Volatility and Correlation Workshop (Part II)

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Correlation

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Introduction

Many institutions have positions on a large number of assets/markets. They are exposed to joint moves of these risk factors. In this talk, we review:

- Background on correlation
- Data visualization
- Data analysis
- Correlation scenarios

Background on Correlation

Definitions

- (X,Y) random variables

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$

Cauchy-Schwarz inequality $\Rightarrow \rho_{X,Y} \in [-1,1]$

- Is $C = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$ a correlation matrix ?

i.e. does (X_i) r.v. exist s.t. $Corr((X_i)) = C$?

Correlation Matrix: a constrained object

- If $\forall i, Var(X_i) = 1$ then

$$Var\left(\sum_i \lambda_i X_i\right) = \sum_i \lambda_i^2 + 2\sum_{i<j} \rho_{ij}\lambda_i\lambda_j = \lambda^T C \lambda > 0$$

$$\Rightarrow \boxed{C \geq 0}$$

- Example: for *N* r.v., if $C = \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho \\ \rho & \cdots & \rho & 1 \end{pmatrix}$ then $\rho \geq -\frac{1}{N-1}$

Dem.: $Var\left(\sum_i X_i\right) = \sum_i 1^2 + 2\sum_{i<j} \rho = N + N(N-1)\rho \geq 0$

Correlation matrix: to handle with care

Correlation matrix:

- Difficult to manipulate
  When bumping one coefficient and its symmetric, C must remain > 0.

- How to compute a correct matrix if asynchronous or missing data? Moreover if we have few data, the matrix will be noisy.

- Computation of implied C may not respect constraints
  $\Rightarrow$ Arbitrage ?

But correlation is a key data in risk management

Correlation Matrix Computation

- Given 2 time series (X_i) and (Y_i) for $i = 1...n$ we want to compute their correlation
- A possible estimator of their variance is:

$$Cov(X,Y) = \frac{N}{N-1}\left(\frac{1}{N}\sum X_i Y_i - \frac{1}{N^2}\left(\sum X_i\right)\left(\sum Y_i\right)\right)$$

- Then the correlation is given by: $Corr(X,Y) = \dfrac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$
- We have:

$$Corr(X,Y) = \frac{\left(N\sum X_i Y_i - \left(\sum X_i\right)\left(\sum Y_i\right)\right)}{\sqrt{\left(N\sum X_i^2 - \left(\sum X_i\right)^2\right)\left(N\sum Y_i^2 - \left(\sum Y_i\right)^2\right)}}$$

Completing data

- In order to construct a correlation matrix, we may have to complete some missing data, or to deal with asynchronous data.

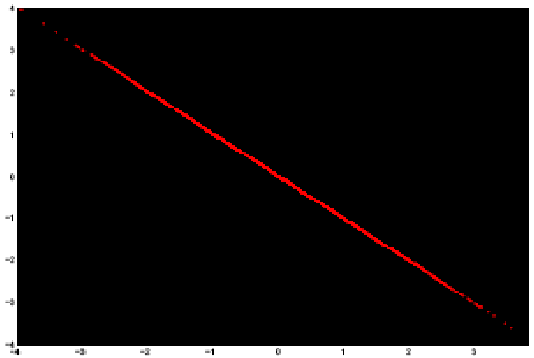
- In the case of missing data, one can use the E.M. (Expectation Maximization) algorithm to complete time series.

- In the case of asynchronous data (ex: closing prices on different markets), that can introduce correlation, distorting the values of portfolios, value at risk measures, and hedge strategies.

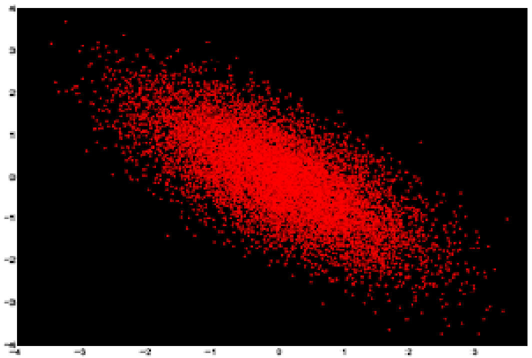
  → Prices can be synchronized by computing estimates of the value of assets even when markets are closed, given information from markets which are open.

Gaussian examples

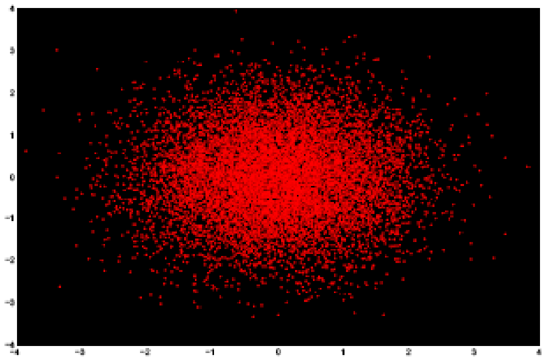
$\rho = -1$



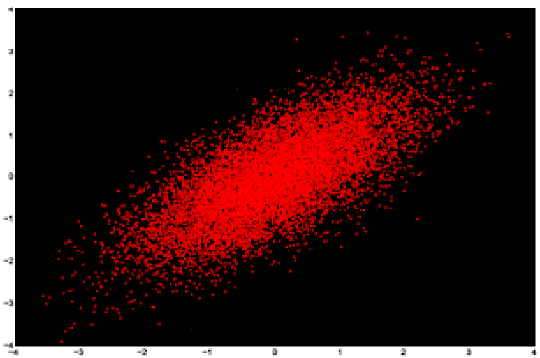
$\rho < 0$



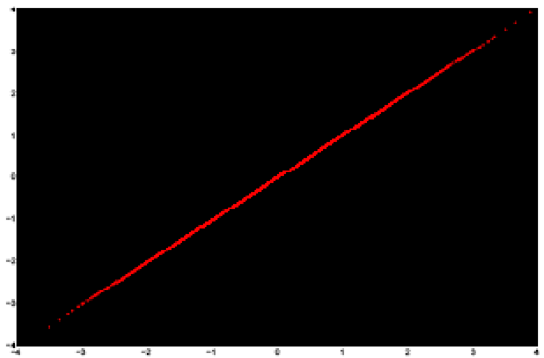
$\rho = 0$



$\rho > 0$

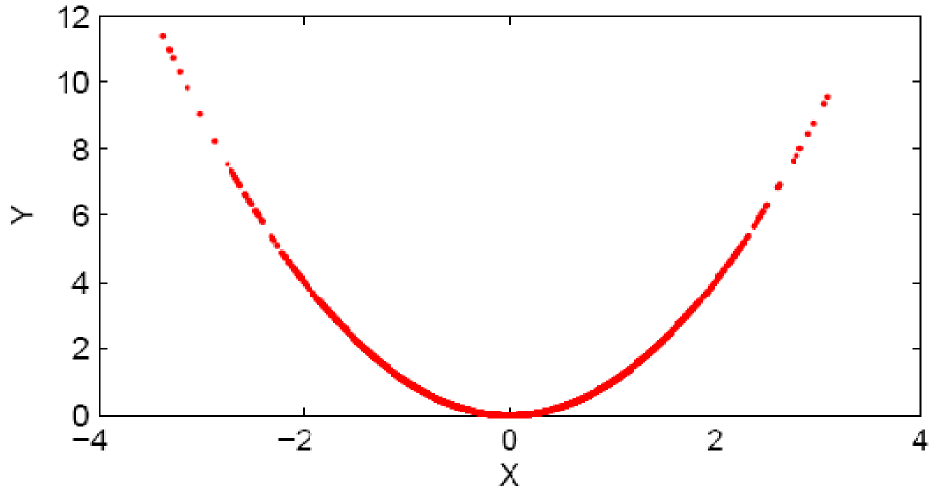


$\rho = 1$



Correlation is NOT Causality

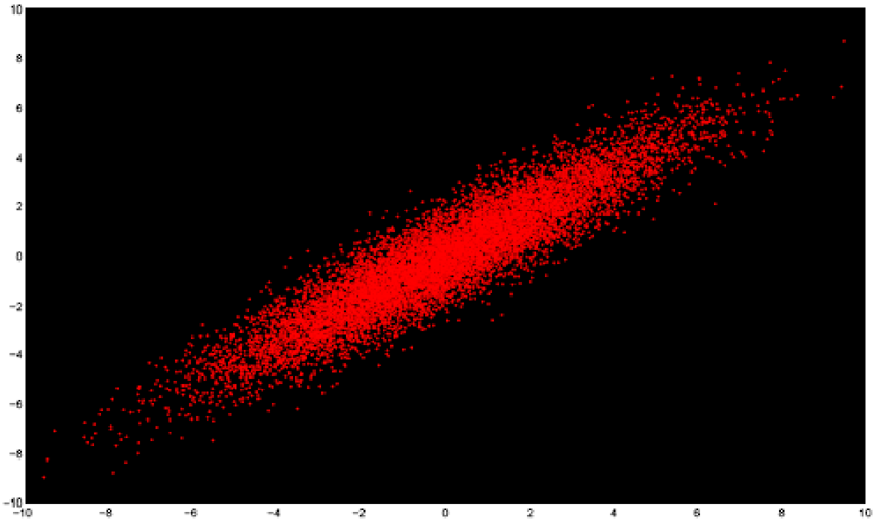
- **!!** ρ captures only <u>linear</u> relationships

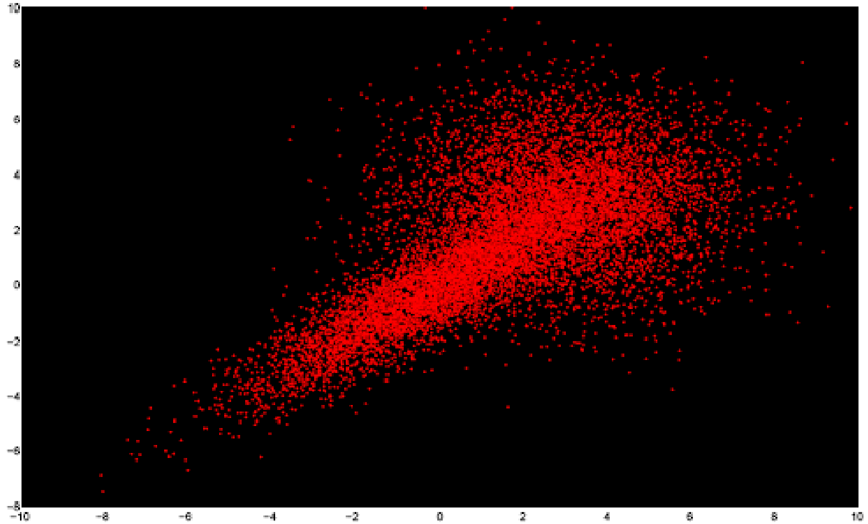


- *X* following uniform law over $\left[-1,1\right]$ and $Y = X^2$
- Information on *X* gives information on *Y*, and vice-versa
- But $Corr(X,Y) = \mathrm{E}[XY] - \mathrm{E}[X]\mathrm{E}[Y] = \mathrm{E}\left[X^3\right] - \mathrm{E}[X]\mathrm{E}\left[X^2\right] = 0$

Correlation is NOT Causality

- Correlation does not distinguish shapes





- To capture conditional correlation: Copulas

High correlation trap

X and *Y* are 2 stocks of same volatility: σ

Very highly correlated: $\rho(X,Y) = 0.99$

Are they almost perfect substitutes?

NO

$$\sigma_{X-Y}^2 = \sigma^2 + \sigma^2 - 2\rho\sigma^2$$

$$\sigma_{X-Y} = \sigma\sqrt{2(1-\rho)} \approx 0.14\sigma$$

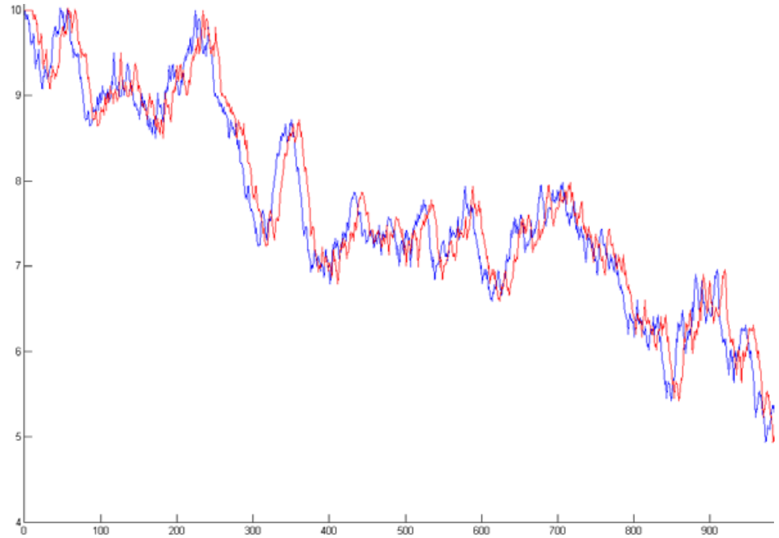
The risk of *X-Y* is still 14% of the initial risk!

Correlating levels/increments

- $X_t = S\&P_t$, $Y_t = S\&P_{t+\delta t}$

  Levels very correlated

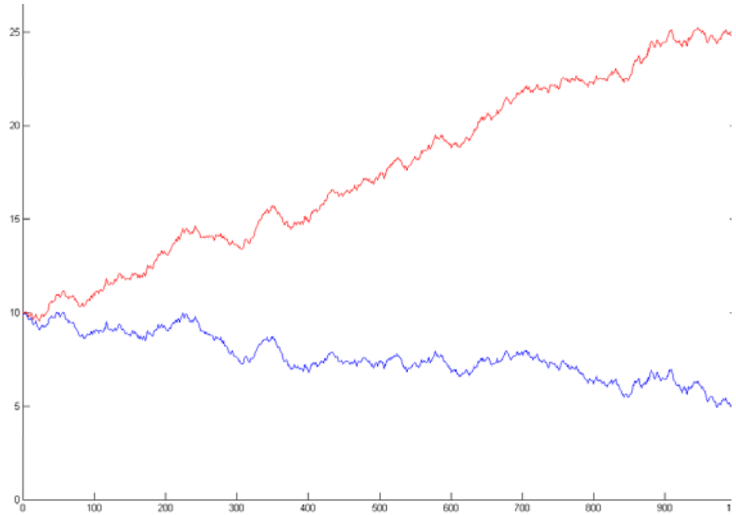
  Increments decorrelated



- $X_t = S\&P_t$, $Y_t = X_t + \alpha t$

  Levels weakly correlated

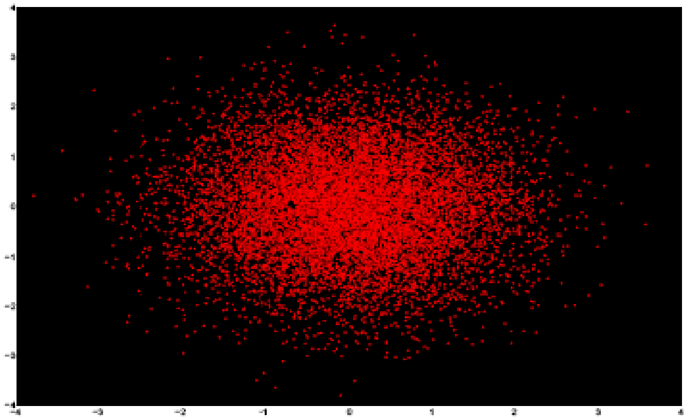
  Increments fully correlated

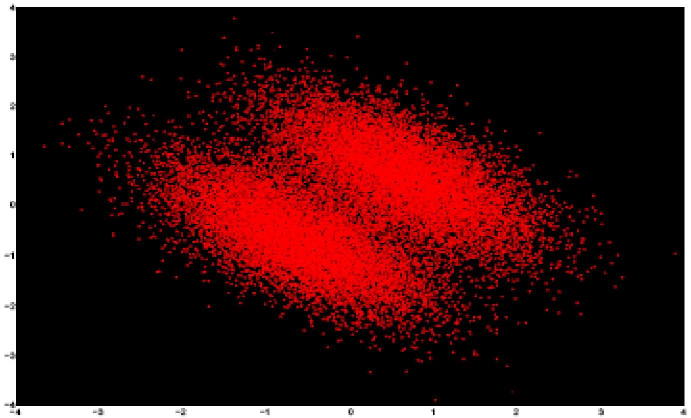


Independent Components Analysis

- Covariance Matrix discards information on joint behavior beyond correlation

- If factors are independent, joint density = product of marginal density





These two distributions have the same covariance matrix

- The purpose of ICA is to try to decompose the signal as a mix of

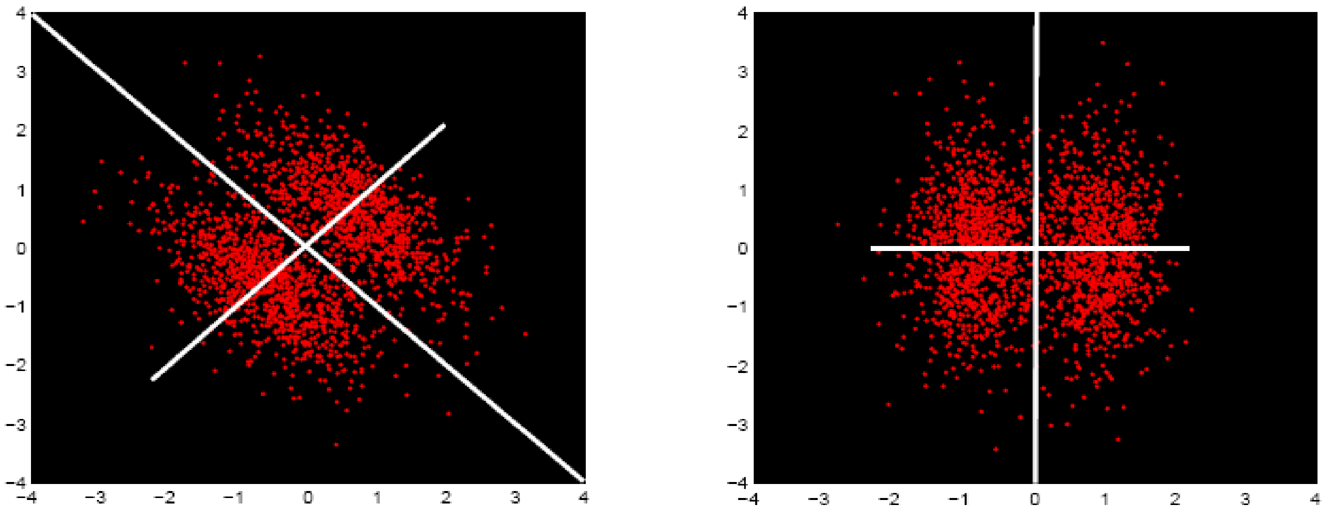
independent factors: $S = Af$

Where *A* is the mixing matrix, and *f* are the independent factors

Independent Components Analysis

- Mix of independent non gaussian random variables is more gaussian (Central Limit Theorem)

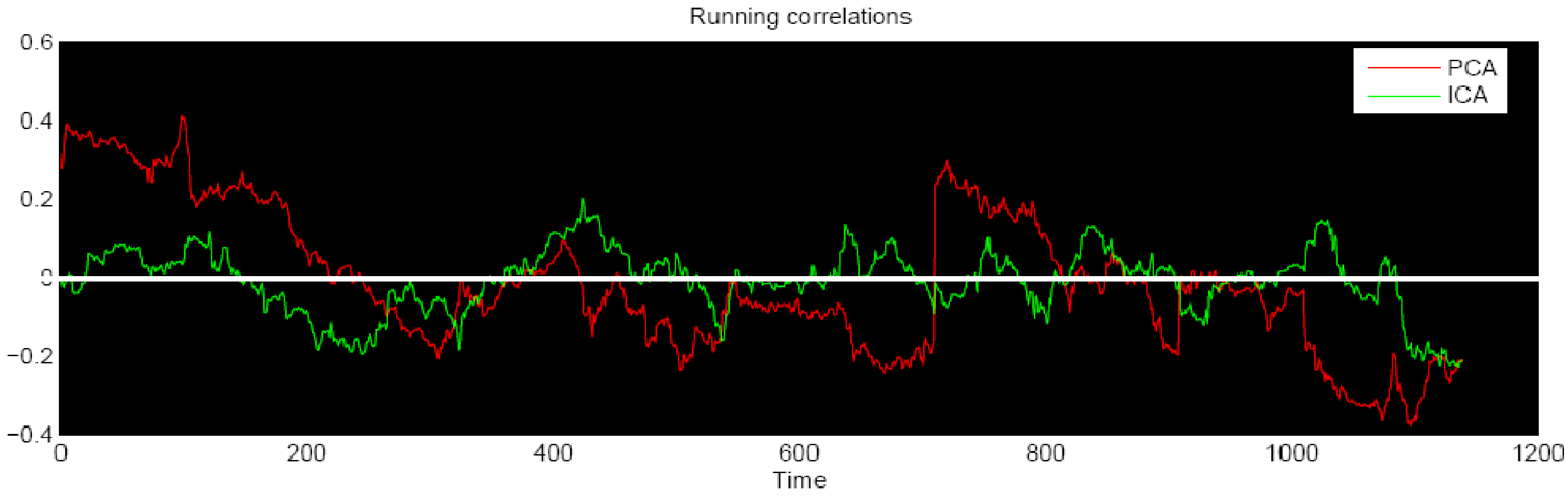
- To recover independent factors, identify non gaussian combinations. For instance use kurtosis as a criterion



- Reconstruction not as good as PCA in terms of variance. But better than PCA in capturing qualitative behavior

Comparison P.C.A. / I.C.A.

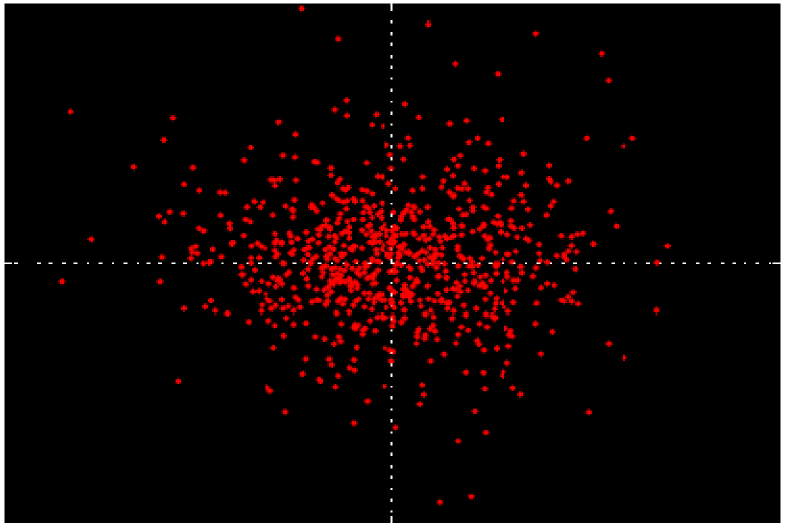
- Both have null correlation between factors over the whole period
- But running correlation is better with ICA (here with a time window of 100)



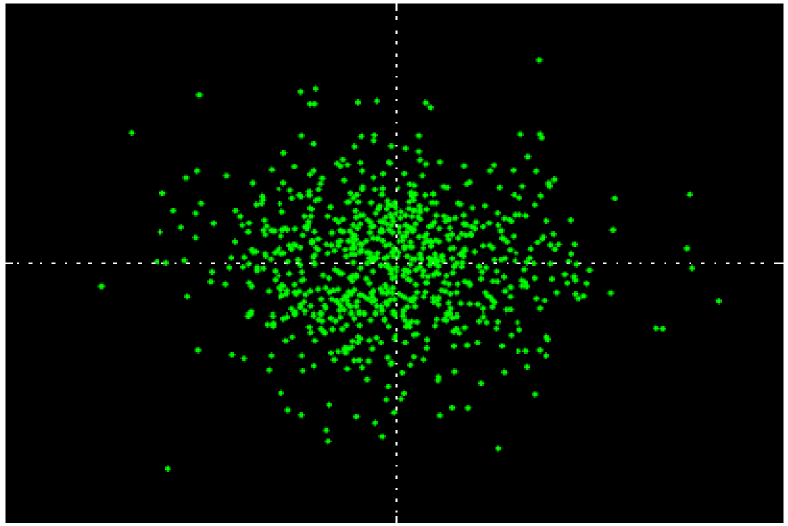
Comparison P.C.A. / I.C.A.

- We can also compare the distribution of the factors given by PCA and by ICA:

PCA – joint



ICA joint



- In fact ICA gives factor less gaussian than PCA:
  - kurtosis(PCA) ~ 5
  - kurtosis(ICA) ~ 20

Correlation scenarios

Correlation Scenarios

- If you want to test simple hypothesis about the evolution of correlation how to modify the current correlation matrix, as it needs to remain non-negative?

- How to increase/decrease the global correlation

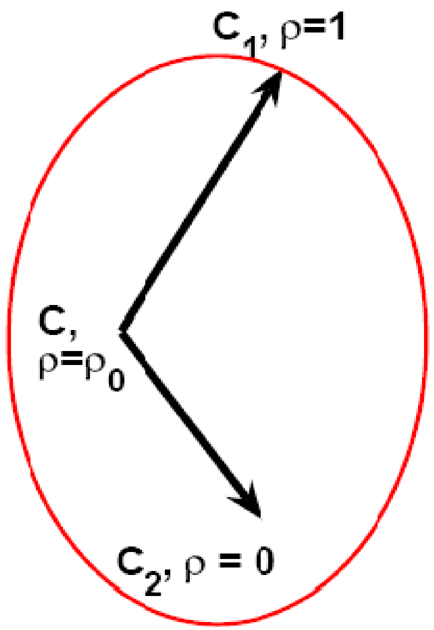
- First, let us define a global correlation measurement:

$$\rho_0 = \frac{\sum_i \sum_{j \neq i} \rho_{ij}}{n(n-1)}$$

- The idea is to move within the set of all the correlation matrices increasing or decreasing the correlation

Correlation Scenarios

- The set of all correlation matrices is convex. Using this property we define the following matrices:



$$C_1 = 1$$

$$C_2 = Id$$

$$C_\rho,\ \rho_0 < \rho \le 1$$

$$C_\rho = \frac{\rho - \rho_0}{1 - \rho_0} C_1 + \frac{1-\rho}{1-\rho_0} C$$

$$C_\rho,\ 0 \le \rho \le \rho_0$$

$$C_\rho = \frac{\rho}{\rho_0} C + \frac{\rho_0 - \rho}{\rho_0} C_2$$

- We have: $\rho_0(C_1) = 1, \rho_0(C_2) = 0, \rho_0(C_\rho) = \rho$

Correlation Matrix Deformation

2 possible dynamical models for correlation:

$$\text{Model 1: } \rho_t = a + \sigma \varepsilon_t$$

$$\text{Model 2: } \rho_t - \rho_{t-1} = \alpha \varepsilon_t$$

Where $\varepsilon_t \sim N(0,1)$. To test these models:

- We sliced out 5 years of data into 20 quarters to get a time series of 20 correlation matrices.
- We computed the covariance matrix of these 20 correlations C_1 and the covariance of the 19 increments C_2 .
- Decision is based upon a comparison of total variance:
  - Model 1: one should have $tr(C_1) < tr(C_2)$
  - Model 2: one should have $tr(C_2) < tr(C_1)$

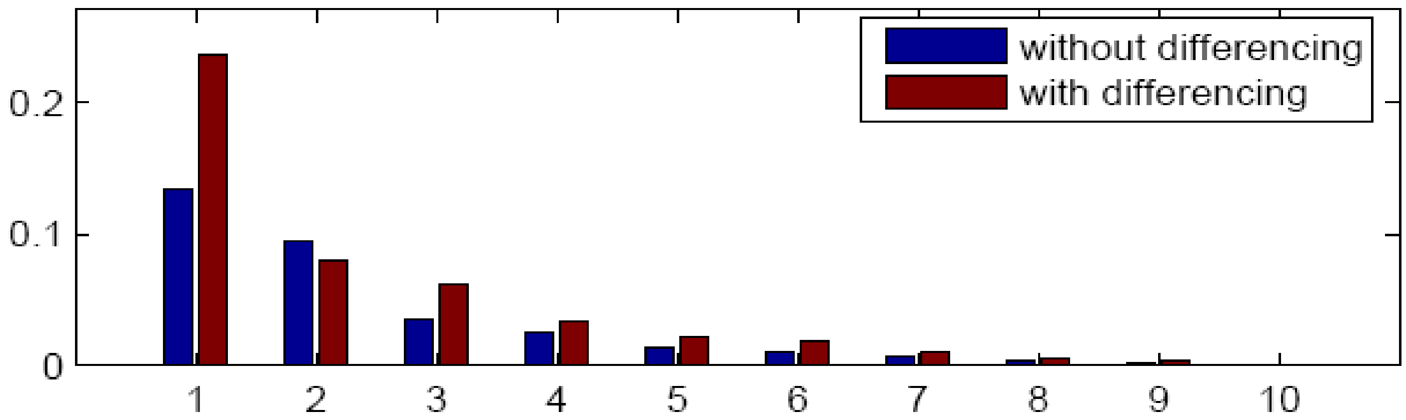
Correlation Matrix Deformation

For the ten largest stocks, we found:

$$tr(C_1) = 0.3289 < 0.4721 = tr(C_2)$$

This supports a model of the first type.

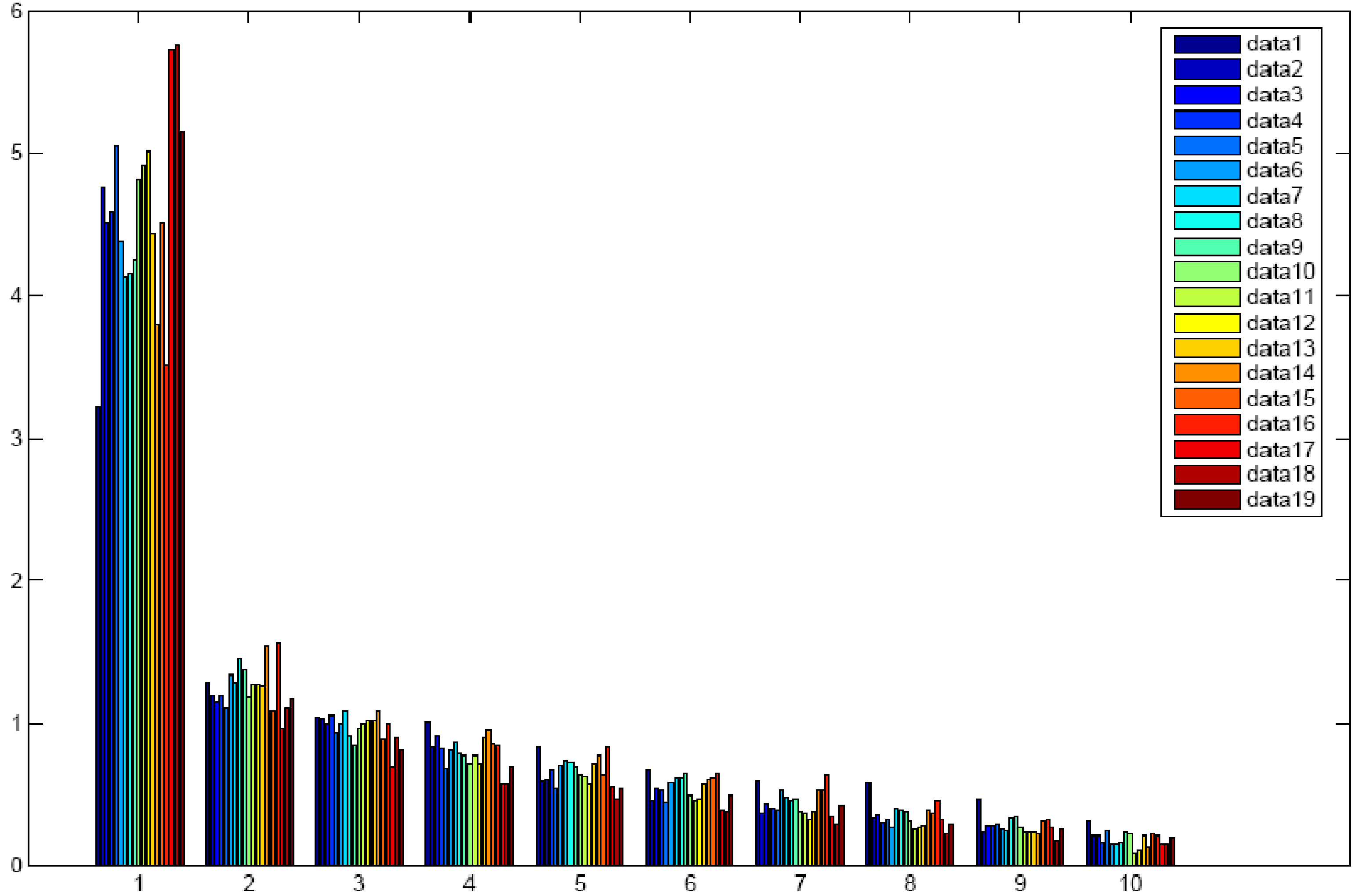
Now we compare the spectrum of the 2 covariance matrices:



Empirical Scenarios for VaR / Stress testing

- Compute *n* correlation matrices over non overlapping time windows
- Reprice the current portfolio with those n correlation matrices
- Retain the (p^{th}) worst results
- For volatility stress: multiply all volatilities by the same λ

19 Spectrums



Conclusion

- The value of large portfolios depends crucially on the covariance matrix
- It is important to synthesize this huge amount of information and to represent it visually
- Developing correlation scenarios is important but requires care
- New techniques are becoming available

Trading Volatility and Correlation

Outline

Trading volatility and correlation

Why trade volatility/correlation?

- Trade volatility spread between two indices
- Trade realised volatility against implied volatilities
- Trade correlation between two underlyings, e.g. interest rates, equity indices, FX
- Buy gamma, cross-gamma or vega for hedging purpose

Notation

Simplification:

- Assume interest rate = 0
- Normal model:

$$dX_t = \sigma_{X,t} dW_t^X$$

$$dY_t = \sigma_{Y,t} dW_t^Y$$

$$E[dW_t^X dW_t^Y] = \rho$$

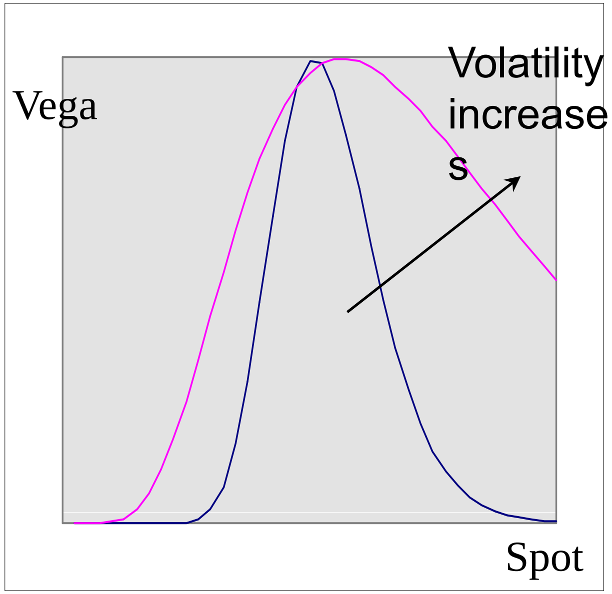
European call and put options

• Trade volatility using delta hedged European call or put option

• Complex exposure to spot and volatility level

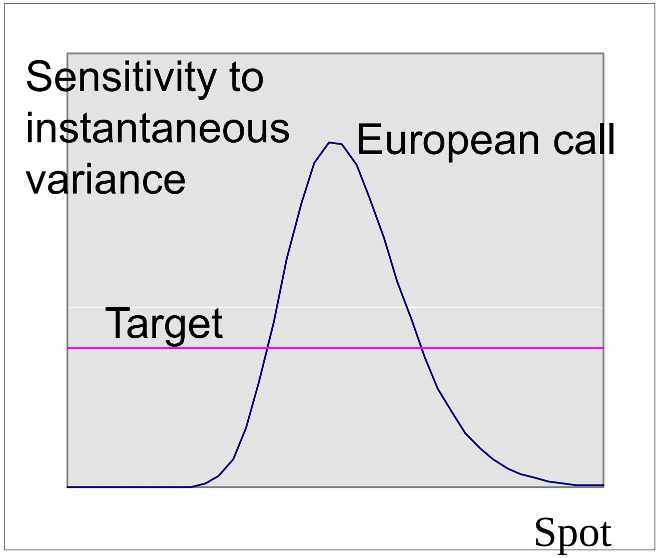
• Take a view on the spot in order to determine the expected variance sensitivity

How to trade volatility with better control on spot sensitivity?



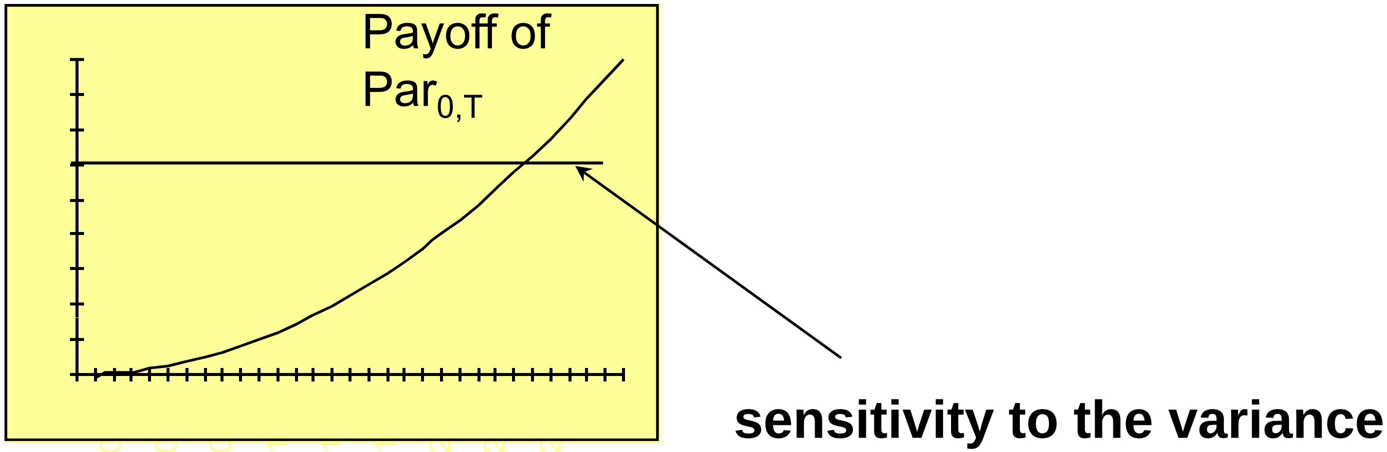
Instantaneous Forward Variance

- Trade instantaneous variance at T
- Requirement: sensitivity to variance independent of spot
- Vega hedging purpose: simple vega
- Arbitrage variance on all possible spot levels



Instantaneous Forward Variance

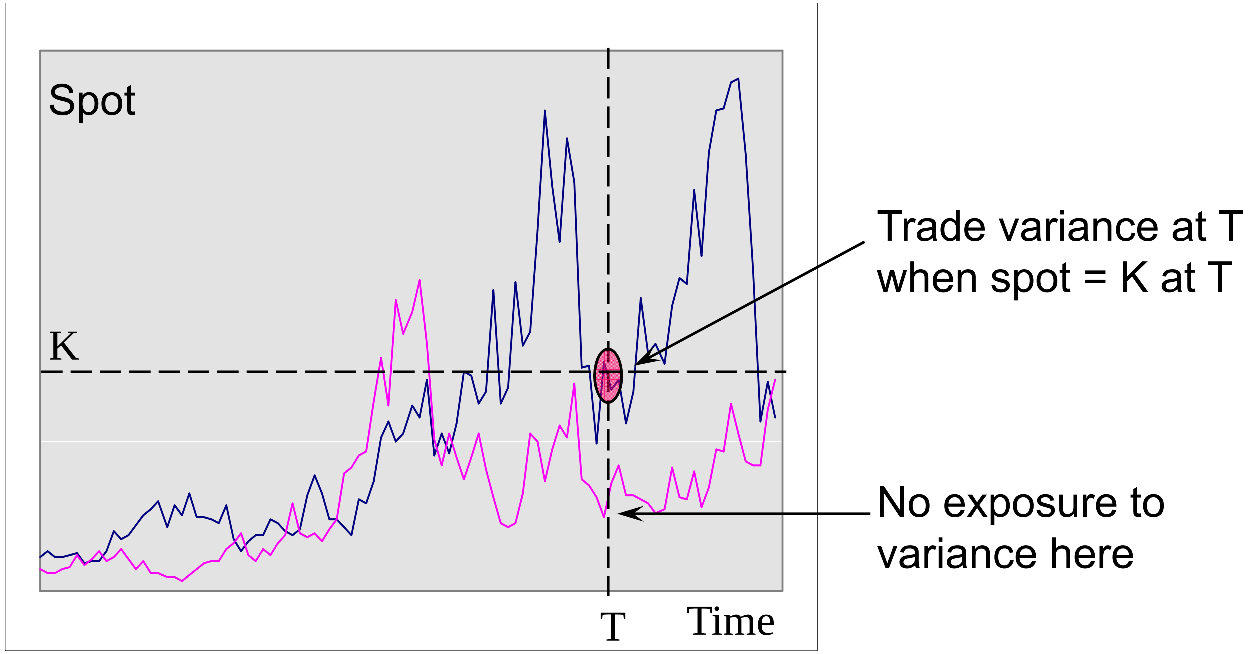
- Par_T = contract that gives S_T^2 at time T
- Constant sensitivity to the variance
- Calendar Spread: $\dfrac{\mathrm{Par}_{T+\Delta T} - \mathrm{Par}_T}{\Delta T}$



Conditional Instantaneous Forward Variance

What is conditional instantaneous variance at T?

Instantaneous variance at T condition on spot at T equal to a particular value



Conditional Instantaneous Forward Variance

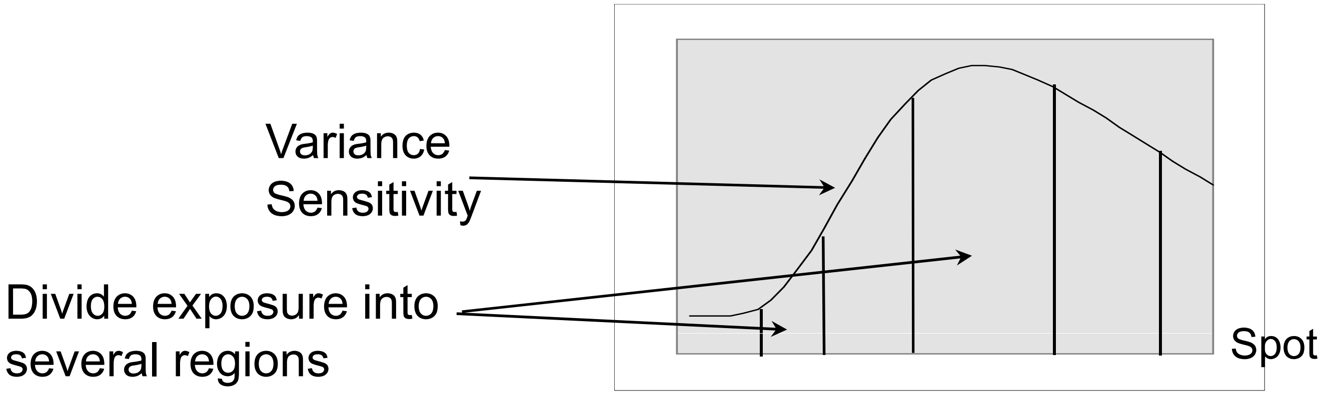
Why trade conditional instantaneous variance?

• Control exposure

$S_T \neq$ target value: no instantaneous variance exposure

• Hedging: Exotics, such as knock-out option, have different variance exposures at different spot levels

• Arbitrage variance only over a particular spot range

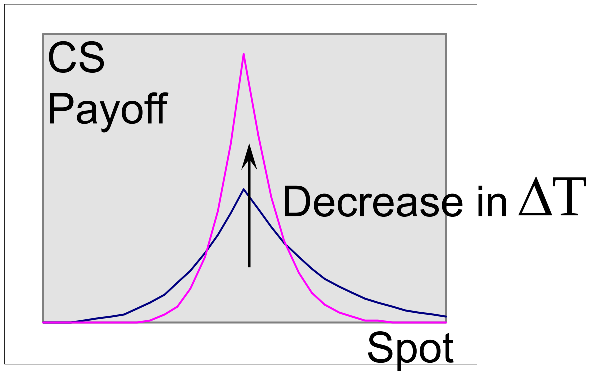


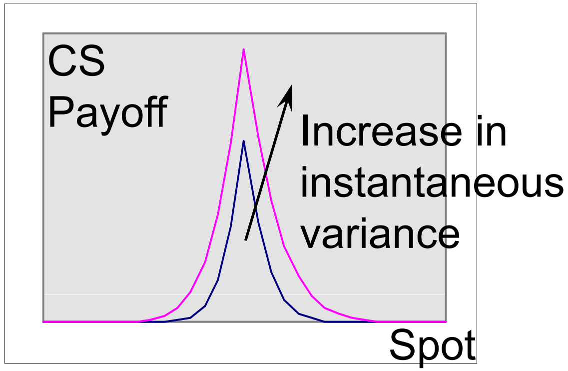
Conditional Instantaneous Forward Variance

Naïve approach: Calendar spread of European Call with strike K and maturity T, i.e

$$\frac{C(K,T+\Delta T)-C(K,T)}{\Delta T}$$

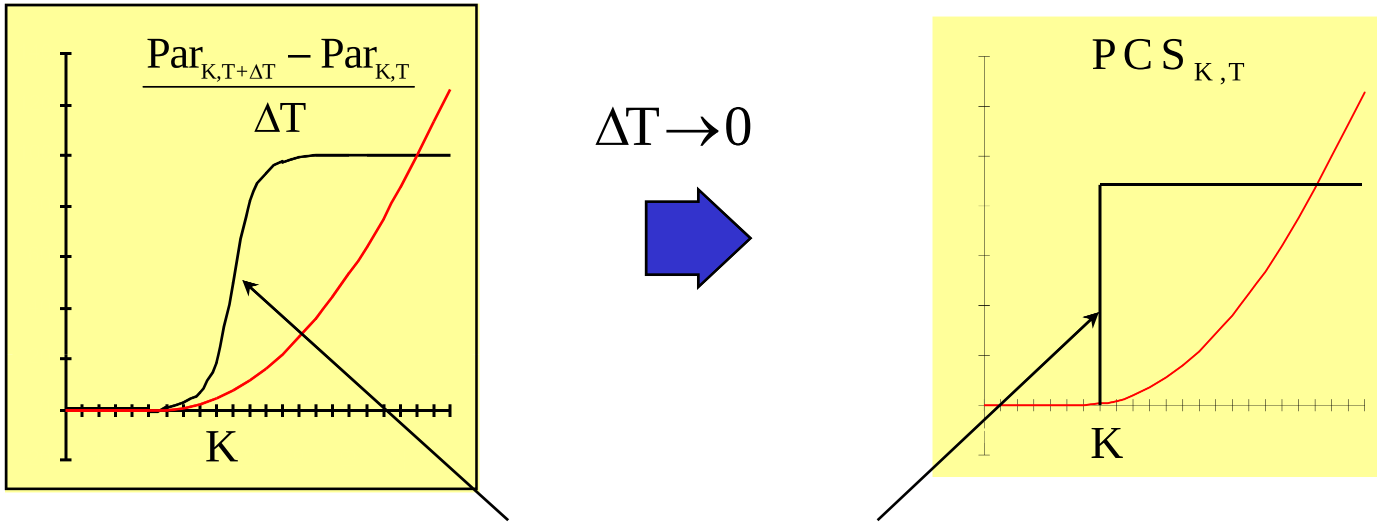
- $\Delta T \to 0$: receive payment only when $S_T = K$
- Width changes as instantaneous variance changes





Conditional Instantaneous Forward Variance

- Truncated parabola contract: $\mathrm{Par}_{\mathrm{K,T}} = \max(S_T - K, 0)^2$
- Non-constant sensitivity to variance
- Parabola Calendar Spread: $\mathrm{PCS}_{\mathrm{K,T}} = \lim_{\Delta T \to 0} \dfrac{\mathrm{Par}_{\mathrm{K,T+\Delta T}} - \mathrm{Par}_{\mathrm{K,T}}}{\Delta \mathrm{T}}$
- Strongly sensitive to variance when $S_T \geq K$

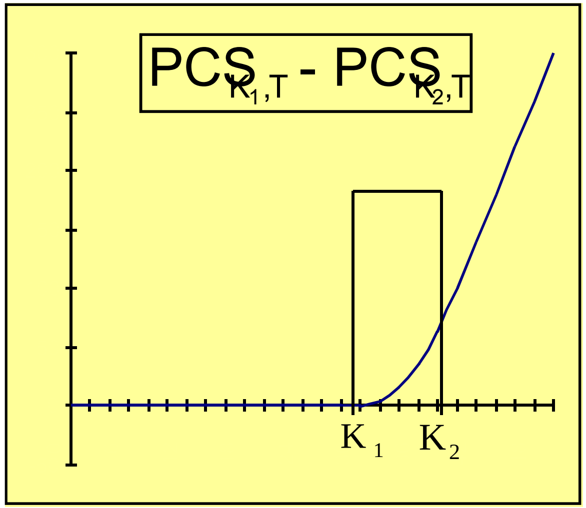


Sensitivity to the variance

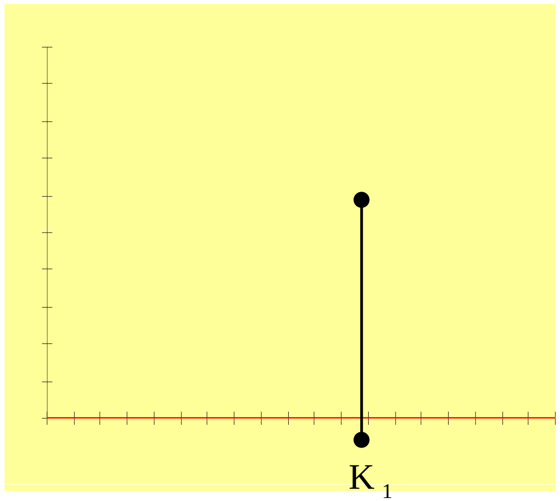
Conditional Instantaneous Forward Variance

- Consider the portfolio: $\dfrac{PCS_{K_1,T} - PCS_{K_2,T}}{K_2 - K_1}$
- It gives the instantaneous variance at T only when

$$K_1 \leq S_T \leq K_2$$



$K_2 \to K_1$

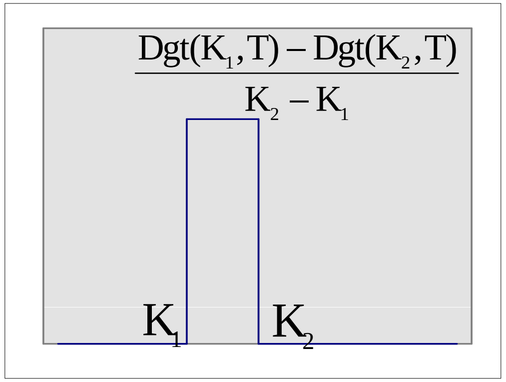


Conditional Instantaneous Forward Variance

- Problem: lose your premium if S_T is not within $[K_1, K_2]$
- Finance your premium using digital spread

$$\alpha \frac{\mathrm{Dgt}(K_1,T) - \mathrm{Dgt}(K_2,T)}{K_2 - K_1} = \frac{PCS_{K_1,T} - PCS_{K_2,T}}{K_2 - K_1}$$

- No need to pay the digital spread when S_T is not within $[K_1, K_2]$



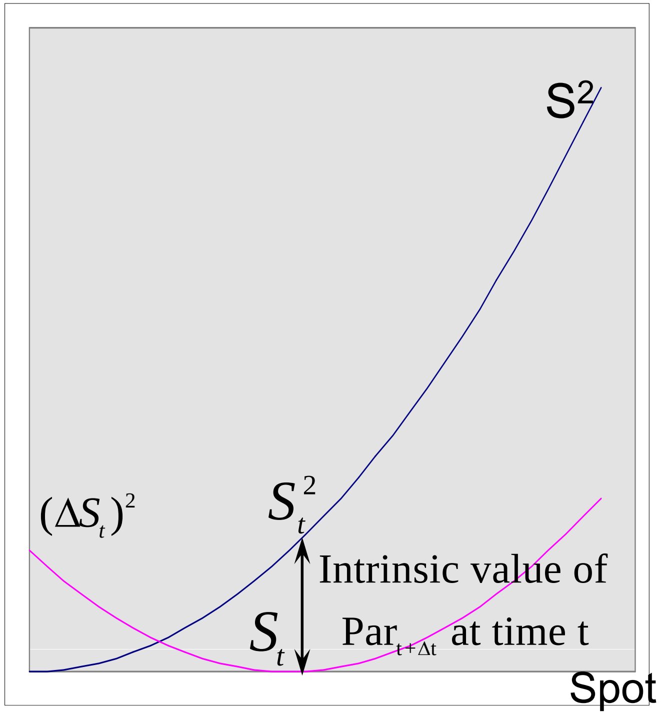
Trade realized variance

How to trade realised variance between T_1 and T_2 ?

- Parabola Calendar Spread: $\mathrm{Par}_{T_2} - \mathrm{Par}_{T_1}$
- Delta-hedge between T_1 and T_2

$$\mathrm{Par}_{t+\Delta t} - \mathrm{Par}_t = \underbrace{2S_t \Delta S_t}_{\text{Spot Dependent}} + \underbrace{(\Delta S_t)^2}_{\text{Quadratic}}$$

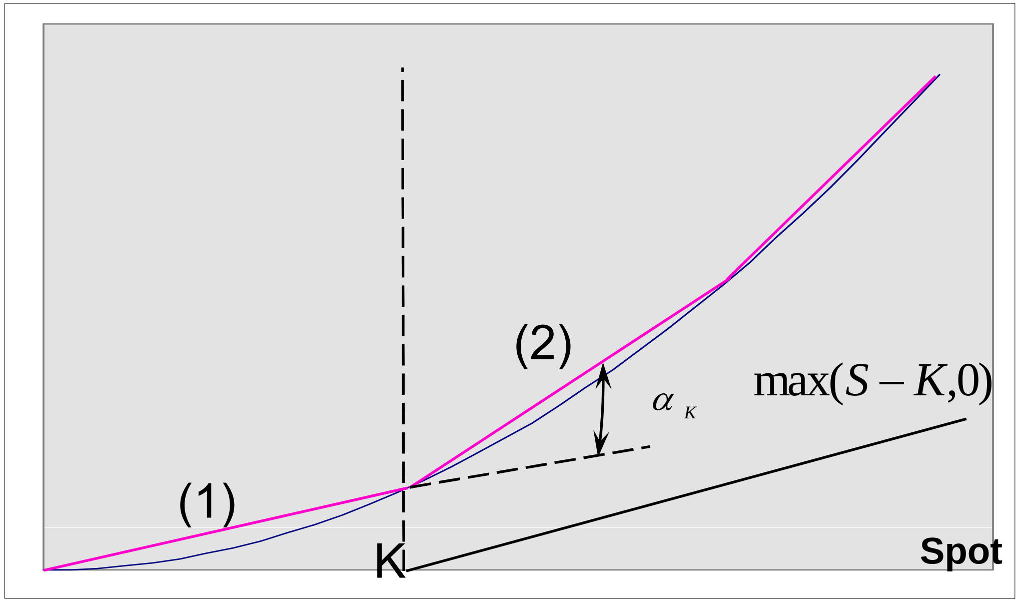
$$\Rightarrow S_{T_2}^2 - S_{T_1}^2 - \underbrace{\sum 2S_t \Delta S_t}_{\text{Delta Hedge with ratio } = 2S_t} = \underbrace{\sum (\Delta S_t)^2}_{\text{Total realized variance}}$$



Replication

We can replicate Parabola contract using a combination of European Call and Put options

- Additional option = Change in slopes
- $\mathrm{Par}_{\mathrm{T}} = \int 2C(K,T)dK$



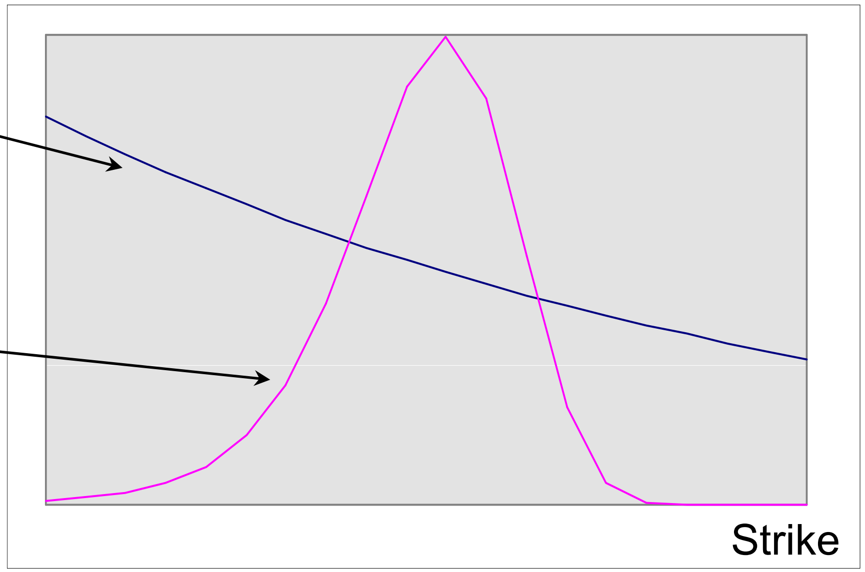
Implied volatility

What is the relationship between implied volatility and the value of $\mathrm{Par}_{T_1} - \mathrm{Par}_T$ at T?

$$\mathrm{Par}_{T_1} - \mathrm{Par}_T = \int 2\mathrm{varga}(K)\sigma_{imp}^2(K,T_1)dK$$

Implied Volatilities at T of European Call matured at T_1

Varga of call options at different strikes

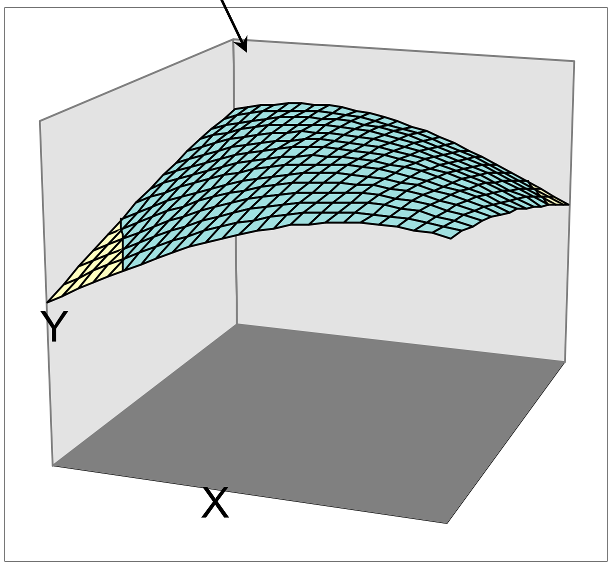


Trading correlation

• Similar to volatility, correlation sensitivity in general depends on spot level

Spot dependent correlation vega of basket option $\max(X_T+Y_T-K,0)$

How to trade correlation with better control on spot sensitivity?



Instantaneous Forward Covariance

Why we want to trade instantaneous covariance at T ?

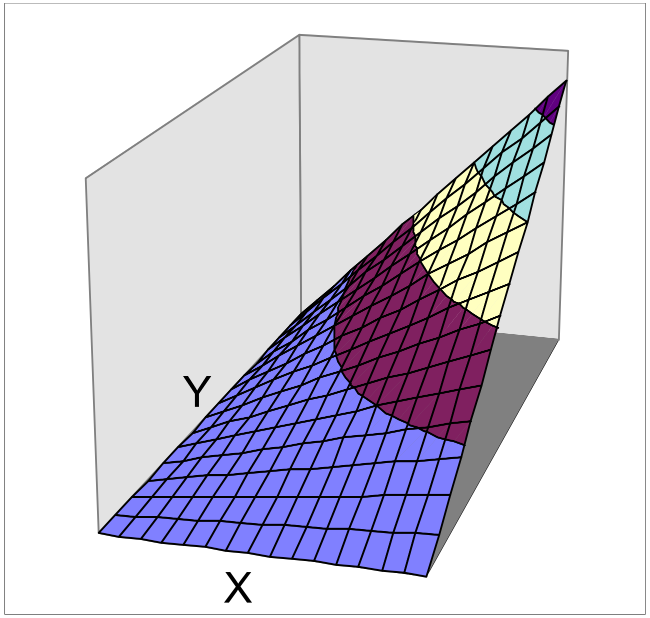
- Spot independent covariance sensitivity
- Simple covariance hedging instrument
- Trade covariance on every possible spot levels

How to lock instantaneous covariance?

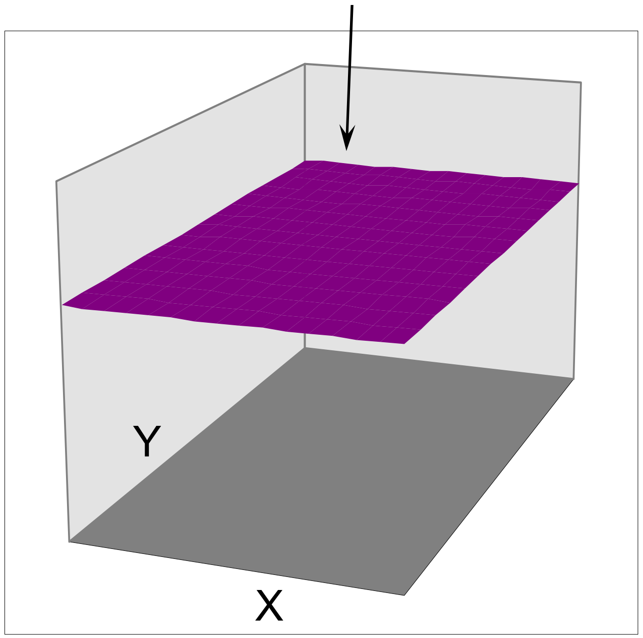
Instantaneous Forward Covariance

- Product contract: Pro gives XY at T
- Calendar spread: $\dfrac{\text{Pro}_{T+\Delta T} - \text{Pro}_T}{\Delta T}$

Product contract payoff function



Constant sensitivity to covariance



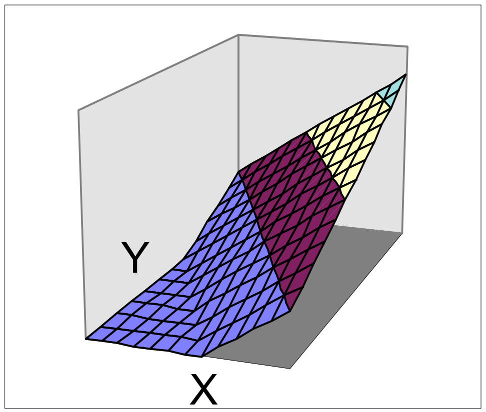
Instantaneous Forward Covariance

• Replicate Product contract using basket options and European call on single assets

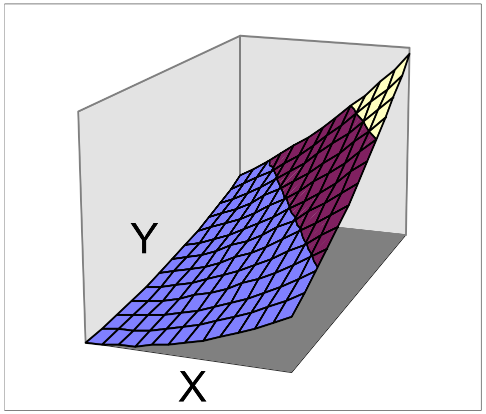
Example: Use Basket option $\mathrm{Bas}(K) = \max(X_T + Y_T - K, 0)$

• Parabola contract $\max(X,0)^2 \longleftrightarrow \max(X+Y,0)^2$

$\mathrm{Bas}(K) = \max(X_T + Y_T - K, 0)$



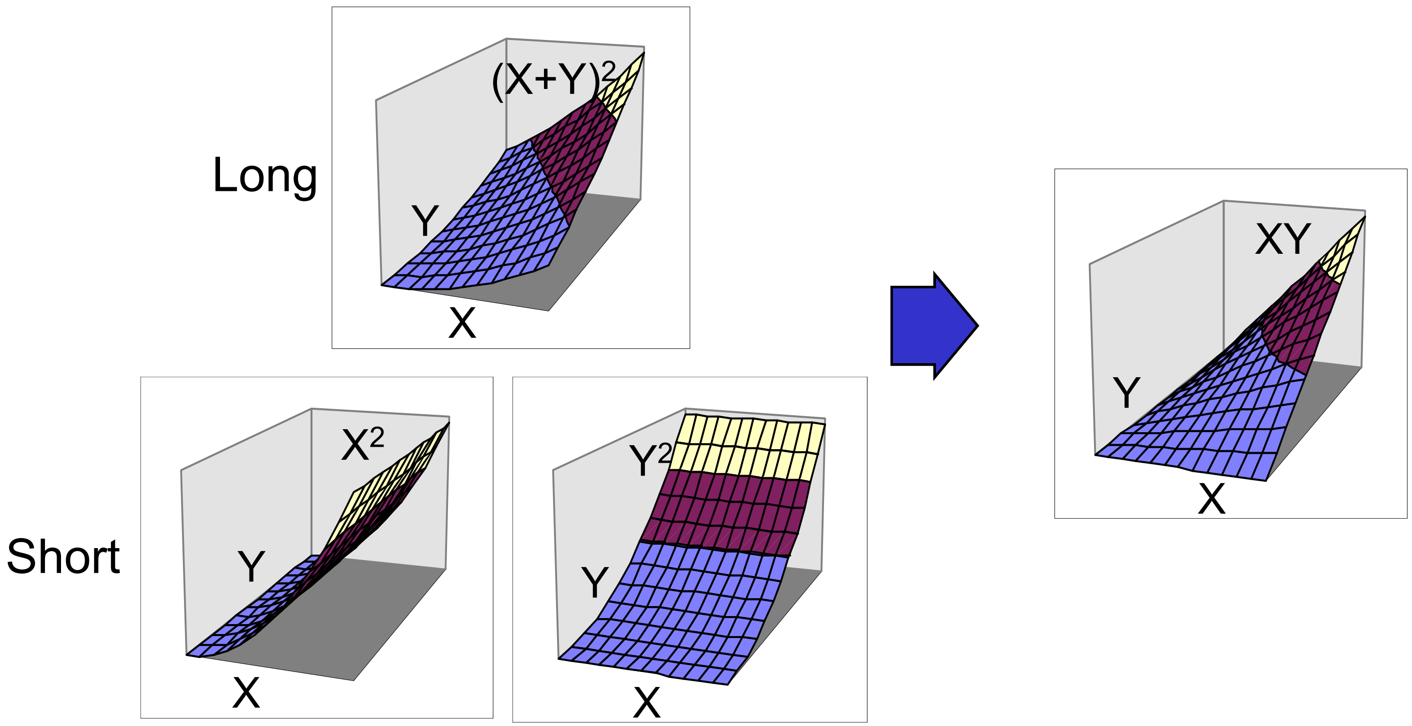
$\mathrm{Par} = (X_T + Y_T)^2$



Instantaneous Forward Covariance

- Long parabola on (X+Y) and short parabola on X and on Y

$$\tfrac{1}{2}(X+Y)^2 - \tfrac{1}{2}X^2 - \tfrac{1}{2}Y^2 = XY$$



Realized covariance

How to trade realised covariance between T_1 and T_2 ?

- Calendar Spread on Product contract: $\mathrm{Pro}_{T_2} - \mathrm{Pro}_{T_1}$
- Delta hedge between T_1 and T_2

Let $T_2 = T_1 + 1$:

$$X_{T_1+1}Y_{T_1+1} - X_{T_1}Y_{T_1} = \underbrace{X_{T_1}\Delta Y_{T_1} + Y_{T_1}\Delta X_{T_1}}_{\text{Spot Dependency}} + \underbrace{\Delta X_{T_1}\Delta Y_{T_1}}_{\text{covariance}}$$

$$\Rightarrow X_{T_2}Y_{T_2} - X_{T_1}Y_{T_1} - \underbrace{\sum X_t \Delta Y_t}_{\text{Delta Hedge Ratio for } Y_t = X_t} - \underbrace{\sum Y_t \Delta X_t}_{\text{Delta Hedge Ratio for } X_t = Y_t} = \underbrace{\sum \Delta X_t \Delta Y_t}_{\text{Total Realised covariance}}$$

Instantaneous Forward Correlation

Can we lock instantaneous forward correlation using model free method ? No!

- Any delta-hedged two-asset derivative

$$\underbrace{\frac{\partial P}{\partial t}}_{\substack{\text{Time value}\\\text{of a delta-}\\\text{hedged}\\\text{option}}} = -\underbrace{\frac{1}{2}\sigma_X^2\Gamma_X}_{\substack{\text{Variance of}\\\text{X * Gamma}\\\text{of X}}} - \underbrace{\frac{1}{2}\sigma_Y^2\Gamma_Y}_{\substack{\text{Variance of}\\\text{Y * Gamma}\\\text{of Y}}} - \underbrace{\sigma_X\sigma_Y\rho\Gamma_{XY}}_{\substack{\text{Covariance *}\\\text{Cross-}\\\text{Gamma}}}$$

Instantaneous Forward Correlation

- Through options, we can only trade variance and covariance
- Correlation $\rho = \dfrac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$

$$\Rightarrow E[\rho] = E\left[\frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{var}(X)\mathrm{var}(Y)}}\right] \neq \frac{E[\mathrm{cov}(X,Y)]}{E[\sqrt{\mathrm{var}(X)}]E[\sqrt{\mathrm{var}(Y)}]}$$

Correlation is a non-linear function of var and cov

➡ Expected correlation depends on volatility of var and cov

Restriction on trading covariance

• Options (except for quantos) capture instantaneous variance and covariance through gamma and cross-gamma

• Restriction on the relationship between gamma and cross-gamma

$$\Gamma_{XY} = \frac{\partial^2 C}{\partial X \partial Y} \Rightarrow \Gamma_X = \frac{\partial^2 C}{\partial X^2} = \frac{\partial}{\partial X}\int \Gamma_{XY}\, dY \text{ and } \Gamma_Y = \frac{\partial^2 C}{\partial Y^2} = \frac{\partial}{\partial Y}\int \Gamma_{XY}\, dX$$

Implication:

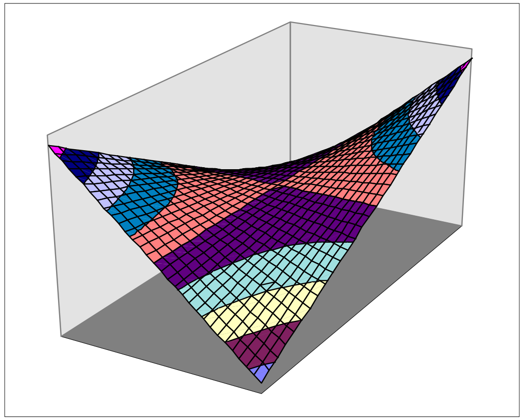
• Lock conditional instantaneous covariance

(e.g. $X_T = K_X \;\&\; Y_T = K_Y$) ? No!

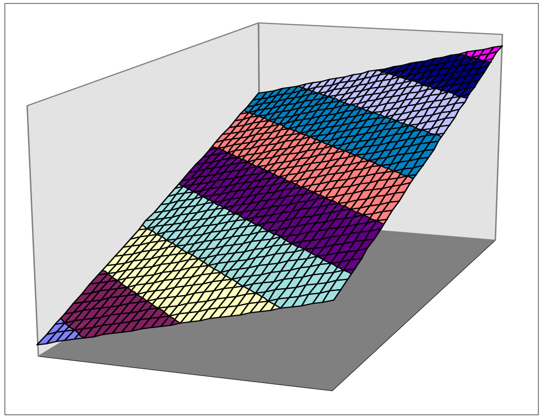
Restriction on trading covariance

Cross-Gamma = 1 when $X = K_X$ and $Y = K_Y$

e.g. $F(X,Y) = (X-K_X)(Y-K_Y)$ locally

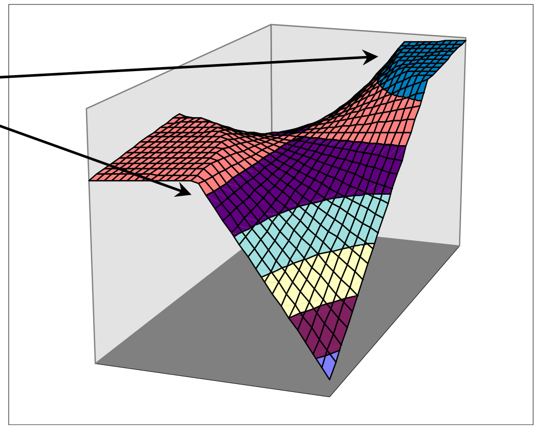


Globally, no Gamma and Cross-Gamma



Change in slopes on the boundary

- Additional convexity
- Additional variance and covariance exposure



Covariance modelling

• Model free method ➡ trade forward covariance

• More flexible covariance trading requires modelling

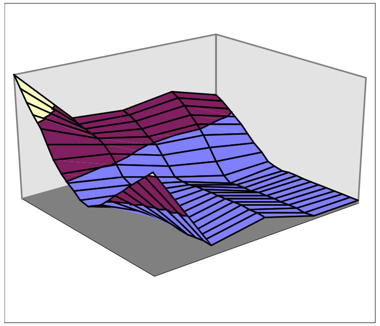
• A good model should generate prices which match option prices traded in liquid market

• Important for hedging complicated options using some simple ones traded in liquid market

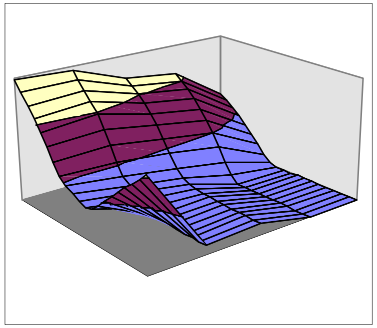
How to achieve this?

Smile model

Call Prices on X



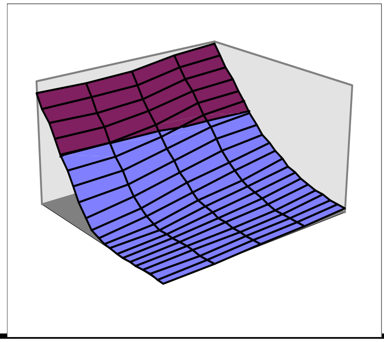
Call Prices on Y



$$\frac{dX_t}{X_t} = rdt + \sigma\left(X_t, t\right)dW_t$$

$$\frac{dY_t}{Y_t} = rdt + \sigma\left(Y_t, t\right)dW_t$$

Prices on two-asset derivatives



$$\rho(X_t, Y_t, t)$$

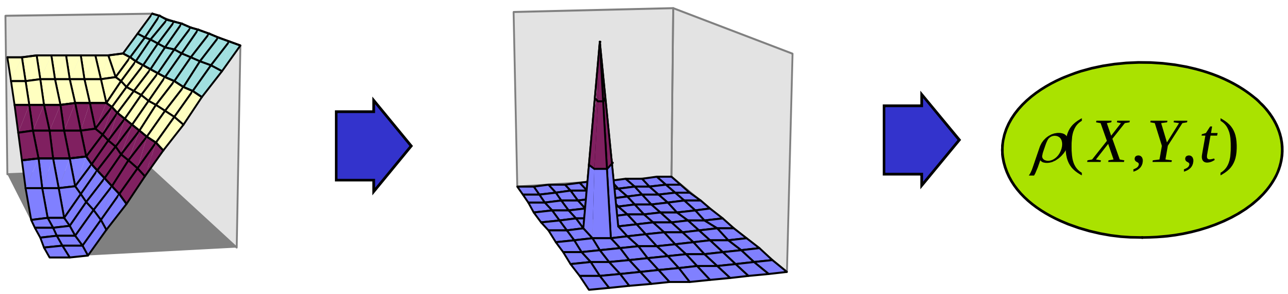
Application: read local correlation

How to read local correlation from MAX Option?

$$M(K_X, K_Y, T) = \max(X_T - K_X, Y_T - K_Y, 0)$$

• First, construct a portfolio of MAX option such that it only has value when X and Y equal to particular values

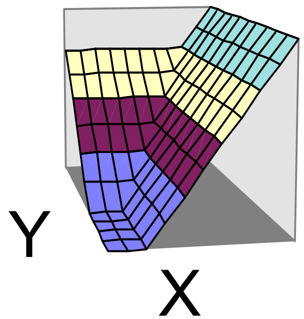
• Then, read correlation by aggregation

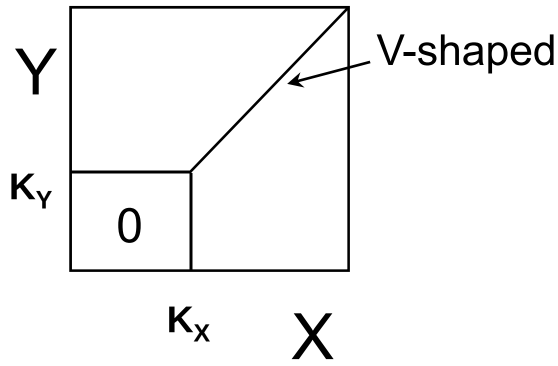


Application: read local correlation

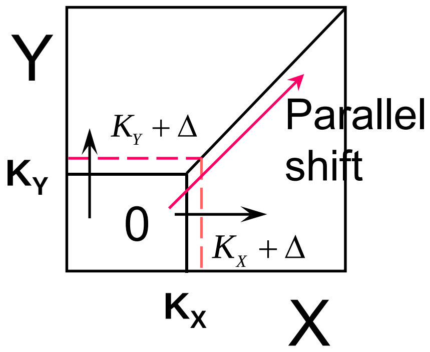
- MAX option $M(K_X,K_Y,T)=\max(X_T-K_X,Y_T-K_Y,0)$

$M(K_X,K_Y,T)$

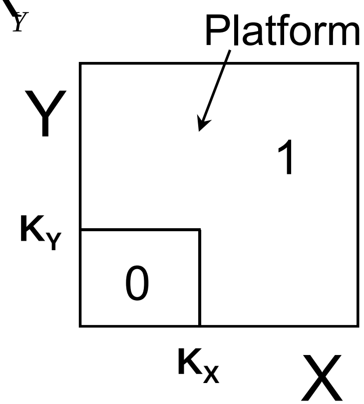




- Create a spread along $X-K_X=Y-K_Y$

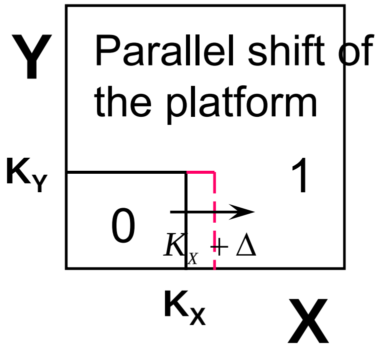


$$SM_{(X+Y)}=\frac{M(K_X,K_Y,T)-M(K_X+\Delta,K_Y+\Delta,T)}{\Delta}$$

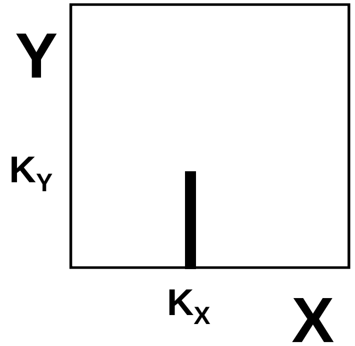


Application: read local correlation

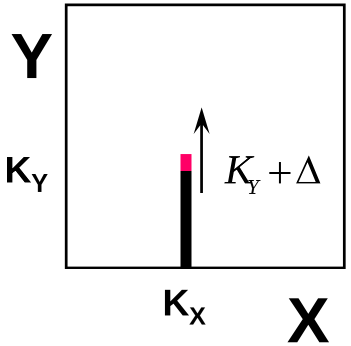
- Create a spread on $SM_{(X+Y)}$ along K_X



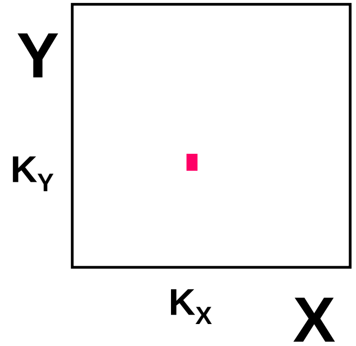
$$SM_{(X+Y)X} = \frac{SM_{(X+Y)}(K_X, K_Y, T) - SM_{(X+Y)}(K_X + \Delta, K_Y, T)}{\Delta}$$



- Create a spread on $SM_{(X+Y)X}$ along K_Y



$$SM_{(X+Y)XY} = \frac{SM_{(X+Y)X}(K_X, K_Y + \Delta, T) - SM_{(X+Y)X}(K_X, K_Y, T)}{\Delta}$$



Application: read local correlation

$$\mathrm{SM}_{(X+Y)XY} = \text{Probability density}(K_X, K_Y, T) = \phi(K_X, K_Y, T)$$

Fokker-Planck equation:

$$\frac{\partial \phi}{\partial T} = \frac{1}{2}\frac{\partial^2 \sigma_X^2 \phi}{\partial X^2} + \frac{1}{2}\frac{\partial^2 \sigma_Y^2 \phi}{\partial Y^2} + \frac{\partial^2 \sigma_X \sigma_Y \rho \phi}{\partial X \partial Y}$$

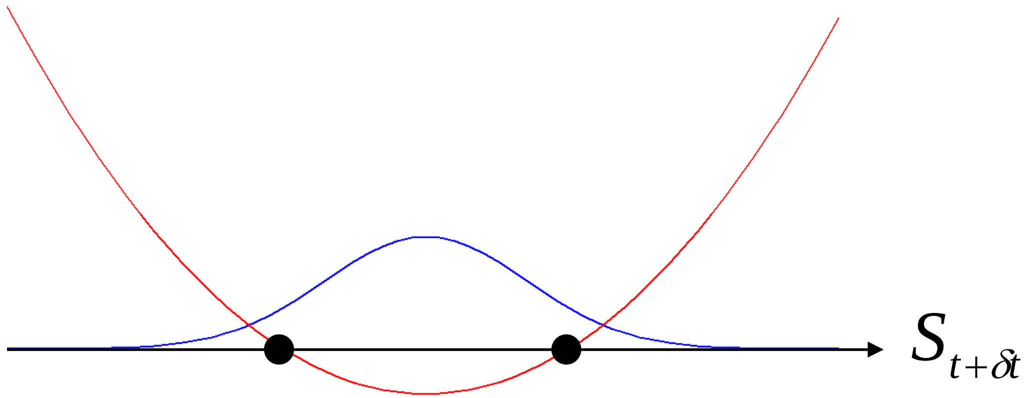
$$\Downarrow$$

$$\rho(K_X, K_Y, T) = \frac{\dfrac{\partial}{\partial T}\displaystyle\int_{X \le K_X}\int_{Y \le K_Y} \phi \, dY dX - \frac{1}{2}\int_{Y \le K_Y} \left.\frac{\partial \sigma_X^2 \phi}{\partial X}\right|_{X=K_X} dY - \frac{1}{2}\int_{X \le K_X} \left.\frac{\partial \sigma_Y^2 \phi}{\partial Y}\right|_{Y=K_Y} dX}{\sigma_X(K_X, T)\sigma_Y(K_Y, T)\phi(K_X, K_Y, T)}$$

Reprise of Break-Even Points

- Break-even points (BEP): price of the underlying(s) that leave PL of Δ-hedged position unaffected.

<u>1D Case</u>



1D BEP are ± 1 SD away from the FWD.
They depend on the price dynamics, not on the option

2D BEP

- 2D BEP are of dimension 1
- At first sight: depends only the Cov matrix

  →WRONG : depends mostly on another quadratic form, the

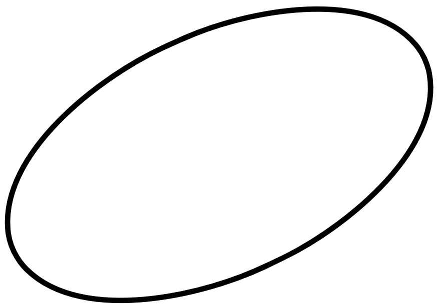
  Hessian of the option price $f(X, Y, t)$: $\begin{pmatrix} f_{XX} & f_{XY} \\ f_{XY} & f_{YY} \end{pmatrix}$

$$\delta PL = \frac{1}{2}\left[f_{XX}\left((\delta X)^2 - \sigma_X^2 \delta t \right) + f_{YY}\left((\delta Y)^2 - \sigma_Y^2 \delta t \right) + 2 f_{XY}\left(\delta X \delta Y - \rho \sigma_X \sigma_Y \delta t \right) \right]$$

2D BEP

- 3 cases for signature of quadratic form:

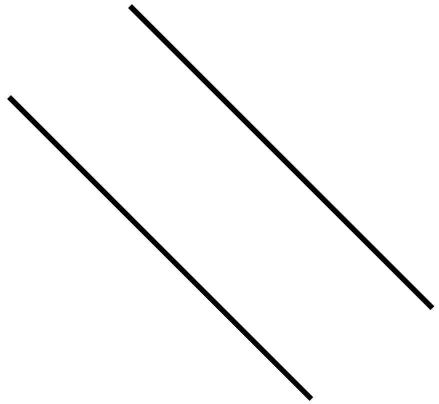
++ :



$$\left(Max\left(X,Y\right)-K\right)^{+}$$

Option on the max

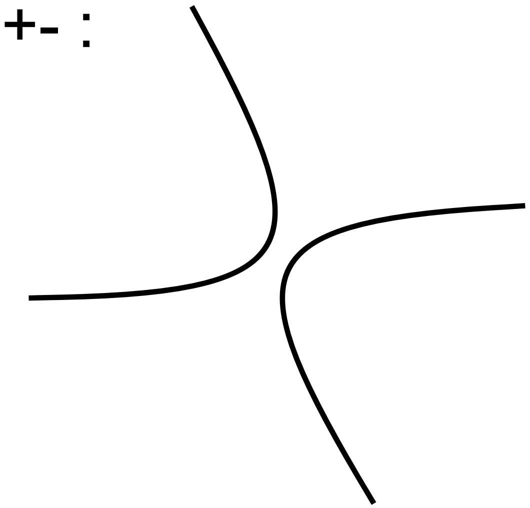
+0 :



$$\left(X+Y-K\right)^{+}$$

Basket option

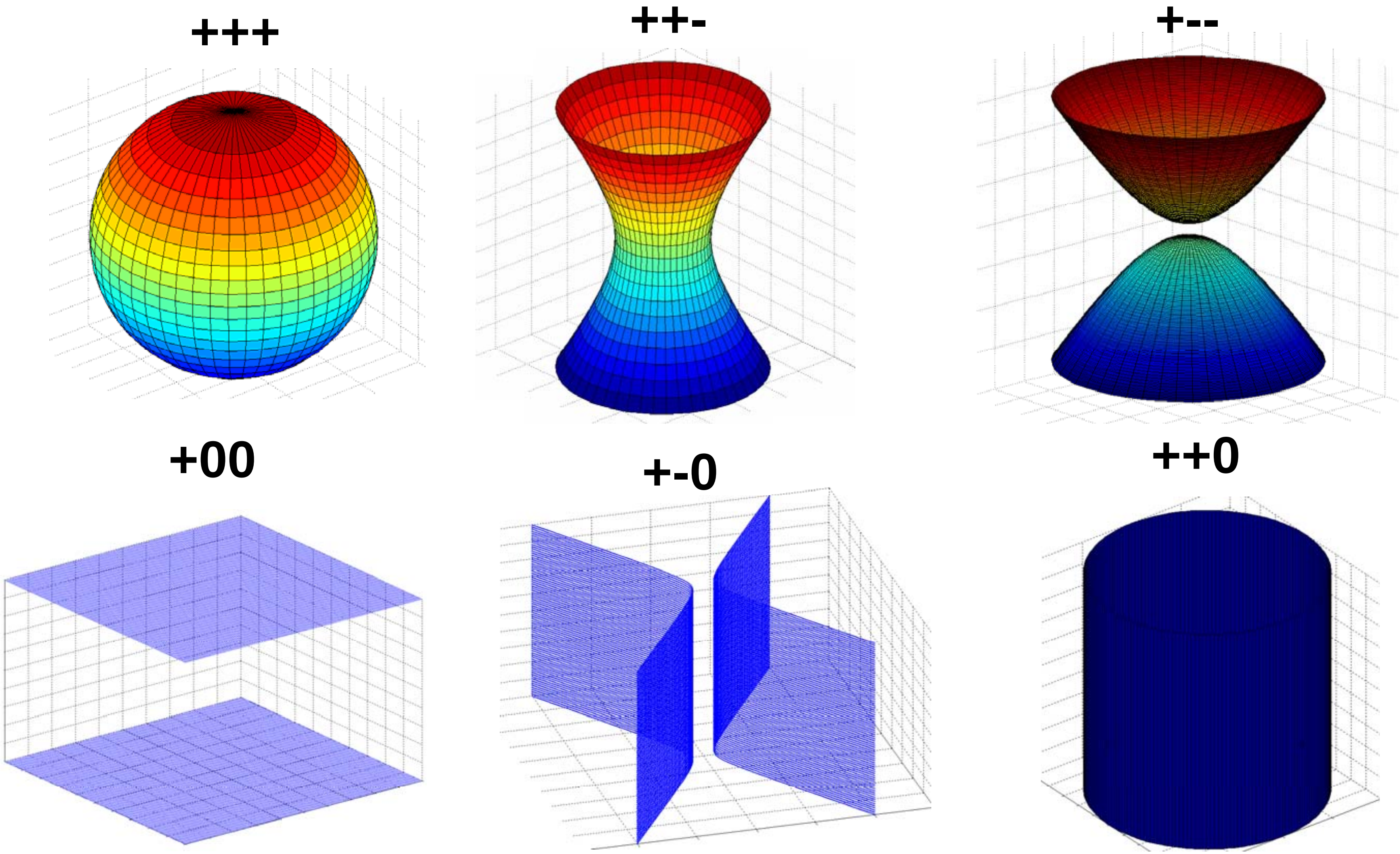
+- :



$$XY$$

Quanto stock

3D BEP



Basket options

Δ-Hedge of Basket Option

- X and Y correlated with ρ
- Option price $C(X,Y,t)$

$$\Delta_X = \frac{\partial C}{\partial X} \qquad \Delta_Y = \frac{\partial C}{\partial Y}$$

Δ hedge = $\Delta_X \tilde{X} + \Delta_Y \tilde{Y} \quad ?$

Or does it depend on correlation?

What if we can only hedge with X?

Γ-Hedge of Basket Option

Option on $X_1, \ldots, X_n$

1 price, n deltas, (n x n) Γ matrix

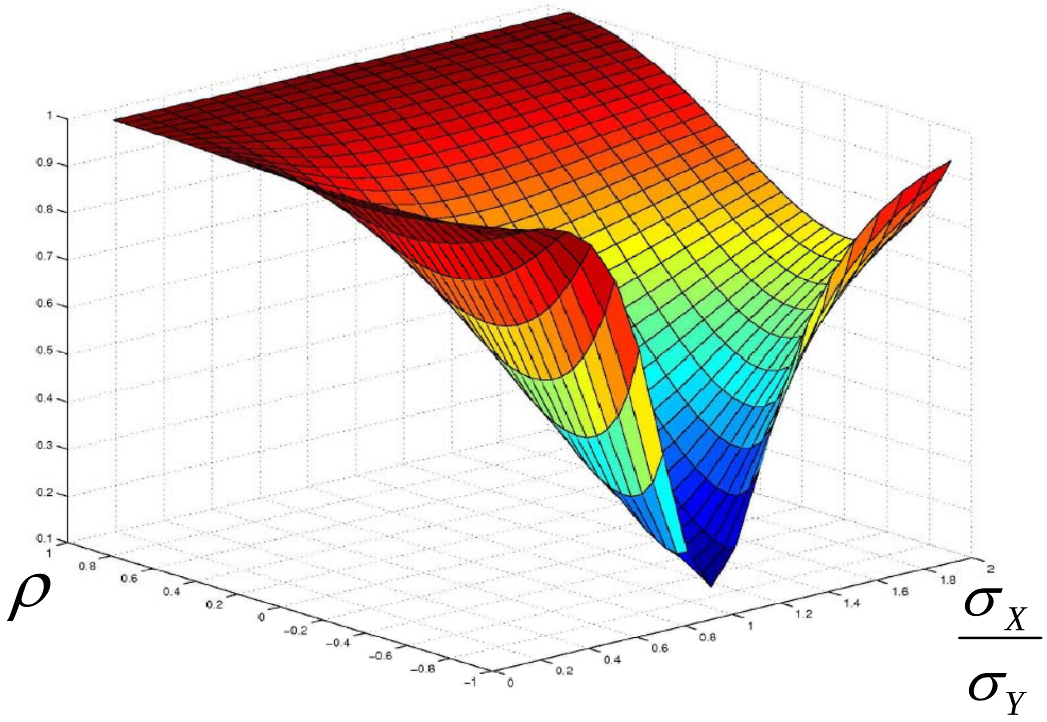
As with Δ:

- If all entries can be hedged, do it
- If only diagonal entries can be hedged, beware of correlation

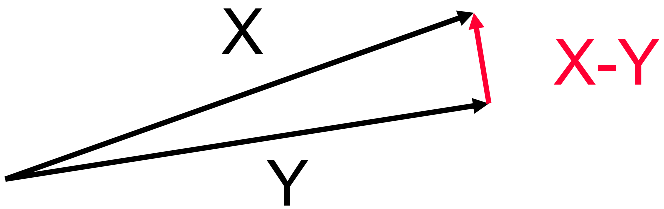
Spread example

$$\left(X - Y - K\right)^{+}$$

If $\sigma_X \sim \sigma_Y$, high correlation, options on X and Y are useless



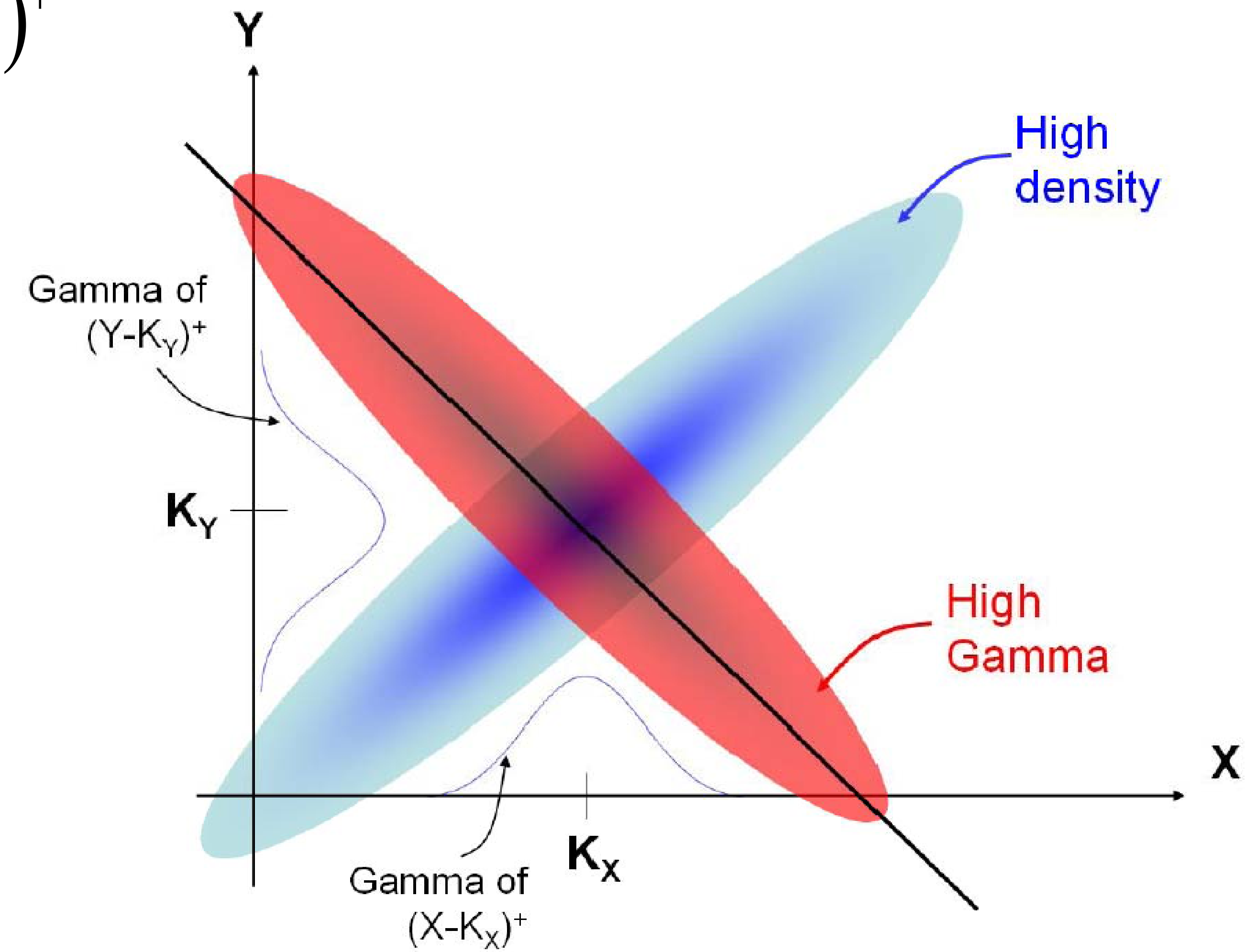
Values of $\mathbb{H}^2$



$\left(X - Y\right)$ almost orthogonal with X and Y

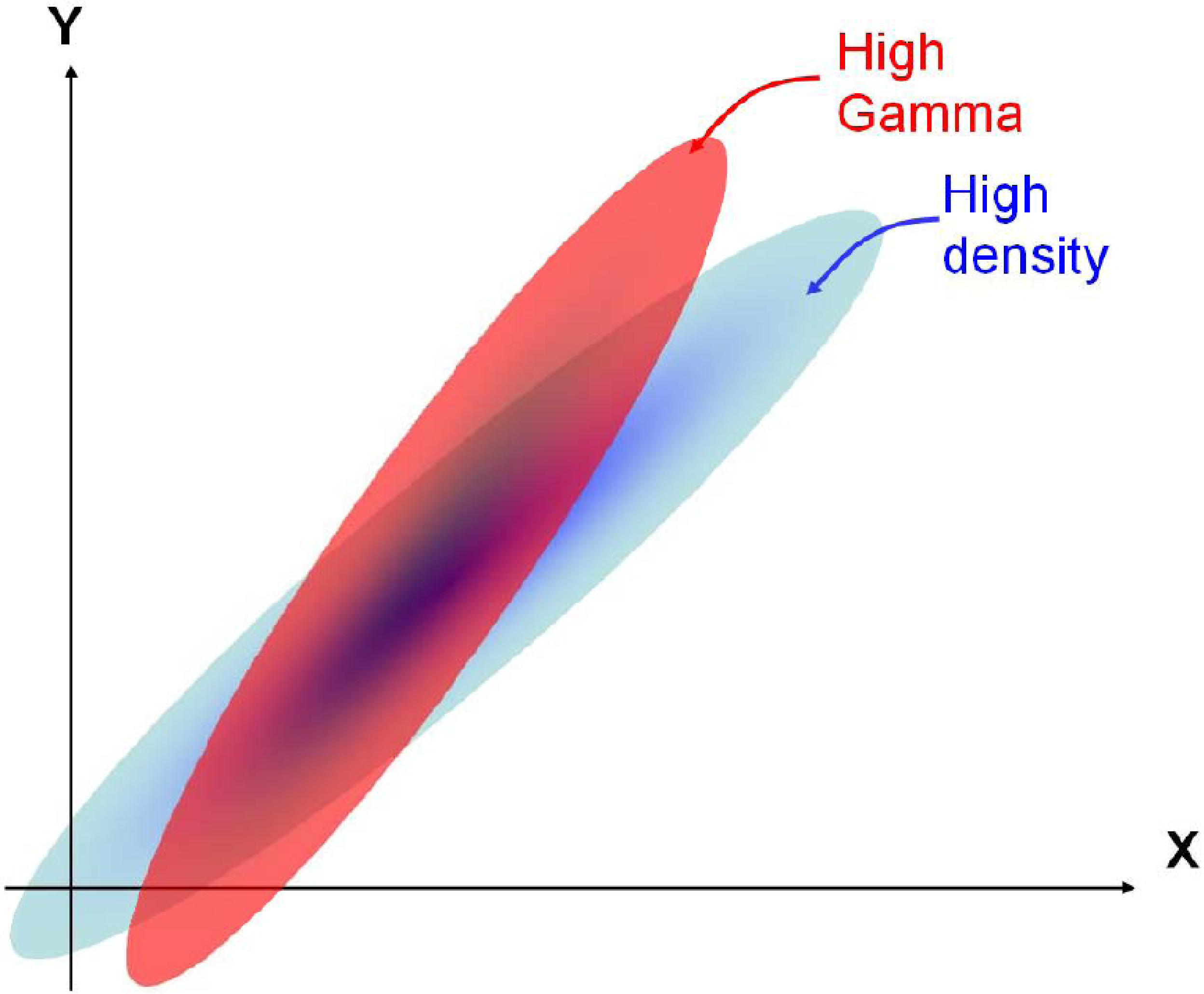
Hedge of Basket Option

$$\left(X+Y-K\right)^{+}$$



Hedge of Spread Option

$$\left(X - Y - K\right)^{+}$$



Basket Correlation Skew

2 reasons for Basket skew:

1) Individual skews
2) State dependent correlation

Individual skew + fixed correlation

- nD LVM, no rates

$$dS_i = \sigma_i(S_i, t)dW_i$$

$$< dW_i, dW_j >= \rho_{ij}\, dt$$

- $I = \sum \alpha_i S_i \qquad dI = \sigma_I\, dW$

with

$$\sigma_I^{\ 2} = \sum\sum \alpha_i \alpha_j \rho_{ij} \sigma_i(S_i, t)\sigma_j(S_j, t) = \sigma_I^{\ 2}(S_1, \ldots, S_n, t)$$

Approximation

Produces same basket option prices as

$$dI = \sigma(I,t)\,dW$$

where

$$\begin{aligned}\sigma(I,t) &\equiv E\left[\sigma_I^2(S_1,\ldots,S_n,t) \mid I_t = I\right] \\ &\sim \sigma_I^2\left(E[S_1 \mid I_t = I],\ldots,E[S_n \mid I_t = I],t\right) \\ &= \sum\sum \alpha_i\alpha_j\, E[S_i \mid I_t = I]\,E[S_j \mid I_t = I]\end{aligned}$$

FWD PDE → Basket option prices and skew

State dependent correlation

- Assume S_i, i=1…n flat Bachelier $dS_i = \sigma_i dW_i$ with σ_i constant

  Correlation matrix at *t* indexed by one variable θ:

$$\rho(S_i, S_j, t) = \rho_{ij}(\theta, t)$$

$$I = \sum \alpha_i S_i \qquad dI = \sigma_I dW$$

$$\sigma_I^2 = \sum\sum \alpha_i \alpha_j \rho_{ij}(\theta, t) = f(\theta, t)$$

- If we know the vanillas on I, we know the local vol $\sigma(I, t)$ and $f(\theta, t) = \sigma^2(I, t)$ can be inverted → $\theta = \theta(I, t)$

- Conclusion: $\begin{cases} dS_i = \sigma_i dW_i \\ \rho_{ij} = \rho_{ij}(\theta(I, t), t) \end{cases}$ ← Instantaneous correlation skew

is a model that fits the skew of I

Barrier option on basket

- If the barrier is triggered by the basket value, one can use the static replication as a basket Call option minus a basket Put option

- In this case, it is better to hedge with vanillas on the components as opposed to barriers on the components

Mountain Range Options and Correlation risk management

Mountain Range Options

- **Altiplano**

$$\begin{cases} \left(\sum_i \frac{S_i(T)}{S_i(0)} - K \right)^+ & \text{if} \quad \min_{i,t} \left(\frac{S_i(t)}{S_i(0)} \right) \le L \\ 1 & \text{else} \end{cases}$$

- **Atlas**

$$\left(\sum_{i=1+n_1}^{n-n_2} \frac{S_i(T)}{S_i(0)} - K \right)^+$$

Where $S_1,\ldots,S_{n_1}$ are the n_1 worst stocks

and $S_{n-n_2+1},\ldots,S_n$ are the n_2 best stocks

Mountain Range Options

- **Everest**

$$\min_i \left(\frac{S_i(T)}{S_i(0)} \right)$$

- **Annapurna**

$$1 \quad \text{if} \quad \min_{i,t} \left(\frac{S_i(t)}{S_i(0)} \right) \geq L$$

- **Himalaya**

$$\sum_i \frac{S_{n(i)}(T_i)}{S_{n(i)}(0)}$$

Where $S_{n(i)}$ is the best remaining stock at time T_i

(and it is then removed from the basket)

Correlation Risk Management

- Option A with price $A(X, Y, \sigma_X, \sigma_Y, \rho, t)$

  with $\mathrm{Rega}(A) = \dfrac{\partial A}{\partial \rho}$

- Hedge with $B(X, Y, \sigma_X, \sigma_Y, \rho, t)$

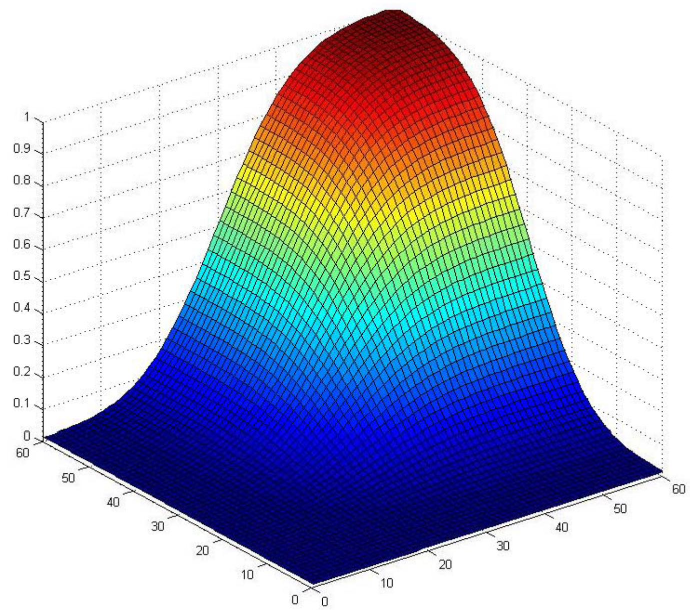
- $\mathrm{Rega}\left(A - \dfrac{\mathrm{Rega}(A)}{\mathrm{Rega}(B)} B \right) = 0$

So selling $\dfrac{\mathrm{Rega}(A)}{\mathrm{Rega}(B)} B$ seems to be a good hedge

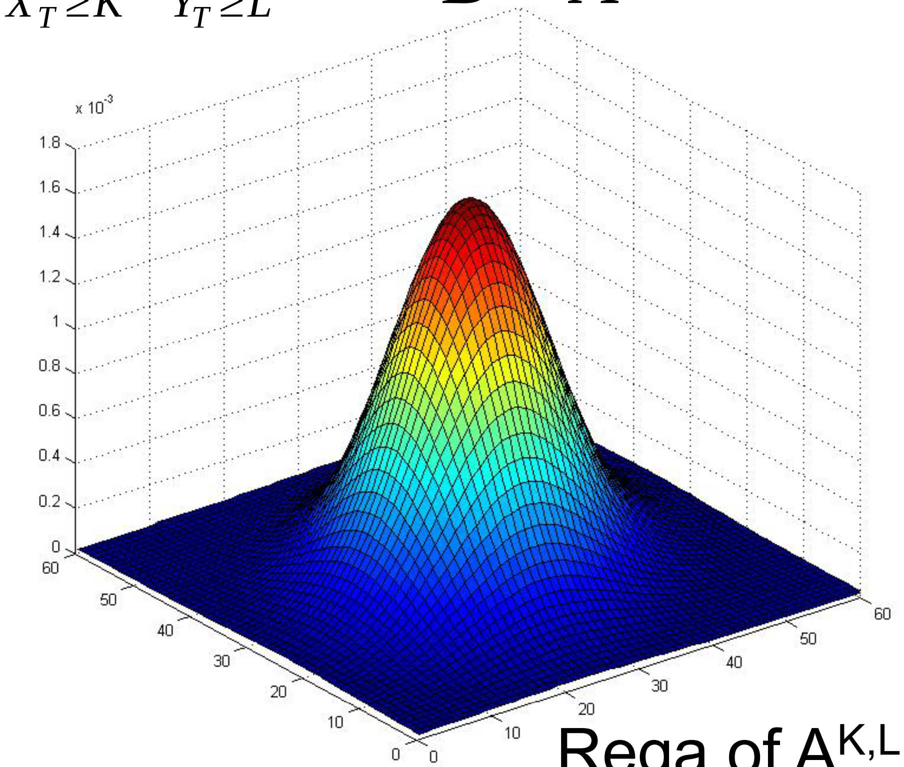
Rega is not a single number

- Cancelling Rega today is no guarantee for the future
- Example: simplified ANNAPURNA

  Payoff $A^{K,L}\left(X_T, Y_T, T\right) = 1_{X_T \geq K} 1_{Y_T \geq L}$ $\qquad B = A^{K',L}$

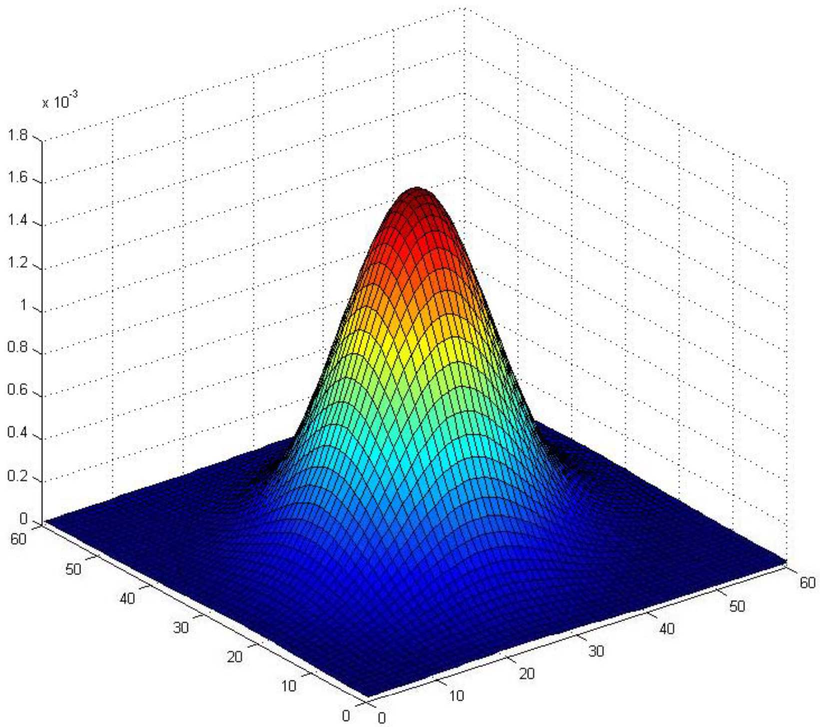


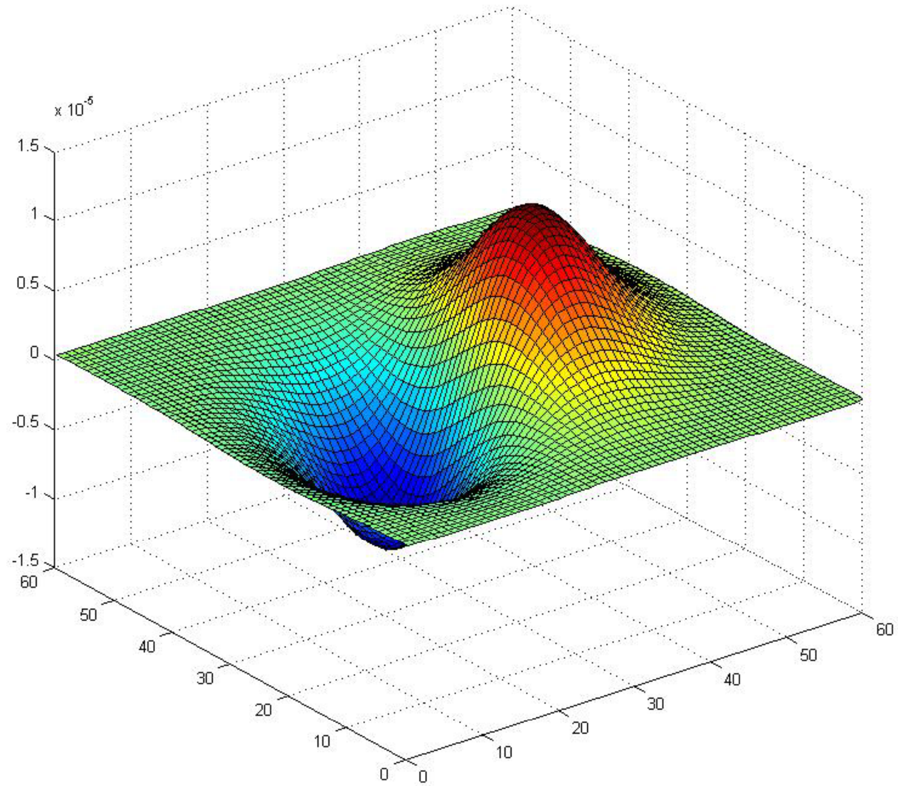
Price of $A^{K,L}$



Rega of $A^{K,L}$

The danger of naïve Rega hedging





Before hedge

$\text{Rega}\left(A^{K,L}\right)$

After hedge

$\text{Rega}\left(A^{K,L} - A^{K',L}\right)$

Sensitivity to correlation

- 3 stocks X_1,X_2 and X_3
- A : Pay-off at T : Second highest value
- We assume $X_1(t) \geq X_2(t) \geq X_3(t)$
- Correlation matrix : $\begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}$
- $\text{Rega} = \dfrac{\partial A}{\partial \rho}$

Rega > 0 ?

- For instance, assume :

$X_1(t)=120;\ X_2(t)=119;\ X_3(t)=80$

Then A~min(X_1,X_2) at T

With $\min(X_1, X_2) = X_1 - (X_1 - X_2)^+$

Short a spread option ➔ Rega>0

Rega < 0 ?

- Now, assume :

$$X_1(t)=120;\ X_2(t)=81;\ X_3(t)=80$$

Then A~max(X_2,X_3) at T

With $\max(X_2, X_3) = X_2 + (X_3 - X_2)^+$

Short a spread option ➔ Rega<0

Correlation arbitrage

I. FX Arbitrage

FX Triangle Arbitrage

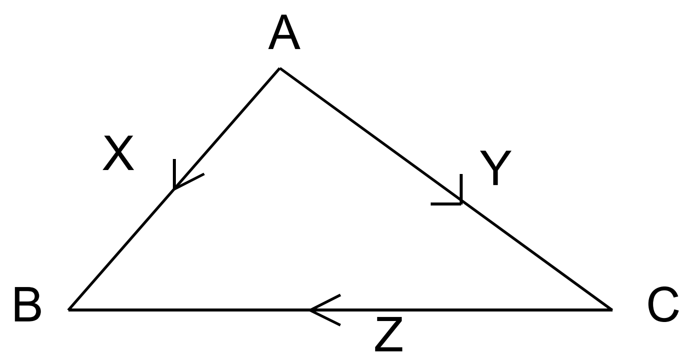
$X \equiv EUR/USD$ $Z \equiv EUR/JPY$ $Y \equiv JPY/USD$

Spot arbitrage: $Z_t = \dfrac{X_t}{Y_t}$

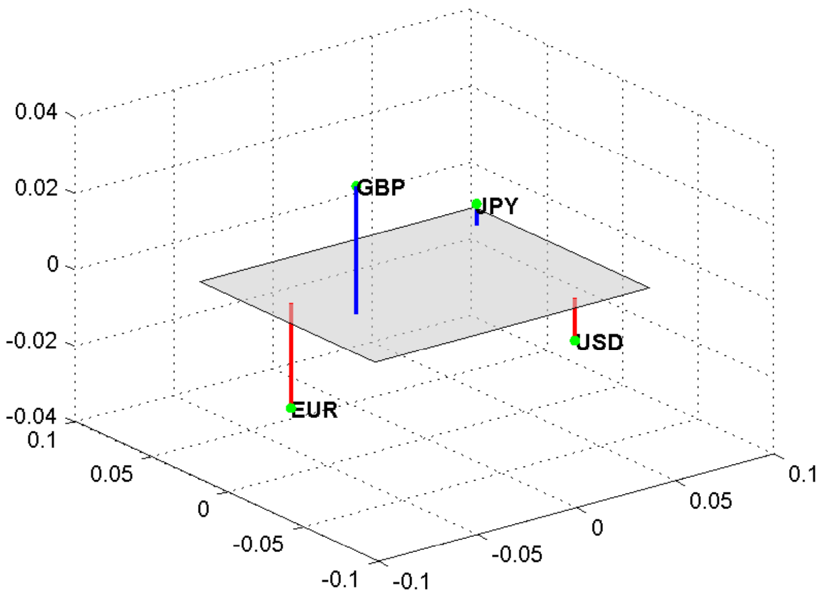
Vol arbitrage: $\sigma_Z^2 = \sigma_{X/Y}^2 = \sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y$

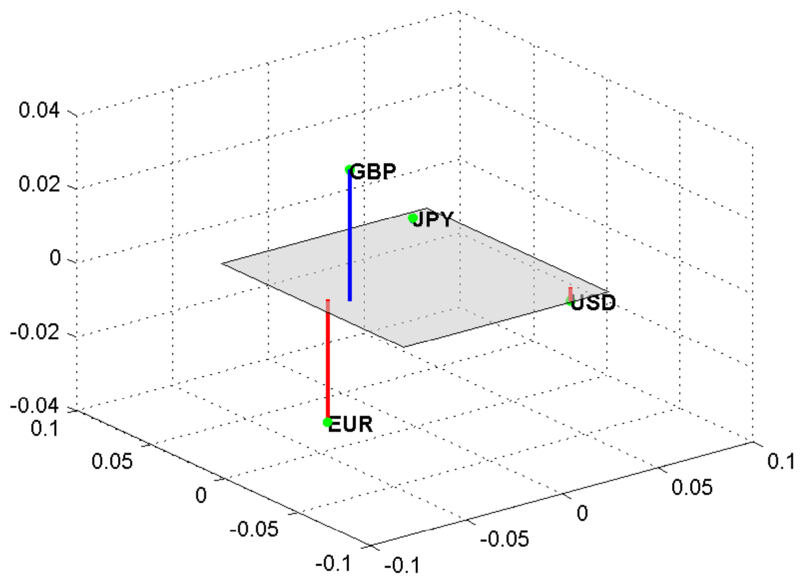
$$\Rightarrow \left|\sigma_X - \sigma_Y\right| \le \sigma_Z \le \sigma_X + \sigma_Y$$

Implemented by: $(X - Z_0 Y)^+ \le (X - X_0)^+ + Z_0(Y_0 - Y)^+$



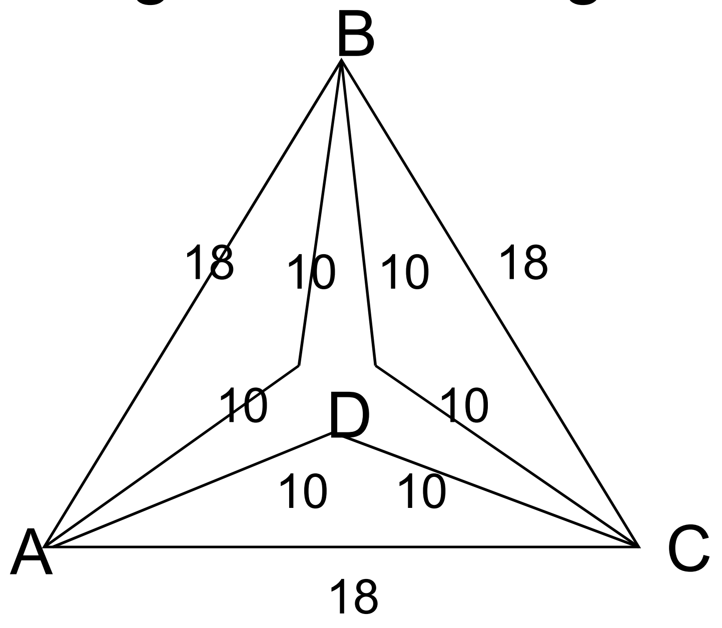
N Currency Case





Tetrahedron Arbitrage

- With 4 currencies, all triangles may be viable but still there is a global arbitrage



- In general, it is possible to “trade” the height of a simplex

nD Arbitrage

The identity

$$\left\|\sum_{i=1}^{n}\alpha_i X_i\right\|^2 = \left(\sum_{i=1}^{n}\alpha_i\right)\sum_{i=1}^{n}\alpha_i\left\|X_i\right\|^2 - \sum_{i=1}^{n}\sum_{j<i}^{n}\alpha_i\alpha_j\left\|X_i - X_j\right\|^2$$

gives, as $\sum_{i=1}^{n}\alpha_i = 1$

$$\sum_{i=1}^{n}\sum_{j<i}^{n}\alpha_i\alpha_j\sigma_{i,j}^2 \le \sum_{i=1}^{n}\alpha_i\sigma_{i,0}^2$$

The difference is minimized by $\alpha = \dfrac{V^{-1}1}{1'V^{-1}1}$ with $v_{i,j} \equiv \left\langle X_i, X_j \right\rangle$

If the simplex is too flat, buy VS on straight pairs and sell VS on crosses (short maturity to cancel the quanto effect)

II. Dispersion Arbitrage

Dispersion Trades

Index $I = \sum \alpha_i S_i, \quad \sum \alpha_i = 1$

Historical σ, ρ: $\sigma_I^2 = \sum \sum \alpha_i \alpha_j \rho_{ij} \sigma_i \sigma_j$

Global historical ρ / $\sigma_I^2 = \sum \sum \alpha_i \alpha_j \rho_{ij} \sigma_i \sigma_j \approx \rho(\sum \alpha_i \sigma_i)^2$ ($\rho = 1$ on diagonal)

$$\rho = \left(\frac{\sigma_I}{\sum \alpha_i \sigma_i}\right)^2$$

Implied ρ: $\hat{\rho} = \left(\frac{\hat{\sigma}_I}{\sum \alpha_i \hat{\sigma}_i}\right)^2$

Usually, $\rho < \hat{\rho} < 1$: buy basket of options, sell Index options.

Correlation / Dispersion

- Index $I=\sum \alpha_i S_i \qquad \sum \alpha_i = 1$
- $Par_i =$ Parabolic profile on S_i
- To lock Dispersion $\left(\sum \alpha_i \sigma_i^2 - \sigma_I^2\right)$
  - Buy $\sum \alpha_i Par_i$
  - Sell Par_I
- To lock Diversification $\left(\left(\sum \alpha_i \sigma_i\right)^2 - \sigma_I^2\right)$
  - Buy $\sum_i \left(\sum_j \alpha_j \sigma_j\right) \frac{\alpha_i}{\sigma_i} Par_i$
  - Sell Par_I

$$\sigma_I \leq \sum \alpha_i \sigma_i \leq \sqrt{\sum \alpha_i \sigma_i^2}$$

Correlation / Dispersion (2)

Dispersion

$$\left(\sum \alpha_i \delta_i\right)^2 = \left(\sum \sqrt{\alpha_i} . \sqrt{\alpha_i} \delta_i\right)^2 \le \left(\sum \alpha_i\right)\left(\sum \alpha_i \delta_i^2\right) = \sum \alpha_i \delta_i{}^2$$

$$Par_I \le \sum \alpha_i Par_i$$

$$\sigma_I^2 \le \sum \alpha_i \sigma_i^2$$

Diversification

$$\left(\sum \alpha_i \delta_i\right)^2 = \left(\sum \sqrt{\alpha_i \sigma_i} . \sqrt{\frac{\alpha_i}{\sigma_i}} \delta_i\right)^2 \le \left(\sum \alpha_i \sigma_i\right)\left(\sum \frac{\alpha_i}{\sigma_i} \delta_i^2\right)$$

$$Par_I \le \left(\sum \alpha_i \sigma_i\right) \sum \frac{\alpha_i}{\sigma_i} Par_i$$

$$\sigma_I^2 \le \left(\sum \alpha_i \sigma_i\right) \sum \frac{\alpha_i}{\sigma_i} \sigma_i^2 = \left(\sum \alpha_i \sigma_i\right)^2 \Rightarrow \sigma_I \le \sum \alpha_i \sigma_i \le \sqrt{\sum \alpha_i \sigma_i^2}$$

Cheapest super-replication of Par_I with a portfolio of Par_i (or Variance Swaps)

Conclusion

- Viewing volatility as an asset class is not a fiction anymore:
  - options capture different flavors of volatility
  - much can be extracted from vanillas
  - volatility linked products are being launched

- The same is becoming true for correlation.
- However, exotics give more information on joint densities