Dealing with Incomplete Market
A Pricing Exercise with CDO Tranche Options

Yadong Li and Ariye Shater

Quant Analytics - Credit
BARCLAYS CAPITAL

Incomplete Market

- Standard method of pricing derivatives: $\frac{V_0}{A_0} = \mathbb{E}^A[\frac{V_T}{A_T}]$
  - Based on the arguments of no-arbitrage and dynamic replication
  - Dynamic Replication cannot be applied to illiquid (incomplete) markets: loans, private equity, CDOs etc
- An alternative view: derivative pricing as an interpolation method
  - Find a price that is the most consistent with existing (incomplete) market information
  - Not replication based, can be applied to illiquid or incomplete markets
  - Need to define the "interpolation" and "consistency" to be useful

Simple Example: European Option

Assume there are the following securities in the market:

- A risk-free deposit that pays e^{rT} at T
- An illiquid asset, whose price is S_0 today

Option price can be derived without the dynamic replication:

- π_i is the price of the state security that pays \$1 if and only if $S(T) = S_i$, note that π_i themselves are not observable.
- From the risk free deposit: $1 = \sum_i \pi_i e^{rT}$, which makes $q_i = \pi_i e^{rT}$ a probability measure.
- From the underlying asset: $S_0 = \sum_i \pi_i S(T) = e^{-rT} \sum_i q_i S(T)$

Black-Scholes without Dynamic Replication

European call option: $V_0 = \sum_i \pi_i (S(T) - K)^+ = e^{-rT} \sum_i q_i (S(T) - K)^+$.

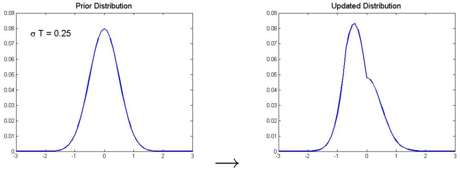
An assumption (or a prior view, a leap-of-faith):

- The log asset return ($\log \frac{S(T)}{S_0}$) is normally distributed with variance of $\sigma^2 T$ in the probability measure of q_i.

The Black-Scholes formula naturally follows:

- The expectation (drift) of the log asset return can be determined by: $S_0 = e^{-rT} \sum_i q_i S(T)$
- Dynamic replication argument is not used: we did not even specify the dynamics of the underlying asset process S_t.

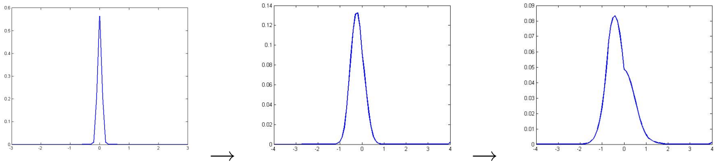
Pricing in an Incomplete Market



- Start with certain prior views on the risk neutral distribution
- Update the view by the observable market prices
  - Many possible solutions → Incomplete Market
  - A unique solution → Complete Market
  - No feasible solution → Inconsistent (Arbitrageable) Market
- Use the updated distribution to price other (non-traded) instruments:
  - A price range can be identified by perturbing the price till a distribution solution can no longer be found

Pricing Uncertainties over Time

Evolvement of the implied distribution over time:



Pricing uncertainty for path dependent instruments (eg, Asian options):

- Different ways of connecting the same distributions over time will result in different prices
- A price range could be obtained by numerical optimizations over all possible Markov chains.

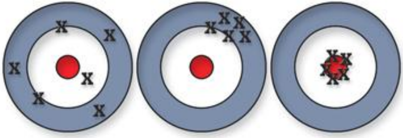
CDO Tranche Options

CDO Tranche Options are relevant because:

- Counterparty risk
- Gap Risk
- Levered Super Senior
- Liquidation Risk

An interesting problem of incomplete market and illiquid instruments

- Exact prices are impossible to obtain
- A pricing range could be a more useful alternative



Objectives

- A generic CDO tranche call option:

$$C = \mathbb{E}[d(0,t)\mathbf{1}_{\tau=t}\max(V_t - K, 0)]$$

- Direct valuation is difficult:
  - Complicated instrument, incomplete market
  - Require a full dynamic model with advanced Monte Carlo simulation
  - Low confidence in the resulting prices due to strong model assumptions
- Instead, we attempt to derive the valuation bounds from the observed tranche prices.
  - Straight forward pricing methodology
  - High confidence in the pricing bounds

Upper Bound

$$
\begin{aligned}
C &= \mathbb{E}[d(0,t)\mathbf{1}_{\tau=t}\max(V_t - K, 0)] \\
&= \mathbb{E}[d(0,t)\mathbf{1}_{\tau=t}\max(\mathbb{E}[\sum_{t_i>t} d(t,t_i)c_i|\mathcal{F}_t] - K, 0)] && : \text{expand } V_t \\
&\leq \mathbb{E}[d(0,t)\mathbf{1}_{\tau=t}\mathbb{E}[\max(\sum_{t_i>t} d(t,t_i)c_i - K, 0)|\mathcal{F}_t]] && : \text{Jensen's inequality} \\
&= \mathbb{E}[\mathbb{E}[d(0,t)\mathbf{1}_{\tau=t}\max(\sum_{t_i>t} d(t,t_i)c_i - K, 0)|\mathcal{F}_t]] && : \mathbf{1}_{\tau=t} \text{ is adapted to } \mathcal{F}_t \\
&= \mathbb{E}[d(0,t)\mathbf{1}_{\tau=t}\max(\sum_{t_i>t} d(t,t_i)c_i - K, 0)] && : \text{iterative expectation}
\end{aligned}
$$

- Corresponds to perfect fore-sight of future cashflows.
- The upper bound can be computed using simple Monte-Carlo simulation

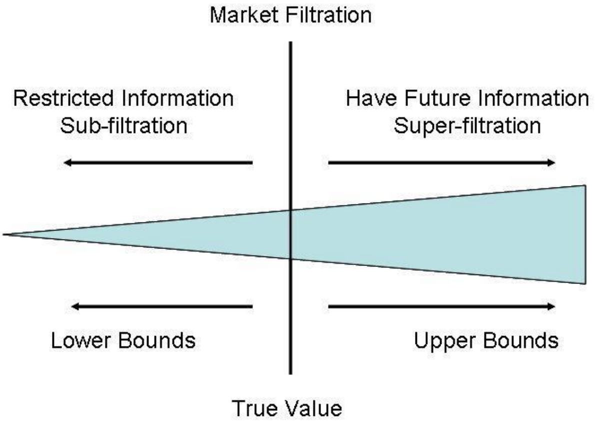
Lower Bound

Assume $\mathcal{Y}_t$ is a sub-filtration of the market filtration $\mathcal{F}_t$:

$$
\begin{aligned}
C &= \mathbb{E}[d(0,t)\mathbf{1}_{\tau=t}\max(V_t - K, 0)] \\
&= \mathbb{E}[\mathbb{E}[d(0,t)\mathbf{1}_{\tau=t}\max(V_t - K, 0)|\mathcal{Y}_t]] && : \text{iterative expectation} \\
&= \mathbb{E}[d(0,t)\mathbf{1}_{\tau=t}\mathbb{E}[\max(V_t - K, 0)|\mathcal{Y}_t]] && : \mathbf{1}_{\tau=t} \text{ is adapted to } \mathcal{Y}_t \\
&\geq \mathbb{E}[d(0,t)\mathbf{1}_{\tau=t}\max(\mathbb{E}[V_t - K|\mathcal{Y}_t], 0)] && : \text{Jensen's inequality} \\
&= \mathbb{E}[d(0,t)\mathbf{1}_{\tau=t}\max(\mathbb{E}[V_t|\mathcal{Y}_t] - K, 0)] && : \text{K is constant}
\end{aligned}
$$

- Corresponds to restricted information.
- The lower bound can be computed using least square Monte Carlo.

Filtration and Option Value



The option value increases with more information to make the exercise decision.

Tranche Loss Option

We consider an option on a tranche's expected loss: $\mathbb{E}[L_T(A,D)|\mathcal{F}_t]$

$$C = \mathbb{E}[\max(\mathbb{E}[L_T(A,D)|\mathcal{F}_t] - K, 0)]$$

where

$$L_T(A,D) = \min(\max(L_T - A, 0), D - A)$$

- The main driver of the tranche PV is its expected loss at maturity.
- Option on tranche loss is a good proxy to the actual CDO tranche option whose underlying is the tranche PV.
- Easier to treat analytically, discount factors can be dropped without the loss of generality

Upper Bounds of Tranche Loss Option

$$
\begin{aligned}
C &= \mathbb{E}[\max(\mathbb{E}[L_T(A,D)|\mathcal{F}_t] - K, 0)] \\
&\leq \mathbb{E}[\max(L_T(A,D) - K, 0)] \\
&= \mathbb{E}[\max(\min(\max(L_T - A, 0), D - A) - K, 0)))] \\
&= \mathbb{E}[\min(\max(L_T - (A+K), 0), D - (A+K))] \\
&= \mathbb{E}[L_T(A+K, D)]
\end{aligned}
$$

- Model Independent, can be obtained from a base correlation model.

Lower Bound of Tranche Loss Option

- Model and filtration dependent

$$C \geq \mathbb{E}[\max(\mathbb{E}[L_T(A, D)|\mathcal{Y}_t] - K, 0)]$$

- Naive Lower Bound

$$\begin{aligned} C &= \mathbb{E}[\max(\mathbb{E}[L_T(A, D)|\mathcal{F}_t] - K, 0)] \\ &\geq \max(\mathbb{E}[\mathbb{E}[L_T(A, D)|\mathcal{F}_t] - K], 0) \\ &= \max(\mathbb{E}[L_T(A, D)] - K, 0) \end{aligned}$$

CDO Models for the Lower Bounds

- Top-down model
  - Only models the aggregated portfolio loss, ignores single name information
  - Portfolio loss distributions are calibrated to index tranche prices
  - Different Markov Chains on the portfolio loss can be built
- Bottom-up model
  - Common market factor is modeled as an increasing process X_t
  - Single name defaults are independent conditioned on X_t
  - The distributions of X_t are calibrated to index tranches
  - Different Markov Chains on X_t can be built across time
  - The model is published in Risk, Jun 2010

Filtrations for the Lower Bounds

Lower bounds depend on the choice of the sub-filtration $\mathcal{Y}_t$:

- $\mathcal{L}_t$: generated by the portfolio loss only (the top-down model)
- $\mathcal{S}_t$: generated by the common factor X_t (the bottom-up model)

Different ways of building Markov chains for portfolio loss loss or X_t:

- Co-monotonic: strongest inter-temporal dependence
- Maximum Entropy: weakest inter-temporal dependence
- LLB: the Markov Chain that minimizes the LB of a given tranche

Tranche Market Inputs

CDX-IG9 on COB Jul 21, 2009:

Table: CDX-IG9 Expected Tranche Loss

Tranches	3Y	5Y	7Y	10Y
0-3%	54.12%	80.19%	86.76%	91.12%
3-7%	17.03%	42.64%	55.16%	66.18%
7-10%	5.36%	20.09%	33.98%	48.18%
10-15%	1.35%	8.17%	15.82%	23.34%
15-30%	0.76%	2.29%	4.81%	7.95%
30-60%	0.49%	1.62%	3.40%	5.31%
60-100%	0.02%	0.42%	0.95%	1.54%

Model Independent Upper Bounds

3Y to 5Y Tranche Loss Option:

Table: Upper Bounds of 3Y-5Y Tranche Loss Option

CDX-IG9 Tranches	Upper Bounds		
	ITM	ATM	OTM
0-3%	43.33%	12.73%	0.00%
3-7%	30.71%	20.57%	4.44%
7-10%	17.35%	14.79%	10.21%
10-15%	7.63%	7.11%	6.14%
15-30%	2.24%	2.19%	2.10%
30-60%	1.61%	1.59%	1.56%
60-100%	0.42%	0.41%	0.41%

Lower Bounds from Top-down Models

Table: Lower Bounds of 3Y-5Y Option from $\mathcal{L}_t$

CDX-IG9	ITM Lower Bounds			ATM Lower Bounds			OTM Lower Bounds		
Tranches	Co-mo	Max-E	LLB	Co-mo	Max-E	LLB	Co-mo	Max-E	LLB
0-3%	43.09%	39.97%	39.97%	12.50%	7.91%	6.40%	0.00%	0.00%	0.00%
3-7%	30.32%	22.34%	21.40%	19.75%	11.74%	7.55%	4.18%	1.82%	1.54%
7-10%	17.01%	11.05%	9.82%	14.39%	7.53%	3.67%	9.35%	3.99%	2.28%
10-15%	7.63%	4.88%	4.20%	7.13%	3.64%	1.07%	6.28%	2.38%	0.94%
15-30%	2.19%	1.48%	1.12%	2.11%	1.24%	0.68%	2.02%	1.00%	0.66%
30-60%	1.60%	1.11%	0.81%	1.57%	0.97%	0.41%	1.52%	0.83%	0.40%
60-100%	0.48%	0.35%	0.28%	0.48%	0.30%	0.07%	0.47%	0.26%	0.03%

- Even the OTM tranche options have non-zero minimum value, more precise than the naive lower bounds
- The LLB is the lowest possible lower bound among all possible ways of connecting loss distributions over time
- The lower bounds from $\mathcal{L}_t$ are very far from the upper bounds

Lower Bounds from the Bottom-up Model

Table: Lower Bounds of 3Y-5Y Option from $\mathcal{S}_t$

CDX-IG9 Tranches	ITM Lower Bounds			ATM Lower Bounds			OTM Lower Bounds		
	Co-mo	Max-E	LLB	Co-mo	Max-E	LLB	Co-mo	Max-E	LLB
0-3%	41.07%	40.93%	40.05%	10.47%	9.53%	8.46%	0.00%	0.00%	0.00%
3-7%	29.15%	26.63%	22.07%	19.13%	15.45%	11.96%	3.19%	2.57%	2.47%
7-10%	16.26%	14.48%	13.12%	13.20%	11.74%	11.20%	8.12%	7.85%	7.58%
10-15%	7.32%	7.26%	6.42%	6.70%	6.64%	5.79%	5.54%	5.47%	4.85%
15-30%	2.19%	2.14%	1.23%	2.12%	2.04%	1.16%	1.99%	1.90%	1.09%
30-60%	1.60%	1.53%	0.82%	1.57%	1.47%	0.65%	1.52%	1.37%	0.73%
60-100%	0.41%	0.39%	0.20%	0.41%	0.38%	0.12%	0.41%	0.35%	0.14%

- The sub-filtration $\mathcal{S}_t$ includes single name information and the systemic factor X_t
- The lower bounds from $\mathcal{S}_t$ are much closer to the upper bounds

Systemic vs Idiosyncratic Dynamics

Table: Lower Bounds from $\mathcal{U}_t$

CDX-IG9 Tranches	Lower Bounds		
	ITM	ATM	OTM
0-3%	41.14%	10.73%	0.00%
3-7%	29.47%	19.15%	3.28%
7-10%	16.37%	13.38%	8.18%
10-15%	7.36%	6.70%	5.69%
15-30%	2.20%	2.13%	2.04%
30-60%	1.60%	1.59%	1.56%
60-100%	0.42%	0.41%	0.41%

- $\mathcal{U}_t$: filtration with $\mathcal{S}_t$ and X_T (perfect foresight of systemic factor)
- LB from $\mathcal{U}_t$ is very close to the UB: idiosyncratic contribution is very small
- LB from Co-monotonic Markov chain is very close to LB from $\mathcal{U}_t$
- Static copula will over-price tranche options

Bounds of Single Default Event Trigger

Upper Bound:

$$
\begin{aligned}
C^U &= \mathbb{E}[\mathbf{1}_{\tau<t} \max(\sum_{t_i>t} d(0,t_i)c_i - K, 0)] \\
&= \mathbb{E}[\mathbb{E}[\mathbf{1}_{\tau<t} \max(\sum_{t_i>t} d(0,t_i)c_i - K, 0)|X_t]] && : \text{Iterative expectation} \\
&= \mathbb{E}[\mathbb{E}[\mathbf{1}_{\tau<t}|X_t]\mathbb{E}[\max(\sum_{t_i>t} d(0,t_i)c_i - K, 0)|X_t]] && : \text{Conditional Independence} \\
&= \mathbb{E}[q(X_t,t)\mathbb{E}[\max(\sum_{t_i>t} d(0,t_i)c_i - K, 0)|X_t]]
\end{aligned}
$$

Lower Bound:

$$
C^L = \mathbb{E}[q(X_t,t)\mathbb{E}[\max(\mathbb{E}[\sum_{t_i>t} d(0,t_i)c_i|\mathcal{Y}_t] - K, 0)|X_t]]
$$

Counterparty Risk

Table: 3Y-5Y ATM Options with Single Default Event Trigger

CDX-IG9 Tranches	Independent		Less Correlated		More Correlated	
	LB	UB	LB	UB	LB	UB
0-3%	0.48%	0.64%	0.69%	0.73%	0.86%	0.87%
3-7%	0.77%	1.03%	1.57%	1.69%	2.14%	2.22%
7-10%	0.59%	0.74%	1.69%	1.75%	2.41%	2.46%
10-15%	0.33%	0.36%	1.37%	1.37%	1.89%	1.90%
15-30%	0.10%	0.11%	0.89%	0.90%	1.09%	1.11%
30-60%	0.07%	0.08%	0.80%	0.81%	0.94%	0.96%
60-100%	0.02%	0.02%	0.22%	0.22%	0.25%	0.26%

- Counterparty risk is a classic example.
- Trigger Default Prob = 5%
- Sensitive to correlation between the common factor and the trigger credit
- Bounds are very narrow.

More Generic Triggers

Do not have analytical solutions, requiring Monte Carlo simulation

Table: Price Bounds of 3Y-5Y Option to Call Tranche

CDX-IG9	$\alpha = 4\%$		$\alpha = 8\%$		$\alpha = 12\%$	
Tranches	LB	UB	LB	UB	LB	UB
15-30%	2.02%	2.22%	1.85%	1.95%	1.08%	1.13%
30-60%	1.45%	1.62%	1.36%	1.47%	0.95%	1.00%
60-100%	0.38%	0.42%	0.36%	0.38%	0.26%	0.27%

- Option to buy tranche protection at original expected loss if portfolio loss reaches a pre-determined level of α

Conclusion

The advantages of using valuation bounds for tranche options

- High confidence in the bounds, minimal model assumptions.
- Computationally efficient.
- Can be effective for managing embedded tranche options.
- The methodology can be used in other asset classes.