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A Comparative Analysis of Correlation Approaches in Finance

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The paper is at www.dersoft.com/Acomparative.pdf
or at SSRN

Heuristic Definition

Financial Correlations measure the co-movement of two or more assets in time

Motivation

"…correlation, while being one of the most ubiquitous concepts in modern finance and insurance, is also one of the most misunderstood concepts." (Embrechts et al. 1999)

Correlations play a key role in

- Investing: From CAPM → An increase in *diversification* increases the Return/Risk ratio. Diversification is synonymous with (negative) *correlation*

- Risk Management: The lower the correlation of the assets in the portfolio, the lower is the risk, derived by any risk measure as VAR (Value at risk), SRM (Spectral risk measures) or ERM (enterprise risk management).

- Trading: Correlation Desk exist at every investment bank

Motivation cont.

Nassim Taleb: Anything that relies on correlation is charlatanism

Rational: Correlations are notoriously instable and cluster in a recession

Charles M. Schulz: No problem is so big that it can’t be run away from

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Modeling Financial Correlations

Statistical Correlation Measures

- Pearson
- Spearman's ρ
- Kendall's τ

Specific Financial Correlation Approaches

Bottom-up Approaches

- Correlating Brownian Motions (Heston 1993)
- Binomial Correlations (Lucas 1995)
- Copulas (Sklar 1959, Li 2000)
- Lattice Models with Dynamic Copulas (Albanese et al 2005, 2007, 2010)
- CID Models (Vasicek 1987 and extensions)
- Contagion correlation approach (Davis and Lo 1999, Jarrow and Yu 2001)
- Hybrids

Top-down Approaches

- Vasicek's 1987 LHP
- Markov Chain Models of Transition Rates
  - Hurd, Kuznetsov (2006a, 2000b)
  - Schönbucher (2006)
- Giesecke et al (2009)

Statistical Correlation Measures

Why not just use Pearson ??

$$\rho_1(X,Y) = \frac{\text{cov}(X,Y)}{\sigma(X)\sigma(Y)} \quad (1)$$

$$\rho_1(X,Y) = \frac{E(XY) - E(X)E(Y)}{\sqrt{E(X^2) - (E(X))^2}\sqrt{E(Y^2) - (E(Y))^2}} \quad (2)$$

1) Linear dependencies as assessed in equations (1) and (2) do not appear often in finance.

2) Linear correlation measures are only natural dependence measures if the joint distribution of the variables is elliptical.

3) Zero correlation derived in equations (1) and (2) does not necessarily mean independence. This is because only the two first moments are considered in (1) and (2). For example, $Y = X^2$ $\{y \neq 0\}$ will lead to $p_1 = 0$, which is arguably misleading.

4) The variances of the variates X and Y have to be finite. However, for distributions with strong kurtosis, for example the student-t distribution with $v \leq 2$, the variance is infinite.

5) Pearson correlation approach is typically not invariant to transformations. For example, the Pearson correlation between pairs X and Y is in general different than the the Pearson correlation between the pairs ln(X) and ln(Y).

Statistical Correlation Measures

What about the ordinal correlation measures Spearman's ρ and Kendall's τ?

They can be useful correlation measure in Finance, if the underlying variables are ordinal as in a transition matrix.

$$\rho_2 = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n(n^2-1)}$$

Spearman's ρ

with i=1,…,n pairs x_i and y_i, numerically ranked by x_i, and $d_i = x_i - y_i$,

$$\rho_3 = \frac{n_c - n_d}{0.5n(n-1)}$$

Kendall's τ

where n_c is the number of concordant data pairs and n_d is the number of discordant pairs.

Specific Financial Correlation Approaches

Correlating Brownian Motions (Heston 1993)

$$\frac{dS(t)}{S(t)} = \mu\, dt + \sigma(t) dz_1(t)$$

$$d\sigma^2(t) = g[\sigma_L^2 - \sigma^2(t)]\, dt + \xi\, \sigma(t)\, dz_2(t)$$

$$\mathbf{Corr[dz_1(t), dz_2(t)] = \rho_4\, dt} \qquad \mathbf{(7)}$$

The definition (7) can be conveniently modeled with the identity

$$dz_1(t) = \sqrt{\rho_4}\, dz_2(t) + \sqrt{1-\rho_4}\, dz_3(t)$$

where $dz_2(t)$ and $dz_3(t)$ are independent, and $dz(t)$ and $dz(t')$ are independent, $t \neq t'$.

Critical Appraisal of Heston 1993 model

The Heston correlation approach is a dynamic, versatile, and mathematically rigorous correlation model.

It allows to positively or negatively correlate stochastic processes and permits dynamic risk management.

Hence it is not surprising that the approach is an integral part of correlation modeling in finance.

There are numerous applications of the Heston model in Finance as the SABR model of Hagan et al (2002), where stochastic interest rates and stochastic volatility are correlated to derive realistic volatility smiles and skews. Further applications are Huang and Yildirim (2008), Langnau (2009),Zhou (2001), Brigo and Pallacinini (2008), Meissner et al 2009.

Binomial Correlation Coefficient

The standard deviation of a one-trial binomial event is $\sqrt{P(X)-(P(X))^2}$ *where P is the prob of outcome X*

Hence the Pearson equation (2) $\rho_1(X,Y)=\dfrac{E(XY)-E(X)E(Y)}{\sqrt{E(X^2)-(E(X))^2}\sqrt{E(Y^2)-(E(Y))^2}}$ *changes to*

$$\rho_5(1_{\{\tau_X\le T\}},1_{\{\tau_Y\le T\}})=\frac{P(XY)-P(X)P(Y)}{\sqrt{(P(X)-(P(X))^2}\sqrt{(P(Y)-(P(Y))^2}} \qquad (9)$$

By construction, equation (9) can only model binomial events, for example default and no default.

Equation (9) is a special case of equation (2). Hence the same benefits and limitations as (2) apply.

Copulas

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Copula Milestones

Founder: Abe Sklar 1959

Sklar's Theorem: $C[F(x),G(y)] = M(x,y;\rho_M)$ **(1)**

Vasicek 1987 derives a 'one-factor Gaussian' CAR:

$$V(X,T) = N\left(\frac{N^{-1}(Q(T)) + \sqrt{\rho}N^{-1}(X)}{\sqrt{1-\rho}}\right)$$

applied in Basel II's IRB approach

David Li 2000 "On Default Correlation: A Copula Function Approach"

"An Introduction to Copulas" Roger Nelson 1998, second ed. 2006

"Copula Methods in Finance" Cherubini et al 2004

Copulas

One factor copulas

- **Gaussian**
  - ρ = correlation
- **Archimedian**
  - $\varphi_a(t)$ = generator
  - Gumbel
  - Clayton
  - Frank

Two factor copulas

- **Student's t**
  - ρ = correlation
  - ν = d.o.f
- **Fréchet**
  - p, q = linear combination
- **Marshall-Olkin**
  - m,n = weight factors

Valuation of CDOs

To value CDOs we need to generate correlated **default times** of the assets in the CDO

The most popular approach to do this is the

Copula approach

Def: A Copula creates a **joint distribution** of two or more variables, by preserving the marginal distributions.

The marginal distributions are typically **mapped** to convenient distributions as the standard normal or student-t. The joint distribution structure of the convenient distribution is then applied.

Gaussian Copula

$$C(u_A, u_B, \ldots, u_n) = M_n[N^{-1}(u_A), N^{-1}(u_B), \ldots, N^{-1}(u_n), \rho_M] \quad (2)$$

- C: Copula
- M_n: n-dimensional cumulative normal distribution
- N^{-1}: Inverse of univariate cumulative standard normal distribution
- u_A: (Uniform)[1)] cumulative marginal distribution u of name A.
- ρ_M: correlation parameter or M-correlation matrix of $u_A, \ldots, u_n$; $-1 \le \rho \le 1$

1) If u is uniform → $N^{-1}(u)$ is standard normal and 'well behaved'
(see http://www.dersoft.com/CopulaVARgeneration.xls) sheet 7, and next 2 slides

In CDO pricing, $u_A = Q_A(t_A)$ where $Q_A(t_A)$ is the cumulative default prob of name A at time t, given from market credit curves. This is why this copula is called Default-time copula.
Before we look at default time copulas, an excurse:

Critical Question: Can we blame the Copula on the global financial crisis 2007/2008??

Some say yes:

Salmon, F. (2009), “Recipe for Disaster: The Formula that Killed Wall Street”, *Wired Magazine,* 2009

Lohr, S. “Wall Street’s Math Wizards Forgot a Few Variables”, *New York Times*, September 12, 2009

Jones, S., “The formula that felled Wall St”, *The Financial Times*, April 24, 2009

Do these heuristic papers have any merit???

Cons of the Copula Model

a) Tail dependence

In a crisis, correlations typically increase, see studies by Das et al (2007) and Duffie et al (2009) and references therein.

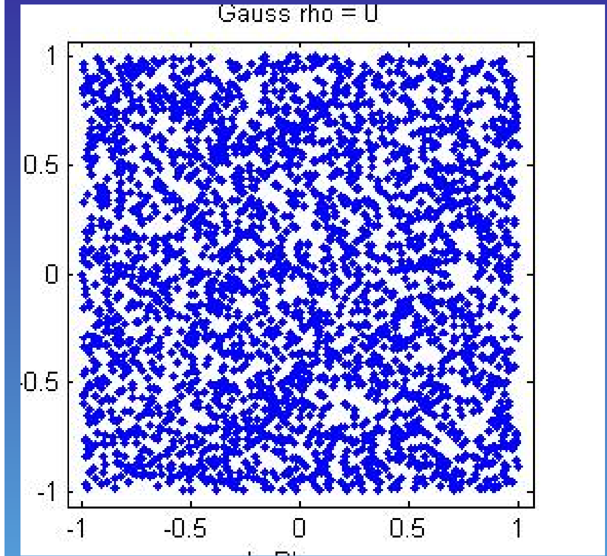
Following the tail dependence definition of Joe (1999), a bivariate copula has lower tail dependence if

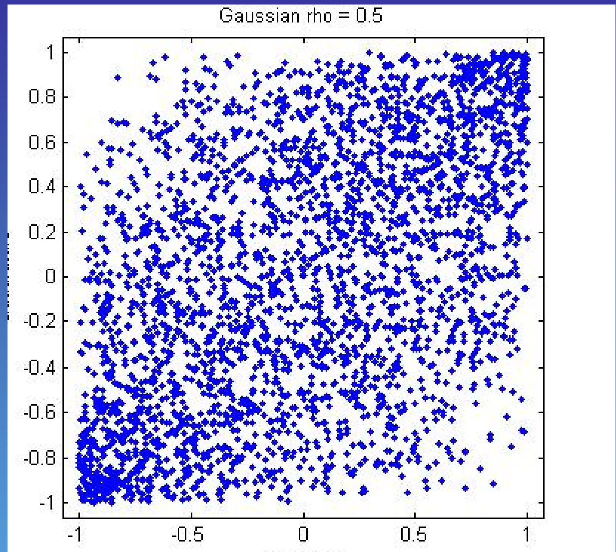
$$\lim_{u_1 \downarrow 0,\, u_2 \downarrow 0} P[(\tau_1 < N_1^{-1}(u_1) \,|\, P(\tau_2 < N_2^{-1}(u_2)] > 0$$

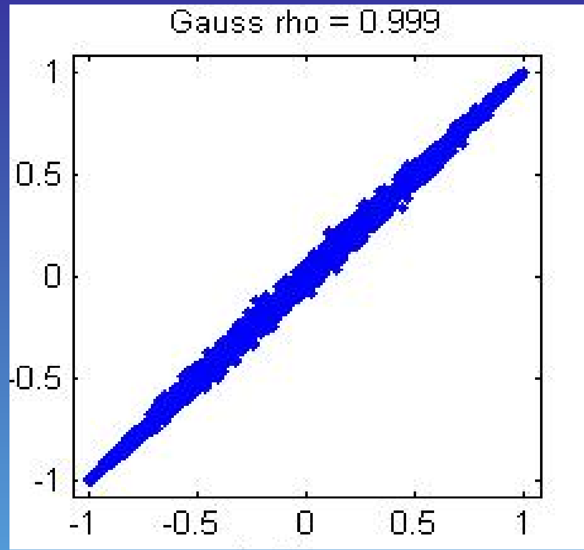
However, it can be easily shown that the Gaussian copula has no tail dependence for any correlation parameter ρ

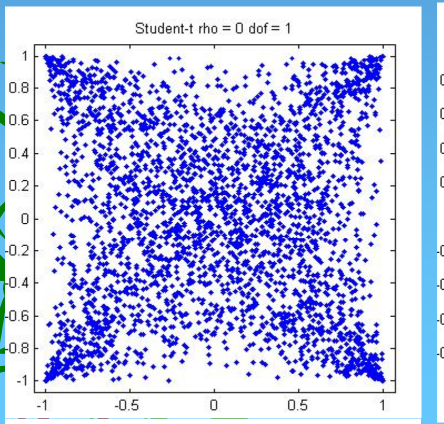
$$\lim_{u_1 \downarrow 0,\, u_2 \downarrow 0} P[(\tau_1 < N_1^{-1}(u_1) \,|\, P(\tau_2 < N_2^{-1}(u_2)] = 0,\ \rho \in \{-1,1\}$$

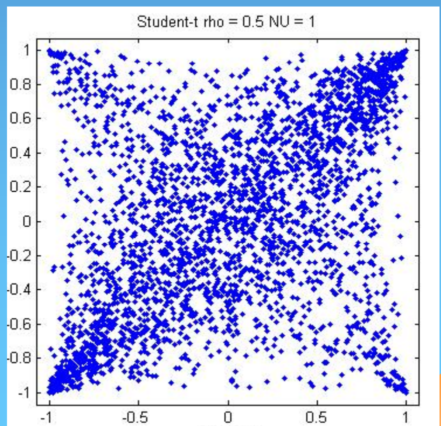
Copulas Tail dependence

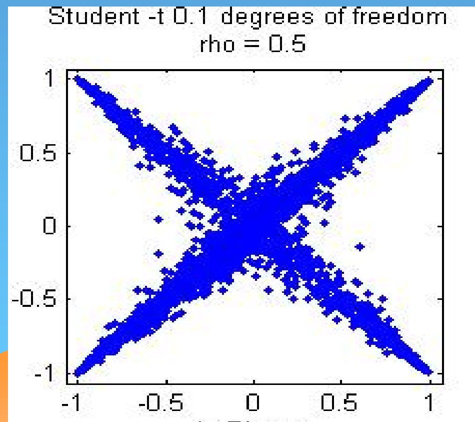


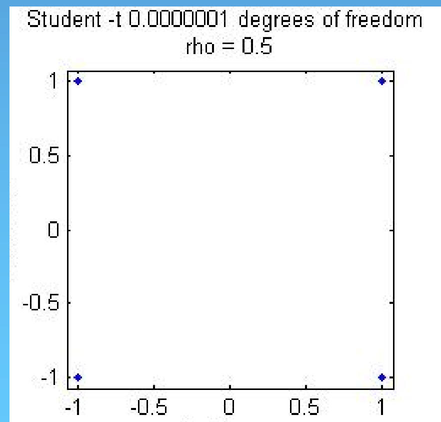




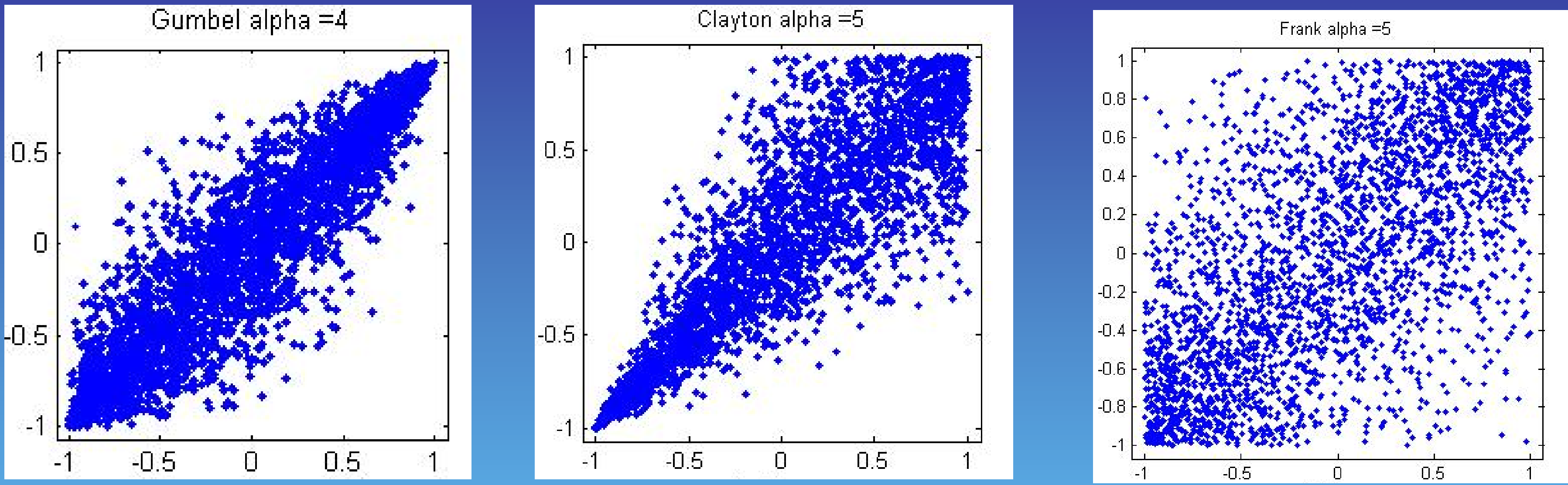








Copulas Tail dependence



Source: MatLab

Conclusion: Student-t and Gumbel might be better choices to model joint correlation behavior.

Cons of the Copula Model

b) Calibration

Another criticism of the Gaussian copula is that for certain parameter constellations it may not be possible to imply a market CDO tranche spread for a correlation parameter between 0 and 1.

Kherraz (2006) and Finger (2009) provide some evidence.

Several other studies, such as Hull and White (2004), Andersen and Sidenius (2004) and Burtschell et al (2008) test the one-factor Gaussian copula as well as other copulas such as Marshall-Olkin, Clayton or double-t. None of the studies finds any calibration failures for these copulas.

Cons of the Copula Model

b) Risk Management

A further criticism of the Copula approach is that the copula model is static and consequently allows only limited risk management, see Finger (2009) or Donnelly and Embrechts (2010).

In particular, there is no stochastic process for the critical underlying variables default intensity and default correlation.

However, even in these early copula formulations, back testing and stress testing the variables for different time horizons can give valuable sensitivities, see Whetten and Adelson (2004) and Meissner et al (2008b).

Conclusion "Can we blame the Copula on the global financial crisis 2007/2008??"

As any model, the Gaussian copula model has its limitations as

- zero-tail dependence when following a definition of Joe (1999)
- Copula models are static, i.e. have a one-period fixed time horizon.

Fact: **Before the 2007/2008 financial crisis, numerous market participants trusted the copula model uncritically and naively.**

However: **The 2007/2008 crisis was less a matter of a particular correlation model, but rather an issue of 'irrational complacency'.**

In the extremely benign time period from 2003 to 2006, proper hedging, proper risk management and stress test results were largely ignored.

The prime example is AIG's London subsidiary, which had sold CDSs and CDOs in an amount of close to $500 billion without conducting any major hedging

For an insightful paper on inadequate risk management leading up to the crisis, see "A personal view of the crisis - Confessions of a Risk Manager" (The Economist 2008).

Conclusion "Can we blame the Copula on the global financial crisis 2007/2008??"

When modeling:

Beware of inputs!

If any credit correlation model is fed with benign input data as low default intensities and low default correlation, the risk output figures will be benign, 'garbage in garbage out' in modeling terminology.

In general:

John Hull: Don't blame models, blame the people that use them

There are no toxic models or toxic products, just toxic people

Primarily Top-down Correlation Models

In a bottom-up model, the distribution of the portfolio intensity is an aggregate of the individual entities' default intensity.

In a top-down model the evolution of the portfolio intensity distribution is derived directly, i.e. abstracting from the individual entities' default intensities.

Top-down models are typically applied in practice if

- The default intensities of the individual entities are unavailable or unreliable.
- The default intensities of the individual entities are unnecessary. This may be the case when evaluating a homogeneous portfolio such as an index of homogeneous entities.
- The sheer size of a portfolio makes the modeling of individual default intensities problematic

Primarily Top-down Correlation Models and stochastic time change

Good old Einstein: **Speed of time s = f (gravity g, traveling speed tr)**

(-) **(-)**

gravity g = f (traveling speed tr)

(+)

Clark (1973) who first applied *stochastic* time processes to financial modeling.

***Stochastic* time processes are applied in Finance by Variance Gamma model of Madan et al (1998), Geman et al (2001), Carr et al (2003) and Cont and Tankov (2004).**

Primarily Top-down Correlation Models and stochastic time change

Hurd and Kuznetsov (2006a, 2006b)

Hurd and Kuznetsov introduce a vector-valued process

$$X_t = \{r_t, u_t, \lambda_t\} \qquad (47)$$

where r_t is the risk-free interest rate, the recovery rate is and, importantly, λ_t is the stochastic migration intensity process

The vector X_t captures macroeconomic data and represents a common factor, which affects all entities.

We introduce a time-changed process, a stochastic clock τ_t, which may have continuous and jump components. τ_t is a function of λ_t,

$$\tau_t = \int_0^t \lambda_s \, ds \qquad (49)$$

Hurd and Kuznetsov (2006a, 2006b)

$$E^Q(Y_{t+dt} = j \mid Y_t = i) = l_{ij}\,\lambda_t\,dt \qquad (50)$$

Since λ_t is an element of the conditioning market factor X_t (see equation 47), the migration processes Y_t in the new process under Q are now dependent.

Specifically, from equations (49) and (50) we observe that default correlation is induced by the speed of the stochastic clock τ_t.

An increase in the speed of the clock increases the speed of migration of all entities and hence increases the probability of simultaneous defaults.

If the stochastic clock jumps, the probability of simultaneous defaults is even higher.

Schönbucher (2006)

Schönbucher (2006) generates different transition and default correlation properties via different transition rate volatilities in a time-inhomogeneous, finite-state Markov chain framework.

Default correlation is induced by the dynamics of the transition volatility .

A higher volatility of means a higher transition rate of all entities n to a lower state, hence a higher default correlation; and vice versa.

We find that the induction of correlation via volatility changes (Schönbucher 2006) and the induction of correlation via stochastic time change have a similar interpretation.

An increase in transition volatility as well as an increase in the stochastic clock both increase the migration within the transition matrix, hence increase the probability of simultaneous defaults, and vice versa.

Some critical questions

1) Should we model bottom-up or top-down?

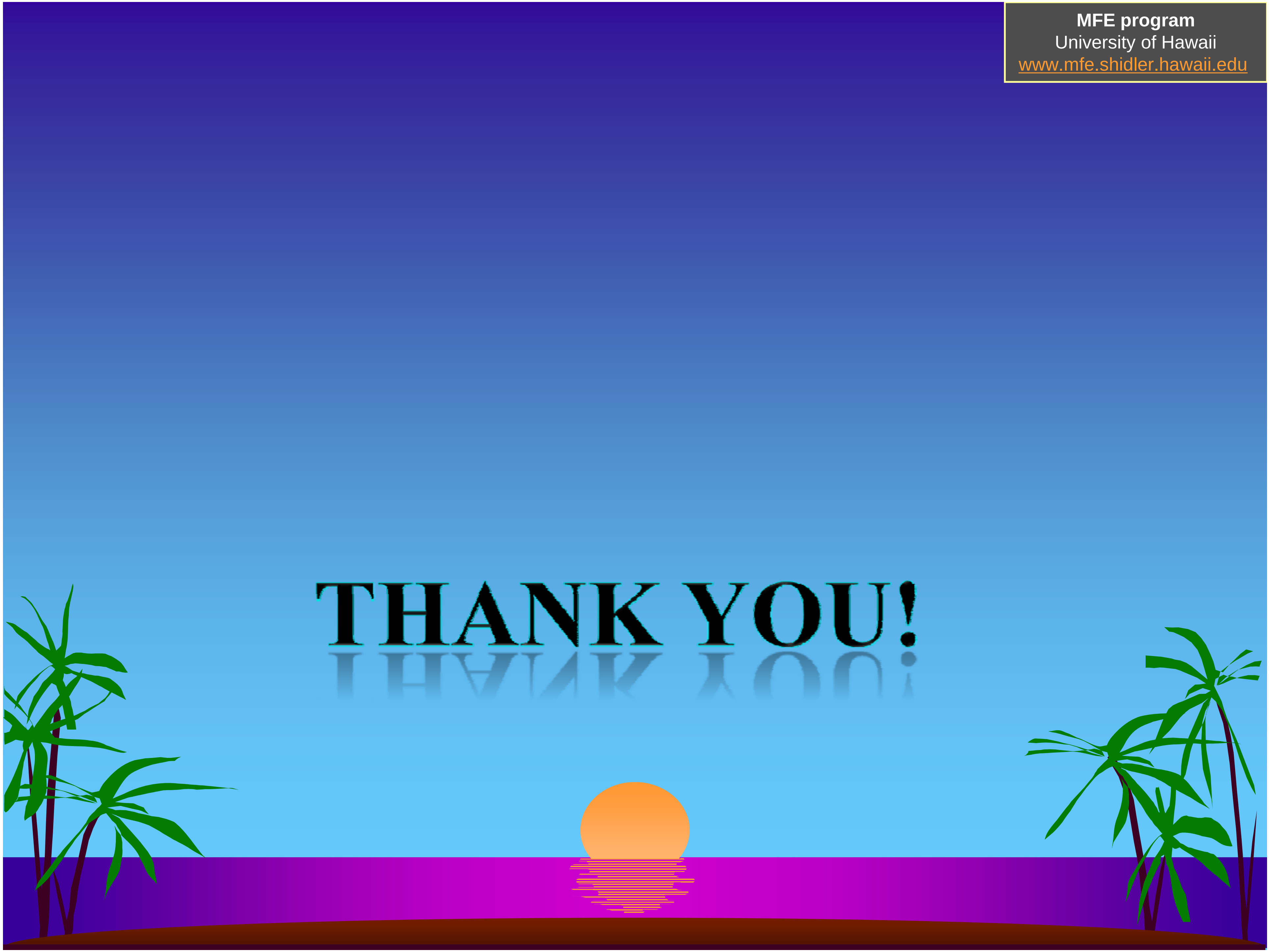
The answer depends on the nature of the underlying portfolio..

2) Can we combine bottom-up top-down models?

Is problematic..

3) Will there be a Black-Scholes-Merton correlation model, which dominates correlation modeling?

Probably not..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THANK YOU!