1 Drift Derivation – The Strategy

To establish notation and the reasoning, let me start from the Libor-in-arrears case, and then move to the Libor-in-double-arrears case. When we agree on this we have all the ingredients we need.

Stratgey:

1. Find a martingale, x, under measure $\mathbb{T}_i$, associated with numeraire P_{i+1}.

2. Choose a different measure, either $\mathbb{T}_{i-k}$, with $k = 1, 2$.

3. Construct Radon-Nikodym derivative, ζ_k, (the change of measure), corresponding to the two new measures.

4. Remember that ζ_k is just given by the ratio of the numeraires:
$$\frac{1}{1+f_i} \text{ or} \frac{1}{1+f_i}\frac{1}{1+f_{i-1}}$$
for the two measures (I have set $\tau = 1$ to keep notation light).

5. Express ζ as
$$\frac{d\zeta}{\zeta} = qdB$$

6. Use the relationship
$$E_0^{\mathbb{T}_i}[x] = E_0^{\mathbb{T}_{i-k}}[x\zeta_k]$$
to find the drift of x under $\mathbb{T}_{i-k}$.

2 The Processes

I assume the following processes:

$$df_i = \mu_i dt + \sigma_i k_i dz_i \tag{1}$$

$$dk_i = [\eta_i dt + \nu dw_i] \tag{2}$$

where σ_i and η_i are $\sigma_i(f, t)$ and $\eta_i(k_i, t)$, respectively.

To keep notation light I have not shown the measure. However, under $\mathbb{T}_i$, k_i is a martingale ($\eta_i = 0$).

3 Libor in Arrears

I assume that the forward-rate drifts are unproblematic. I go straight to the volatility drift. The martingale under $\mathbb{T}_i$ is k_i so, the find the drfit of k_i under $\mathbb{T}_{i-1}$ I need to consider $E_0^{\mathbb{T}_{i-1}}[k_i\zeta_1]$. To lighten notation I also drop the subscript on ζ.

Evaluate

$$d(k_i\zeta) = \zeta dk_i + k_i d\zeta + dk_i d\zeta =$$
$$\zeta\left[\eta_i dt + \nu dw_i\right] + k_i d\zeta + \left[\eta_i dt + \nu dw_i\right] d\zeta$$

Now,

$$\zeta = \frac{1}{1+f_i}$$
$$d\zeta = \zeta q dB = \zeta q dz_i$$

Therefore

$$d(k_i\zeta) = \zeta\left[\eta_i dt + \nu dw_i\right] + k_i d\zeta + \left[\eta_i dt + \nu dw_i\right] d\zeta =$$
$$\zeta\left[\eta_i dt + \nu dw_i\right] + k_i\zeta q dz_i + \left[\eta_i dt + \nu dw_i\right]\zeta q dz_i =$$
$$\zeta\left[\eta_i dt + \nu dw_i\right] + k_i\zeta q dz_i + \nu\zeta q R_{ii} dt =$$
$$\zeta\left[\eta_i + \nu q R_{ii}\right] dt + k_i\zeta q dz_i$$

Now I impose that $k_i\zeta$ is a martingale under $\mathbb{T}_{i-1}$:

$$\eta_i + \nu q R_{ii} = 0 \implies \eta_i = -\nu q R_{ii}$$

I only need the 'volatility' of ζ. This is obtained via Ito's lemma on $y = y(f_i) = \frac{1}{1+f_i}$. Focussing just on the volatility term I get

$$\zeta q = \frac{\partial y}{\partial f_i}\sigma_i k_i$$
$$\frac{\partial y}{\partial f_i} = -\frac{1}{(1+f_i)^2}$$

and therefore

$$\zeta q = \frac{1}{1+f_i} q = -\frac{1}{(1+f_i)^2}\sigma_i k_i \Longrightarrow q = -\frac{1}{1+f_i}\sigma_i k_i$$

which gives

$$\eta_i = \frac{\sigma_i k_i \nu R_{ii}}{1+f_i}$$

which coincides with Equation 4.89. wioth $R_{ii} = \rho_i$.

4 Twice-in-Arrears Case

Now the chosen measure is $\mathbb{T}_{i-2}$, and I want timpose that $k_i\zeta$ is a martingale under $\mathbb{T}_{i-1}$, with ζ now given by

$$\zeta = \frac{1}{1+f_i}\frac{1}{1+f_{i-1}}$$

Everything proceeds as above:

$$d(k_i\zeta) = \zeta dk_i + k_i d\zeta + dk_i d\zeta =$$

$$\zeta\left[\eta_i dt + \nu dw_i\right] + k_i d\zeta + \left[\eta_i dt + \nu dw_i\right] d\zeta$$

However, we have to be a bit more careful with the term $d\zeta = \zeta q dB$. Previoiusly, ζ was just a functioj of f_i, and therefore $dB = dz_i$. Now we have to use Ito's lemma and ficus on the volatility part. Consider

$$y = y(f_i, f_{i-1}) = \frac{1}{1+f_i}\frac{1}{1+f_{i-1}}$$

with

$$df_i = \mu_i dt + \sigma_i k_i dz_i$$

$$df_{i-1} = \mu_i dt + \sigma_i k_i dz_{i-1}$$

and

$$dz_i dz_{i-1} = \rho_{i,i-1}$$

where now ρ is the correlation *among forward rates.* So

$$\zeta q dB = \frac{\partial y}{\partial f_i}\sigma_i k_i dz_i + \frac{\partial y}{\partial f_{i-1}}\sigma_{i-1}k_{i-1}dz_{i-1}$$

Proceeding as above I have

$$d(k_i\zeta) = \zeta\left[\eta_i dt + \nu dw_i\right] + k_i d\zeta + \left[\eta_i dt + \nu dw_i\right] d\zeta =$$

$$\zeta\left[\eta_i dt + \nu dw_i\right] + k_i\zeta q dB + \left[\eta_i dt + \nu dw_i\right]\zeta q dB =$$

$$\zeta\left[\eta_i dt + \nu dw_i\right] +$$

$$k_i\zeta q dB +$$

$$\left[\eta_i dt + \nu dw_i\right]\zeta q dB =$$

$$\zeta\left[\eta_i dt+\nu dw_i\right]+$$
$$k_i\left(\frac{\partial y}{\partial f_i}\sigma_i k_i dz_i+\frac{\partial y}{\partial f_{i-1}}\sigma_{i-1}k_{i-1}dz_{i-1}\right)+$$
$$\left[\eta_i dt+\nu dw_i\right]\left(\frac{\partial y}{\partial f_i}\sigma_i k_i dz_i+\frac{\partial y}{\partial f_{i-1}}\sigma_{i-1}k_{i-1}dz_{i-1}\right)$$

Now

$$\frac{\partial y}{\partial f_i}=\frac{\partial}{\partial f_i}\left[\frac{1}{1+f_i}\frac{1}{1+f_{i-1}}\right]=-\frac{1}{(1+f_i)^2}\frac{1}{1+f_{i-1}}=-\zeta\frac{1}{1+f_i}$$

$$\frac{\partial y}{\partial f_{i-1}}=\frac{\partial}{\partial f_{i-1}}\left[\frac{1}{1+f_i}\frac{1}{1+f_{i-1}}\right]=-\frac{1}{(1+f_{i-1})^2}\frac{1}{1+f_i}=-\zeta\frac{1}{1+f_{i-1}}$$

and therefore

$$d(k_i\zeta)=$$
$$\zeta\left[\eta_i dt+\nu dw_i\right]+$$
$$-k_i\zeta\left(\frac{1}{1+f_i}\sigma_i k_i dz_i+\frac{1}{1+f_{i-1}}\sigma_{i-1}k_{i-1}dz_{i-1}\right)+$$
$$\left[\eta_i dt+\nu dw_i\right]\zeta\left(\frac{1}{1+f_i}\sigma_i k_i dz_i+\frac{1}{1+f_{i-1}}\sigma_{i-1}k_{i-1}dz_{i-1}\right)$$

This gives me

$$d(k_i\zeta)=$$
$$\zeta\left[\eta_i dt+\nu dw_i\right]+$$
$$-k_i\zeta\left(\frac{1}{1+f_i}\sigma_i k_i dz_i+\frac{1}{1+f_{i-1}}\sigma_{i-1}k_{i-1}dz_{i-1}\right)+$$
$$\zeta\left(\frac{1}{1+f_i}\sigma_i k_i\nu R_{ii}+\frac{1}{1+f_{i-1}}\sigma_{i-1}k_{i-1}\nu R_{i,i-1}\right)dt$$

Rearranging terms I get

$$d(k_i\zeta)=$$
$$\zeta\left[\eta_i+\frac{1}{1+f_i}\sigma_i k_i\nu R_{ii}+\frac{1}{1+f_{i-1}}\sigma_{i-1}k_{i-1}\nu R_{i,i-1}\right]dt+...$$

and therefore

$$\eta_i=-\nu\left(\frac{\sigma_i k_i R_{ii}}{1+f_i}+\frac{\sigma_{i-1}k_{i-1}R_{i,i-1}}{1+f_{i-1}}\right)$$