Quadratic Variance Swap Term Structure Models

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Outline

Variance Swap Term Structure

Quadratic Term Structure Models

Model Estimation

Optimal Investment

Outline

Term Structure Models

- Variance swaps are traded OTC at many different maturities
- Variance swap curve moves randomly
- Need to capture the term structure of volatility risk

$\Rightarrow$ Design and estimate variance swap term-structure models

- Related literature: Buehler (06), Egloff et al. (10), Cont and Kokholm (09), a.o.

Underlying Price Process

- Price process (e.g. S&P 500 index): semimartingale S

$$\frac{dS_t}{S_{t-}} = r_t\,dt + \sigma_t\,dW_t^* + \int_{\mathbb{R}} (\mathrm{e}^x - 1)\,(\mu(dt,dx) - \nu_t(dx)dt)$$

- In particular,

$$\Delta \log S_t = \int_{\mathbb{R}} x\,\mu(dt,dx)$$

- Hence the annualized realized variance equals

$$\mathrm{RV}(t,T) = \frac{1}{T-t}\left(\int_t^T \sigma_s^2\,ds + \int_t^T \int_{\mathbb{R}} x^2\,\mu(ds,dx)\right)$$

Forward Variance

- Define the spot variance

$$v_t = \sigma_t^2 + \int_{\mathbb{R}} x^2 \, \nu_t(dx)$$

- Define the forward variance

$$f(t, T) = \mathbb{E}_{\mathbb{Q}} \left[v_T \mid \mathcal{F}_t \right]$$

- Then the variance swap rate equals

$$\mathrm{VS}(t, T) = \mathbb{E}_{\mathbb{Q}} \left[\mathrm{RV}(t, T) \mid \mathcal{F}_t \right] = \frac{1}{T - t} \int_t^T f(t, s) \, ds$$

- Note the analogy to interest rates (spot, forward, yield)

Outline

Quadratic State Diffusion

- State space $\mathcal{X} \subseteq \mathbb{R}^m$
- Diffusion state process under $\mathbb{Q}$

$$dX_t = \mu(X_t)\,dt + \Sigma(X_t)\,dW_t^*$$

- X is quadratic if its drift and diffusion functions are linear and quadratic in the state variable:

$$\mu(x) = b + \beta\,x$$
$$\Sigma(x)\Sigma(x)^\top = a + \sum_{k=1}^m \alpha^k x_k + \sum_{k,l=1}^m A^{kl} x_k x_l$$

Quadratic Variance Swap Model

- A quadratic variance swap model is obtained under the assumption that the spot variance is a quadratic function of the latent state variable X_t:

$$v_t = \phi_0 + \psi_0^\top X_t + X_t^\top \pi_0 X_t$$

Quadratic Term Structure Theorem

Under the above assumptions, the quadratic variance swap model admits a quadratic term structure:

$$f(t,T) = \phi(T-t) + \psi(T-t)^\top X_t + X_t^\top \pi(T-t)\, X_t$$

where the functions ϕ, ψ, and π satisfy the linear system of ODEs

$$\begin{aligned}
\frac{d\phi(\tau)}{d\tau} &= b^\top \psi(\tau) + \mathrm{tr}(a\,\pi(\tau)), && \phi(0) = \phi_0 \\
\frac{d\psi(\tau)}{d\tau} &= \beta^\top \psi(\tau) + 2\pi(\tau) b + \alpha \cdot \pi(\tau), && \psi(0) = \psi_0 \\
\frac{d\pi(\tau)}{d\tau} &= \beta^\top \pi(\tau) + \pi(\tau)\beta + A \bullet \pi(\tau), && \pi(0) = \pi_0
\end{aligned}$$

where $(\alpha \cdot \pi)_k = \mathrm{tr}(\alpha^k\,\pi)$ and $(A \bullet \pi)_{kl} = \mathrm{tr}(A^{kl}\,\pi)$.

Model Identification

- The quadratic property of X and the term structure is invariant w.r.t. affine transformations

$$X \mapsto c + \gamma X$$

- Need canonical representation of X for econometric model identification
- Exhaustive specification analysis of multi-factor quadratic models difficult (needs classification of zero level sets of quadratic forms on $\mathbb{R}^m$)
- Strategy:
  1. Exhaustive specification analysis of univariate case is possible
  2. Discuss specific bivariate extension

Bivariate Model Specification

- Assume

$$dX_{1t} = (b_1 + \beta_{11}\, X_{1t} + \beta_{12}\, X_{2t})\, dt + \sqrt{a_1 + \alpha_1 X_{1t} + A_1 X_{1t}^2}\, dW_{1t}^*$$

$$dX_{2t} = (b_2 + \beta_{22}\, X_{2t})\, dt + \sqrt{a_2 + \alpha_2 X_{2t} + A_2 X_{2t}^2}\, dW_{2t}^*$$

- Spot variance is given by

$$v_t = \phi_0 + \psi_0 X_{1t} + \pi_0 X_{1t}^2$$

- Interpretation: one-factor quadratic variance swap model extended by stochastic mean reversion level

$$\frac{b_1 + \beta_{12}\, X_{2t}}{|\beta_{11}|} \geq 0$$

Explicit Forward Variance Swaps

- $f(t,T) = \phi(T-t) + \psi(T-t)^\top X_t + X_t^\top \pi(T-t)\, X_t$
- Linear ODEs for ϕ, ψ, and π can be vectorized by setting

$$q(\tau) = \begin{pmatrix}\phi(\tau) & \psi_1(\tau) & \psi_2(\tau) & \pi_{11}(\tau) & \pi_{12}(\tau) & \pi_{22}(\tau)\end{pmatrix}^\top$$

- The above system then reads

$$\frac{dq(\tau)}{d\tau} = \begin{pmatrix} 0 & b_1 & b_2 & a_1 & 0 & a_2 \\ 0 & \beta_{11} & \beta_{21} & 2b_1 + \alpha_1 & 2b_2 & 0 \\ 0 & \beta_{12} & \beta_{22} & 0 & 2b_1 & 2b_2 + \alpha_2 \\ 0 & 0 & 0 & 2\beta_{11} + A_1 & 2\beta_{21} & 0 \\ 0 & 0 & 0 & \beta_{12} & \beta_{11} + \beta_{22} & \beta_{21} \\ 0 & 0 & 0 & 0 & 2\beta_{12} & 2\beta_{22} + A_2 \end{pmatrix} q(\tau)$$

$$q(0) = \begin{pmatrix}\phi_0 & \psi_0 & 0 & \pi_0 & 0 & 0\end{pmatrix}^\top.$$

Outline

Data

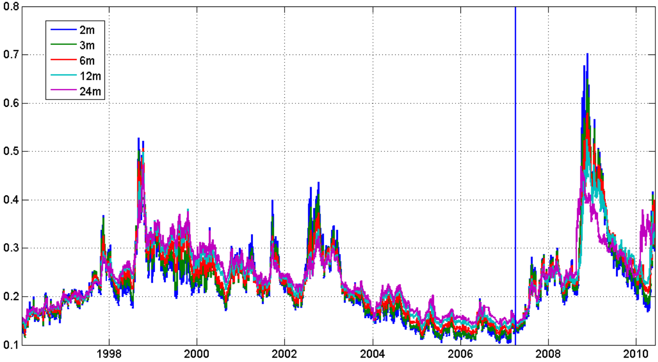

Figure: Variance swap rates $\sqrt{\mathrm{VS}(t, t+\tau)}$ on the S&P 500 index from Jan 4, 1996 to Jun 7, 2010. Source: Bloomberg

- In-sample (pre-crisis): Jan 4, 1996 to Apr 2, 2007
- Out-of-sample: Apr 3, 2007 to Jun 7, 2010

Principal Component Analysis

- PCA of variance swap curve $\tau \mapsto \sqrt{\mathrm{VS}(t, t+\tau)}$
- One major factor (level), explains 96% **of variance**
- Second factor (slope), explains 3% **of variance**

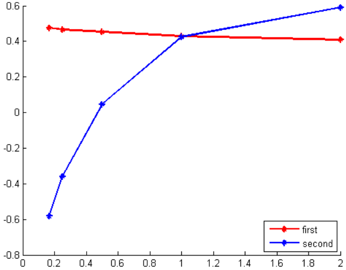

Figure: First two variance swap curve loadings

Estimation Results: Bivariate Model

- Best fit for

$$dX_{1t} = (\ell + (\lambda + \beta_{11})\, X_{1t} + \beta_{12}\, X_{2t})\, dt + \sqrt{1 + A_1 X_{1t}^2}\, dW_{1t}$$

$$dX_{2t} = (b_2 + \beta_{22}\, X_{2t})\, dt + \sqrt{X_{2t} + A_2 X_{2t}^2}\, dW_{2t}$$

- Recall spot variance $v_t = \phi_0 + \psi_0 X_{1t} + \pi_0 X_{1t}^2$

β_{11}	β_{12}	b_2	β_{22}	A_1	A_2
-5.1720	4.2324	0.1824	-0.2483	3.3895	0.0985
(0.0903)	(0.2346)	(0.0322)	(0.0021)	(0.1206)	(0.0001)

ϕ_0	ψ_0	π_0	**MPR**	ℓ	λ
0.0175	0.0130	0.0283		-0.1770	-0.0021
(0.0002)	(0.0008)	(0.0004)		(0.0190)	(0.0868)

Table: Estimated parameters (robust standard errors into parentheses)

In-Sample Analysis: Variance Risk Premium

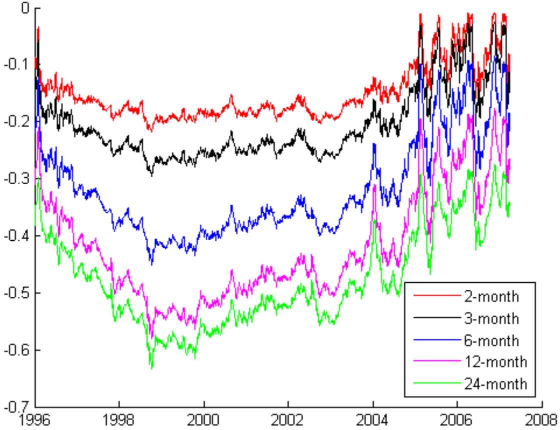

Figure: The variance risk premium for period $[t, t+\tau]$ is defined as $100 \times \left(\sqrt{\mathbb{E}_{\mathbb{P}}\left[\mathrm{RV}(t, t+\tau) \mid \mathcal{F}_t\right]} - \sqrt{\mathbb{E}_{\mathbb{Q}}\left[\mathrm{RV}(t, t+\tau) \mid \mathcal{F}_t\right]}\right)$.

In-Sample Analysis: Variance Risk Premium

- The VRP grows systematically with the time to maturity: long term variance swap rates command more risk premium.

- It is always negative across time and across maturities (in contrast to findings of Carr and Wu (09), who replace conditional $\mathbb{P}$-expected by ex-post realized variance)

In-Sample Analysis: Filtered Factors

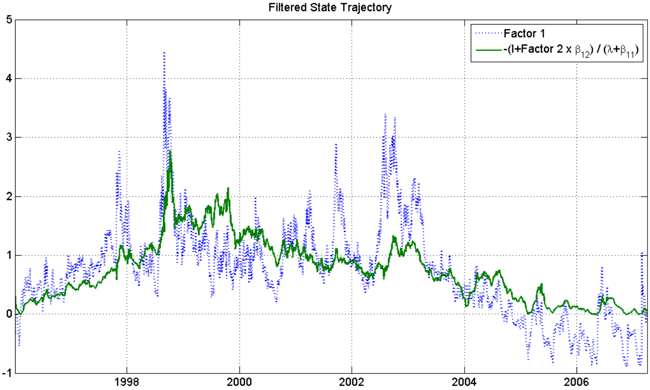

Figure: Filtered factors X_1 vs. stochastic mean reversion level $\frac{\ell+\beta_{12}X_2}{-(\lambda+\beta_{11})}$.

In-Sample Analysis: Fitting VS

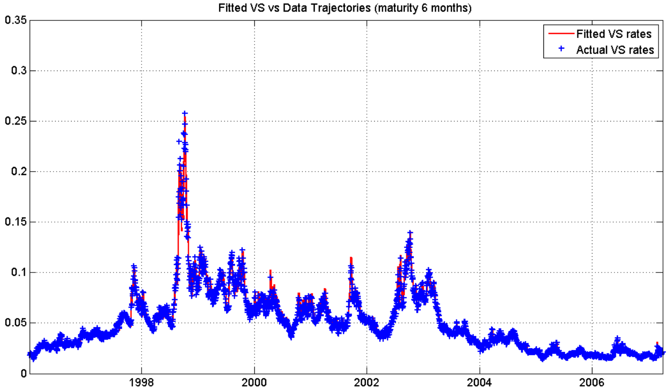

Figure: Very good fit of variance swap rates for all (here 6 months) maturities.

Out-of-Sample Analysis: Predicted VS

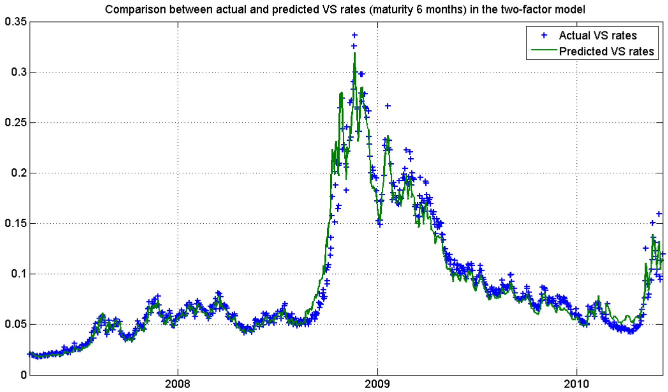

Figure: Out-of-sample predicted variance swap rates vs. data for 6 months maturity. The quadratic diffusion model captures extreme movements and spikes.

Out-of-Sample Analysis: Pricing Errors

	One-factor			Two-factor		
	Mean	RMSE	Max	Mean	RMSE	Max
2	0.0035	0.02	0.1459	-0.0010	0.0067	0.0433
3	0.0026	0.0102	0.0577	-0.0002	0.0053	0.0169
6	0.0007	0.0093	0.0231	0.0016	0.0095	0.0295
12	-0.0034	0.0152	0.0401	0.0004	0.0124	0.0421
24	-0.0048	0.0286	0.0847	-0.0014	0.0159	0.0793
Average	-0.0003	0.01666	0.0703	-0.0001	0.0099	0.0422

Table: Summary statistics of the model pricing errors on the variance swap rates, Mean is sample average, RMSE root mean square error, Max maximum absolute error. Average denote the mean of previous summary statistics. The pricing error are defined as the difference between model-implied and actual variance swap rates, both in volatility terms.

Out-of-Sample Analysis: Pricing Errors

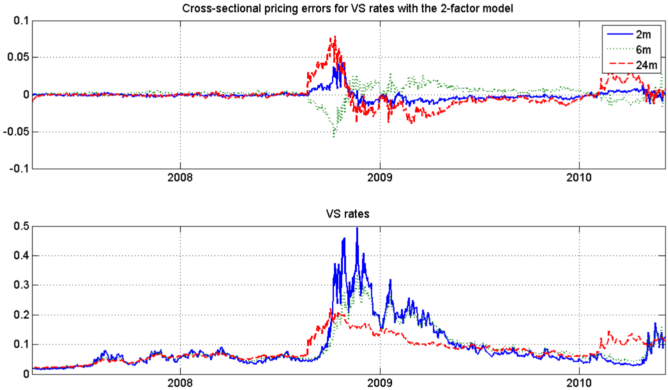

Figure: Out-of-sample model pricing errors for bivariate model. The quadratic diffusion model captures extreme movements and spikes of the entire term structure.

Outline

Investment Problem

- Optimal investment in
  - variance swap contracts
  - S&P 500 index
  - risk-free bond

Setup

- m-dimensional diffusion state process under $\mathbb{Q}$

$$dX_t = \mu(X_t)\,dt + \Sigma(X_t)\,dW_t^*$$

- Spot variance $v_t = g_0(X_t)$
- Non-annualized variance swap rate

$$\overline{\mathrm{VS}}(t,T) = \int_t^T \mathbb{E}_{\mathbb{Q}}\left[g_0(X_s) \mid \mathcal{F}_t\right] ds = G(T-t, X_t)$$

- S&P 500 index

$$\frac{dS_t}{S_t} = r\,dt + \sqrt{g_0(X_t)}\,dB_t^*$$

- Correlation vector $\mathbf{R}_{it} = \frac{d\langle W_i^*, B^*\rangle}{dt}$, $i = 1, \ldots, m$

Investment in Variance Swaps

- Fixed maturity $[t_0, T_0]$-variance swap return

$$
\begin{aligned}
d\mathcal{M}_t &= \frac{\mathrm{e}^{-r(T_0-t)}}{T_0 - t_0}\left(v_t\,dt + d\overline{\mathrm{VS}}(t, T_0)\right) \\
&= \frac{\mathrm{e}^{-r(T_0-t)}}{T_0 - t_0}\,\nabla_x G(T_0 - t, X_t)\Sigma(X_t)\,dW_t^*
\end{aligned}
$$

- Sliding maturity $[t, t+\tau]$-variance swap return

$$
\begin{aligned}
d\mathcal{M}_t &= \frac{\mathrm{e}^{-r\tau}}{\tau}\left(v_t\,dt + d\overline{\mathrm{VS}}(t, T)|_{T=t+\tau}\right) \\
&= \frac{\mathrm{e}^{-r\tau}}{\tau}\,\nabla_x G(\tau, X_t)\Sigma(X_t)\,dW_t^*
\end{aligned}
$$

Wealth Dynamics

- Wealth process $V = V^{n,w}$

$$\frac{dV_t}{V_t} = \mathbf{n}_t^\top d\mathcal{M}_t + w_t \frac{dS_t}{S_t} + (1 - w_t) r \, dt$$

- Investment strategy:
  - $\mathbf{n}_t$ = fractions of wealth allocated to variance swaps
  - w_t = fraction of wealth invested in S&P500 index
  - $1 - w_t$ = fraction of wealth invested in bond

Optimization Problem

- Optimize terminal wealth w.r.t. power utility function

$$u(v) = \frac{v^{1-\eta}}{1-\eta}$$

- Constant relative risk aversion level $\eta > 0$
- Maximization problem:

$$\max_{w,\mathbf{n}} \mathbb{E}_{\mathbb{P}} \left[u(V_T) \right]$$

Solution via Martingale Method

- Complete market: optimal terminal wealth is

$$V_T^* = \left(u'\right)^{-1}\left(\lambda \mathrm{e}^{-rT}\frac{d\mathbb{Q}}{d\mathbb{P}}\,|_{\mathcal{F}_T}\right)$$

- λ: Lagrangian such that $\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-rT}V_T^*\right] = V_0$

Optimal Trading Strategy

- Optimal fraction invested in stock (closed form)

$$w_t^* = \frac{1}{1 - \|\mathbf{R}_t\|^2} \left(\frac{\Lambda^S(X_t)}{\eta\sqrt{g_0(X_t)}} - \frac{\mathbf{R}_t^\top \Lambda^X(X_t)}{\eta\sqrt{g_0(X_t)}} \right)$$

- Optimal fractions allocated to variance swaps

$$\mathbf{n}_t^* = \left(\Sigma(X_t)^\top \nabla_x \mathcal{G}(t, X_t)^\top\right)^{-1} \left(\frac{1}{\eta}\Lambda^X(X_t) - w_t^* \sqrt{g_0(X_t)}\,\mathbf{R}_t \right.$$
$$\left. + \Sigma(X_t)^\top \frac{\nabla_x F(T-t, X_t)}{F(T-t, X_t)} \right)$$

- To be computed numerically:

$$F(T-t, X_t) = \mathbb{E}_{\widehat{\mathbb{Q}}}\left[\exp\left(\frac{1}{2\eta} \left(\frac{1}{\eta} - 1 \right) \int_t^T \|\Lambda(X_s)\|^2 \, ds \right) \mid \mathcal{F}_t \right]$$

Special Case

- Assume
$$\|\Lambda(X_t)\|^2 \equiv \text{constant}$$
- Then $F(T-t, X_t)$ does not depend on X_t:
$$\nabla_x F(T-t, X_t) \equiv 0$$
- Consequence: closed form optimal allocation to variance swaps:
$$\mathbf{n}_t^* = \left(\Sigma(X_t)^\top \nabla_x \mathcal{G}(t, X_t)^\top\right)^{-1} \left(\frac{1}{\eta}\Lambda^X(X_t) - w_t^* \sqrt{g_0(X_t)}\,\mathbf{R}_t\right)$$

Bivariate Quadratic Example

- Bivariate quadratic model

$$
\begin{aligned}
dX_{1t} &= (\beta_{11}\, X_{1t} + \beta_{12}\, X_{2t})\, dt + \sqrt{1 + A_1 X_{1t}^2}\, dW_{1t}^* \\
dX_{2t} &= (b_2 + \beta_{22}\, X_{2t})\, dt + \sqrt{X_{2t} + A_2 X_{2t}^2}\, dW_{2t}^*
\end{aligned}
$$

- Spot variance $v_t = g_0(X_t) = \phi_0 + \psi_0 X_{1t} + \pi_0 X_{1t}^2$
- Market price of risk vector $\Lambda(X_t) = \begin{pmatrix} \frac{\ell + \lambda X_{1t}}{\sqrt{1 + A_1 X_{1t}^2}} \\ 0 \\ \Lambda_3(X_t) \end{pmatrix}$
- $\Lambda_3(X_t)$ such that $\|\Lambda(X_t)\|^2 = \kappa_0 + \kappa_1 X_{1t}^2$
- Closed form first order expansion available for $F(T - t, X_t)$

Calibration of Equity Risk Premium $\Lambda^S(X_t)$

- Set correlation vector $\mathbf{R}_t = (\rho_t, 0)^\top$
- Equity risk premium $\Lambda^S(X_t) = \rho_t \Lambda_1(X_t) + \sqrt{1-\rho_t^2}\,\Lambda_3(X_t)$
- Stylized fact (leverage effect), see e.g. Ait-Sahalia et al. [1]:

$$\mathrm{Corr}\left(\frac{dS_t}{S_t}, dv_t\right) = \frac{\left\langle \frac{dS_t}{S_t}, dv_t \right\rangle}{\sqrt{\left\langle \frac{dS_t}{S_t} \right\rangle \langle dv_t \rangle}} \approx -0.7$$

- In our bivariate quadratic example we have

$$\frac{\left\langle \frac{dS_t}{S_t}, dv_t \right\rangle}{\sqrt{\left\langle \frac{dS_t}{S_t} \right\rangle \langle dv_t \rangle}} = \frac{\nabla_x g_0(X_t)^\top \Sigma(X_t)}{\|\nabla_x g_0(X_t)^\top \Sigma(X_t)\|}\,\mathbf{R}_t = \mathrm{sign}\left(\psi_0 + 2\pi_0 X_{1t}\right)\rho_t$$

- Hence we calibrate $\rho_t = -\mathrm{sign}\left(\psi_0 + 2\pi_0 X_{1t}\right) \times 0.7$

Conclusion

- Introduced a quadratic variance swap term structure model
- Analytically tractable: closed form curves, explicit conditional moments
- Captures nonlinear phenomena: rare events and volatility clustering
- Optimal investment in bond, S&P500 index, and variance swaps w.r.t. power utility in quasi closed form

- Thanks for your attention !

Conclusion

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- Thanks for your attention !

References I

Y. Ait-Sahalia, M. Karaman, and L. Mancini.
Variance swap, jumps and risk premiums.
EPFL Working Paper, 2011.

H. Buehler.
Consistent variance curve models.
Finance Stoch., 10(2):178–203, 2006.

P. Carr and L. Wu.
Variance risk premiums.
The Review of Financial Studies, 22(3):1311–1341, 2009.

R. Cont and T. Kokholm.
A Consistent Pricing Model for Index Options and Volatility Derivatives.
SSRN eLibrary, 2009.

References II

D. Egloff, M. Leippold, and L. Wu.
The term structure of variance swap rates and optimal variance swap investments.
Journal of Financial annd Quantitative Analysis, forthcoming, 2010.

D. Filipović, E. Gourier, and L. Mancini.
Quadratic variance swap models.
EPFL Working Paper, 2011.