Variance derivatives: Pricing and convergence

John Crosby

UBS, London
and Centre for Economic and Financial Studies,
Department of Economics,
Glasgow University

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- We denote the initial time (today) by $t_0 \equiv 0$. We consider a stock whose price, at time t, is $S(t)$. We consider a time interval $[t_0, T]$ which is partitioned into N time periods (not necessarily equal in length) whose end-points are t_j, $j = 1, 2, ..., N$, where $0 \equiv t_0 < t_1 < ... < t_{j-1} < t_j < ... < t_N \equiv T$.
- What difference does it make if realised variance is measured by log changes squared (i.e. $\sum_{i=1}^{N}(\log(S(t_i)/S(t_{i-1})))^2)$ or by proportional differences squared (i.e. $\sum_{i=1}^{N}((S(t_i)/S(t_{i-1})) - 1)^2)$?
- What impact does monitoring frequency (i.e. the value of N above) have on the measurement of realised variance?
- What impact do jumps in the underlying stock price have on the measurement of realised variance?
- Building on Broadie and Jain (2008), Carr and Lee (2009) and Hong (2004), we will try to answer these questions.

- Our results have two important applications:
- 1./ The pricing (under an equivalent martingale measure (EMM) $\mathbb{Q}$) of variance swaps which pay $\sum_{i=1}^{N}(\log(S(t_i)/S(t_{i-1})))^2$ (which is how the payoffs are usually defined in practice) and of proportional variance swaps which pay $\sum_{i=1}^{N}((S(t_i)/S(t_{i-1}))-1)^2$ at maturity T. In particular, we consider the case when N is infinite (continuously monitored) and the case when N is finite (discretely monitored - as they must always be in practice).
- 2./ Given observations of $S(t_i)$ for times t_i, $i = 1, 2, ..., N$ (from historical data under the real-world physical measure $\mathbb{P}$), what can we say about the process which generated this data? We are thinking, in particular, of high-frequency data (at least several, perhaps, a few hundred observations per day).
- We will only look at the first.

- Nearly all papers on variance swaps have focussed on the log-contract replication approach (eg. Neuberger (1990), Dupire (1993), Derman et al. (1999)) but it does assume continuous monitoring of the variance and continuous sample paths for the underlying stock.
- There is a completely different approach (see Hong (2004) and Broadie and Jain (2008)) which utilises characteristic functions and assumes **neither**. We build upon this approach. However, firstly, we discuss the assumed stock price dynamics.

- We construct the stock price process by assuming that the log of the stock price is a time-changed Lévy process (allows a very generic process which includes (nearly) all models seen in the literature).
- We have a Lévy process (eg Brownian motion, Kou (2002) jump-diffusion, Variance Gamma or CGMY) denoted by X_t, satisfying $X_{t_0} = 0$. We assume that we mean-correct X_t so that $\exp(X_t)$ is a (non-constant) martingale (under $\mathbb{Q}$) - with respect to the natural filtration generated by X_t i.e. that $\mathbb{E}^{\mathbb{Q}}_{t_0}[\exp(X_t)] = \exp(X_{t_0}) = 1$ for all $t \geq t_0$.
- Define (minus) the (mean-corrected) characteristic exponent $\overline{\psi}_X(z)$ via $\mathbb{E}^{\mathbb{Q}}_{t_0}[\exp(iuX_t)] \equiv \exp(-(t-t_0)\overline{\psi}_X(u))$. The Lévy-Khinchin formula implies there is an analytical formula for $\overline{\psi}_X(z)$.

  For future reference, $'$ denotes differentiation i.e. $\overline{\psi}'_X(z) \equiv \partial\overline{\psi}_X(z)/\partial z$, $\overline{\psi}''_X(z) \equiv \partial^2\overline{\psi}_X(z)/\partial z^2$ and $\overline{\psi}'''_X(z) \equiv \partial^3\overline{\psi}_X(z)/\partial z^3$.
- For the case of Brownian motion, "$X_t = -\frac{1}{2}\sigma^2 t + \sigma W(t)$ where $W(t)$ is standard (driftless) Brownian motion".

- We assume that we have a non-decreasing, continuous time-change process denoted by Y_t. We normalise so that $Y_{t_0} = t_0 \equiv 0$.
- In general, Y_t may be correlated with X_t.
- Our assumption, for example, allows Y_t to be of the form $Y_t = \int_{t_0}^{t} y_s ds$ where the activity rate y_t (which must be non-negative) follows, for example, a Heston (1993) square-root process, a non-Gaussian OU process (Barndorff-Nielsen and Shephard (2001)) or it could follow the Heston (1993) plus jumps process of Duffie et al. (2000). In the latter two cases, y_t is discontinuous but Y_t is always continuous.
- (The time-change will allow us to model stochastic volatility / leverage / volatility clustering type effects).

- We time-change the Lévy process X_t by Y_t to get a process X_{Y_t}, with $X_{Y_{t_0}} = 0$.
- The stock price $S(t)$, at time t, is assumed to have the following dynamics (under $\mathbb{Q}$):

$$S(t) = S(t_0) \exp\left(\int_{t_0}^{t} (r(s) - q(s))ds + X_{Y_t}\right).$$

- Here, $r(t)$ is the risk-free interest-rate and $q(t)$ is the dividend yield (assumed finite and deterministic), at time t.
- To lighten notation, I will henceforth write equations as if $r(t) - q(t) \equiv 0$ for all t (or equivalently work with forward or future prices - the paper considers the general case). Hence, $S(t) = S(t_0) \exp(X_{Y_t})$.

- We now define, for all $t \geq t_0$:

$$\Xi_t(u) \equiv \exp(iuX_{Y_t} + Y_t\overline{\psi}_X(u)).$$

Since the mean-corrected characteristic exponent $\overline{\psi}_X(u)$ is defined via: $\mathbb{E}^{\mathbb{Q}}_{t_0}[\exp(iuX_t)] = \exp(-(t-t_0)\overline{\psi}_X(u))$, then $\exp(iuX_t + (t-t_0)\overline{\psi}_X(u))$ is a martingale, under $\mathbb{Q}$, with respect to the natural filtration generated by X_t.

- By a "randomising time" (Optional Stopping Theorem) argument, for any u, $\Xi_t(u)$ is a martingale, under $\mathbb{Q}$, with respect to the filtration generated by $\mathcal{F}_t \equiv \sigma(X_{Y_u}, u \leq t)$.

- In particular,

$$\mathbb{E}^{\mathbb{Q}}_{t_{j-1}}[\frac{\Xi_{t_j}(u)}{\Xi_{t_{j-1}}(u)}] = \mathbb{E}^{\mathbb{Q}}_{t_{j-1}}[\exp(iu(X_{Y_{t_j}} - X_{Y_{t_{j-1}}}) + (Y_{t_j} - Y_{t_{j-1}})\overline{\psi}_X(u))] = 1.$$

- We now introduce what we call the joint extended characteristic function $\Phi(z;j)$, which we define, for each j, $j = 1, \ldots, N$, by:

$$\begin{aligned}
\Phi(z;j) &\equiv \mathbb{E}^{\mathbb{Q}}_{t_0}[\exp(iz \log \frac{S(t_j)}{S(t_{j-1})})] = \mathbb{E}^{\mathbb{Q}}_{t_0}[\exp(iz(X_{Y_{t_j}} - X_{Y_{t_{j-1}}}))] \\
&= \mathbb{E}^{\mathbb{Q}}_{t_0}[\exp(iz(X_{Y_{t_j}} - X_{Y_{t_{j-1}}}) + (Y_{t_j} - Y_{t_{j-1}})\overline{\psi}_X(z)) \exp(-(Y_{t_j} - Y_{t_{j-1}})\overline{\psi}_X(z))] \\
&= \mathbb{E}^{\mathbb{Q}}_{t_0}[\mathbb{E}^{\mathbb{Q}}_{t_{j-1}}[\frac{\Xi_{t_j}(z)}{\Xi_{t_{j-1}}(z)} \exp(-(Y_{t_j} - Y_{t_{j-1}})\overline{\psi}_X(z))]].
\end{aligned}$$

- (Note as an aside, $\Phi(z;j)$ is "a kind of forward characteristic function". One can compute $\Phi(z;j)$, for cases of interest, via conditioning arguments and by using results in Carr and Wu (2004) and Duffie et al. (2000), so we will say nothing more about this.)

- We note that the joint extended characteristic function $\Phi(z; j)$ allows us to immediately evaluate the price of a discretely monitored proportional variance swap. We let $iz = 2$ in the equation for $\Phi(z; j)$, then sum over j and simplify.
- $\Rightarrow$: The price $\mathrm{PVS}(t_0, T, N)$, at time t_0, of a (discretely monitored) proportional variance swap (paying $\sum_{i=1}^{N}((S(t_i)/S(t_{i-1})) - 1)^2$ at time T) is:

$$\mathrm{PVS}(t_0, T, N) = P(t_0, T)\Big(\sum_{j=1}^{N}\big(\Phi(-2i; j) - 1\big)\Big).$$

  Here, $P(t_0, T)$ is the price of a zero-coupon bond, at time t_0, that matures at time T.
- We will examine the limit as $N \to \infty$ of this equation later.

- Now we differentiate $\Phi(z;j)$ with respect to z and divide by i:

$$\begin{aligned}\frac{1}{i}\frac{\partial\Phi(z;j)}{\partial z} &= \mathbb{E}^{\mathbb{Q}}_{t_0}[\log\frac{S(t_j)}{S(t_{j-1})}\exp(iz\log\frac{S(t_j)}{S(t_{j-1})})] \\ &= \mathbb{E}^{\mathbb{Q}}_{t_0}[\mathbb{E}^{\mathbb{Q}}_{t_{j-1}}[\frac{\Xi_{t_j}(z)}{\Xi_{t_{j-1}}(z)}\exp(-(Y_{t_j}-Y_{t_{j-1}})\overline{\psi}_X(z)) \\ & \quad \Big(\varpi^{(j)}(iz)+\big((X_{Y_{t_j}}-X_{Y_{t_{j-1}}})-i\overline{\psi}'_X(z)(Y_{t_j}-Y_{t_{j-1}})\big)\Big)]], \quad \text{where} \\ \varpi^{(j)}(iz) &\equiv i\overline{\psi}'_X(z)(Y_{t_j}-Y_{t_{j-1}}).\end{aligned}$$

- It is now straightforward to value log-forward-contracts (paying $\log(S(t_N)/S(t_0))$ at time T). We set $iz=0$, then we sum from $j=1$ to N and then simplify. The price $\mathrm{LFC}(t_0,T)$, at time t_0, of a log-forward-contract is:

$$\mathrm{LFC}(t_0,T) = P(t_0,T)i\overline{\psi}'_X(0)\mathbb{E}^{\mathbb{Q}}_{t_0}[Y_T-Y_{t_0}] \equiv P(t_0,T)m_X\mathbb{E}^{\mathbb{Q}}_{t_0}[Y_T-Y_{t_0}].$$

Note m_X defined by $m_X\equiv i\overline{\psi}'_X(0)$ is real.

- We differentiate again with respect to z and again divide by i:

$$
\begin{aligned}
-\frac{\partial^2\Phi(z;j)}{\partial z^2} &= \mathbb{E}^{\mathbb{Q}}_{t_0}[(\log\frac{S(t_j)}{S(t_{j-1})})^2\exp(iz\log\frac{S(t_j)}{S(t_{j-1})})] \\
&= \mathbb{E}^{\mathbb{Q}}_{t_0}[\mathbb{E}^{\mathbb{Q}}_{t_{j-1}}[\frac{\Xi_{t_j}(z)}{\Xi_{t_{j-1}}(z)}\exp(-(Y_{t_j}-Y_{t_{j-1}})\overline{\psi}_X(z)) \\
& \Big(\varpi^{(j)\,2}(iz) \\
& +\Big\{2\varpi^{(j)}(iz)\Big((X_{Y_{t_j}}-X_{Y_{t_{j-1}}})-i\overline{\psi}'_X(z)(Y_{t_j}-Y_{t_{j-1}})\Big)\Big\} \\
& +\Big((X_{Y_{t_j}}-X_{Y_{t_{j-1}}})-i\overline{\psi}'_X(z)(Y_{t_j}-Y_{t_{j-1}})\Big)^2-\overline{\psi}''_X(z)(Y_{t_j}-Y_{t_{j-1}}) \\
& +\overline{\psi}''_X(z)(Y_{t_j}-Y_{t_{j-1}})\Big)]].
\end{aligned}
$$

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- The price, at time t_0, of a variance swap $\mathrm{VS}(t_0, T, N)$ can be obtained by setting $iz = 0$, summing from $j = 1$ to N and simplifying: The price $\mathrm{VS}(t_0, T, N)$ is:

$$
\begin{aligned}
& \mathrm{VS}(t_0, T, N) \\
= \; & P(t_0, T)\mathbb{E}^{\mathbb{Q}}_{t_0}[\sum_{j=1}^{N} \mathbb{E}^{\mathbb{Q}}_{t_{j-1}}[\varpi^{(j)\,2}(0)]] \\
+ \; & P(t_0, T)\mathbb{E}^{\mathbb{Q}}_{t_0}[\sum_{j=1}^{N} \mathbb{E}^{\mathbb{Q}}_{t_{j-1}}[2m_X(Y_{t_j} - Y_{t_{j-1}})\big((X_{Y_{t_j}} - X_{Y_{t_{j-1}}}) - m_X(Y_{t_j} - Y_{t_{j-1}})\big)]] \\
+ \; & P(t_0, T)\overline{\psi}''_X(0)\mathbb{E}^{\mathbb{Q}}_{t_0}[\sum_{j=1}^{N}(Y_{t_j} - Y_{t_{j-1}})]. \qquad (1)
\end{aligned}
$$

- Note that $\varpi^{(j)}(0)$ is the drift of log of the stock price (over the time interval t_{j-1} to t_j) (it is real and for Brownian motion and a deterministic time-change it is "$(r - q - \frac{1}{2}\sigma^2)(t_j - t_{j-1})$").
- Here $m_X \equiv i\overline{\psi}'_X(0)$ (note m_X is real and for Browian motion it is "$-\frac{1}{2}\sigma^2$").
- Lets look at each of the three lines of equation (1) in turn.

- Again, $\mathrm{VS}(t_0, T, N)$

$$
\begin{aligned}
&= P(t_0,T)\mathbb{E}^{\mathbb{Q}}_{t_0}[\sum_{j=1}^{N}\mathbb{E}^{\mathbb{Q}}_{t_{j-1}}[\varpi^{(j)\,2}(0)]] \\
&+ P(t_0,T)\mathbb{E}^{\mathbb{Q}}_{t_0}[\sum_{j=1}^{N}\mathbb{E}^{\mathbb{Q}}_{t_{j-1}}[2m_X(Y_{t_j}-Y_{t_{j-1}})\big((X_{Y_{t_j}}-X_{Y_{t_{j-1}}})-m_X(Y_{t_j}-Y_{t_{j-1}})\big)]] \\
&+ P(t_0,T)\overline{\psi}''_X(0)\mathbb{E}^{\mathbb{Q}}_{t_0}[\sum_{j=1}^{N}(Y_{t_j}-Y_{t_{j-1}})].
\end{aligned}
$$

- Note that, with a deterministic time-change, $\varpi^{(j)\,2}(0)$ is $O(1/N^2)$. Broadie and Jain (2008) show that it is $O(1/N^2)$ if the activity-rate of the time-change is Heston (1993). In the paper, we show that it is $O(1/N^2)$ for "almost any" time-change.
- Hence the first line is $O(1/N)$ and $\rightarrow 0$ as $N \rightarrow \infty$.
- Since $\varpi^{(j)}(0)$ is real, $\varpi^{(j)\,2}(0)$ is definitely non-negative and zero only if the drift of the log of the stock price is identically equal to zero.

- Again, the second line is:

$$P(t_0,T)\mathbb{E}^{\mathbb{Q}}_{t_0}[\sum_{j=1}^{N}\mathbb{E}^{\mathbb{Q}}_{t_{j-1}}[2m_X(Y_{t_j}-Y_{t_{j-1}})\big((X_{Y_{t_j}}-X_{Y_{t_{j-1}}})-m_X(Y_{t_j}-Y_{t_{j-1}})\big)]].$$

- Note $\mathbb{E}^{\mathbb{Q}}_{t_{j-1}}[(X_{Y_{t_j}}-X_{Y_{t_{j-1}}})-m_X(Y_{t_j}-Y_{t_{j-1}})]\equiv 0$ (by construction it is a martingale eg the whole term is standard Brownian motion).
- Therefore, if X_t and Y_t are independent, the second line is identically equal to zero.
- m_X is always negative (eg for Browian motion it is "$-\frac{1}{2}\sigma^2$"). Therefore, if X_t and Y_t are negatively correlated, the second term is positive.
- Results in Broadie and Jain (2008) show, for Heston (1993) that the (absolute value of the) second line is $O(1/N)$. In the paper, we show that it is $O(1/N)$ for any Lévy process and "almost any" time-change.

- Again, VS(t_0, T, N)

$$
\begin{aligned}
&= P(t_0,T)\mathbb{E}^{\mathbb{Q}}_{t_0}[\sum_{j=1}^{N} \mathbb{E}^{\mathbb{Q}}_{t_{j-1}}[\varpi^{(j)\,2}(0)]] \\
&+ P(t_0,T)\mathbb{E}^{\mathbb{Q}}_{t_0}[\sum_{j=1}^{N} \mathbb{E}^{\mathbb{Q}}_{t_{j-1}}[2m_X(Y_{t_j} - Y_{t_{j-1}})\big((X_{Y_{t_j}} - X_{Y_{t_{j-1}}}) - m_X(Y_{t_j} - Y_{t_{j-1}})\big)]] \\
&+ P(t_0,T)\overline{\psi}''_X(0)\mathbb{E}^{\mathbb{Q}}_{t_0}[Y_T - Y_{t_0}].
\end{aligned}
$$

- The term $\mathbb{E}^{\mathbb{Q}}_{t_0}[\sum_{j=1}^{N}(Y_{t_j} - Y_{t_{j-1}})] = \mathbb{E}^{\mathbb{Q}}_{t_0}[Y_T - Y_{t_0}]$ due to a telescoping sum.
- The third line is the price of the continuously monitored version of the variance swap.

- Again, $\mathrm{VS}(t_0, T, N)$

$$
\begin{aligned}
&= P(t_0,T)\mathbb{E}^{\mathbb{Q}}_{t_0}[\sum_{j=1}^{N}\mathbb{E}^{\mathbb{Q}}_{t_{j-1}}[\varpi^{(j)\,2}(0)]] \\
&+ P(t_0,T)\mathbb{E}^{\mathbb{Q}}_{t_0}[\sum_{j=1}^{N}\mathbb{E}^{\mathbb{Q}}_{t_{j-1}}[2m_X(Y_{t_j}-Y_{t_{j-1}})\big((X_{Y_{t_j}}-X_{Y_{t_{j-1}}})-m_X(Y_{t_j}-Y_{t_{j-1}})\big)]] \\
&+ P(t_0,T)\overline{\psi}''_X(0)\mathbb{E}^{\mathbb{Q}}_{t_0}[Y_T-Y_{t_0}].
\end{aligned}
$$

- The price of a (discretely monitored) variance swap is the sum of three terms: A non-negative "drift-related" term, a "covariance" term which is non-negative (respectively, zero) if $\mathrm{Correl}(X_t, Y_t)$ is negative (respectively, zero) and the price of the continuously monitored version of the variance swap.

- In particular, if the "covariance" term is non-positive, a discretely monitored variance swap is always worth than its continuously monitored counterpart.

- Convergence is always $O(1/N)$.

- We saw earlier that the price $\mathrm{PVS}(t_0, T, N)$, at time t_0, of a (discretely monitored) proportional variance swap (paying $\sum_{i=1}^{N}((S(t_i)/S(t_{i-1})) - 1)^2$ at time T) is:

$$\mathrm{PVS}(t_0, T, N) = P(t_0, T)\Big(\sum_{j=1}^{N}\big(\Phi(-2i; j) - 1\big)\Big).$$

- Hence:

$$\begin{aligned}
\lim_{N\to\infty} \mathrm{PVS}(t_0, T, N) &= \lim_{N\to\infty} P(t_0, T)\Big(\sum_{j=1}^{N}\big(\Phi(-2i; j) - 1\big)\Big) \\
&= P(t_0, T) \lim_{N\to\infty} \sum_{j=1}^{N} \mathbb{E}_{t_0}^{\mathbb{Q}}[\frac{\Xi_{t_j}(-2i)}{\Xi_{t_{j-1}}(-2i)}(\exp(-(Y_{t_j} - Y_{t_{j-1}})\overline{\psi}_X(-2i)) - 1)] \\
&= -P(t_0, T)\overline{\psi}_X(-2i)\mathbb{E}_{t_0}^{\mathbb{Q}}[Y_T - Y_{t_0}] + O(1/N).
\end{aligned}$$

- Hence, the price of the continuously monitored version of the proportional variance swap is $-P(t_0, T)\overline{\psi}_X(-2i)\mathbb{E}_{t_0}^{\mathbb{Q}}[Y_T - Y_{t_0}]$.
- Convergence is also $O(1/N)$.

- From the previous slide,

$$\begin{aligned}\mathrm{PVS}(t_0,T,N) &= P(t_0,T)\Big(\sum_{j=1}^{N}\big(\Phi(-2i;j)-1\big)\Big) \text{ with}\\ \Phi(-2i;j) &= \mathbb{E}^{\mathbb{Q}}_{t_0}[\frac{\Xi_{t_j}(-2i)}{\Xi_{t_{j-1}}(-2i)}\exp(-(Y_{t_j}-Y_{t_{j-1}})\overline{\psi}_X(-2i))].\end{aligned}$$

  Hence, it is clear (since $\overline{\psi}^{(k)}_X(-2i) < 0$ eg. for Brownian motion $\overline{\psi}^{(k)}_X(-2i) = -\sigma^2$) that when X_t and Y_t are positively correlated then the price of a discretely monitored proportional variance swap is higher than the price of the same discretely monitored proportional variance swap under the assumption that they are independent (the opposite way round to a variance swap).

- Under the assumption of independence, a discretely monitored proportional variance swap is always worth at least as much as an otherwise identical continuously monitored proportional variance swap (the same way round as a variance swap).

- We have explicit expressions for the prices of variance swaps and proportional variance swaps (both discretely monitored and continuously monitored). Discretely monitored prices tend to their continuously monitored counterparts as $O(1/N)$ (for both variance swaps and proportional variance swaps).
- In the paper, we prove $O(1/N)$ convergence is also true for discontinuous time-changes.
- In the paper, we prove $O(1/N)$ convergence is also true for gamma swaps, self-quantoed variance swaps and skewness swaps.
- The prices of continuously monitored variance swaps and proportional variance swaps (and also gamma swaps and skewness swaps) do **NOT** depend upon $\mathrm{Correl}(X_t, Y_t)$.
- Can easily see dependence of discretely monitored versions of these swaps on $\mathrm{Correl}(X_t, Y_t)$.
- In particular,

$$\mathrm{VS}(t_0, T, N) \geq \mathrm{VS}(t_0, T, \infty) \text{ provided } \mathrm{Correl}(X_t, Y_t) \leq 0,$$

(and a non-positive correlation seems most likely in practice).

- The price of a continuously monitored proportional variance swap is:
  $\mathrm{PVS}(t_0, T, \infty) = -P(t_0, T)\overline{\psi}_X(-2i)\mathbb{E}^{\mathbb{Q}}_{t_0}[Y_T - Y_{t_0}]$.
- The price of a continuously monitored variance swap is:
  $\mathrm{VS}(t_0, T, \infty) = P(t_0, T)\overline{\psi}''_X(0)\mathbb{E}^{\mathbb{Q}}_{t_0}[Y_T - Y_{t_0}]$.
- The price of a log-forward-contract is:
  $\mathrm{LFC}(t_0, T) = P(t_0, T)i\overline{\psi}'_X(0)\mathbb{E}^{\mathbb{Q}}_{t_0}[Y_T - Y_{t_0}] \equiv P(t_0, T)m_X\mathbb{E}^{\mathbb{Q}}_{t_0}[Y_T - Y_{t_0}]$.
- Hence:
  $$\frac{\mathrm{VS}(t_0, T, \infty)}{\mathrm{LFC}(t_0, T)} = \frac{\overline{\psi}''_X(0)}{m_X}, \qquad \frac{\mathrm{PVS}(t_0, T, \infty)}{\mathrm{LFC}(t_0, T)} = \frac{-\overline{\psi}_X(-2i)}{m_X}.$$
  Carr and Lee (2009) have already proven the left-hand-side equation (i.e. for variance swaps (VS)) by a different method. In the paper, we show similar analogous results, not only for proportional variance swaps, but also for other types of variance derivatives.
- Hence, given vanilla prices, can price variance swaps and proportional variance swaps independent of any assumption on Y_t (and therefore robust to model (mis-)specification).

- For the case, when X_t is Brownian motion with volatility σ:
  We have: $\overline{\psi}_X(z) = \sigma^2(z^2 + iz)/2$, $m_X = -\sigma^2/2$, $\overline{\psi}''_X(0) = \sigma^2$, $\overline{\psi}''_X(-i) = \sigma^2$, $\overline{\psi}'''_X(0) = 0$ and $\overline{\psi}_X(-2i) = -\sigma^2$.

$$\frac{\mathrm{VS}(t_0, T, \infty)}{-\mathrm{LFC}(t_0, T)} = 2, \qquad \frac{\mathrm{PVS}(t_0, T, \infty)}{-\mathrm{LFC}(t_0, T)} = 2.$$

- The left-hand-side equation restates Neuberger (1990), Dupire (1993) and Derman et al. (1999):
  The price of a variance swap equals (minus) two times the price of a log-forward-contract (with the assumption of continuous sample paths (i.e. the log of the stock price is time-changed Brownian motion)).
- The right-hand-side equation says that it makes **no difference** if realised variance is measured by log changes squared (i.e. $\sum_{i=1}^{N}(\log(S(t_i)/S(t_{i-1})))^2)$ or by proportional differences squared (i.e. $\sum_{i=1}^{N}((S(t_i)/S(t_{i-1})) - 1)^2)$ **when there are no jumps (i.e. continuous sample paths) and when $N = \infty$ (i.e. continuously monitored)**.

- For the case, when X_t is a compound Poisson process with a fixed jump amplitude a (and with no diffusion component), then we have:

$$\begin{aligned}\frac{\mathrm{VS}(t_0,T,\infty)}{-\mathrm{LFC}(t_0,T)} &= \frac{a^2}{(\exp(a)-1-a)} \approx 2\left(1-\frac{a}{3}\right),\\ \frac{\mathrm{PVS}(t_0,T,\infty)}{-\mathrm{LFC}(t_0,T)} &= \frac{(\exp(a)-1)^2}{(\exp(a)-1-a)} \approx 2\left(1+\frac{2a}{3}\right),\end{aligned}$$

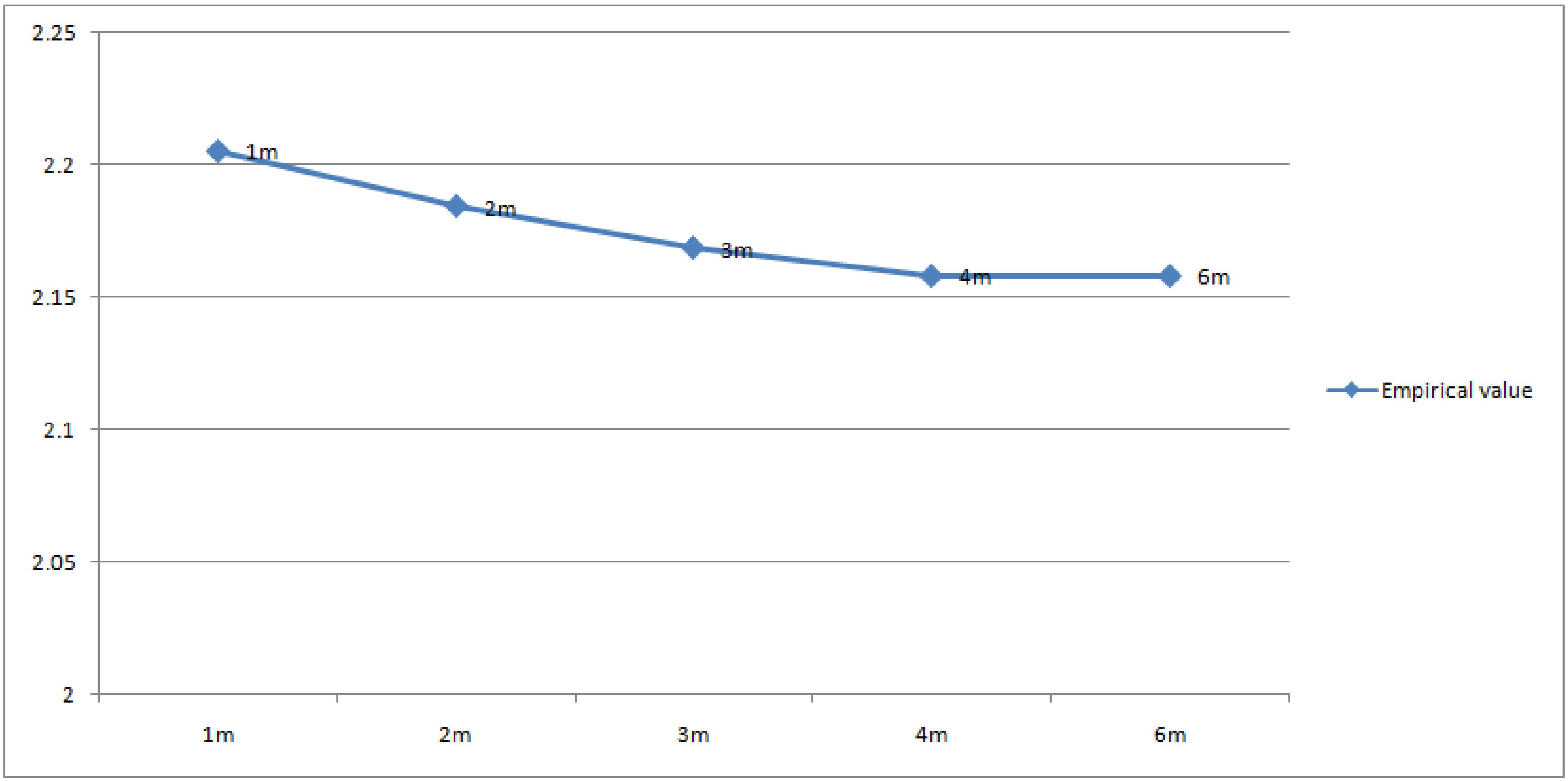
  where, in each part, the first term is exact and the second term is the expansion of the first term to leading order when $|a|$ is small.

- $\Rightarrow$: The prices of variance swaps and proportional variance swaps have the **opposite sensitivities** to jumps (and the impact will be larger in magnitude (perhaps, twice as large) for proportional variance swaps).

- The right-hand-side equation suggests that it will make a **big difference** if realised variance is measured by log changes squared (i.e. $\sum_{i=1}^{N}(\log(S(t_i)/S(t_{i-1})))^2)$ or by proportional differences squared (i.e. $\sum_{i=1}^{N}((S(t_i)/S(t_{i-1}))-1)^2)$ **when there are (large) jumps**.

- Traders report that, before the global financial crisis of Autumn 2008, empirical values of $\frac{\mathrm{VS}(t_0,T,\infty)}{-\mathrm{LFC}(t_0,T)}$ were approximately two - say, between 1.96 and 2.04 (presumably, this might have been a self-fulfilling prophecy?).
- What about since the aftermath of the global financial crisis?

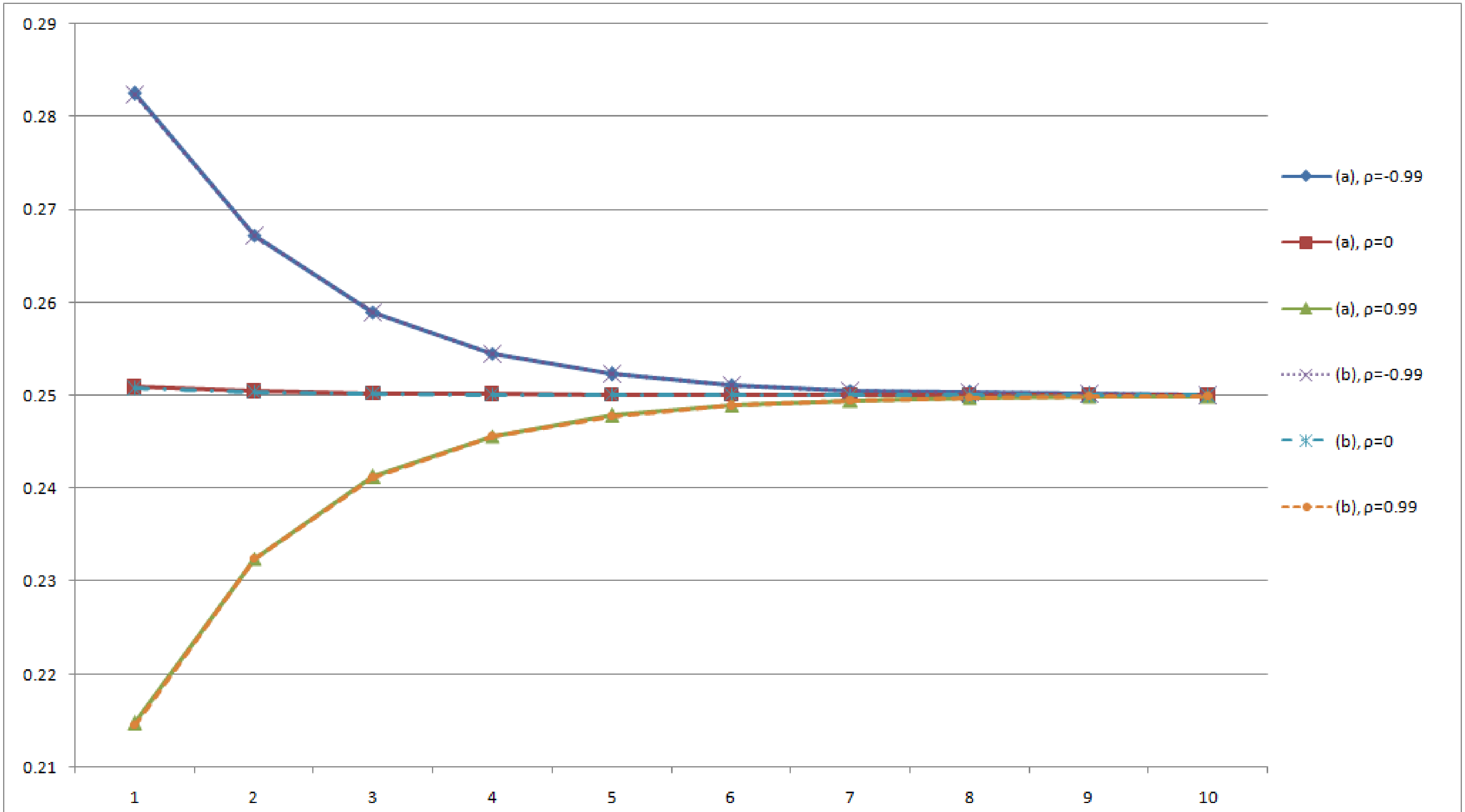
- Empirical values of $\frac{\mathrm{VS}(t_0,T,\infty)}{-\mathrm{LFC}(t_0,T)}$ for Nikkei-225 stock index as of 10th December 2010.



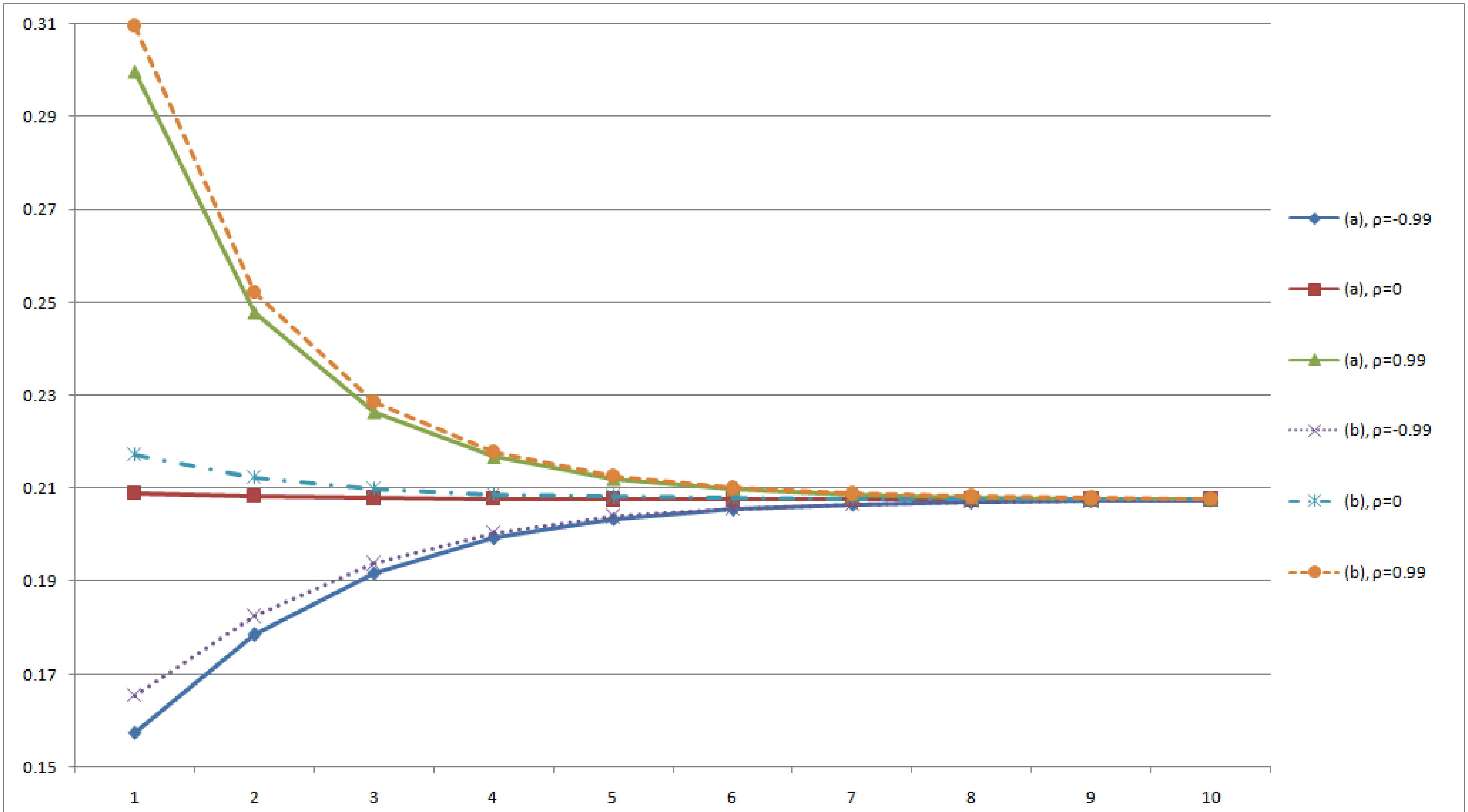
- Since the global financial crisis, empirical values of $\frac{\mathrm{VS}(t_0,T,\infty)}{-\mathrm{LFC}(t_0,T)}$ for variance swaps written on most major stock indices have been consistently been in the range 2.1 to 2.25.
- This suggests that traders are now pricing in the possibility of large downward jumps in the underlying stock.

- We now consider some numerical examples.
- We consider variance swaps and proportional variance swaps, with maturity $T = 0.5$, and with N (equally-spaced) monitoring times where $N = 2^{(J-1)}$, for $J = 1, 2, ..., 10$.
- We consider a generalised CGMY process (with a diffusion component) time-changed by a Heston (1993) activity rate (parameters from calibration to the market prices of vanilla options on the S & P500 stock index).
- To see effect of drift and correlation:
- We consider two possible choices, labelled (a) and (b) for the values of the interest-rate $r(t)$ and the dividend yield $q(t)$. In the first choice (a), $r(t) = 0$, $q(t) = 0$, for all t. In the second choice (b), $r(t) = 0.065$, $q(t) = 0.015$, for all t.
- We consider three different combinations for the correlation ρ between the activity rate and the diffusion component of the CGMY process: Namely, $\rho = -0.99$, $\rho = 0$ and $\rho = 0.99$.

- Variance swap rates (expressed in volatility terms as a decimal)



- Proportional variance swap rates (expressed in volatility terms as a decimal)



- Generally speaking, discrete monitoring makes little difference to the prices of variance swaps and proportional variance swaps (in the paper, we show more or less the same story for self-quantoed variance swaps, gamma swaps and skewness swaps). This means they are also little affected by the value of Correl(X_t, Y_t).
- Jumps in the underlying dynamics make a lot of difference (there are more examples in the paper) - this is especially true with asymmetric jumps.
- This motivates empirical studies which try to determine how much of the negative skewness seen in stock price returns (under both $\mathbb{P}$ and $\mathbb{Q}$) comes from a negatively skewed Lévy process and how much comes from a negative value of Correl(X_t, Y_t).
- The paper ("Variance derivatives: Pricing and convergence") on which this talk is based is on my website:
  http://www.john-crosby.co.uk .

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