ECON2125/8013

Lecture 3

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Semester 1, 2015

Constrained Optimization

A major focus of econ: the optimal allocation of scarce resources

- Optimal means optimization, scarce means constrained

Standard constrained problems:

- Maximize utility given budget
- Maximize portfolio return given risk constraints
- Minimize cost given output requirement

Example. Maximization of utility subject to budget constraint

$$\max u(x_1, x_2) \;\; \text{s.t.} \;\; p_1x_1 + p_2x_2 \leq m$$

Here

- p_i is the price of good i, assumed > 0
- m is the budget, assumed > 0
- $x_i \geq 0$ for $i = 1, 2$

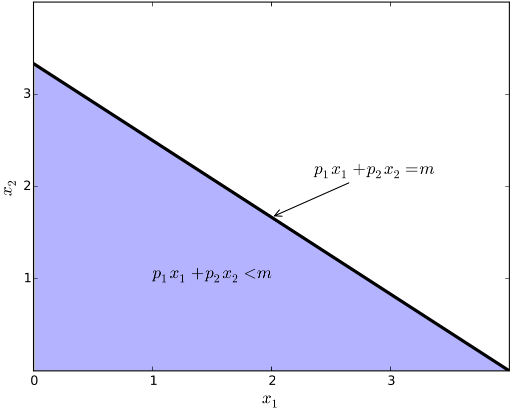

Figure : Budget set when, $p_1 = 1$, $p_2 = 1.2$, $m = 4$

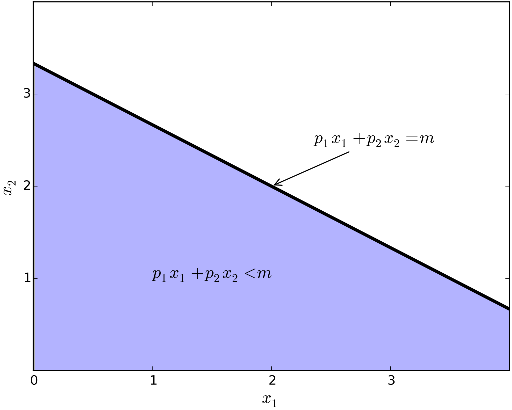

Figure : Budget set when, $p_1 = 0.8$, $p_2 = 1.2$, $m = 4$

Example. Suppose we want to solve

$$\max u(x_1, x_2) \;\; \text{s.t.} \;\; p_1 x_1 + p_2 x_2 \leq m$$

Let's assume that

$$u(x_1, x_2) = \alpha \log(x_1) + \beta \log(x_2)$$

where

- $0 < \alpha, \beta$

Let's recall the utility function's shape

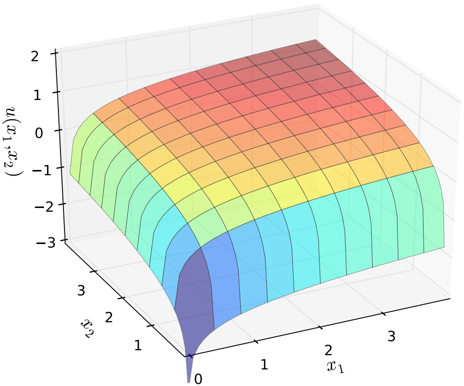

Figure : Log utility with $\alpha = 0.4$, $\beta = 0.5$

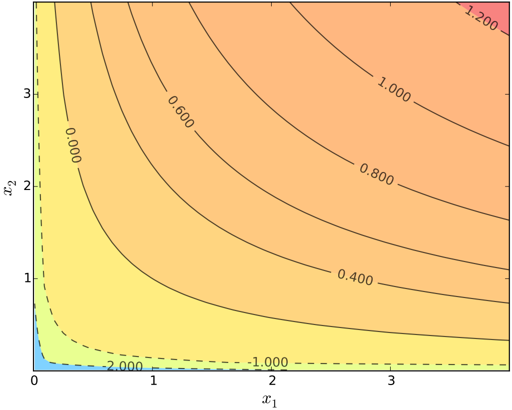

Figure : Log utility with $\alpha = 0.4$, $\beta = 0.5$

We seek a bundle (x_1^*, x_2^*) that maximizes u over the budget set

That is,

$$\alpha \log(x_1^*) + \beta \log(x_2^*) \geq \alpha \log(x_1) + \beta \log(x_2)$$

for all (x_1, x_2) satisfying $x_i \geq 0$ for each i and

$$p_1 x_1 + p_2 x_2 \leq m$$

Visually, here is the budget set and objective function:

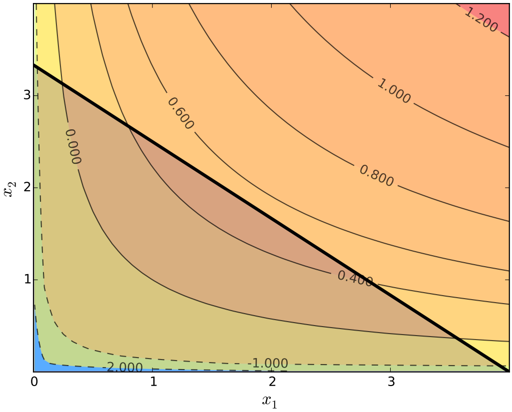

Figure : Utility max for $p_1 = 1$, $p_2 = 1.2$, $m = 4$, $\alpha = 0.4$, $\beta = 0.5$

First observation: $u(0, x_2) = u(x_1, 0) = u(0, 0) = -\infty$

- Hence we need consider only strictly positive bundles

Second observation: $u(x_1, x_2)$ is strictly increasing in both x_i

- Never choose a point (x_1, x_2) with $p_1 x_1 + p_2 x_2 < m$
- Otherwise can increase $u(x_1, x_2)$ by small change

Hence we can replace $\leq$ with $=$ in the constraint

$$p_1 x_1 + p_2 x_2 \leq m \quad \text{becomes} \quad p_1 x_1 + p_2 x_2 = m$$

Implication: Just search along the budget line

Substitution Method

Substitute constraint into objective function

This changes

$$\max_{x_1,x_2} \{\alpha \log(x_1) + \beta \log(x_2)\} \text{ s.t. } p_1x_1 + p_2x_2 = m$$

into

$$\max_{x_1} \{\alpha \log(x_1) + \beta \log((m - p_1x_1)/p_2)\}$$

Since all candidates satisfy $x_1 > 0$ and $x_2 > 0$, the domain is

$$0 < x_1 < \frac{m}{p_1}$$

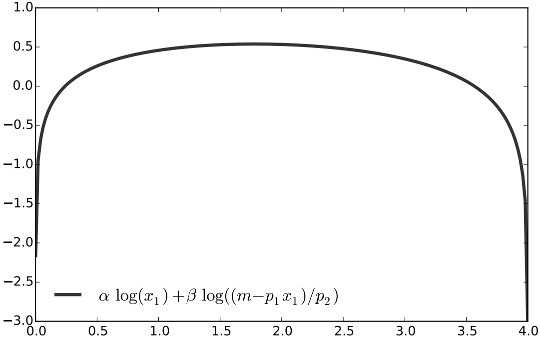

Figure : Utility max for $p_1 = 1$, $p_2 = 1.2$, $m = 4$, $\alpha = 0.4$, $\beta = 0.5$

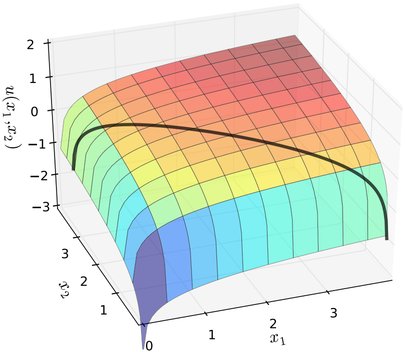

Figure : Utility max for $p_1 = 1$, $p_2 = 1.2$, $m = 4$, $\alpha = 0.4$, $\beta = 0.5$

First order condition for

$$\max_{x_1} \{\alpha \log(x_1) + \beta \log((m - p_1 x_1)/p_2)\}$$

gives

$$x_1^* = \frac{\alpha}{\alpha + \beta} \cdot \frac{m}{p_1}$$

Ex. Verify

Ex. Check second order condition (strict concavity)

Substituting into $p_1 x_1^* + p_2 x_2^* = m$ gives

$$x_2^* = \frac{\beta}{\beta + \alpha} \cdot \frac{m}{p_2}$$

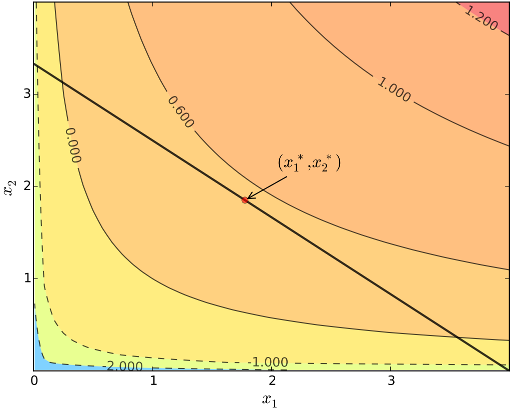

Figure : Maximizer for $p_1 = 1$, $p_2 = 1.2$, $m = 4$, $\alpha = 0.4$, $\beta = 0.5$

Substitution Method Cookbook

How to solve

$$\max_{x_1, x_2} \ f(x_1, x_2)$$

$$\text{s.t. } g(x_1, x_2) = 0$$

Steps:

1. Write constraint as $x_2 = h(x_1)$ for some function h
2. Solve univariate problem $\max_{x_1} f(x_1, h(x_1))$ to get x_1^*
3. Plug x_1^* into $x_2 = h(x_1)$ to get x_2^*

Example. (Minimization)

Consider the simple problem

$$\min_{x_1,x_2} \{x_1^2 + x_2^2\}$$

$$\text{s.t. } x_1 + x_2 - 10 = 0$$

1. Write constraint as $x_2 = 10 - x_1$
2. Solve $\min_{x_1}\{x_1^2 + (10 - x_1)^2\}$ to get $x_1^* = 5$
3. Plug $x_1^* = 5$ into $x_1 + x_2 = 10$ to get $x_2^* = 5$

Limitations

Substitution doesn't always work

Example. Suppose that $g(x_1, x_2) = x_1^2 + x_2^2 - 1$

Step 1 from the cookbook says

use $g(x_1, x_2) = 0$ to write x_2 as a function of x_1

But x_2 has two possible values for each $x_1 \in (-1, 1)$

$$x_2 = \pm\sqrt{1 - x_1^2}$$

In other words, x_2 is not a well defined function of x_1

As we meet more complicated constraints such problems intensify

- Constraint defines complex curve in (x_1, x_2) space
- Inequality constraints, etc.

We need more general, systematic approaches too

Leads to next discussion

Tangency

Consider again the problem

$$\max_{x_1,x_2} \{\alpha \log(x_1) + \beta \log(x_2)\}$$

$$\text{s.t. } p_1 x_1 + p_2 x_2 = m$$

Turns out that the maximizer has the following property:

- budget line is tangent to an indifference curve at maximizer

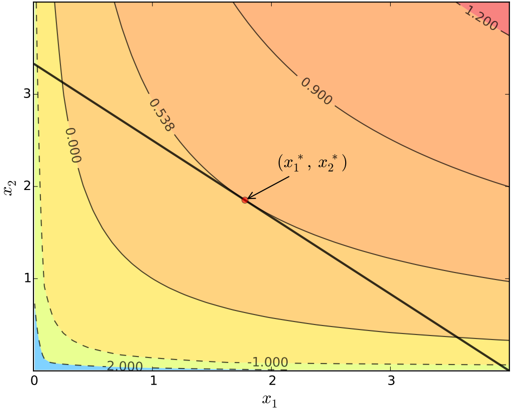

Figure : Maximizer for $p_1 = 1$, $p_2 = 1.2$, $m = 4$, $\alpha = 0.4$, $\beta = 0.5$

In fact this is an instance of a general pattern

Notation: Let's call (x_1, x_2) interior to the budget line if $x_i > 0$ for $i = 1, 2$

- Not a "corner" solution

In general, any interior maximizer (x_1^*, x_2^*) of differentiable utility function u has the property

budget line is tangent to a contour line at (x_1^*, x_2^*)

Otherwise we can do better:

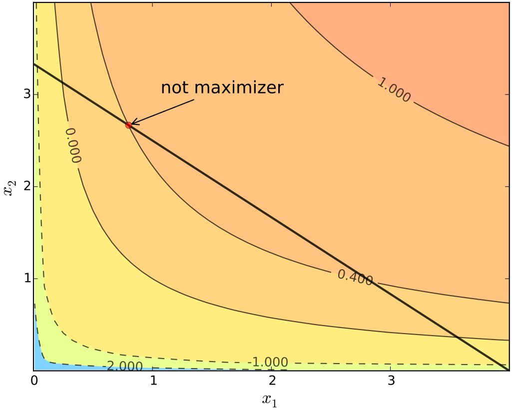

Figure : When tangency fails we can do better

Necessity of tangency often rules out a lot of points

Can we exploit this fact to

- Build intuition
- Develop more general methods?

The answer is yes

Using Tangency: Relative Slope Conditions

- Relies on tangency idea discussed above
- Generalizes nicely

Consider the smooth, equality constrained optimization problem

$$\max_{x_1,x_2} \; f(x_1,x_2)$$

$$\text{s.t. } g(x_1,x_2)=0$$

How to develop necessary conditions for optima via tangency?

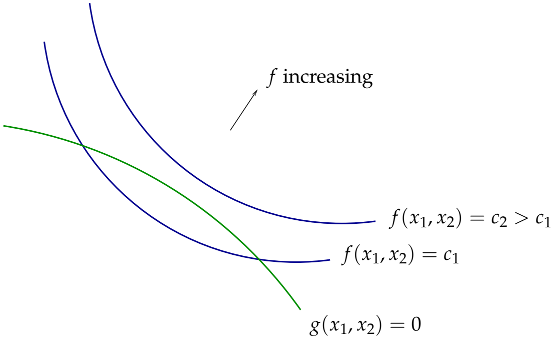

Figure : Contours for f and g

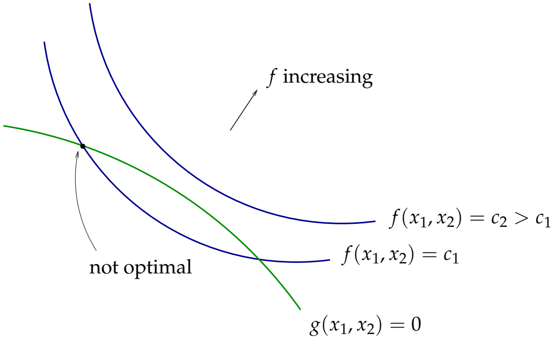

Figure : Contours for f and g

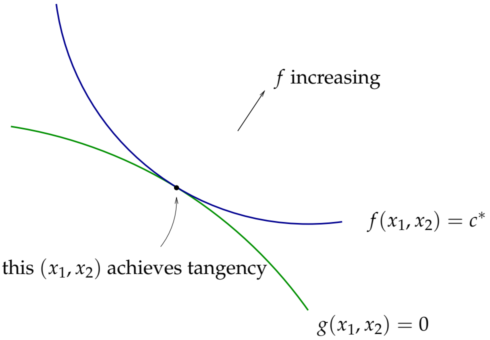

Figure : Tangency necessary for optimality

How do we locate such an (x_1, x_2) pair?

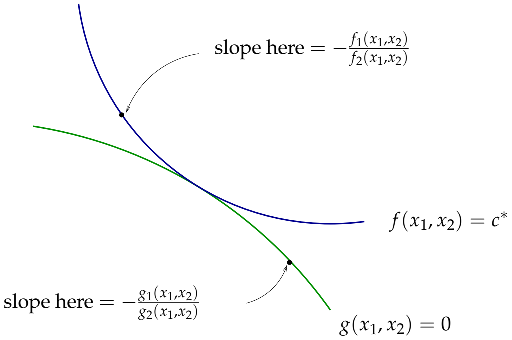

Figure : Slope of contour lines

Sketch of proof for case of f

Let's fix c and vary x_2 with x_1 to maintain $f(x_1, x_2) = c$

This implicitly defines x_2 as a function of x_1

The slope of this function is what we're after

Differentiating $f(x_1, x_2(x_1)) = c$ with respect to x_1 gives

$$f_1(x_1, x_2) + f_2(x_1, x_2)x_2'(x_1) = 0$$

Solving gives slope $= x_2'(x_1) = -f_1(x_1, x_2)/f_2(x_1, x_2)$

Proper proof: See formula for implicit differentiation

Now let's choose (x_1, x_2) to equalize the slopes

That is, choose (x_1, x_2) to solve

$$-\frac{f_1(x_1, x_2)}{f_2(x_1, x_2)} = -\frac{g_1(x_1, x_2)}{g_2(x_1, x_2)}$$

Equivalent:

$$\frac{f_1(x_1, x_2)}{f_2(x_1, x_2)} = \frac{g_1(x_1, x_2)}{g_2(x_1, x_2)}$$

Also need to respect $g(x_1, x_2) = 0$

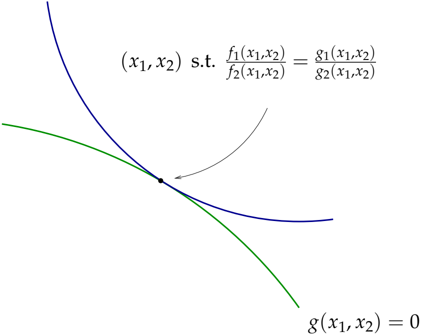

Figure : Condition for tangency

Tangency Condition Cookbook

In summary, when f and g are both differentiable functions, to find candidates for optima in

$$\max_{x_1, x_2} f(x_1, x_2)$$

$$\text{s.t. } \; g(x_1, x_2) = 0$$

1. (Impose slope tangency) Set

$$\frac{f_1(x_1, x_2)}{f_2(x_1, x_2)} = \frac{g_1(x_1, x_2)}{g_2(x_1, x_2)}$$

2. (Impose constraint) Set $g(x_1, x_2) = 0$
3. Solve simultaneously for (x_1, x_2) pairs satisfying these conditions

Example. Consider again

$$\max_{x_1,x_2} \{\alpha \log(x_1) + \beta \log(x_2)\}$$

$$\text{s.t.} \;\; p_1x_1 + p_2x_2 - m = 0$$

Then

$$\frac{f_1(x_1,x_2)}{f_2(x_1,x_2)} = \frac{g_1(x_1,x_2)}{g_2(x_1,x_2)} \quad \Longleftrightarrow \quad \frac{\alpha}{\beta}\frac{x_2}{x_1} = \frac{p_1}{p_2}$$

Solving simultaneously with $p_1x_1 + p_2x_2 = m$ gives

$$x_1^* = \frac{\alpha}{\alpha+\beta}\cdot\frac{m}{p_1} \quad \text{and} \quad x_2^* = \frac{\beta}{\beta+\alpha}\cdot\frac{m}{p_2}$$

Same as before...

Slope Conditions for Minimization

Good news: The conditions are exactly the same

In particular:

- Lack of tangency means not optimizer
- Constraint must be satisfied

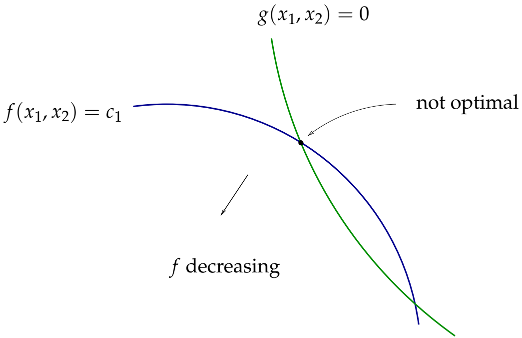

Figure : Lack of tangency

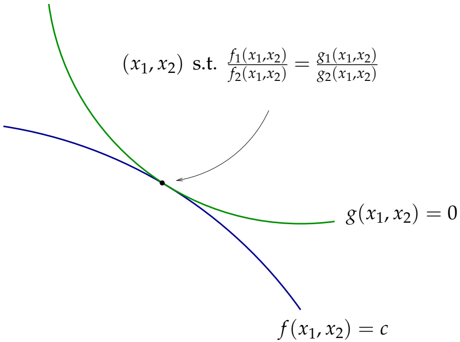

Figure : Condition for tangency

Example. Minimize cost for a given level of production q

$$\min_{k,\ell}\{rk + w\ell\}$$
$$\text{s.t.} \quad Ak^{\alpha}\ell^{\beta} \geq q$$

All parameters assumed to be strictly positive

Since inputs are costly, any minimizer will produce exactly q

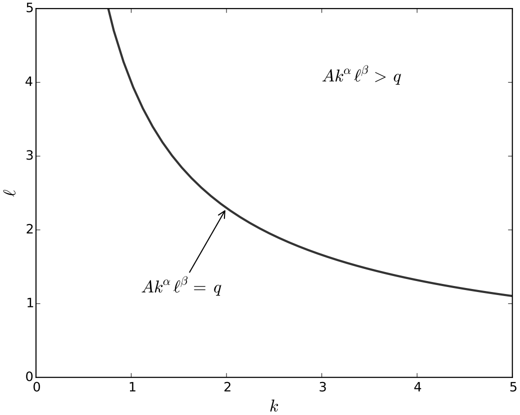

Figure : Constraint set when $A = 2.0$, $\alpha = 0.4$, $\beta = 0.5$, $q = 4.0$

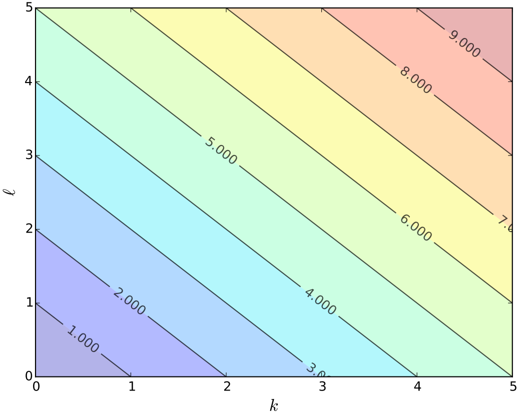

Figure : Objective function $rk + w\ell$ when $w = r = 1.0$

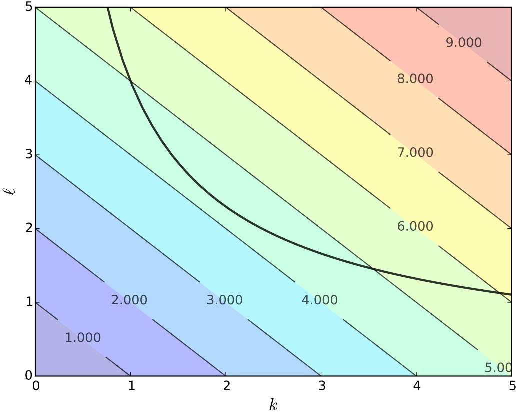

Figure : Together, same parameters

To apply the method we set

$$g(k, \ell) := Ak^{\alpha}\ell^{\beta} - q = 0$$

and

$$f(k, \ell) = rk + w\ell$$

Now let's apply the conditions to obtain candidates for minima

Slope tangency condition:

$$\frac{f_1(k,\ell)}{f_2(k,\ell)} = \frac{g_1(k,\ell)}{g_2(k,\ell)} \iff \frac{r}{w} = \frac{A\alpha k^{\alpha-1}\ell^{\beta}}{Ak^{\alpha}\beta\ell^{\beta-1}} = \frac{\alpha\ell}{\beta k}$$

Combine this with the constraint

$$g(k,\ell) = Ak^{\alpha}\ell^{\beta} - q = 0$$

to get

$$k^* = \left[\frac{q}{A}\left(\frac{w\alpha}{r\beta}\right)^{\beta}\right]^{1/(\alpha+\beta)} \quad \text{and} \quad \ell^* = \left[\frac{q}{A}\left(\frac{r\beta}{w\alpha}\right)^{\alpha}\right]^{1/(\alpha+\beta)}$$

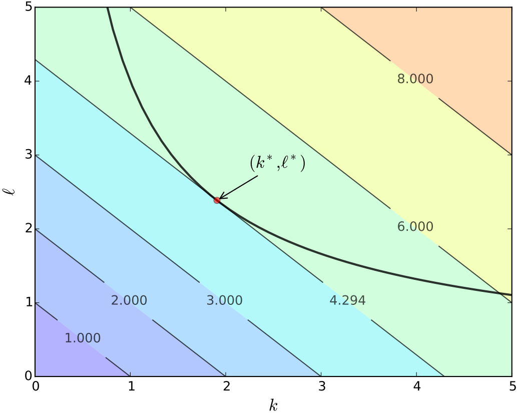

Figure : Minimizer (k^*, ℓ^*)

Example. Intertemporal problem

$$\max_{c_1, c_2} U(c_1, c_2) := u(c_1) + \beta u(c_2)$$
$$\text{s.t. } \; c_2 \leq (1+r)(w - c_1)$$

where

- r = interest rate, w = wealth, β = discount factor
- all parameters > 0 and u strictly increasing

Write constraint as $g(c_1, c_2) := (1+r)(w - c_1) - c_2 = 0$

Any interior solution must satisfy tangency condition

$$\frac{U_1(c_1, c_2)}{U_2(c_1, c_2)} = \frac{g_1(c_1, c_2)}{g_2(c_1, c_2)} \quad \iff \quad \frac{u'(c_1)}{u'(c_2)} = \beta(1+r)$$

Method of Lagrange

The "standard machine" for optimization with equality constraints

$$\max_{x_1,x_2} f(x_1,x_2) \ \text{ s.t. } \ g(x_1,x_2)=0$$

Set

$$\mathcal{L}(x_1,x_2,\lambda) := f(x_1,x_2) + \lambda g(x_1,x_2)$$

and solve

$$\frac{\partial}{\partial x_1}\mathcal{L}=0, \quad \frac{\partial}{\partial x_2}\mathcal{L}=0, \quad \frac{\partial}{\partial \lambda}\mathcal{L}=0$$

simultaneously

Since $\mathcal{L}(x_1, x_2, \lambda) := f(x_1, x_2) + \lambda g(x_1, x_2)$ we have

$$\frac{\partial}{\partial x_i}\mathcal{L}(x_1, x_2, \lambda) = 0 \iff f_i(x_1, x_2) = -\lambda g_i(x_1, x_2),\ i = 1, 2$$

Hence $\frac{\partial}{\partial x_1}\mathcal{L} = \frac{\partial}{\partial x_2}\mathcal{L} = 0$ gives

$$\frac{f_1(x_1, x_2)}{f_2(x_1, x_2)} = \frac{g_1(x_1, x_2)}{g_2(x_1, x_2)}$$

Finally

$$\frac{\partial}{\partial \lambda}\mathcal{L}(x_1, x_2, \lambda) = 0 \iff g(x_1, x_2) = 0$$

Hence the method leads us to the same conditions

Extensions

Let's look at some problems and extensions

Remark 1: The direct tangent slope condition can fail if we're dividing by zero in

$$\frac{f_1(x_1, x_2)}{f_2(x_1, x_2)} = \frac{g_1(x_1, x_2)}{g_2(x_1, x_2)}$$

In this case try the more general Lagrange conditions

$$f_1(x_1, x_2) + \lambda g_1(x_1, x_2) = 0$$

$$f_2(x_1, x_2) + \lambda g_2(x_1, x_2) = 0$$

Remark 2: Consider the two optimization problems

$$\max_{x_1,x_2} \{\alpha \log(x_1) + \beta \log(x_2)\}$$

$$\text{s.t. } p_1x_1 + p_2x_2 = m$$

and

$$\max_{x_1,x_2} x_1^\alpha x_2^\beta$$

$$\text{s.t. } p_1x_1 + p_2x_2 = m$$

The tangency conditions are identical

$$\frac{\alpha}{\beta}\frac{x_2}{x_1} = \frac{p_1}{p_2} \quad \text{and} \quad = \frac{\alpha x_1^{\alpha-1}x_2^\beta}{\beta x_1^\alpha x_2^{\beta-1}} = \frac{\alpha}{\beta}\frac{x_2}{x_1} = \frac{p_1}{p_2}$$

More generally, maximizers are unchanged by increasing transformations

- Can be useful to simplify your problem

On the other hand maximum values are changed, of course

More on this later...

Corner Solutions

So far all our solutions have been interior ($x_i > 0$ for $i = 1, 2$)

Such solutions can be tracked down by the tangency conditions

However sometimes solutions are naturally on the boundaries

Example. Maximize $x_1 + \log(x_2)$ subject to

$$p_1 x_1 + p_2 x_2 = m \quad \text{and} \quad x_1, x_2 \geq 0$$

Let's try the tangency approach with $p_1 = p_2 = 1$ and $m = 0.4$

Tangency condition is

$$\frac{1}{1/x_2} = \frac{p_1}{p_2} \quad \Longleftrightarrow \quad x_2 = \frac{p_1}{p_2} = 1$$

Applying the budget constraint gives

$$x_1 + x_2 = 0.4 \quad \text{and hence} \quad x_1 = -0.6$$

Meaning: There is no tangent point in

$$D := \{(x_1, x_2) : x_1 \geq 0,\ x_2 \geq 0 \text{ and } p_1 x_1 + p_2 x_2 = m\}$$

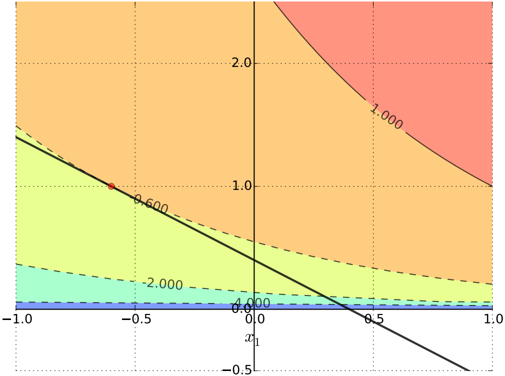

Figure : Tangent point is infeasible

Interpretation: No interior solution

Put differently

- At every interior point on the budget line you can do better
- Hence solution must be on the boundary

Since $x_2 = 0$ implies $x_1 + \log(x_2) = -\infty$, solution is

- $x_1^* = 0$
- $x_2^* = m/p_2 = 0.4$

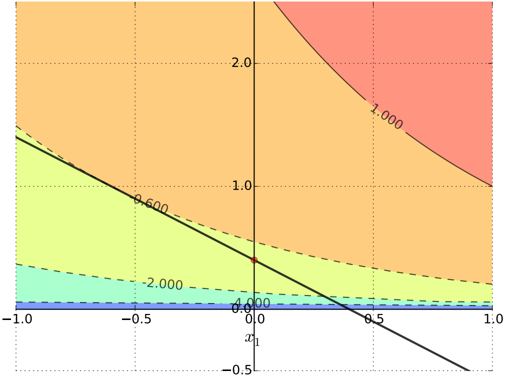

Figure : Corner solution