Pricing CMS With Smile

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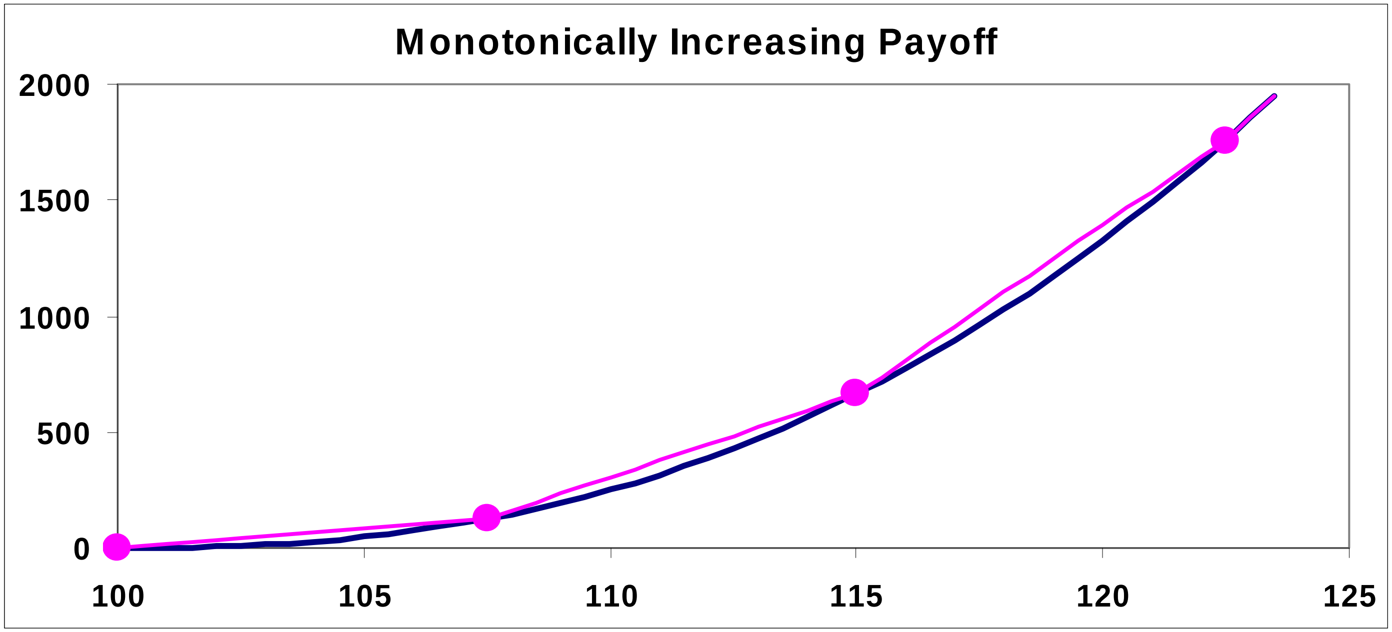
1. Replication Principles

- **Replication (pricing) has largely been associated with Static Hedge (practice);**
- **Comparing to dynamic hedge, Static Hedge is more robust by construction. However, the product range suited for Static Hedge is rather limited;**
- **In the presence of pronounced volatility smile/skew, Replication, which allows one to calibrate to the smile/skew, is a very valuable tool;**
- **Let's review some key principles:**

Replication Formula 1

- **If $f(S)$ is monotonically increasing, and $f(K_0) = 0$, then:**

$$f(S) \approx \sum_i w(K_i) \cdot (S - K_i)^+$$



Payoff Replicated by Calls

- **The weightings are:**

$$w(K_0) = f'(K_0)$$

$$w(K_1) = f'(K_1) - f'(K_0) = f''(K_1) \cdot \Delta K$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$w(K_i) = f'(K_i) - f'(K_{i-1}) = f''(K_i) \cdot \Delta K$$

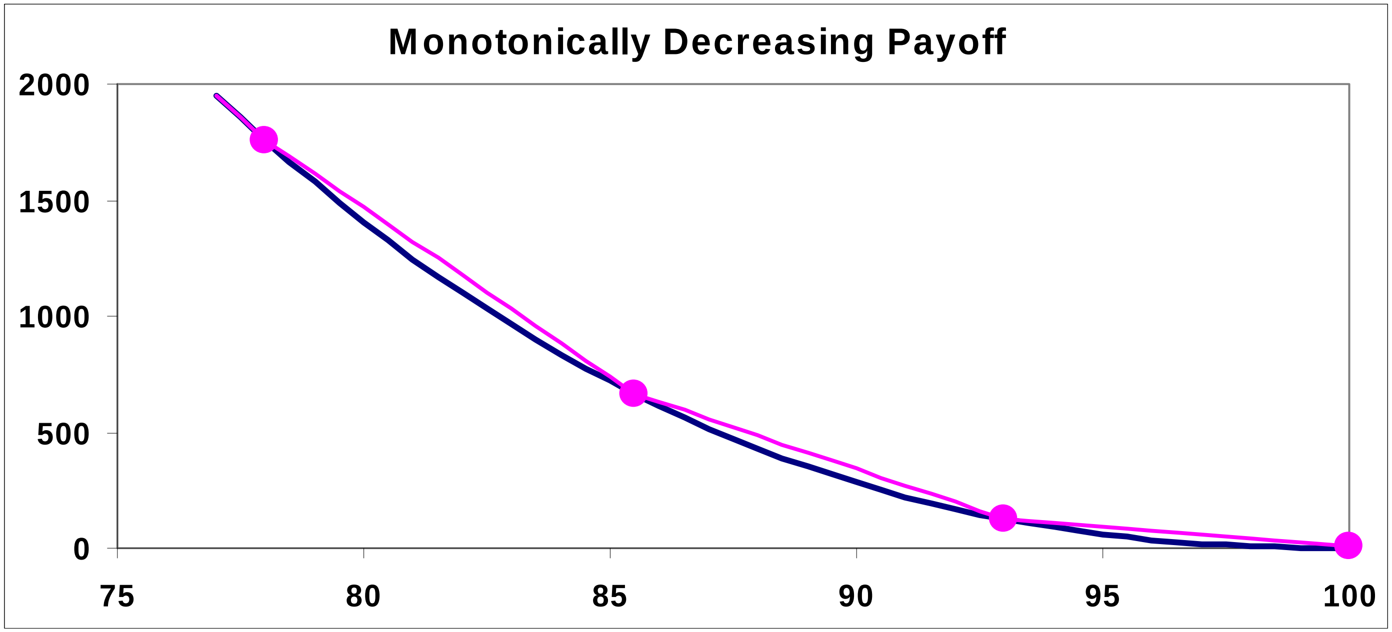
- **When** $\Delta K \to 0$, **Golden Formula 1:**

$$\boxed{f(S) = f'(K_0)(S - K_0)^+ + \int_{K_0}^{\infty} f''(K) \cdot (S - K)^+ \, dK}$$

Replication Formula 2

- **If $f(S)$ is monotonically decreasing, and $f(K_0) = 0$, then:**

$$f(S) \approx \sum_i w(K_i) \cdot (K_i - S)^+$$



Payoff Replicated by Puts

- **The weightings are:**

$$w(K_0) = -f'(K_0)$$
$$w(K_1) = -f'(K_1) + f'(K_0) = -f''(K_1) \cdot \Delta K$$

....

$$w(K_i) = -f'(K_i) + f'(K_{i-1}) = -f''(K_i) \cdot \Delta K$$

- **When** $\Delta K \to 0$, **Golden Formula 2:**

$$\boxed{f(S) = -f'(K_0)(K_0 - S)^+ + \int_0^{K_0} f''(K) \cdot (K - S)^+ dK}$$

Golden Formulae Applications

- **Armed with Golden Formulae, one can replicate (or convexity adjust) many payoffs in all asset classes!**

- **Applying Golden F1 on:** $f(S) = [(S - K_0)^+]^2$

$$[(S - K_0)^+]^2 = 2\int_{K_0}^{\infty} (S - K)^+ \, dK$$

$$\left\langle [(S - K_0)^+]^2 \right\rangle^Q = 2\int_{K_0}^{\infty} \left\langle (S - K)^+ \right\rangle^Q dK$$

Replication Example 1: FX Self-Quanto

- **Payoff in foreign Ccy, equivalent to the following payoff converted to home Ccy:**

$$
\begin{aligned}
(S_T - K_0)^+ \cdot S_T &= (S_T - K_0)^+ \cdot (S_T - K_0 + K_0) \\
&= [(S_T - K_0)^+]^2 + (S_T - K_0)^+ \cdot K_0
\end{aligned}
$$

- **Applying Golden F1:**

$$
\left\langle (S_T - K_0)^+ \cdot S_T \right\rangle = 2\int_{K_0}^{\infty} \left\langle (S_T - K)^+ \right\rangle \cdot dK + \left\langle (S_T - K_0)^+ \right\rangle \cdot K_0
$$

Replication Example 2: Equity Variance Swap

- **Payoff as log contract:**

$$\int_0^T \sigma_t^2 dt = 2 \cdot \int_0^T dS_t / S_t - 2 \cdot \ln\left(S_T / S_0\right)$$

$$\left\langle \frac{1}{T} \int_0^T \sigma_t^2 dt \right\rangle = \frac{2}{T} \left\langle \ln\left(F / S_T\right) \right\rangle$$

- **Applying Golden F1 & F2:**

$$VS = \frac{2}{T}\left[\int_0^F \frac{1}{K^2} \left\langle (K - S_T)^+ \right\rangle dK + \int_F^\infty \frac{1}{K^2} \left\langle (S_T - K)^+ \right\rangle dK \right]$$

Replication Example 3: In-Arrear Cap/Swap

- **The *i*-th caplet payoff:**

$$
\begin{aligned}
& D(0,T_i)\cdot(F_{i,i+1}-K)^+\cdot\tau_i \\
&= D(0,T_{i+1})\cdot\frac{1}{P(T_i,T_{i+1})}(F_{i,i+1}-K)^+\cdot\tau_i \\
&= D(0,T_{i+1})\cdot\{\tau_i^2\cdot[(F_{i,i+1}-K)^+]^2+(\tau_i+\tau_i^2\cdot K)\cdot(F_{i,i+1}-K)^+\}
\end{aligned}
$$

$$
\left\langle D(0,T_i)\cdot(F_{i,i+1}-K)^+\cdot\tau_i\right\rangle^{T_{i+1}} = P(0,T_{i+1})\cdot\tau_i^2\cdot\left\langle[(F_{i,i+1}-K)^+]^2\right\rangle^{T_{i+1}}+\cdots
$$

- **When $K = 0$, it's in-arrear swap (floating leg).**

Replication Example 4: CMS

- **The *i*-th cash flow (*x*-year swap index):**

$$S_{i,x}(t) = \frac{P(t,T_i) - P(t,T_{i+x})}{A_i(t)}$$

where the annuity:

$$A_i(t) = \sum_{j}^{x} \tau_j \cdot P(t,T_j)$$

Under $T_i + \delta$ **measure, CMS rate is:** $\left\langle S_{i,x}(t) \right\rangle^{T_i + \delta}$

CMS Replication

- **Radon-Nikodym, changing *T*-measure to *A*-measure:**

$$\left\langle S_{i,x}(T_i)\right\rangle^{T_i+\delta} = \frac{A_i(0)}{P(0,T_i+\delta)}\cdot\left\langle S_{i,x}(T_i)\cdot\frac{P(T_i,T_i+\delta)}{A_i(T_i)}\right\rangle^{A_i}$$
$$= \frac{A_i(0)}{P(0,T_i+\delta)}\cdot\left\langle S_{i,x}^2(T_i)\cdot\frac{P(T_i,T_i+\delta)}{1-P(T_i,T_i+x)}\right\rangle^{A_i}$$

- **The marked convex function can be approximated and differentiated with respect to $S_{i,x}$;**

- **Golden F1 can then be applied to replicate numerically.**

2. CMS Calibration in SABR Framework

- **All Replications share the same issue: volatility wings!**
- **CMS replication is no exception:**
  - **Needs to integrate along the relevant swaption volatility curve across all strikes;**
  - **Volatility wings will impact the numerical integration and result;**
- **Let's examine CMS replication under the SABR functional vol curve:**

SABR Functional Vol Curve

- **SABR parameters:**
  - α **: ATM driver;**
  - β **: related to smile/skew dynamics, hence it should not be used "too much" for CMS calibration;**
  - ρ **: correlation of fwd and vol, skew tilting;**
  - ν **: vol-on-vol, smile;**
- **All parameters affect the middle part of the vol curve, which will impact European prices!**

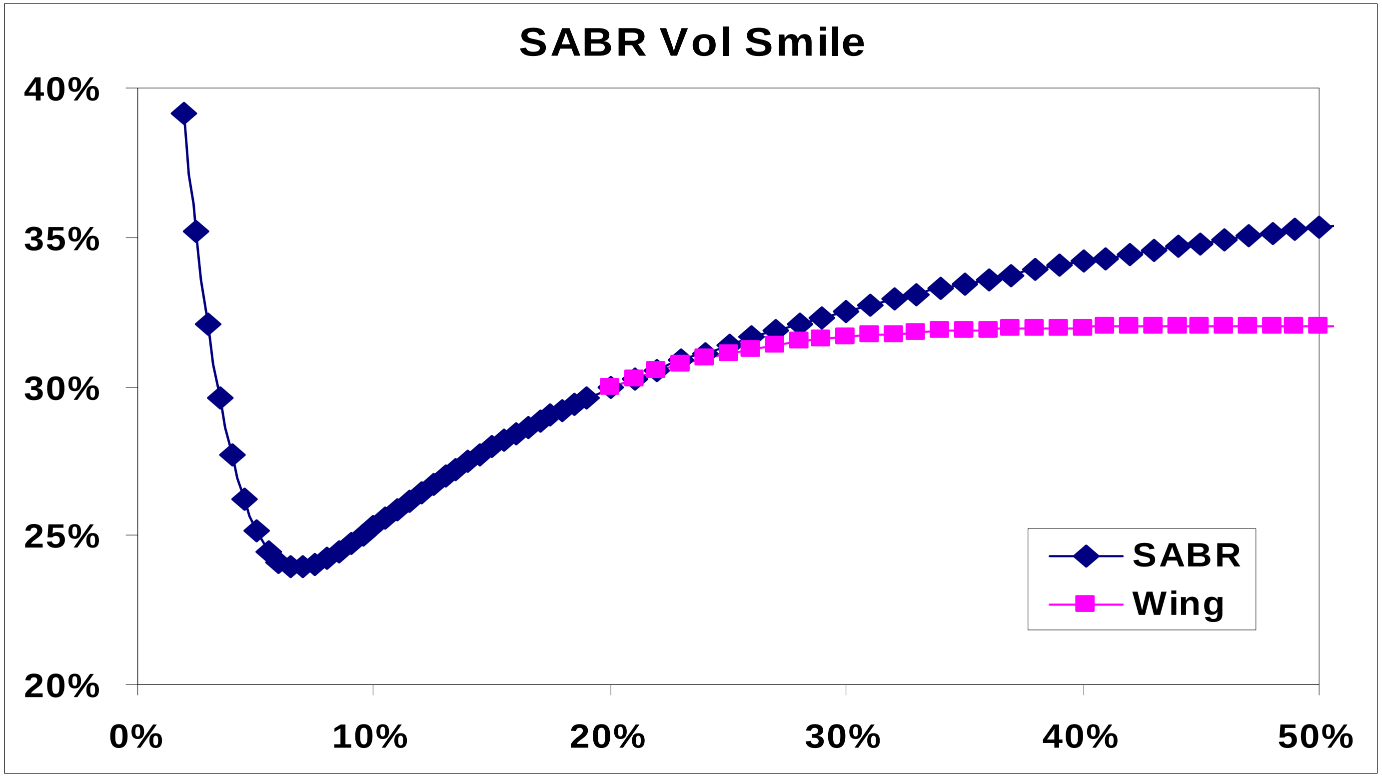
Volatility Wings

- **Therefore it's better to treat Wings separately:**
  - **To avoid impacting on European prices;**
  - **To avoid impacting on volatility smile/skew dynamics, embedded in β ;**

- **Stitching a Wing function [*f (K, A, B)*] to the vol curve:**
  - **The function needs to behave well when *K* ➔ infinity;**
  - **By matching both value and 1st derivative at a chosen high K_h, one can easily solve for *A* and *B*;**
  - **K_h provides a degree of freedom to calibrate CMS.**

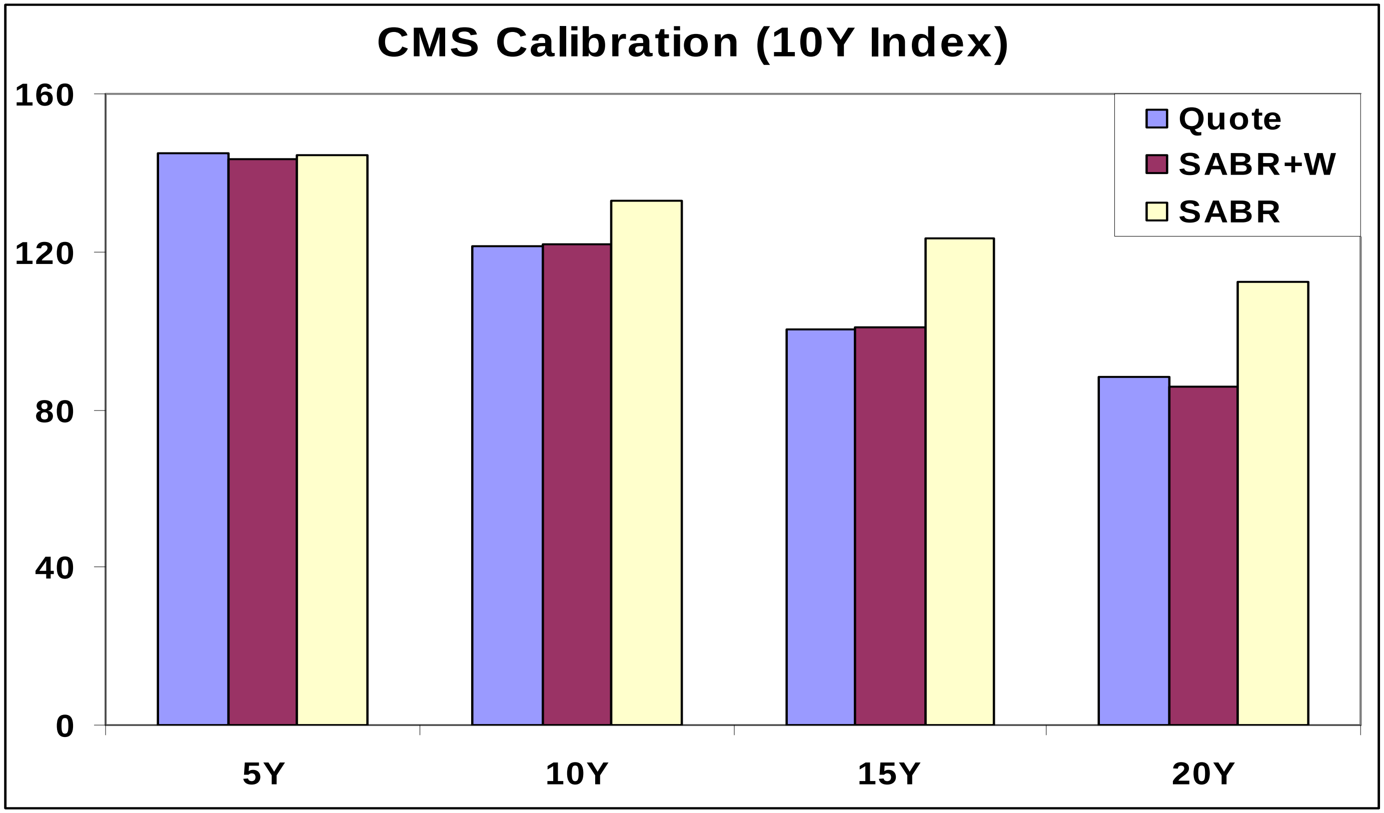
SABR + Wing

- **SABR with a Wing:**

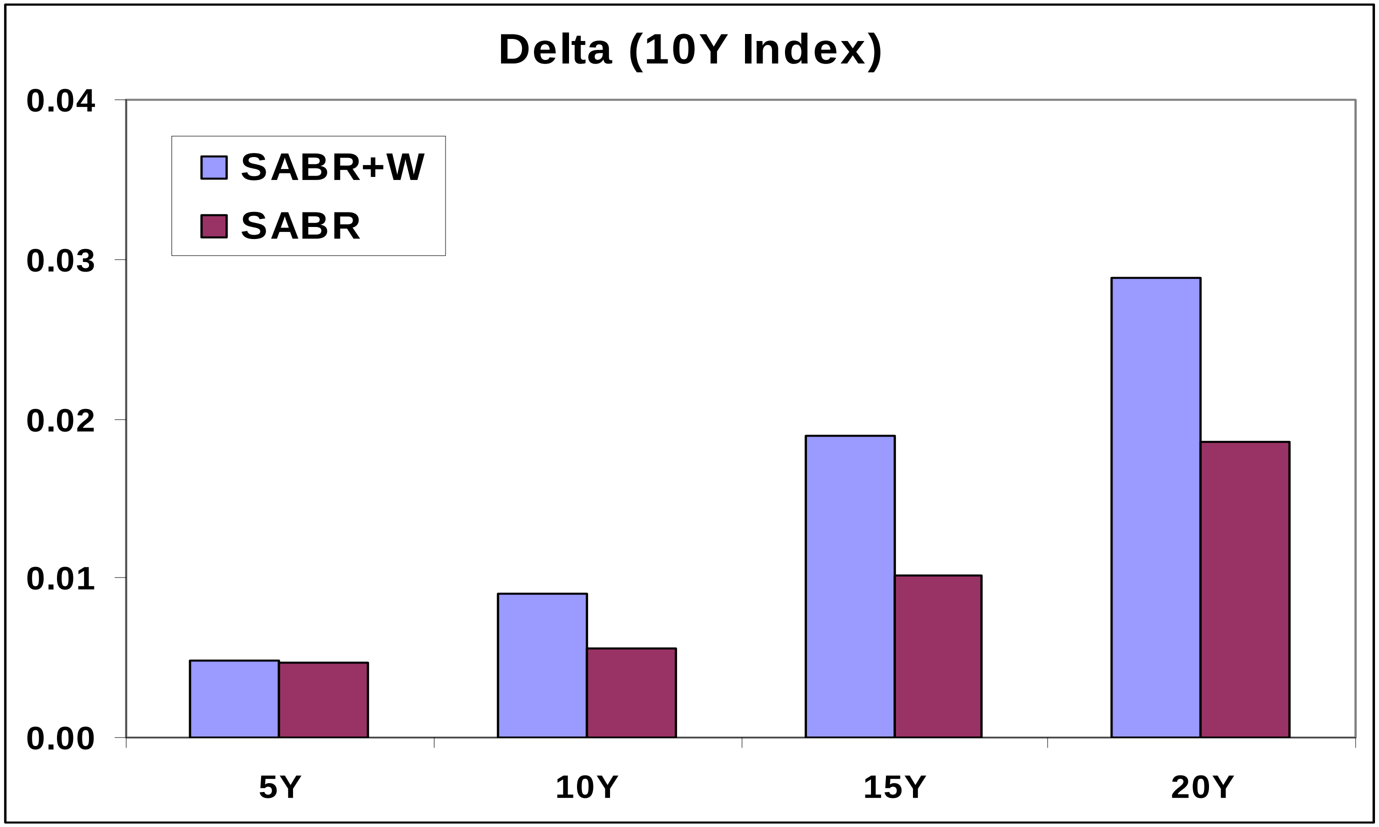


CMS Calibration Example 1

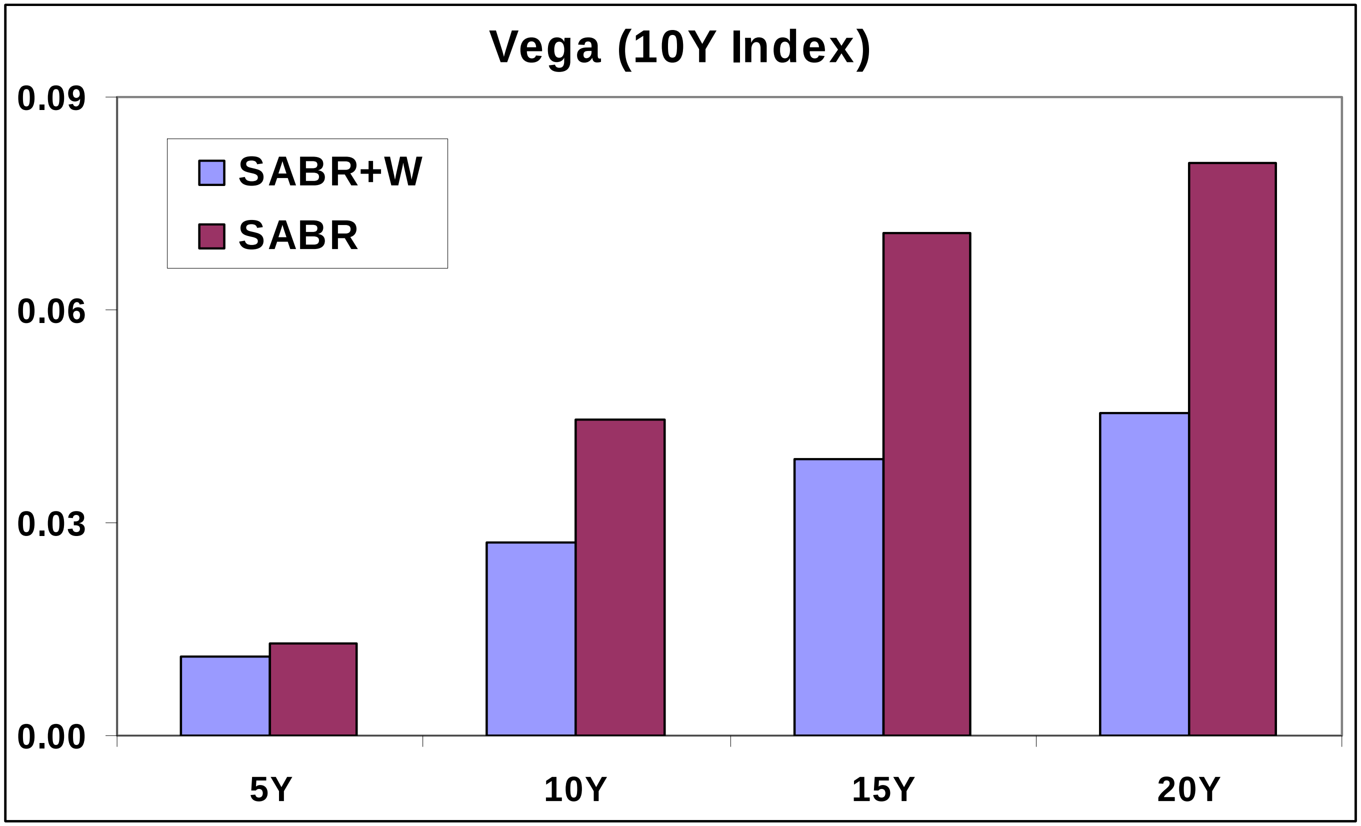
- **10Y index with $K_h = 2.3*Fwd$:**



Delta (Bumping Zero Curve)

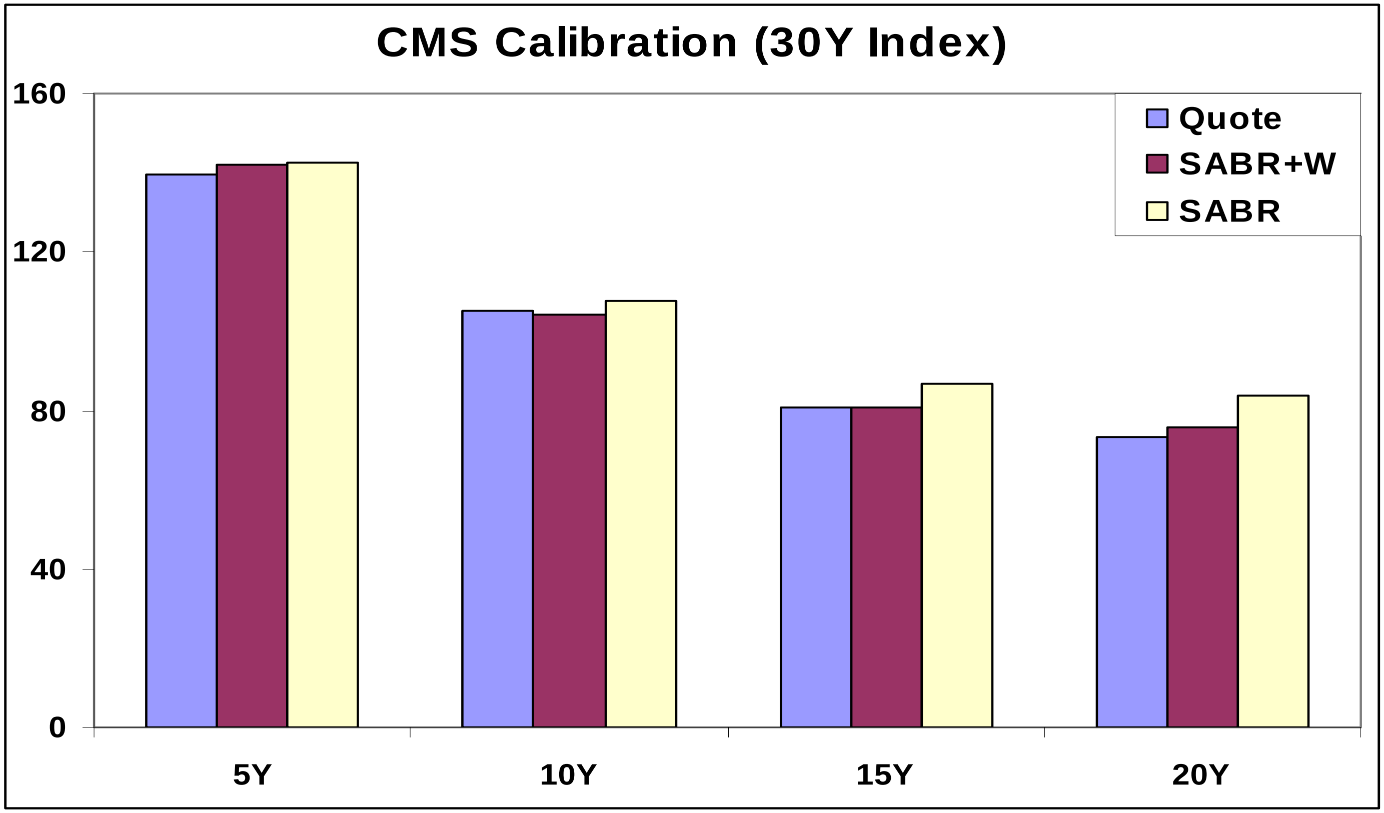


Vega (Bumping ATM Vol)

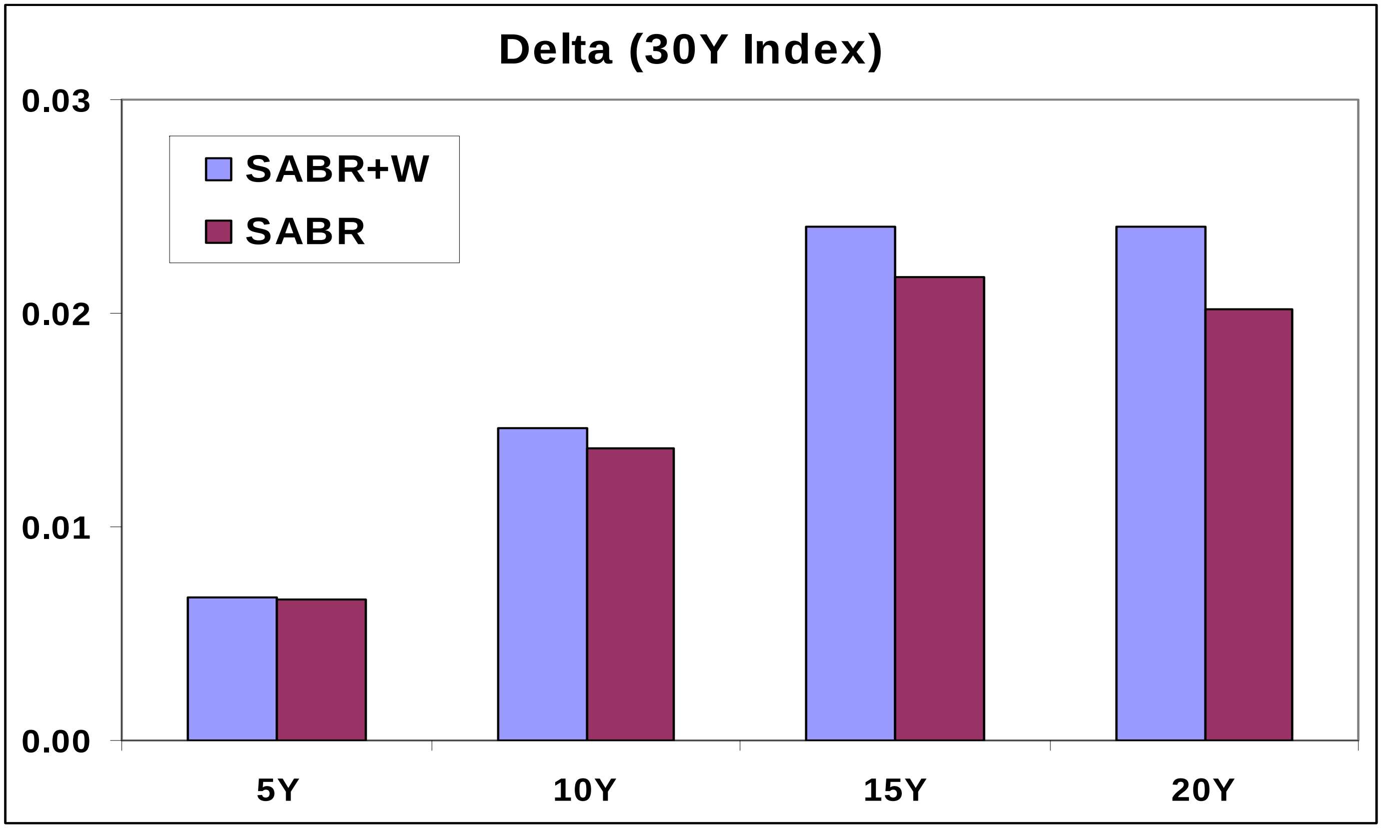


CMS Calibration Example 2

- **30Y index with $K_h = 1.9 * Fwd$:**



Delta (Bumping Zero Curve)



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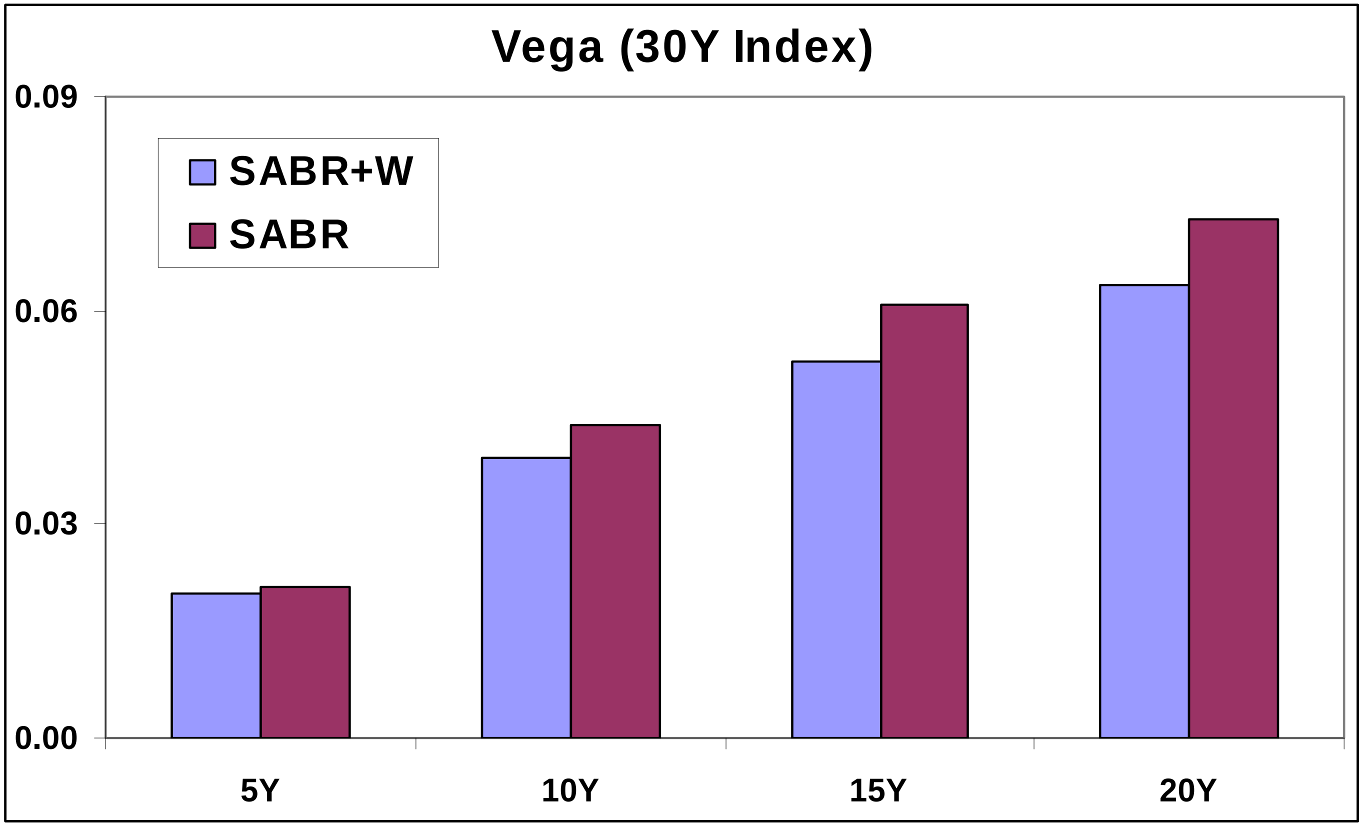
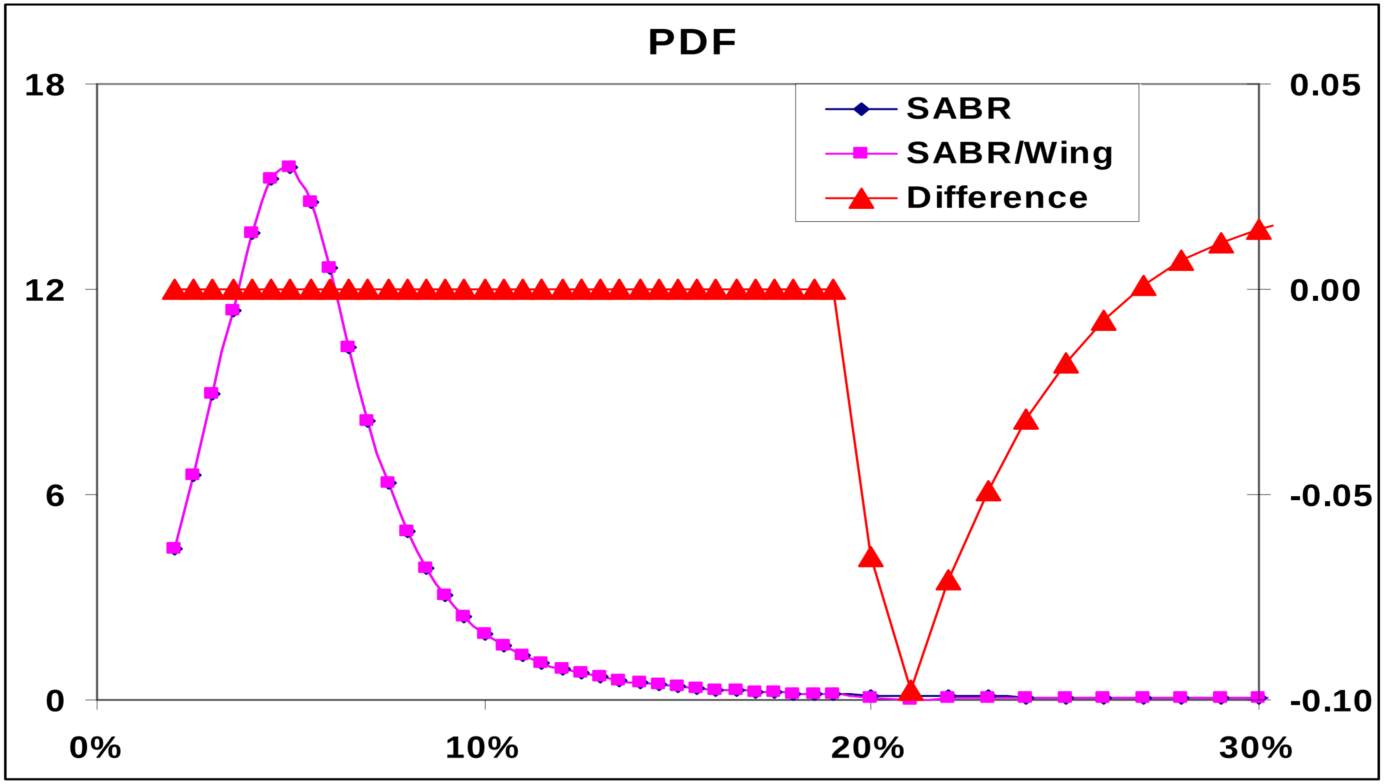


SABR + Wing: PDF

- **The Wing has little impact on PDF:**

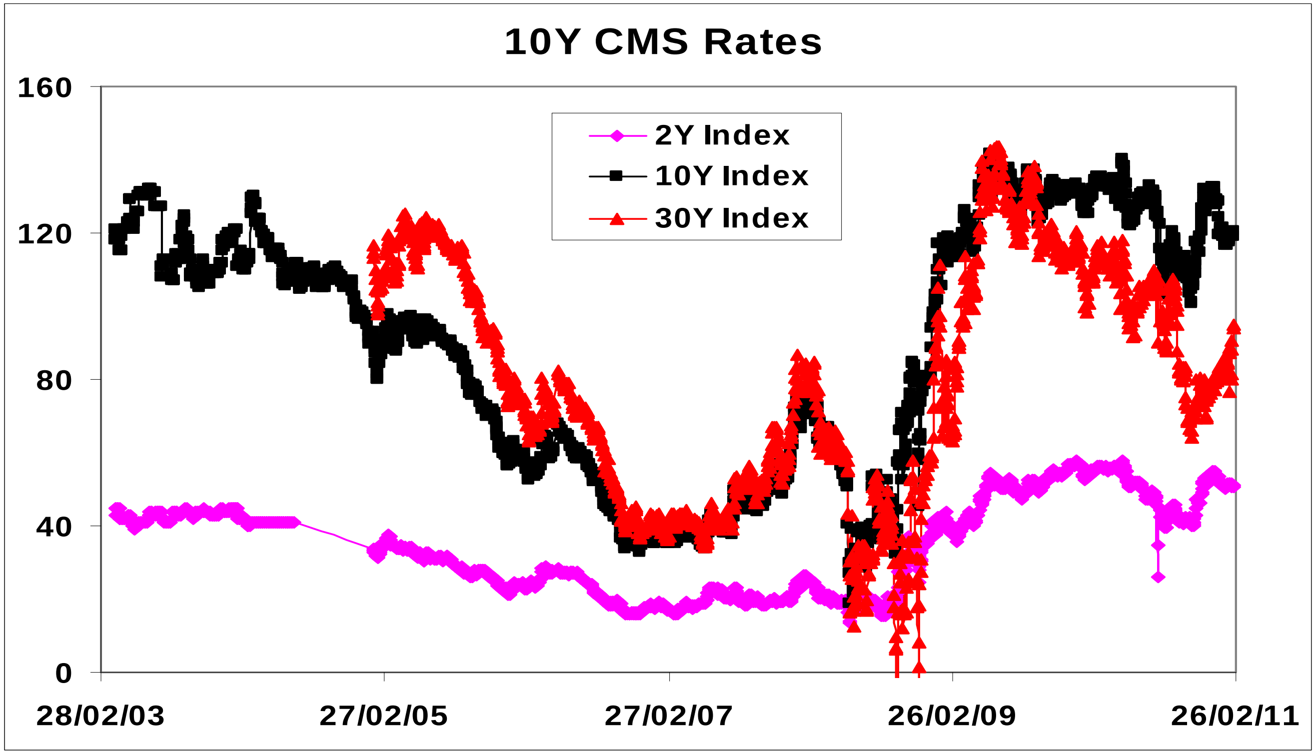


3. Marginal Distributions of CMS Rates

- **CMS rates, though derived from the same yield curve and the same swaption vols, can be viewed as correlated individual components in a basket;**
- **The marginal distributions of CMS rates are important in vol smile/skew calibration and analysing joint statistical characteristics;**
- **This section examines:**
  - **Some historical statistical behaviors (e.g. PDFs);**
  - **Implied PDFs, calculated by using different methodologies;**

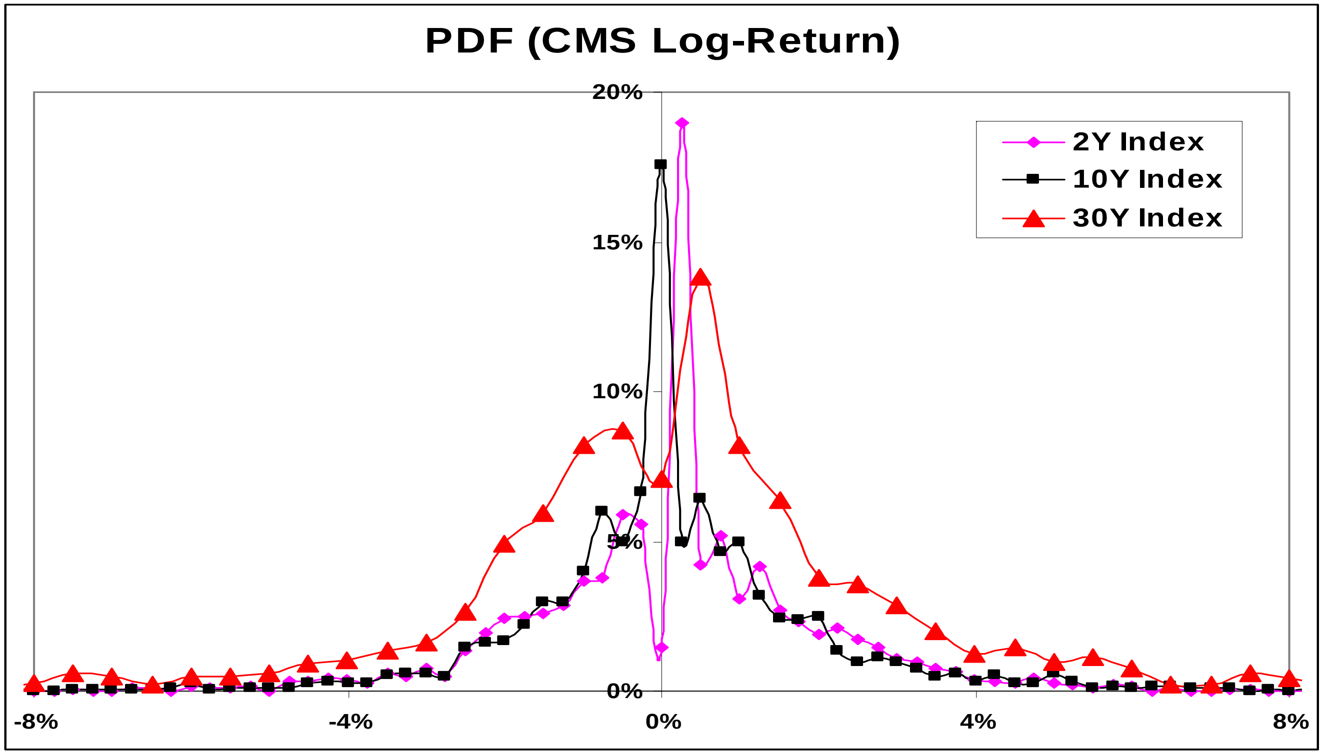
Historical CMS Rates

- **Historical 10Y CMS rates for 2Y, 10Y and 30Y index:**



Historical Distributions

- **Historical 10Y CMS rates for 2Y, 10Y and 30Y index:**



Historical Correlation

- **10Y Swap:**

Index	2Y	10Y	30Y	30s2s
2Y		~ 95%	~ 75%	
10Y			~ 85%	
30Y				
10s2s				~ 63%

- **Correlation seems to follow the “distance rule”, the shorter the distance, the more correlated they are. Spread correlation is less “predictable”.**

Historical Correlation

- **30Y Swap:**

Index	2Y	10Y	30Y	30s2s
2Y		~ 90%	~ 62%	
10Y			~ 82%	
30Y				
10s2s				~ 76%

- **Similar pattern as for the 10Y swap.**

Implied Marginal Distributions of CMS

- **Continuum of CMS digitals ➔ CDF & PDF;**
- **Two different methods of calculating CMS digitals:**

  1. **Full replication: replicating CMS cap/floor/digital pricing formulae fully;**
  2. **Black CMS: replicating CMS ➔ re-basing swaption vols to CMS forward ➔ using Black cap/floor/digital formula;**

- **What are the differences in practice?**

M1: Full Replication

- **Radon-Nikodym, *T*-measure to *A*-measure, Cap:**

$$\left\langle (S_{i,x}(T_i)-K)^+ \right\rangle^{T_i+\delta} = \frac{A_i(0)}{P(0,T_i+\delta)} \cdot \left\langle (S_{i,x}(T_i)-K)^+ \cdot \frac{P(T_i,T_i+\delta)}{A_i(T_i)} \right\rangle^{A_i}$$

$$= \frac{A_i(0)}{P(0,T_i+\delta)} \cdot \left\langle S_{i,x}(T_i) \cdot (S_{i,x}(T_i)-K)^+ \cdot \frac{P(T_i,T_i+\delta)}{1-P(T_i,T_i+x)} \right\rangle^{A_i}$$

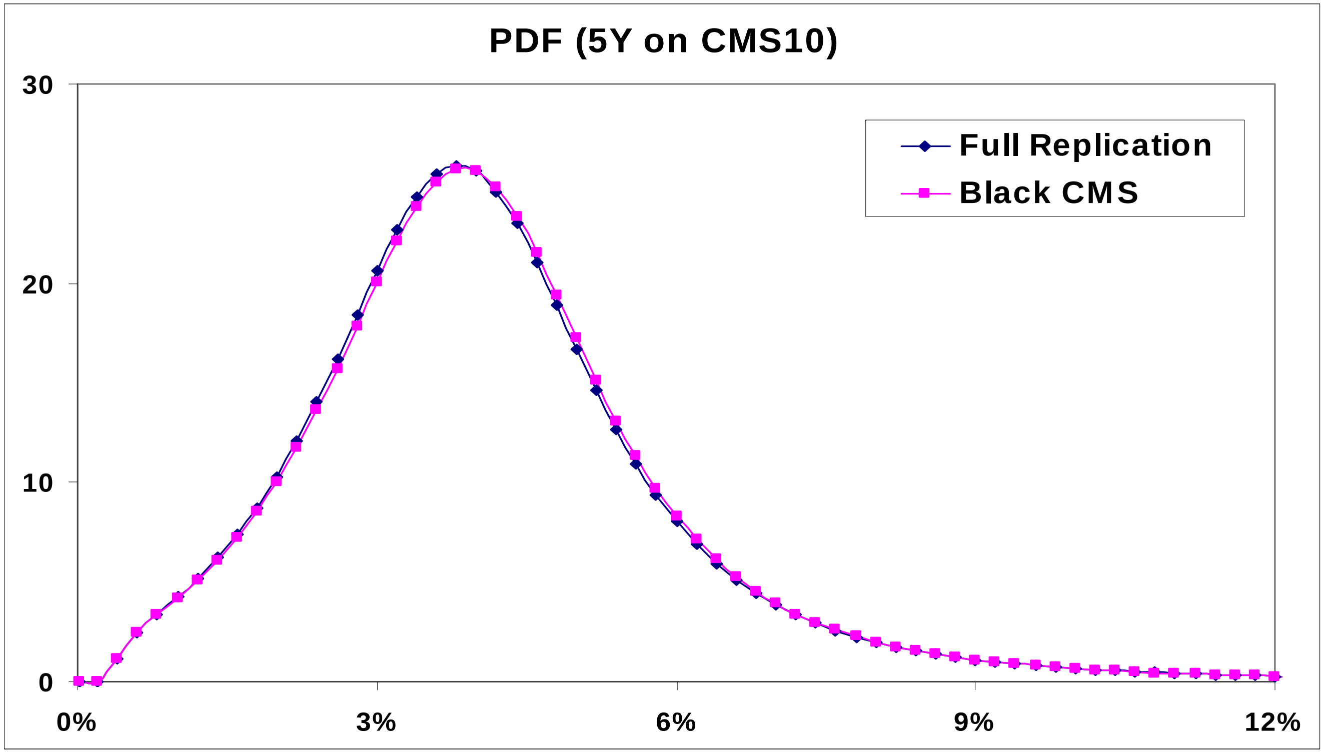
- **Digital:**

$$\left\langle 1_{|S_{i,x}(T_i)\geq K} \right\rangle^{T_i+\delta} = \frac{A_i(0)}{P(0,T_i+\delta)} \cdot \left\langle 1_{|S_{i,x}(T_i)\geq K} \cdot \frac{P(T_i,T_i+\delta)}{A_i(T_i)} \right\rangle^{A_i}$$

$$= \frac{A_i(0)}{P(0,T_i+\delta)} \cdot \left\langle S_{i,x}(T_i)_{|S_{i,x}(T_i)\geq K} \cdot \frac{P(T_i,T_i+\delta)}{1-P(T_i,T_i+x)} \right\rangle^{A_i}$$

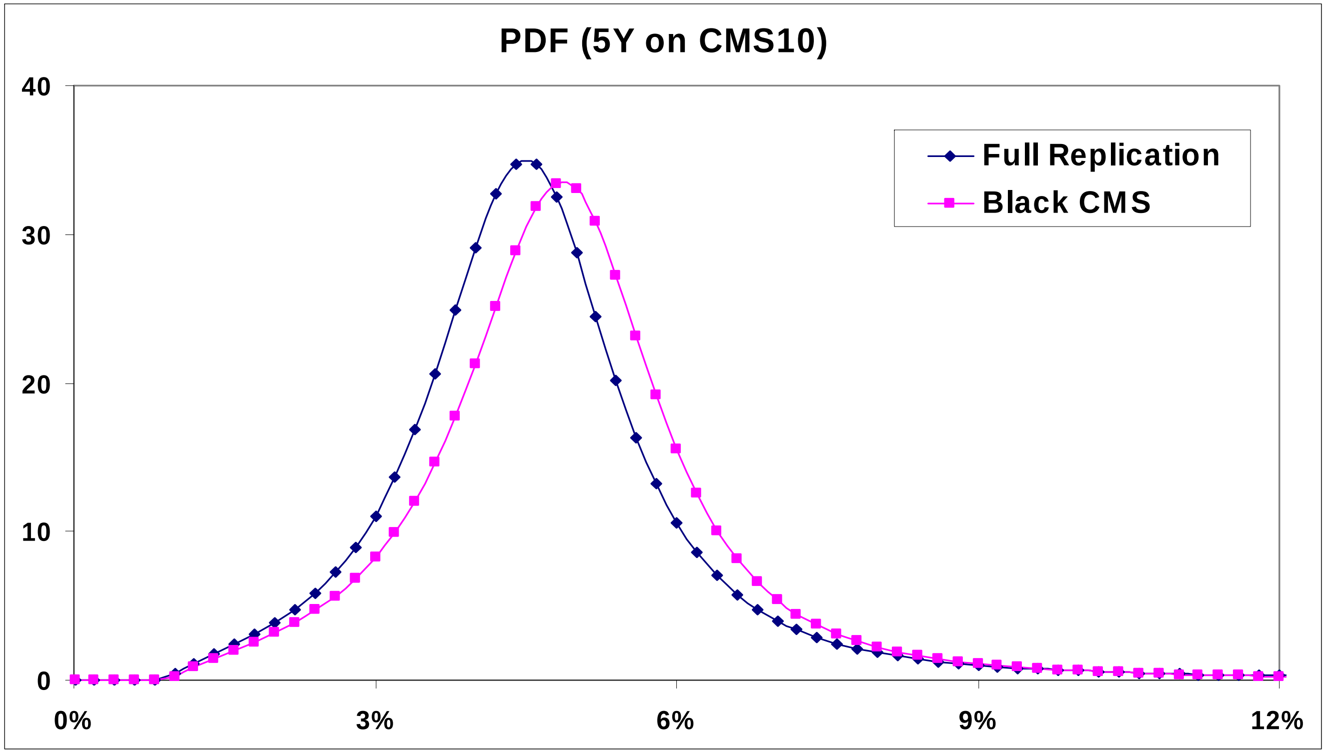
Implied PDFs – Visible Difference

- **5Y PDF for a 2Y CMS index (comparison of 2 methods):**



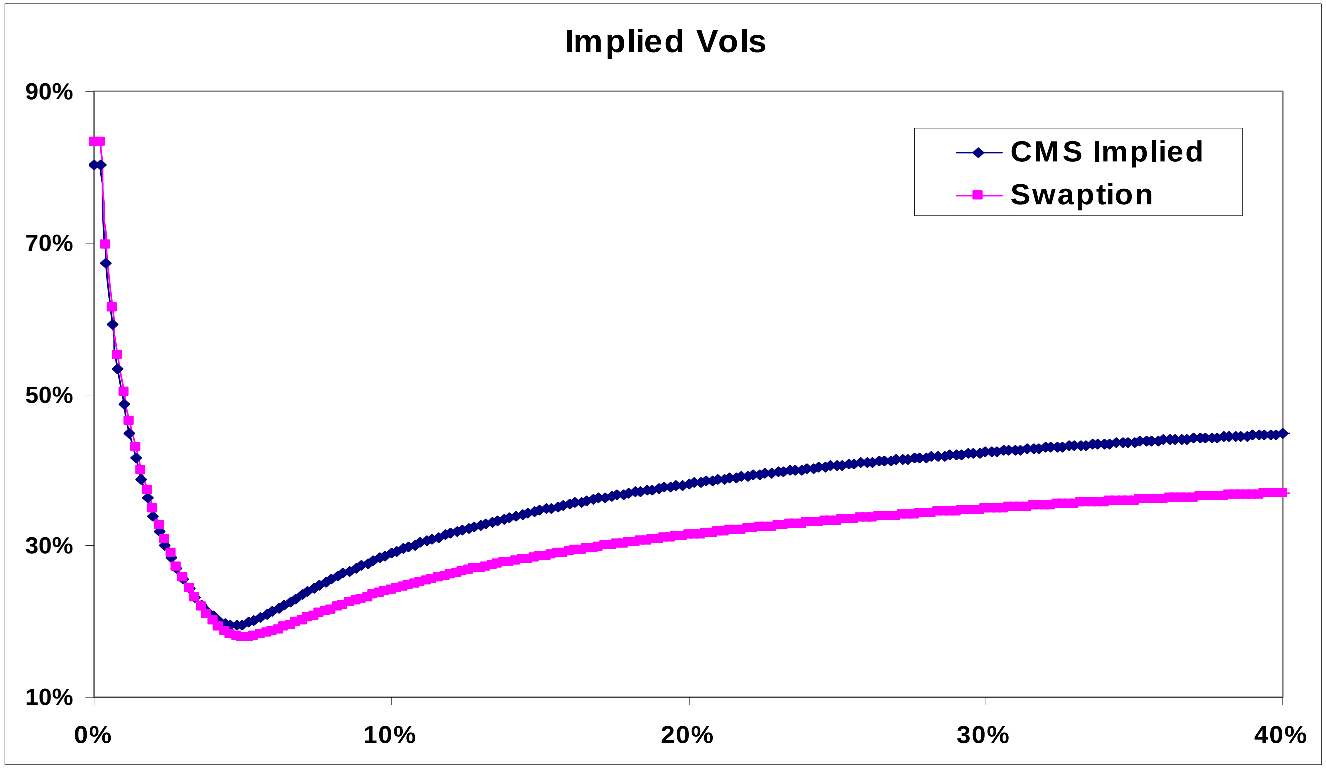
Implied PDFs – Large Difference

- **5Y PDF for 10Y CMS index (comparison of 2 methods):**



CMS Implied Vols

- **5Y on 10Y index (comparison of equivalent swaption vols):**



4. Consistent Pricing of CMS Spread Options

- **There have been various models used in practice for pricing CMS spread options, ranging from Black, to bi-variate, to multi-factor term structure models;**

- **A good model, however, should exhibit:**

  - **Self-consistency to the vanilla markets, in terms of underlyings calibrations as well as their marginal distributions (volatility smile/skew);**

  - **Simple and transparent specification of co-dependence between S_1 and S_2;**

  - **Simple and stable numerical implementation scheme.**

Example CMS Spread Option Payoffs

- **Standard Spread Option:**

$$\max(w_1 S_1 - w_2 S_2 - K, 0)$$

- **Curve Cap:**

$$\max\left[w_1 (S_1 - S_2), w_2 S_2 + c\right]$$

- **Floored Curve Cap:**

$$\min\left[\max(w_1 S_1 - w_2 S_2 + c_1, K_1), \max(w_3 S_2 + c_2, K_2)\right]$$

Pricing Using Copula

- **To price them using copula, one can utilise the hard work (results) already done in Section 3:**
  - **Calibration of underlyings & calculation of correct marginals;**

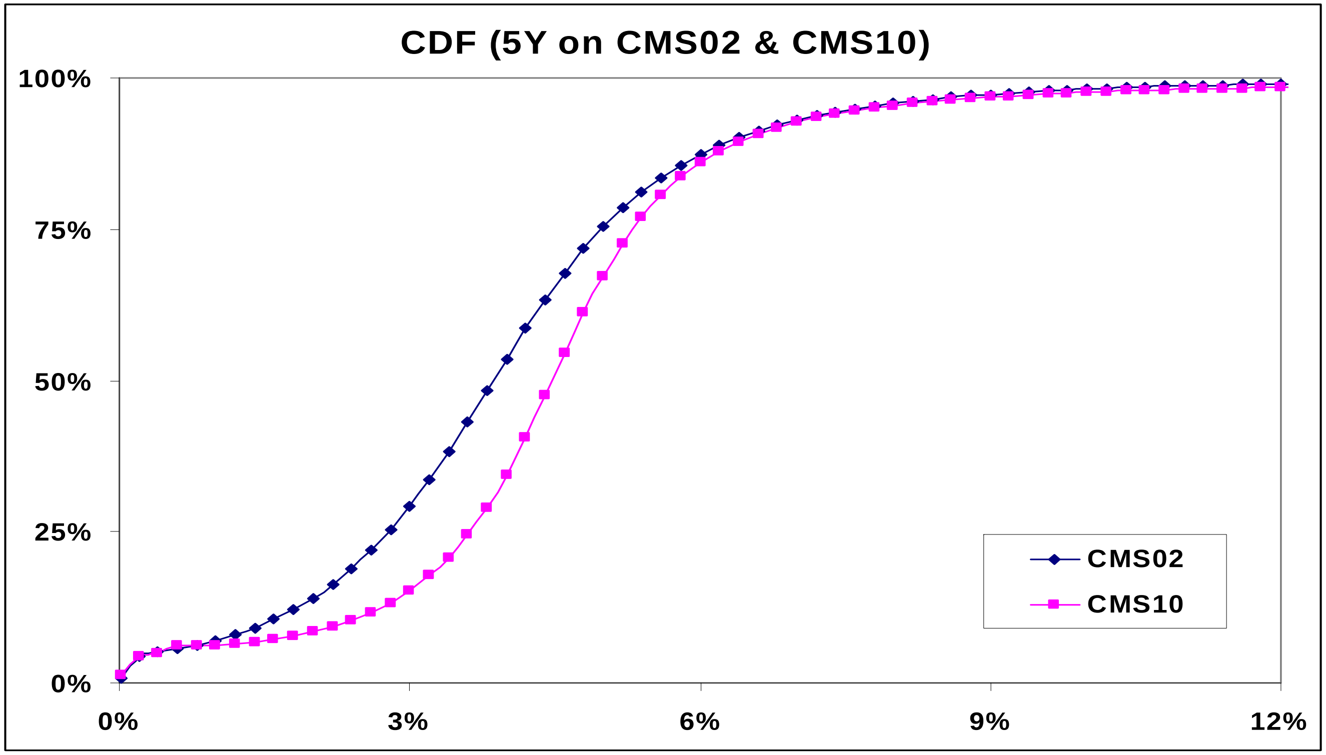
- **Apply Sklar's theorem: given marginals in uniforms, there exists a copula that binds the marginal uniforms to give the joint distribution of the multivariates;**

$$F(X_1, X_2, \cdots, X_n) = C(G(X_1), G(X_2), \cdots, G(X_n))$$

- **Either numerical integration or 1-step Monte Carlo can then be used to price the spread payoffs.**

Marginal CDFs

- **5Y CDF for 2Y and 10Y CMS index:**



Copula (1-Step MC)

- **Generate *n* independent random Gaussian numbers (g_j);**
- **Correlate the *n* Gaussian numbers:**

$$G_i = \sum_j \rho_{ij} \cdot g_j$$

- **Convert the correlated Gaussian numbers back to uniformly distributed numbers by inverse Wiener process:**

$$U_i = W^{-1}(G_i)$$

- **U_i [0, 1] can be used to sample CDFs to obtain S_i, which are subsequently used in calculating spread payoffs.**

Numerical Corrections

- **For Monte Carlo, it's important to correct S_i for all MC runs, to ensure correct spread forward;**

- **Average spread forward (prior to correction) is given by:**

$$F^i = S_1^i - S_2^i \qquad F^N = \sum_{i=1}^{N} F_i / N$$

- **Given we know exactly what the spread forward S_F is, we can calculate a correction factor C , which can then be used to modify all simulated S_i:**

$$C = F / F^N \qquad S_i^{'} = C \cdot S_i$$

Blending Copula With Replication

- **Advantages:**
  - **Fully consistent with the underlying CMS market (yield curve + swaption vols);**
  - **Fully consistent with the marginal distributions of relevant CMS rates (volatility smile/skew);**
  - **“Terminal correlation” is explicitly expressed;**
- **Of course, one has to calibrate to the (correlation) market quotes of “standard” spread options themselves;**
- **Copula is more stable and consistent, and should do better in the overall calibration to relevant markets.**

Important Comments

- **An interesting model paradigm:**
  - **Purely from a map of vanilla prices ➔ Exotics**
- **Does such paradigm exist (along the line of hedging)?**
- **Is this possible without going through the underlying processes?**
- **Must be product dependent. CMS spread options?**

❖ **The author thanks D. Zhu, R. Sesodia and structured derivatives trading for valuable discussions.**