New Classes of Markovian Factor Models with Stochastic Volatility and Jumps for Commodity Futures

Alex Levin

alex.levin@rbccm.com

Global Derivatives Trading & Risk Management
Paris, April 2011

Outline

- **Properties of Crude Oil, Natural Gas and other Commodity futures price curves**
- **Extension of PCA/factor approach to HGM-type futures curve modeling**
- **Levin (2004) family of no-arbitrage co-integrated Markovian factor models for futures curve and corresponding spot models**
  - **A family of four multi-factor futures curve models with Gaussian factors**
  - **Linear and Exponential models**
    - **Example of two-factor co-integrated Exponential model for Crude Oil equivalent to Schwartz-Smith model**
    - **Extension for seasonality and time-dependent futures curve volatility**
    - **Jump-diffusion futures curve model driven by Lévy processes**
  - **New Normal CDF (NCDF)-transformation model with bounded futures prices**
    - **European futures and spot option pricing in Levin (2004) NCDF model**
    - **Locally bounded extension of NCDF model with time-depended bounds**
  - **New Modified Dowson Integral (MDI)-transformation model**
- **Family of affine stochastic volatility Markovian factor models for futures curve**
- **Simulation in historical measure for Commodity Risk Management systems**
- **Possible applications to interest rate, FX, equity and volatility futures curves**

References

1. Amin, K., Ng, V., and Pirrong, S. (1995) Valuing Energy Derivatives. In: Managing Energy Price Risk. Risk Publications, pp. 57-70.
2. Andersen, L. (2008) Markov Models for Commodity Futures: Theory and Practice. *Quantitative Finance*, 10, pp. 831–854.
3. Andersen, L., and Piterbarg, V. (2007) Moment Explosions in Stochastic Volatility Models. *Finance and Stochastics*, 11, pp. 29-50.
4. Bachelier, L. (1900) Theory of Speculation. Ph.D. Thesis. English translation in: The Random Character of Stock Market Prices. P.H. Cootner (ed.). MIT Press, 1964.
5. Bergomi, L. (2005) Smile Dynamics II. *Risk*, October, pp. 67-73.
6. Björk, T., and Landén, C. (2000) On the Term Structure of Futures and Forward Prices. In: Mathematical Finance - Bachelier Congress 2000. H. Geman et al (ed.), Springer, pp. 111-149.
7. Black, F. (1976) The Pricing of Commodity Contracts. *Journal of Financial Economics*, 3, 167-179.
8. Carr, P., and Madan, D. (1999) Option Valuation Using the Fast Fourier Transform. *Journal of Computational Finance*, 2, pp. 61-73.
9. Casassus, J., and Collin-Dufresne, P. (2005) Stochastic Convenience Yield Implied from Commodity Futures and Interest Rates. *Journal of Finance*, 60, pp. 2283-2331.

10. Cheridito, P., Filipovic, D. and Kimmel, R. (2007) Market Price of Risk Specifications for Affine Models: Theory and Evidence. *Journal of Financial Economics*, 83, 123-170.

11. Cortazar, G., and Schwartz, E.S. (1994) The Valuation of Commodity Contingent Claims. *Journal of Derivatives*, Summer, pp. 27-39.

12. Duffie, D., Pan J., and Singleton, K. (2000) Transform Analysis and Asset Pricing for Affine Jump Diffusions. *Econometrica*, 68, pp. 1343-1376.

13. Engle, R., and Granger, C. (1987) Co-Integration and Error Correction: Representation, Estimation, and Testing. *Econometrica*, 55, pp. 251-276.

14. Eydeland, A., and Geman, H. (1998) Pricing Power Derivatives. *Risk*, 11, pp. 71-73.

15. Gibson, R., and Schwartz, E.S. (1990) Stochastic Convenience Yield and the Pricing of Oil Contingent Claims. *Journal of Finance*, 45, pp. 959-976.

16. Heston, S. (1993) A Closed-form Solution of Options with Stochastic Volatility and Applications to Bond and Currency Options. *Review of Financial Studies*, 6, 327-343.

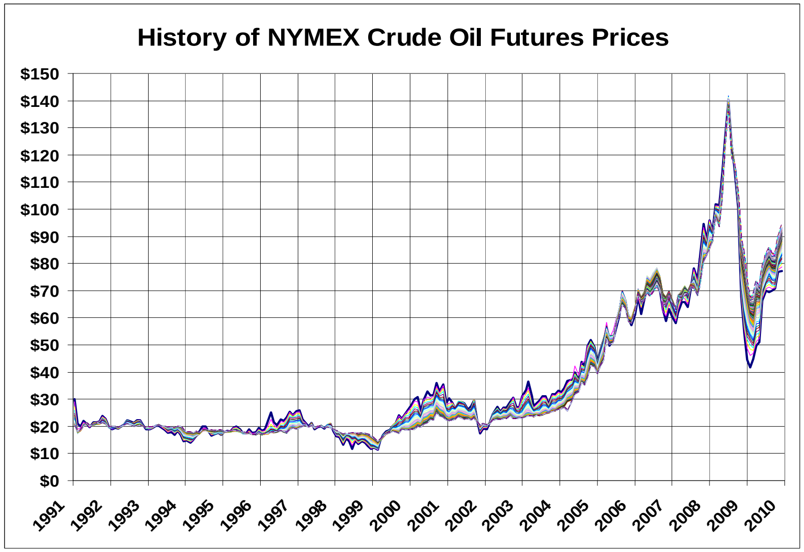
17. Ingersoll, J. (1997) Valuing Foreign Exchange Rate Derivatives with a Bounded Exchange Process. *Review of Derivatives Research*, 1, pp. 159–181.

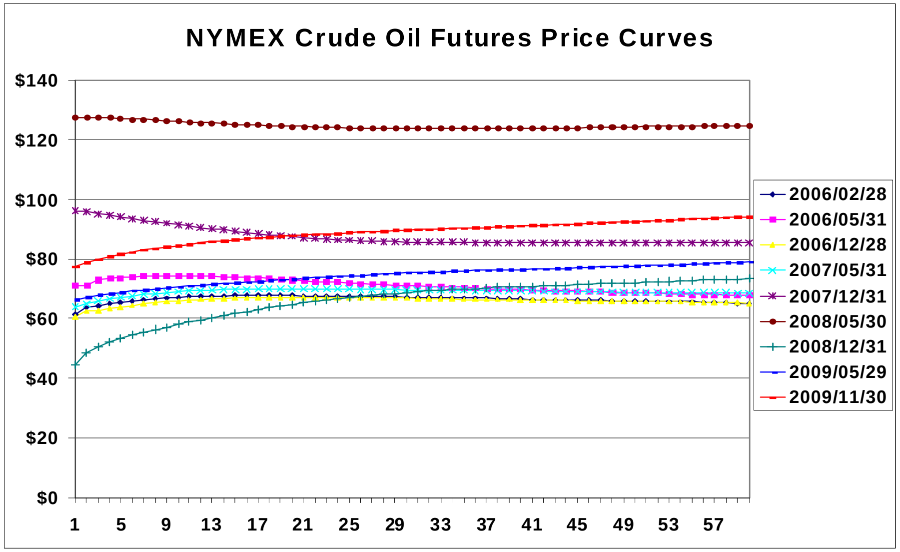
18. Levin, A. (2002) Markovian HJM-Type Futures Curve Model for Long-Term Simulation in Credit Risk Systems. Presentation at the Risk Conference “Quantitative Finance’2002, New York”.

19. Levin, A. (2004) Mean-Reverting and Co-Integrated Energy Futures Curve Models for Pricing and Risk Management. Presentation at the Third World Congress of the Bachelier Finance Society, Chicago.
20. Levin, A. (2005) Classes of Multifactor HJM-Type Futures Curve Models. Presentation at the Conference “Quantitative Methods in Finance’2005, Sydney”.
21. Miltersen, K., and Schwartz, E.S. (1998) Pricing of Options on Commodity Futures with Stochastic Term Structures of Convenience Yields and Interest Rates. *Journal of Financial and Quantitative Analysis*, 33, pp. 33-59.
22. Reisman, H. (1991) Movements of the Term Structure of Commodity Futures and the Pricing of Commodity Claims. Working paper, Technion - Israel Institute of Technology, Haifa.
23. Schwartz, E. (1997) The Stochastic Behavior of Commodity Prices: Implications for Valuation and Hedging. *Journal of Finance*, 52, pp. 923-972.
24. Schwartz, E., and Smith, J. (2000) Short-Term Variations and Long-Term Dynamics in Commodity Prices. *Management Science*, 46, pp. 893-911.
25. Tompaidis, S., and Manoliu, M. (2002) Energy Futures Prices: Term Structure Models with Kalman Filter Estimation. *Applied Mathematical Finance*, 9, pp. 21-43.
26. Trolle, A., and Schwartz, E. (2009) Unspanned Stochastic Volatility and the Pricing of Commodity Derivatives. *Review of Financial Studies*, 22, pp. 4423-4461.
27. Wong, H.Y., and Lo, Y.W. (2009) Option Pricing with Mean Reversion and Stochastic Volatility. *European Journal of Operational Research*, 197, pp. 179-187.

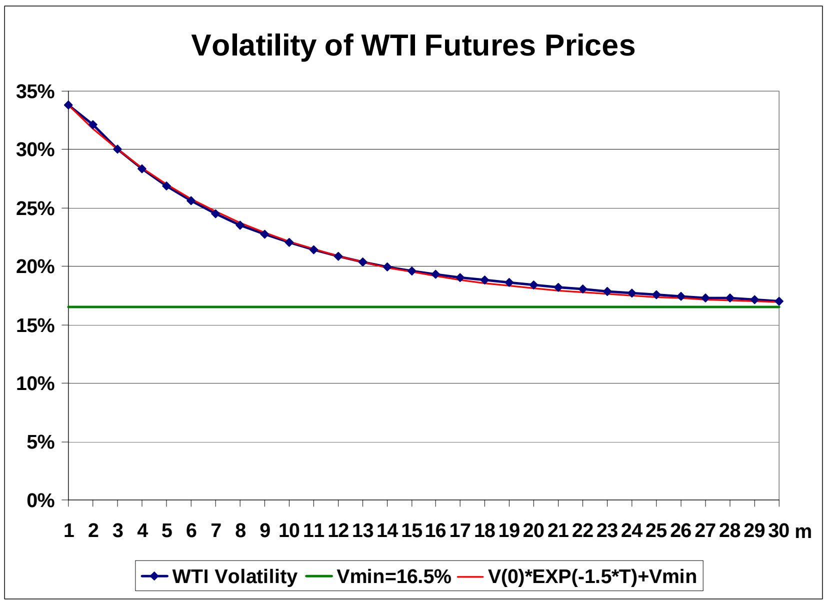
Properties of Crude Oil futures price and implied volatility curves

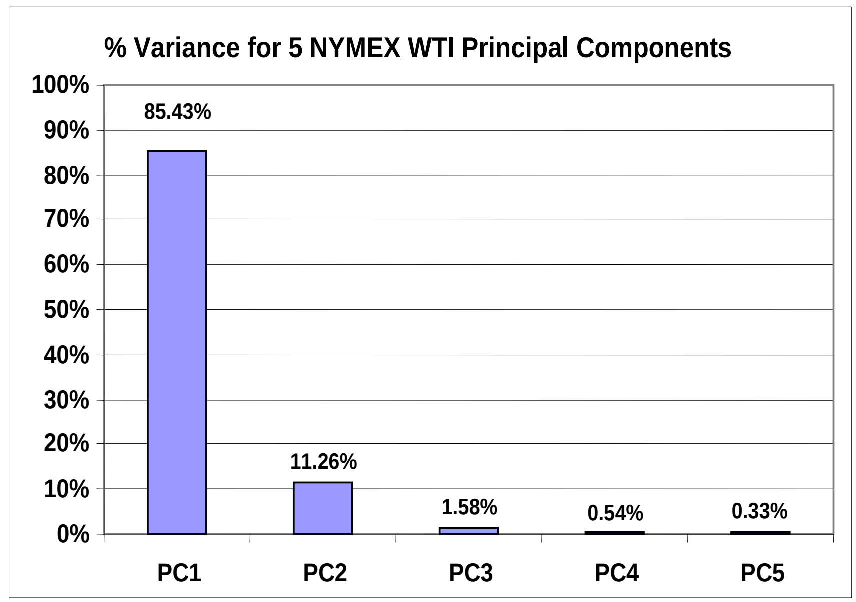
- Co-integrated "log-normal – mean-reverting" stochastic dynamics when Crude Oil futures prices for different tenors fluctuate ("mean revert") around some level that itself changes stochastically like "Geometric Brownian Motion".
- Futures curves have horizontal asymptotes for long tenors. Crude Oil futures price curves are not seasonal, and the volatility of the futures prices is not seasonal.



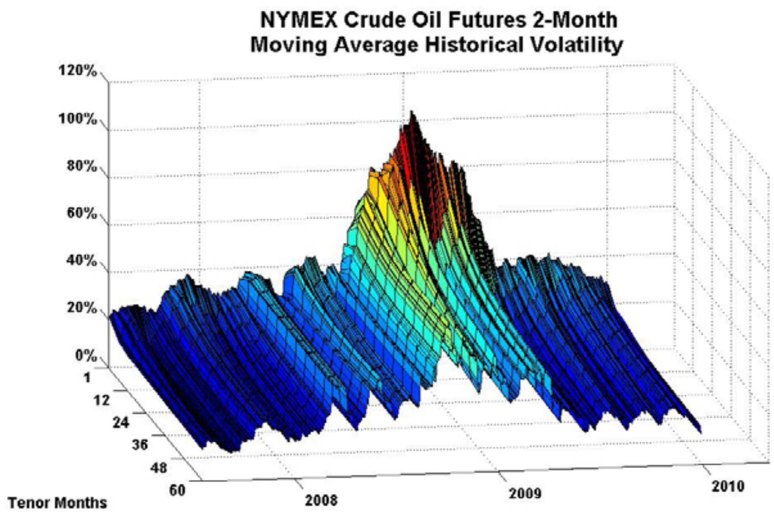


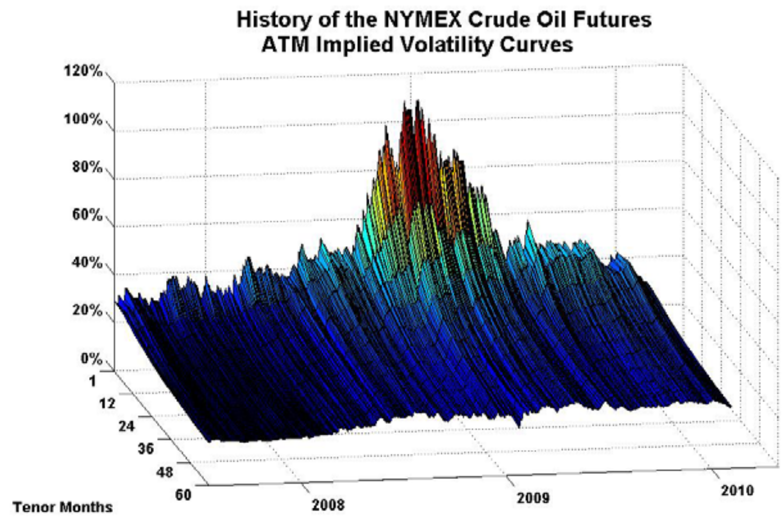
- Average futures price historical volatility term structure is well-described by exponentially decreasing to some positive level function of the tenor.
- Important property: futures volatility does not go to zero for long tenors, but has a horizontal positive asymptote.
- 96% of total variance for NYMEX Crude Oil futures curve is explained by only 2 Principal Components.



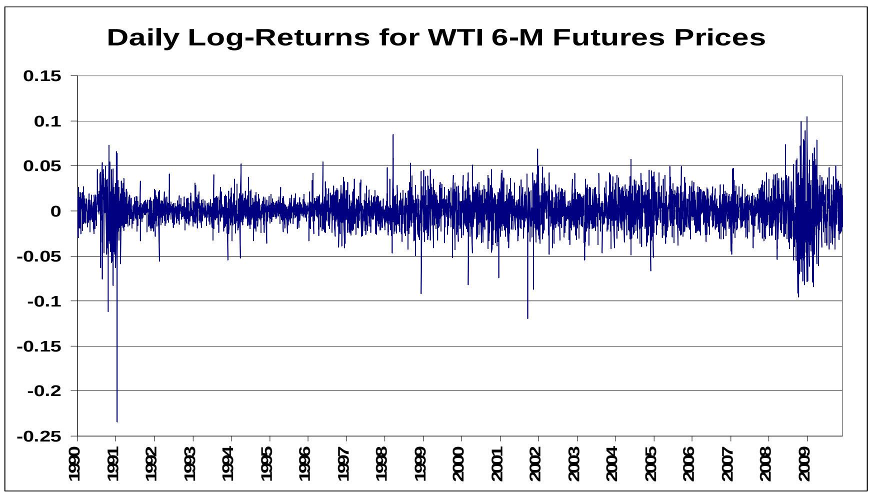


- Volatility of the futures curves is stochastic with the dynamics fully consistent with the stochastic mean-reverting dynamics for the ATM implied volatility curves.



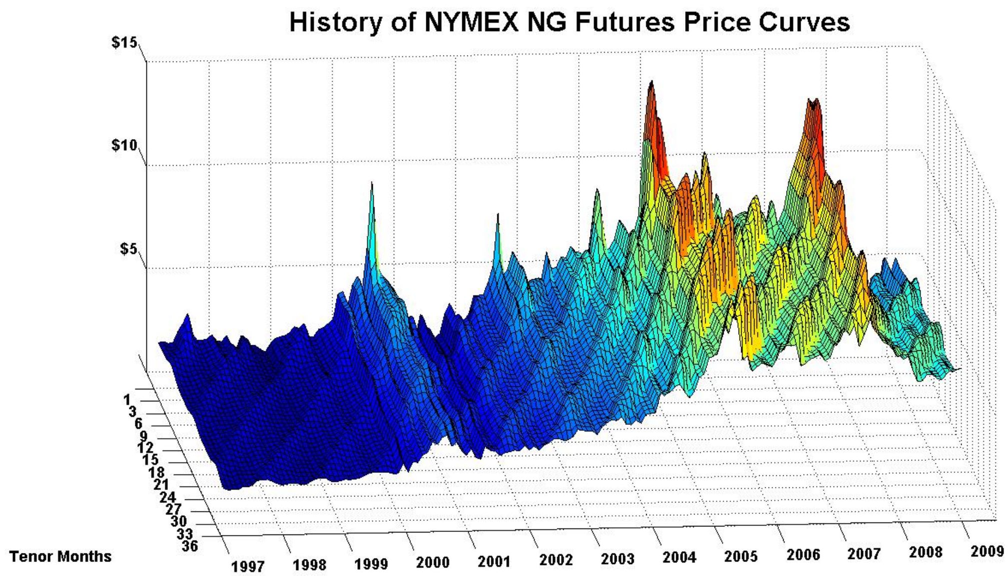


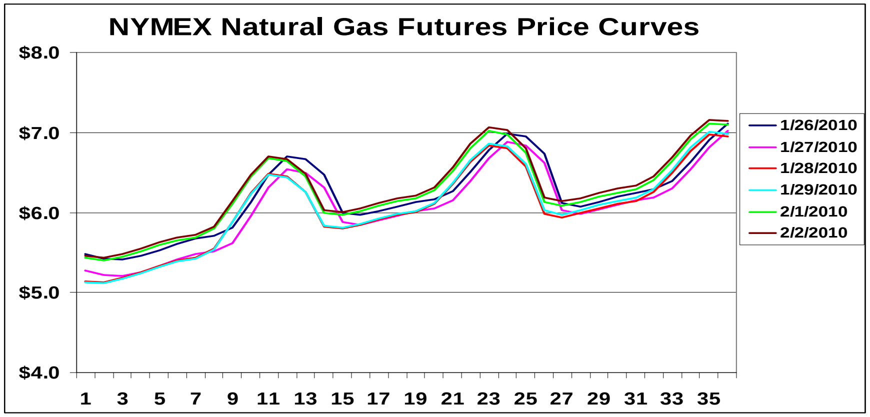
- There are large jumps in the futures prices. There is volatility clustering.



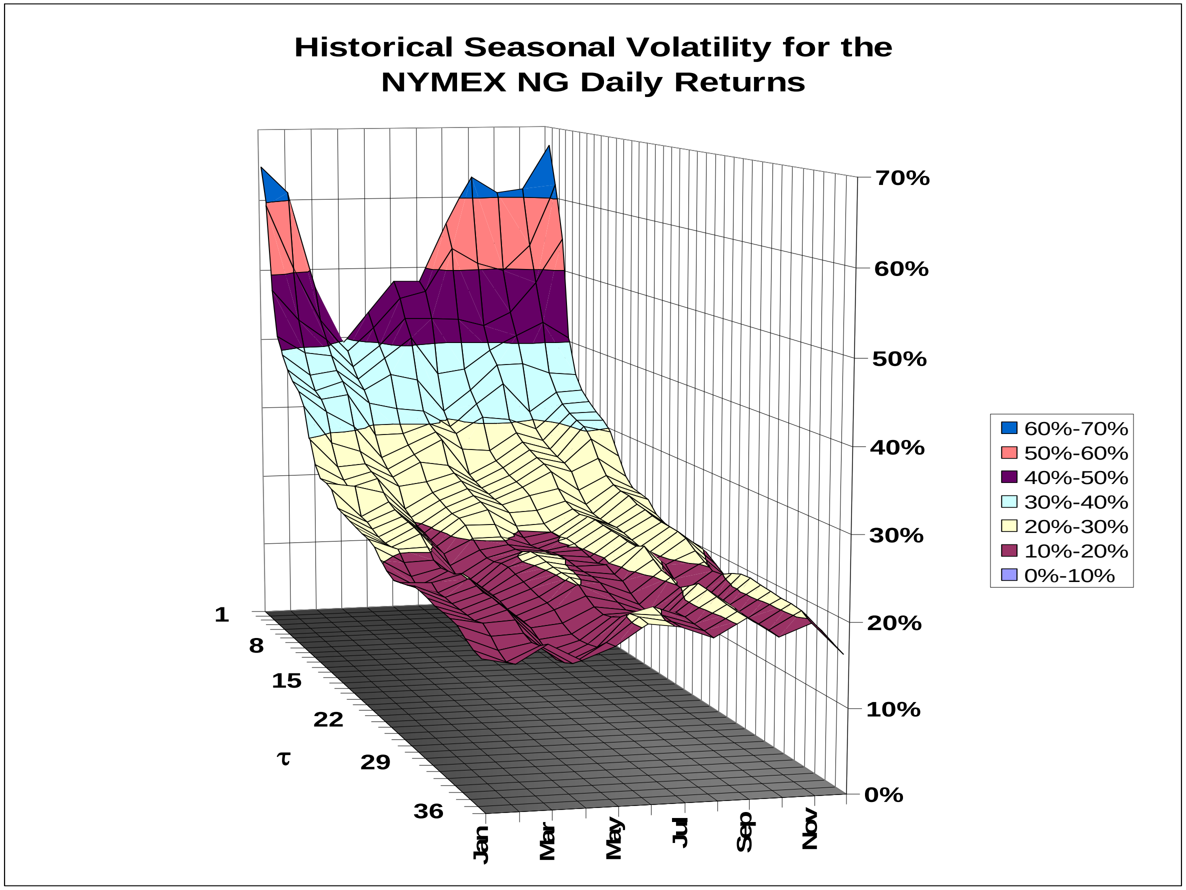
Properties of Natural Gas futures price and implied volatility curves

- Co-integrated “log-normal – mean-reverting” NG futures curve stochastic dynamics.
- Natural Gas futures price curves are seasonal with annual periodicity.

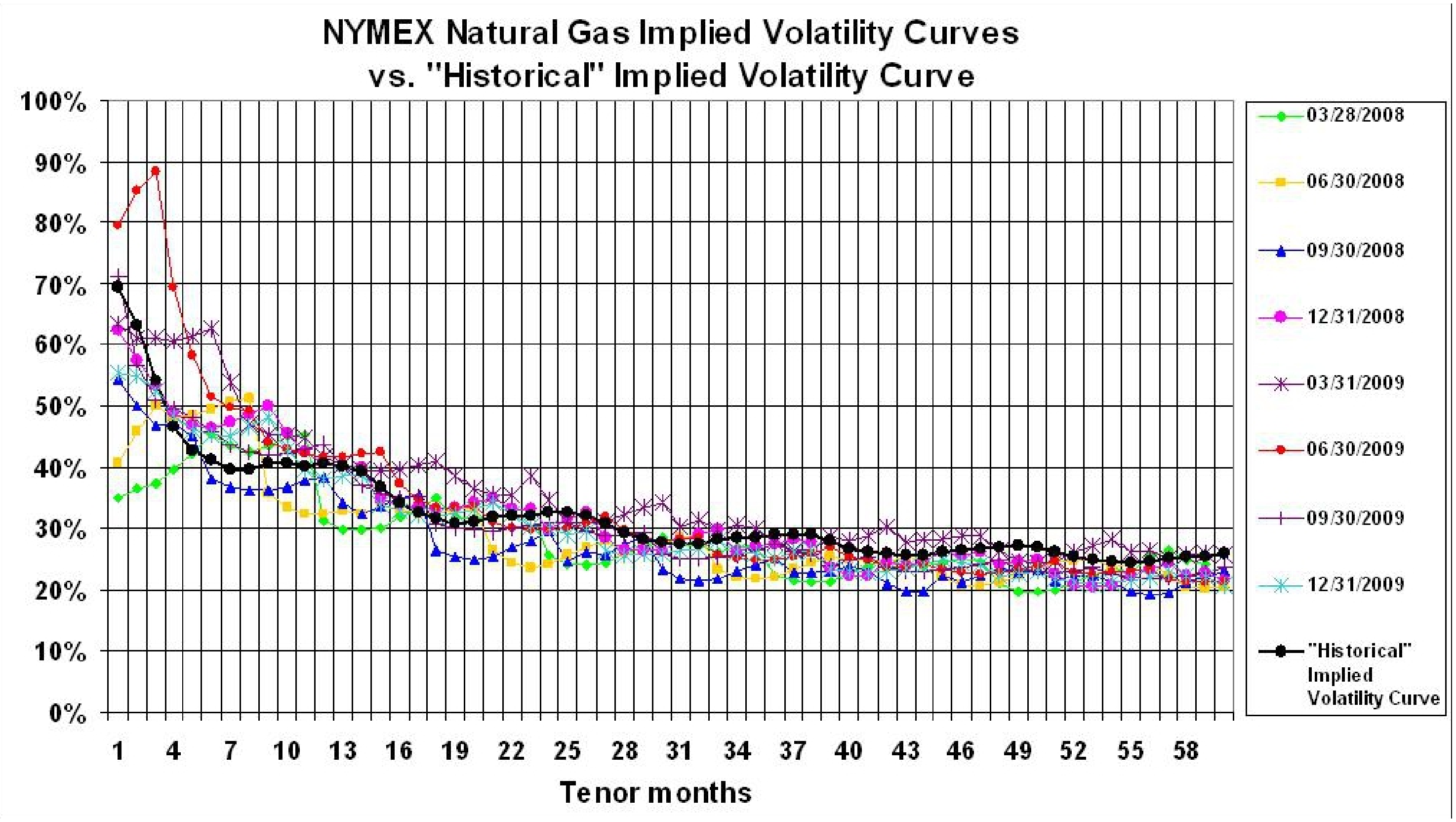




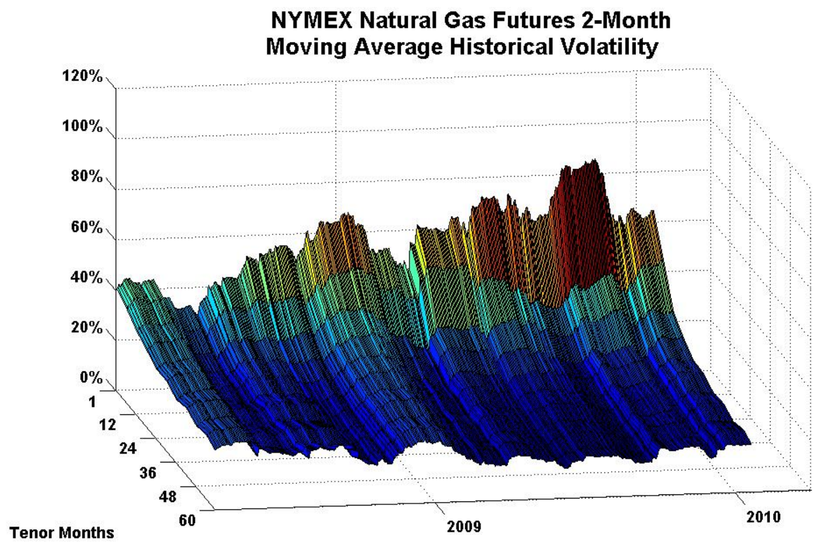
- Historical volatility of the NYMEX Natural Gas futures curves is also seasonal: higher in winter and lower in summer.
- Historical volatility has higher seasonality for short-term futures and almost no seasonality for long-term futures.
- For each month of the year, the average historical volatility term structure for this month is well described by exponentially decreasing to some positive level function.

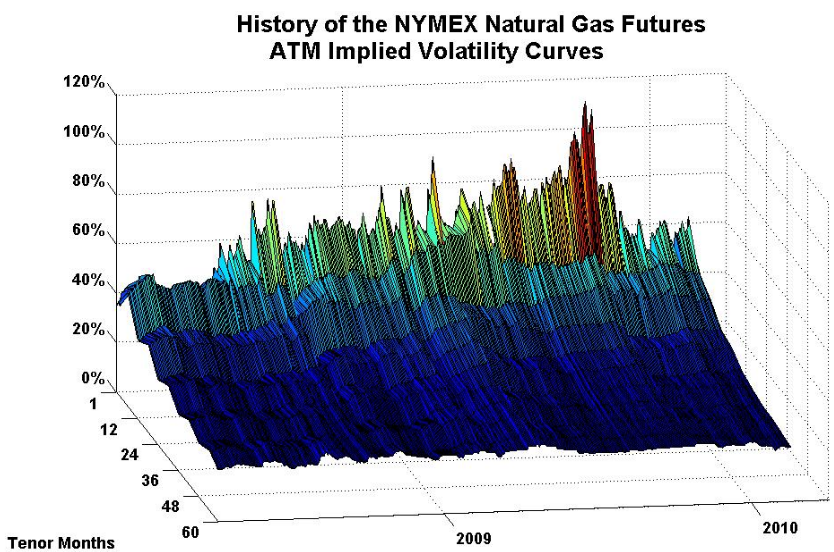


- Consistently with a seasonal Natural Gas futures historical volatility, the Natural Gas ATM implied volatility curves are "seasonal with exponentially decreasing dumping factor": they have periodically humped term structure with decreasing amplitudes of humps vanishing for long maturities.
- Natural Gas ATM implied volatility curves go to horizontal asymptotes for long maturities consistently with the term structure of the historical volatility.



- Volatility of the Natural Gas futures curves is stochastic in full consistency with the stochastic mean-reverting dynamics for the NG ATM implied volatility curves.





Some additional properties of Commodity futures prices

- **Electricity price** stochastic dynamics exhibits multi-level seasonality, significant jumps, spikes, and stochastic volatility.
- **Electricity prices** in some countries and US states are capped from the above resulting in **bounded electricity futures prices**. A satisfactory no-arbitrage model for bounded futures prices is needed for pricing electricity derivatives in risk neutral measure!
- **Temperature** exhibits seasonal mean-reverting stochastic behavior within some natural physical bounds. For temperature derivatives, it is also desirable to have some reasonable no-arbitrage model with bounded futures prices.

Extension of PCA/factor approach to HGM-type futures curve modeling

We consider a continuous futures curve model, i.e. assume the futures contracts are traded for all maturities $T \geq t_0$. Time t_0 is viewed as the current time. Let $F(t,T)$ be the price at time $t \geq t_0$ of a futures contract for delivery at maturity $T > t$. If spot commodity is traded then, from the no-arbitrage condition, the spot price is $S(t) = F(t,t)$. The initial futures curve $F(t_0,T)$ is given from the market.

We consider a dynamics of the futures curves under equivalent martingale measure. Futures prices for all fixed maturities $T > t$ are martingales under the risk neutral measure, i.e. the HJM model SDE for the futures price has zero drift:

$$dF(t,T) = \sum_{j=1}^{M} \tilde{\sigma}_j^F(t,T)\, dw_j(t) \qquad (1)$$

The choice of the volatility functions completely determines the particular HJM model.

Our goal is to extend PCA approach for modeling HJM-type risk neutral dynamics of entire futures curve using Markovian mean-revering/co-integrated factors $z(t)$ together with time-homogeneous (dependent only on the tenor $\tau = T - t$) volatility functions ("Principal Components"), i.e. to model the futures curve in the following form:

$$F(t,T) = \Phi\left(\sum_{j=1}^{M} \eta_j(T-t)\, z_j(t) + \ldots \right)$$

Levin (2004) family of no-arbitrage co-integrated Markovian factor models

For non-seasonal commodity, Levin (2004) considered a class of risk neutral futures curve models corresponding to some static one-to-one transformation $\Phi(x)$ of a linear combination of the time-homogeneous driver functions $\eta_j(T-t)$ with the stochastic coefficients $z_j(t)$ that follow a general co-integrated Wiener – mean-reversion Gaussian system with constant coefficients:

$$F(t,T)=\Phi\big(X(t,T)\big), \qquad \Phi'(x)\neq 0 \tag{2}$$

$$X(t,T)=\sum_{j=1}^{M}\eta_j(T-t)\,z_j(t)+\Psi(t,T)=\eta(T-t)^{*}z(t)+\Psi(t,T) \tag{3}$$

$$d\,z(t)=-\mathrm{K}\,z(t)\,dt+B\,dw, \quad z(t)=(z_1(t),\ldots,z_M(t))^{*}, \quad z(t_0)=0 \tag{4}$$

$$X(t_0,T)=\Psi(t_0,T)=\Phi^{-1}\big(F(t_0,T)\big) \tag{5}$$

A deterministic function $\Psi(t,T)$ is in fact Ito martingale correction for a non-linear transformation $\Phi(x)$. It is assumed that the eigenvalues of a general matrix K of mean-reversion and cross-mean reversion speeds have non-negative real parts and the covariance matrix $C=BB^{*}$ for instantaneous returns of the state vector $z(t)$ is positive definite. The model automatically fits into the initial market futures price curve.

<u>The following theorem completely characterizes the considered above class of models.</u>

Theorem 1 (Levin 2004). *The necessary and sufficient conditions for the no-arbitrage dynamics of the futures price curve* $F(t,T)$ *of the form (2) - (5) are as follows*:

a) A one-to-one static transformation $\Phi(x)$ *is only of the form*

$$\Phi(x) = \pm\int^{x} e^{\frac{1}{2}as^2+bs+c}\,ds + l \tag{6}$$

with four constant parameters a, b, c, l. ***This means there are only four different types of models in this family: Linear*** *(* $a=b=0$ *),* ***Exponential*** *(* $a=0$ *),* ***Normal Cumulative Distribution Function (NCDF)*** *(* $a<0$ *),* ***and Dawson Integral-type*** *(* $a>0$ *)* ***models.***

b) Time-homogenous functions $\eta(\tau)$ *satisfy a system of (non-linear for* $a \neq 0$*) ODEs:*

$$\eta'(\tau) = -\mathrm{K}^{*}\eta(\tau) + \tfrac{1}{2}a\left[\eta(\tau)^{*}C\eta(\tau)\right]\eta(\tau), \qquad \tau = T-t>0, \qquad \eta(0)=\eta_0 \neq 0 \tag{7}$$

c) For all fixed $T>t_0$*, the function* $\Psi(t,T)$ *satisfies the following ODE:*

$$\frac{\partial \Psi(t,T)}{\partial t} = -\tfrac{1}{2}\left(a\Psi(t,T)+b\right)\left[\eta(T-t)^{*}C\eta(T-t)\right], \qquad \Psi(t_0,T) = X(t_0,T) \tag{8}$$

A sketch of the proof. Calculating $dF(t,T)$ for the model (2)-(5) with fixed T using Ito's Lemma and equating the drift term to zero as in (1), one can obtain the following equality that should be satisfied for all $t,\ T > t$ and arbitrary vector z (with $\tau = T - t$):

$$\partial\Psi(t,T)/\partial t - \eta'(\tau)^* z - \eta(\tau)^* \mathrm{K} z + \frac{1}{2}\frac{\Phi''\left(\eta(\tau)^* z + \Psi(t,T)\right)}{\Phi'\left(\eta(\tau)^* z + \Psi(t,T)\right)}\left[\eta(\tau)^* C \eta(\tau)\right] \equiv 0 \tag{9}$$

Because $\Phi(x)$ is a static transformation, the left-hand side of (9) is identical zero <u>if and only if</u> the following equality holds with some constants a and b:

$$\Phi''(x)/\Phi'(x) = a\,x + b \tag{10}$$

This proves the main result (6) of Theorem 1 stating that **only four types of non-linear transformations $\Phi(x)$ satisfy no-arbitrage conditions for Levin (2004) class of futures curve models**. Equations (7) and (8) immediately follow from the substitution of (10) into (9). Note that no-arbitrage conditions fully determine the driver functions $\eta(\tau)$!

Proposition 1. *HJM futures price dynamics for the model (2)-(5) is described by SDE:*

$$dF(t,T) = \Phi'\left(\Phi^{-1}\left(F(t,T)\right)\right)\sigma(T-t)^* dw$$

with time-homogeneous volatility functions $\sigma(\tau)$ expressed in terms of $\eta(\tau)$ as:

$$\sigma(\tau) = B^* \eta(\tau)$$

Theorem 2. *Given four constant parameters* $a, b, c,$ *and* l *in the transformation* $\Phi(x)$ *and a vector of initial values* η_0 *in (7), a closed-form solution for the vector of the driver functions* $\eta(\tau)$ *is given by the following formula*:

$$\eta(\tau) = \left(1 - a\omega(\tau)\right)^{-1/2} e^{-\mathrm{K}^*\tau}\eta_0, \qquad \omega(\tau) = \int_0^\tau \eta_0{}^* e^{-\mathrm{K}s} C e^{-\mathrm{K}^*s}\,\eta_0\,ds \tag{11}$$

Function $\Psi(t,T)$ *is given by the formula:*

$$\Psi(t,T) = X(t_0,T) + \frac{1}{2}b\left[\omega(T-t) - \omega(T-t_0)\right], \qquad \text{if} \quad a = 0 \tag{12}$$

$$\Psi(t,T) = \left(X(t_0,T) + \frac{b}{a}\right)\left(\frac{1 - a\omega(T-t_0)}{1 - a\omega(T-t)}\right)^{1/2} - \frac{b}{a}, \qquad \text{if} \quad a \neq 0 \tag{13}$$

It is well known that matrix exponents $e^{-\mathrm{K}^*\tau}$ in (11) are in general represented as linear combinations of products of exponents, polynomials and sine/cosine functions. Periodic sine/cosine waves in the commodity futures PCs are not observed in the market even for seasonal commodities like Natural Gas. Therefore, we assume all eigenvalues of the matrix K are real. Products of exponents and polynomials appear in the driver functions only if positive eigenvalues have multiplicity more than 1. However, it is a singular case improbable in practice. Therefore we assume distinct single positive eigenvalues with the corresponding exponential $e^{-\kappa_i\tau}$ components in the driver functions $\eta(\tau)$.

Linear and Exponential models

Analysis of zero eigenvalue (i.e., zero mean reversion speed) case corresponding to Wiener factors $z_i(t)$ in (4) leads us to a very important co-integration property in the sense of Engle and Granger (1987) for the Linear and Exponential models. If multiplicity of zero eigenvalue was more than 1 then the corresponding driver function would be a polynomial of non-zero order resulting in the infinitely increasing volatility of the futures prices for long tenors. Therefore, it can be only one non-mean-reverting Wiener factor $z_0(t)$ in (4) with the corresponding flat "Black's" volatility driver function $\eta(\tau) \equiv 1$.

Proposition 2. *Without loss of generality,* ***Linear*** *model can be defined by the linear transformation* $\Phi(x) = x$ *(* $a = b = c = l = 0$ *),* ***Exponential*** *model – by shifted exponential transformation* $\Phi(x) = \pm e^x + l$ *(* $a = c = 0$ *,* $b = 1$ *), while both models have a diagonal matrix* K *with all different mean-reversion speeds* $\kappa_i \geq 0$ *for the Gaussian factors* $z_i(t)$ *and the corresponding exponential driver functions*

$$\eta_i(\tau) = e^{-\kappa_i \tau}$$

Linear model from this family is a multi-factor mean-reverting or co-integrated extension of Gaussian Hull – White-type and Bachelier (1900) models of the form:

$$F(t,T) = \sigma_1 dw_1(t) + \sum_{i=2}^{M} \sigma_i e^{-\kappa_i (T-t)} dw_i(t)$$

Exponential model from the family is a multi-factor mean-reverting or co-integrated extension of the form

$$F(t,T) = \exp\left(\sum_{i=1}^{M} \eta_i(T-t)\, z_i(t) + \Psi(t,T)\right), \qquad \eta_i(T-t) = e^{-\kappa_i (T-t)}, \quad \kappa_i \ge 0$$

$$d\,z(t) = -diag(\kappa)\, z(t)\, dt + B\, dw, \quad z(t) = (z_1(t),\dots, z_M(t))^*, \quad z(t_0) = 0$$

for the log-normal Black's (1997) model $dF(t,T)/F(t,T) = \sigma\, dw(t)$,

Schwartz (1997) mean-reverting model $dF(t,T)/F(t,T) = \sigma\, e^{-\kappa(T-t)}\, dw(t)$,

and Schwartz - Smith (2000) two-factor co-integrated model

$$dF(t,T)/F(t,T) = \sigma_1\, d\tilde{w}_1(t) + \sigma_2\, e^{-\kappa(T-t)}\, d\tilde{w}_2(t)$$

Schwartz - Smith two-factor co-integrated model sufficiently accurately describes the observed behavior of the NYMEX Crude Oil futures curves (see figures on pp. 6 – 7).

<u>The important no-arbitrage condition for the considered Exponential family stating that the mean reversion speed</u> $\kappa_i \ge 0$ <u>of the factor</u> $z_i(t)$ <u>is exactly equal to the decay factor in the corresponding exponential driver function</u> $\eta_i(\tau) = e^{-\kappa_i \tau}$ <u>is nicely verified by the presence of the “log-normal” Black's component (</u> $\kappa_1 = 0$ <u>) in the history of WTI prices in the figure on p. 6 and flat positive component in the volatility term structure on p. 7 !</u>

Extension for seasonality and time-dependent futures curve volatility

The model (2)-(5) automatically fits into the initial (possibly seasonal) term structure of the futures prices. However, the need to calibrate the model to given ATM implied volatilities and necessity to reproduce seasonal futures curve volatilities observed, for example, for NYMEX Natural Gas futures (see figure on p. 10) require an extension of the model for time-dependent and periodic futures curve volatilities. We consider the extension of Exponential (and Linear) models by time-dependent (including periodic) volatilities of the factors $z(t)$ similar to the one from Tompaidis and Manoliu (2002):

$$F(t,T) = \Phi\big(X(t,T)\big), \qquad \Phi(x) = \exp(x) \quad or \quad \Phi(x) = x \tag{14}$$

$$X(t,T) = \eta(T-t)^* z(t) + \Psi(t,T), \qquad \Psi(t_0,T) = \Phi^{-1}\big(F(t_0,T)\big) \tag{15}$$

$$d\,z(t) = -diag(\kappa)\,z(t)\,dt + B(t)\,dw, \quad z(t_0) = 0 \tag{16}$$

The statement of Theorem 2 remains valid with the same exponential $\eta_i(\tau) = e^{-\kappa_i \tau}$ and

$$\partial\Psi(t,T)/\partial t = -\tfrac{1}{2}\big[\eta(T-t)^* B(t)B(t)^* \eta(T-t)\big]\Phi''(x)\Phi'(x)^{-1}, \quad \Psi(t_0,T) = \Phi^{-1}\big(F(t_0,T)\big) \tag{17}$$

For example, the following Seasonal Two-Factor Schwartz-Smith-type model accurately describes the observed NYMEX NG historical seasonal volatility surface on p. 10:

$$dF(t,T)/F(t,T) = \sigma_1 d\tilde{w}_1(t) + \sigma_2(t)\,e^{-\kappa_2 (T-t)}\,d\tilde{w}_2(t) \tag{18}$$

Jump-diffusion Exponential and Linear futures curve models driven by Lévy processes

A jump-diffusion extension of the Exponential and Linear futures curve models to the factor models driven by Lévy processes with Lévy measure of pure jump part $\nu(dy)$ is:

$$F(t,T) = \Phi\big(X(t,T)\big), \qquad \Phi(x) = \exp(x) \ \ or \ \ \Phi(x) = x \tag{19}$$

$$X(t,T) = \eta(T-t)^{*} z(t) + \Psi(t,T), \qquad \Psi(t_0,T) = \Phi^{-1}\big(F(t_0,T)\big) \tag{20}$$

$$d\,z(t) = -diag(\kappa)z(t)\,dt + B\,dw + dL, \qquad z(t_0) = 0 \tag{21}$$

For simplicity of presentation, we consider a one-dimensional example. From a general results of Björk and Landén (2000), one can derive the following integro-differential equation of no-arbitrage futures curve dynamics:

$$\left\{\frac{\partial\Psi(t,T)}{\partial t}\eta(\tau) + \big[x - \Psi(t,T)\big]\big[\eta'(\tau) + \kappa\eta(\tau)\big]\right\}\Phi'\big(x\big) + \frac{B^2\eta(\tau)^3}{2}\Phi''\big(x\big) +$$
$$\eta(\tau)\int_{-\infty}^{+\infty}\big[\Phi\big(x + y\eta(\tau)\big) - \Phi\big(x\big)\big]\nu(dy) \equiv 0 \qquad \forall t,\ T > t,\ \tau = T - t,\ x \in R \tag{22}$$

It is shown in Levin (2004) that linear and exponential $\Phi(x)$ are valid solutions of (22). S. Levendorskii has proved (in private communication) that equation (22) does not admit bounded solutions for transformation $\Phi(x)$ contrary to pure diffusion case in Theorem 1.

The driver functions for jump-diffusion model are the same exponential functions $\eta_i(\tau) = e^{-\kappa_i \tau}$ as for the diffusion model, and $\Psi(t,T)$ is defined by the following ODE:

Linear model:
$$\frac{\partial \Psi(t,T)}{\partial t} = \eta(T-t) \int_{-\infty}^{+\infty} y \nu(dy) \tag{23}$$

Exponential model:
$$\frac{\partial \Psi(t,T)}{\partial t} = -\tfrac{1}{2} B^2 \eta(T-t)^2 + \int_{-\infty}^{+\infty} \left[\exp(y\eta(T-t)) - 1\right] \nu(dy) \tag{24}$$

Modeling of Electricity price "spikes" in Exponential jump-diffusion model

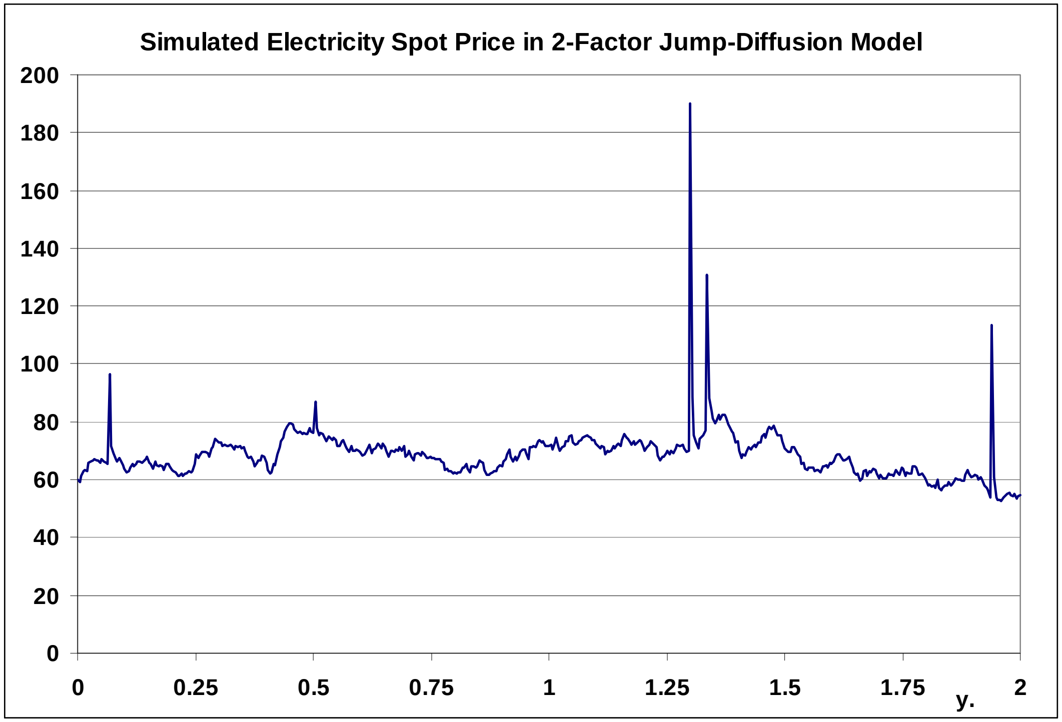
Multi-factor Exponential jump-diffusion futures curve model above, where one of the factors $z_1(t)$ is driven by large positive Poisson jumps dL_1 and has very high mean reversion speed (e.g., $\kappa_1 = 250$) and the corresponding rapidly decreasing exponential driver function $\eta_1(\tau) = e^{-\kappa_1 \tau}$, naturally describes "spikes" in the spot electricity prices when the spot price jumps up with the jump of the factor $z_1(t)$ and practically immediately "mean reverts" back (e.g., "in one day" for mean reversion speed $\kappa_1 = 250$). Other factors can be driven by jump-diffusions with "normal" mean reversion speeds.

Example of 2-factor jump-diffusion model with spikes for Electricity

As an example of jump-diffusion Exponential model with spikes for Electricity, consider a two-factor model with one Gaussian slow mean-reverting factor ($\kappa_2 = 4$) and one fast mean-reverting factor ($\kappa_1 = 200$) driven by Compound Poison process $L(t)$ with positive exponentially distributed jump sizes:

$$F(t,T) = \exp\left(e^{-\kappa_1 (T-t)} z_1(t) + e^{-\kappa_2 (T-t)} z_2(t) + \Psi(t,T)\right), \qquad S(t) = F(t,t)$$

$$d\,z_1(t) = -\kappa_1 z_1(t)\,dt + dL, \qquad d\,z_2(t) = -\kappa_2 z_2(t)\,dt + B\,dw$$



Levin (2004) Normal Cumulative Distribution Function (NCDF)-transformation model with bounded futures prices

Proposition 3. *Without loss of generality, bounded **NCDF** model can be defined by non-linear transformation* $\Phi(x)$ *in (6) with* $a=-1$ *and* $b=0$ *in the following form:*

$$\Phi(x)=\int_{-\infty}^{x} e^{\frac{1}{2}as^2+bs+c}\,ds+l=h\,N(x)+l, \qquad h=u-l \tag{25}$$

where N(x) is a standard Normal CDF, and $l<u$ *are lower and upper bounds for the futures (and spot) commodity prices. We assume the diagonal matrix* K *has all different mean-reversion speeds* $\kappa_i \geq 0$ *for the factors* $z_i(t)$*, and the corresponding driver functions are defined by (11) in Theorem 2 with* $\eta_0=1$*, while* $\Psi(t,T)$ *is defined by (13).*

For simplicity of presentation, we consider a one-dimensional NCDF model as example:

$$F(t,T)=h\,N\big(\eta(T-t)z(t)+\Psi(t,T)\big)+l \tag{26}$$

$$d\,z(t)=-\kappa\,z(t)\,dt+\sigma\,dw, \qquad z(t_0)=0 \tag{27}$$

$$\eta(\tau)=e^{-\kappa\tau}/\sqrt{\beta(\tau)}, \qquad \Psi(t,T)=N^{-1}\big((F(t_0,T)-l)/h\big)\left(\frac{\beta(T-t_0)}{\beta(T-t)}\right)^{1/2} \tag{28}$$

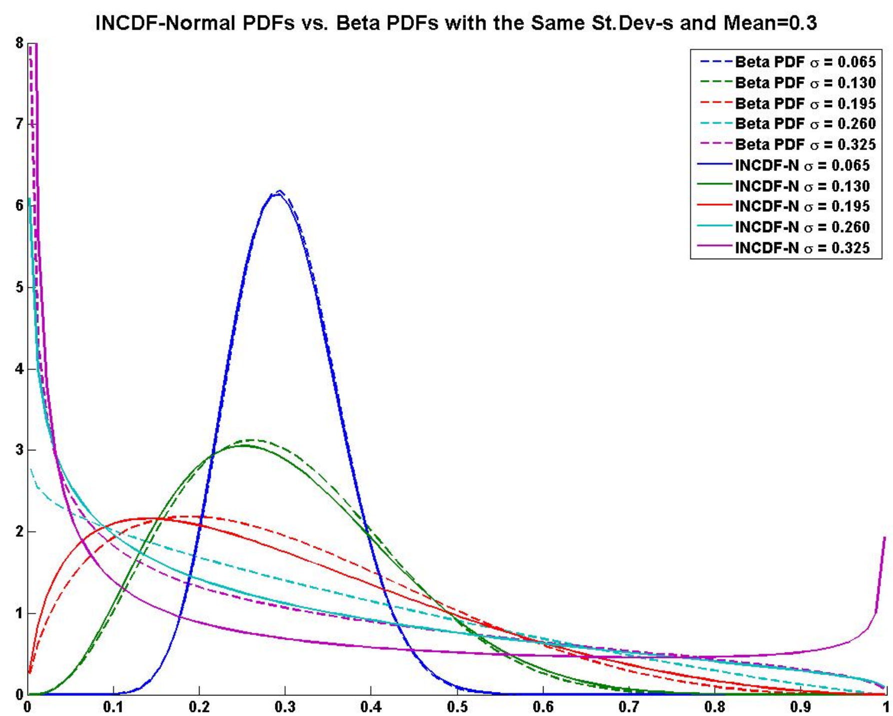
$$\beta(\tau)=1+\sigma^2(1-\exp(-2\kappa\tau))/(2\kappa), \quad \kappa>0; \qquad \beta(\tau)=1+\sigma^2\tau, \quad \kappa=0 \tag{29}$$

Denote the mean and standard deviation of $X(t,T)$ by $m = \Psi(t,T)$ and

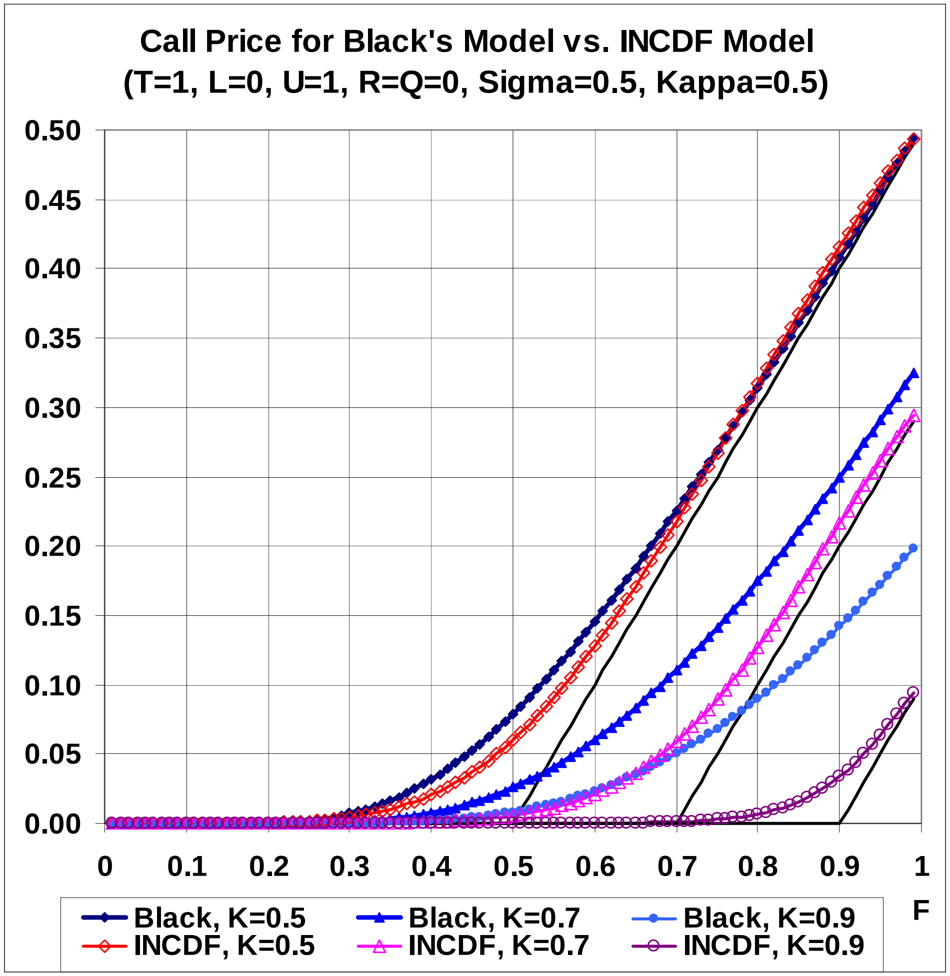
$$s^2 = \sigma^2(1-e^{-2\kappa(t-t_0)})/(2\kappa) \quad \text{if} \quad \kappa > 0; \qquad s^2 = \sigma^2(t-t_0) \quad \text{if} \quad \kappa = 0$$

Then, PDF of the futures price $F(t,T)$ is the following "Beta distribution-like" function:

$$f(y) = \frac{1}{hs}\exp\left(-\frac{(1-s^2)\left[N^{-1}\left((y-l)/h\right)\right]^2 - 2mN^{-1}\left((y-l)/h\right) + m^2}{2s^2}\right) \tag{30}$$



Prices of European futures options in NCDF model are given by mathematical expectations of the pay-offs with respect to the risk neutral density (30). For example, prices of European Call options for the NCDF model vs. Black's model are presented in the figure below.



Locally bounded extension of NCDF model with time-depended bounds

A drawback of the Levin (2004) Bounded NCDF model is constant values of the lower and upper bounds that do not allow, for example, exponentially increasing initial futures (forward) price curve $F(t_0,T)$ due to compounding with positive interest rate. We present "Locally Bounded" extension of the model with time-dependent bounds.

Let us consider a non-linear transformation in the general framework (2)-(5) of the form:

$$F(t,T)=\Phi\bigl(T,X(t,T)\bigr), \qquad \Phi'_x(T,x)\neq 0 \tag{31}$$

$$X(t,T)=\eta(T-t)^{*}z(t)+\Psi(t,T), \qquad \Psi(t_0,T)=\Phi^{-1}\bigl(T,F(t_0,T)\bigr) \tag{32}$$

$$dz(t)=-\mathrm{K}\,z(t)\,dt+B\,dw, \qquad z(t_0)=0 \tag{33}$$

Following the same reasoning as in Theorem 1, one can prove that transformation Φ is:

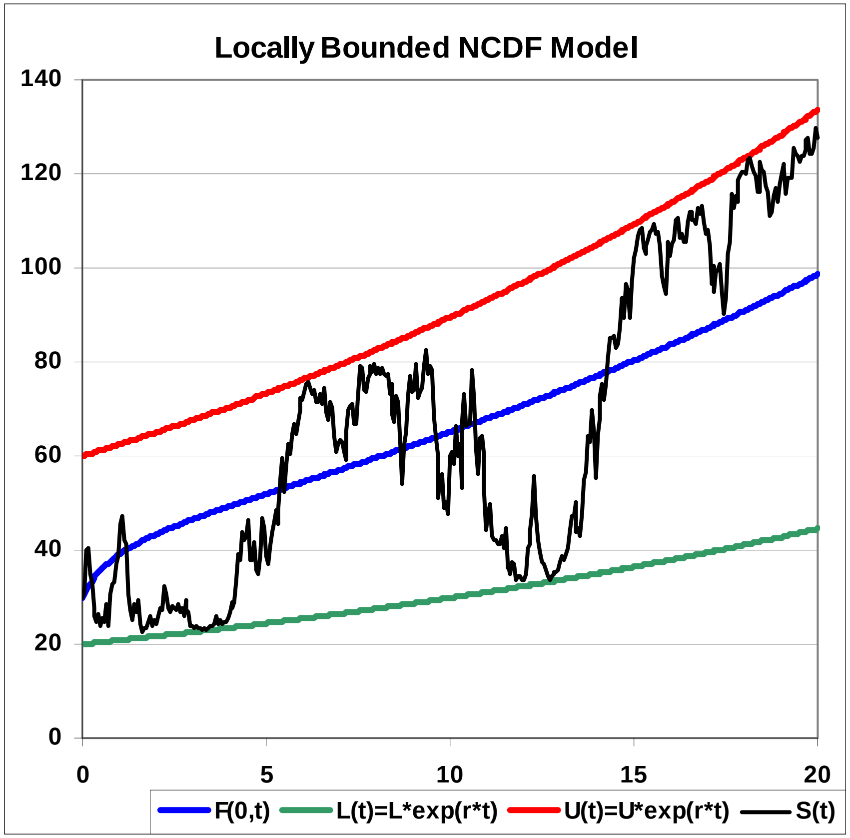
$$\Phi''(T,x)/\Phi'(T,x)=a\,x+b \quad\Rightarrow\quad \Phi(T,x)=\pm\int^{x} e^{\frac{1}{2}as^2+bs+c(T)}ds+l(T) \tag{34}$$

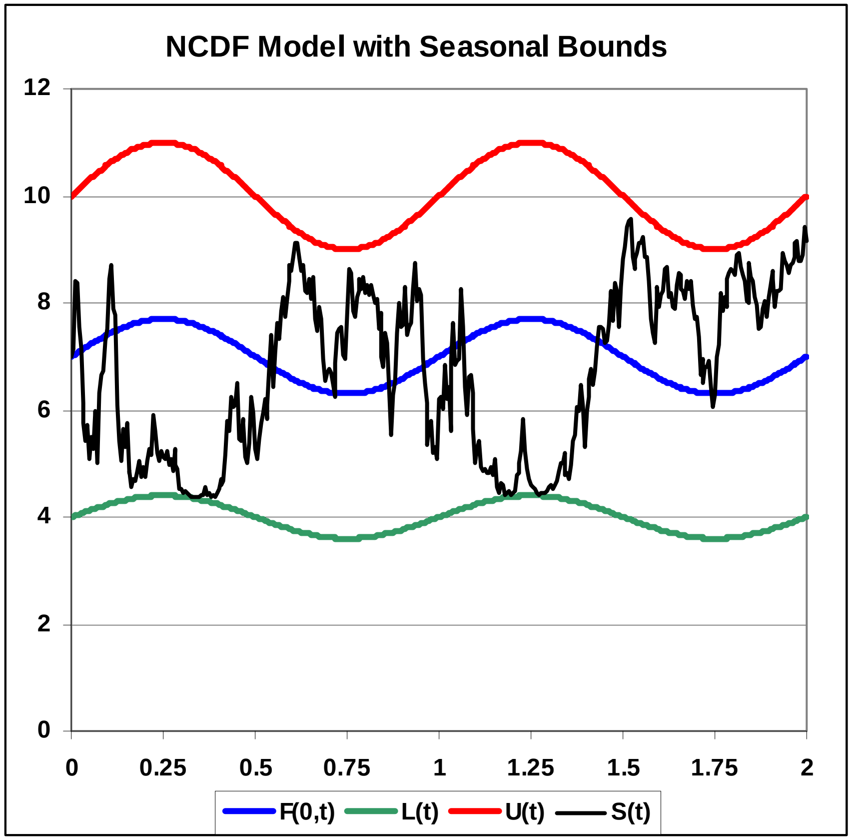
Resulting for NCDF model (with $a=-1$ and $b=0$) in time-depended bounds:

$$\Phi(T,x)=h(T)\,N(x)+l(T), \qquad h(T)=u(T)-l(T) \tag{35}$$

Driver functions $\eta(\tau)$ in this extension are the same as in the original NCDF model.

Figures below illustrate a new Locally Bounded NCDF futures curve model.





Levin (2004) Modified Dawson Integral-transformation model

The final fourth type of models from the considered family is presented by the Dowson Integral-type transformation $\Phi(x)$ with $a=1$ and $b=c=l=0$:

$$\Phi(x)=\int_0^x e^{\frac{1}{2}s^2}\,ds, \qquad x\in R \tag{36}$$

This transformation results in positive and negative futures price values. The driver function for this model is as follows:

$$\eta(\tau)=e^{-\kappa\tau}/\sqrt{\beta(\tau)} \tag{37}$$

$$\beta(\tau)=1-\sigma^2\frac{(1-\exp(-2\kappa\tau))}{2\kappa} \quad \text{for } \kappa>0; \qquad \beta(\tau)=1-\sigma^2\tau \quad \text{for } \kappa=0$$

Unfortunately, function $\Phi(x)$ goes to infinity so rapidly that the corresponding futures curve model "explodes" (futures prices diverge on finite tenor interval) for the Wiener factor $z(t)$ and even for mean-reverting factors for some values of parameters (see (37)).

Family of affine stochastic volatility factor models for futures curve

As we see from the graphs on pp. 8 and 12, the volatility of the Crude Oil and Natural Gas futures curves is stochastic that is fully consistent with the stochastic behavior of the ATM implied volatility curves. We present a multi-factor affine extension of the Exponential family of the futures curve models (2)-(5) by the Heston stochastic volatility similar to the construction in Andersen (2008). For simplicity of presentation, we illustrate the proposed framework on the example of two-factor co-integrated Schwartz - Smith model. Fully according to the construction (2)-(5), the three-factor model with one common Heston stochastic volatility is:

$$F(t,T) = \exp\left(X^H(t,T)\right) \tag{38}$$

$$X^H(t,T) = \eta^V(T-t)V(t) + \eta_2^{\ H}(T-t)x_2^H(t) + \eta_3^{\ H}(T-t)x_3^H(t) + \Psi^H(t,T) \tag{39}$$

$$dV = \kappa^V[\theta^V(t) - V]dt + \sigma^V\sqrt{V}[1\,dW^V + 0\,dW_2^{\ H} + 0\,dW_3^{\ H}], \qquad V(t_0) = V_0 \tag{40}$$

$$d\,x_2^H = \left(-c_2^H V - \kappa_2\,x_2^H\right)dt + \sigma_2^{\ H}\sqrt{V}[\rho^{2,V}dW^V + a_{22}^H dW_2^{\ H} + 0dW_3^{\ H}], \quad x_2^H(t_0) = 0 \tag{41}$$

$$dx_3^H = -\frac{(\sigma_3^{\ H})^2}{2}Vdt + \sigma_3^{\ H}\sqrt{V}[\rho^{3,V}dW^V + a_{32}^H dW_2^H + a_{33}^H dW_3^{\ H}], \qquad x_3^H(t_0) = 0 \tag{42}$$

$$\Psi(t_0,T) = \ln\left(F(t_0,T)\right) \tag{43}$$

The particular models from this family differ by the price of volatility risk c_2^H for the mean reversion factor and value at zero of the driver function $\eta^V(\tau)$.

The following theorem completely characterizes the considered above class of models.

Theorem 3. *The necessary and sufficient conditions for the no-arbitrage dynamics of the futures price curve* $F(t,T)$ *of the form (38) - (43) are as follows*:

a) Time-homogenous functions $\eta_2^H(\tau)$ *and* $\eta_3^H(\tau)$ *are of the same form as for the Exponential model:*

$$\eta_2^H(\tau) = \exp(-\kappa_2\tau), \qquad \eta_3^H(\tau) = \exp(-0\tau) \equiv 1 \tag{44}$$

b) Time-homogenous function $\eta^V(\tau)$ *satisfies the following Riccati equation:*

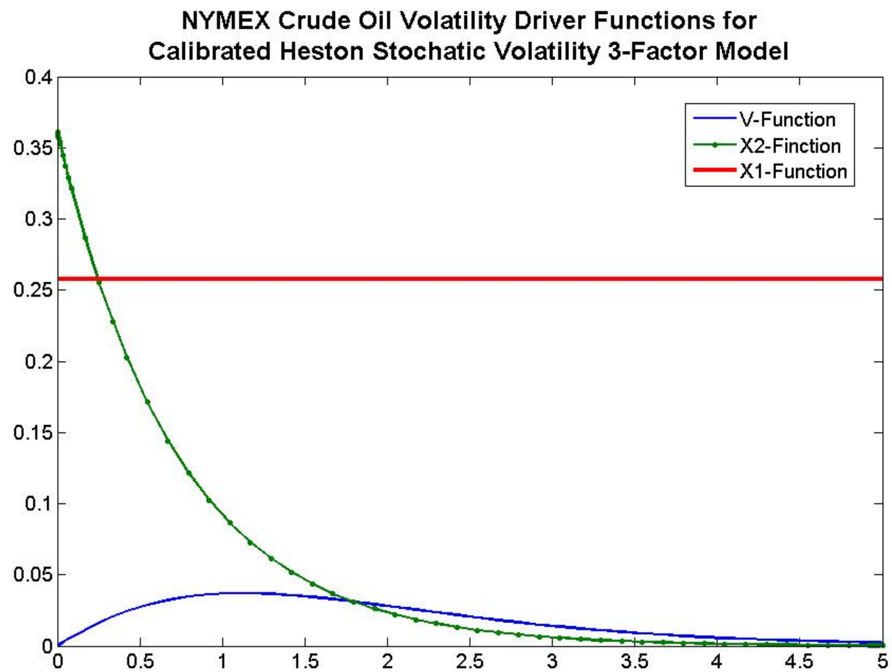
$$\begin{aligned}\eta^V(\tau)' &= \frac{(\sigma^V)^2}{2}\eta^V(\tau)^2 + [\sigma^V\sigma_3^H\rho^{3,V} + \sigma^V\sigma_2^H\rho^{2,V}e^{-\kappa_2\tau} - \kappa^V]\eta^V(\tau) + \\ &\left[\frac{(\sigma_2^H)^2}{2}e^{-2\kappa_2\tau} + (\sigma^V\sigma_3^H\rho^{3,V} - c_2^H)\right], \qquad \eta(0) = \eta_0 \neq 0\end{aligned} \tag{7}$$

c) For all fixed $T > t_0$, *the function* $\Psi(t,T)$ *is given by the formula:*

$$\Psi(t,T) = \ln(F(t_0,T)) - \eta^V(T-t_0)V_0 - \kappa^V\int_{t_0}^{t}\theta_V(s)\eta^V(T-s)ds \tag{8}$$

Driver function $\eta^V(\tau)$ has a very complicated formula via incomplete hypergeometric functions. We calculate it numerically by Runge – Kutta method. For some values of the parameters and the price of volatility risk c_2^H, the driver function $\eta^V(\tau)$ can go to infinity at finite tenors (i.e., the model has explosions similar to Andersen and Piterbarg (2007)).

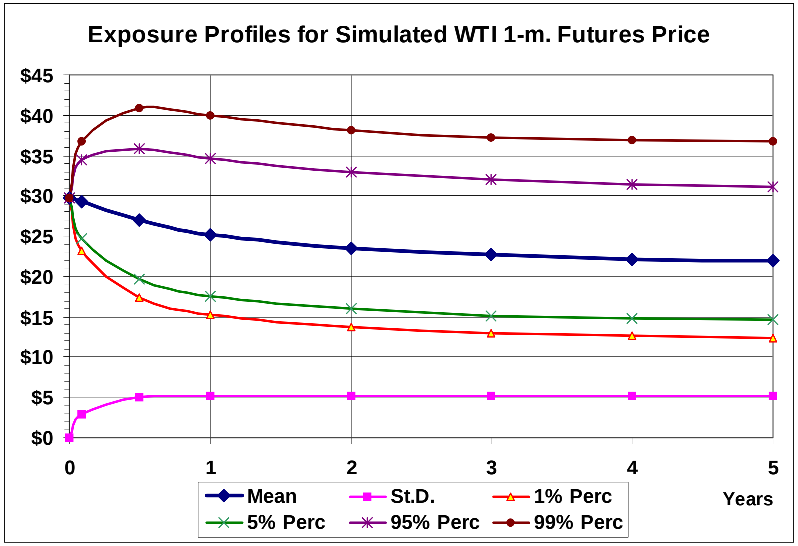
European futures option prices can be calculated from the available multivariate Characteristic Function based on general formulas for affine jump-diffusions in Duffie, Pan and Singleton (2000) or Carr and Madan (1999) Fourier transform method.

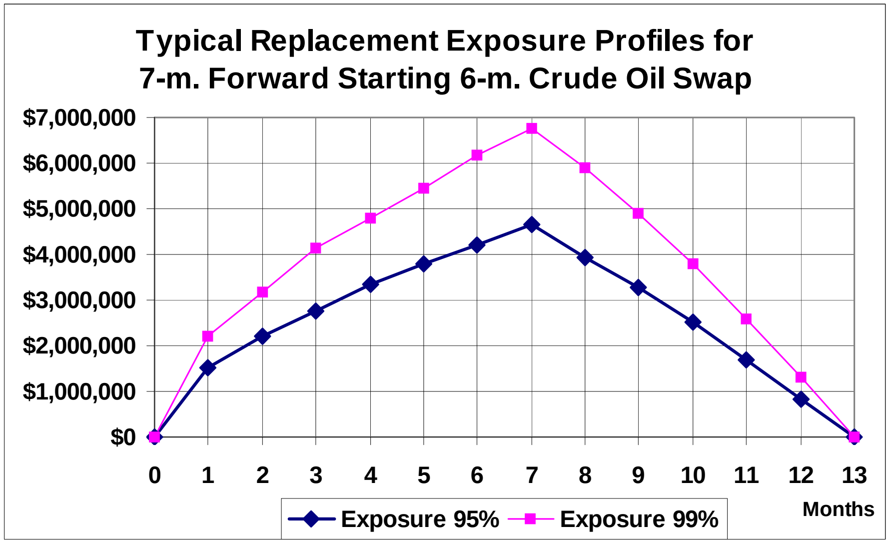


Simulation in historical measure for Commodity Risk Management systems

Simulation of the futures curves in historical measure, for example, in Commodity Risk Management systems is performed by adding the corresponding prices of risk to the processes for risk factors and stochastic volatility (see Cheridito, Filipovic and Kimmel (2007)).

A considered family of Markovian factor futures curve models is extremely effective for long-term Monte Carlo simulation in Commodity Credit Risk systems.





Possible applications to interest rate, FX, equity and volatility futures

- **Volatility modeling:** Bergomi (2005) two-factor mean-reverting model with exponential driver functions for stochastic volatility term structure is a special case of the Exponential volatility futures (forward) curve model.
- **Interest rate modeling**: It is well known that Black – Karasinski short rate model in continuous time produces infinite bond prices because of log-normal short rate. This model corresponds to a spot model of Exponential futures curve model. The spot model for the short rate corresponding to Levin (2004) Bounded NCDF interest rate futures curve model (with zero lower bound and a very high upper bound, say 100%) is a natural theoretically self-consistent replacement for the BK model, because bond prices are well defined in continuous time. Similar to BK model, bond and other interest rate derivative prices can be calculated using trees.
- **FX modeling**: Author of the presentation is aware only of one model for bounded futures and spot prices – Ingersoll (1997) bounded FX model. However, the Ingersoll model has some deficiency in the futures and spot price distributions: they gradually localize near the bounds. To the contrary, a proposed new Locally Bounded NCDF mean-reverting model has natural stationary distributions, and it allows for bounded FX rate for any finite interval of time even if the interest rate differential does not go to zero at infinity (see figure on p. 28).
- **Equity futures modeling**: Exponential jump-diffusion model is directly applicable to the Equity futures and spot derivative modeling.