ECON2125/4021/8013

Lecture 20

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Introduction

In this lecture we continue to study nonlinear equations

- Which problems have solutions?
- When do we have uniqueness?
- How can we compute solutions?
- How can we apply these ideas?

We will study these problems from the perspective of fixed point theory

- An important branch of analysis

Fixed Points

Let $T \colon S \to S$ where $S \subset \mathbb{R}^K$

- The function T is a "self-mapping" because it maps S to S
- We write $T\mathbf{x}$ instead of $T(\mathbf{x})$ below

A point $\mathbf{x}^* \in S$ is called a **fixed point** of T if

$$T\mathbf{x}^* = \mathbf{x}^*$$

Related to

- optimization because $\mathbf{x}^*$ solves $\min_{\mathbf{x} \in S} \|T\mathbf{x} - \mathbf{x}\|$
- zeros because $\mathbf{x}^*$ solves $H(\mathbf{x}) = \mathbf{0}$ for $H(\mathbf{x}) := T\mathbf{x} - \mathbf{x}$

Example. If $f\colon \mathbb{R} \to \mathbb{R}$ is the identity $f(x) = x$, then every $x \in \mathbb{R}$ is a fixed point

Example. If $f\colon \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x + 1$, then no $x \in \mathbb{R}$ is a fixed point

Example. Let $f\colon [0,1] \to [0,1]$ be defined by

$$f(x) = 4x(1-x)$$

Then $x = \frac{3}{4}$ is a fixed point of f because

$$f\left(\frac{3}{4}\right) = 4\frac{3}{4}\left(1 - \frac{3}{4}\right) = \frac{3}{4}$$

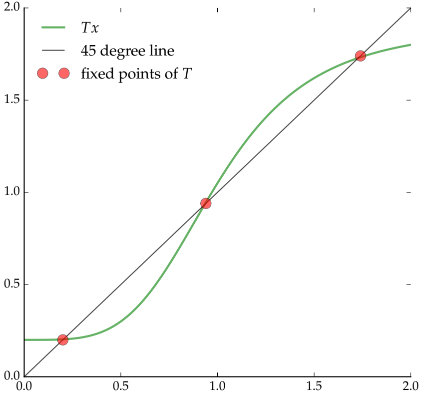


Figure : Fixed points in one dimension

Brouwer's Fixed Point Theorem

Fact. If $S \subset \mathbb{R}^K$ is closed, bounded and convex and $T \colon S \to S$ is continuous, then T has at least one fixed point in S

Proof for case $S = [0,1]$

Let

- T be a continuous function from $[0,1]$ to $[0,1]$
- $f(x) := x - Tx$

Ex. Show that f is continuous on $[0,1]$ and $f(0) \leq 0 \leq f(1)$

Result now follows from the Intermediate Value Theorem

General proof: Quite long, omitted

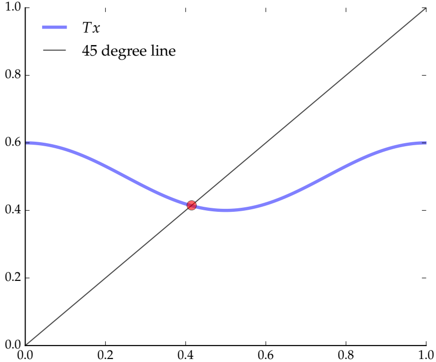


Figure : Brouwer fixed point theorem in one dimension

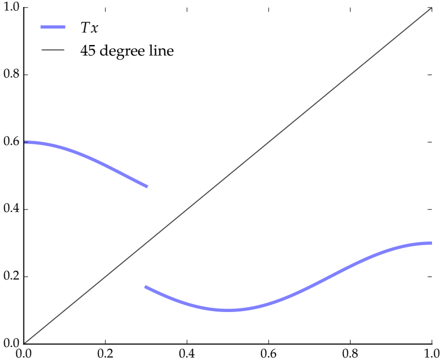


Figure : When continuity fails the theorem does not apply

Contractions

Like the Intermediate Value Theorem, Brouwer's fixed point theorem can give us existence

But do we have uniqueness?

Uniqueness is important in practice

- "My model predicts this..."
    - or this...
        - or this...

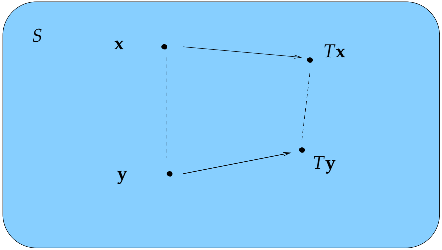
Also important is finding that fixed point

Let's look at a method that makes strong assumptions but gives us uniqueness and a way to find the fixed point

Let $S \subset \mathbb{R}^K$ and let $T \colon S \to S$

T is called a **contraction mapping** on S if

$$\exists\, \beta < 1 \quad \text{s.t.} \quad \|T\mathbf{x} - T\mathbf{y}\| \leq \beta \|\mathbf{x} - \mathbf{y}\| \quad \text{for all} \quad \mathbf{x}, \mathbf{y} \in S$$



Example. Let $T \colon \mathbb{R} \to \mathbb{R}$ be defined by

$$Tx = ax + b$$

where a and b are parameters

For any $x, y \in \mathbb{R}$ we have

$$\begin{aligned} |Tx - Ty| &= |ax + b - ay - b| \\ &= |ax - ay| \\ &= |a(x - y)| \\ &= |a||x - y| \end{aligned}$$

Hence $|a| < 1 \implies T$ is a contraction mapping on $\mathbb{R}$

Fact. If T is a contraction mapping on S then T is continuous on S

Proof: Pick

- any $\mathbf{x} \in S$
- any sequence $\{\mathbf{x}_n\}$ with $\mathbf{x}_n \to \mathbf{x}$

Since T is a contraction on S, we can find a $\beta < 1$ with

$$\|T\mathbf{x}_n - T\mathbf{x}\| \leq \beta \|\mathbf{x}_n - \mathbf{x}\| \quad \forall \, n \in \mathbb{N}$$

Since $\|\mathbf{x}_n - \mathbf{x}\| \to 0$ we see that $\|T\mathbf{x}_n - T\mathbf{x}\| \to 0$

Hence $T\mathbf{x}_n \to T\mathbf{x}$, and T is continuous as claimed

Banach Contraction Mapping Theorem

Fact. If S is closed and T is a contraction mapping on S then

1. T has a unique fixed point $\bar{\mathbf{x}} \in S$
2. $T^n\mathbf{x} \to \bar{\mathbf{x}}$ as $n \to \infty$ for any $\mathbf{x} \in S$

Proof of uniqueness: Suppose that $\mathbf{x}, \mathbf{y} \in S$ with

$$T\mathbf{x} = \mathbf{x} \quad \text{and} \quad T\mathbf{y} = \mathbf{y}$$

Then

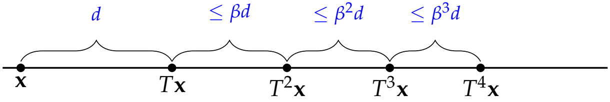
$$\|\mathbf{x} - \mathbf{y}\| = \|T\mathbf{x} - T\mathbf{y}\| \leq \beta\|\mathbf{x} - \mathbf{y}\|$$

Since $\beta < 1$, it must be that $\|\mathbf{x} - \mathbf{y}\| = 0$, and hence $\mathbf{x} = \mathbf{y}$

Sketch of existence proof: Fix $\mathbf{x} \in S$ and let

$$d := \|T\mathbf{x} - \mathbf{x}\|$$

It can be shown that $\|T^{n+1}\mathbf{x} - T^n\mathbf{x}\| \leq \beta^n d$ for all n



One can then show that $\{\mathbf{x}_n\} := \{T^n\mathbf{x}\}$ is Cauchy

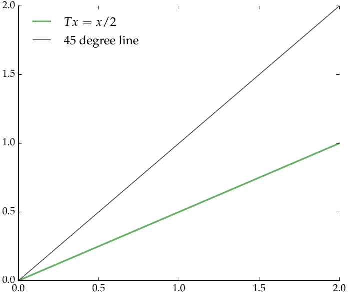
The Cauchy property implies convergence to some $\bar{\mathbf{x}} \in S$

It can then be shown that $\bar{\mathbf{x}}$ is a fixed point

By the way, why does S need to be closed?

An example of failure when S is not closed:

$$Tx = x/2 \quad \text{and} \quad S = (0, \infty)$$



Example. Recall: If $\mathbf{b} \in \mathbb{R}^N$ and $\mathbf{A}$ is $N \times N$ with $\|\mathbf{A}\| < 1$ then $\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{b}$ has a unique solution

- this is part of the Neumann series lemma

One proof: Define $T\colon \mathbb{R}^N \to \mathbb{R}^N$ by $T\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{b}$

A fixed point of $T \iff$ a solution of $\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{b}$

For any $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^N$ we have

$$\begin{aligned} \|T\mathbf{x} - T\mathbf{y}\| &= \|\mathbf{A}\mathbf{x} - \mathbf{A}\mathbf{y}\| \\ &= \|\mathbf{A}(\mathbf{x} - \mathbf{y})\| \\ &\leq \|\mathbf{A}\| \|\mathbf{x} - \mathbf{y}\| \end{aligned}$$

A contraction on $\mathbb{R}^N$ with $\beta := \|\mathbf{A}\|$

Comments on Iteration

Suppose that

- T is a contraction mapping on closed set S
- $\bar{\mathbf{x}}$ is the unique fixed point

We know that for any $\mathbf{x} \in S$ we have $T^n \mathbf{x} \to \bar{\mathbf{x}}$

This means that we can compute the fixed point "iteratively"

1. Pick any $\mathbf{x} \in S$
2. Let $\mathbf{y} = T\mathbf{x}$
3. Set $\mathbf{x} = \mathbf{y}$ and go to step 2

This generates the sequence $\mathbf{x}, T\mathbf{x}, T^2\mathbf{x}, \ldots$

Application: Job Search Again

Let's apply these ideas to solving the the McCall job search model

We seek a $\bar{w}$ that solves the reservation wage equation

$$\bar{w} = c(1 - \beta) + \beta \sum_{k=1}^{K} \max\{w_k, \bar{w}\}\ p_k \tag{$\star$}$$

Here $c > 0$, $\beta \in (0, 1)$ and $p_1, \ldots, p_K$ is a pmf

Note that $\bar{w}$ solves $(\star)$ if and only if it is a fixed point of

$$Tx = c(1 - \beta) + \beta \sum_{k=1}^{K} \max\{w_k, x\}\ p_k$$

Ex. Check it

Claim: The operator T defined by

$$Tx = c(1-\beta) + \beta \sum_{k=1}^{K} \max\{w_k, x\}\, p_k$$

is a contraction mapping on $S := [0, \infty)$

To check this we'll use two facts:

Fact. If $x_1, \ldots, x_K$ are any K numbers, then $\left|\sum_{k=1}^K x_k\right| \leq \sum_{k=1}^K |x_k|$

- Any extension of the triangle inequality to K numbers

Fact. For any a, x, y in $\mathbb{R}$, $|\max\{a, x\} - \max\{a, y\}| \leq |x - y|$

- Draw a picture, check the different possibilities

Proof: For any $x, y \in S$, we have

$$|Tx - Ty| = \left|\beta \sum_{k=1}^{K} \max\{w_k, x\} p_k - \beta \sum_{k=1}^{K} \max\{w_k, y\} p_k\right|$$

$$= \beta \left|\sum_{k=1}^{K} [\max\{w_k, x\} - \max\{w_k, y\}]\ p_k\right|$$

$$\leq \beta \sum_{k=1}^{K} |\max\{w_k, x\} - \max\{w_k, y\}|\ p_k$$

$$\leq \beta \sum_{k=1}^{K} |x - y|\, p_k = \beta |x - y|$$

Since $\beta < 1$, we see that T is indeed a contraction on S

Equivalent Norms

Recall that $\|\mathbf{x}-\mathbf{y}\|$ is a measure of “distance” between $\mathbf{x}$ and $\mathbf{y}$

- called the **Euclidean distance** between $\mathbf{x}$ and $\mathbf{y}$

There are other notions of distance that are also useful

This leads us to introduce the family of ***p*-norms**

$$\|\mathbf{x}\|_p := \left(\sum_{k=1}^{K} |x_k|^p\right)^{1/p} \quad \text{if} \quad 1 \leq p < \infty$$

and

$$\|\mathbf{x}\|_\infty := \max_{1 \leq k \leq K} |x_k|$$

If $p = 2$ then this is the Euclidean norm

Let $p \in [1, \infty]$ and let $\{\mathbf{x}_n\}$ be a sequence in $\mathbb{R}^K$

We say that $\mathbf{x}_n \to \mathbf{x}$ in p-norm if

$$\|\mathbf{x}_n - \mathbf{x}\|_p \to 0 \quad \text{as} \quad n \to \infty$$

If $p = 2$ this is ordinary Euclidean convergence

The next fact generalizes an earlier result about Euclidean distance

Fact. A sequence in $\mathbb{R}^K$ converges in p-norm $\iff$ each component sequence converges in $\mathbb{R}$

That is, for any $p \in [1, \infty]$ and sequence $\{\mathbf{x}_n\}$ we have

$$\|\mathbf{x}_n - \mathbf{x}\|_p \to 0 \quad \iff \quad |\mathbf{e}_k' \mathbf{x}_n - \mathbf{e}_k' \mathbf{x}| \to 0 \text{ in } \mathbb{R} \text{ for all } k$$

We give the proof for $p < \infty$ and leave $p = \infty$ as an **Ex.**

Proof:

($\Longrightarrow$) Suppose first that $\|\mathbf{x}_n - \mathbf{x}\|_p \to 0$

Then, fixing any k in $1, \ldots, K$, we have

$$|\mathbf{e}_k' \mathbf{x}_n - \mathbf{e}_k' \mathbf{x}| = |x_n^k - x^k| \leq \|\mathbf{x}_n - \mathbf{x}\|_p \to 0$$

Ex. Confirm the inequality in the last expression

($\Longleftarrow$) Suppose instead that $|x_n^k - x^k| \to 0$ for all k

Then $|x_n^k - x^k|^p \to 0$ for all k by continuity of $g(x) = x^p$

$$\therefore \quad z_n := |x_n^1 - x^1|^p + \cdots + |x_n^K - x^K|^p \to 0$$

$$\therefore \quad \|\mathbf{x}_n - \mathbf{x}\|_p = z_n^{1/p} \to 0$$

There is an important implication of this result

Fact. For any $p \in [1, \infty]$ and any sequence $\{\mathbf{x}_n\}$,

$$\mathbf{x}_n \to \mathbf{x} \text{ in } p\text{-norm} \quad \iff \quad \mathbf{x}_n \to \mathbf{x} \text{ in Euclidean norm}$$

Proof: Fix $p \in [1, \infty]$ and sequence $\{\mathbf{x}_n\}$

We have

$$\begin{aligned} \mathbf{x}_n \to \mathbf{x} \text{ in } p\text{-norm} \quad &\iff \quad \text{every component sequence converges} \\ &\iff \quad \mathbf{x}_n \to \mathbf{x} \text{ in Euclidean norm} \end{aligned}$$

Here's a nice example of why p-norms are important

Fact. The conclusions of the Banach contraction mapping theorem continue to hold if T is a contraction with respect to any p-norm

Thus, if S is closed and there exists a $p \in [1, \infty]$ and $\beta < 1$ with

$$\|T\mathbf{x} - T\mathbf{y}\|_p \leq \beta \|\mathbf{x} - \mathbf{y}\|_p$$

for all $\mathbf{x}, \mathbf{y} \in S$, then

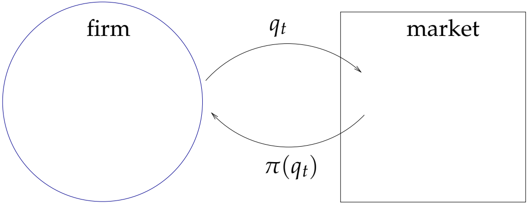
1. T has a unique fixed point $\bar{\mathbf{x}} \in S$
2. $T^n\mathbf{x} \to \bar{\mathbf{x}}$ as $n \to \infty$ for any $\mathbf{x} \in S$

Implication: When we try to show the contraction property, we can pick the most convenient p to work with

Application: A Planning Problem

A firm

- owns stock s_t of a natural resouce (e.g., oil)
- supplies q_t at time t and gets current profit $\pi(q_t)$



- stock next period is $s_{t+1} = s_t - q_t$

Suppose that $t = 0$, current stock is s_0

Given supply sequence $\{q_t\}_{t=0}^{\infty}$, net present value of profits flow is

$$\text{NPV} = \sum_{t=0}^{\infty} \beta^t \pi(q_t) \quad \text{where} \quad \beta := \frac{1}{1+r}$$

Assume the resource is nonrenewable, so

$$\text{sequence } \{q_t\} \text{ feasible} \quad \iff \quad \sum_{t=0}^{\infty} q_t \leq s_0$$

Suppose that

- s_t and q_t take integer values
- $\pi(q) = q^\alpha$ for some $\alpha \in (0,1)$

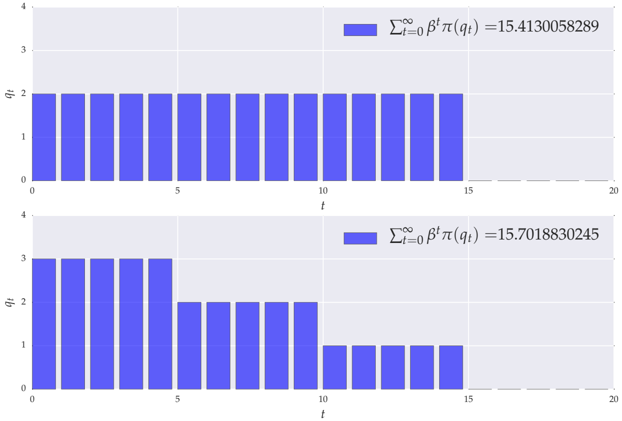


Figure : Present value of different $\{q_t\}$ sequences ($\alpha = 0.5$, $r = 0.05$)

Assume that the firm chooses $\{q_t\}$ to maximize NPV

Let $v^*(s)$ be the NPV corresponding to

- current stock s_0 equal to s
- an optimal supply sequence choice given $s = s_0$

$$v^*(s) = \sup \left\{ \sum_{t=0}^{\infty} \beta^t \pi(q_t) \; : \; \sum_{t=0}^{\infty} q_t \leq s \right\}$$

Thus $v^*(s)$ is the "market value of the firm with current stock s"

How to compute $v^*(s)$ for all $s \leq N =$: some max level of stock?

It turns out that v^* satisfies the equation

$$v^*(s) = \max_{0 \leq q \leq s} \{\pi(q) + \beta v^*(s-q)\} \qquad (s = 0, \ldots, N)$$

Intuition: Max value attained if current q chosen to trade off

- current profits $\pi(q)$
- depletion of stock to $s - q$ weighted by future value

Proof: Omitted — see Bellman's principle of optimality

More intuition / examples of these kinds of recursions coming later

Remark: We're restricting q to be an integer for simplicity

Let $\mathbf{v} = (v(0), \ldots, v(N))$ be any vector in $\mathbb{R}^{N+1}$

Consider creating a new vector $\hat{\mathbf{v}} \in \mathbb{R}^{N+1}$ from $\mathbf{v}$ via

$$\hat{v}(s) = \max_{0 \leq q \leq s} \{\pi(q) + \beta v(s-q)\} \qquad (s = 0, \ldots, N)$$

- $\hat{v}(0) = \max_{0 \leq q \leq 0} \{\pi(q) + \beta v(0-q)\} = \pi(0) + \beta v(0)$
- $\hat{v}(1) = \max_{0 \leq q \leq 1} \{\pi(q) + \beta v(1-q)\} = \cdots$
- $\cdots$

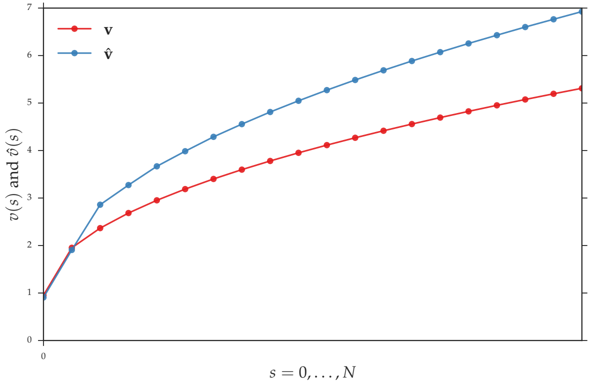


Figure : Creating $\hat{\mathbf{v}}$ from given $\mathbf{v}$

We've specified a rule that creates a new vector $\hat{\mathbf{v}}$ from any existing vector $\mathbf{v}$

We can think of this operation $\mathbf{v} \mapsto \hat{\mathbf{v}}$ as a mapping

Let T be the mapping defined in this way

That is, $\hat{\mathbf{v}} = T\mathbf{v}$ where

$$Tv(s) = \max_{0 \leq q \leq s} \{\pi(q) + \beta v(s-q)\} \qquad (s = 0, 1, \ldots, N)$$

T is a well-defined mapping from $\mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$

Recall that

$$v^*(s) = \max_{0 \le q \le s} \{\pi(q) + \beta v^*(s-q)\}$$

and that $T\colon \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ maps $\mathbf{v}$ to $\hat{\mathbf{v}}$ by

$$Tv(s) = \max_{0 \le q \le s} \{\pi(q) + \beta v(s-q)\}$$

It follows that

$$Tv^*(s) = \max_{0 \le q \le s} \{\pi(q) + \beta v^*(s-q)\} = v^*(s)$$

That is, $T\mathbf{v}^* = \mathbf{v}^*$

Thus, solving for $\mathbf{v}^*$ is the same as finding a fixed point of T

Claim: T is a contraction on $\mathbb{R}^{N+1}$ with p-norm $\| \cdot \|_\infty$

Proof: Pick any $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^{N+1}$ and any s in $0, 1, \ldots, N$

By definition,

$$|Tv(s) - Tw(s)| =$$

$$\left| \max_{0 \leq q \leq s} \{\pi(q) + \beta v(s-q)\} - \max_{0 \leq q \leq s} \{\pi(q) + \beta w(s-q)\} \right|$$

Recall now the rule

$$|\sup_{x \in A} f(x) - \sup_{x \in A} g(x)| \leq \sup_{x \in A} |f(x) - g(x)|$$

Hence

$$
\begin{aligned}
|Tv(s) - Tw(s)| &\leq \max_{0\leq q\leq s} |\pi(q) + \beta v(s-q) - (\pi(q) + \beta w(s-q))| \\
&= \beta \max_{0\leq q\leq s} |v(s-q) - w(s-q)| \\
&\leq \beta \max_{0\leq u\leq N} |v(u) - w(u)| \\
&= \beta \|\mathbf{v} - \mathbf{w}\|_\infty
\end{aligned}
$$

Since the last term is an upper bound on $|Tv(s) - Tw(s)|$, we have

$$
\|T\mathbf{v} - T\mathbf{w}\|_\infty \leq \beta \|\mathbf{v} - \mathbf{w}\|_\infty
$$

What we know so far

- T has a unique fixed point in $\mathbb{R}^{N+1}$
- that fixed point is $\mathbf{v}^*$, the object we want to compute
- If $\mathbf{v}$ is any point in $\mathbb{R}^{N+1}$, then $T^k\mathbf{v} \to \mathbf{v}^*$

So let's pick $\mathbf{v}$ and iterate with T

In practice we

1. Iterate until $\|T^k\mathbf{v} - T^{k+1}\mathbf{v}\|_\infty < \epsilon :=$ small error tolerance
2. Take the final $T^k\mathbf{v}$ as approximate solution for $\mathbf{v}^*$

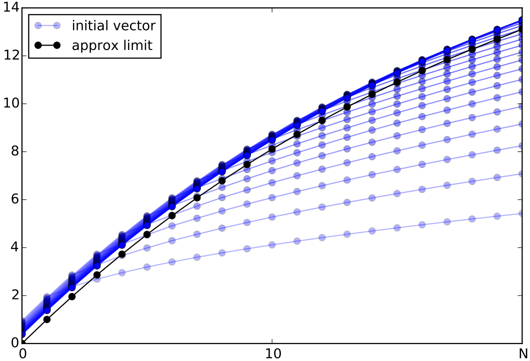


Figure : The sequence $\mathbf{v}, T\mathbf{v}, T^2\mathbf{v}, \ldots$ and limit

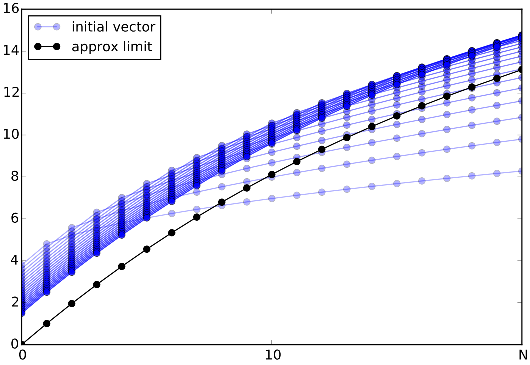


Figure : Iterates with alternative initial condition

Comparative Statics

Next we usually look at the properties of the solution

Example. How is the value of the firm affected by r?

Intuitively, higher interest rate decreases net present value

Let's

- compute approximate $\mathbf{v}^*$ associated with different r
- see whether they do go down as r goes up

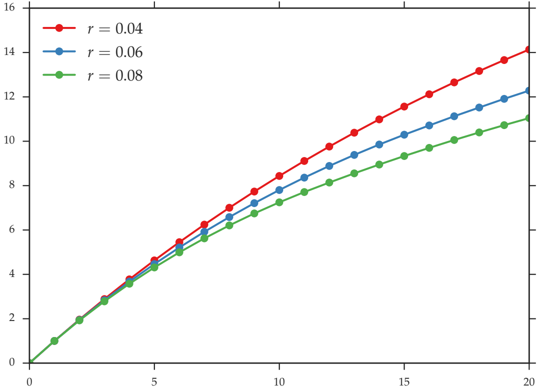


Figure : The vector $\mathbf{v}^*$ computed at different values of r