A Local Correlation Model :
Motivation and Practical Implementation

Exotic Pricing and Hedging

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GROUPE BPCE

Contents

1. **Motivation**
2. **Local Correlation Model**
3. **Pricing, Hedging Methodology**
   - Langnau Method
   - Calibration Using Fixed-Point Algorithm
   - Model Estimation Using Envelop Approach
4. **Conclusion**

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1 Motivation

Literature I – on local correlation

1. A-M. Avenalleda, D. Boyer-Olson, J. Busca and P.Friz, « Reconstructing Volatility », October 2002

2. B-V. Durrleman + N. El-Karoui, « Basket Skew », April 2007

3. C-Bruno Dupire « Basket Skew Asymptotics » working paper 2004

4. D-X. Burtschell, J. Gregory and J-P. Laurent, « Beyond the Gaussian Copula: Stochastic and Local Correlation » , Working Paper, 2005

5. E-A. Langnau, « Introduction Into Local Correlation Modelling » , September 2009.

6. F-B. Jourdain, Mohamed Sbai “Coupling Index and stocks” 2009

Literature II – some comments

1. A-It gives the framework for calibrating baskets and numerical algorithms for short term asymptotics for pricing

2. B-C-It provides a good grasp of the phenomenology with model free approach
3. Good for the phenomenology

4. D-Simple idea to expand the dimension and obtain stochastic correlation at a cheap cost (is used for the local correlation model)

5. E-Simplest (sufficient conditions) local volatility extension plus direct calibration formulae and model risk illustration through the chewing gum effect

6. F- Nice numerical method – particle method, specific to baskets

Literature II

It provides a good grasp of the phenomenology (model-wise + chewing-gum effect)

However,

> **Only a short-term asymptotic formula for basket options is available.**
>
> **It assumes that basket options are traded.**
>
> **Numerical pricing with such models is extremely cumbersome. It requires to perform one Cholesky decomposition at each time step of a Monte-Carlo Path.**

Remaining Issues

1. **How to deal with long-term maturity contracts?**

2. **How to deal with worst-of options?**

3. **How to handle cases where basket implied dynamics are not observable?**

4. **How to price, risk-manage and stress-test correlation in a trading environment?**

Motivation I

$$\frac{d\sigma_t}{\sigma_t} = 2\sum_{i=1}^{n} \beta_t^i dq_t^i + 2\sum_{i=1}^{n} \beta_t^i q_t^i \frac{d\sigma_t^i}{\sigma_t^i} + \frac{2}{\sigma_t^2} \sum_{1\le i<j\le n} q_t^i q_t^j \sigma_t^i \sigma_t^j d\rho_t^{ij} + \{\ldots\}dt$$

With $B_t = \sum_{i=1}^{n} w_i S_t^i$, $q_t^i = \frac{w_i S_t^i}{B_t}$ and $\beta_t^i = \frac{1}{\sigma_t^2 dt} Cov\left(\frac{dS_t^i}{S_t^i}, \frac{dB_t}{B_t}\right)$

Three factors contribute to the basket skew generation:

- **S^i's relative weight q^i in the basket;**
- **S^i's volatility σ^i (individual stock skew and their interactions);**
- **S^i vs. S^j's stock correlation ρ^{ij}.**

$$\begin{aligned} u_{SV} &= u_{BS} \\ &+ (T-t)\left\{\sum_{i,j} \frac{1}{12}\alpha_i\alpha_j\sigma_i(t)\sigma_j(t)\rho_{i,j}^{\sigma,\sigma}\frac{\partial u_0}{\partial(\sigma_i\sigma_j)}\right\} \\ &+ (T-t)\left\{\sum_{i,j} \frac{1}{6}\alpha_i\alpha_j\sigma_i(t)\sigma_j(t)\rho_{i,j}^{\sigma,\sigma}\frac{\partial^2 u_0}{\partial\sigma_i\partial\sigma_j}\right\} + (T-t)\left\{\sum_{i,j} \frac{1}{2}\sigma_i(t)\alpha_j\sigma_j(t)\rho_{i,j}^{S,\sigma} S_i \frac{\partial^2 u_0}{\partial S_i\partial\sigma_j}\right\} \end{aligned}$$

Motivation I – illustration on SX5E

Without stochastic correlation, we can't fit empirical index skews, much steeper than the reconstructed one.

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Motivation II

In the local volatility framework, carry drives daily P&L on a delta-neutral portfolio Π :

$$\Delta \mathrm{PL}_t \approx \frac{1}{2}\sum_{1\le i,j\le n}\frac{\partial^2\Pi}{\partial S_t^i \partial S_t^j} S_t^i S_t^j \left(\frac{\Delta S_t^i}{S_t^i}\frac{\Delta S_t^j}{S_t^j} - \rho_t^{ij}\sigma_i\left(t,S_t^i\right)\sigma_j\left(t,S_t^j\right)\Delta t\right)$$

Under stress market conditions, realized covariance might exceed expectation…

$$\rho_t^{ij}\sigma_i\left(t,S_t^i\right)\sigma_j\left(t,S_t^j\right)\Delta t << \frac{\Delta S_t^i}{S_t^i}\frac{\Delta S_t^j}{S_t^j}$$

… Damaging the risk-management of trading books embedding short-correlation exposure.

Assuming well-marked volatilities, correlation need to be shifted by Δρ so that

$$\left(\rho+\Delta\rho\right)_t^{ij}\sigma_i\left(t,S_t^i\right)\sigma_j\left(t,S_t^j\right)\Delta t = E\left[\frac{\Delta S_t^i}{S_t^i}\frac{\Delta S_t^j}{S_t^j}\right]$$

Incurring a brutal (∂Π/∂ρ).Δρ to stop the bleeding.

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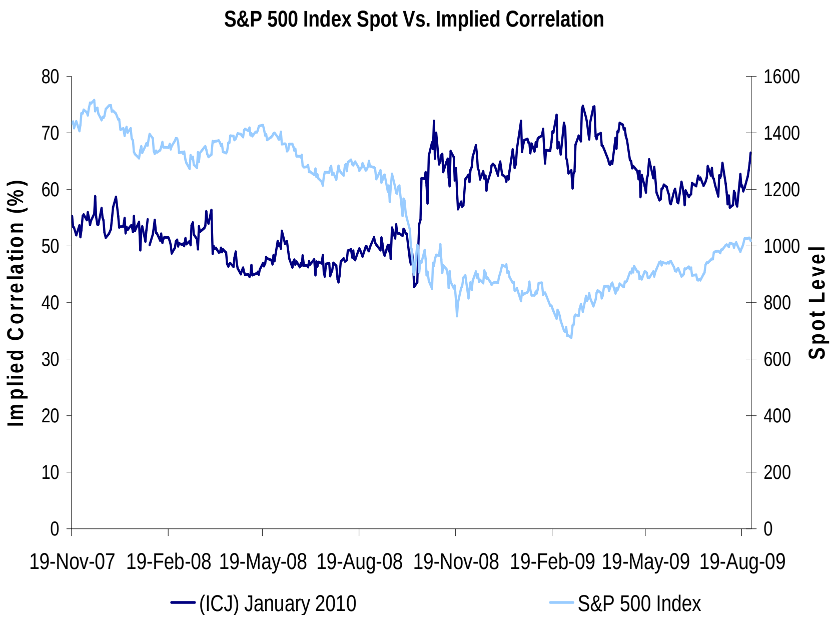
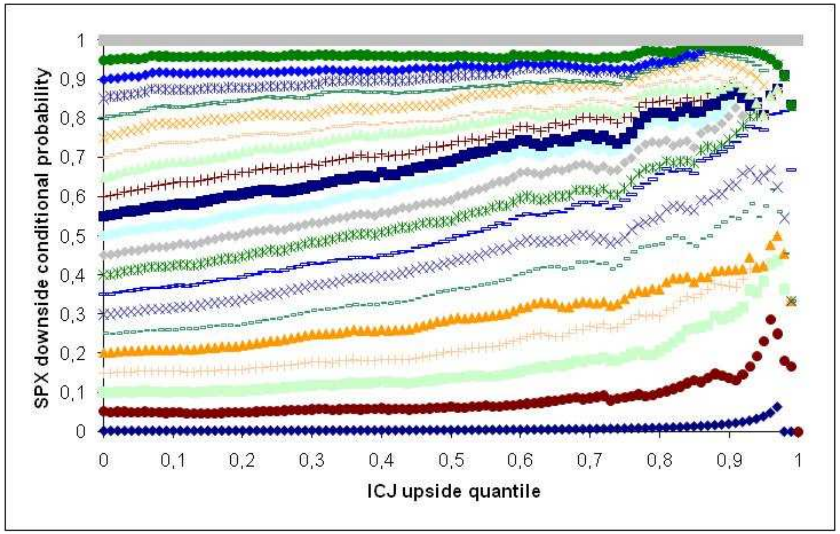
Motivation III

Historical correlation measures (ICJ) implied from past prices of S&P500 index option expiring on January 2010





$$P\left(r_{SPX} \leq F_{SPX}^{-1}(u_0) \middle| r_{JCJ} \geq F_{JCJ}^{-1}(x)\right)$$

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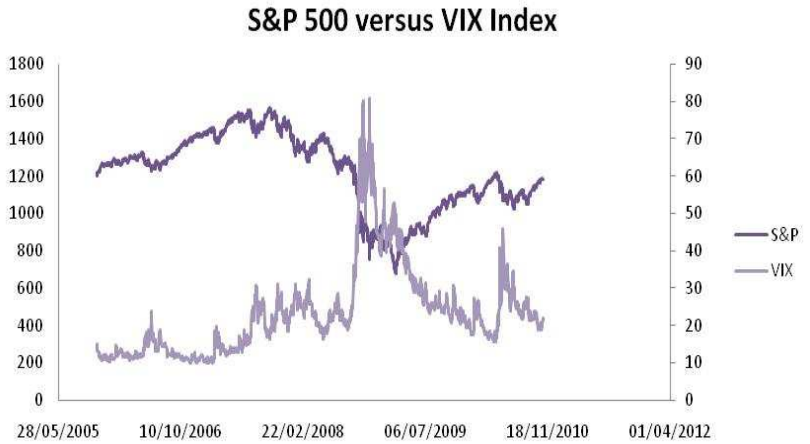
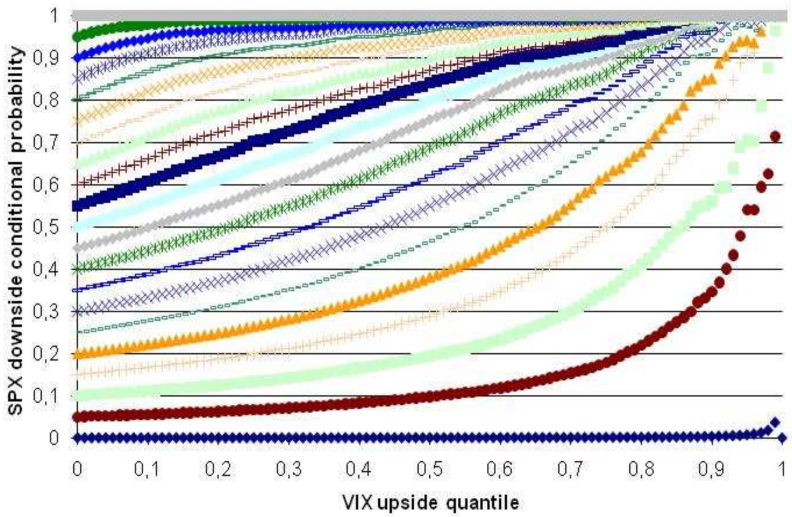
Motivation IV

Measures of Risk are correlated So it is good to model correlation linked to volatility

BUT what do you do when markets are dislocated





$$P\left(r_{SPX} \leq F_{SPX}^{-1}\left(u_0\right) \middle| r_{VIX} \geq F_{VIX}^{-1}(x)\right)$$

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Motivation V

Incorporating correlation risk in the risk-management of multi-underlying options is highly advised.

But implementing stochastic correlation is complicated in practice:

- **Ill-posed calibration problem with very few market information compared to the number of parameters to determine;**

- **Time-consuming diffusion scheme given that a matrix factorization is required at each time step of the discretization grid.**

2 Local Correlation Model

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Models Up and Running In The Banking Industry

Bank/ Model	Black Pricing	Black Risk Mgt	Black Stress Test	Local Vol Pricing	Local Vol Risk Mgt	Local Vol Stress Test
Top Tier*	✓	✓	✓	✓	✓	✓
Second Tier	✓	✓	✓	✓	X	X
Third Tier	✓	✓	✓	X	X	X

*** "Top Tier" at the technical level.**

Incremental progress

Spot dynamics mostly used by banks are as follows:

$$\begin{cases} \dfrac{dS_t^1}{S_t^1} & = & \sigma_1\left(t, S_t^1\right) dW_t^{\rho,1} \\ \vdots & \vdots & \vdots \\ \dfrac{dS_t^n}{S_t^n} & = & \sigma_n\left(t, S_t^n\right) dW_t^{\rho,n} \end{cases}$$

$$< dW_t^{\rho,i}, dW_t^{\rho,j} >= \rho_{i,j}(t)dt$$

A significant enhancement is to consider a local correlation depending on two variables : Time t + an aggregator L.

More precisely,

$$\rho_{ij}^{-} \leftarrow \rho_{ij} * (1-\lambda)$$

$$\rho_{ij}^{+} \leftarrow \rho_{ij} * (1-\lambda) + \lambda$$

$$\lambda\left(t, S_t^1, \ldots, S_t^n\right) = f\left(t, L_t\right)$$

$$L_t = \sum_{i=1}^{n} w^{(i)} \frac{S_t^{(i)}}{S_0^{(i)}}$$

Local Correlation Model

Instantaneous correlation becomes

$$\rho^{+,\lambda}\left(t;S_t^1,\cdots,S_t^k\right)=\left(1-\lambda\left(t;L_t\right)\right)\cdot\boldsymbol{\rho}+\lambda\left(t;L_t\right)\cdot\begin{bmatrix}1 & \cdots & 1\\ \vdots & \ddots & \vdots\\ 1 & \cdots & 1\end{bmatrix}$$

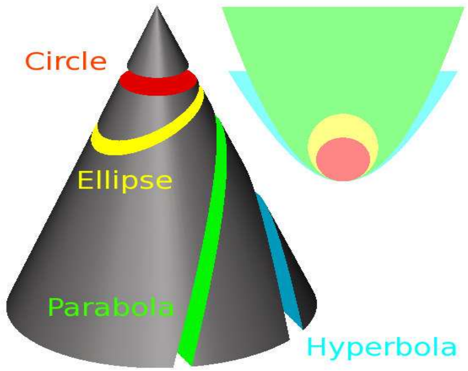
$$\rho^{-,\lambda}\left(t;S_t^1,\cdots,S_t^k\right)=\left(1-\lambda\left(t;L_t\right)\right)\cdot\boldsymbol{\rho}$$

Related Brownian is easily simulated with the following dimension extension:

$$W^{\rho^{\lambda,+}}(t)=\sqrt{\left(1-\lambda\left(t;L_t\right)\right)}\cdot W^{\rho}(t)+\sqrt{\lambda\left(t;L_t\right)}\cdot W^{\perp}(t)$$

$$W^{\rho^{\lambda,-}}(t)=\sqrt{\left(1-\lambda\left(t;L_t\right)\right)}\cdot W^{\rho}(t)+\sqrt{\lambda\left(t;L_t\right)}\cdot W^{Id}(t)$$

Similar idea to Pascal's classification of conics:



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Instantaneous moment matching of variance of basket with constituants gives:

$$\sum_{i=1}^{N} \rho_{ij} c_{ij} = \sigma_B^2(t,B)B^2$$

With

$$c_{ij} := \omega_i \omega_j S_i S_j \sigma_i(t,S_i)\sigma_j(t,S_j)$$

So we have the following local correlation model

$$\lambda(t,\overrightarrow{S}) = \frac{\sigma_B^2 B^2 - \sum_{ij} \rho_{ij}^{\text{fixed}} c_{ij}}{\sum_{ij}(1-\rho_{ij}^{\text{fixed}})c_{ij}}$$ **(to increase correlation)**

$$\lambda(t,\overrightarrow{S}) = \frac{\sigma_B^2 B^2 - \sum_{ij} \rho_{ij}^{\text{fixed}} c_{ij}}{\sum_{ij}(\frac{1}{N-1}+\rho_{ij}^{\text{fixed}})c_{ij}}$$ **(to decrease correlation)**

Provided that:

$$\boxed{\sigma_B^2(t,B)B^2 \le \sum_{ij} c_{ij}}$$

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Bad-Press On Local Volatility Model. Yet…

Widely used by investment banks to manage their trading books.

Main drawbacks are

- **Model-Wise**
  Its dynamics underestimate
  - **forward skew level**
  - **vovol convexity**

- **Numerical**
  Unstable implementation

But :

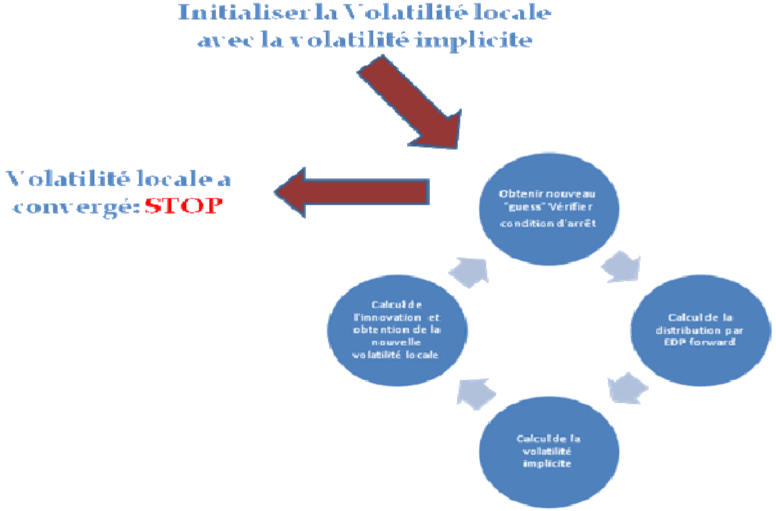
- **Its robustness can be improved with a Fixed-Point Approach.**

- **The "Super Vega Bucket" allows the projection of exotic volatility risk in the vanilla world.**

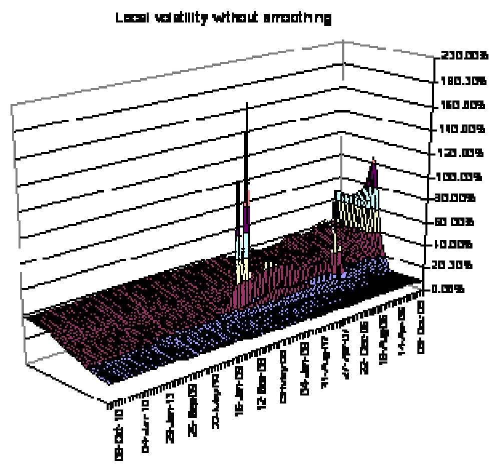
Fixed Point Algorithm

Calibrate with the Fixed-Point method to generate a smoother surface avoiding numerical issues

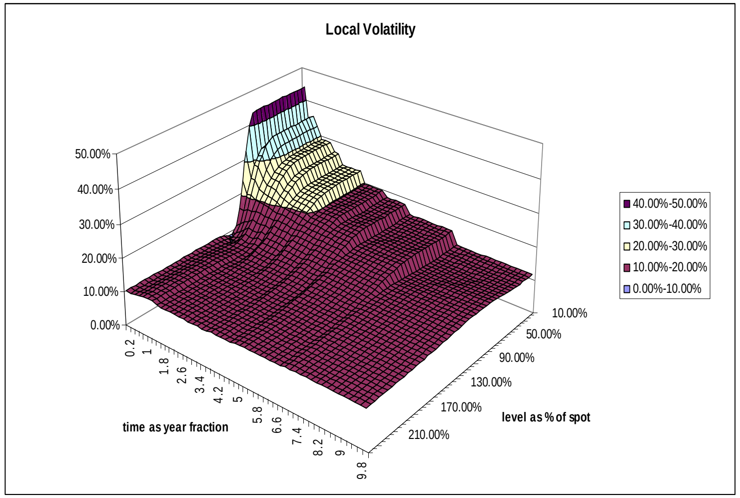
$$\{\sigma_{i+1}\} = \{\sigma_i\}\left\{\frac{\Sigma_{market,\,fwd}}{\Phi(\sigma_i)}\right\}$$



Standard Dupire's Formula



Fixed Point Algorithm



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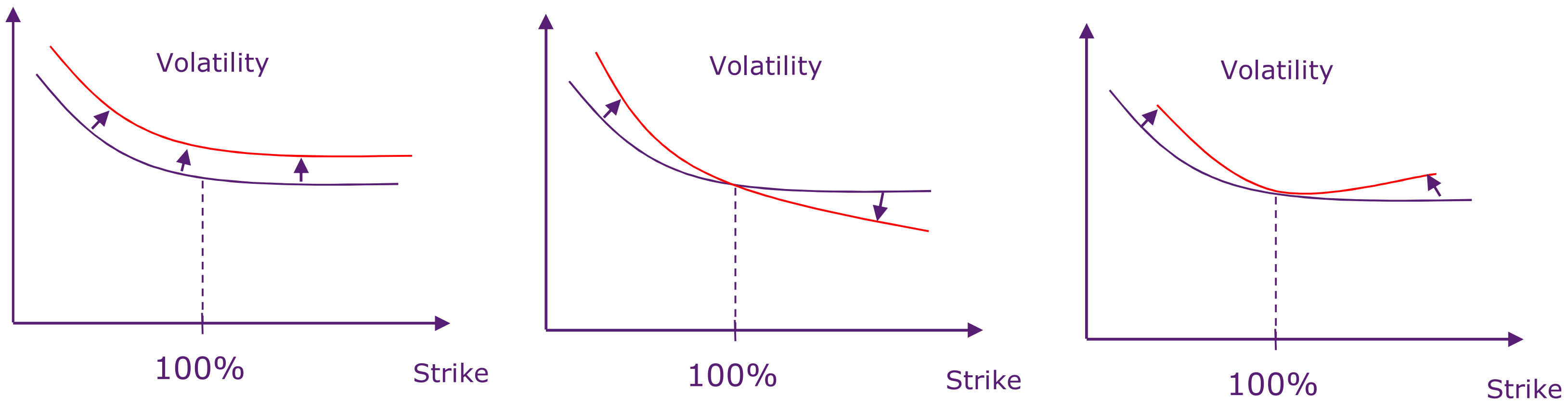
Vega Scenario Hedging

Risk Management

- **On Exotics, it is essential to hedge the distortions of the implied volatility surface.**
- **They are mainly due to :**
  - **Parallel shift**
  - **Rotation of the smile**
  - **Change in convexity**

Vega
Vega$_{Smile}$
Vega$_{curve}$



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But It Is Not Enough…

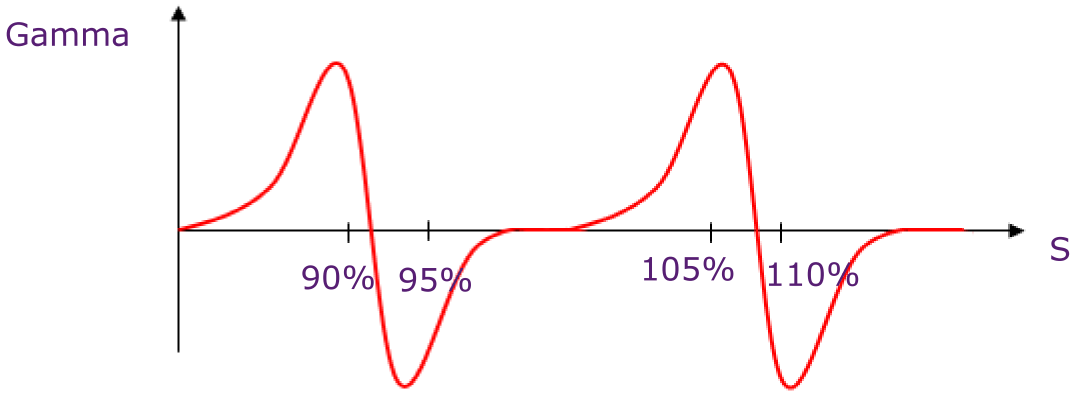
Example

Consider a call spread with a 90%-down strike and a 110%-up strike to hedge

$$CS(90\%,110\%) = Call(90\%) - Call(110\%)$$

With the parametric volatility approach (*Vega, $Vega_{Smile}$, $Vega_{curve}$*), we could sell a 95%-105% call spread which features roughly the same sensitivity to volatility deformations.

Gamma of the new portfolio *CS(90%,110%)- CS(95%,105%)* would be:



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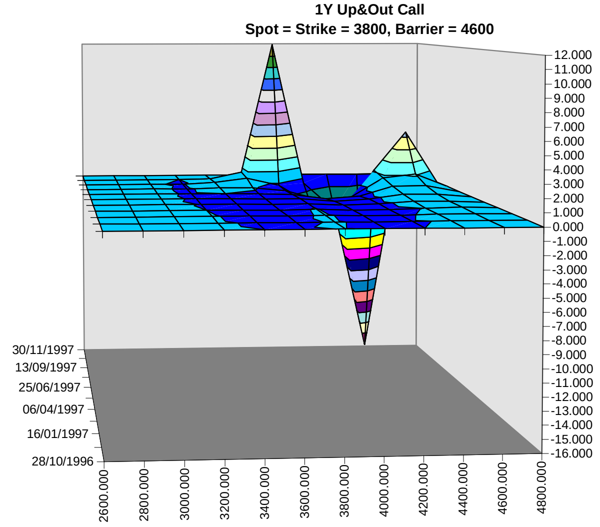
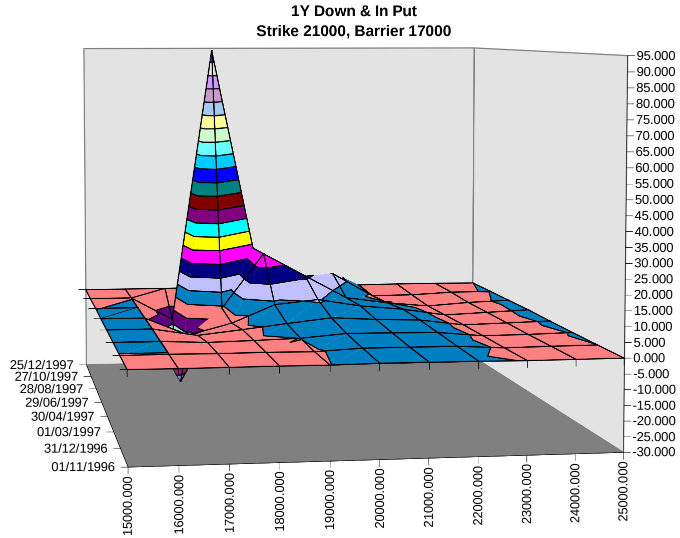
Local Volatility Allows Super Vega Bucket

Super Vega Bucket enables to Gamma hedge everywhere on the (K,T)-Grid



Call Up & Out



Put Down & In

3 Parameters Estimation

Parameter Estimation

The local lambda functional is fine-tuned according to

1. **Payoff**
2. **Available implied information**

We propose two approaches:

- **The implied approach (world indices, liquid stocks) infers λ from basket option implied volatilities quotes in the market;**

- **The statistical Estimation approach analyzes the statistical distributions of the correlation with respect to the basket components as a whole or in some directions.**

3.1 Implied Approach

Calibration Using Fixed-Point Algorithm

The algorithm works exactly as in the one factor local volatility model described above:

1. **Initialization**

   Naïve $\lambda_0 = 0$

   Based on Asymptotic approaches

2. **Lambda update**

$$\lambda_{i+1} = \lambda_i + \ln\left(\frac{\Sigma_{\text{basket,Market}}}{\Phi(\lambda_i)}\right)$$

$\Phi(\lambda)$: Application that transforms the local lambda function into the implied volatility surface of the underlying basket.

Initial guess : Ito and Short Term Asymptotic

Result

$$X_t = B_T^{\frac{1}{\sigma_B}}$$

$$dX_t = \frac{\sum_{i=1}^{n} \sigma_i \omega_i dW_{i,t}}{\sigma_B} - \frac{1}{2} \ln X_t \frac{d\sigma_B^2}{\sigma_B^2} + \theta_t dt$$

$$\frac{d\sigma_B^2}{\sigma_B^2} = \sum_{i,j=1}^{n} \underbrace{\beta_{i,j} \frac{d\rho_{i,j}}{\rho_{i,j}}}_{\text{correlation dynamic}} + 2\sum_{i=1}^{n} \underbrace{\beta_i \frac{d\sigma_i}{\sigma_i}}_{\text{volatility dynamic}} + 2\sum_{i=1}^{n} \underbrace{\beta_i \frac{d\omega_i}{\omega_i}}_{\text{weight's variability}}$$

We recover MEK and BD results:

Three terms contributing to the distortion from a log normal

- **(1) Weights variability**
- **(2) Each underlying own distortion**
- **(3) Correlation skew**

Initial Guess : Pat Hagan formula recovered

We look at a one stoch vol model - keep one underlying :

$$X_t = B_T^{\frac{1}{\sigma_B}}$$

$$\frac{dX_t}{X_t} = \sigma(X_t) dZ_t$$

$$\sigma^2(X_t) = 1 + \alpha^2 \ln^2(X_t) - 2\rho_{S,\sigma}\alpha \ln(X_t)$$

This becomes a local volatility model for which the implied volatility is given by the classical BBF formula.

Recover easily the Pat Hagan formula.

Initial Guess : Multi Stoch vol with local correlation

We keep contribution from each underlying smile

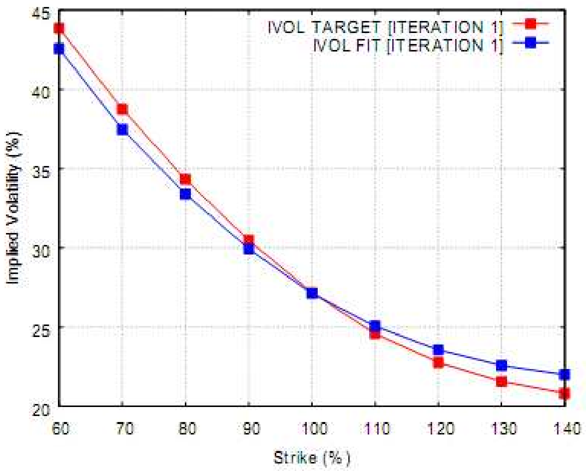
We neglect the variability of the weights (in practice it is negligible)

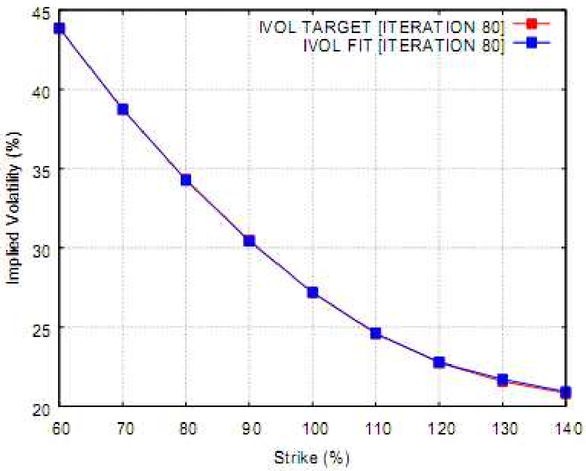
$$
\begin{aligned}
\sigma^2(X_t) \quad = \quad & 1 && \textbf{No Correl} && \textbf{Extra term with Correl} \\
- \quad & 2\ln(X_t) && \left\{ \sum_{i,j=1}^{n} \frac{\sigma_i}{\sigma_B} w_i \beta_j \alpha_j \rho_{i,j}^{S,\sigma} \right. && \left. -\frac{1}{2}\sigma_B \frac{d\lambda}{dB} \sum_{i_1,j_1}^{n} \beta_{i_1 j_1} \frac{1-\rho_{i_1 j_1}}{\rho_{i_1 j_1}(1-\lambda)+\lambda} \right\} \\
+ \quad & \ln^2(X_t) && \left\{ \sum_{i,j=1}^{n} \beta_i \alpha_i \beta_j \alpha_j \rho_{i,j}^{\sigma,\sigma} \right. && \left. + \ldots \right\}
\end{aligned}
$$

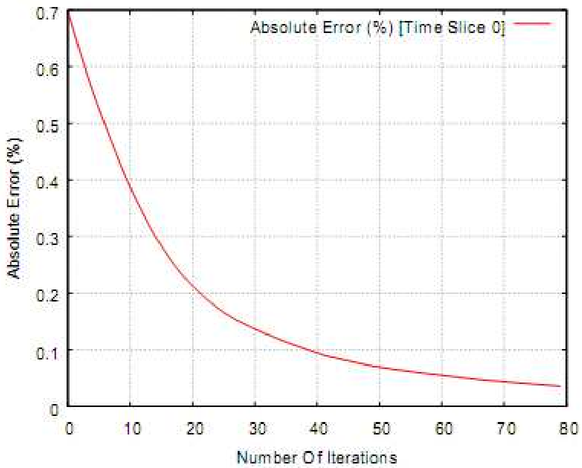
Calibration Using Fixed-Point Algorithm

Enable to price and manage multi-underlying options with a correlation model fitted on a basket implied volatility surface quoted in the market.

Instantaneous correlation controlled by current {Time + Basket Level}.







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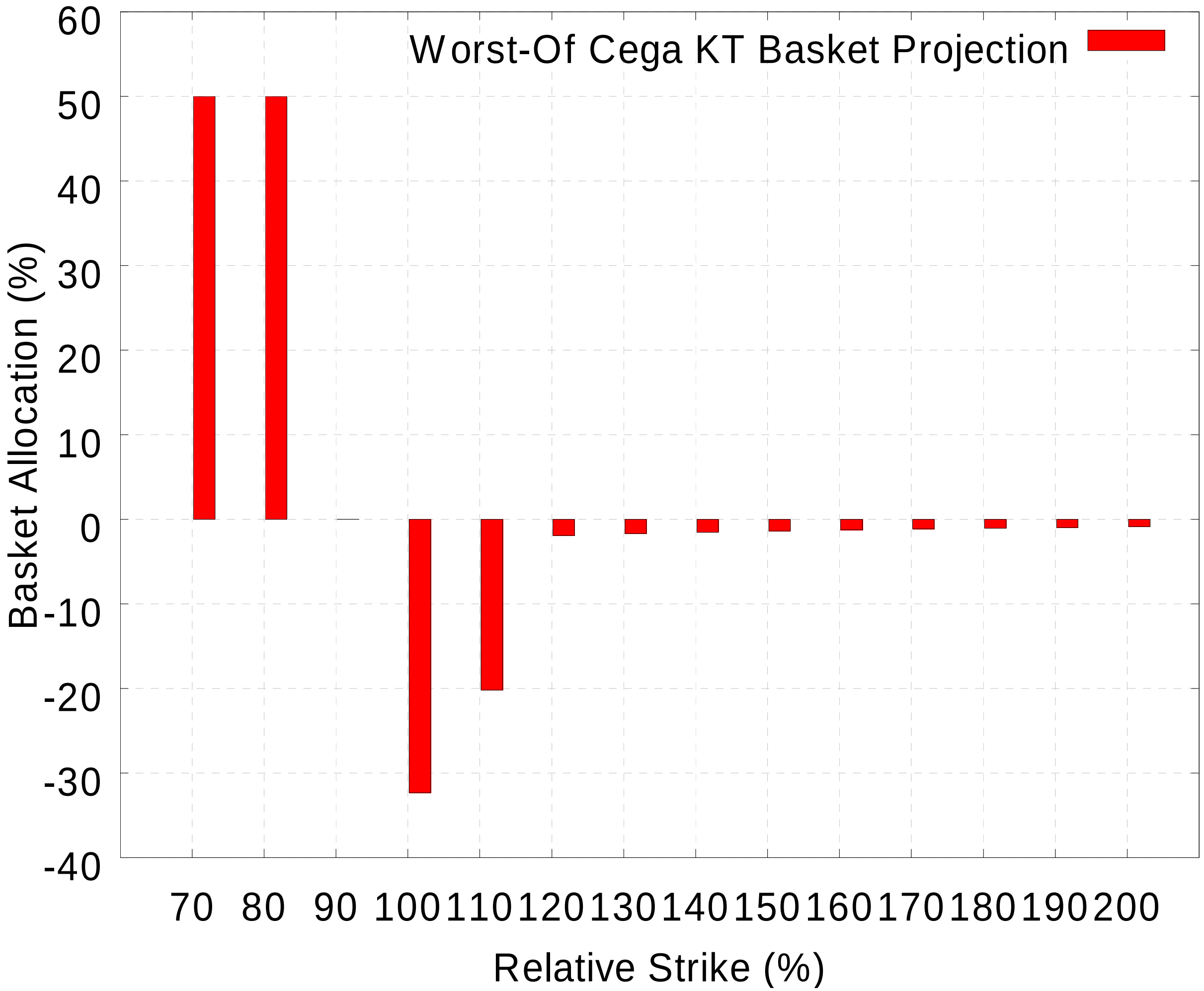
Cega KT



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3.2 Statistical Approach

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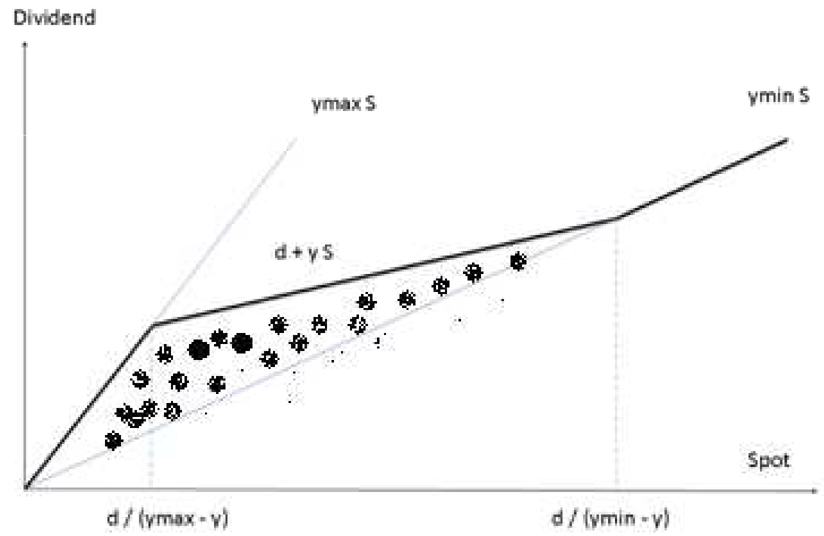
Model Estimation Using Envelop Approach I

Description

- **Models the upper/lower bound of a random quantity Y with a hyperplane X.**
- **Successfully applied to model dividends with respect to spot:**



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Future on Dividend example

- **For large movements Spot is a good hedge for dividends**

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Model Estimation Using Envelop Approach II

Empirical estimation of a correlation envelop with respect to spot levels to define the local lambda function λ.

Depends on the product under consideration.

- **On Best-Of puts, we look for a parametrization which singles out the best performance:**

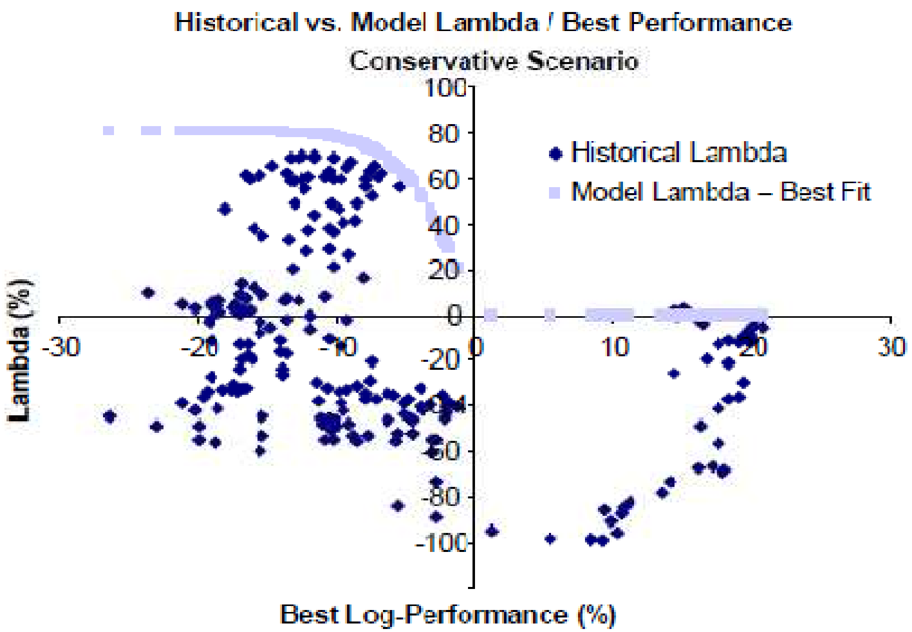
$$\lambda\left(S_t^1, \cdots, S_t^k\right) = \max\left(-\lambda_0 \times \tanh\left(\frac{s}{\lambda_0} \times \max_{i=1,\cdots,n} \ln\left(\frac{S_t^i}{S_0^i}\right)\right), \lambda_{\min}\right)$$

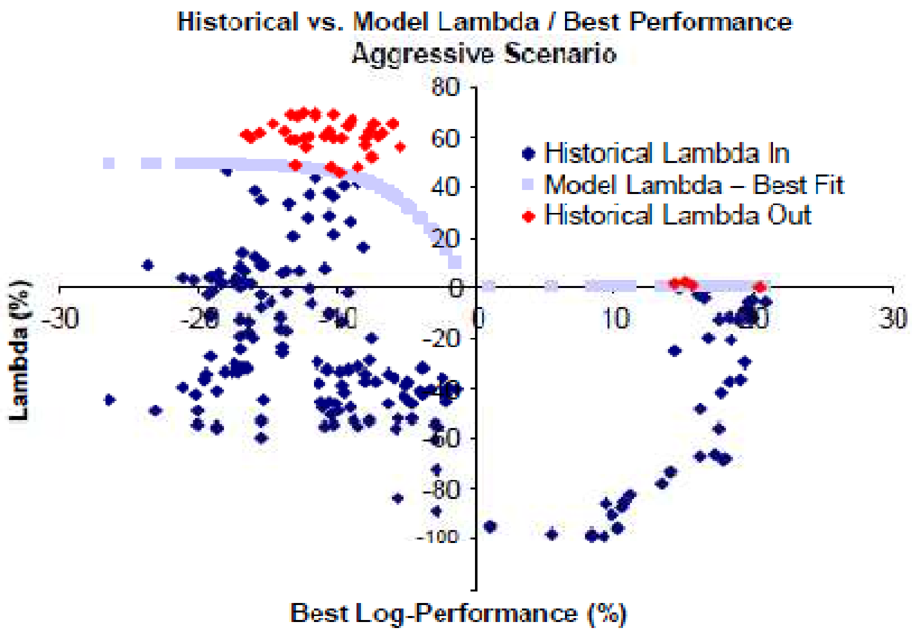
- **λ_0, λ_{min} and s are estimated from historical worst-case scenarios.**

Model Estimation Using Envelop Approach III

Find { λ_0, λ_{min}, s } : P($\lambda_{histo} < \lambda_{model}$ | Best Perf. < 1) = p

- **Conservative scenario : p = 100%**
- **Aggressive scenario : p = 80%**





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Delta Hedging Improvement

In the Local Correlation model, Delta becomes

$$\underbrace{\Delta_{i,LVLC}}_{\text{LocalVolLocalCorrelDelta}} = \underbrace{\Delta_{i,LV}}_{\text{LocalVolatilityDelta}} + \frac{\partial \pi}{\partial \lambda}\frac{\partial \lambda}{\partial S_i}$$

- **The delta hedges the change of correlation**
- **No PnL impact with correlation remarking**

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4 Conclusion

Conclusion

1. **We proposed a local correlation Model which is an incremental enhancement of the local volatility model**

2. **We presented pricing and hedging methodologies suited to available information**

3. **We showed robust numerical techniques to calibrate or estimate this model**

4. **We discussed a smart implementation that enables to run the model into the pricing libraries of an investment bank**

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Questions

Thank You For Your Attention

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