ECON2125/4021/8013

Lecture 15

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Semester 1, 2015

Announcements

- This week's Thursday lecture will be shifted to Friday
  - 9am on 23/04/2015 to 10am on 24/04/2015
  - Same location
  - To let people focus on exam preparation
- Preliminary date for final exam is June 11
  - Still subject to change

Convergence in Distribution

Let

- $\{F_n\}_{n=1}^{\infty}$ be a sequence of cdfs
- F be any cdf

We say that $\{F_n\}_{n=1}^{\infty}$ **converges weakly** to F if

$$F_n(x) \to F(x) \quad \text{as} \quad n \to \infty$$

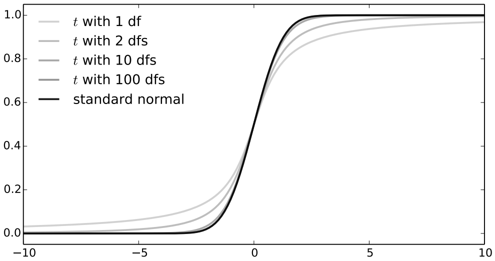
for any x such that F is continuous at x

- In essence, F_n gets close to F when n is large

Example. **Student's t-density** with n degrees of freedom is

$$p_n(x) := \frac{\Gamma\left(\frac{n+1}{2}\right)}{(n\pi)^{1/2}\Gamma\left(\frac{n}{2}\right)}\left(1 + n^{-1}x^2\right)^{-(n+1)/2}$$

It's well known that the corresponding cdfs F_n converge weakly to the standard normal cdf



We say that $\{X_n\}_{n=1}^{\infty}$ converges to X **in distribution** if

1. $X_n \sim F_n$,
2. $X \sim F$ and
3. $F_n \to F$ weakly

In this case we write $X_n \xrightarrow{d} X$

- In short, the distribution of X_n converges to that of X

Fact. If $X_n \xrightarrow{p} X$, then $X_n \xrightarrow{d} X$

Example. If X is any RV and $X_n := X + \frac{1}{n}$ then $X_n \xrightarrow{d} X$

Proof: Let F and F_n be the cdfs of X and X_n respectively

Observe that, $\forall\, x \in \mathbb{R}$,

$$F_n(x) = \mathbb{P}\left\{X + \frac{1}{n} \leq x\right\} = \mathbb{P}\left\{X \leq x - \frac{1}{n}\right\} = F\left(x - \frac{1}{n}\right)$$

Suppose that F is continuous at x

Since $x - \frac{1}{n} \to x$, we have

$$F\left(x - \frac{1}{n}\right) \to F(x)$$

(By the def of continuity — more on this later)

Fact. If $g\colon \mathbb{R} \to \mathbb{R}$ is continuous, then

1. $X_n \xrightarrow{d} X \implies g(X_n) \xrightarrow{d} g(X)$
2. $X_n \xrightarrow{p} X \implies g(X_n) \xrightarrow{p} g(X)$

Remark: This fact is called the **continuous mapping theorem**

Example. If α is constant and $X_n \xrightarrow{d} X$, then

- $X_n + \alpha \xrightarrow{d} X + \alpha$
- $\alpha X_n \xrightarrow{d} \alpha X$
- etc.

The Central Limit Theorem

Let $\{X_i\}_{i=1}^{\infty} \stackrel{\text{IID}}{\sim} F$ with

- $\mu := \mathbb{E}[X_i] = \int xF(dx)$
- $\sigma^2 := \text{var}[X_i] = \int (x-\mu)^2 F(dx)$, assumed finite

Fact. In this setting we have

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2) \quad \text{as} \quad n \to \infty$$

- $\bar{X}_n := \frac{1}{n} \sum_{i=1}^{n} X_i$
- $\xrightarrow{d}$ means the cdf of LHS $\to$ weakly to the $N(0, \sigma^2)$ cdf

Proof: Omitted

Alternative version: Under the same conditions we have

$$\sqrt{n}\left\{\frac{\bar{X}_n - \mu}{\sigma}\right\} \xrightarrow{d} N(0,1)$$

To see this let $Y \sim N(0,\sigma^2)$, so that $\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} Y$

Applying the continuous mapping theorem gives

$$\sqrt{n}\left\{\frac{\bar{X}_n - \mu}{\sigma}\right\} \xrightarrow{d} \frac{Y}{\sigma}$$

Clearly Y/σ is normal, with

$$\mathbb{E}\left[\frac{Y}{\sigma}\right] = \frac{1}{\sigma}\mathbb{E}\left[Y\right] = 0 \quad \text{and} \quad \operatorname{var}\left[\frac{Y}{\sigma}\right] = \frac{1}{\sigma^2}\operatorname{var}[Y] = 1$$

Discussion: The CLT tells us about distribution of $\bar{X}_n$ when

- sample is IID
- n large

Informally,

$$\sqrt{n}(\bar{X}_n - \mu) \approx Y \sim N(0, \sigma^2)$$

$$\therefore \quad \bar{X}_n \approx \frac{Y}{\sqrt{n}} + \mu \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

Thus, $\bar{X}_n$ approximately normal, with

- mean equal to μ, and
- variance $\to 0$ at rate proportional to $1/n$

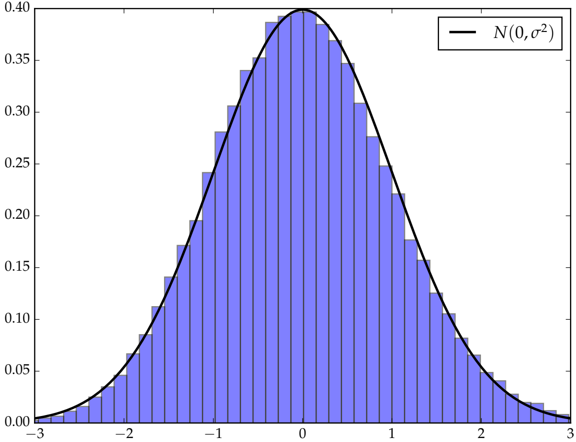
Illustrating the CLT

We can illustrate the CLT with simulations by

1. choosing an arbitrary cdf F for X_n and a large value for n
2. generating independent draws of $Y_n := \sqrt{n}(\bar{X}_n - \mu)$
3. using these draws to compute some measure of their distribution, such as a histogram
4. comparing the latter with $N(0, \sigma^2)$

We do this for

- $F(x) = 1 - e^{-\lambda x}$ (exponential distribution)
- $n = 250$


0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
−3
−2
−1
0
1
2
3
$N(0,\sigma^2)$

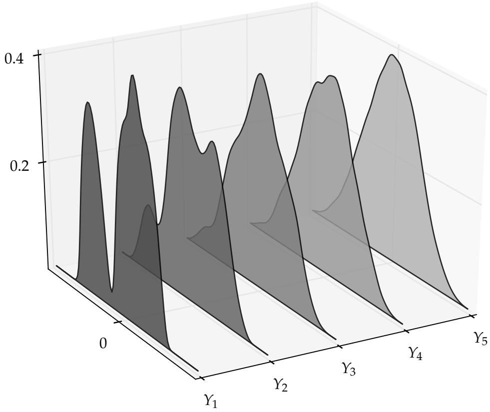
Another way we can illustrate the CLT:

Numerically compute the distributions of

1. $Y_1 = \sqrt{1}(\bar{X}_1 - \mu) = X_1 - \mu$
2. $Y_2 = \sqrt{2}(\bar{X}_2 - \mu) = \sqrt{2}(X_1/2 + X_2/2 - \mu)$
3. $Y_3 = \cdots$

The distribution of each Y_n can be calculated once the distribution F of X_n is specified

The next figure shows these distributions for arbitrarily chosen F


0.4
0.2
0
Y_1
Y_2
Y_3
Y_4
Y_5

Conditional Expectation

Let X and Y be two random variables

To economize on notation we overload the p symbol by writing

- $p(x,y)$ for the joint density
- $p(y \mid x)$ for the conditional density of y given x, etc.

Example. If on a computer we draw

1. $X \sim U[0,1]$
2. and then $Y \sim N(\mu, \sigma^2)$ with μ set to X

then

$$p(y \mid x) = p(y \mid X = x) = N(x, \sigma^2)$$

The **conditional expectation** of Y given X is then defined as

$$\mathbb{E}[Y \mid X] = \int y \, p(y \mid X) dy$$

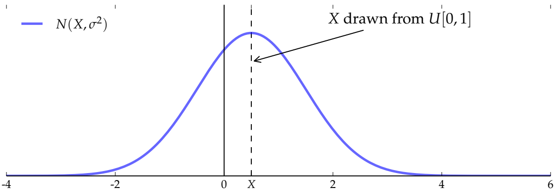
- Notation: Here and below, by convention, $\int := \int_{-\infty}^{\infty}$

The right hand side contains X, so it is a random variable!

In general,

- $\mathbb{E}[Y \mid X]$ is the "best predictor of Y given X"
- A rule that maps X into a prediction of Y
- And therefore a function of X
- And therefore random

Example. As before we draw $X \sim U[0,1]$ and then $Y \sim N(X, \sigma^2)$



We want a rule that maps X to a prediction of Y

Intuition suggests that the best guess of Y given X is just X

Let's make sure this checks out

$$\mathbb{E}[Y \mid X] = \int y\, p(y \mid X) dy$$

For this case we saw that

$$p(y \mid x) = N(x, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ \frac{-(y-x)^2}{2\sigma^2} \right\}$$

$$\therefore \quad p(y \mid X) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ \frac{-(y-X)^2}{2\sigma^2} \right\}$$

$$\therefore \quad \mathbb{E}[Y \mid X] = \int y \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ \frac{-(y-X)^2}{2\sigma^2} \right\} dy$$

This is just the mean of $N(X, \sigma^2)$, which is X

Also intuitive: when X and Y are independent, X is no help in predicting Y

- the same as predicting Y with no information

Since $\mathbb{E}[Y]$ = best guess of Y with no information, this suggests

$$\mathbb{E}[Y \mid X] = \mathbb{E}[Y]$$

The conjecture checks out too, since for this case we have

$$p(y \mid X) = \frac{p(y, X)}{p(X)} = \frac{p(y)p(X)}{p(X)} = p(y)$$

Hence

$$\mathbb{E}[Y \mid X] = \int y p(y \mid X) dy = \int y p(y) dy = \mathbb{E}[Y]$$

Sometimes we want to compute the conditional expectation of a function $f(X, Y)$ depending on both X and Y

Example. Suppose that

- Y is the payoff from a foreign asset, random
- $r(X)$ is an exchange rate, depending on some random X
- return in domestic currency is $f(X, Y) = r(X)Y$

What is the expectation of $f(X, Y)$ given X?

The general definition is

$$\mathbb{E}\left[f(X, Y) \mid X\right] = \int f(X, y) p(y \mid X) dy$$

For the preceding example this gives

$$\mathbb{E}\left[r(X)Y \mid X\right] = \int r(X)\, y\, p(y \mid X) dy$$

Since $r(X)$ doesn't depend on y it can pass out of the integral

Hence

$$\mathbb{E}\left[r(X)Y \mid X\right] = r(X) \int y\, p(y \mid X) dy$$

That is,

$$\mathbb{E}\left[r(X)Y \mid X\right] = r(X)\mathbb{E}\left[Y \mid X\right]$$

This is a general rule — when conditioning on X, RVs depending only on X can be passed out of the expectation

The Multivariate Case

We can condition on $X_1, \ldots, X_K$ using

$$p(y \mid \mathbf{x}) = p(y \mid x_1, x_2, \ldots, x_K)$$
$$= p(y \mid X_1 = x_1, X_2 = x_2, \ldots, X_K = x_K)$$

Then we set

$$\mathbb{E}[Y \mid \mathbf{X}] := \int y\, p(y \mid \mathbf{X}) dy$$
$$= \int y\, p(y \mid X_1, X_2, \ldots, X_K) dy$$

- $\mathbf{X}$ can be a matrix: we condition on all X_{ij} in $\mathbf{X}$

We can also extend the definition the case where $\mathbf{X}$ and $\mathbf{Y}$ are matrices

Given

$$\mathbf{Y} = \begin{pmatrix} Y_{11} & \cdots & Y_{1K} \\ \vdots & \vdots & \vdots \\ Y_{N1} & \cdots & Y_{NK} \end{pmatrix}$$

we set

$$\mathbb{E}\left[\mathbf{Y} \mid \mathbf{X}\right] = \begin{pmatrix} \mathbb{E}\left[Y_{11} \mid \mathbf{X}\right] & \cdots & \mathbb{E}\left[Y_{1K} \mid \mathbf{X}\right] \\ \vdots & \vdots & \vdots \\ \mathbb{E}\left[Y_{N1} \mid \mathbf{X}\right] & \cdots & \mathbb{E}\left[Y_{NK} \mid \mathbf{X}\right] \end{pmatrix}$$

We have provided some intuition for the following key facts

Fact. If $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ are random matrices and $\mathbf{A}$ and $\mathbf{B}$ are constant matrices, then, assuming conformability,

1. $\mathbb{E}\left[\mathbf{AX}+\mathbf{BY} \,|\, \mathbf{Z}\right] = \mathbf{A}\mathbb{E}\left[\mathbf{X} \,|\, \mathbf{Z}\right] + \mathbf{B}\mathbb{E}\left[\mathbf{Y} \,|\, \mathbf{Z}\right]$
2. If $\mathbf{X}$ and $\mathbf{Y}$ are independent, then $\mathbb{E}\left[\mathbf{Y} \,|\, \mathbf{X}\right] = \mathbb{E}\left[\mathbf{Y}\right]$
3. If $G(\mathbf{X})$ is a matrix depending only on $\mathbf{X}$, then
    - $\mathbb{E}\left[G(\mathbf{X})\,\mathbf{Y} \,|\, \mathbf{X}\right] = G(\mathbf{X})\mathbb{E}\left[\mathbf{Y} \,|\, \mathbf{X}\right]$
    - $\mathbb{E}\left[\mathbf{Y}\,G(\mathbf{X}) \,|\, \mathbf{X}\right] = \mathbb{E}\left[\mathbf{Y} \,|\, \mathbf{X}\right] G(\mathbf{X})$
4. $\mathbb{E}\left[\mathbf{Y} \,|\, \mathbf{Z}\right]' = \mathbb{E}\left[\mathbf{Y}' \,|\, \mathbf{Z}\right]$
5. $\mathbb{E}\left[\mathbb{E}\left[\mathbf{Y} \,|\, \mathbf{X}\right]\right] = \mathbb{E}\left[\mathbf{Y}\right]$

Let's just check that $\mathbb{E}\left[\mathbb{E}\left[Y \mid X\right]\right] = \mathbb{E}\left[Y\right]$ in the scalar case

We have

$$
\begin{aligned}
\mathbb{E}\left[\mathbb{E}\left[Y \mid X\right]\right] &= \mathbb{E}\left[\int y\, p(y \mid X) dy\right] \\
&= \int \left[\int y\, p(y \mid x) dy\right] p(x) dx \\
&= \int y \left[\int p(y \mid x) p(x) dx\right] dy \\
&= \int y\, p(y) dy = \mathbb{E}\left[Y\right]
\end{aligned}
$$

New Topic

ANALYSIS

Motivation

We looked at linear systems carefully, but how about nonlinear systems?

- Solving nonlinear equations
- Optimization problems

How are these problems different?

What mathematics do we need to study them?

An example problem:

Let f be a given nonlinear function

Does there exist an $\bar{x}$ such that $f(\bar{x}) = 0$?

Examples.

- F is a profit function, $f = F'$, we're looking for stationary points of the profit function
- We want to solve an equation $g(\bar{x}) = y$ for $\bar{x}$
  - Set $f(x) = g(x) - y$

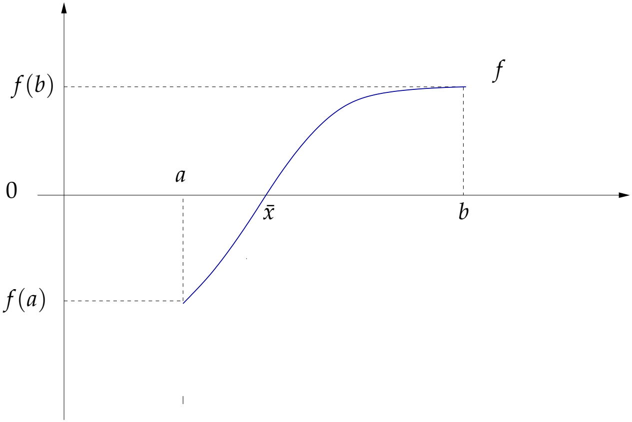


Figure : Existence of a root

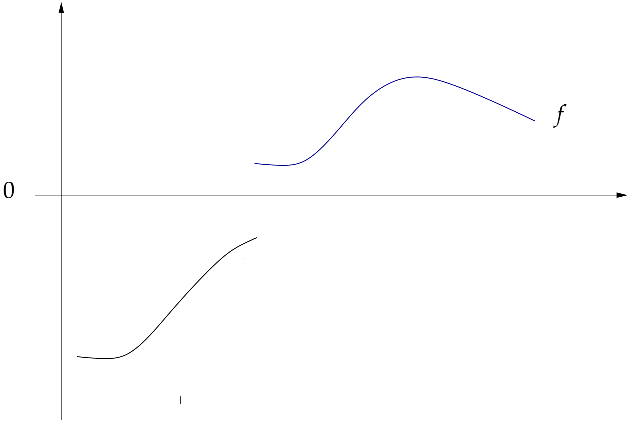


Figure : Non-existence of a root

One answer: a solution exists under certain conditions including continuity

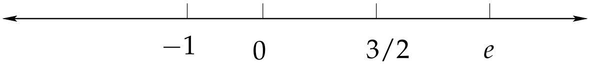
Questions:

- So how can I tell if f is continuous?
- Can we weaken the continuity assumption?
- Does this work in multiple dimensions?
- When is the root unique?
- How can we compute it?
- Etc.

These are typical problems in analysis

Analysis on the Line

Recall that $\mathbb{R}$ denotes the continuous real line



Can be thought of as $\mathbb{Q} \cup \mathbb{I}$ where

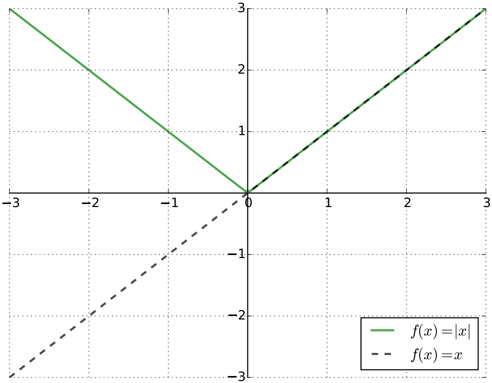
- $\mathbb{Q}$ is the rational numbers
- $\mathbb{I}$ is the irrational numbers

Facts

- Between any two real numbers $a < b$ there exists a rational number
- Between any two real numbers $a < b$ there exists an irrational number

Thus, the rationals and irrationals are "all mixed together"

If $x \in \mathbb{R}$ then $|x| := \max\{x, -x\}$ called its **absolute value**



Fact. For any $x, y \in \mathbb{R}$, the following statements hold

1. $|x| \leq y$ if and only if $-y \leq x \leq y$
2. $|x| < y$ if and only if $-y < x < y$
3. $|x| = 0$ if and only if $x = 0$
4. $|xy| = |x||y|$
5. $|x + y| \leq |x| + |y|$

Last inequality is called the **triangle inequality**

Ex. Show that if $x, y, z \in \mathbb{R}$, then

1. $|x - y| \leq |x| + |y|$
2. $|x - y| \leq |x - z| + |z - y|$

Bounded sets

$A \subset \mathbb{R}$ is called **bounded** if $\exists\, M \in \mathbb{R}$ s.t. $|x| \leq M$, all $x \in A$

Example. (a, b) is bounded for any a, b

$\because$ Each $x \in (a, b)$ satisfies $|x| \leq M := \max\{|a|, |b|\}$

Example. $\mathbb{N}$ is unbounded

$\because$ For any $M \in \mathbb{R}$ there is an n that exceeds it

Example. Every finite subset A of $\mathbb{R}$ is bounded

$\because$ Set $M := \max\{|a| : a \in A\}$

Fact. If A and B are bounded sets then so is $A \cup B$

Proof: Let A and B be bounded sets and let $C := A \cup B$

By definition, $\exists\, M_A$ and M_B with

$$|a| \leq M_A, \text{ all } a \in A, \qquad |b| \leq M_B, \text{ all } b \in B$$

Let $M_C := \max\{M_A, M_B\}$ and fix any $x \in C$

$$x \in C \implies x \in A \text{ or } x \in B$$

$$\therefore \quad |x| \leq M_A \quad \text{or} \quad |x| \leq M_B$$

$$\therefore \quad |x| \leq M_C$$

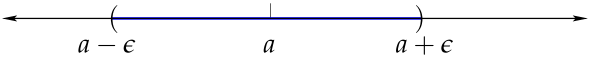
ϵ-balls

Given $\epsilon > 0$ and $a \in \mathbb{R}$, the **ϵ-ball** around a is

$$B_\epsilon(a) := \{x \in \mathbb{R} : |a - x| < \epsilon\}$$

Equivalently,

$$B_\epsilon(a) = \{x \in \mathbb{R} : a - \epsilon < x < a + \epsilon\}$$



Ex. Check equivalence

Fact. If x is in every ϵ-ball around a then $x = a$

Proof:

Suppose to the contrary that

1. x is in every ϵ-ball around a
2. $x \neq a$

Since x is not a we must have $|x - a| > 0$

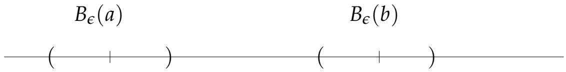
Set $\epsilon := |x - a|$

Since $x \in B_\epsilon(a)$, we have $|x - a| < \epsilon$

That is, $|x - a| < \epsilon = |x - a|$

Contradiction

Fact. If $a \neq b$, then $\exists\, \epsilon > 0$ s.t. $B_\epsilon(a)$ and $B_\epsilon(b)$ are disjoint



Proof: Let $a, b \in \mathbb{R}$ with $a \neq b$, so that $|a - b| > 0$

Set $\epsilon = |a - b|/2$

For this ϵ we can't have $x \in B_\epsilon(a)$ and $x \in B_\epsilon(b)$ because then

$$|x - a| < |a - b|/2 \quad \text{and} \quad |x - b| < |a - b|/2$$

and hence

$$|a - b| \leq |a - x| + |x - b| < |a - b|/2 + |a - b|/2 = |a - b|$$

Contradiction

Sequences

A **sequence** is a function from $\mathbb{N}$ to $\mathbb{R}$

- to each $n \in \mathbb{N}$ we associate one $x_n \in \mathbb{R}$

Typically written as $\{x_n\}_{n=1}^{\infty}$ or $\{x_n\}$ or $\{x_1, x_2, x_3, \ldots\}$

Examples.

- $\{x_n\} = \{2, 4, 6, \ldots\}$
- $\{x_n\} = \{1, 1/2, 1/4, \ldots\}$
- $\{x_n\} = \{1, -1, 1, -1, \ldots\}$
- $\{x_n\} = \{0, 0, 0, \ldots\}$

Sequence $\{x_n\}$ is called

- **bounded** if $\{x_1, x_2, \ldots\}$ is a bounded set
- **monotone increasing** if $x_{n+1} \geq x_n$ for all n
- **monotone decreasing** if $x_{n+1} \leq x_n$ for all n
- **monotone** if it is either monotone increasing or monotone decreasing

Examples.

- $x_n = 1/n$ is monotone decreasing, bounded
- $x_n = (-1)^n$ is not monotone but is bounded
- $x_n = 2n$ is monotone increasing but not bounded