Bespoke Model Validation: Applying Hedging Strategies to Estimate Model Risk

Alberto Elices

Model Validation Group
Area of Methodology
Santander

Eduard Giménez

Model Development Group
Front Office
Caixabank

Global Derivatives, Trading and Risk Management 2011
Paris, April 12th – 16th, 2011

Santander

Outline

- **Introduction.**
- **Model validation philosophy.**
- **Reconcile FO and Risk interests: provisions.**
- **Estimation of model risk applying hedging strategies:**
  - Formulation of hedging strategy.
  - Case study: double-no-touch option.
- **Justification of findings.**
- **Conclusions.**

Introduction

- **After the crisis in the 2nd half of 2007, a big concern about pricing models has been raised.**
- **Risk management and model validation raise now considerably more attention.**
- **Model validation:**
  - Validation of model implementation is no longer enough.
  - Periodic and comprehensive review of pricing models.
  - Estimation of model risk.
- **Risk management:**
  - Calculate and apply provisions.
  - Limit model risk exposure (reduce volume of operations).

Model Validation Philosophy

- **Validation process:**
  - Background and motivation.
  - Model testing:
    - Model adequacy analysis.
    - Test of complex models in simple cases.
    - Premium tests: implementation, convergence, robustness, life cycle.
    - Greek and stability analysis.
  - Integration in corporate systems.
  - Tests to estimate model risk.

Model Validation Philosophy

- **Model risk estimation (premium based):**
  - **Premium sensitivity** to non-calibrated (unobserved) or innaccurately calibrated parameters: e.g. correlations, dividends.
  - **Comparison with other models** with more accurate or simply different hypothesis:
    - Compare same product with different models available in FO.
    - Development of “toy” models:

      Get sets of model parameters calibrated manually to market.

      Generate market and stressed market scenarios.

      Compare model under validation with “toy” model valuation.
  - **Simulation of hedging strategies**: either back test with real or “toy” model market data.

How to reconcile FO and Risk department interests?: provisions

- **The provision should cover the expected hedging loss and its uncertainty:**
  - When hedging is carried out with a model with aggresive prices, the expected hedging loss is the fair minus the aggresive price: that difference plus a cushion for its uncertainty is the provision.

- **Provisions as a means to approve campaigns using limited models with controllable risk:**
  - A provision allows accomplishing campaigns which would not possible with a slower more sophisticated model.

- **Provisions to foster improvement of FO models:**
  - Models with limitations should be given provisions which should be released the more the model is improved.

How to reconcile FO and Risk department interests?: provisions

- **Provision calculation philosophy:**
  - They should be transparent, easy to compute.
  - They should be dynamic, stable, with smooth evolution through time (they should decrease approaching expiry).
  - They should balance risk limitation and trading mitigation.
  - Front Office should be able to reproduce them.

How to reconcile FO and Risk department interests?: provisions

- **How provisions can be calculated:**
  - Use provision tables calculated from studies.
  - Use FO pricing models to estimate model risk:
    - – Changing unobserved or non-calibrated model parameters (mean reversion, correlations, etc).
    - – Compare prices of deals valued with different FO models (better models might take too long on a daily basis).
  - Simulate or back test portfolio hedging: sometimes impractical.

Outline

- Introduction.
- Model validation philosophy.
- Reconcile FO and Risk interests: provisions.
- **Estimation of model risk applying hedging strategies:**
  - Formulation of hedging strategy.
  - Case study: double-no-touch option.
- **Justification of findings.**
- **Conclusions.**

Estimation of model risk applying hedging strategies

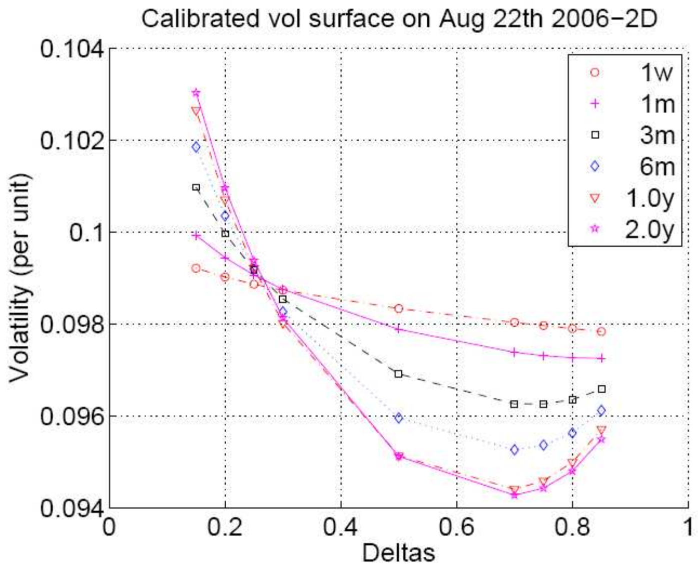
- **Assume that market is driven by Heston's dynamics.**
- **Simulate hedging strategy with different pricing models.**
- **Look at profit and loss (P&L) distribution of hedging strategy at maturity:**
  - What is the expected hedging P&L?
  - What is the uncertainty (e.g. StdDev) of hedging P&L?
- **Calculate Fair Value Adjustment (FVA) to account for:**
  - Expected hedging losses.
  - Uncertainty of those losses: a number of StdDev of hedge loss.
- **Which model is better?**

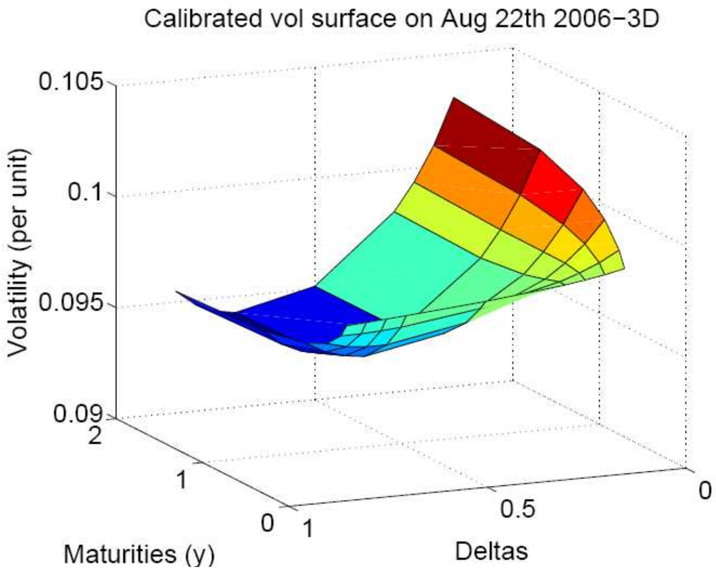
Formulation of the hedging strategy

- **Hypothesis: market is driven by Heston's dynamics:**

$$\frac{dS_t}{S_t} = \left(r_t^d - r_t^f\right) dt + \sqrt{v_t} dW_t \qquad d\langle W_t, V_t\rangle = \rho dt$$
$$dv_t = \kappa\left(\theta - v_t\right) dt + \eta\sqrt{v_t} dV_t$$

- **Heston's parameters are calibrated to 1y EUR/USD:**



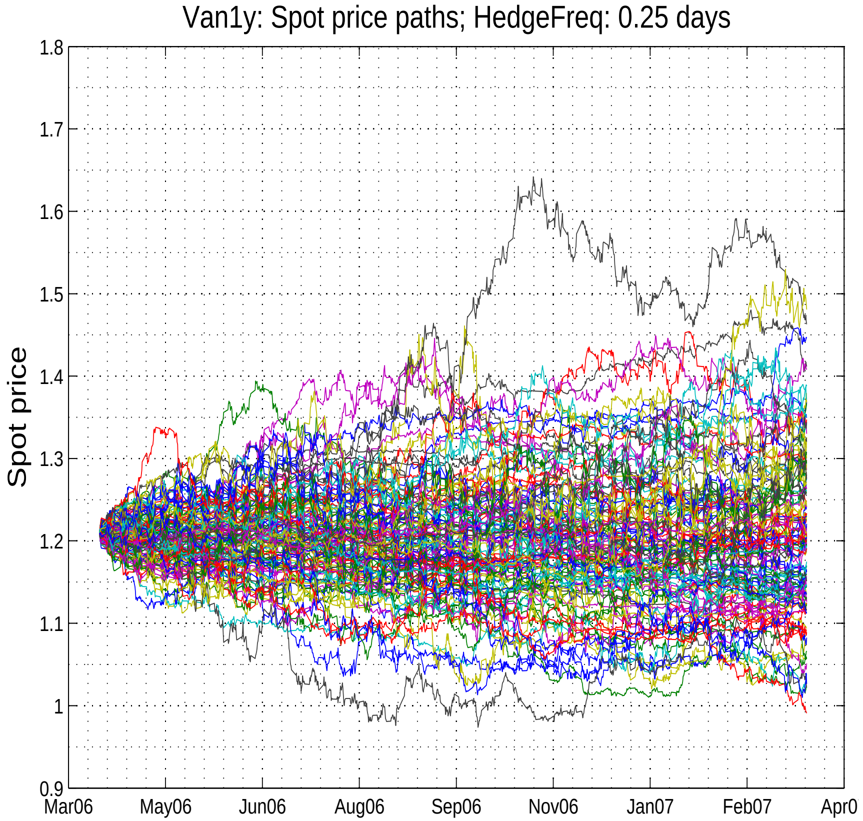


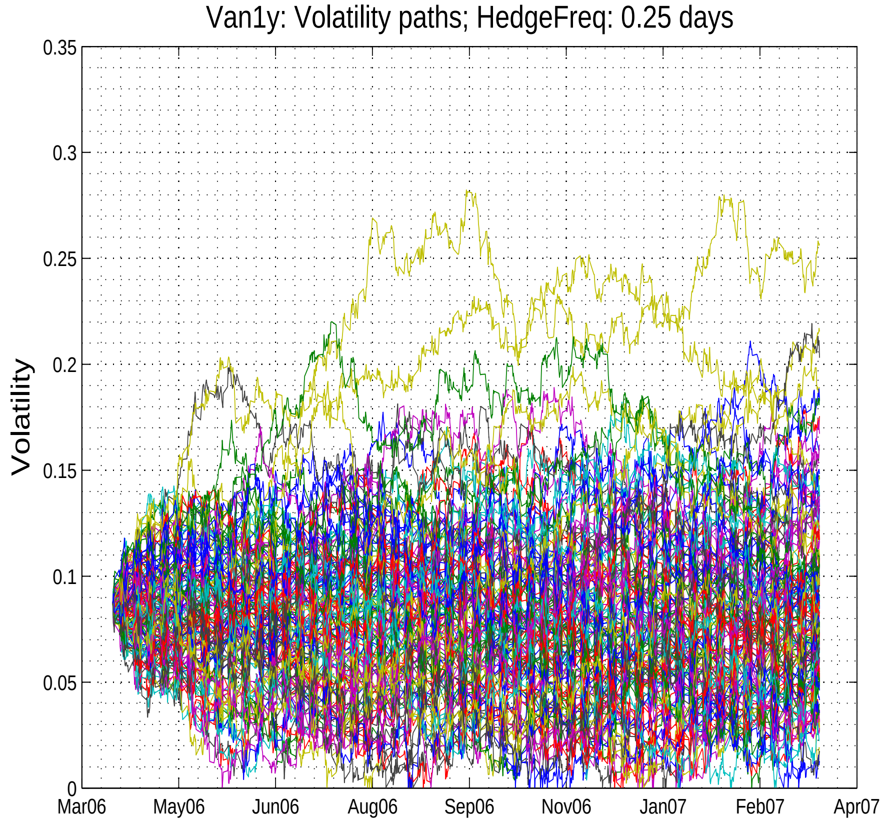
$$v_0 = 0.0097,\ \kappa = 1.1,\ \theta = 0.0097,\ \eta = 0.14 \text{ and } \rho = 0.14.$$

Santander

Formulation of the hedging strategy

- **Heston's two factors (spot and variance) are simulated:**





Formulation of the hedging strategy

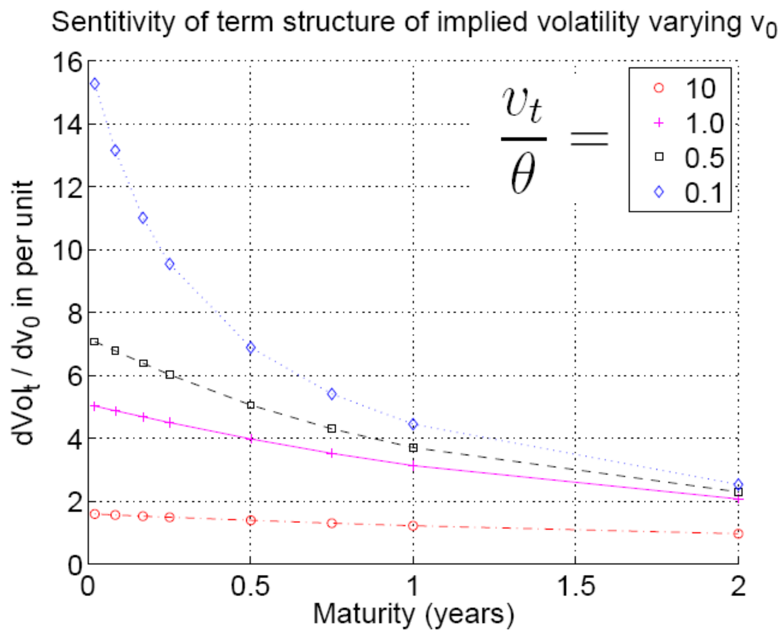
- **On each simulated step:**
  - Calculate vol surf from simulated S_t , v_t & Heston parameters.
  - Sensitivities: after each factor change, vol surf is re-built.
  - Delta and vega are hedged with underlying and a 6m vanilla.

For Heston:
- Vol surf does not change with spot.
- Vol surf has only term structure change varying variance.

$$\Delta = \frac{\partial P}{\partial S_t} = \frac{P(S_t+\delta_S)-P(S_t-\delta_S)}{2\delta_S}$$

$$\vartheta = \frac{\partial P}{\partial v_t} = \frac{P(v_t+\delta_v)-P(v_t)}{\delta_v}$$

$$\vartheta = \frac{dP}{dv_t} = \sum_{i=1}^{N} \frac{\partial P}{\partial \sigma_i}\frac{\partial \sigma_i}{\partial v_t}$$

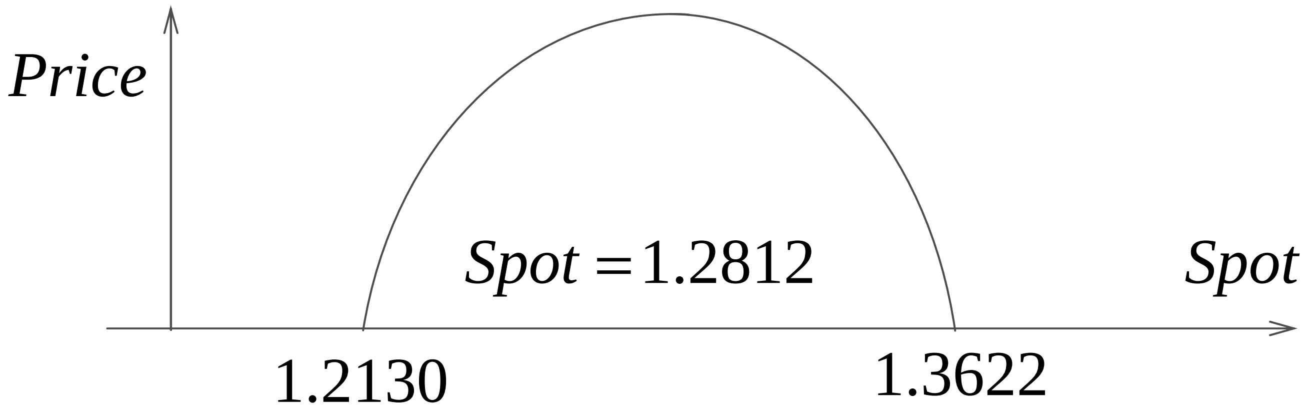


Santander

Case study: double-no-touch option (DNT)

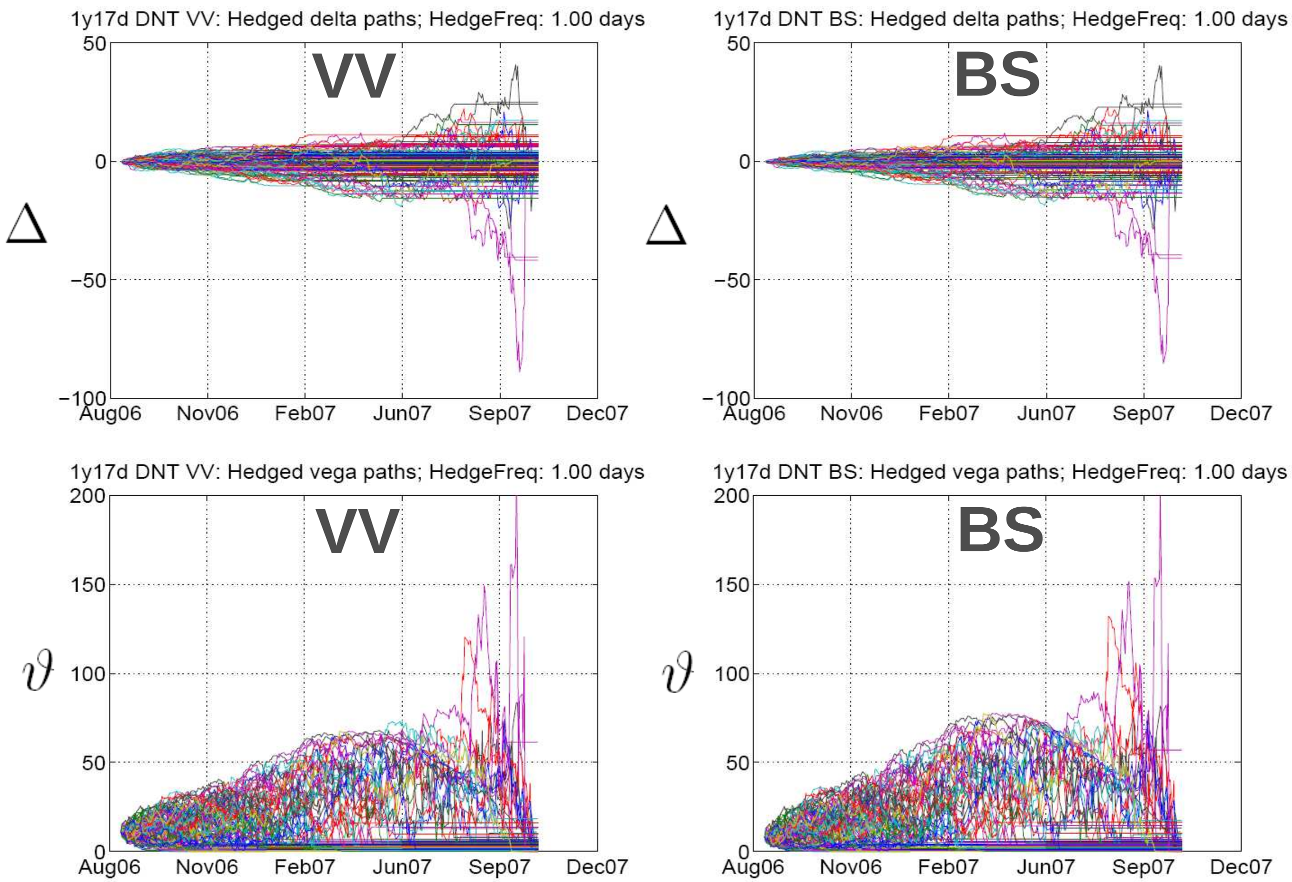
- **A 1y double-no-touch option is considered.**
- **Estimation and comparison of model risk for:**
  - VV: Volga-vanna heuristic model.
  - BS: Black Scholes with at-the-money volatility.

VV	BS	Heston
0.0839	0.0466	0.1122



Case study: double-no-touch option (DNT)

- **Qualitatively similar hedge ratios Δ and ϑ :**



Santander

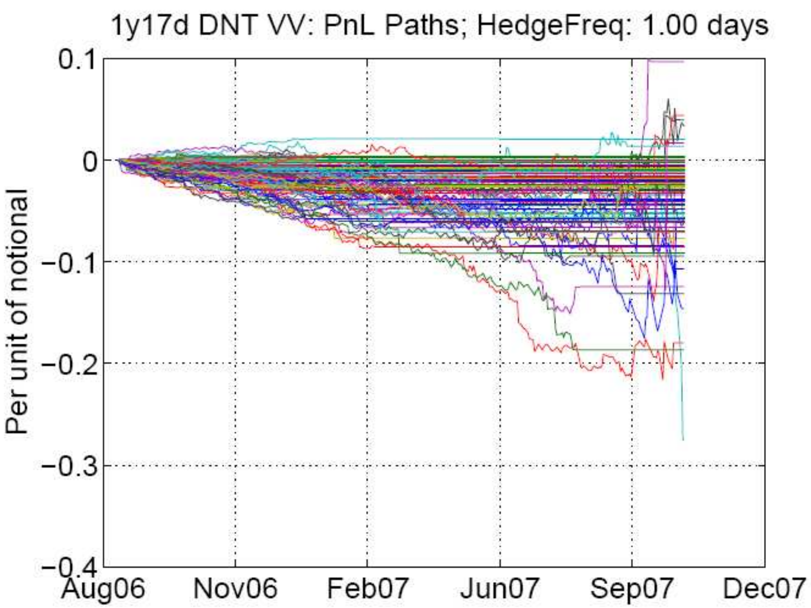
Should not the P&L be also very similar?

Case study: double-no-touch option (DNT)

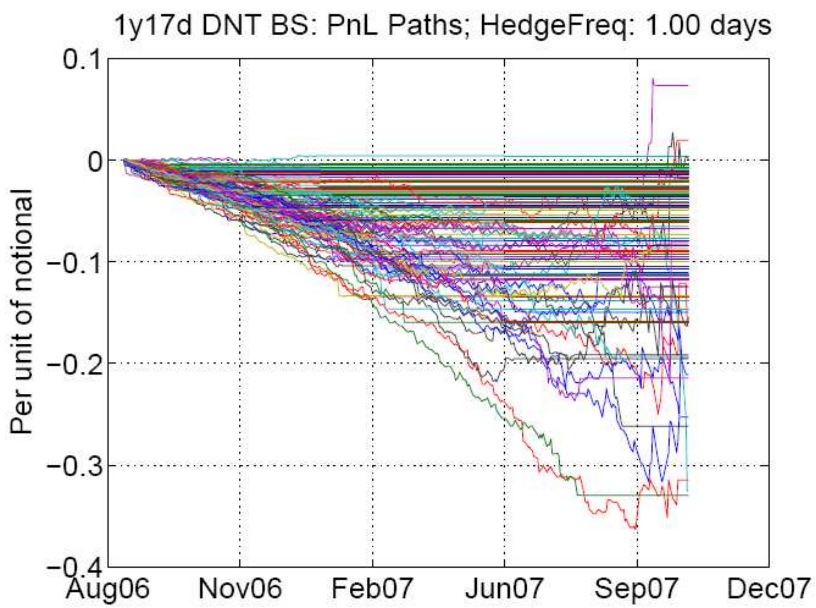
- **Consistent hedging losses much higher for BS:**

P&L paths of hedging strategy

VV

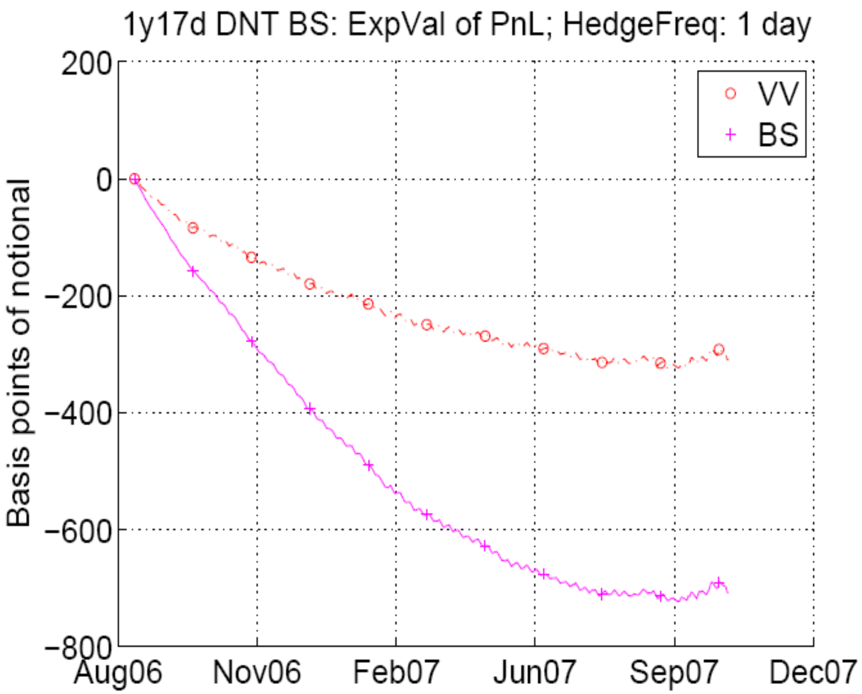


BS

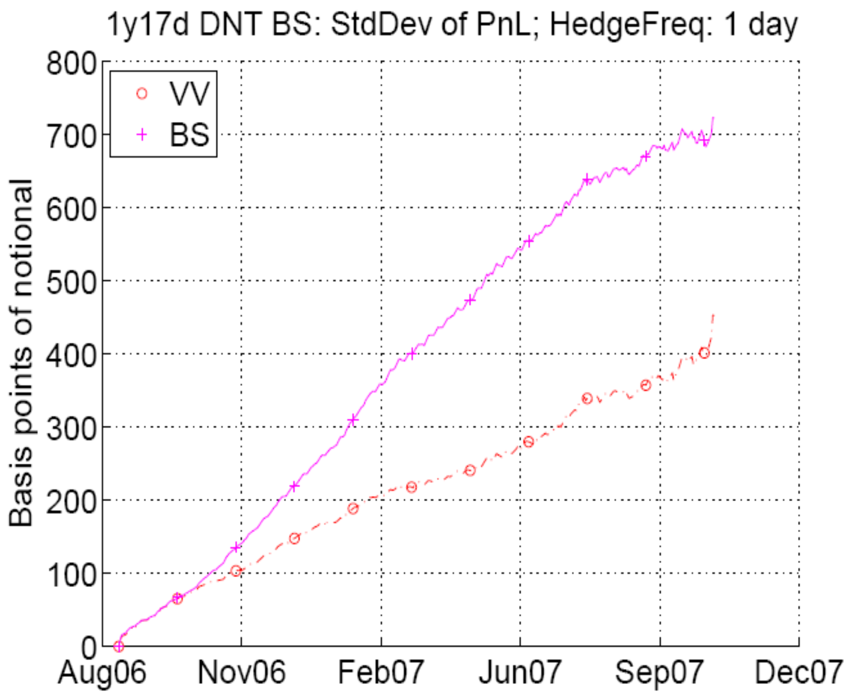


Case study: double-no-touch option (DNT)

- **Evolution through time of hedge P&L distribution:**
  - VV has lower expected loss and lower StdDev.



Expected hedging P&L



StdDev of hedging P&L

Case study: double-no-touch option (DNT)

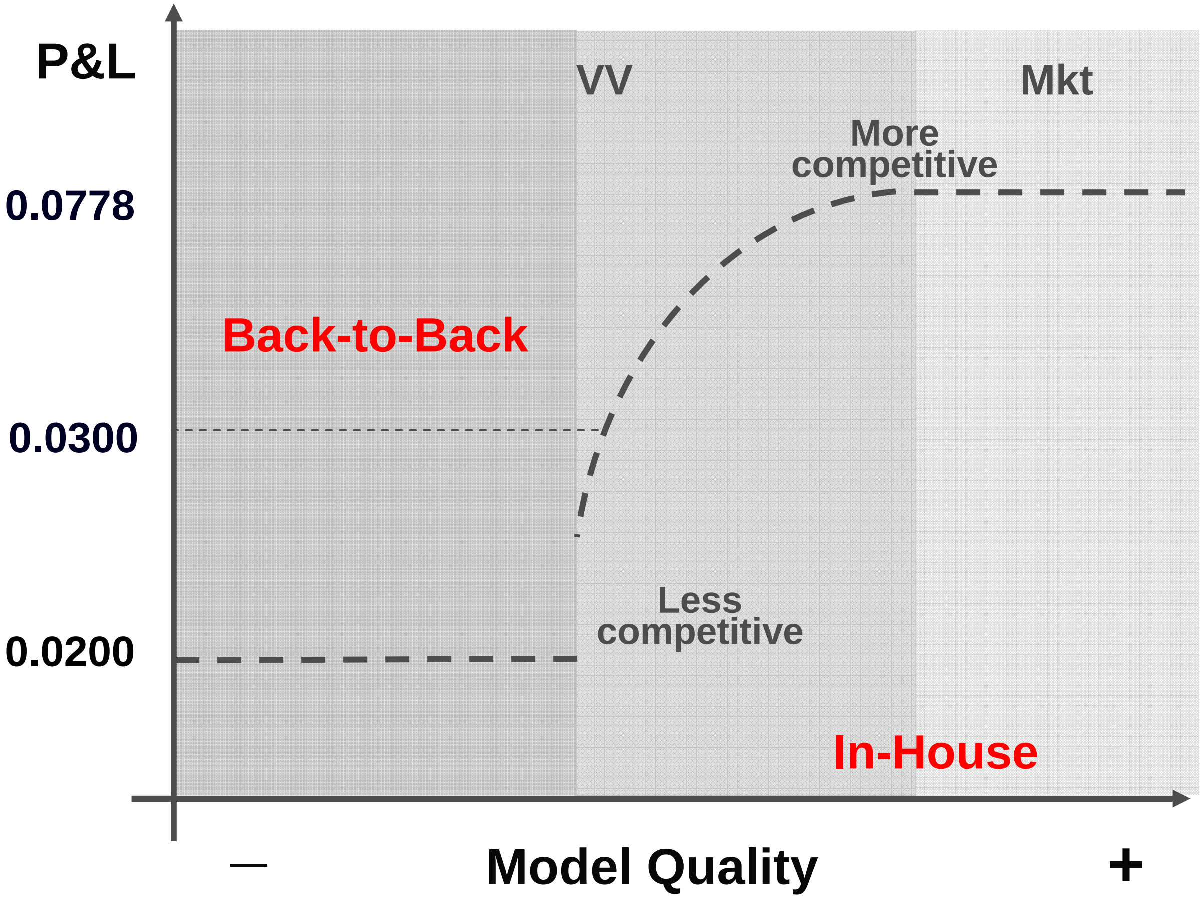
$$\text{ModelRisk} = \mathbf{E}\left[\text{HedgingLoss}\right] + a \cdot \textbf{StdDev}\left[\text{HedgingLoss}\right]$$

$$P_{market} = P_{model} + \mathbf{E}\left[\text{HedgingLoss}\right]$$

VV		BS
0.0839	Initial model price: P_0	0.0466
0.0309	Expected hedging cost: EHC	0.0707
0.1148	Price including hedging cost: $P_0 + EHC$	0.1173
0.1122	Heston's price	0.1122
0.0452	StdDev of hedging cost: $SDHC$	0.0723
0.0761	Model Risk: $MR = EHC + 1 \cdot SDHC$	0.1430
0.1600	Final price: $P_0 + MR$	0.1896

Case study: double-no-touch option (DNT)

- **Back-to-Back versus In-House.**



Corporate client price: 0.19

B2B, Mkt+CVA: 0.17
Margin = 0.02
With BS model: 0.1896
Margin = 0.0004
With VV model: 0.1600
Margin = 0.03
With Mkt model: 0.1122
Margin = 0.0778

BS worse than B2B

VV better than B2B

The better model the more competitive price

Case study: double-no-touch option (DNT) : conclusions

- **Double-no-touch options have huge model risk.**
- **Model Risk measure: a accounts for risk aversion to uncertainty of hedging loss.**

$$\text{ModelRisk} = \mathbf{E}\left[\text{HedgingLoss}\right] + a \cdot \mathbf{StdDev}\left[\text{HedgingLoss}\right]$$

- **BS and VV models are compared under this measure. VV performs better than BS:**
  - VV has less expected hedging loss.
  - VV has less uncertainty of hedging loss.

- **The provision is equal to the model risk measure, adjusting a for a given risk aversion view.**

Justification of findings: Expected loss = market price – model price

- **Definition of total Π_t^{Tot} and hedging Π_t^{Hedge} portfolios and hedging position $\mathrm{H}_{t_j}^{t_i}$:**

$$\Pi_t^{Tot} = \Pi_t + \Pi_t^{Hedge}$$

$$\Pi_t^{Hedge} = B_t + \alpha_t \cdot S_t B_t^f + \beta_t \cdot C_t$$

$$\mathrm{H}_{t_j}^{t_i} = \alpha_{t_i} \cdot S_{t_j} B_{t_j}^f + \beta_{t_i} \cdot C_{t_j}$$

Π_t Option price to hedge. α_t Amount of underlying.

B_t Domestic bank account. β_t Amount of vanilla option.

B_t^f Foreign bank account. C_t Price of vanilla option.

Justification of findings: Expected loss = market price – model price

- **Construction of time evolution of hedging portfolio:**

$$\Pi_{t_0}^{Hedge} = \left(-\Pi_{t_0} - \mathrm{H}_{t_0}^{t_0}\right) + \alpha_{t_0} S_{t_0} B_{t_0}^f + \beta_{t_0} C_{t_0}$$

$$\Pi_{t_1}^{Hedge} = \left(-\frac{\Pi_{t_0}}{P_{t_0,t_1}^d} - \frac{\mathrm{H}_{t_0}^{t_0}}{P_{t_0,t_1}^d} + \mathrm{H}_{t_1}^{t_0} - \mathrm{H}_{t_1}^{t_1}\right) + \alpha_{t_1} S_{t_1} B_{t_1}^f + \beta_{t_1} C_{t_1}$$

$$\begin{aligned}\Pi_{t_N}^{Hedge} = &\left(\frac{-\Pi_{t_0}}{P_{t_0,t_N}^d} + \sum_{i=1}^{N}\left(\frac{\mathrm{H}_{t_i}^{t_{i-1}}}{P_{t_i,t_N}^d} - \frac{\mathrm{H}_{t_{i-1}}^{t_{i-1}}}{P_{t_{i-1},t_N}^d}\right) - \mathrm{H}_{t_N}^{t_N}\right) \\ &+\alpha_{t_N} S_{t_N} B_{t_N}^f + \beta_{t_N} C_{t_N}\end{aligned}$$

P_{t_i,t_j}^d Domestic zero coupon.

Justification of findings: Expected loss = market price – model price

- **The expected total portfolio value at maturity does not depend on the hedge ratios (the numeraire is P^d_{t,t_N}):**

$$\mathbf{E}^{mkt}_{t_0}\left[\Pi^{Tot}_{t_N}\right] = P^d_{t_0,t_N}\mathbf{E}^{mkt}_{t_0}\left[\Pi_{t_N}\right] + P^d_{t_0,t_N}\mathbf{E}^{mkt}_{t_0}\left[\Pi^{Hedge}_{t_N}\right]$$
$$= \Pi^{mkt}_{t_0} - P^d_{t_0,t_N}\frac{\Pi_{t_0}}{P^d_{t_0,t_N}} = \Pi^{mkt}_{t_0} - \Pi_{t_0}$$

$$\mathbf{E}^{mkt}_{t_0}\left[\Pi^{Hedge}_{t_N}\right] = \frac{-\Pi_{t_0}}{P^d_{t_0,t_N}} + \overbrace{\mathbf{E}^{mkt}_{t_0}\left[\sum_{i=1}^{N}\left(\mathbf{E}^{mkt}_{t_{i-1}}\left[\frac{\mathrm{H}^{t_{i-1}}_{t_i}}{P^d_{t_i,t_N}}\right] - \frac{\mathrm{H}^{t_{i-1}}_{t_{i-1}}}{P^d_{t_{i-1},t_N}}\right)\right]}^{\text{Equal to 0}}$$

$$\mathbf{E}^{mkt}_{t_{i-1}}\left[\frac{S_{t_i}B^f_{t_i}}{P^d_{t_i,t_N}}\right] = \frac{S_{t_{i-1}}B^f_{t_{i-1}}}{P^d_{t_{i-1},t_N}} \qquad \mathbf{E}^{mkt}_{t_{i-1}}\left[\frac{C_{t_i}}{P^d_{t_i,t_N}}\right] = \frac{C_{t_{i-1}}}{P^d_{t_{i-1},t_N}}$$

Justification of findings: why is there a consistent loss drift?

- Evolution of total portfolio when all risks are hedged:

$$d\Pi_t^{Tot} = d\Pi_t + d\Pi_t^{Hedge} = r_t^d \left(\Pi_t + \Pi_t^{Hedge}\right) dt$$

- Evolution of any pricing model given Heston's market:

$$\begin{aligned} df = & \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S_t} dS_t + \frac{\partial f}{\partial v_t} dv_t + \\ & + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2} d\langle S_t, S_t\rangle + \frac{1}{2}\frac{\partial^2 f}{\partial v_t^2} d\langle v_t, v_t\rangle + \frac{\partial^2 f}{\partial v_t \partial S_t} d\langle v_t, S_t\rangle \end{aligned}$$

- Evolution of option to hedge Π_t and vanilla option C_t :

$$\begin{aligned} d\Pi_t &= \mathcal{L}^{mkt}\Pi_t dt + \Delta_t^{\Pi} S_t \sqrt{v_t} dW_t + \vartheta_t^{\Pi} \eta \sqrt{v_t} dV_t \\ dC_t &= \mathcal{L}^{mkt} C_t dt + \Delta_t^{C} S_t \sqrt{v_t} dW_t + \vartheta_t^{C} \eta \sqrt{v_t} dV_t \end{aligned}$$

$\mathcal{L}^{mkt}$: Infinitesimal generator of Heston's dynamics (market).

Santander

Justification of findings: why is there a consistent loss drift?

- Evolution of the hedge porfolio:

$$dB_t = r_t^d B_t dt$$
$$dB_t^f = r_t^f B_t^f dt$$

$$d\Pi_t^{Hedge} = dB_t + \alpha_t d\left(S_t B_t^f\right) + \beta_t dC_t =$$
$$= \left(r_t^d B_t + \alpha_t r_t^d S_t B_t^f\right) dt + \alpha_t \sqrt{\upsilon_t} S_t B_t^f dW_t + \beta_t dC_t$$

- Evolution of the total portfolio minus risk free return:

$$d\Pi_t + d\Pi_t^{Hedge} - r_t^d \left(\Pi_t + \Pi_t^{Hedge}\right) dt$$
$$= \left(\mathcal{L}^{mkt}\Pi_t - r_t^d \Pi_t\right) dt + \beta_t \left(\mathcal{L}^{mkt} C_t - r_t^d C_t\right) dt +$$
$$\left(\Delta_t^{\Pi} + \alpha_t B_t^f + \beta_t \Delta_t^C\right) S_t \sqrt{\upsilon_t} dW_t + \left(\vartheta_t^{\Pi} + \beta_t \vartheta_t^C\right) \eta \sqrt{\upsilon_t} dV_t$$

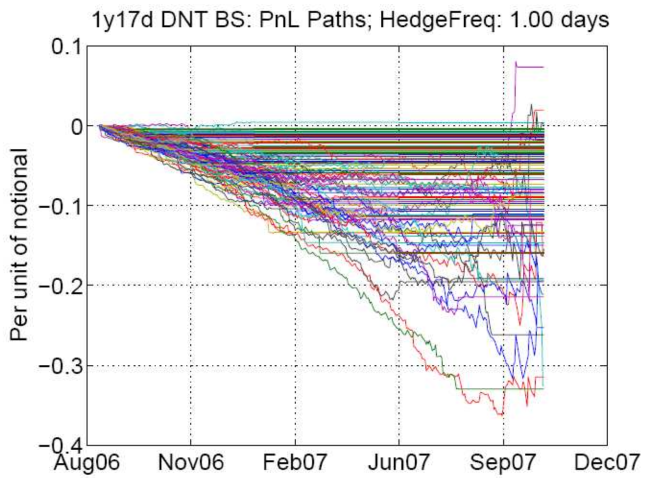
- Hedge ratios are chosen to eliminate randomness:

$$\alpha_t = -\frac{\Delta_t^{\Pi} + \beta_t \Delta_t^C}{B_t^f} \qquad \beta_t = -\frac{\vartheta_t^{\Pi}}{\vartheta_t^C}$$

Justification of findings: why is there a consistent loss drift?

▪ Why is there a consistent loss drift?

$$d\Pi_t^{Tot} - r_t^d \Pi_t^{Tot} dt = \left(\mathcal{L}^{mkt} \Pi_t - r_t^d \Pi_t \right) dt$$



- $\mathcal{L}^{mkt} \Pi_t - r_t^d \Pi_t$ = 0: Market dynamics equal to model dynamics. No drift.
- $\mathcal{L}^{mkt} \Pi_t - r_t^d \Pi_t$ > 0: Positive drift => consistent profit.
- $\mathcal{L}^{mkt} \Pi_t - r_t^d \Pi_t$ < 0: Negative drift => consistent loss.

Conclusions: looking at model risk from a hedging perspective

- **Two sources of model risk from a hedging perspective:**
  - **Expected hedging loss**:
    - It depends on model price but not on its hedge ratios.
    - It can be estimated by comparing with a better (usu. slower) model or moving non-calibrated or unobserved parameters.
  - **Uncertainty of hedging loss** (e.g. measured by its StdDev) :
    - It depends on hedge ratios given by the model.
    - More difficult to estimate: hedging simulation or back-test studies.

- **Model Risk measure: a measures risk aversion to uncertainty of hedging loss.**

$$\mathrm{ModelRisk} = \mathbf{E}\left[\mathrm{HedgingLoss}\right] + a \cdot \mathbf{StdDev}\left[\mathrm{HedgingLoss}\right]$$

Conclusions: looking at model risk from a hedging perspective

- **No one knows market dynamics but,**
  - There are hypothesis more plausible than others.
  - There are proxy models used by many participants.
  - A good model should monitor market prices as close as possible.

- **Covering model risk with Fair Value Adjustment (FVA):**
  - The expected hedging loss can be accounted for until expiry, on the date of deal closing.
  - The uncertainty of hedging loss needs a study for each product.
  - Benefits of portfolio effect need simulation of the whole portfolio.
  - The factor a allows customizing model risk aversion.

For more details, look at the paper:

Elices A., Giménez E.,"Applying hedging strategies to estimate model risk and provision calculation", available on ArXiv at "http://arxiv.org/abs/1102.3534".

Santander

Dow Jones Sustainability Indexes
Member 2008/09

FTSE4Good Index Series