BNP PARIBAS | CORPORATE & INVESTMENT BANKING

Multiple Curve Construction for Interest Rate Derivatives

Overcoming the Challenges of Modelling Interdependent Curve Systems

Igor Smirnov

BNP Paribas

Global Derivatives & Risk Management 2011

Overview

The views presented here are those of the author and not of BNP Paribas. All mistakes & omissions are also my own.

- From textbooks to reality - the need for multiple related curves
- Distinguishing between instrument & curve dependence
- Pricing & risk consequences of interpolation choices in multi-curve systems
- Calibration: Speed and accuracy
- Applications to funding modeling

From Textbooks to Reality

Classical Interest Rate curve treatment in finance postulates a single, arbitrage-free curve

- Relies on fungibility of all market instruments, complete markets
- 1Y rate given by 12M loan, compound 6M or 3M loans equivalent
- Neglects liquidity, credit, market accessibility & other factors.

Not satisfactory even before crisis*

? Basis swaps

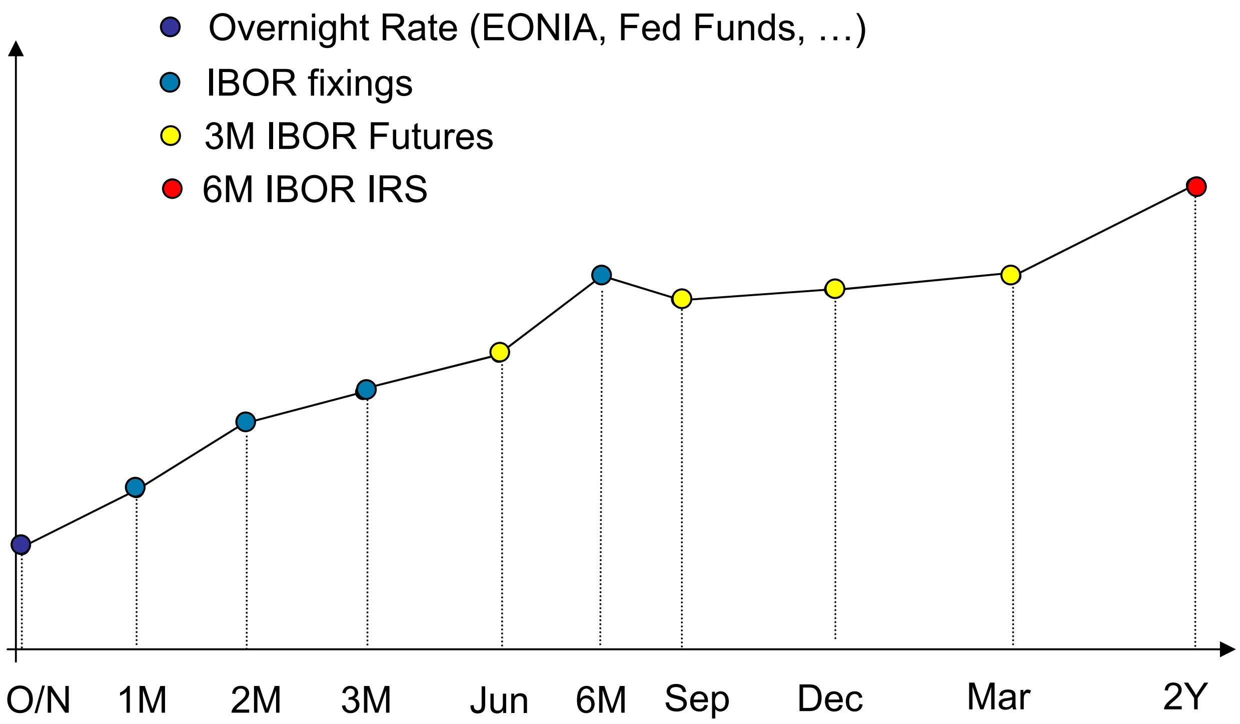
? FX swaps / deposits

? Collateralized vs. uncollateralized cost of funds**

? … etc

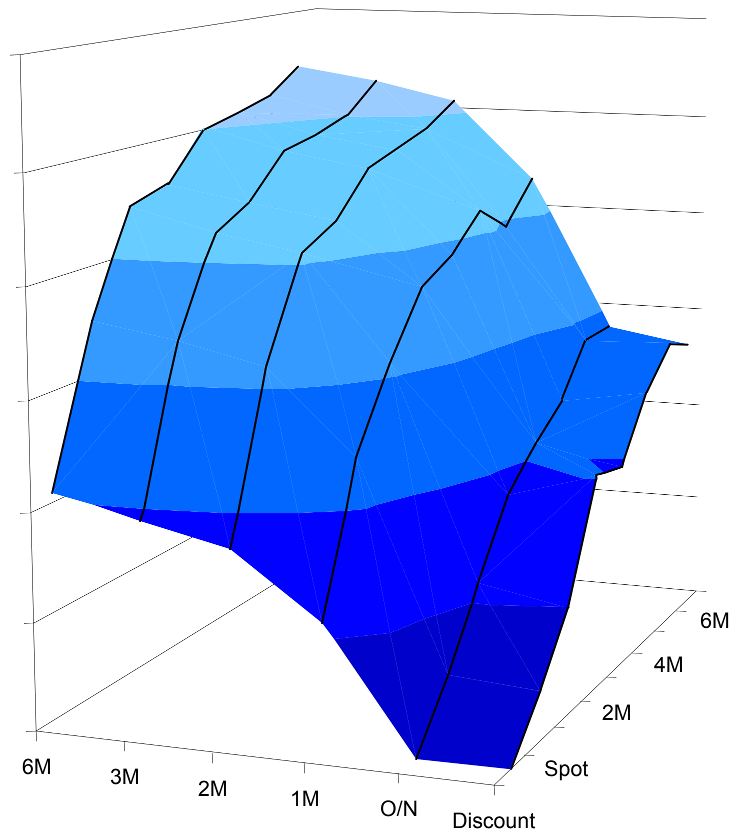
** For example see ref. 1, though earlier papers also were in public domain*
*** see ref. 2*

Classical Curve



- 4 types of instruments
- 6 flavours of liquidity

Separating Underlying Assets



- Curves split by underlying class
- 6 flavours of liquidity = 6 curves

Instrument Dependencies

Every market instrument is sensitive to one or more curves –

- 6M / Fixed IRS sensitive to 6M fixing, discount curves
- 3M IBOR Future sensitive to 3M fixing curve
- 3M/6M basis swap sensitive to 3M, 6M fixing as well as discount curves
- etc..

as well as other risk factors.

Volatility Dependencies

Many of the instruments used for curve construction may have an element of volatility dependence, for example

- 3M IBOR futures, due to margining
- Cross-currency basis swaps, due to MTM resets
- Collateralised products, due to margining

So it is necessary to decide whether this dependence is

- Expressed directly, requiring simultaneous calibration of volatility models and interest rate curves
- Expressed indirectly - via “convexity adjustments” kept constant (or approximated using simple models) during interest rate calibration. Much faster but potentially less accurate
- Ignored (e.g. by LCH for futures!)

This choice also dictates different respective risk interpretation.

Curve Dependencies

A dependence between two curves in a multi-curve system may arise from a number of factors:

- Direct model implied dependence
  - e.g. 6M curve modelled as term-structure of spread to 3M
- Direct instrument implied dependence
  - e.g. 6M curve calibrated to 3M/6M basis swaps so also sensitive to 3M and discount curves
- Indirect instrument or model implied dependence
  - either of the above inherited via intermediate steps
  - e.g. 6M curve as above also dependent on the overnight curve, as discount curve is modelled as spread to overnight

So resulting dependence graph can become very complex indeed.

Interpolations

Interpolation choices in a multi-curve system have a number of indirect consequences. Before we consider these more deeply let's look at the two broad classes of interpolations:

- **Outright** interpolations are those where the curve structure calibrated to market instruments can be written without reference to other curves.
- **Spread** interpolations are those where the calibrated curve structure is directly dependent on another curve

There are a large number of examples of each of those types.

Outright Interpolations

The more frequently encountered group, examples include

- Linear in zero coupon rate $D(t) = \exp[-R(t)\cdot t]$
- Linear in Libor forwards $D(t) = \left[\prod (1 + L(t)\cdot \delta t)\right]^{-1}$
- Cubic spline of zero coupon rates
- Polynomial in instantaneous rates $D(t) = \exp\left[-\int_{x=0}^{t} r(x)\,dx\right]$
- Vasicek – Fong
- Step in overnight rates, explicit basis splines, etc…

- All share important property: only instrument related dependence on other curves, no model related dependence
- Also means related curve structure must be defined consistently using same control points, or risk systemic differences.

Spread Interpolations

Where one curve is expected to closely follow another, modelling spreads between the two can be a natural choice.

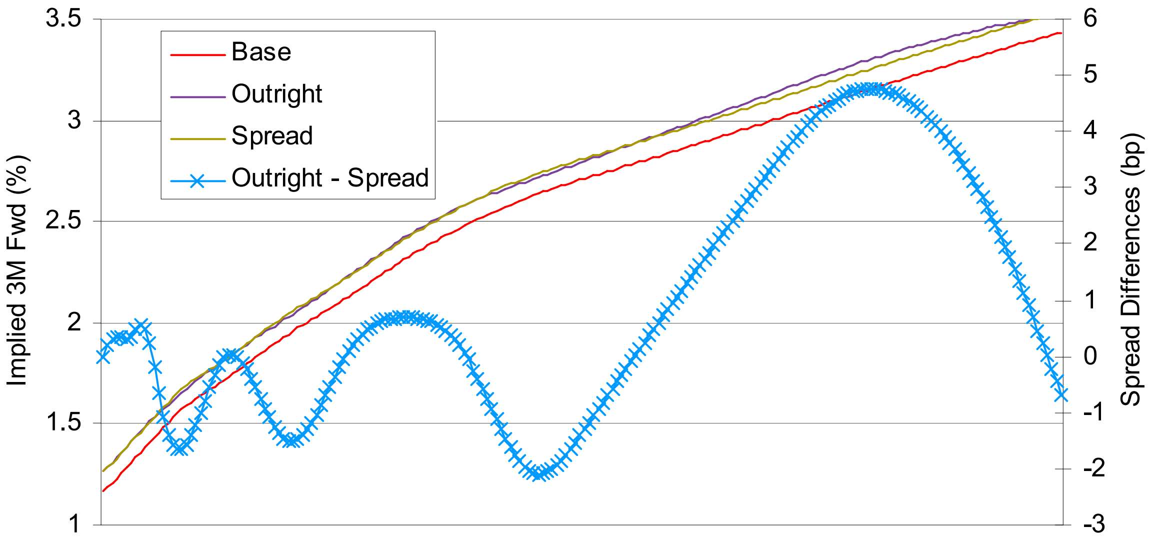
- Linear in Libor forward spreads $D(t) = \left[\prod \left(1 + \left(L_{BASE}(t) + s(t)\right) \cdot \delta t\right)\right]^{-1}$
- Polynomial in instantaneous spreads $D(t) = \exp\left[-\int_{x=0}^{t} s(x)\,dx\right] \cdot D_{BASE}(t)$
- Multiplicative inst. spreads $D(t) = \exp\left[-\int_{x=0}^{t} r_{BASE}(x) \cdot s(x)\,dx\right]$
- explicit basis splines of spreads, etc…
- Can be based on any outright method $D(t) = f(t) \cdot D_{BASE}(t)$

- Model related as well as instrument related dependence on other curves
- Curve structure need not be defined in a homogenous manner

Spread Vs Outright

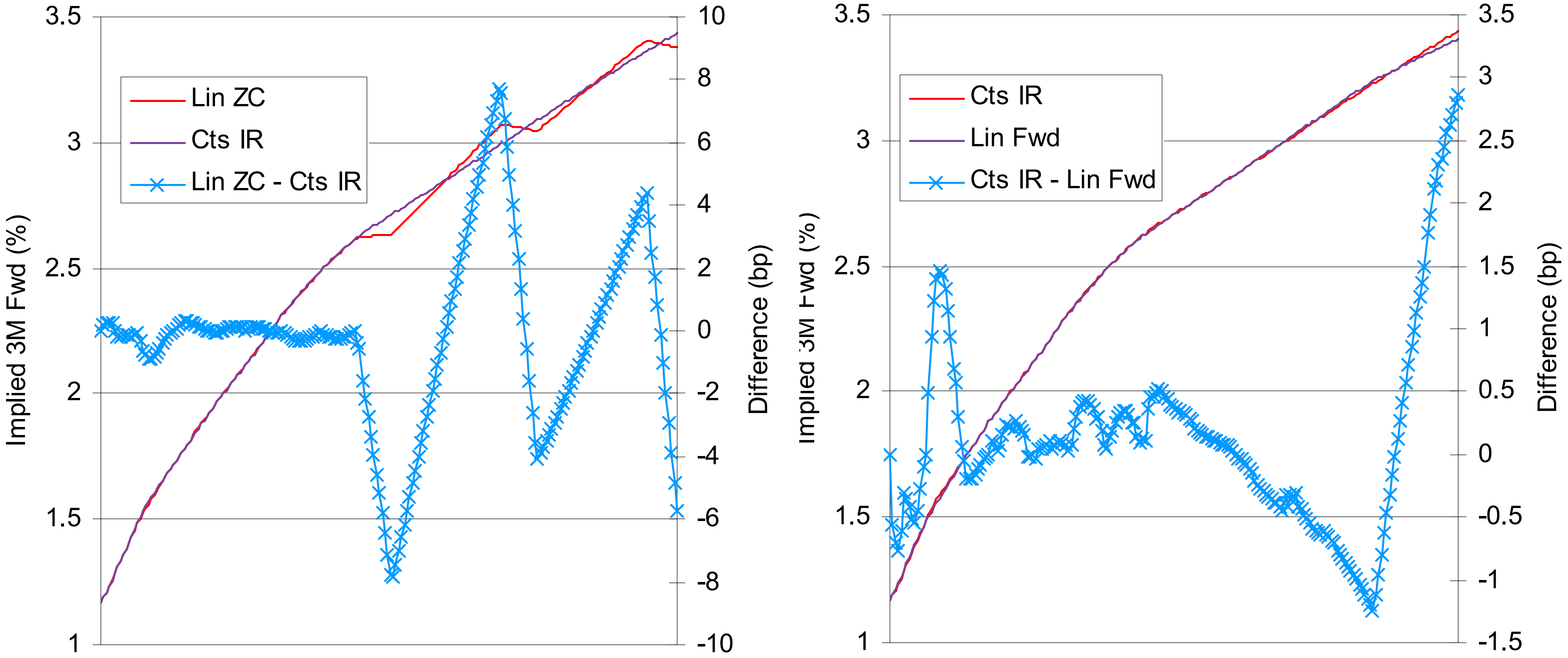
Let's consider a simple example – derived curve defined using basis spread instruments, constant 10bp market spread inputs

- With spread curve tenors matching base curve, no difference between spread & outright interpolations
- Significant differences when we remove every other basis swap



Pricing Impact

- Usually fit all control points (reprice calibration inputs)
- Intermediate behaviour defined by interpolation method
  - Pricing importance depends on control point density, convexity
  - Impact can be large, especially in presence of discontinuities



Risk Impact

- Usually more important than pricing
- Methods yielding materially the same pricing may significantly differ in implied risk dynamics
  - Particularly when curves are less convex
  - Method choice should be benchmarked to desired risk properties
    - this includes risk stability over time!
- Broadly speaking a question of tradeoff

Smoothness ⟷ Localised Risk

  - actually a lot of subtlety possible in optimising for both
  - amongst other things control point positioning can be critical*
  - many ways to obtain visually smooth Libor forwards!

* *see appendix*

Calibration

How do we calibrate our system of curves?

In general

- Identify most liquid instruments
- Establish enough constraints along each term structure
  - Optimal choice depends on curve models chosen, risk preferences, available data, …
- Fit models to selected constraints (market instruments)
  - Usually exact fit, to allow perturbative risk
  - Very high dimensional non-linear root finding problem

Without further rationalisation, impractical for real-time pricing/risk!

Practical Calibration

A range of optimization steps are necessary to achieve speed and accuracy needed when calibrating complex markets.

- Identify smallest independent curve subgroups
  - Breaks down high-dimensionality simultaneous root finding into sequence of smaller problems solved serially
  - Much faster as for most methods algorithmic complexity increases super-linearly in number of dimensions
  - Curve/instrument dependence should also be reduced where possible at market design stage

Practical Calibration II

- Identify appropriate fitting procedures
  - "Bootstrap" (iterative 1d) methods still possible if -
    - Interpolation method chosen has directional dependence
    - No factors introduce cross-dependence on given curve
    - Fitting a single curve (or fitting a set of curves iteratively)
  - Iterative vs. Simultaneous fitting of curve sets
    - Simultaneous fitting is $\sum_j n_j$ – dimensional problem
    - Guaranteed to converge for globally convergent algos
    - Iterative is iterated sequence of n_j –dim problems
    - Not guaranteed to converge but can be faster
  - Choose best performing nonlinear root finding algorithms

Practical Calibration III

- Efficiently implement methods & calibration processes
  - Mathematical optimisation e.g. $D(t) = e^{-\sum \int_{t_i}^{t_{i+1}} r_{BASE}(x) \cdot s_i \, dx} = \prod \left[\frac{D_{BASE}(t_{i+1})}{D_{BASE}(t_i)} \right]^{s_i}$
  - Caching
    - Discount factors / model parameters
    - Schedules, etc…
  - Efficient implementation technology
  - Partial recalibration only during risk

Taken together, these approaches allow even complex interdependent multi-curve systems to be calibrated very quickly!

Modelling Funding

Previous discussion applies to challenges known for a number of years. Recent advances in funding modelling result in even more complex, interdependent curve systems

- For trades under each bilateral CSA
  - Funding curve for each type of admissible collateral
  - Assets spanning cash, bonds etc. in different currencies
- Uncollateralized funding curves for uncollateralized trades, asymmetric CSA's, no-collateral windows, etc…
- For clearing house trades, curves reflecting margining terms

So potentially multiplying the number of curves needed …

Modelling Funding II

Earlier we talked about simultaneously fitting curve systems implicitly restricting ourselves to curves in the same currency.

- Now need curves implied by collateral in other currencies
    - Can try to use FX markets to project into target ccy
        - Chicken and egg – ccy1 curves / instruments cross dependent on ccy2
    - Why not consider all curves / currencies simultaneously?
        - Same calibration principles apply
        - Total dimensionality lower than splitting by currency
        - Lazy evaluation can keep performance manageable
- A number of issues arise, including
    - What funding curves to use for calibration instruments?
    - Non-cash collateral & repo terms, collateralization of FX, …

Modelling Funding III

Now we can add to the long list of modelling elements that may be important and possibly can be accurately integrated in calibration and perturbative risk, but are usually approximated:

- Exact CSA terms of calibration instruments
- Funding spreads volatilities
- Funding spreads / rates correlations
- FX / rates correlations
- CSA's with no collateral substitution
- Small no-collateral windows
- etc…

Summary

We have discussed the reasons for modelling accurately term structures corresponding to different assets, and some of the implications on calibration, pricing and risk.

- Multiple curve systems are necessary for accurate pricing / risk
- Importance increased further with volatile, wide basis spreads
- CSA, funding model developments increase complexity further
- Fundamental change – from single curves to multi-currency, interdependent curve systems
- Technical challenges can be solved with the right methods
- Many further improvements possible incorporating volatilities, correlations in calibration & pricing of flow products

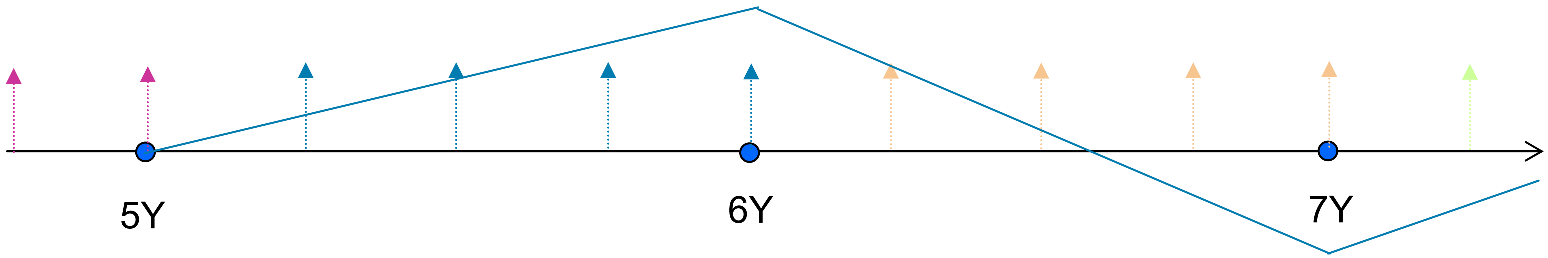
Appendix – Knot Point Positioning

Something as simple as knot point positioning can make all the difference – consider linear interpolation of forwards

- Judged unstable (see ref. 3)
- Can achieve good stability / risk properties by considering point position choice

Suppose we are calibrating a Libor forward interpolation to swaps

What would a shift of 1bp in 6Y imply, with knots = maturities?



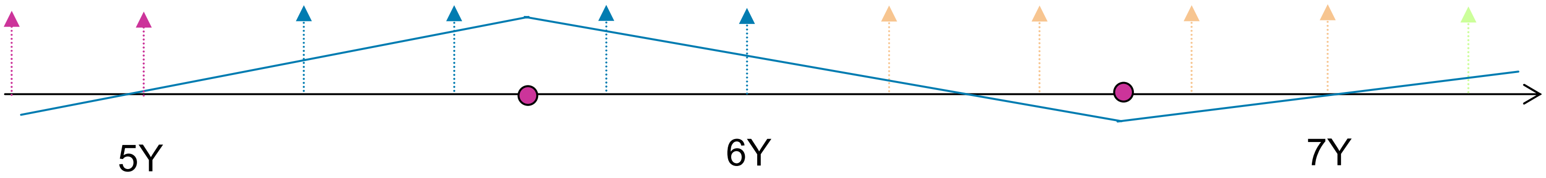
interpolating using end points of Libor periods

Appendix – Knot Point Positioning II

Oscillations were the result of moving 6Y “too high”

- Influence of control point on individual fixings asymmetric
- To compensate 7Y control point must move too low, etc.

What if we placed knot points where the moment of fixings to the left of the knot point equals the moment of fixings to the right?



➢ No overshooting to fit, so lower propagation, smooth curve

This “Midpoints” method can be applied for any interpolation...

… e.g. for flat rates, optimum points = maturities!

References

1. “**Interest Rate Parity, Money Market Basis Swaps, and Cross Currency Basis Swaps**”, B. Tuckman, P. Porfirio, June 2003, Lehman Brothers research note
2. “**Funding beyond discounting: collateral agreements and derivatives pricing**” V. Piterbarg, February 2010, RISK Magazine
3. “**Interpolation Methods for Curve Construction**”, P. Hagan, G. West, June 2006, Applied Mathematical Finance