Stochastic Volatility Models with Jumps

Efficient and accurate pricing of derivatives

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Overview of the Talk

- Motivation
- The Model
- Pricing vanillas
- Calibration example: FX market
- Pricing volatility derivatives, and other exotics

Motivation

Purpose of the model:

Understanding the risk of portfolios of derivative securities for

- Pricing
- Hedging
- Risk Management

Features that the model ideally should possess:

- Jumps (Gamma Regime)
- Stochasticity of Volatility (Vega Regime, Volatility Clustering)
- Analytical Tractability (Calibration, Hedging and Risk Managemement)
- Parsimony (limited number of parameters)

Motivation

We focus on two features:

- **Regime shifts**
  - Key observation: Significant shifts to parameters (vol, trend, etc.) tend to happen simultaneously.
  - Anecdotal evidence & empirical studies (e.g. Hamilton (1990))
- **Gap risk**
  - Sudden changes (jumps) in the price of the underlying which cannot be captured by standard delta hedging.

FX Model with regime-shifts and jumps

- Domestic and foreign money market accounts (MMA):

$$B_t^D := \exp\left(\int_0^t R_D(Z_s)\mathsf{d}s\right), \qquad B_t^F := \exp\left(\int_0^t R_F(Z_s)\mathsf{d}s\right).$$

- Model for the foreign exchange rate $S = (S_t)_{t \geq 0}$ is given by

$$\begin{aligned} S_t &:= S_0 \exp(X_t) \quad \text{where} \quad S_0 \in (0, \infty) \quad \text{and} \\ X_t &:= \sum_{i \in E} \int_0^t I_{\{Z_s = i\}} \mathsf{d}X_s^i. \end{aligned}$$

-
  - $R_D, R_F : E \to \mathbb{R}$ functions
  - Z is a finite state Markov chain with state space E — models the regime shifts
  - X^i are independent Lévy processes — model the jumps

FX model with regime shifts and jumps

- Q: How to choose the Markov chain Z?
- A: two approaches:
  - As approximations to general stochastic volatility models with jumps (the chain Z has many states).
  - As parsimonious descriptions of risk-neutral probability laws implied by the markets (the chain Z has two or three states).
- In this talk we restrict to the seconde approach.

Lévy processes

- **Lévy processes**: processes with stationary and independent increments
- Examples: CGMY model, VG model, Kou model.
- Calibrate well across strikes, term structure of moments restrictive
- For any $i \in E$ consider a Lévy process $X^i = (X^i_t)_{t \geq 0}$ with characteristic exponent $\psi_i : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}\left[\mathrm{e}^{\mathtt{i} u X^i_t}\right] = \mathrm{e}^{t\psi_i(u)},$$

with the Lévy-Khintchine representation

$$\psi_i(u) = \mathtt{i}\mu_i u - \frac{\sigma_i^2}{2}u^2 + \int_{-\infty}^{\infty} [\mathrm{e}^{\mathtt{i}ux} - 1 - \mathtt{i}uxI_{\{|x|\leq 1\}}]\nu_i(\mathsf{d}x),$$

where $\sigma_i, \mu_i \in \mathbb{R}$ are constants and ν_i is the Lévy measure.

The Markov chain of regime shifts

- Finite state-space of regimes: $E := \{1, \ldots, N\}$, of a continuous-time Markov chain Z.
- Generator of Z is an $N \times N$ matrix Q.
- Notation: we write

  $$M(i,j) = M_{ij}$$

  for the ijth element of the matrix M.
- Chain jumps from regime i to regime j at rate $Q(i,j)$, i.e.

  $$P(Z_{t+\delta} = j | Z_t = i) = Q(i,j)\delta + o(\delta),$$

  as $\delta \to 0$.

The model: basic observations

- The process X is not Markovian!
- The pair (X, Z), is Markov and the task is to understand its law!
- Let J^i, $i \in E^0$, be independent pure jump Lévy processes (i.e. with characteristic triplets $(0, 0, \nu_i)$ and $W = (W)_{t \geq 0}$ standard Brownian motion. Then the process $\widetilde{X}$, defined by

$$\widetilde{X}_t := \int_0^t \mu(Z_s)\mathrm{d}s + \int_0^t \sigma(Z_s)\mathrm{d}W_s + \sum_{i \in E^0} \int_0^t I_{\{Z_s = i\}}\mathrm{d}J_s^i,$$

has the same law as X.

FX Model with regime shifts and jumps

- The price at time s of a **zero coupon bond** maturing at $t \geq s$

$$\mathbb{E}_i \left[\frac{1}{B_t^D} \middle| \mathcal{F}_s^{(X,Z)} \right] = \frac{1}{B_s^D} \cdot \left(\exp((t-s)(Q - \Lambda_D))\mathbf{1} \right)(Z_s),$$

where $\mathcal{F}_s^{(X,Z)} = \sigma\left((X_u, Z_u) \ : \ u \in [0,s] \right)$.

The characteristic matrix exponent

The *characteristic matrix exponent* $K : \mathbb{R} \to \mathbb{C}^{N_0 \times N_0}$ of (X, Z) is

$$K(u) := Q + \Lambda(u), \quad \text{where } \Lambda(u)(i,i) = \psi_i(u),\ i \in E^0,$$

$\Lambda(u)$ is a diagonal matrix and Q the generator of Z.
Define diagonal matrices Λ_D and Λ_F by

$$\Lambda_D(i,i) := R_D(i), \qquad \Lambda_F(i,i) := R_F(i).$$

Theorem 1 *The discounted characteristic function of Markov process (X, Z):*

$$\mathbb{E}_{x,i}\left[\frac{\exp(\mathtt{i}uX_t)}{B_t^D} I_{\{Z_t=j\}}\right] = \exp(\mathtt{i}ux)\cdot\exp(t(K(u)-\Lambda_D))(i,j), \quad u \in \mathbb{R},$$

where $\mathbb{E}_{x,i}[\cdot]$ denotes the conditional expectation $\mathbb{E}[\cdot|X_0 = x, Z_0 = i]$.

Regime switching Lévy model

- Under a pricing measure, to avoid arbitrage we need to ensure that

$$(S_t B_t^F / B_t^D)_{t\geq 0}$$

is a positive martingale.

- A restriction on the parameters that guarantees this is:

$$\begin{aligned}
\Lambda(-\mathtt{i}) &= \Lambda_D - \Lambda_F \\
\Leftrightarrow \psi_j(-\mathtt{i}) &= R_D(j) - R_F(j) \\
\Longrightarrow \mathbb{E}_{i,x}[S_t B_t^F / B_t^D] &= \mathrm{e}^x \left[\exp\left(tQ\right)\mathbf{1}\right](i) = S_0 B_0^F / B_0^D
\end{aligned}$$

for all $S_0 = \mathrm{e}^x \in (0,\infty)$. This, together with Markov property of (X,Z), implies that $(S_t B_t^F / B_t^D)_{t\geq 0}$ is a martingale.

Regime switching Lévy model

- (X, Z) is a Markov process
- Infinitesimal generator $\mathcal{L}$ is for sufficiently smooth functions $f : \mathbb{R} \times E \to \mathbb{R}$ given by

$$\begin{aligned}
\mathcal{L}f(x,i) &= \frac{\sigma^2(i)}{2} f''(x,i) + \mu(i) f'(x,i) \\
&\quad + \int_{\mathbb{R}} \left[f(x+z,i) - f(x,i) - f'(x,i) z I_{\{|z| \le 1\}} \right] \nu_i(\mathrm{d}z), \\
&\quad + \sum_{j \in E^0} Q(i,j) [f(x,j) - f(x,i)].
\end{aligned}$$

Pricing European options

A call option struck at K with expiry T is defined as

$$C_T(K) := C(S_0, i, K, T) := \mathbb{E}_{x,i}\left[(B_T^D)^{-1}(S_T - K)^+\right].$$

- Fourier transform c_T^* in log-strike $k = \log K$ of $C_T(K)$ is

$$c_T^*(\xi) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\xi k} C_T(\mathrm{e}^k)\mathsf{d}k \quad \text{where} \quad \Im(\xi) < 0.$$

- Let $\xi \in \mathbb{C}\backslash\{0, \mathtt{i}\}$, $x \in \mathbb{R}$, $j \in E^0$. Define

$$D(\xi, x, j) := \frac{\mathrm{e}^{(1+\mathtt{i}\xi)x}}{\mathtt{i}\xi - \xi^2} \cdot \left[\exp\left\{T(K(1 + \mathtt{i}\xi) - \Lambda_D)\right\}\mathbf{1}\right](j).$$

- If $\Im(\xi) < 0$, then for $x = \log S_0$ and $Z_0 = j$, it holds

$$c_T^*(\xi) = D(\xi, x, j)$$

The implied volatility surface

IVol surface is a graph of a function $(K, T) \mapsto \sigma(K, T)$ defined implicitly by the equation

$$C^{\mathrm{BS}}(S_0, K, T, \sigma(K, T)) \quad = \quad C(K, T),$$

where $C(K, T)$ are the market/model specified call option prices and $C^{\mathrm{BS}}(S_0, K, T, \cdot)$ is the Black-Scholes formula.

- $C(K_{ij}, T_i)$, $i = 1, \ldots, n$, $j = 1, 2, 3$, are the most liquid derivative instruments in the financial markets.
- Knowing σ is equivalent to knowing the one-dimensional marginals in a risk-neutral measure of the underlying process.
- To calibrate to the observed IVol surface the model needs to have stochastic volatility AND jumps.
- If $n = 2$ (i.e. two maturities) typically time-dependence of parameters is needed for calibration.

Calibration study: two states

- $N = 2$ (two states only!)
- Lévy processes: Kou model (double exponential jumps)
- $\Lambda(u)$ a 2×2 diagonal matrix with the i-th diagonal element

$$\psi_i(u) := u\mu_i + \sigma_i^2 u^2/2 + \lambda_i p_i \left(\frac{\alpha_i^+}{\alpha_i^+ - u} - 1 \right) + \lambda_i (1 - p_i) \left(\frac{\alpha_i^-}{\alpha_i^- + u} - 1 \right).$$

- Recall $\Lambda_D := \mathrm{diag}(R_D)$, $\Lambda_F := \mathrm{diag}(R_F)$ and

$$\mathbb{E}_{0,i} \left[\frac{\exp(uX_t)}{B_t^D} I_{\{Z_t = j\}} \right] \quad = \quad \left[\exp(t(Q + \Lambda(u) - \Lambda_D)) \right] (i, j).$$

- A risk-neutral drift $\mu : E^0 \to \mathbb{R}$ is given by the formula

$$\Lambda(1) = \Lambda_D - \Lambda_F.$$

Markov additive model – calibration of stochastic rates

- For maturities $T_1 < T_2$ market implies two pairs P^D_{0,T_k}, P^F_{0,T_k}, $k = 1, 2$, of domestic and foreign zero coupon bond prices.
- In our model we have

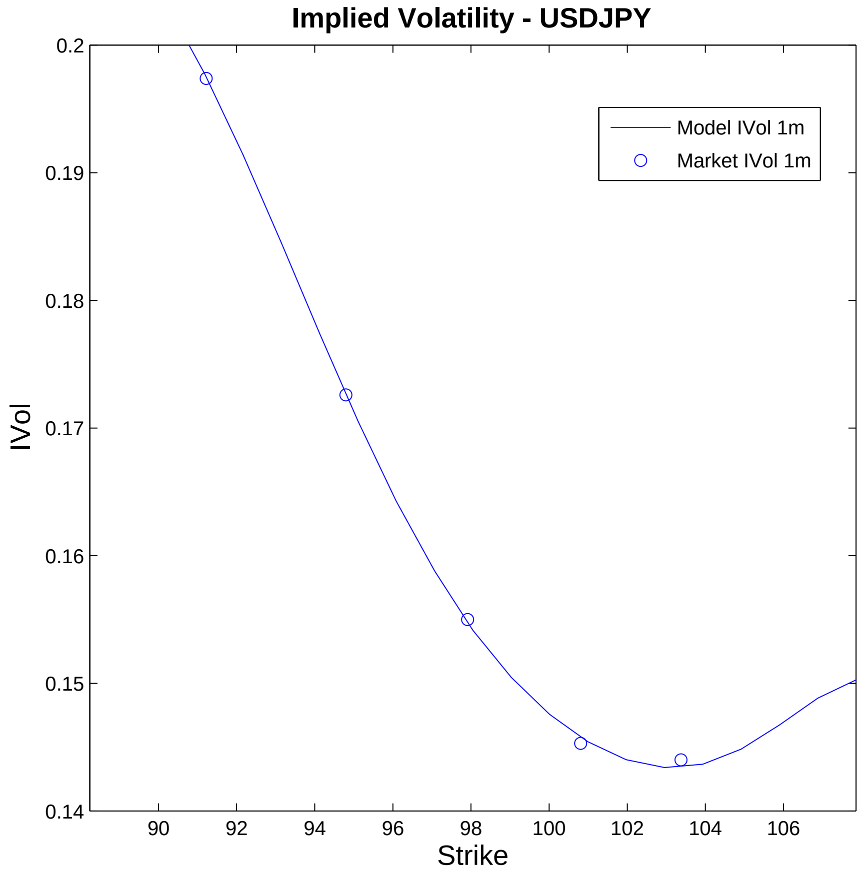
$$P^F_{0,T_k} = \mathbb{E}_{x,i}[(B^D_{T_k})^{-1} S_{T_k}]/S_0 \quad \text{and} \quad P^D_{0,T_k} = \mathbb{E}_{x,i}[(B^D_{T_k})^{-1}].$$

- To calibrate R_D, R_F solve the system:

$$\begin{aligned} P^D_{0,T_k} &= e_i' \exp\left((Q - \Lambda_D)T_k\right) \mathbf{1}, \\ P^F_{0,T_k} &= e_i' \exp\left((Q - \Lambda_F)T_k\right) \mathbf{1}, \end{aligned}$$

  where $k = 1, 2$ and $\Lambda_D = \mathrm{diag}(R_D), \Lambda_F = \mathrm{diag}(R_F)$.
- Since $N_0 = 2$, this system determines the risk-neutral drift of S, is independent of the calibration to option prices and can be solved accurately very fast.

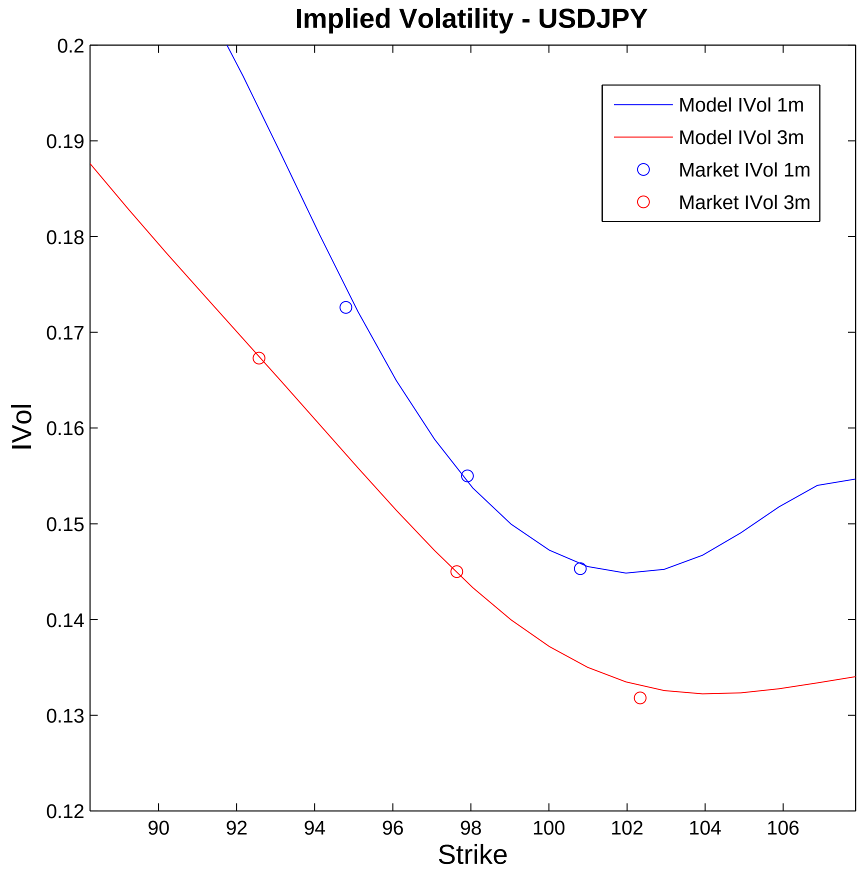
USDJPY – one maturity



Market data: $S_0 = 98.05$, domestic rate $r_d = -0.00036$, foreign rate $r_f = 0.0045$, maturity $T = 1/12$.

Model parameters: $N = 2$, $q_1 = 12, q_2 = 6$, $B_m(1) = \mathrm{diag}(-45, -300)$, $B_p(1) = -100$,$b_m(1) = (0.12, 0.88)$, $\lambda_2 = 0$ (chosen), $\sigma = (\sigma_1, \sigma_2), \lambda_1, p_1$ (calibrated).

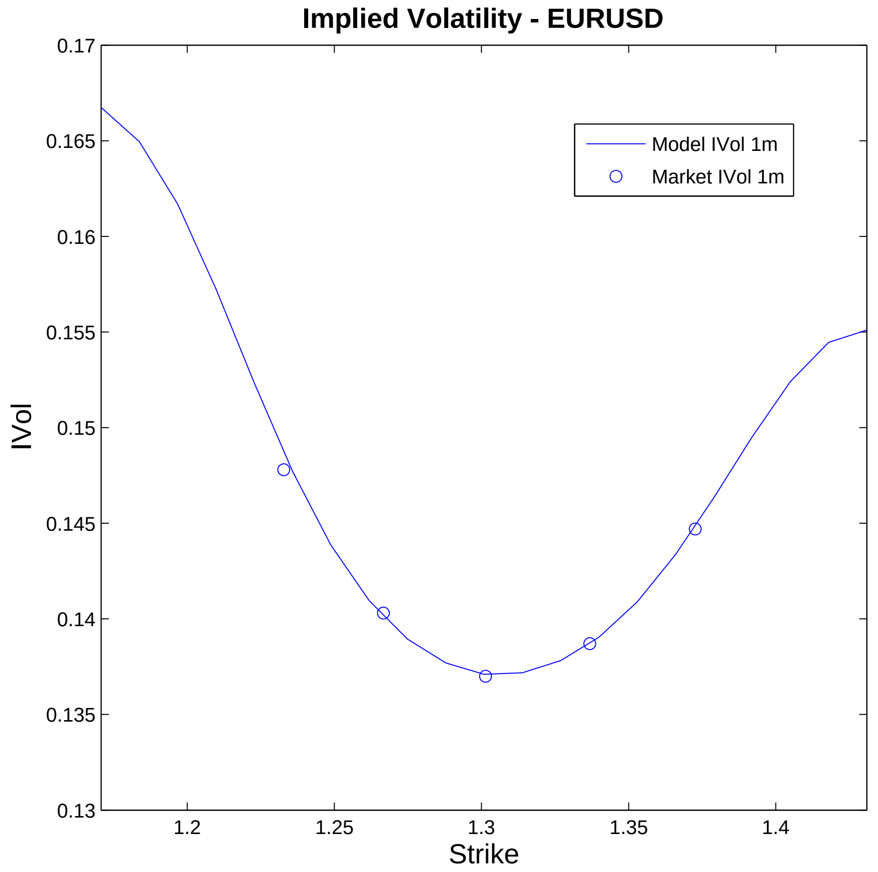
USDJPY – two maturities



Market data: $S_0 = 98.05$, domestic interest rate $r_d = (-0.00036, 0.005)$, foreign interest rate $r_f = (0.0045, 0.0111)$, maturity $T = (1/12, 3/12)$.

Model parameters: $N = 2$, $q_1 = 12, q_2 = 6$, $B_m(1) = \mathrm{diag}(-45, -300)$, $b_m(1) = (0.12, 0.88)$, $B_m(2) = -50$, $B_p(1) = -130$, $p_2 = 0$ (chosen), $\sigma = (\sigma_1, \sigma_2)$, λ_1, λ_2, p_1 (calibrated)

EURUSD – one maturity

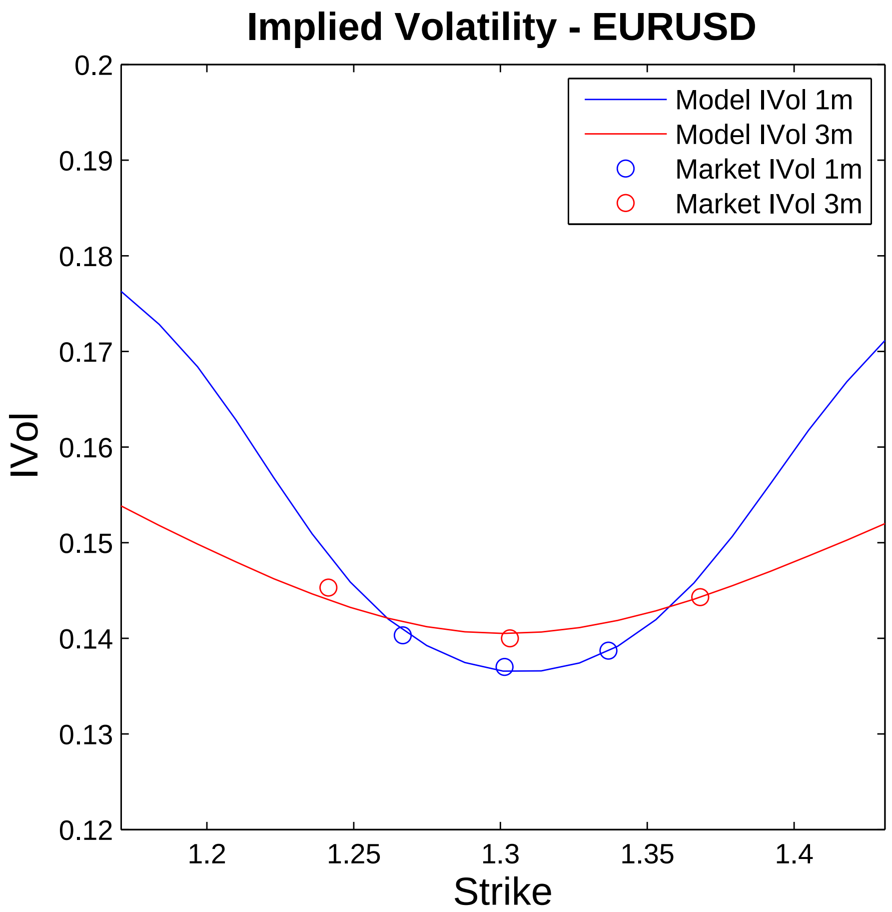


Market data: spot $S_0 = 1.3009$, domestic interest rate $r_d = 0.0045$, foreign interest rate $r_f = 0.0084$, maturity $T = 1/12$.

Model parameters: $N = 2$, $q_1 = 12, q_2 = 6$, $B_m(1) = \mathrm{diag}(-45, -300)$, $b_m(1) = (0.1, 0.9)$, $B_p(1) = -130$, $\lambda_2 = 0$ (chosen)

$\sigma = (\sigma_1, \sigma_2)$, λ_1, p_1 (calibrated)

EURUSD – two maturities



Market data: $S_0 = 1.3009$, domestic rate $r_d = (0.0045, 0.0111)$, foreign rate $r_f = (0.0084, 0.0139)$, maturity $T = (1/12, 3/12)$.

Model parameters: $N = 2$, $q_1 = 12, q_2 = 6$, $B_m(1) = -70$, $B_p(1) = -70$, $B_m(2) = -30$, $B_p(2) = -30$, $p_2 = 0.5$ (chosen)

$\sigma = (\sigma_1, \sigma_2)$, λ_1, λ_2, p_1 (calibrated)

Implied volatility at extreme strikes

The *implied volatility* $\sigma_{x,i}(K,T)$ in (X,Z) satisfies

$$C^{\mathsf{BS}}(\mathrm{e}^x, K, T, \sigma_{x,i}(K,T)) = \mathbb{E}_{x,i}\left[(B_T^D)^{-1}(S_T - K)^+\right].$$

For fixed maturity T define the quantities $F_T := \mathbb{E}_{x,i}[S_T]$ and

$$\begin{aligned} q_+ &:= \sup\left\{u : \mathbb{E}_{x,i}\left[\mathrm{e}^{(1+u)X_T}\right] < \infty \quad \text{for all} \quad i \in E^0\right\}, \\ q_- &:= \sup\left\{u : \mathbb{E}_{x,i}\left[\mathrm{e}^{-uX_T}\right] < \infty \quad \text{for all} \quad i \in E^0\right\}. \end{aligned}$$

Lee formula (under some assumptions):

$$\begin{aligned} \lim_{K\to\infty} \frac{T\sigma_{x,i}(K,T)^2}{\log(K/F_T)} &= 2 - 4\left(\sqrt{q_+^2 + q_+} - q_+\right), \\ \lim_{K\to 0} \frac{T\sigma_{x,i}(K,T)^2}{|\log(K/F_T)|} &= 2 - 4\left(\sqrt{q_-^2 + q_-} - q_-\right). \end{aligned}$$

Forward starting options

A payoff of T_1-*forward starting call option* with maturity $T_2 > T_1$ is

$$(S_{T_2} - \kappa S_{T_1})^+, \qquad\qquad \kappa \in \mathbb{R}_+.$$

- The Fourier transform in the forward log-strike of $F_{T_1,T_2}(\kappa) = \mathbb{E}_{x,i}\left[(B^D_{T_2})^{-1}(S_{T_2} - \kappa S_{T_1})^+\right]$ is defined by

$$F^*_{T_1,T_2}(\xi) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\xi k} F_{T_1,T_2}(\mathrm{e}^k)\mathsf{d}k, \quad \text{where} \quad \Im(\xi) < 0.$$

- For $x = \log S_0$, $Z_0 = j$ and ξ with $\Im(\xi) < 0$ it holds that

$$F^*_{T_1,T_2}(\xi) = \frac{\mathrm{e}^{(1+\mathtt{i}\xi)x}}{\mathtt{i}\xi - \xi^2} \cdot \Big[\exp(T_1(Q - \Lambda_F)) \exp\left\{(T_2 - T_1)(K(1+\mathtt{i}\xi) - \Lambda_D)\right\}\mathbf{1}\Big](j).$$

The forward smile

The *forward implied volatility* $\sigma^{fw}_{x,i}(S_T, \kappa, T)$ at a future time T:

$$C^{\mathsf{BS}}(S_{T_1}, \kappa S_{T_1}, T_2 - T_1, \sigma^{fw}_{x,i}(S_{T_1}, \kappa, T_1)) = \mathbb{E}_{x,i}\left[\frac{B^D_{T_1}}{B^D_{T_2}}(S_{T_2} - \kappa S_{T_1})^+ \middle| S_{T_1}\right],$$

where C^{BS} the Black-Scholes formula with strike κS_{T_1} and spot S_{T_1}.

$$\mathbb{E}_{x,i}\left[\frac{B^D_{T_1}}{B^D_{T_2}}(S_{T_2} - \kappa S_{T_1})^+ \middle| S_{T_1}\right] = S_{T_1} f^{x,i}(X_{T_1}, T_1)' C_{T_2 - T_1}(\kappa, 1), \quad \text{where}$$

$$f^{x,i}_j(y, T) := \mathbb{P}_{x,i}\left[Z_T = j \middle| X_T = y\right] = \frac{q^{x,i}_T(y, j)}{q^{x,i}_T(y)} \qquad \text{and ...}$$

The forward smile

... the joint distribution $q_T^{x,i}(y,j) = \frac{\mathsf{d}}{\mathsf{d}y}\mathbb{P}_{x,i}[X_T \le y, Z_T = j]$ at time T of (X_T, Z_T) is given by

$$q_T^{x,i}(y,j) \quad = \quad \frac{1}{2\pi}\int_{\mathbb{R}} e^{\mathtt{i}\xi(x-y)} \exp\left(K\left(\xi\right)T\right)(i,j)\,\mathsf{d}\xi, \qquad y \in \mathbb{R}, i,j \in E^0.$$

X_T is a continuous random variable with probability density function $q_T^{x,i}(y) = \frac{\mathbb{P}_{x,i}[X_T \in \mathsf{d}y]}{\mathsf{d}y}$ given by

$$q_T^{x,i}(y) \quad = \quad \frac{1}{2\pi}\int_{\mathbb{R}} e^{\mathtt{i}\xi(x-y)} \left[\exp\left(K\left(\xi\right)T\right)\mathbf{1}\right](i)\,\mathsf{d}\xi, \qquad y \in \mathbb{R}, i \in E^0.$$

Proof. The characteristic function is in $L^1(\mathbb{R})$.

Volatility derivatives

Refining sequence of partitions $(\Pi_n)_{n\in\mathbb{N}}$ of $[0,T]$: $\Pi_n \subset \Pi_{n+1}$, $\Pi_n = \{t_0^n \leq \ldots \leq t_n^n\}$ s.t. $\lim_{n\to\infty} \max\{|t_i^n - t_{i-1}^n| : 1 \leq i \leq n\} = 0$.

- *Quadratic variation* Σ_T of $X = \log S$:

$$\Sigma_T \quad := \quad \lim_{n\to\infty} \sum_{t_i^n \in \Pi_n, i\geq 1} \log\left(\frac{S_{t_i^n}}{S_{t_{i-1}^n}}\right)^2 .$$

- The sequence converges in probability, uniformly on $[0,T]$.
- The limit is given by

$$\Sigma_T = \int_0^T \sigma(Z_t)^2 \mathrm{d}t + \sum_{i\in E^0} \sum_{t\leq T} I_{\{Z_t=i\}} (\Delta X_t^i)^2,$$

where $\Delta X_t^i := X_t^i - X_{t-}^i$.

Volatility derivatives

$(\Sigma_t)_{t\geq 0}$ is the *quadratic variation* (*realized variance*) *process* of X.

- A buyer of a swap on the realized variance pays a premium (the swap rate) to receive at maturity T a pay-off $\phi(\Sigma_T)$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a measurable payoff function.
- Most common examples of ϕ are
  - (i) variance swap: $\phi(x) = x/T$.
  - (ii) volatility swap: $\phi(x) = \sqrt{x/T}$.
  - (iii) option on variance: $\phi(x) = (x-\kappa)^+$, where $\kappa \in \mathbb{R}_+$.
- The swap rate for the payoff ϕ is $\mathbb{E}_i\left[\phi(\Sigma_T)/B_T^D\right]$.

Volatility derivatives

$(\Sigma_t)_{t\geq 0}$ is a regime-switching Lévy process with

$$\Sigma_t = \int_0^t \sigma(Z_s)^2 \mathrm{d}s + \sum_{i\in E^0} \int_0^t I_{\{Z_s=i\}} d\widetilde{X}^i_s,$$

where $\widetilde{X}^i$, $i \in E^0$, is a pure-jump subordinator with

$$\begin{aligned} \nu^\Sigma(\mathrm{d}x) &= I_{(0,\infty)}(x)[-\mathrm{d}\overline{\nu}(\sqrt{x}) + \mathrm{d}\underline{\nu}(-\sqrt{x})] && \text{(Lévy measure)} \\ \psi_i^\Sigma(u) &= u\sigma_i^2 + \int_{\mathbb{R}_+} (1-\mathrm{e}^{-ux})\nu_i^\Sigma(\mathrm{d}x) \\ &= u\sigma_i^2 + \int_{\mathbb{R}} (1-\mathrm{e}^{-uy^2})\nu_i(\mathrm{d}y) && \text{(characteristic exponent of } \widetilde{X}^i\text{).} \end{aligned}$$

Recall: $\psi_i^\Sigma(u) = -\log \mathbb{E}[\mathrm{e}^{-u\widetilde{X}^i_1}]$, $\overline{\nu}(x) = \nu([x,\infty)), \underline{\nu}(x) = \nu(-\infty, x])$.

Volatility derivatives

The Laplace transform of Σ_t is given by

$$\mathbb{E}_i\left[\exp(-u\Sigma_t)\right] = \left[\exp(tK_\Sigma(u))\mathbf{1}\right](i), \qquad u > 0,$$

where

- the characteristic matrix $K_\Sigma(u)$ is given by

$$K_\Sigma(u) := Q + \Lambda_\Sigma(u) \qquad \text{and}$$

- $\Lambda_\Sigma(u)$ is an $N_0 \times N_0$ diagonal matrix with

$$\Lambda_\Sigma(u)(i,i) = \psi_i^\Sigma(u) = -\log \mathbb{E}[\mathrm{e}^{-u\widetilde{X}_1^i}], \quad i \in E^0.$$

Volatility derivatives

X^i jump-diffusion with double phase-type jumps. Then

- $\widetilde{X}^i$ is a compound Poisson process with intensity λ_i
- with positive jump sizes K_i with probability density

$$g_i(x) = \frac{1}{2\sqrt{x}} \left[p_i \beta_i^+ \mathrm{e}^{\sqrt{x} B_i^+}(-B_i^+)\mathbf{1} + (1-p_i)\beta_i^- \mathrm{e}^{\sqrt{x} B_i^-}(-B_i^-)\mathbf{1} \right] I_{(0,\infty)}(x).$$

- $\Phi(x) := \exp(x^2/2)\mathcal{N}(x)$, $\mathcal{N}$ normal cdf. Then $\mathbb{E}\left[\exp(-uK_i)\right]$ is

$$\sqrt{\frac{\pi}{u}} \left[p_i \beta_i^+ \Phi\left(\frac{1}{\sqrt{2u}} B_i^+\right)(-B_i^+) + (1-p_i)\beta_i^- \Phi\left(\frac{1}{\sqrt{2u}} B_i^-\right)(-B_i^-) \right] \mathbf{1}$$

- and the characteristic exponent of $\widetilde{X}^i$ equals

$$\psi_i^{\Sigma}(u) := u\sigma_i^2 + \lambda_i \left(1 - \mathbb{E}\left[\exp(-uK_i)\right]\right).$$

Volatility derivatives - the pricing formulae

Assume $R_D \equiv$ const (to simplify the formulae) and $Z_0 = i$.

- $\varsigma_{var}(T,j) = \mathbb{E}_i\left[\Sigma_T/T\right]$ and $\varsigma_{vol}(T,j) = \mathbb{E}_i\left[\sqrt{\Sigma_T/T}\right]$ are

$$\begin{aligned}\varsigma_{var}(T,j) &= \frac{1}{T}\left[\int_0^T \mathrm{e}^{Qt} V \mathrm{d}t\right](j),\\ \varsigma_{vol}(T,j) &= \frac{1}{2\sqrt{\pi T}}\int_0^\infty \left\{\left[I - \exp(TK_\Sigma(u))\right]\mathbf{1}\right\}(j)\frac{\mathrm{d}u}{u^{3/2}},\end{aligned}$$

  where $V \in \mathbb{R}^{N_0}$ with $V(i) = (\psi_i^\Sigma)'(0) = \sigma_i^2 + \int_{\mathbb{R}} y^2 \nu_i(\mathrm{d}y)$.

- $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ and $\exists a > 0$ s.t the Fourier transform ϕ_a^* of $\phi_a(x) = \mathrm{e}^{ax}\phi(x)$ is in $L^1(\mathbb{R})$. Then the ϕ-swap rate is

$$\varsigma_\phi(T,j) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \phi_a^*(\xi)\left[\exp(T(K_\Sigma(a - \mathtt{i}\xi) - \Lambda_D))\mathbf{1}\right](j)\mathrm{d}\xi.$$

Barrier contracts

A *barrier contract* with expiry $T > 0$ pays the random cash flow

$$g(S_T)\mathbf{I}_{\{\tau_A > T\}} + h(S_{\tau_A})\mathbf{I}_{\{\tau_A \le T\}}, \quad \text{where} \quad \tau_A = \inf\{t \ge 0 : S_t \in A\},$$

- knock-out set $A = (0, \ell] \cup [u, \infty), \quad 0 \le \ell < u \le \infty$;
- $g, h : (0, \infty) \to \mathbb{R}_+$ payoff and rebate functions respectively.

Examples:

- knock-out double barrier ($0 < \ell, u < \infty, h \equiv 0$);
- down-and-out ($u = \infty$, $h \equiv 0$), up-and-out ($\ell = 0$, $h \equiv 0$);
- rebate ($g \equiv 0$), European ($0 = \ell, u = \infty$).

Double-no-touch options

Double-no-touch (or **range bet**) pays one unit of domestic currency at T if FX rate S stays in (ℓ, u) during $[0, T]$ and zero else.

- DNTs are the most liquid exotic options in financial markets.
- Hence DNTs should be used for calibration of the model S.
- The arbitrage-free price in a model S of a double-no-touch:

$$
\begin{aligned}
D_{S_0}(T) &= \mathbb{E}_{S_0}\left[\frac{I_{\{\tau_{\ell u} > T\}}}{B_T^D}\right], \quad \text{where} \\
\tau_{\ell u} &:= \inf\{t \,:\, S_t \notin (\ell, u)\}.
\end{aligned}
$$

Warning: price of DNT involves joint law of max and min of S.

Wiener-Hopf factorisation for Brownian motion X

Let e_q be exponential rv, $\mathbb{E}[e_q] = 1/q$, independent of X.

$$\frac{q}{q - u^2/2} \quad = \quad \frac{\rho_+(q)}{\rho_+(q) + u} \cdot \frac{\rho_-(q)}{\rho_-(q) - u}, \quad \text{where} \quad \rho_\pm(q) = \pm\sqrt{2q}$$

are the largest and smallest root of the characteristic equation

$$q - \frac{u^2}{2} = 0.$$

Define $\overline{X}_t = \max\{X_s \ : \ s \in [0,t]\}, \quad \underline{X}_t = \min\{X_s \ : \ s \in [0,t]\}$.
Moment generating function of $\overline{X}_{e_q}$, $\underline{X}_{e_q}$ are

$$\mathbb{E}\left[\exp(-u\overline{X}_{e_q})\right] = \frac{\rho_+(q)}{\rho_+(q) - u}, \quad \mathbb{E}\left[\exp(u\underline{X}_{e_q})\right] = \frac{\rho_-(q)}{u + \rho_-(q)}, \quad u \geq 0.$$

Wiener-Hopf factorisation for Brownian motion X

Therefore $\overline{X}_{e_q}$, $-\underline{X}_{e_q}$ are geometric rvs with params $\rho_+(q)$, $-\rho_-(q)$.
Let $\tau_u := \min\{t \geq 0 \ : \ X_t \geq u\}$ and $\tau_\ell := \min\{t \geq 0 \ : \ X_t \leq \ell\}$.

$$\{\tau_u < t\} = \{\overline{X}_t > u\}, \quad \{\tau_\ell < t\} = \{\underline{X}_t < \ell\} \quad \forall t \in \mathbb{R}_+.$$

Hence

$$\begin{aligned} \mathbb{E}[e^{-q\tau_u}] &= \mathbb{E}\left[\int_0^\infty I_{\{\tau_u < t\}} q e^{-qt} dt\right] = \mathbb{P}(\tau_u < e_q) = e^{-u\rho_+(q)} \\ \mathbb{E}[e^{-q\tau_\ell}] &= e^{\ell\rho_-(q)}. \end{aligned}$$

An application of Doob's optional stopping theorem yields a closed form for the Laplace transform for the two-sided first passage time

$$\tau_{\ell u} := \inf\{t \ : \ X_t \notin (\ell, u)\}.$$

Matrix Wiener-Hopf factorisation

In the general case of the Markov additive process the steps are similar (but the details are very different):

- Fluid-embedding: embed the jumps to get a continuous Markov additive process (phase-type distribution of jumps is used in this step).
- The characteristic equation becomes a quadratic matrix equation.
- The Wiener-Hopf factors can be inverted analytically.
- Closed-form formula for Laplace transform of the one-sided first passage time can be obtained.
- Doob's optional stopping theorem gives a closed-form formula for the Laplace transform of the two-sided first passage time.

Further reading

Papers that can be downloaded from SSRN:

- Exotic Derivatives under Stochastic Volatility Models with Jumps (with A. Mijatovic)
- Continuously Monitored Barrier Options Under Markov Processes (with A. Mijatovic)
- Pricing and Hedging Barrier Options in a Hyper-Exponential Additive Model (with M. Jeannin)
- The Valuation of Structured Products Using Markov Chain Models (with D Madan and W Schoutens)