ECON2125/8013

Lecture 6

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Announcements

- None

New Topic

LINEAR ALGEBRA

Motivation

Linear algebra is used to study linear models

Foundational for many disciplines related to economics

- Economic theory
- Econometrics and statistics
- Finance
- Operations research

Example

Equilibrium in a single market with price p

$$q_d = a + bp$$

$$q_s = c + dp$$

$$q_s = q_d$$

What price p clears the market, and at what quantity $q = q_s = q_d$?

Remark: Here a, b, c, d are the model **parameters** or **coefficients**

Treated as fixed for a single computation but might vary between computations to better fit the data

Example

Determination of income

$$C = a + b(Y - T)$$

$$E = C + I$$

$$G = T$$

$$Y = E$$

Solve for Y as a function of I and G

Bigger, more complex systems found in problems related to

- Regression and forecasting
- Portfolio analysis
- Ranking systems
- Etc., etc. — any number of applications

A general system of equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1K}x_K &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2K}x_K &= b_2 \\ &\vdots \\ a_{N1}x_1 + a_{N2}x_2 + \cdots + a_{NK}x_K &= b_N \end{aligned}$$

Typically

- the a_{nm} and b_n are exogenous / given / parameters
- the values x_n are endogenous

Key question

- What values of $x_1, \ldots, x_K$ solve this system?

We often write this in **matrix form**

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1K} \\ a_{21} & a_{22} & \cdots & a_{2K} \\ \vdots & \vdots & & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NK} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_K \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_K \end{pmatrix}$$

or

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

And we solve it on a computer

```
In [1]: import numpy as np

In [2]: from scipy.linalg import solve

In [3]: A = [[0, 2, 4],
   ...:       [1, 4, 8],
   ...:       [0, 3, 7]]

In [4]: b = (1, 2, 0)

In [5]: A, b = np.asarray(A), np.asarray(b)

In [6]: solve(A, b)
Out[6]: array([ 0. ,  3.5, -1.5])
```

This tells us that the solution is

```
array([ 0. ,  3.5, -1.5])
```

That is,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3.5 \\ -1.5 \end{pmatrix}$$

Hey, this is easy — what do we need to study for?

But now let's try this similar looking problem

```
In [1]: import numpy as np

In [2]: from scipy.linalg import solve

In [3]: A = [[0, 2, 4],
   ...:      [1, 4, 8],
   ...:      [0, 3, 6]]

In [4]: b = (1, 2, 0)

In [5]: A, b = np.asarray(A), np.asarray(b)

In [6]: solve(A, b)
```

This is the output that we get

```
LinAlgError          Traceback (most recent call last)
<ipython-input-8-4fb5f41eaf7c> in <module>()
----> 1 solve(A, b)
/home/john/anaconda/lib/python2.7/site-packages/scipy/linal
     97          return x
     98      if info > 0:
---> 99          raise LinAlgError("singular matrix")
    100      raise ValueError('illegal value in %d-th argume
LinAlgError: singular matrix
```

What does this mean? How can we fix it?

Moral: We still need to understand the concepts

Vector Space

Recall that $\mathbb{R}^N :=$ set of all N-vectors

An N-vector $\mathbf{x}$ is a tuple of N real numbers:

$$\mathbf{x} = (x_1, \ldots, x_N) \quad \text{where} \quad x_n \in \mathbb{R} \text{ for each } n$$

We can also write $\mathbf{x}$ vertically, like so:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

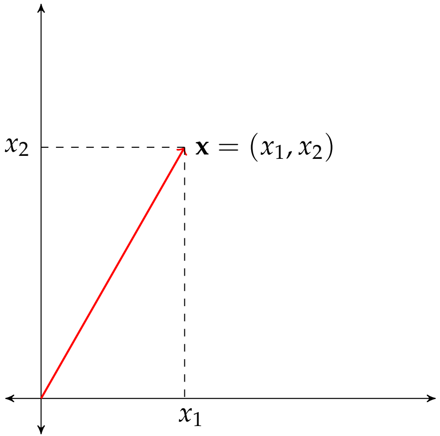


Figure : Visualization of vector $\mathbf{x}$ in $\mathbb{R}^2$

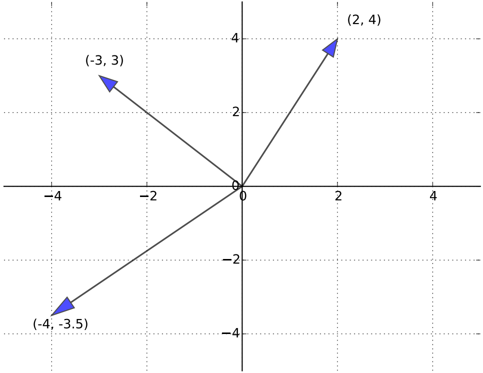


Figure : Three vectors in $\mathbb{R}^2$

The vector of ones will be denoted $\mathbf{1}$

$$\mathbf{1} := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

Vector of zeros will be denoted $\mathbf{0}$

$$\mathbf{0} := \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Linear Operations

Two fundamental algebraic operations:

1. Vector addition
2. Scalar multiplication

1. **Sum** of $\mathbf{x} \in \mathbb{R}^N$ and $\mathbf{y} \in \mathbb{R}^N$ defined by

$$\mathbf{x} + \mathbf{y} =: \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} := \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_N + y_N \end{pmatrix}$$

Example 1:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \\ 6 \\ 8 \end{pmatrix} := \begin{pmatrix} 3 \\ 6 \\ 9 \\ 12 \end{pmatrix}$$

Example 2:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} := \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$$

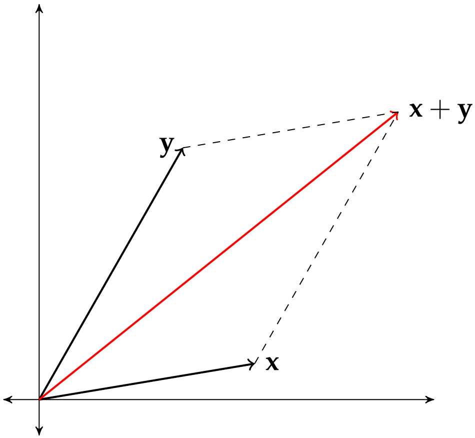


Figure : Vector addition

2. **Scalar product** of $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^N$ defined by

$$\alpha \mathbf{x} = \alpha \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} := \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_N \end{pmatrix}$$

Example 1:

$$0.5 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} := \begin{pmatrix} 0.5 \\ 1.0 \\ 1.5 \\ 2.0 \end{pmatrix}$$

Example 2:

$$-1 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} := \begin{pmatrix} -1 \\ -2 \\ -3 \\ -4 \end{pmatrix}$$

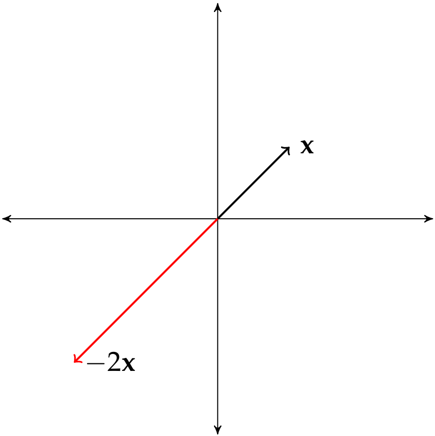


Figure : Scalar multiplication

Subtraction performed element by element, analogous to addition

$$\mathbf{x} - \mathbf{y} := \begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \\ \vdots \\ x_N - y_N \end{pmatrix}$$

Def can be given in terms of addition and scalar multiplication:

$$\mathbf{x} - \mathbf{y} := \mathbf{x} + (-1)\mathbf{y}$$

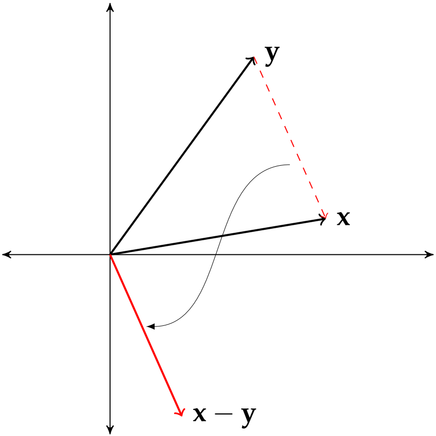


Figure : Difference between vectors

Incidentally, most high level numerical libraries treat vector addition and scalar multiplication in the same way — elementwise

```
In [1]: import numpy as np

In [2]: x = np.array((2, 4, 6))

In [3]: y = np.array((10, 10, 10))

In [4]: x + y  # Vector addition
Out[4]: array([12, 14, 16])

In [6]: 2 * x  # Scalar multiplication
Out[6]: array([ 4,  8, 12])
```

A **linear combination** of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_K$ in $\mathbb{R}^N$ is a vector

$$\mathbf{y} = \sum_{k=1}^{K} \alpha_k \mathbf{x}_k = \alpha_1 \mathbf{x}_1 + \cdots + \alpha_K \mathbf{x}_K$$

where $\alpha_1, \ldots, \alpha_K$ are scalars

Example.

$$0.5 \begin{pmatrix} 6.0 \\ 2.0 \\ 8.0 \end{pmatrix} + 3.0 \begin{pmatrix} 0 \\ 1.0 \\ -1.0 \end{pmatrix} = \begin{pmatrix} 3.0 \\ 4.0 \\ 1.0 \end{pmatrix}$$

Inner Product

The **inner product** of two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^N$ is

$$\mathbf{x}'\mathbf{y} := \sum_{n=1}^{N} x_n y_n$$

Example: $\mathbf{x} = (2,3)$ and $\mathbf{y} = (-1,1)$ implies that

$$\mathbf{x}'\mathbf{y} = 2 \times (-1) + 3 \times 1 = 1$$

Example: $\mathbf{x} = (1/N)\mathbf{1}$ and $\mathbf{y} = (y_1, \ldots, y_N)$ implies

$$\mathbf{x}'\mathbf{y} = \frac{1}{N} \sum_{n=1}^{N} y_n$$

```
In [1]: import numpy as np

In [2]: x = np.array((1, 2, 3, 4))

In [3]: y = np.array((2, 4, 6, 8))

In [6]: np.sum(x * y)  # Inner product
Out[6]: 60
```

Fact. For any $\alpha, \beta \in \mathbb{R}$ and any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$, the following statements are true:

1. $\mathbf{x}'\mathbf{y} = \mathbf{y}'\mathbf{x}$
2. $(\alpha\mathbf{x})'(\beta\mathbf{y}) = \alpha\beta(\mathbf{x}'\mathbf{y})$
3. $\mathbf{x}'(\mathbf{y} + \mathbf{z}) = \mathbf{x}'\mathbf{y} + \mathbf{x}'\mathbf{z}$

For example, item 2 is true because

$$(\alpha\mathbf{x})'(\beta\mathbf{y}) = \sum_{n=1}^{N} \alpha x_n \beta y_n = \alpha\beta \sum_{n=1}^{N} x_n y_n = \alpha\beta(\mathbf{x}'\mathbf{y})$$

Ex. Use above rules to show that $(\alpha\mathbf{y} + \beta\mathbf{z})'\mathbf{x} = \alpha\mathbf{x}'\mathbf{y} + \beta\mathbf{x}'\mathbf{z}$

The next result is a generalization

Fact. Inner products of linear combinations satisfy

$$\left(\sum_{k=1}^{K} \alpha_k \mathbf{x}_k\right)' \left(\sum_{j=1}^{J} \beta_j \mathbf{y}_j\right) = \sum_{k=1}^{K} \sum_{j=1}^{J} \alpha_k \beta_j \mathbf{x}_k' \mathbf{y}_j$$

Norms and Distance

The (Euclidean) **norm** of $\mathbf{x} \in \mathbb{R}^N$ is defined as

$$\|\mathbf{x}\| := \sqrt{\mathbf{x}'\mathbf{x}} = \left(\sum_{n=1}^{N} x_n^2 \right)^{1/2}$$

Interpretation:

- $\|\mathbf{x}\|$ represents the "length" of $\mathbf{x}$
- $\|\mathbf{x} - \mathbf{y}\|$ represents distance between $\mathbf{x}$ and $\mathbf{y}$

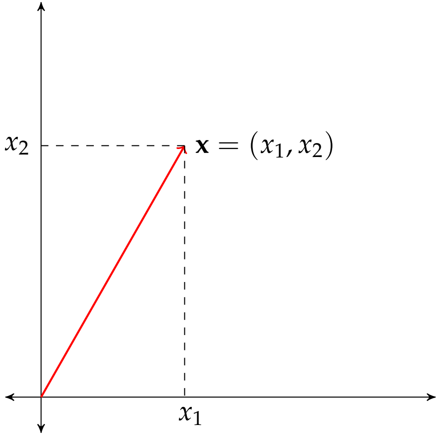
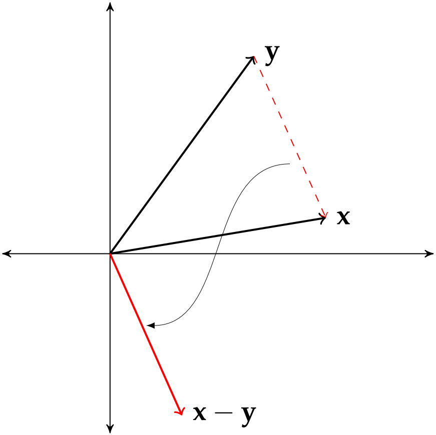


Figure : Length of red line $= \sqrt{x_1^2 + x_2^2} =: \|\mathbf{x}\|$

$\|\mathbf{x} - \mathbf{y}\|$ represents distance between $\mathbf{x}$ and $\mathbf{y}$



Fact. For any $\alpha \in \mathbb{R}$ and any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$, the following statements are true:

1. $\|\mathbf{x}\| \geq 0$ and $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$
2. $\|\alpha \mathbf{x}\| = |\alpha| \|\mathbf{x}\|$
3. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (**triangle inequality**)
4. $|\mathbf{x}'\mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|$ (**Cauchy-Schwarz inequality**)

For example, let's show that $\|\mathbf{x}\| = 0 \iff \mathbf{x} = \mathbf{0}$

First let's assume that $\|\mathbf{x}\| = 0$ and show $\mathbf{x} = \mathbf{0}$

Since $\|\mathbf{x}\| = 0$ we have $\|\mathbf{x}\|^2 = 0$ and hence $\sum_{n=1}^{N} x_n^2 = 0$

That is $x_n = 0$ for all n, or, equivalently, $\mathbf{x} = \mathbf{0}$

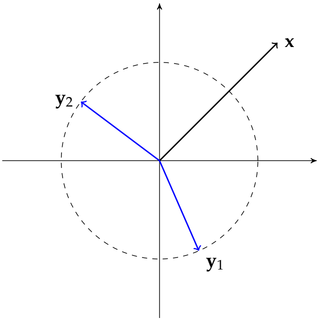
Next let's assume that $\mathbf{x} = \mathbf{0}$ and show $\|\mathbf{x}\| = 0$

This is immediate from the definition of the norm

Fact. If $\mathbf{x} \in \mathbb{R}^N$ is nonzero, then the solution to the optimization problem

$$\max_{\mathbf{y}} \mathbf{x}'\mathbf{y} \quad \text{subject to} \quad \mathbf{y} \in \mathbb{R}^N \text{ and } \|\mathbf{y}\| = 1$$

is $\hat{\mathbf{x}} := \mathbf{x} / \|\mathbf{x}\|$



Proof: Fix nonzero $\mathbf{x} \in \mathbb{R}^N$

Let $\hat{\mathbf{x}} := \mathbf{x} / \|\mathbf{x}\| := \alpha \mathbf{x}$ when $\alpha := 1 / \|\mathbf{x}\|$

Evidently $\|\hat{\mathbf{x}}\| = 1$

Pick any other $\mathbf{y} \in \mathbb{R}^N$ satisfying $\|\mathbf{y}\| = 1$

The Cauchy-Schwarz inequality yields

$$\mathbf{y}'\mathbf{x} \leq |\mathbf{y}'\mathbf{x}| \leq \|\mathbf{y}\| \|\mathbf{x}\| = \|\mathbf{x}\| = \frac{\mathbf{x}'\mathbf{x}}{\|\mathbf{x}\|} = \hat{\mathbf{x}}'\mathbf{x}$$

Hence $\hat{\mathbf{x}}$ is the maximizer, as claimed

Span

Let $X \subset \mathbb{R}^N$ be any nonempty set

Set of all possible linear combinations of elements of X is called the **span** of X, denoted by $\mathrm{span}(X)$

For finite $X := \{\mathbf{x}_1, \ldots, \mathbf{x}_K\}$ the span can be expressed as

$$\mathrm{span}(X) := \left\{ \text{all } \sum_{k=1}^{K} \alpha_k \mathbf{x}_k \text{ such that } (\alpha_1, \ldots, \alpha_K) \in \mathbb{R}^K \right\}$$

We are mainly interested in the span of finite sets...

Example

Let's start with the span of a singleton

Let $X = \{\mathbf{1}\} \subset \mathbb{R}^2$, where $\mathbf{1} := (1, 1)$

The span of X is all vectors of the form

$$\alpha \mathbf{1} = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix} \quad \text{with} \quad \alpha \in \mathbb{R}$$

Constitutes a line in the plane that passes through

- the vector $\mathbf{1}$ (set $\alpha = 1$)
- the origin $\mathbf{0}$ (set $\alpha = 0$)

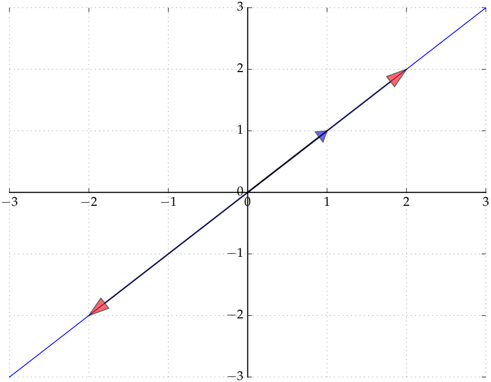


Figure : The span of $\mathbf{1} := (1, 1)$ in $\mathbb{R}^2$

Example

Let $\mathbf{x}_1 = (3, 4, 2)$ and let $\mathbf{x}_2 = (3, -4, 0.4)$

By definition, the span is all vectors of the form

$$\mathbf{y} = \alpha \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ -4 \\ 0.4 \end{pmatrix} \quad \text{where} \quad \alpha, \beta \in \mathbb{R}$$

It turns out to be a plane that passes through

- the vector $\mathbf{x}_1$
- the vector $\mathbf{x}_2$
- the origin $\mathbf{0}$

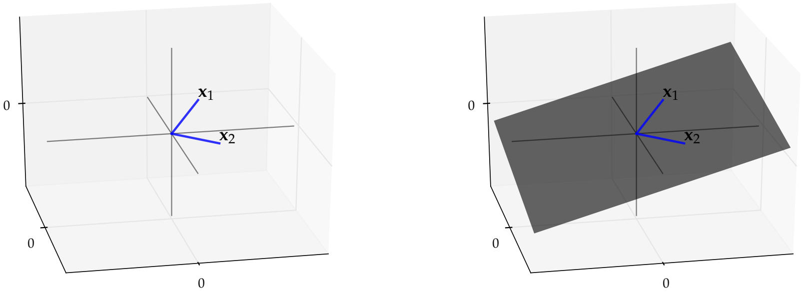


Figure : Span of $\mathbf{x}_1, \mathbf{x}_2$

Fact. If $X \subset Y$, then $\text{span}(X) \subset \text{span}(Y)$

To see this, pick any nonempty $X \subset Y \subset \mathbb{R}^N$

Letting $\mathbf{z} \in \text{span}(X)$, we have

$$\mathbf{z} = \sum_{k=1}^{K} \alpha_k \mathbf{x}_k \text{ for some } \mathbf{x}_1, \ldots, \mathbf{x}_K \in X, \; \alpha_1, \ldots, \alpha_K \in \mathbb{R}$$

Since $X \subset Y$, each $\mathbf{x}_k$ is also in Y, giving us

$$\mathbf{z} = \sum_{k=1}^{K} \alpha_k \mathbf{x}_k \text{ for some } \mathbf{x}_1, \ldots, \mathbf{x}_K \in Y, \; \alpha_1, \ldots, \alpha_K \in \mathbb{R}$$

Hence $\mathbf{z} \in \text{span}(Y)$

Let Y be any subset of $\mathbb{R}^N$, and let $X := \{\mathbf{x}_1, \ldots, \mathbf{x}_K\}$

If $Y \subset \mathrm{span}(X)$, we say that the vectors in X **span the set** Y

Alternatively, we say that X is a **spanning set** for Y

A nice situation: Y is large but X is small

$\implies$ large set Y "described" by the small number of vectors in X

Example

Consider the vectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\} \subset \mathbb{R}^N$, where

$$\mathbf{e}_1 := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 := \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \mathbf{e}_N := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

That is, $\mathbf{e}_n$ has all zeros except for a 1 as the n-th element

Vectors $\mathbf{e}_1, \ldots, \mathbf{e}_N$ called the **canonical basis vectors** of $\mathbb{R}^N$

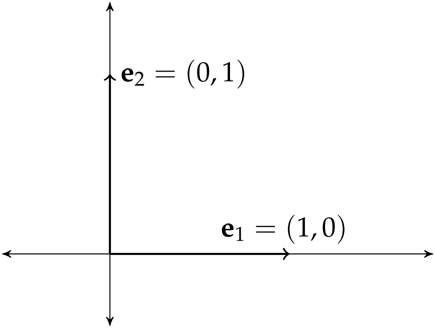


Figure : Canonical basis vectors in $\mathbb{R}^2$

Fact. The span of $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ is equal to all of $\mathbb{R}^N$

Proof for $N = 2$:

Pick any $\mathbf{y} \in \mathbb{R}^2$

We have

$$\mathbf{y} := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y_1 \end{pmatrix}$$

$$= y_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = y_1 \mathbf{e}_1 + y_2 \mathbf{e}_2$$

Thus, $\mathbf{y} \in \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2\}$

Since $\mathbf{y}$ arbitrary, we have shown that $\operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2\} = \mathbb{R}^2$

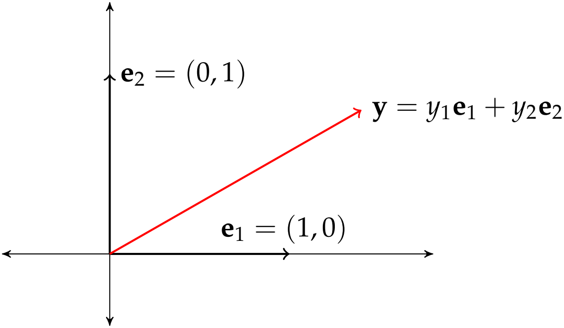
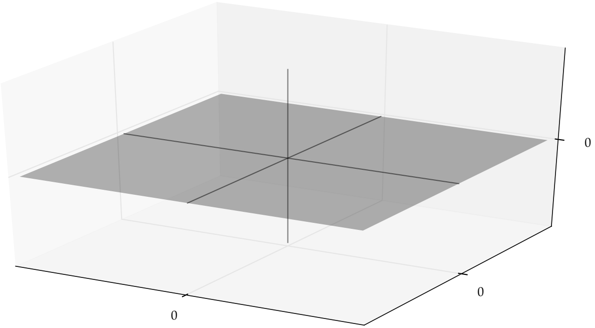


Figure : Canonical basis vectors in $\mathbb{R}^2$

Example. Consider the set

$$P := \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1, x_2 \in \mathbb{R}\}$$

Graphically, $P =$ flat plane in $\mathbb{R}^3$, where height coordinate $= 0$



Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be the canonical basis vectors in $\mathbb{R}^3$

Claim: $\text{span}\{\mathbf{e}_1, \mathbf{e}_2\} = P$

Proof:

Let $\mathbf{x} = (x_1, x_2, 0)$ be any element of P

We can write $\mathbf{x}$ as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$$

In other words, $P \subset \text{span}\{\mathbf{e}_1, \mathbf{e}_2\}$

Conversely (check it) we have $\text{span}\{\mathbf{e}_1, \mathbf{e}_2\} \subset P$

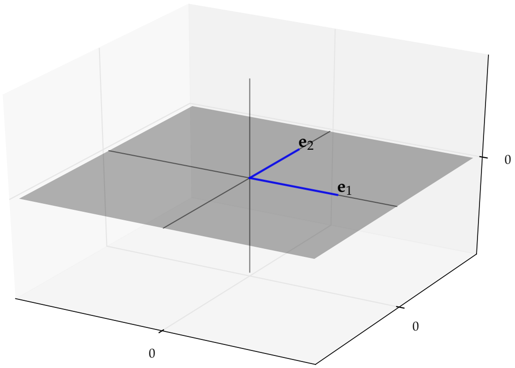


Figure : $\mathrm{span}\{\mathbf{e}_1, \mathbf{e}_2\} = P$

Linear Subspaces

A nonempty $S \subset \mathbb{R}^N$ called a **linear subspace** of $\mathbb{R}^N$ if

$$\mathbf{x}, \mathbf{y} \in S \text{ and } \alpha, \beta \in \mathbb{R} \quad \Longrightarrow \quad \alpha\mathbf{x} + \beta\mathbf{y} \in S$$

In other words, $S \subset \mathbb{R}^N$ is "closed" under vector addition and scalar multiplication

Note: Sometimes we just say **subspace**...

Example. $\mathbb{R}^N$ itself is a linear subspace of $\mathbb{R}^N$

Example

Fix $\mathbf{a} \in \mathbb{R}^N$ and let $A := \{\mathbf{x} \in \mathbb{R}^N : \mathbf{a}'\mathbf{x} = 0\}$

Fact. The set A is a linear subspace of $\mathbb{R}^N$

Proof: Let $\mathbf{x}, \mathbf{y} \in A$ and let $\alpha, \beta \in \mathbb{R}$

We must show that $\mathbf{z} := \alpha\mathbf{x} + \beta\mathbf{y} \in A$

Equivalently, that $\mathbf{a}'\mathbf{z} = 0$

True because

$$\mathbf{a}'\mathbf{z} = \mathbf{a}'(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha\mathbf{a}'\mathbf{x} + \beta\mathbf{a}'\mathbf{y} = 0 + 0 = 0$$

Fact. If Z is a nonempty subset of $\mathbb{R}^N$, then $\mathrm{span}(Z)$ is a linear subspace

Proof: If $\mathbf{x}, \mathbf{y} \in \mathrm{span}(Z)$, then $\exists$ vectors $\mathbf{z}_k$ in Z and scalars γ_k and δ_k such that

$$\mathbf{x} = \sum_{k=1}^{K} \gamma_k \mathbf{z}_k \quad \text{and} \quad \mathbf{y} = \sum_{k=1}^{K} \delta_k \mathbf{z}_k$$

$$\therefore \quad \alpha \mathbf{x} = \sum_{k=1}^{K} \alpha \gamma_k \mathbf{z}_k \quad \text{and} \quad \beta \mathbf{y} = \sum_{k=1}^{K} \beta \delta_k \mathbf{z}_k$$

$$\therefore \quad \alpha \mathbf{x} + \beta \mathbf{y} = \sum_{k=1}^{K} (\alpha \gamma_k + \beta \delta_k) \mathbf{z}_k$$

This vector clearly lies in $\mathrm{span}(Z)$

Fact. If S and S' are two linear subspaces of $\mathbb{R}^N$, then $S \cap S'$ is also a linear subspace of $\mathbb{R}^N$.

Proof: Let S and S' be two linear subspaces of $\mathbb{R}^N$

Fix $\mathbf{x}, \mathbf{y} \in S \cap S'$ and $\alpha, \beta \in \mathbb{R}$

We claim that $\mathbf{z} := \alpha \mathbf{x} + \beta \mathbf{y} \in S \cap S'$

- Since $\mathbf{x}, \mathbf{y} \in S$ and S is a linear subspace we have $\mathbf{z} \in S$
- Since $\mathbf{x}, \mathbf{y} \in S'$ and S' is a linear subspace we have $\mathbf{z} \in S'$

Therefore $\mathbf{z} \in S \cap S'$

Other examples of linear subspaces

- The singleton $\{\mathbf{0}\}$ in $\mathbb{R}^N$
- Lines through the origin in $\mathbb{R}^2$ and $\mathbb{R}^3$
- Planes through the origin in $\mathbb{R}^3$

Ex. Let S be a linear subspace of $\mathbb{R}^N$. Show that

1. $\mathbf{0} \in S$
2. If $X \subset S$, then $\operatorname{span}(X) \subset S$
3. $\operatorname{span}(S) = S$