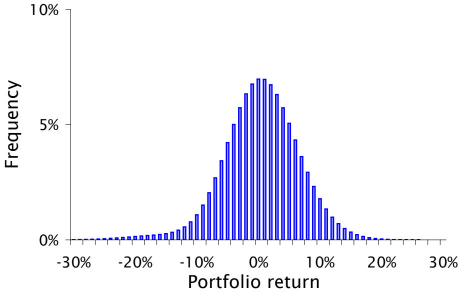
Example 2: Stock, Bond and Twelve Options

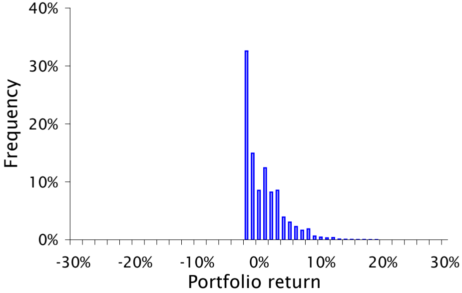




- Time horizon: $T = 1/12$, interest rate $r = 2\%$
- Market prices in red
- Real-world view: $\mu = 5\%$, higher volatility
- Constraint: $\sigma_{\text{RW}} \leq 3\%$ (monthly, i.e., 10% annual)
- Optimal portfolio includes puts and calls of all available strikes
- Solution has straddle-like payoff, with real-world expected return: 7% (annual)

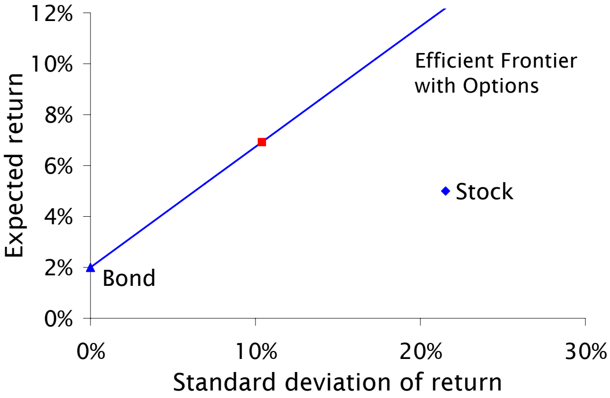
Example 2: Stock, Bond and Twelve Options





- Real-world view: $\mu = 5\%$, lower volatility
- Constraint: $\sigma_{\mathrm{RW}} \leq 3\%$ (monthly, i.e., 10% annual)
- Optimal portfolio includes puts and calls of all available strikes
- Solution has straddle-like payoff, with real-world expected return: 7% (annual)

Efficient Frontier: Stock, Bond and Twelve Options



- Real-world view: $\mu = 5\%$, higher volatility
- Efficient frontier dominates stock-bond combinations

Scenario Approach to Option Portfolio Optimization

- Prices
  - Market prices are given inputs
  - Automatically identifies trades to take advantage of mispricing
  - Bid-offer spreads
- Preferences
  - Risk aversion: mean-variance or expected utility
  - Skewness, kurtosis and higher moments easily handled
- Probabilities
  - View specified using real-world parameters of option pricing model

Scenario optimization approach finds option portfolios consistent with market prices and investor views and preferences