ECON2125/4021/8013

Lecture 16

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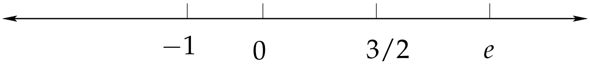
Analysis on the Line

Recall that $\mathbb{R}$ denotes the continuous real line



Can be thought of as $\mathbb{Q} \cup \mathbb{I}$ where

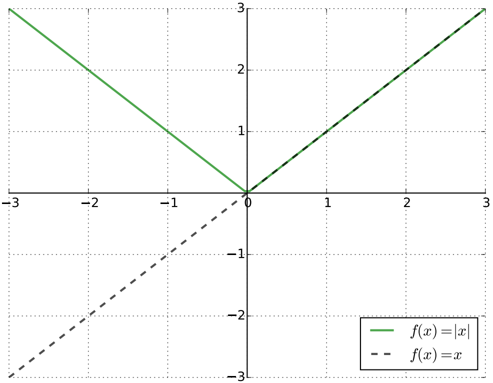
- $\mathbb{Q}$ is the rational numbers
- $\mathbb{I}$ is the irrational numbers

Facts

- Between any two real numbers $a < b$ there exists a rational number
- Between any two real numbers $a < b$ there exists an irrational number

Thus, the rationals and irrationals are "all mixed together"

If $x \in \mathbb{R}$ then $|x| := \max\{x, -x\}$ called its **absolute value**



Fact. For any $x, y \in \mathbb{R}$, the following statements hold

1. $|x| \leq y$ if and only if $-y \leq x \leq y$
2. $|x| < y$ if and only if $-y < x < y$
3. $|x| = 0$ if and only if $x = 0$
4. $|xy| = |x||y|$
5. $|x + y| \leq |x| + |y|$

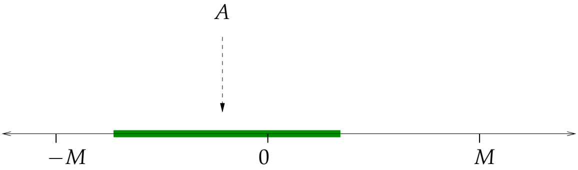
Last inequality is called the **triangle inequality**

Ex. Using these rules, show that if $x, y, z \in \mathbb{R}$, then

1. $|x - y| \leq |x| + |y|$
2. $|x - y| \leq |x - z| + |z - y|$ (Hint: $x - y = x - z + z - y$)

Bounded sets

$A \subset \mathbb{R}$ is called **bounded** if $\exists\, M \in \mathbb{R}$ s.t. $|x| \leq M$, all $x \in A$



Example. Every finite subset A of $\mathbb{R}$ is bounded

$\because$ Set $M := \max\{|a| : a \in A\}$

Example. $\mathbb{N}$ is unbounded

$\because$ For any $M \in \mathbb{R}$ there is an n that exceeds it

Example. (a,b) is bounded for any a,b

$\because$ Each $x \in (a,b)$ satisfies $|x| \leq M := \max\{|a|,|b|\}$

Ex. Check it

Fact. If A and B are bounded sets then so is $A \cup B$

Proof: Let A and B be bounded sets and let $C := A \cup B$

By definition, $\exists\, M_A$ and M_B with

$$|a| \leq M_A, \text{ all } a \in A, \qquad |b| \leq M_B, \text{ all } b \in B$$

Let $M_C := \max\{M_A, M_B\}$ and fix any $x \in C$

$$x \in C \implies x \in A \text{ or } x \in B$$

$$\therefore \quad |x| \leq M_A \quad \text{or} \quad |x| \leq M_B$$

$$\therefore \quad |x| \leq M_C$$

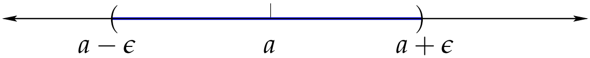
ϵ-balls

Given $\epsilon > 0$ and $a \in \mathbb{R}$, the **ϵ-ball** around a is

$$B_\epsilon(a) := \{x \in \mathbb{R} : |a - x| < \epsilon\}$$

Equivalently,

$$B_\epsilon(a) = \{x \in \mathbb{R} : a - \epsilon < x < a + \epsilon\}$$



Ex. Check equivalence

Fact. If x is in every ϵ-ball around a then $x = a$

Proof:

Suppose to the contrary that

- x is in every ϵ-ball around a and yet $x \neq a$

Since x is not a we must have $|x - a| > 0$

Set $\epsilon := |x - a|$

Since $\epsilon > 0$, we have $x \in B_\epsilon(a)$

This means that $|x - a| < \epsilon$

That is, $|x - a| < |x - a|$ — contradiction

Fact. If $a \neq b$, then $\exists\, \epsilon > 0$ s.t. $B_\epsilon(a)$ and $B_\epsilon(b)$ are disjoint

Proof: Let $a, b \in \mathbb{R}$ with $a \neq b$

If we set $\epsilon := |a-b|/2$, then $B_\epsilon(a)$ and $B_\epsilon(b)$ are disjoint

To see this, suppose to the contrary that $\exists\, x \in B_\epsilon(a) \cap B_\epsilon(B)$

Then $|x-a| < |a-b|/2$ and $|x-b| < |a-b|/2$

But then

$$|a-b| \leq |a-x| + |x-b| < |a-b|/2 + |a-b|/2 = |a-b|$$

Contradiction

Sequences

A **sequence** is a function from $\mathbb{N}$ to $\mathbb{R}$

- to each $n \in \mathbb{N}$ we associate one $x_n \in \mathbb{R}$

Typically written as $\{x_n\}_{n=1}^{\infty}$ or $\{x_n\}$ or $\{x_1, x_2, x_3, \ldots\}$

Examples.

- $\{x_n\} = \{2, 4, 6, \ldots\}$
- $\{x_n\} = \{1, 1/2, 1/4, \ldots\}$
- $\{x_n\} = \{1, -1, 1, -1, \ldots\}$
- $\{x_n\} = \{0, 0, 0, \ldots\}$

Sequence $\{x_n\}$ is called

- **bounded** if $\{x_1, x_2, \ldots\}$ is a bounded set
- **monotone increasing** if $x_{n+1} \geq x_n$ for all n
- **monotone decreasing** if $x_{n+1} \leq x_n$ for all n
- **monotone** if it is either monotone increasing or monotone decreasing

Examples.

- $x_n = 1/n$ is monotone decreasing, bounded
- $x_n = (-1)^n$ is not monotone but is bounded
- $x_n = 2n$ is monotone increasing but not bounded

Convergence

Let $a \in \mathbb{R}$ and let $\{x_n\}$ be a sequence

Suppose, for any $\epsilon > 0$, we can find an $N \in \mathbb{N}$ with

$$x_n \in B_\epsilon(a) \text{ for all } n \geq N$$

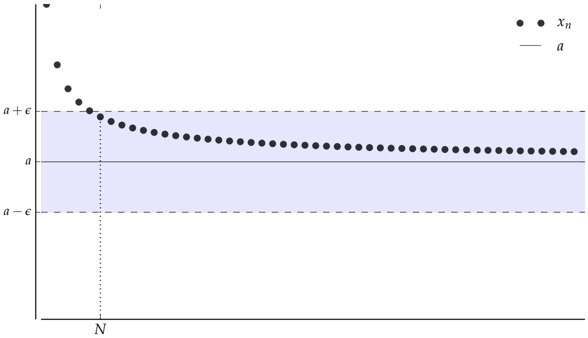
Then $\{x_n\}$ is said to **converge** to a

Convergence to a in symbols,

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n \geq \mathbb{N} \implies x_n \in B_\epsilon(a)$$

"$\{x_n\}$ is eventually in any ϵ-ball around a"

The sequence $\{x_n\}$ is eventually in this ϵ-ball around a



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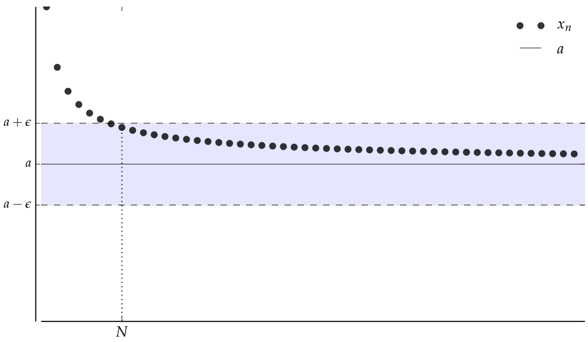

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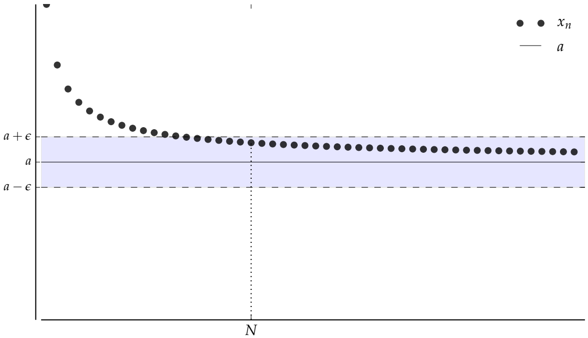

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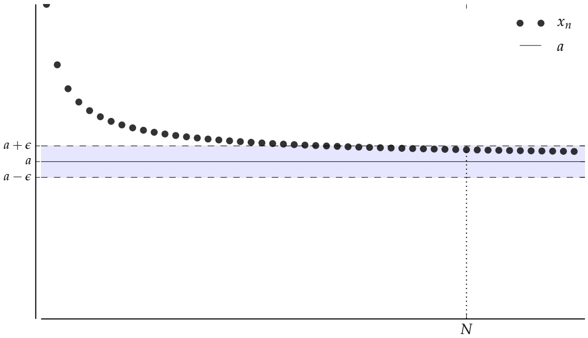


The point a is called the **limit** of the sequence, and we write

$$x_n \to a \quad \text{as} \quad n \to \infty$$

or

$$\lim_{n \to \infty} x_n = a$$

We call $\{x_n\}$ **convergent** if it converges to some limit in $\mathbb{R}$

Example. $\{x_n\}$ defined by $x_n = 1 + 1/n$ converges to 1

To prove this must show that $\forall\, \epsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$n \geq N \implies |x_n - 1| < \epsilon \tag{$\star$}$$

To show this formally we need to come up with an "algorithm"

1. You give me any $\epsilon > 0$
2. I respond with an N such that $(\star)$ holds

In general, as ϵ shrinks, N will have to grow

Here's how to do this for the case $1+1/n$ converges to 1

First pick an arbitrary $\epsilon > 0$

Now we have to come up with an N such that

$$n \geq N \implies |1+1/n-1| < \epsilon \qquad (\star)$$

Let N be the first integer greater than $1/\epsilon$

Then

$$n \geq N \implies n > 1/\epsilon \implies 1/n < \epsilon \implies |1+1/n-1| < \epsilon$$

Remark: Any $N' > N$ would also work

Example. The sequence $x_n = 2^{-n}$ converges to 0

Proof: Must show that, $\forall\, \epsilon > 0$, $\exists\, N \in \mathbb{N}$ such that

$$n \geq N \implies |2^{-n} - 0| < \epsilon \tag{$\star$}$$

So pick any $\epsilon > 0$, and observe that

$$|2^{-n} - 0| < \epsilon \iff 2^{-n} < \epsilon \iff n > -\frac{\ln \epsilon}{\ln 2}$$

Hence we take N to be the first integer greater than $-\ln \epsilon / \ln 2$

Then

$$n \geq N \implies n > -\frac{\ln \epsilon}{\ln 2} \implies (\star)$$

What if we want to show that $x_n \to a$ fails?

To show convergence fails we need to show the negation of

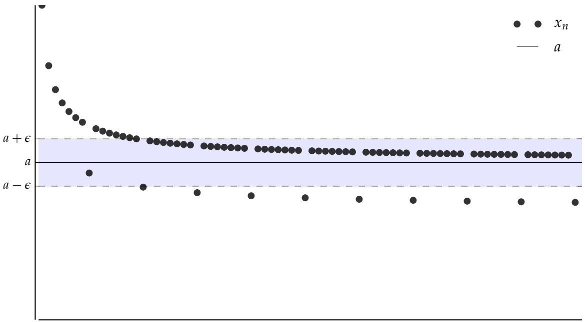
$$\forall\ \epsilon > 0,\ \exists\ N \in \mathbb{N}\ \text{ s.t. }\ n \geq N \implies x_n \in B_\epsilon(a)$$

Negation: there is an $\epsilon > 0$ where we can't find any such N

More specifically, $\exists\, \epsilon > 0$ such that, which ever $N \in \mathbb{N}$ we look at, there's an $n \geq N$ with x_n outside $B_\epsilon(a)$

One way to say this: There exists a $B_\epsilon(a)$ such that $x_n \notin B_\epsilon(a)$ infinitely often

This is the kind of picture we're thinking of



Example. The sequence $x_n = (-1)^n$ does <u>not</u> converge to 1

Proof: This is what we want to show

$$\exists\ \epsilon > 0 \ \text{ s.t. } \quad \text{s.t. } x_n \notin B_\epsilon(1) \text{ infinitely often}$$

Since it's a "there exists", we need to come up with such an ϵ

Let's try $\epsilon = 0.5$, so that

$$B_\epsilon(1) = \{x \in \mathbb{R} : |x - 1| < 0.5\} = (0.5, 1.5)$$

If n is odd then $x_n = -1$, which is not in $(0.5, 1.5)$

Hence $\{x_n\}$ not in $B_\epsilon(1)$ infinitely often

An Equivalence

Let $\{x_n\}$ be a sequence in $\mathbb{R}$ and let $a \in \mathbb{R}$

Fact. $x_n \to a$ if and only if $|x_n - a| \to 0$

Proof: Compare the definitions:

- $x_n \to a \iff \forall\, \epsilon > 0, \exists\, N \in \mathbb{N}$ s.t. $|x_n - a| < \epsilon$
- $|x_n - a| \to 0 \iff \forall\, \epsilon > 0, \exists\, N \in \mathbb{N}$ s.t. $||x_n - a| - 0| < \epsilon$

Clearly these statements are equivalent

Fact. Each sequence in $\mathbb{R}$ has at most one limit

Proof: Suppose instead that $x_n \to a$ and $x_n \to b$ with $a \neq b$

Take disjoint ϵ-balls around a and b

Since $x_n \to a$ and $x_n \to b$,

- $\exists\, N_a$ s.t. $n \geq N_a \implies x_n \in B_\epsilon(a)$
- $\exists\, N_b$ s.t. $n \geq N_b \implies x_n \in B_\epsilon(b)$

But then $n \geq \max\{N_a, N_b\} \implies x_n \in B_\epsilon(a)$ and $x_n \in B_\epsilon(b)$

Contradiction of disjoint

Fact. Every convergent sequence is bounded

Proof: Let $\{x_n\}$ be convergent with $x_n \to a$

Fix any $\epsilon > 0$ and choose N s.t. $x_n \in B_\epsilon(a)$ when $n \geq N$

Regarded as sets,

$$\{x_n\} \subset \{x_1, \ldots, x_{N-1}\} \cup B_\epsilon(a)$$

Both of these sets are bounded

- First because finite sets are bounded
- Second because $B_\epsilon(a)$ is bounded

Moreover, finite unions of bounded sets are bounded

Limits vs Algebra

Here are some basic tools for working with limits

Facts If $x_n \to x$ and $y_n \to y$, then

1. $x_n + y_n \to x + y$
2. $x_n y_n \to xy$
3. $x_n / y_n \to x / y$ when y_n and y are $\neq 0$
4. $x_n \leq y_n$ for all $n \implies x \leq y$

Let's check that $x_n \to x$ and $y_n \to y$ implies $x_n + y_n \to x + y$

Proof: Fix $\epsilon > 0$

Need to find $N \in \mathbb{N}$ such that

$$n \geq N \implies |(x_n + y_n) - (x + y)| < \epsilon \qquad (\star)$$

Note that

- $|(x_n + y_n) - (x + y)| \leq |x_n - x| + |y_n - y|$
- $\exists N_x \in \mathbb{N}$ such that $n \geq N_x \implies |x_n - x| < \epsilon/2$
- $\exists N_y \in \mathbb{N}$ such that $n \geq N_y \implies |y_n - y| < \epsilon/2$

Ex. Show $N := \max\{N_x, N_y\}$ satisfies $(\star)$

Let's also check the claim that $x_n \to x$, $y_n \to y$ and $x_n \leq y_n$ for all $n \in \mathbb{N}$ implies $x \leq y$

Proof: Suppose instead that $x > y$

Take disjoint ϵ-balls $B_\epsilon(x)$ and $B_\epsilon(y)$ around these points

Exists an n such that $x_n \in B_\epsilon(x)$ and $y_n \in B_\epsilon(y)$

But then $x_n > y_n$, a contradiction

In words: "Weak inequalities are preserved under limits"

Here's another property of limits, called the "squeeze theorem"

Fact. Let $\{x_n\}$ $\{y_n\}$ and $\{z_n\}$ be sequences in $\mathbb{R}$. If

1. $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$
2. $x_n \to a$ and $z_n \to a$

then $y_n \to a$ also holds

Proof: Pick any $\epsilon > 0$

We can choose an

- $N_x \in \mathbb{N}$ such that $n \geq N_x \implies x_n \in B_\epsilon(a)$
- $N_z \in \mathbb{N}$ such that $n \geq N_z \implies z_n \in B_\epsilon(a)$

Ex. Show that $n \geq \max\{N_x, N_z\} \implies y_n \in B_\epsilon(a)$

Infinite Sums

Let $\{x_n\}$ be a sequence in $\mathbb{R}$

Then

$$\sum_{n=1}^{\infty} x_n := \lim_{k\to\infty} \sum_{n=1}^{k} x_n$$

Thus, $\sum_{n=1}^{\infty} x_n$ is defined, if it exists, as the limit of $\{y_k\}$ where

$$y_k := \sum_{n=1}^{k} x_n$$

Other notation:

$$\sum_{n} x_n, \qquad \sum_{n\geq 1} x_n, \qquad \sum_{n\in\mathbb{N}} x_n, \qquad \text{etc.}$$

Example. If $x_n = \alpha^n$ for $\alpha \in (0,1)$, then

$$\sum_{n=1}^{\infty} x_n = \lim_{k\to\infty} \sum_{n=1}^{k} \alpha^n = \lim_{k\to\infty} \alpha \frac{1-\alpha^k}{1-\alpha} = \frac{\alpha}{1-\alpha}$$

Example. If $x_n = (-1)^n$ the limit fails to exist because

$$y_k = \sum_{n=1}^{k} x_n = \begin{cases} 0 & \text{if } k \text{ is even} \\ -1 & \text{otherwise} \end{cases}$$

Fact. If $\{x_n\}$ is nonnegative and $\sum_n x_n < \infty$, then $x_n \to 0$

Proof: Suppose to the contrary that $x_n \to 0$ fails

Then

$$\exists\, \epsilon > 0 \text{ such that } x_n \notin B_\epsilon(0) \text{ infinitely often}$$

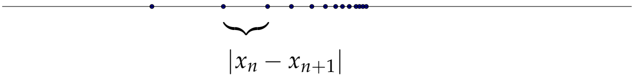
Since x_n is nonnegative,

$$\exists\, \epsilon > 0 \text{ such that } x_n \text{ exceeds } \epsilon \text{ infinitely often}$$

But then $\sum_n x_n$ cannot be finite — contradiction

Cauchy Sequences

Informal def: Cauchy sequences are those where $|x_n - x_{n+1}|$ gets smaller and smaller



Example. Sequences generated by iterative methods for solving nonlinear equations often have this property

Cauchy sequences "look like" they are converging to something

A key axiom of analysis is that such sequences do converge to something — details follow

A sequence $\{x_n\}$ is called **Cauchy** if $\forall\, \epsilon > 0, \;\; \exists\, N \in \mathbb{N}$ such that

$$n \geq N \text{ and } j \geq 1 \quad \Longrightarrow \quad |x_n - x_{n+j}| < \epsilon \qquad (\star)$$

Example. $\{x_n\}$ defined by $x_n = \alpha^n$ where $\alpha \in (0,1)$ is Cauchy

Proof: For any n, j we have

$$|x_n - x_{n+j}| = |\alpha^n - \alpha^{n+j}| = \alpha^n |1 - \alpha^j| \leq \alpha^n$$

Fix $\epsilon > 0$

Ex. Show that $n > \epsilon / \log(\alpha) \implies \alpha^n < \epsilon$

Hence any integer $N > \epsilon / \log(\alpha)$ makes $(\star)$ hold

Fact. For any sequence, convergent $\iff$ Cauchy

Proof of $\implies$:

Let $\{x_n\}$ be a sequence converging to some $a \in \mathbb{R}$

Fix $\epsilon > 0$

We can choose N s.t.

$$n \geq N \implies |x_n - a| < \frac{\epsilon}{2}$$

For this N we have $n \geq N$ and $j \geq 1$ implies

$$|x_n - x_{n+j}| \leq |x_n - a| + |x_{n+j} - a| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Proof of $\Longleftarrow$:

This is basically an axiom in the definition of $\mathbb{R}$

Either

1. We assume it, or
2. We assume something else that's essentially equivalent

We'll go for option 1

Implications:

- There are no "gaps" in the real line
- To check $\{x_n\}$ converges to something we just need to check Cauchy property

Fact. Every bounded monotone sequence in $\mathbb{R}$ is convergent

Sketch of proof:

Suffices to show that $\{x_n\}$ is Cauchy

Suppose not

Then no matter how far we go down the sequence we can find another jump of size $\epsilon > 0$

Since monotone, all the jumps are in the same direction

But then $\{x_n\}$ not bounded — a contradiction

Full proof: See any text on analysis

Subsequences

A sequence $\{x_{n_k}\}$ is called a **subsequence** of $\{x_n\}$ if

1. $\{x_{n_k}\}$ is a subset of $\{x_n\}$
2. the indices n_k are strictly increasing

Example.

$$\{x_n\} = \{x_1, x_2, x_3, x_4, x_5, \ldots\}$$

and

$$\{x_{n_k}\} = \{x_2, x_4, x_6, x_8 \ldots\}$$

In this case

$$\{n_k\} = \{n_1, n_2, n_3, \ldots\} = \{2, 4, 6, \ldots\}$$

More Examples.

1. $\{\frac{1}{1}, \frac{1}{3}, \frac{1}{5}, \ldots\}$ is a subsequence of $\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\}$
2. $\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\}$ is a subsequence of $\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\}$
3. $\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots\}$ is **not** a subsequence of $\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\}$

Fact. Every sequence has a monotone subsequence

Proof: Omitted

Example. The sequence $x_n = (-1)^n$ has monotone subsequence

$$\{x_2, x_4, x_6, \ldots\} = \{1, 1, 1, \ldots\}$$

This leads us to the famous **Bolzano–Weierstrass theorem**, to be used later when we discuss optimization

Fact. Every bounded sequence in $\mathbb{R}$ has a convergent subsequence

Proof: Let $\{x_n\}$ be a bounded sequence

There exists a monotone subsequence

- which is itself a bounded sequence (why?)
- and hence both monotone and bounded

Every bounded monotone sequence converges

Hence $\{x_n\}$ has a convergent subsequence

Derivatives

Let $f\colon (a,b) \to \mathbb{R}$ and let $x \in (a,b)$

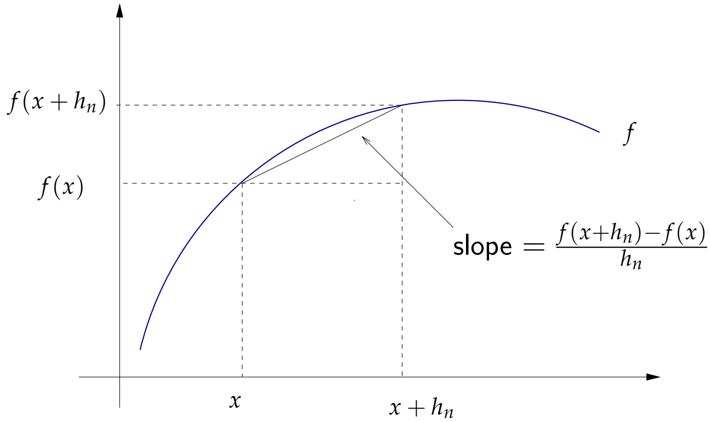
Let H be all sequences $\{h_n\}$ such that $h_n \neq 0$ and $h_n \to 0$

If there exists a constant $f'(x)$ such that

$$\frac{f(x+h_n)-f(x)}{h_n} \to f'(x)$$

for every $\{h_n\} \in H$, then

- f is said to be **differentiable** at x
- $f'(x)$ is called the **derivative** of f at x


$f(x+h_n)$
$f(x)$
f
slope $= \frac{f(x+h_n)-f(x)}{h_n}$
x
$x+h_n$

Example. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$

Fix any $x \in \mathbb{R}$ and any $h_n \to 0$

We have

$$\begin{aligned}\frac{f(x+h_n) - f(x)}{h_n} &= \frac{(x+h_n)^2 - x^2}{h_n} \\ &= \frac{x^2 + 2xh_n + h_n^2 - x^2}{h_n} = 2x + h_n\end{aligned}$$

$$\therefore \quad f'(x) = \lim_{n\to\infty}(2x + h_n) = 2x$$

Example. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x|$

This function is not differentiable at $x = 0$

Indeed, if $h_n = 1/n$, then

$$\frac{f(0+h_n) - f(0)}{h_n} = \frac{|0+1/n| - |0|}{1/n} \to 1$$

On the other hand, if $h_n = -1/n$, then

$$\frac{f(0+h_n) - f(0)}{h_n} = \frac{|0-1/n| - |0|}{-1/n} \to -1$$

Useful for intuition: Taylor series

Loosely speaking, if $f\colon \mathbb{R} \to \mathbb{R}$ is suitably differentiable at a, then

$$f(x) \approx f(a) + f'(a)(x-a)$$

for x very close to a,

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

on a slightly wider interval, etc.

These are the 1st and 2nd order **Taylor series approximations** to f at a respectively

As the order goes higher we get better and better approximation (see any text on calculus)

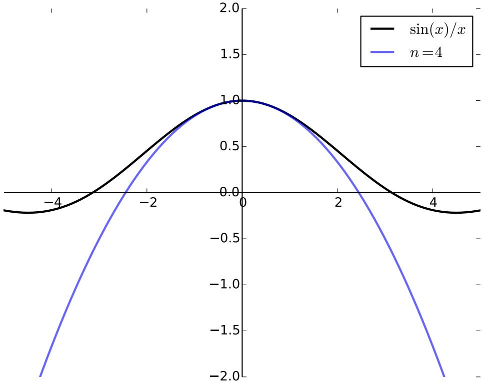


Figure : 4th order Taylor series for $f(x) = \sin(x)/x$ at 0

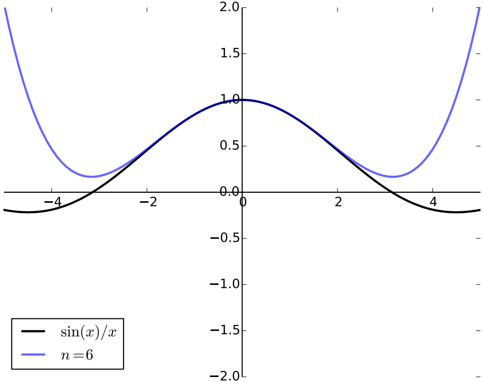


Figure : 6th order Taylor series for $f(x) = \sin(x)/x$ at 0

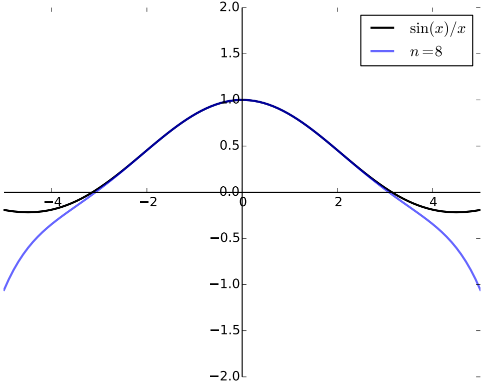


Figure : 8th order Taylor series for $f(x) = \sin(x)/x$ at 0

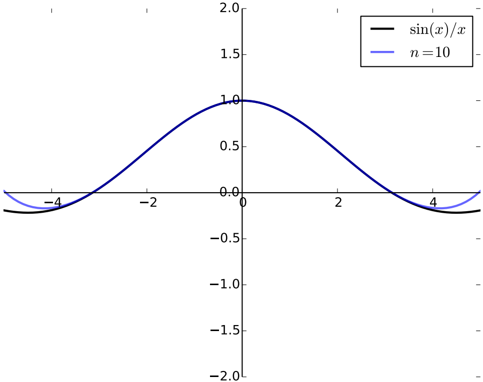


Figure : 10th order Taylor series for $f(x) = \sin(x)/x$ at 0

Key result: **Fundamental Theorem of Calculus**

Fact. Let f and F be real-valued functions on $[a, b]$ such that $F' = f$. If f is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

In fact this is true under slightly weaker assumptions on f