Option Pricing and Hedging by Risk Minimization (with Multiple Factors and Transaction Costs)

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Outline

- Motivation: option pricing as risky business
- Pricing and hedging by risk minimization
- Indifference pricing
- Transaction costs
- Examples
- Summary

Option pricing as risky business

- The classical Black-Scholes approach relies on three key assumptions:
- Markets are complete (perfect replication is possible)
- Continuous rebalancing of hedge portfolio
- Zero transaction costs
- The BS theory results in options being redundant instruments with zero risk and unique price
- None of the above assumptions hold in practice
- That is, perfect replication is either impossible or prohibitively costly, option prices are not unique, and residual risk remains

Option pricing in practice

- Practitioners correct for deficiencies of the BS model and its extensions using a combination of various semi-empirical methods and rules of thumb:

- Keeping track of residual risks (gamma, vega, etc.) using the same BS framework: hedges/risk limits

- Various methods to price in expected transaction costs

- Reserves

But:

- It would be desirable to have a framework where risk inherent in options would be recognized from the beginning rather than added as a post-factum overhead

- Don't even need to switch to fat-tail models, the problem arises even in a lognormal setting once the possibility of continuous rebalancing is eliminated

Hedging under market incompleteness

- In the BS model, risk is completely eliminated by a unique choice of the hedging strategy. The option price is the price of the hedge portfolio (price of hedge).

- When the residual risk of a hedged option cannot be completely eliminated, the optimal hedging strategy is the one that minimizes this residual risk, under a chosen metric

- As there are multiple ways to choose the measure for risk (e.g. volatility of total P&L, VaR, expected shortfall etc), the notion of optimal hedging strategy is only meaningful when the risk measure is specified first

- Hedging comes ahead of pricing. The option price can only be determined after the hedge is specified, hence it will be different for different dealers, depending on both the risk metric, optimal hedge strategy, and risk preferences of the dealer

Pricing under market incompleteness

- Even if two institutions agree on the risk metric and optimal hedge strategy, they can differ regarding the option price because of different risk appetites (different premia for bearing a residual risk in the option position)

- In the BS model, the option price is determined as as a price that makes the total P&L at $t = T$ equal zero. In the BS model, the P&L is a delta-function, but in all realistic cases it has a non-trivial distribution

- We can specify the option price in terms of lowest moments of this distribution. In particular, the following ansatz has the right BS limit ($\bar{V}$ is the mean hedge price, and λ is a constant)

$$C = \bar{V} + \lambda \mathbb{E}\left[\left(V - \bar{V} \right)^2 \right]$$

  (But how about pricing higher moments? We'll see later...)

- Option pricing effectively becomes a portfolio problem as the option price now depends on what else is in the portfolio

Hedging by risk minimization

- Assume we short an option and hedge it with the stock. The number of shares we hold between time t and $t+1$ is θ_{t+1} (set at t)

- The wealth (P&L) change from time t to $t+1$ before the next re-hedge is

$$\Delta W_t = W_{t+1} - W_t = -C_{t+1} + \theta_{t+1} S_{t+1} - (-C_t + \theta_{t+1} S_t)$$

- Hedging can be done by minimizing $Var(\Delta W_t)$ for future intervals $[t, t+1]$ as seen today. Alternatively, one can minimize the variance of the terminal wealth W_T

- If price increments are independent, then $Var(\Delta W_T) = \sum_i Var(\Delta W_i)$, i.e. the global and local risk minimizations are equivalent

- References: Bouchaud and Sornette (1994), Schweizer (1995), Hedged Monte Carlo (HMC) method of Bouchaud and Potters (2000). More recent work: Kapoor et al (2008, 2009)

Optimal hedge by multi-period risk minimization

- Work backward in time for $t = T, T-1, \ldots$
- At each time step, minimize the quadratic risk

$$R = \mathbb{E}_t \left[\left(e^{-r\Delta t} C_{t+1} - C_t - \theta_{t+1} \left[e^{-r\Delta t} S_{t+1} - S_t \right] \right)^2 \right]$$

- Expectation is taken wrt measure $\mathbb{P}$ or $\mathbb{Q}$
- Minimization wrt θ_{t+1} and C_t yields (here $\Delta S_t = e^{-r\Delta t} S_{t+1} - S_t$)

$$\theta_{t+1} = \frac{Cov_t \left(e^{-r\Delta t} C_{t+1}, \Delta S_t \right)}{Var_t \left(\Delta S_t \right)}$$

Emphasizes the pricing perspective

and

$$C_t = \mathbb{E}_t \left[e^{-r\Delta t} C_{t+1} \right] - \theta_{t+1} \mathbb{E}_t \left[e^{-r\Delta t} S_{t+1} - S_t \right]$$

Equivalently:

$$C_t = \theta_{t+1} S_t + e^{-r\Delta t} \mathbb{E}_t \left[C_{t+1} - \theta_{t+1} S_{t+1} \right]$$

Emphasizes the hedging perspective

The Black-Scholes limit

- If $\Delta t \to 0$, we can use the first-order Taylor expansion $C_{t+1} = C_t + \frac{\partial C_t}{\partial S_t}\Delta S + \ldots$. Plugging into the formula for θ_{t+1}, we get the BS delta

$$\theta_{t+1} = \lim_{\Delta t \to 0} \frac{Cov_t\left(e^{-r\Delta t}C_{t+1}, \Delta S_t\right)}{Var_t\left(\Delta S_t\right)} = \frac{\partial C_t}{\partial S_t}$$

- The pricing equation produces the risk-neutral valuation formula in the limit $\Delta t \to 0$ as the second term in

$$C_t = E_t\left[e^{-r\Delta t}C_{t+1}\right] - \theta_{t+1}E_t\left[e^{-r\Delta t}S_{t+1} - S_t\right]$$

vanishes:

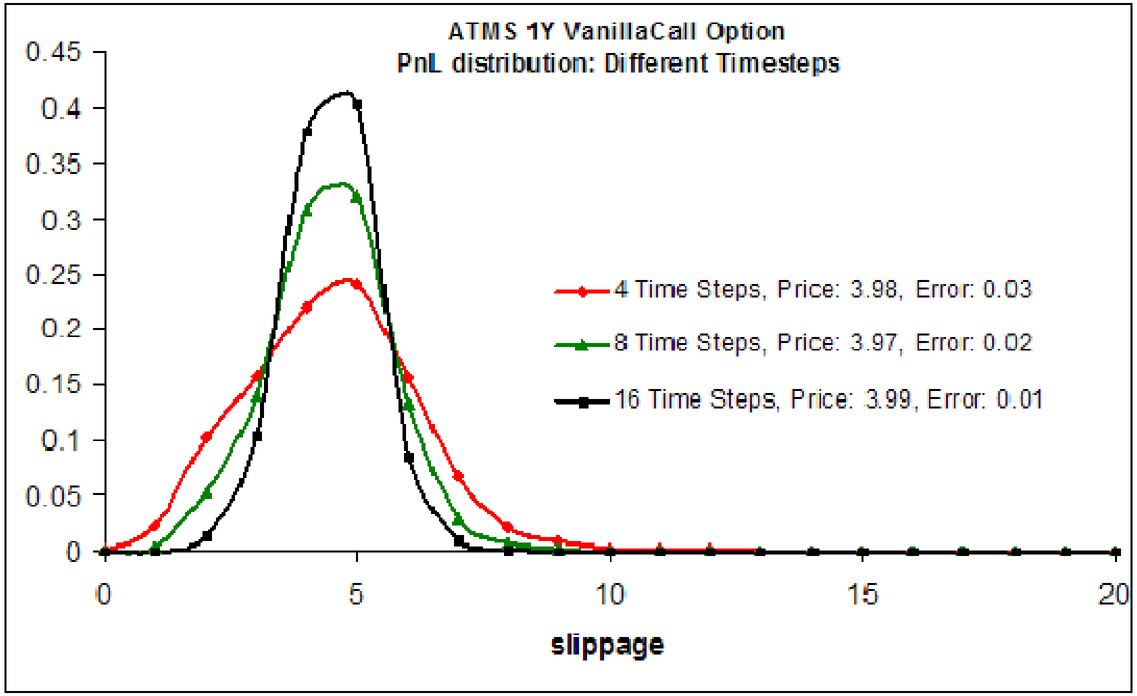
$$\theta_{t+1}S_t\left(e^{(\mu-r)\Delta t} - 1\right) = O(\Delta t)$$

and the first expectation becomes identical to the risk-neutral expectation (as the impact of the drift vanishes in the limit $\Delta t \to 0$)

Illustration: convergence of P&L distribution

- P&L distribution for a hedged European call option with $S_0 = K = 100, \sigma = 10\%, r = 0, T = 1$

- As re-hedging frequency increases, P&L distribution approaches a delta-function.



Risk minimization in a multi-factor setting

- Formulas for the optimal hedge and option price are very similar to their 1D counterparts:

$$\theta_{i,t+1} = \sum_{j=1}^{D} V_{ij,t}^{-1}\, Cov\left(e^{-r\Delta t} C_{t+1}, \Delta S_{j,t} \middle| \mathcal{F}_t\right)$$

$$C_t(S_t) = \mathbb{E}\left[e^{-r\Delta t} C_{t+1} - \sum_{i=1}^{D} \theta_{i,t+1}\left(e^{-r\Delta t} S_{i,t+1} - S_{i,t}\right) \middle| \mathcal{F}_t \right]$$

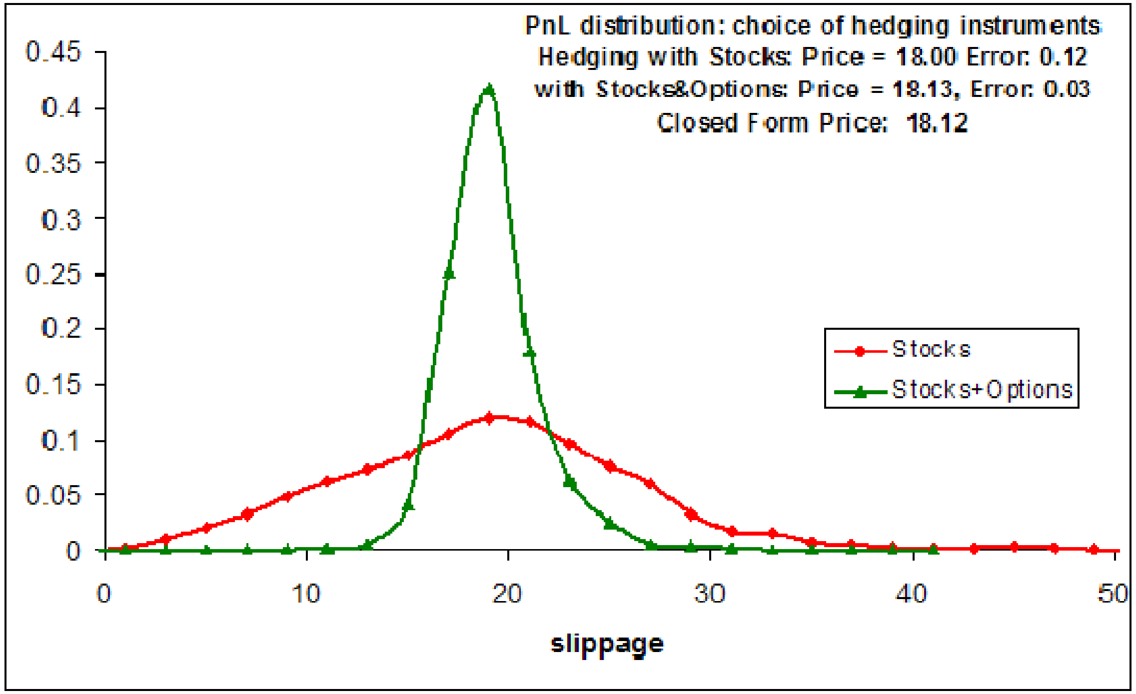
where V^{-1} stands for the inverse of the covariance matrix of increments:

$$V_{ij,t} = \mathbb{E}\left[\Delta S_{i,t} \Delta S_{j,t} | \mathcal{F}_t\right] - \mathbb{E}\left[\Delta S_{i,t} | \mathcal{F}_t\right] \mathbb{E}\left[\Delta S_{j,t} | \mathcal{F}_t\right]$$

Illustration: choice of hedging instruments

- P&L distribution for a hedged exchange option with payoff $\max(S_1 - S_2, 0)$ with with $S_{10} = S_{20} = 100, \sigma_1 = 10\%, \sigma_2 = 50\%, \rho = 50\%, r = 0, T = 1, \Delta T = 0.25$

- We can hedge with the underlyings only, or e.g. with underlyings plus options with ATM strikes of 100



Negative option prices in quadratic risk minimization scheme

- Plugging the expression for θ_{t+1} into the formula for C_t and re-arranging:

$$C_t = e^{-r\Delta t} \mathbb{E}^{\hat{\mathbb{Q}}} \left[C_{t+1} | \mathcal{F}_t \right]$$

where $\hat{\mathbb{Q}}$ is a signed measure with probabilities

$$\hat{q}(S_{t+1}|S_t) = p(S_{t+1}|S_t) \left[1 - \frac{\left(\Delta S_t - \mathbb{E}_t \left[\Delta S_t\right]\right) \mathbb{E}_t \left[\Delta S_t\right]}{Var_t \left(\Delta S_t\right)} \right]$$

where $p(S_{t+1}|S_t)$ are transition probabilities under the real measure $\mathbb{P}$.

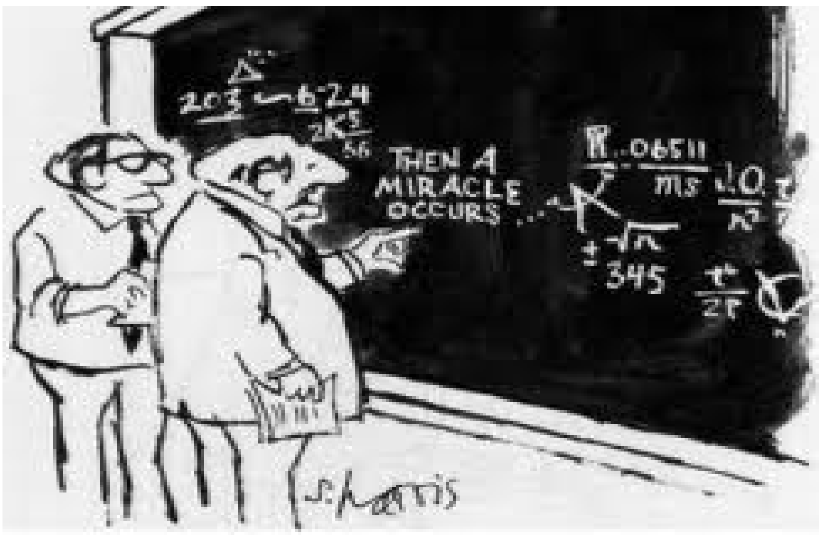
- For large enough moves ΔS_t, this expression can become negative, which exactly means that $\hat{\mathbb{Q}}$ is only a *signed* measure rather than a genuine probability measure.

- Emergence of negative option prices is a well known drawback of the quadratic risk hedging method

- Note that negative prices do not occur if we optimize under the risk-neutral measure where $\mathbb{E}_t \left[\Delta S_t\right] = 0$

How to avoid negative option prices

- Three ways to avoid negative option prices:
- Add a risk premium on top of the fair option price C_t to make the option price positive (Bouchaud et al, Kapoor et al)
- (Almost the same) Introduce risky discounting/funding
- Switch to indifference pricing formalism: this brings $\mathbb{Q}$ back...



"I think you should be more explicit here in step two."

Risky discounting for incremental wealth

- In the BS analysis, ΔW_t is riskless, therefore it should earn a risk-free interest. This results in the Black-Scholes equation

- In the current setting, risk can only be minimized but not completely eliminated. This can be compensated for by adding an extra spread on top of the risk-free rate. The drift condition on ΔW_t becomes

$$E_t\left[W_{t+1} - W_t\right] = \left(e^{(r+s)\Delta t} - 1\right) W_t$$

- This translates into the recursive pricing formula

$$C_t = E_t\left[e^{-(r+s)\Delta t} C_{t+1}\right] - \theta_{t+1} E_t\left[e^{-(r+s)\Delta t} S_{t+1} - S_t\right]$$

- We could choose s by imposing a CAPM-like relation

$$s = \frac{\lambda \sigma_W}{\sqrt{\Delta t}}$$

  where λ is a risk-aversion parameter, and σ_W is the residual P&L volatility. The flip side is that hedging/pricing becomes non-linear and hedging can no longer be done ahead of pricing

Utility-based pricing and hedging with exponential utility

- The indifference price $C_t(B)$ of a claim B is such that writer of the option is indifferent whether to add it to the optimal portfolio:
$$V(x,t) = V^B(x + C_t(B), t)$$
where V and V^B are value functions and x is the initial wealth:
$$\begin{aligned} V(x,t) &= \sup_{\theta\in\Theta} \mathbb{E}^P\left[-e^{-\gamma X^{x,\theta}(T)}\middle| X^{x,\theta}(t) = x, \mathcal{F}_t\right] \\ V^B(x,t) &= \sup_{\theta\in\Theta} \mathbb{E}^P\left[-e^{-\gamma (X^{x,\theta}(T)-B)}\middle| X^{x,\theta}(t) = x, \mathcal{F}_t\right] \end{aligned}$$
Here $\theta = \theta_t$ is the investment strategy

- The duality formula replaces the real-measure maximization wrt strategies Θ by maximization wrt equivalent martingale measures $\mathcal{M}_e$ of $\mathbb{P}$:
$$\sup_{\theta\in\Theta} \mathbb{E}^P\left[-e^{-\gamma (X^{x,\theta}(T)-B)}\middle| \mathcal{F}_t\right] = -e^{-\gamma x} \exp\left(\sup_{M\in\mathcal{M}_e}\left\{-H_t(M|P) + \gamma\mathbb{E}^M[B|\mathcal{F}_t]\right\}\right)$$
where $H_t(M|P)$ stands for the conditional cross entropy of measures M and P.

Indifference pricing in a multi-period setting

- A multi-period recursive version involving only the MEMM $\mathbb{Q}$ was worked out by Lim (2005):

$$C_t = \frac{1}{\gamma}\log \mathbb{E}^{Q}_{(t)}\left[e^{\gamma(C_{t+1}-\theta^T_{t+1}\Delta S_t)}\right] = \theta^T_{t+1}S_t + \frac{1}{\gamma}\log \mathbb{E}^{Q}_{(t)}\left[e^{\gamma(C_{t+1}-\theta^T_{t+1}S_{t+1})}\right]$$

for $t = 0, \ldots, T-1$ and $C_{T,j} = B_j$, where the optimal hedge is defined as follows

$$\theta_{t+1} = \arg\min_{\theta} \mathbb{E}^{Q}_{(t)}\left[e^{\gamma(C_{t+1}-\theta^T\Delta S_t)}\right]$$

(we set $r = 0$ here $\Leftrightarrow$ discounting is implicit)

Certainty equivalent of hedge slippage

- In the limit $\gamma \to 0$ we recover quadratic hedging formulae:

$$\begin{aligned}\lim_{\gamma\to 0} C_t &= \theta^T_{t+1}S_t + \mathbb{E}^{Q}_{(t)}\left[C_{t+1} - \theta^T_{t+1}S_{t+1}\right] \\ &= E_t\left[C_{t+1}\right] - \theta_{t+1}E_t\left[S_{t+1} - S_t\right]\end{aligned}$$

$$\lim_{\gamma\to 0}\theta_{t+1} \equiv \theta^{(0)}_{t+1} = \frac{Cov^Q_t\left[C^{(0)}_{t+1}, \Delta S_t\right]}{Var^Q_t\left[\Delta S_t\right]}$$

Expansions for small risk aversion

- Look for optimal hedge ratios and option prices in terms of expansions in powers of γ:

$$\begin{aligned}\theta_{t+1} &= \theta_{t+1}^{(0)} + \gamma\theta_{t+1}^{(1)} + \gamma^2\theta_{t+1}^{(2)} + \dots \\ C_t &= C_t^{(0)} + \gamma C_t^{(1)} + \gamma^2 C_t^{(2)} + \dots\end{aligned}$$

- Get explicit formulas for hedges and prices. In particular:

$$C_t = \mathbb{E}_t^Q\left[C_{t+1}\right] + \frac{1}{2}\gamma\mathbb{E}_t^Q\left[\left(\tilde{C}_{t+1} - \bar{\tilde{C}}_{t+1}\right)^2\right] + \frac{1}{3!}\gamma^2\mathbb{E}_t^Q\left[\left(\tilde{C}_{t+1} - \bar{\tilde{C}}_{t+1}\right)^3\right] + O(\gamma^2)$$

where $\tilde{C}_{t+1}$ is the hedge slippage

$$\tilde{C}_{t+1} = C_{t+1} - \theta_{t+1}^{;T}\Delta S_t \ , \ \bar{\tilde{C}}_{t+1} = \mathbb{E}_t^Q\left[\tilde{C}_{t+1}\right]$$

- This is an expansion of the option price in terms of moments of the slippage distribution. All moments are priced in terms of the same parameter γ. Option prices are non-negative by construction.

Transaction costs

- Assuming the proportional transaction costs model, the wealth balance becomes

$$W_{t+1} - W_t = C_t - C_{t+1} + \theta_{t+1}\left(S_{t+1} - S_t\right) - \nu S_t |\theta_{t+1} - \theta_t|$$

  This breaks the backward induction scheme as at time t the previous optimal hedge θ_t is fixed but unknown

- One solution (Bouchaud and Potters) is to assume the time step Δt is small and expand

$$\theta_{t+1} - \theta_t \simeq \frac{\partial \theta_{t+1}(S_t)}{\partial S_t}(S_t - S_{t-1})$$

- Because now we have to estimate both θ_{t+1} and $\partial\theta_{t+1}/\partial S_t$, BP use a parametric form of function $\theta_{t+1}(x)$ instead of a basis expansion, and optimize over parameters

- Two challenges with this approach: 1) amounts to a non-linear optimization at each step, and 2) generalization to abritrary payoffs and/or multiple dimensions is unclear

A self-consistent scheme for transaction costs

- Instead of assuming small time steps, we can derive a new equation for the option price imposing self-consistency
- Work to the leading order of the small γ-expansion
- Use the fact that the optimal hedge at the previous step is given by the same formula as before:

$$\theta_t = \frac{Cov_{t-1}\left(e^{-r\Delta t}C_t, \Delta S_{t-1}\right)}{Var_{t-1}\left(\Delta S_{t-1}\right)}$$

- Plug this into the recursive relation for the option price:

$$\begin{aligned} C_t &= E_t\left[e^{-r\Delta t}C_{t+1}\right] - \theta_{t+1}E_t\left[\Delta S_t\right] \\ &+ \nu S_t \left| \theta_{t+1}(S_t) - \frac{Cov_{t-1}\left(e^{-r\Delta t}C_t, \Delta S_{t-1}\right)}{Var_{t-1}\left(\Delta S_{t-1}\right)} \right| \end{aligned}$$

- This is a non-linear integral equation, can be solved iteratively

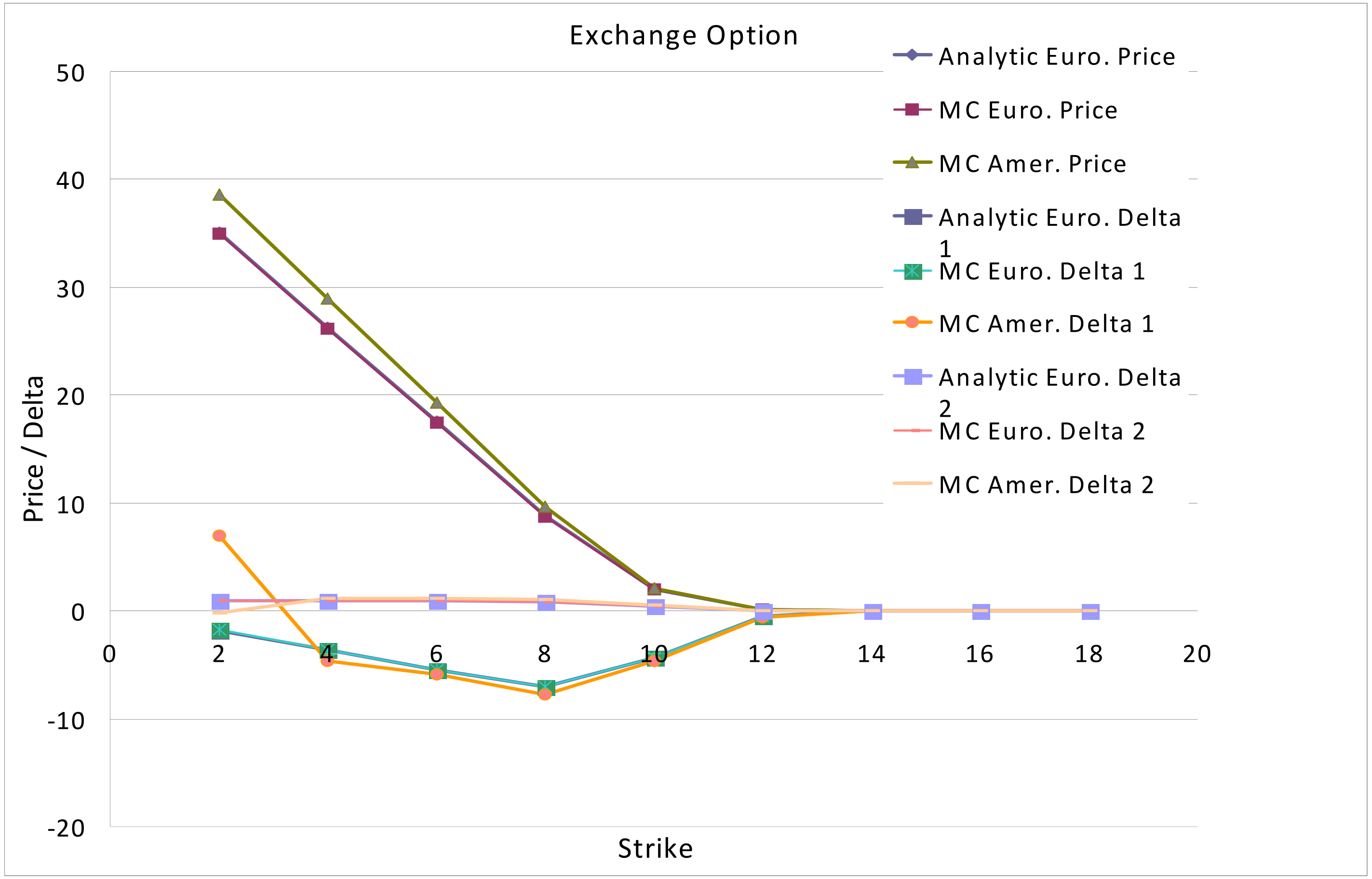
Example: pricing of exchange option with transaction costs

- Option to exchange two assets, the payoff $(S_{1,T} - KS_{2,T})^{+}$.
- Choose $S_{1,0} = S_{2,0} = 36.0$, $K = 1$, $r = 6\%, \sigma_1 = 35\%, \sigma_2 = 50\%$, $\rho = 80\%$
- Use proportional costs model $\nu|\theta_{t+1} - \theta_t|$. For zero transaction costs, can compare with the exact answer (Margrabe formula)

	Exact	**MC** $(\nu = 0.0)$	**MC** $(\nu = 0.01)$
Price	4.3512	4.3389	4.7909
Delta 1	0.5604	0.5639	0.5854
Delta 2	- 0.4396	-0.4426	-0.4481

Example: American vs European exchange option

- $S_{1,0} = 48.25, S_{2,0} = 4.825, \sigma_1 = 30.25\%, \sigma_2 = 25.5\%, \rho = 93.0\%, T = 1Y, r = 0.25\%$



Summary

- Multi-factor Monte Carlo framework
- Can be used to price and hedge options of arbitrary complexity (American, path-dependent, ...)
- Largely agnostic about the underlying model
- Incorporates transaction costs
- Can be used to optimize hedging strategies under transaction costs
- Calibration through market-implied risk aversion parameters

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