ECON2125/4021/8013

Lecture 22

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Formal Concepts and Definitions

Formalization — what's a difference equation?

A **dynamical system** is a pair (S, g), where

1. S is a nonempty subset of $\mathbb{R}^K$
2. g is a function mapping S into itself (a **self-mapping** on S)

These objects are used to represent the difference equation

$$\mathbf{x}_{t+1} = g(\mathbf{x}_t) \quad \text{where} \quad g \colon S \to S$$

The set S is called the **state space**

The function g is called the **transition rule** or **law of motion**

Example. Let $g(k) = sAk^{\alpha} + (1-\delta)k$ with

- $A > 0$
- $0 < s, \alpha, \delta < 1$

The pair $([0,\infty), g)$ is a dynamical system

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Example. Let $g\colon x \mapsto 2x$

The pair $([0,1], g)$ is not a dynamical system

For example, $g(1) = 2 \notin [0,1]$

(Hence g is not a self-mapping on $[0,1]$)

Let (S, g) be a dynamical system and consider the sequence generated recursively by

$$\mathbf{x}_{t+1} = g(\mathbf{x}_t), \quad \text{where } \mathbf{x}_0 = \text{ some given point in } S$$

Not that for this sequence we have

$$\mathbf{x}_2 = g(\mathbf{x}_1) = g(g(\mathbf{x}_0)) =: g^2(\mathbf{x}_0)$$

and, more generally,

$$\mathbf{x}_t = g^t(\mathbf{x}_0) \quad \text{where} \quad g^t = \underbrace{g \circ g \circ \cdots \circ g}_{t \text{ compositions of } g}$$

The sequence $\{g^t(\mathbf{x}_0)\}_{t \geq 0}$ is called the **trajectory** of $\mathbf{x}_0 \in S$

We will also call it a **time series**

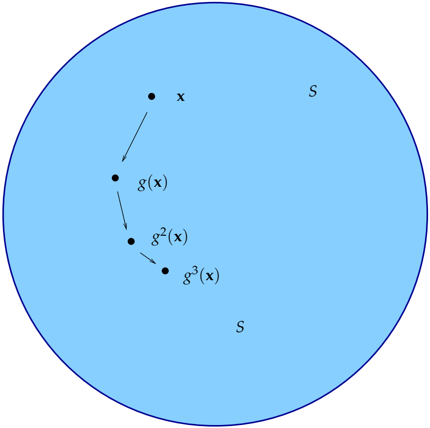

Figure : The trajectory of $\mathbf{x}$ under g

Fact. If g is increasing on S and $S \subset \mathbb{R}$, then every trajectory is monotone (either increasing or decreasing)

Proof: Pick any $x \in S$

Either $x \leq g(x)$ or $g(x) \leq x$ — let's treat the first case

Since g is increasing and $x \leq g(x)$ we have $g(x) \leq g^2(x)$

Putting these inequalities together gives

$$x \leq g(x) \leq g^2(x)$$

Continuing in this way gives

$$x \leq g(x) \leq g^2(x) \leq g^3(x) \leq \cdots$$

Steady States

Let (S, g) be a dynamical system

Suppose that $\mathbf{x}^*$ is a fixed point of g, so that

$$g(\mathbf{x}^*) = \mathbf{x}^*$$

Then, for any trajectory $\{\mathbf{x}_t\}$ generated by g,

$$\mathbf{x}_t = \mathbf{x}^* \quad \implies \quad \mathbf{x}_{t+1} = g(\mathbf{x}_t) = g(\mathbf{x}^*) = \mathbf{x}^*$$

In other words, if we ever get to $\mathbf{x}^*$ we stay there

As a result, in this context, a fixed point of g in S is also called a **steady state**

- Just a fixed point, not a new concept mathematically

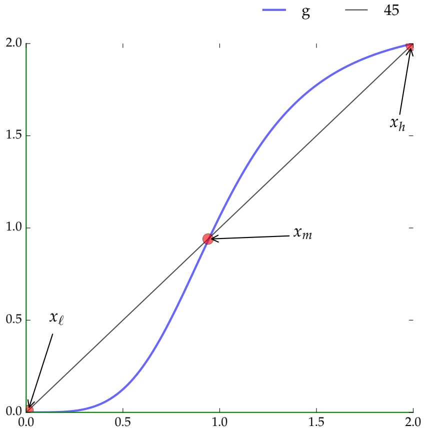

Figure : Steady states of $g(x) = 2.125/(1+x^{-4})$ and $g(0) = 0$

Example. Recall the Solow-Swan growth model

$$k_{t+1} = g(k_t) \quad \text{where} \quad g(k) := sAk^{\alpha} + (1-\delta)k$$

Assume that

1. $S = (0, \infty)$
2. $A > 0$ and $0 < s, \alpha, \delta < 1$

The system (S, g) has a steady state given by the solution to

$$k = sAk^{\alpha} + (1-\delta)k$$

Ex. Solve this equation for k to get steady state

$$k^* := \left(\frac{sA}{\delta}\right)^{1/(1-\alpha)}$$

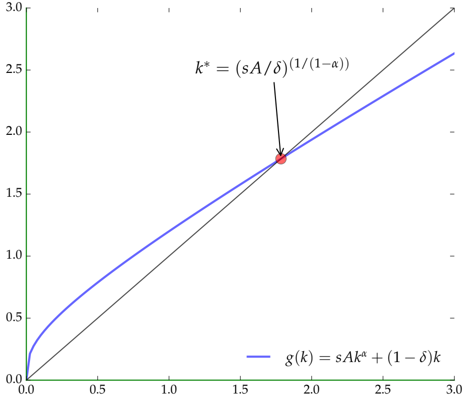

Figure : Steady state of the Solow model

Example. Let's modify the Solow-Swan model to

$$k_{t+1} = g(k_t) \quad \text{where} \quad g(k) = sA(k)k^{\alpha} + (1-\delta)k$$

In the Azariadis-Drazen growth model A takes the form

$$A(k) = \begin{cases} A_1 & \text{if } 0 < k < k_b \\ A_2 & \text{if } k_b \leq k < \infty \end{cases}$$

The value k_b is a "threshold" value of capital stock

- Assume $0 < A_1 < A_2$, so more productive above k_b
- As usual, $0 < s, \alpha, \delta < 1$

This is a dynamical system with

- $S = (0, \infty)$
- $g(k) = sA(k)k^\alpha + (1 - \delta)k$

Let

$$k_i^* := \left(\frac{sA_i}{\delta}\right)^{1/(1-\alpha)} \quad \text{for} \quad i = 1, 2$$

Suppose that $k_1^* < k_b < k_2^*$

Ex. Show that (S, g) has two steady states, given by k_1^* and k_2^*

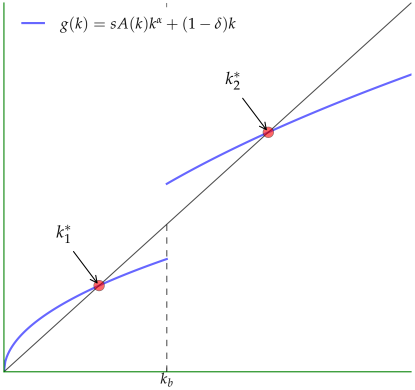

Figure : The threshold model when $k_1^* < k_b < k_2^*$

Stability: Intuition

In some settings trajectories converge

Example. Graphical analysis suggests all trajectories converge for the Solow-Swan model (see above)

Let's look at some more pictures illustrating stability

We focus on the system (S, g) where $S = [0, 2]$ and

$$g(x) = \begin{cases} 2.125/(1 + x^{-4}) & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

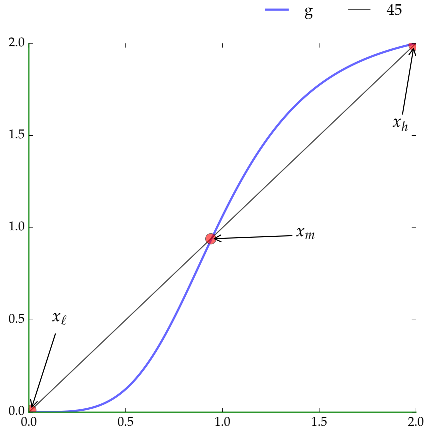

Figure : Steady states of $g(x) = 2.125/(1 + x^{-4})$ and $g(0) = 0$

These steady states appear to have different stability properties

1. x_ℓ is "locally stable"
   - nearby points converge to it
2. x_m is "unstable"
   - nearby points diverge from it
3. x_h is "locally stable"
   - nearby points converge to it

The "basin of attraction" for

- x_ℓ is $[x_\ell, x_m)$
- x_h is $(x_m, x_h]$

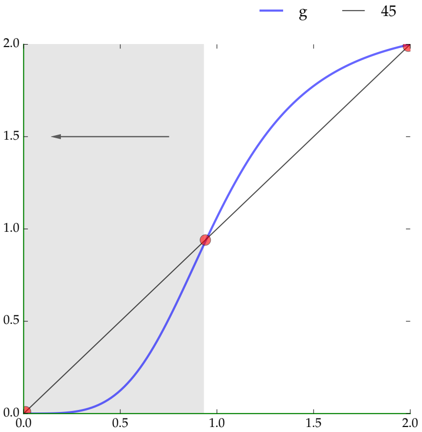

Figure : Basin of attraction for x_ℓ

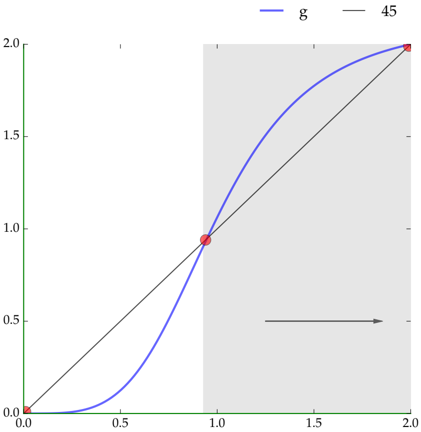

Figure : Basin of attraction for x_h

Let's try to formalize these ideas...

Local Stability

Let $\mathbf{x}^*$ be a fixed point of g on S

The **stable set** of $\mathbf{x}^*$ is

$$\mathcal{O}(\mathbf{x}^*) := \{\mathbf{x} \in S : g^t(\mathbf{x}) \to \mathbf{x}^* \text{ as } t \to \infty\}$$

This set is nonempty (why?)

The point $\mathbf{x}^*$ called **locally stable** or an **attractor** if there exists an $\epsilon > 0$ such that

$$\mathbf{x} \in S \text{ and } \|\mathbf{x} - \mathbf{x}^*\| < \epsilon \implies \mathbf{x} \in \mathcal{O}(\mathbf{x}^*)$$

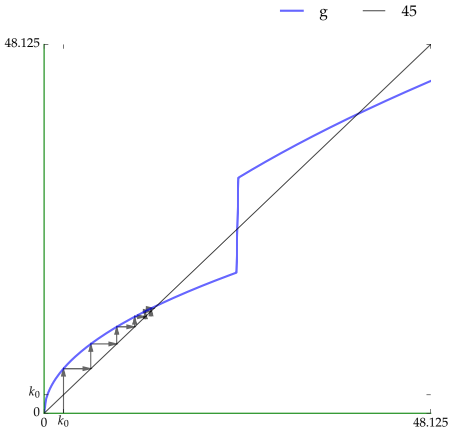

Figure : A poverty trap in the Azariadis–Drazen threshold model

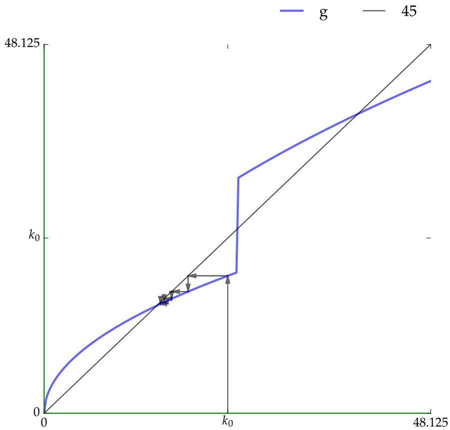

Figure : A poverty trap in the Azariadis–Drazen threshold model

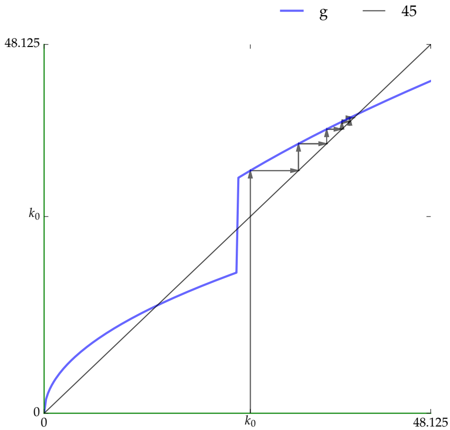

Figure : The higher steady state is also an attractor

Let $S \subset \mathbb{R}$ and let $x^* \in S$ be a steady state of (S, g)

Fact. If g is continuously differentiable at x^* and $|g'(x^*)| < 1$, then x^* is locally stable for (S, g)

Proof (omitted) shows that g is "locally a contraction" near x^* under this condition

Ex. Recall the Azariadis-Drazen growth model with steady states

$$k_i^* := \left(\frac{sA_i}{\delta}\right)^{1/(1-\alpha)} \quad \text{for} \quad i = 1, 2$$

Under the assumptions given above, show that k_1^* and k_2^* are both locally stable

Global Stability

Dynamical system (S, g) is called **globally stable** if

1. g has a fixed point $\mathbf{x}^*$ in S
2. $\mathbf{x}^*$ is the only fixed point of g in S
3. $g^t(\mathbf{x}) \to \mathbf{x}^*$ as $t \to \infty$ for all $\mathbf{x} \in S$

Example. If g is a contraction mapping and S closed then (S, g) globally stable

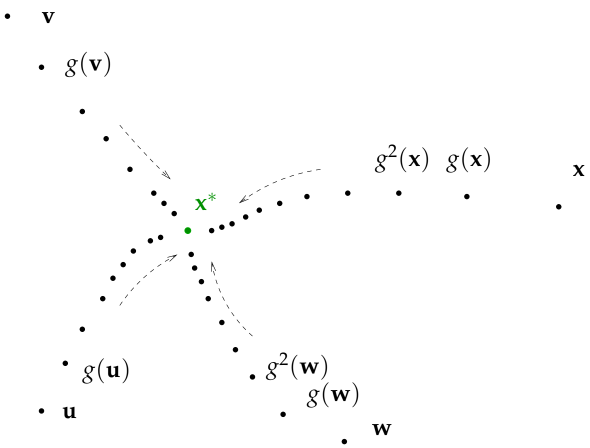

Figure : A contraction mapping

Example. Recall the Solow-Swan growth model where

$$k_{t+1} = g(k_t) \quad \text{for} \quad g(k) = sAk^\alpha + (1-\delta)k$$

with

1. $S = (0, \infty)$
2. $A > 0$ and $0 < s, \alpha, \delta < 1$

The system (S, g) is globally stable with unique fixed point

$$k^* := \left(\frac{sA}{\delta}\right)^{1/(1-\alpha)}$$

Proof: Simple algebra shows that for $k > 0$ we have

$$k = sAk^{\alpha} + (1-\delta)k \quad \iff \quad k = \left(\frac{sA}{\delta}\right)^{1/(1-\alpha)}$$

Hence (S, g) has unique steady state k^*

It remains to show that $g^t(k) \to k^*$ for every $k \in S := (0, \infty)$

Let's show this for any $k \leq k^*$, leaving $k^* \leq k$ as an exercise

Since calculating $g^t(k)$ directly is messy, let's try another strategy

Claim: If $0 < k \leq k^*$, then $\{g^t(k)\}$ is increasing and bounded

Proof increasing: Since g increasing $\{g^t(k)\}$ is monotone

From $k \leq k^*$ and some algebra (exercise) we get

$$k \leq \left(\frac{sA}{\delta}\right)^{1/(1-\alpha)} \implies g(k) \geq k \implies \{g^t(k)\} \text{ increasing}$$

Proof bounded: From $k \leq k^*$ and the fact that g is increasing,

$$g(k) \leq g(k^*) = k^*$$

Applying g to both sides gives $g^2(k) \leq k^*$ and so on

Hence both bounded and increasing

To complete the proof we use the following fact

Fact. If $g^t(k) \to \hat{k}$ for some $k, \hat{k} \in S$ and g is continuous at $\hat{k}$, then $\hat{k}$ is a fixed point of g

Now fix $k \leq k^*$ and recall that $\{g^t(k)\}$ is bounded, increasing

Hence $g^t(k) \to \hat{k}$ for some $\hat{k} \in S$

Because g is continuous, we know that $\hat{k}$ is a fixed point

But k^* is the only fixed point of $k = g(k)$ as discussed above

Hence $\hat{k} = k^*$

In other words, $g^t(k) \to k^*$ as claimed

Example. Consider again the Solow-Swan growth model

$$k_{t+1} = g(k_t) \quad \text{for} \quad g(k) := sAk^{\alpha} + (1 - \delta)k$$

where parameters are as before

If $S = [0, \infty)$ then the same model (S, g) is not globally stable

- We showed above that g has a fixed point k^* in $(0, \infty)$
- However, 0 is also a fixed point of g on $[0, \infty)$
- Hence (S, g) has two steady states in $S = [0, \infty)$

Moral: The state space matters for dynamic properties

Periodic Points and Cycles

If $\mathbf{x}^*$ is a steady state of (S, g) then

$$g^k(\mathbf{x}^*) = \mathbf{x}^* \quad \text{for all } k \in \mathbb{N}$$

However, some (S, g) have points $\mathbf{x}^*$ such that

$$g^k(\mathbf{x}^*) = \mathbf{x}^* \quad \text{for some but not all } k \in \mathbb{N}$$

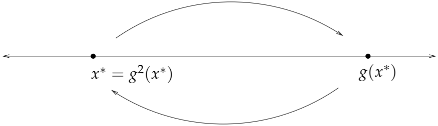

Figure : Here $g(x^*) \neq x^*$ but $g^2(x^*) = x^*$

A point $\mathbf{x}^* \in S$ is called **periodic** for dynamical system (S, g) if

$$g^k(\mathbf{x}^*) = \mathbf{x}^* \quad \text{for some } k \in \mathbb{N}$$

Example. Every steady state of (S, g) is periodic (set $k = 1$)

Example. If $S = \mathbb{R}$ and $g(x) = -x$ then 1 is periodic because

$$g^2(1) = g(g(1)) = -(-1) = 1$$

The **period** of $\mathbf{x}^*$ is the smallest $k \in \mathbb{N}$ such that $g^k(\mathbf{x}^*) = \mathbf{x}^*$

Example. In the previous example, 1 has period 2

Example. Let $S = [0,1]$ and let g be the **logistic** map

$$g(x) = 3.5x(1-x)$$

The second composition g^2 has the form

$$\begin{aligned} g^2(x) &= 3.5g(x)(1-g(x)) \\ &= 3.5^2 x(1-x)(1-3.5x(1-x)) \end{aligned}$$

It has two fixed points that are not fixed points of g

These points are periodic with period 2

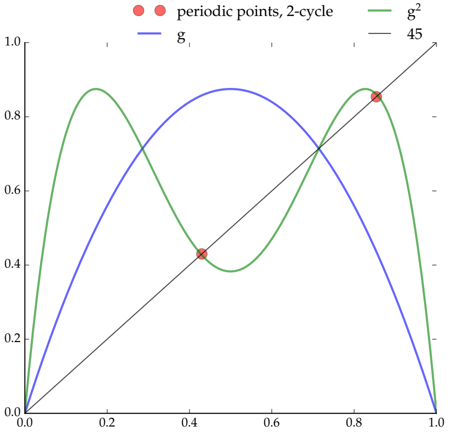

Figure : Logistic map $g(x) = 3.5x(1-x)$ and second iterate g^2

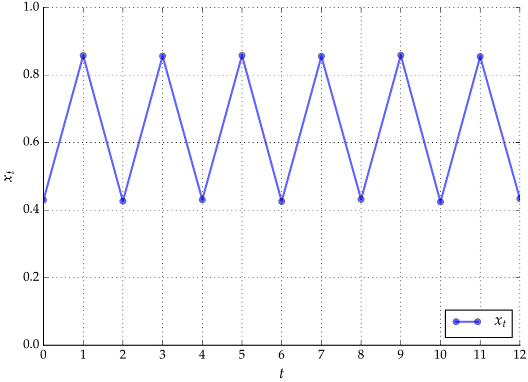

Figure : Time series of logistic map $g(x) = 3.5x(1 - x)$

Chaotic Dynamics

Some simple systems generate complicated time series

Classic example is (some of) the logistic maps

These are systems of the form (S, g) where $S := [0, 1]$ and

$$g(x) = rx(1 - x), \qquad r \in [0, 4] \tag{1}$$

Arise mainly in biological models

Let's consider the case $r = 4$

Then almost all starting points generate "complicated" trajectories

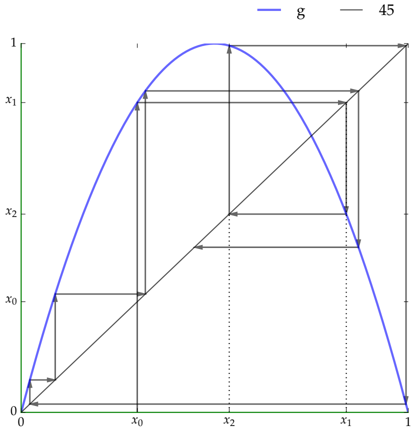

Figure : Logistic map $g(x) = 4x(1-x)$ with $x_0 = 0.3$

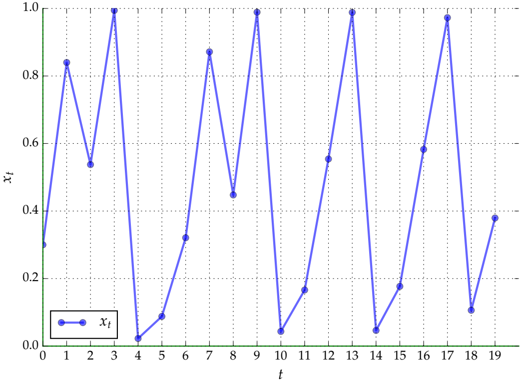

Figure : The corresponding time series

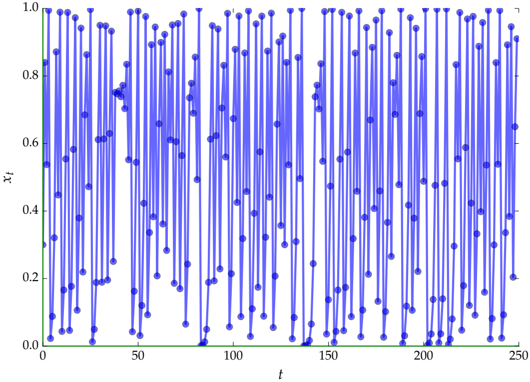

Figure : A longer time series

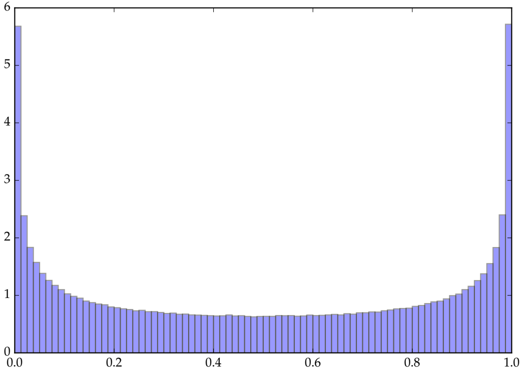

Figure : A long time series, histogram of values

How does the logisitic map behave when we let the multiplicative parameter take values other than 4?

Consider the more general map

$$h(x) = rx(1-x), \quad 0 \leq r \leq 4$$

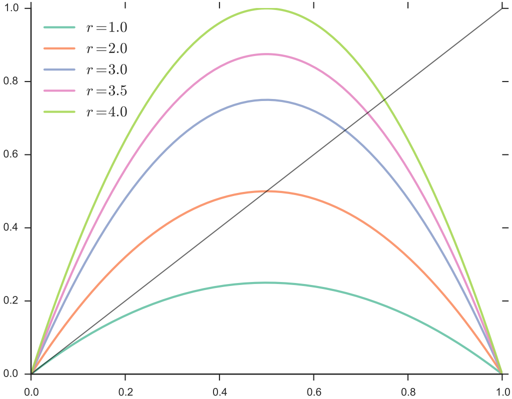

Figure : Logistic maps, $r \in [0, 4]$

For some values of r this system is globally stable and for others, like 4, the behavior is highly complex.

Next slide shows a "bifurcation diagram" which helps to understand long-run behavior at each r.

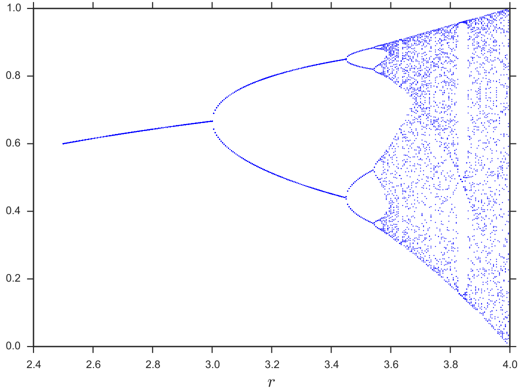

Figure : Bufircation Diagram

Incidentally, what's the difference between "chaotic" and "random" sequences

According to A.N. Kolmogorov, the difference is in degree to which they can be compressed

Degree of compression means shortest "program" that generates them

A chaotic sequence can be compressed without loss of information

The information in $\{x_t\}$ is summarized in g and x_0

A truely random sequence $\{z_t\}$ cannot be reduced to an algorithm in the same way

To store the information in $\{z_t\}$ we need to store the sequence