ECON2125/8013

Lecture 4

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Announcements and Reminders

- First tutorial tomorrow
- Extra tutorial on the way (11am Fridays?)

Optimization and Computers

Some optimization problems are pretty easy

- All functions are differentiable
- Few choice variables (low dimensional)
- Concave (for max) or convex (for min)
- First order / tangency conditions relatively simple

Textbook examples often chosen to have this structure

In reality many problems don't have this structure

- Can't take derivatives
- Many choice variables (high dimensional)
- Neither concave nor convex — local maxima and minima

Moreover, even if we can use derivative conditions they can be useless

- For N choice variables, FOCs are a nonlinear system in $\mathbb{R}^N$

Can Computers Save Us?

For any function we can always try brute force optimization

Here's an example for the following function

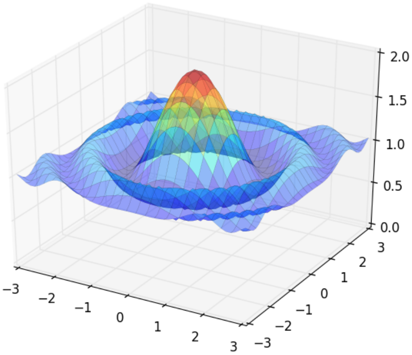


Figure : The function to maximize

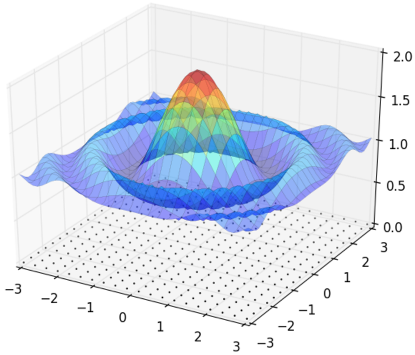


Figure : Grid of points to evaluate the function at

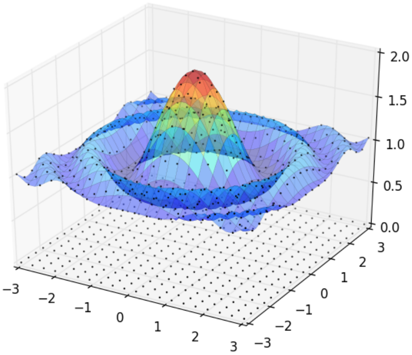


Figure : Evaluations

Grid size $= 20 \times 20 = 400$

Outcomes

- Number of function evaluations $= 400$
- Time taken $=$ almost zero
- Maximal value recorded $= 1.951$
- True maximum $= 2$

Not bad and we can easily do better

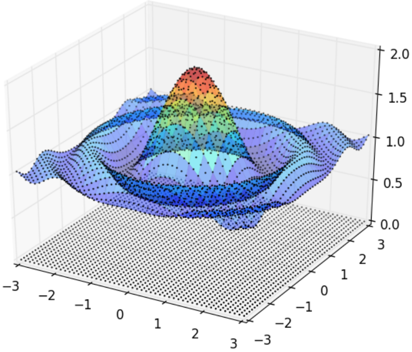


Figure : $50^2 = 2500$ evaluations

- Number of function evaluations $= 50^2$
- Time taken $= 101$ microseconds
- Maximal value recorded $= 1.992$
- True maximum $= 2$

So why even study optimization?

The problem is mainly with larger numbers of choice variables

- 3 vars: $\max_{x_1,x_2,x_3} f(x_1, x_2, x_3)$
- 4 vars: $\max_{x_1,x_2,x_3,x_4} f(x_1, x_2, x_3, x_4)$
- $\cdots$

If we have 50 grid points per variable and

- 2 variables then evaluations $= 50^2 = 2500$
- 3 variables then evaluations $= 50^3 = 125,000$
- 4 variables then evaluations $= 50^4 = 6,250,000$
- 5 variables then evaluations $= 50^5 = 312,500,000$
- $\cdots$

Example. Recent study: Optimal placement of drinks across vending machines in Tokyo

Approximate dimensions of problem:

- Number of choices for each variable $= 2$
- Number of choice variables $= 1000$

Hence number of possibilities $= 2^{1000}$

How big is that?

```
In [10]: 2**1000
Out[10]:
107150860718626732094842504906000181056140481170
553360744375038837035105112493612249319837881569
585812759467291755314682518714528569231404359845
775746985748039345677748242309854210746050623711
418779541821530464749835819412673987675591655439
460770629145711964776865421676604298316526243868
37205668069376
```

Let's say my machine can evaluate about 1 billion possibilities per second

How long would that take?

```
In [16]: (2**1000 / 10**9) / 31556926  # In years
Out[16]:
3395478403651443492780079558636357072806789899995
8993494625396619335961465717339269652558613648540
0602869857073269915919013110292446394538059880920
4593307265745511992438123507294154933231019938830
1571394569707026437986448403352049168514244509939
8167906015686216612651741700199135889415960
```

What about high performance computing?

- more powerful hardware
- faster CPUs
- GPUs
- vector processors
- cloud computing
- massively parallel supercomputers
- $\ldots$

Let's say speed up is 10^{12} (wildly optimistic)

```
In [19]: (2**1000 / 10**(9 + 12)) / 31556926
Out[19]:
339547840365144349278007955863635707280678989995899349462539661933596146571733926965255861364854060286985707326991591901311029244639453805988092045933072657455119924381235072941549332310199388301571394569707026437986448403352049168514244509939816790601568621661265174170019
```

For comparison:

```
In [20]: 5 * 10**9 # Expected lifespan of sun
Out[20]: 5000000000
```

Message: There are serious limits to computation

What's required is clever analysis

Exploit what information we have

- without information (oracle) we're stuck
- with information / structure we can do clever things

Examples later on...

New Topic

ELEMENTS OF SET THEORY

Elements of Set Theory

We now turn to more formal / foundational ideas

- sets
- functions
- logic
- proofs

Mainly review of key ideas

Common Symbols

- $P \implies Q$ means "P implies Q"
- $P \iff Q$ means "$P \implies Q$ and $Q \implies P$"
- $\exists$ means "there exists"
- $\forall$ means "for all"
- s.t. means "such that"
- $\because$ means "because"
- $\therefore$ means "therefore"
- $a := 1$ means "a is defined to be equal to 1"
- $\mathbb{R}$ means all real numbers
- $\mathbb{N}$ means the natural numbers $\{1, 2, \ldots\}$

Logic

Let P and Q be statements, such as

- x is a negative integer
- x is an odd number
- the area of any circle in the plane is -17

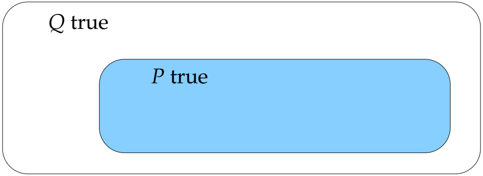
Law of the excluded middle: Every mathematical statement is either true or false

Statement "$P \implies Q$" means "P implies Q"

Example. k is even $\implies k = 2n$ for some integer n

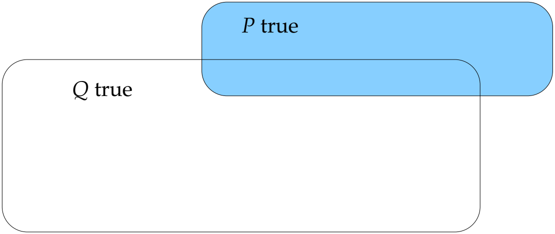
Equivalent forms of $P \implies Q$:

1. If P is true then Q is true
2. P is a sufficient condition for Q
3. Q is a necessary condition for P
4. If Q fails then P fails



Equivalent ways of saying $P \implies Q$ is <u>not</u> true:

1. P does not imply Q
2. P is not sufficient for Q
3. Q is not necessary for P
4. Even if Q fails, P can still hold



Example

Let

- $P :=$ "$n \in \mathbb{N}$ and even"
- $Q :=$ "n even"

Then

1. $P \implies Q$
2. P is sufficient for Q
3. Q is necessary for P
4. If Q fails then P fails

Example

Let

- $P :=$ "R is a rectangle"
- $Q :=$ "R is a square"

Then

1. $P \not\Rightarrow Q$
2. P is not sufficient for Q
3. Q is not necessary for P
4. Just because Q fails does not mean that P fails

Proof by Contradiction

Suppose we wish to prove a statement such as $P \implies Q$

A proof by contradiction starts by assuming

1. P holds
2. and yet Q fails

We then show that this scenario leads to a contradiction

Examples.

- $1 < 0$
- 10 is an odd number

We conclude that $P \implies Q$ is valid after all

Example. Suppose that island X is populated only by pirates and knights

- pirates always lie
- knights always tell the truth

Claim to prove: If person Y says “I’m a pirate” then person Y is <u>not</u> a native of island X

Strategy for the proof:

1. Suppose person Y is a native of the island
2. Show that this leads to a contradiction
3. Conclude that Y is not a native of island X, as claimed

Proof:

Suppose to the contrary that person Y is a native of island X

Then Y is either a pirate or a knight

Suppose first that Y is knight

- Y is a knight who claims to be a pirate

This is impossible, since knights always tell the truth

Suppose next that Y is pirate

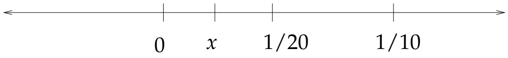
- Y is a pirate who claims to be a pirate

Since pirates always lie, they would not make such a statement

Either way we get a contradiction

Example. There is <u>no</u> $x \in \mathbb{R}$ such that $0 < x < 1/n$, $\forall\, n \in \mathbb{N}$

Proof: Suppose to the contrary that such an x exists



Since $x > 0$ the number $1/x$ exists, is finite

Let N be the smallest integer such that $N \geq 1/x$

- If $x = 0.3$ then $1/x = 3.333\cdots$ so set $N = 4$

Since $N \geq 1/x$ we also have $1/N \leq x$

On the other hand, since $N \in \mathbb{N}$, we have $x < 1/N$

But then $1/N < 1/N$, which is impossible — a contradiction

Example. Let $n \in \mathbb{N}$

Claim: n^2 odd $\implies$ n odd

Proof: Suppose to the contrary that

1. $n \in \mathbb{N}$ and n^2 is odd
2. but n is even

Then $n = 2k$ for some $k \in \mathbb{N}$

Hence $n^2 = (2k)^2$

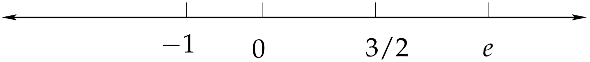
But then $n^2 = 2m$ for $m := 2k^2 \in \mathbb{N}$

Contradiction

Sets

Will often refer to the **real numbers**, $\mathbb{R}$

Understand it to contain "all of the numbers" on the "real line"



Contains both the rational and the irrational numbers

$\mathbb{R}$ is an example of a **set**

A set is a collection of objects viewed as a whole

(In case of $\mathbb{R}$, the objects are numbers)

Other examples of sets:

- set of all rectangles in the plane
- set of all prime numbers
- set of monkeys in Japan

Notation:

- Sets: A, B, C
- Elements: x, y, z

Important sets:

- $\mathbb{N} := \{1, 2, 3, \ldots\}$
- $\mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- $\mathbb{Q} := \{p/q : p, q \in \mathbb{Z},\ q \neq 0\}$
- $\mathbb{R} := \mathbb{Q} \cup \{ \text{ irrationals } \}$

Intervals of $\mathbb{R}$

Common notation:

$$(a,b) := \{x \in \mathbb{R} : a < x < b\}$$

$$(a,b] := \{x \in \mathbb{R} : a < x \leq b\}$$

$$[a,b) := \{x \in \mathbb{R} : a \leq x < b\}$$

$$[a,b] := \{x \in \mathbb{R} : a \leq x \leq b\}$$

$$[a,\infty) := \{x \in \mathbb{R} : a \leq x\}$$

$$(-\infty,b) := \{x \in \mathbb{R} : x < b\}$$

Etc.

Let A and B be sets

Statement $x \in A$ means that x is an element of A

$A \subset B$ means that any element of A is also an element of B

Examples.

- $\mathbb{N} \subset \mathbb{Z}$
- irrationals are a subset of $\mathbb{R}$

$A = B$ means that A and B contain the same elements

- Equivalently, $A \subset B$ and $B \subset A$

Let S be a set and A and B be subsets of S

Union of A and B

$$A \cup B := \{x \in S : x \in A \text{ or } x \in B\}$$

Intersection of A and B

$$A \cap B := \{x \in S : x \in A \text{ and } x \in B\}$$

Set theoretic difference:

$$A \setminus B := \{x \in S : x \in A \text{ and } x \notin B\}$$

In other words, all points in A that are not points in B

Examples.

- $\mathbb{Z} \setminus \mathbb{N} = \{\dots, -2, -1, 0\}$
- $\mathbb{R} \setminus \mathbb{Q} =$ the set of irrational numbers
- $\mathbb{R} \setminus [0, \infty) = (-\infty, 0)$
- $\mathbb{R} \setminus (a, b) = (-\infty, a] \cup [b, \infty)$

Complement of A is all elements of S that are not in A:

$$A^c := S \setminus A =: \{x \in S : x \notin A\}$$

Remarks:

- Need to know what S is before we can determine A^c
- If not clear better write $S \setminus A$

Example. $(a, \infty)^c$ generally understood to be $(-\infty, a]$

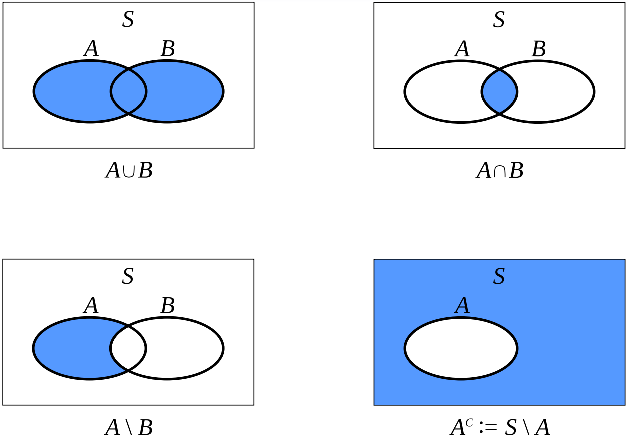


Figure : Unions, intersections and complements

```
In [1]: set_1 = {'green', 'eggs', 'ham'}

In [2]: set_2 = {'red', 'green'}

In [3]: set_1.intersection(set_2)
Out[3]: {'green'}

In [4]: set_1.difference(set_2)
Out[4]: {'eggs', 'ham'}

In [5]: set_1.union(set_2)
Out[5]: {'eggs', 'green', 'ham', 'red'}
```

Set operations:

If A and B subsets of S, then

1. $A \cup B = B \cup A$ and $A \cap B = B \cap A$
2. $(A \cup B)^c = B^c \cap A^c$ and $(A \cap B)^c = B^c \cup A^c$
3. $A \setminus B = A \cap B^c$
4. $(A^c)^c = A$

The **empty set** $\emptyset$ is the set containing no elements

If $A \cap B = \emptyset$, then A and B said to be **disjoint**

Infinite Unions and Intersections

Given a family of sets $K_\lambda \subset S$ with $\lambda \in \Lambda$,

$$\bigcap_{\lambda \in \Lambda} K_\lambda := \{x \in S : x \in K_\lambda \text{ for all } \lambda \in \Lambda\}$$

$$\bigcup_{\lambda \in \Lambda} K_\lambda := \{x \in S : \text{there exists an } \lambda \in \Lambda \text{ such that } x \in K_\lambda\}$$

- "there exists" means "there exists <u>at least</u> one"

Example. Let $A := \cap_{n \in \mathbb{N}} (0, 1/n)$

Claim: $A = \emptyset$

Proof: We need to show that A contains no elements

Suppose to the contrary that $x \in A = \cap_{n \in \mathbb{N}} (0, 1/n)$

Then x is a number satisfying $0 < x < 1/n$ for all $n \in \mathbb{N}$

No such x exists

Contradiction

Example. For any $a < b$ we have $\cup_{\epsilon>0} [a+\epsilon, b) = (a, b)$

Proof: Pick any $a < b$

Suppose first that $x \in \cup_{\epsilon>0} [a+\epsilon, b)$

This means there exists $\epsilon > 0$ such that $a + \epsilon \leq x < b$

Clearly $a < x < b$, and hence $x \in (a, b)$

Conversely, if $a < x < b$, then $\exists\, \epsilon > 0$ s.t. $a + \epsilon \leq x < b$

Hence $x \in \cup_{\epsilon>0} [a+\epsilon, b)$

Ex. Show that $\cup_{n\in\mathbb{N}} (-n, n) = \mathbb{R}$

Let S be any set

Let $K_\lambda \subset S$ for all $\lambda \in \Lambda$

de Morgan's laws state that:

$$\left[\bigcup_{\lambda\in\Lambda} K_\lambda\right]^c = \bigcap_{\lambda\in\Lambda} K_\lambda^c \qquad \text{and} \qquad \left[\bigcap_{\lambda\in\Lambda} K_\lambda\right]^c = \bigcup_{\lambda\in\Lambda} K_\lambda^c$$

Let's prove that $A := (\cup_{\lambda \in \Lambda} K_\lambda)^c = \cap_{\lambda \in \Lambda} K_\lambda^c =: B$

Suffices to show that $A \subset B$ and $B \subset A$

Let's just do $A \subset B$

Must show that every $x \in A$ is also in B

Fix $x \in A$

Since $x \in A$, it must be that x is not in $\cup_{\lambda \in \Lambda} K_\lambda$

$$\therefore \quad x \text{ is not in any } K_\lambda$$

$$\therefore \quad x \in K_\lambda^c \text{ for each } \lambda \in \Lambda$$

$$\therefore \quad x \in \cap_{\lambda \in \Lambda} K_\lambda^c =: B$$

Tuples

We often organize collections with natural order into “tuples”

A **tuple** is

- a finite sequence of terms
- denoted using notation such as (a_1, a_2) or (x_1, x_2, x_3)

Example. Flip a coin 10 times and let

- 0 represent tails and 1 represent heads
- b_n be result of n-th flip

Typical outcome $(1, 1, 0, 0, 0, 0, 1, 0, 1, 1)$

Generic outcome $(b_1, b_2, \dots, b_{10})$ for $b_n \in \{0, 1\}$

Cartesian Products

We make collections of tuples using Cartesian products

The **Cartesian product** of $A_1, \ldots, A_N$ is the set

$$A_1 \times \cdots \times A_N := \{(a_1, \ldots, a_N) : a_n \in A_n \text{ for } n = 1, \ldots, N\}$$

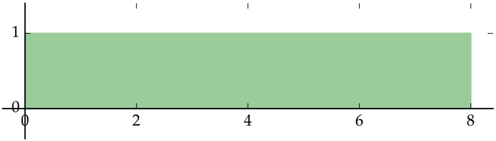
Example. Set of all outcomes from flip experiment is

$$B := \{(b_1, \ldots, b_{10}) : b_n \in \{0,1\} \text{ for } n = 1, \ldots, 10\}$$

$$= \{0,1\} \times \cdots \times \{0,1\} \quad (10 \text{ products})$$

Example.

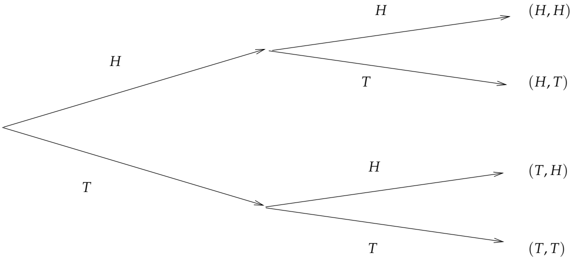
$$[0,8] \times [0,1] = \{(x_1, x_2) : 0 \leq x_1 \leq 8,\ 0 \leq x_2 \leq 1\}$$



Example. $\mathbb{R}^N =$ all tuples $(x_1, \ldots, x_N)$ with $x_n \in \mathbb{R}$

Counting Finite Ordered Tuples

Number of possible tuples = product of the number of possibilities for each element



General rule: $\#(A_1 \times \cdots \times A_N) = (\#A_1) \times \cdots \times (\#A_N)$

Example. Number of possible distinct outcomes sequences if we flip a coin 10 times is

$$\#[\{0,1\} \times \cdots \times \{0,1\}] = 2 \times \cdots \times 2 = 2^{10}$$

Example. Number of possible distinct outcomes from 2 rolls of a fair dice is

$$6 \times 6 = 36$$

Example. Number of 10 letter passwords from the lowercase letters `a, b, ..., z` is

$$26^{10} = 141,167,095,653,376$$