2 - HIGHER-ORDER, CASCADED, ACTIVE FILTERS

In the previous chapter we have shown how to design second-order active filters. However, most filter applications require an order higher than two. The objective of this section will be to show how to use the first- and second-order filters to achieve higher order filters. We shall also introduce a general, second-order stage called the biquad. The biquad is useful as a general component in filter realizations.

Higher-order active filters as presented in this section is strictly a design activity. We will begin by understanding how a filter is specified. A great deal of tabular information is available as a starting point for the design. This tabular information is generally presented in the form of a normalized low-pass filter as shown in Fig. 2-1.

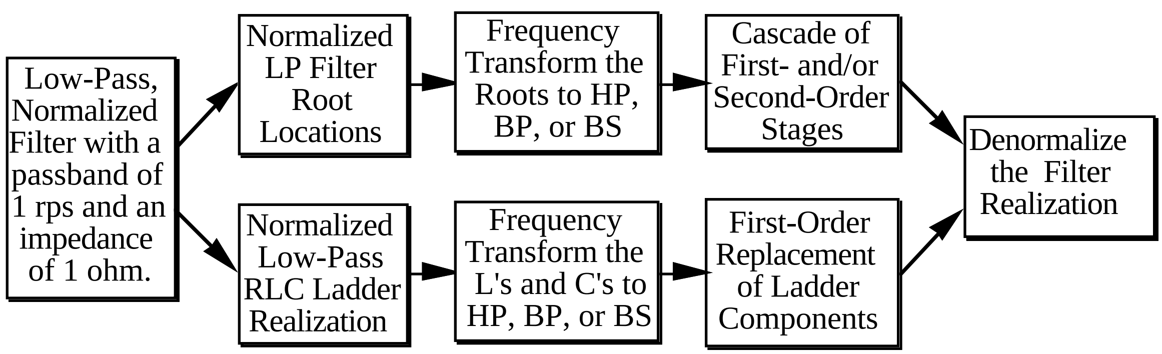

Figure 2-1 - General design approach for active filters of Chapters 2 and 3.

Two design approaches are used design higher-order active filters. One is based on realizing the root locations of the filter with cascaded first- and second-order stages and the second is based on replacing components of a passive RLC ladder filter with first-order stages. We will postpone our discussion of ladder filter design to the next section. If the filter is to be other than low-pass, then it is necessary to transform the filter design to a high-pass (HP), bandpass (BP), or bandstop (BS). The last step in higher-order, low-pass filter design is to denormalize the design to achieve the actual performance specifications.

The subject of active filters is an extensive one and encompasses more material than can be presented in this section. We shall attempt to give an overview and illustrate some of the basic concepts and design procedures. For more information, the reader is referred to the many excellent texts that cover this subject in much more detail†. In the simulation part of this section, we shall illustrate one of the computer-aided design approaches presently available for the design of active filters.

Ideal Filters

Filters are generally classified by their magnitude response in the frequency domain only. However, there are some filters, which will not be considered here, that are characterized by both their magnitude and phase response. The magnitude response of an ideal filter will be divided into two types of regions. One region has a gain of unity and is called the *passband*. Even though filters can have gains greater or less than unity in the passband, we shall consider the passband gain unity for purposes of simplicity. The second region has a gain of zero and is called the *stopband*. We shall also assume that the ideal filter can have the passband adjacent to the stopband and that ω_T is the frequency where the transition is made from one band to the other.

Fig. 2-2 shows the four categories of filters that we shall consider. The first category is called a *low-pass filter* and has a gain of 1 from 0 to ω_T and a gain of zero for all frequencies greater than ω_T. The ideal low-pass filter is illustrated by Fig. 2-2a. The magnitude response of this filter is given as

$$T_{LP}(j\omega)| = \begin{cases} 1 & 0 \le \omega \le \omega_T \\ 0 & \omega_T < \omega < \infty \end{cases} \quad . \tag{2-1}$$

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† A. Budak, *Passive and Active Network Analysis and Synthesis*, Houghton Mifflin Co., Boston, 1974.
J.L. Hilburn and D.E. Johnson, *Manual of Active Filter Design*, McGraw-Hill Book Co., NY, 1973.
L.P. Huelsman and P.E. Allen, *Introduction to the Theory and Design of Active Filters*, McGraw-Hill Book Co., NY, 1980.
R. Schaumann, M. S. Ghausi, and K.R. Laker, *Design of Analog Filters*, Prentice-Hall, Englewood Cliffs, NJ, 1990.
A.S. Sedra and P.O. Brackett, *Filter Theory and Design: Active and Passive*, Matrix Publishers Inc., Portland, OR, 1977.
M.E. Van Valkenburg, *Introduction to Modern Network Synthesis*, John Wiley & Sons, Inc. NY, 1960.
L. Weinburg, *Network Analysis and Synthesis*, McGraw-Hill Book Co., 1962, R.E. Krieger Publishing Co., Huntington, NY, 1975.
A.I. Zverev, *Handbook of Filter Synthesis*, John Wiley & Sons, Inc., NY, 1967.

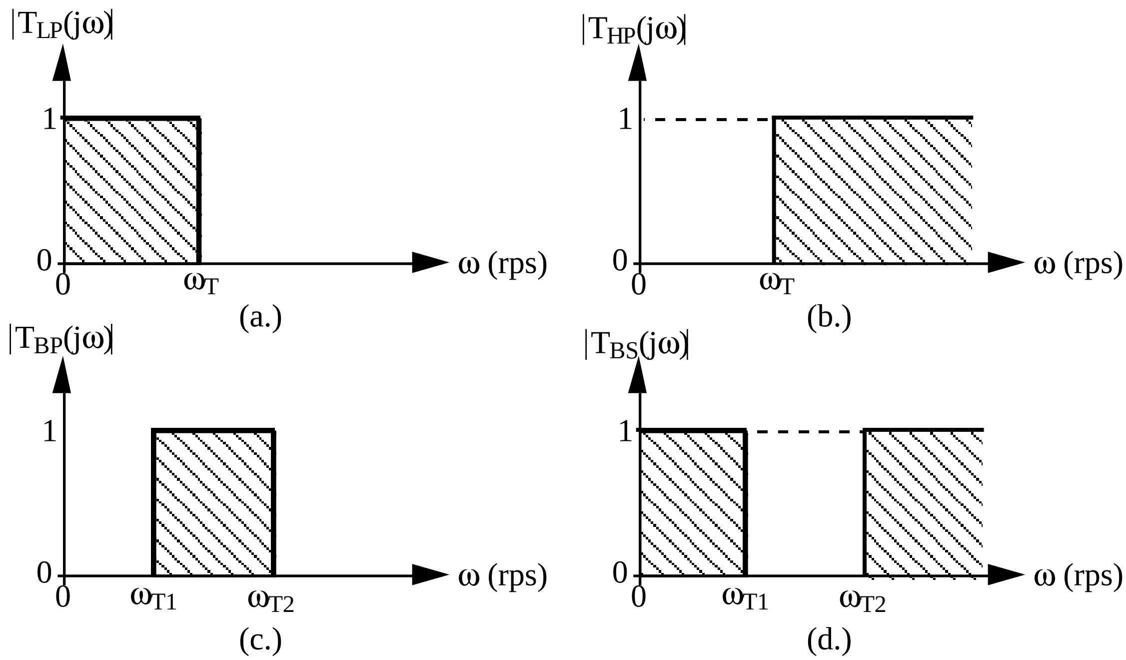

Figure 2-2 - Ideal magnitude responses of (a.) low-pass, (b.) high-pass, (c.) bandpass, and (d.) bandstop filter.

The second category of ideal filter is the *high-pass filter* and has a gain of 0 from 0 to ω_T and a gain of 1 for all frequencies greater than or equal to ω_T. The ideal high-pass filter is shown in Fig. 2-2b. The magnitude response of the high-pass filter is given as

$$T_{HP}(j\omega)| = \begin{cases} 0 & 0 \leq \omega < \omega_T \\ 1 & \omega_T \leq \omega < \infty \end{cases} \quad . \tag{2-2}$$

The third category of ideal filters is the *bandpass filter* and has a gain of 1 between the frequencies of ω_{T1} and ω_{T2} and a gain of zero elsewhere. The magnitude response of an ideal bandpass filter is shown in Fig. 2-2c and is mathematically expressed as

$$T_{BP}(j\omega)| = \begin{cases} 0 & 0 \leq \omega < \omega_{T1} \\ 1 & \omega_{T1} \leq \omega < \omega_{T2} \\ 0 & \omega_{T2} < \omega < \infty \end{cases} \quad . \tag{2-3}$$

The fourth and last category of ideal filters is the *bandstop filter* and has a gain of 0 between the frequencies of ω_{T1} and ω_{T2} and a gain of 1 elsewhere. The magnitude response of an ideal bandpass filter is shown in Fig. 2-2d and is mathematically expressed as

$$T_{BP}(j\omega)| = \begin{cases} 1 & 0 \leq \omega < \omega_{T1} \\ 0 & \omega_{T1} \leq \omega < \omega_{T2} \\ 1 & \omega_{T2} < \omega < \infty \end{cases} \quad . \tag{2-4}$$

The phase response of each of these ideal filters is exactly the same. It is given as

$$\mathrm{Arg}[T(j\omega)] = -\omega T_d\,, \quad 0 \leq \omega \leq \infty \quad . \tag{2-5}$$

We see the ideal phase shift is a straight line passing through 0 degrees at $\omega = 0$ and having a slope of $-T_d$. Because the time delay of a filter is equal to the negative derivative of the phase shift with respect to frequency, T_d is called the *time delay* of the filter and is constant for all frequencies.

Practical Filters

Practical filters cannot have a continuous band of zero gain nor can they have an instantaneous transition from the passband to the stopband. Consequently, we need to extend our ideal filter specifications to include practical filters. This is done by defining the filter magnitude response in terms of constraints separated by finite-width transition regions. Fig. 2-3 shows the magnitude specifications for the four types of filters that can be realized by active and/or passive circuits.

The low-pass filter specification of Fig. 2-3a has been divided into three frequency regions. From 0 to ω_{PB} the magnitude must be between 1 and T_{PB}. This is passband region. From ω_{SB} to infinity, the magnitude must be less than T_{SB}. This is the stopband region. The region between ω_{PB} and ω_{SB} is called the *transition region*. The magnitude of the filter is unspecified in this region, although a good approximation would be expected to closely follow a straight line drawn from point A to point B. Any filter design which stays within the shaded regions and approximates a straight-line from A to B in the transition region will satisfy the filter specifications. The line shown on Fig. 2-3a is an example of one possible filter realization whose magnitude response would satisfy the specifications.

It is not necessary that the magnitude response oscillate as shown, but this characteristic permits practical filters to have a smaller transition region than filter realizations whose slope is monotonic.

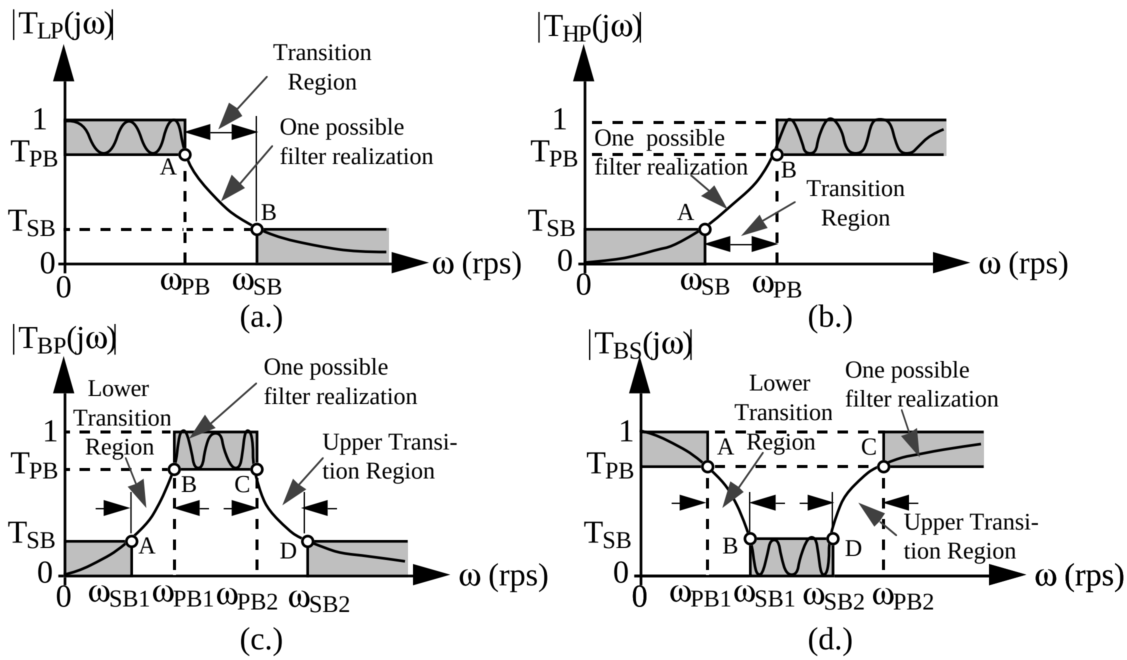

Figure 2-3 - Practical magnitude responses of (a.) low-pass, (b.) high-pass, (c.) bandpass, and (d.) bandstop filter.

Figs. 2-3b through 2-d are practical magnitude specifications for the high-pass, bandpass, and bandstop filters, respectively. In each case, the magnitude of the filter must fall within the shaded areas. The magnitude of the filter in the transition region(s) should approximate a straight-line through points A and B or C and D, in the case of the bandpass and bandstop filters.

In order to simplify the filter design procedure, all filter design begins with a normalized, low-pass filter specification. The normalized low-pass filter is a structure from which all other filters can be derived by denormalization or transformation. The low-pass filter is normalized so that ω_{PB} of Fig. 2-3a is unity. The low-pass filter has also been normalized to an impedance level of 1 ohm. Fig. 2-1 illustrates the low-pass, normalized filter is the starting point of all filter design. The normalizations used in the low-pass, filter are defined as follows. The normalized frequency variable, s_n, is defined as

$$s_n = \frac{s}{\omega_{PB}} \tag{2-6}$$

where we use s_n to indicate the frequency normalized complex variable. The normalization impedance, Z_n, used in the low-pass filter is defined as

$$Z_n = \frac{Z}{z_o} \tag{2-7}$$

where Z is the unnormalized impedance and z_o is a unitless impedance scaling constant. In RLC passive filters, the impedance level is important. However, in active filter design, the impedance denormalization amounts to a simple impedance scaling constant. We must remember that the normalization definitions in Eqs. (2-6) and (2-7) apply only to the low-pass filter of Fig. 2-3a.

Often, the entire design of an active filter is done using the normalized complex frequency variable, s_n, and the normalized impedance, Z_n. At the end of the design procedure, as indicated in Fig. 2-1, the filter is denormalized to achieve the desired frequency range and to scale the passive component to values which are more convenient. This denormalization is simply the inverse of Eqs. (2-6) and (2-7). Table 2-1 gives the influence of the frequency and impedance denormalization is illustrated on the normalized passive components and the normalized root locations of a filter.

<u>Example 2-1 - Application of the Denormalizing Factors to the Low-Pass Filter</u>

Suppose that a normalized, second-order, low-pass filter is characterized by the circuit of Fig. 2-4 or by the following pole locations $p_{n1}, p_{n2} = -\frac{1}{\sqrt{2}} \pm j\frac{1}{\sqrt{2}}$.

Find the denormalized circuit and pole locations if the filter is to have a passband frequency of 1 kHz and the resistors in the realization should be 10 kΩ.

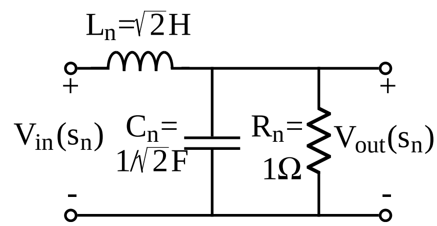

Figure 2-4 - A normalized, second-order, low-pass filter.

Solution

From the information given, ω_{PB} is $2\pi x10^3$ rps. z_o is 10^4 because the normalized 1 Ω in Fig. 2-4 is to become 10 kΩ. Therefore, the denormalized pole locations are

$$p_1, p_2 = -(0.707)(2\pi x10^3) \pm j\,(0.707)(2\pi x10^3) = -4{,}443 \pm j4{,}443 \text{ rps .}$$

The denormalized components of Fig. 2-4 become

$$L = \frac{10^4\sqrt{2}}{6283.2} = 2.251 \text{ H},$$

$$C = \frac{1}{\sqrt{2}(6283.2)(10^4)} = 0.1125 \text{ nF},$$

and

$$R = 10 \text{ k}\Omega.$$

Denormalization ↓	Denormalized Resistance, R	Denormalized Capacitance, C	Denormalized Inductance, L	Denormalized Pole, p, or Zero, z
Frequency - $s = \omega_{PB}s_n$ $= \omega_{PB}p$	$R = R_n$	$C = \dfrac{C_n}{\omega_{PB}}$	$L = \dfrac{L_n}{\omega_{PB}}$	$p = \omega_{PB}p_n$ $z = \omega_{PB}z_n$
Impedance - $Z = z_oZ_n$	$R = z_oR_n$	$C = \dfrac{C_n}{z_o}$	$L = z_oL_n$	$p = p_n$ $z = z_n$
Frequency and Impedance - $Z(s) = z_oZ_n(\omega_{PB}s_n)$	$R = z_oR_n$	$C = \dfrac{C_n}{z_o\omega_{PB}}$	$L = \dfrac{z_oL_n}{\omega_{PB}}$	$p = \omega_{PB}p_n$ $z = \omega_{PB}z_n$

Table 2-1 - Influence of the frequency and impedance denormalizations on the passive components and root locations of a filter.

Attenuation Viewpoint of Practical Filter Specifications

Fig. 2-5a shows the specifications of a low-pass filter where the vertical axis has been plotted in terms of dB. If we normalize the frequency axis by ω_{PB}, then the normalized passband frequency is now 1 rps and the normalized stopband frequency, Ω_n, is given as

$$\Omega_n = \frac{\omega_{SB}}{\omega_{PB}} \quad . \tag{2-8}$$

The realizations of this normalized, low-pass filter must remain in the shaded areas and approximate a straight-line drawn through points A and B in the transition region. A possible filter realization is shown on Fig. 2-5a.

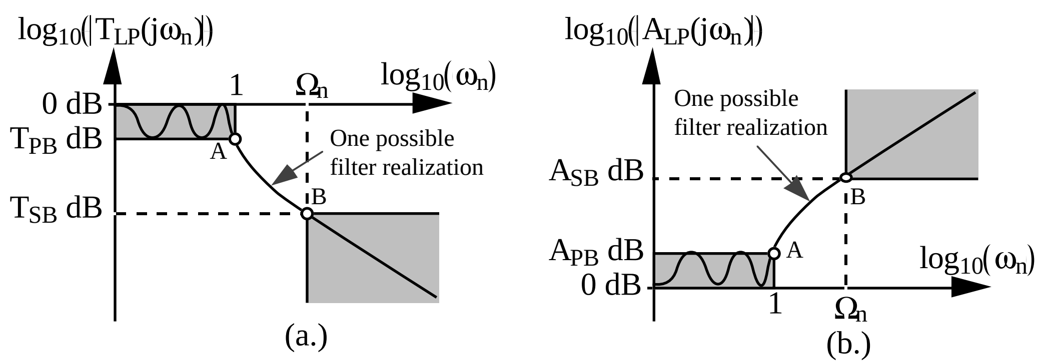

Figure 2-5 - Normalized, low-pass filter. (a.) Gain in dB. (b.) Attenuation in dB.

Because we have normalized the gain of the low-pass filter to unity, the dB values are all negative. Sometimes, it is convenient to view the normalized, low-pass filter from attenuation which is the reciprocal of gain. An attenuation plot in dB for the normalized, low-pass filter of Fig. 2-5a is shown in Fig. 2-5b. In some references, A_{PB} is called A_{MAX} and A_{SB} is called A_{Ω} or A_{MIN}.

The specification for the normalized, low-pass filter is completely described by three parameters. These parameters are T_{PB}, T_{SB}, and Ω_n or A_{PB}, A_{SB}, and Ω_n. Once these parameter values are known, then we must determine the order of the filter necessary to satisfy the specification. However, the order is determined by the type of filter approximation used to realize the specifications. While there are numerous filter approximations, we shall discuss two of the more popular ones next. They are the maximally flat magnitude or Butterworth approximation and the equal passband ripple or Chebyshev approximation.

Normalized, Low-Pass, Butterworth Filter Approximation

One of the more useful filter approximations to the normalized low-pass filter is called the Butterworth† filter approximation. The magnitude of the Butterworth filter approximation is maximally flat at low frequencies ($\omega \rightarrow 0$) and monotonically rolls off to a value approaching zero at high frequencies ($\omega \rightarrow \infty$). The magnitude of the normalized, Butterworth, low-pass filter approximation can be expressed as

$$|T_{LPn}(j\omega_n)| = \frac{1}{\sqrt{1 + \varepsilon^2 \omega_n^{2N}}} \tag{2-9}$$

where N is the order of the filter approximation and ε is defined in Fig. 2-6. Fig. 2-6 shows the magnitude response of the Butterworth filter approximation for several values of N.

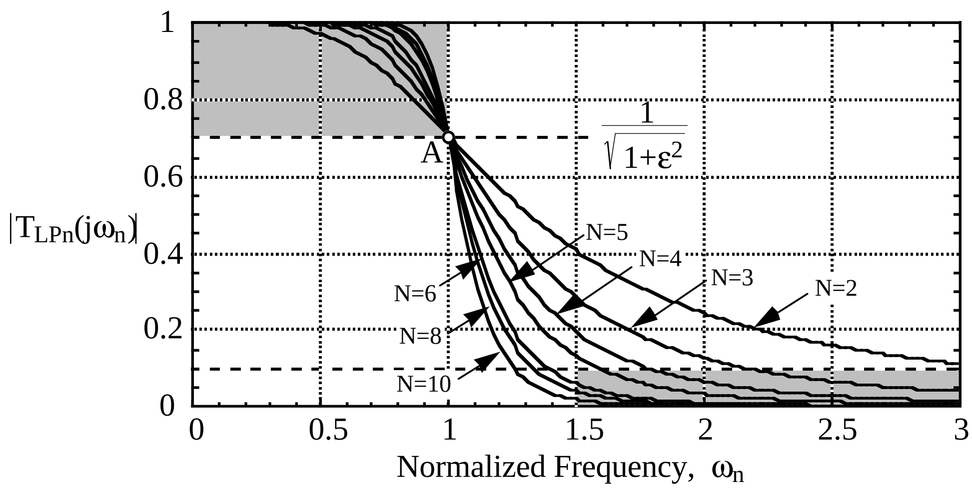

Figure 2-6 - Magnitude response of a normalized Butterworth low-pass filter approximation for various orders, N, and for ε = 1.

† S. Butterworth was a British engineer who described this type of filter approximation in conjunction with electronic amplifiers in his paper "On The Theory of Filter Amplifiers," *Wireless Engineer*, vol. 7, 1930.

The shaded area on Fig. 2-6 corresponds to the shaded area in the passband region of Figs. 2-3a and 2-5a. It is characteristic of all filter approximations that they pass through the point A as illustrated on Fig. 2-6. The value of ε can be used to adjust the width of the shaded area in Fig. 2-6. Normally, Butterworth filter approximations are given for an ε of unity as illustrated on Fig. 2-6. We see from Fig. 2-6 that the higher the order of the filter approximation, the smaller the transition region for given value of T_{SB}. For example, if T_{PB} = 0.707 (ε = 1), T_{SB} = 0.1 and Ω_n = 1.5 (illustrated by the both shaded areas of Fig. 2-6), then the order of the Butterworth filter approximation must be 6 or greater to satisfy the specifications. Note that the order must be an integer which means that even though N = 6 exceeds the specification it must be used because N = 5 does not meet the specification. The magnitude of the Butterworth filter approximation at ω_{SB} can be expressed from Eq. (2-9) as

$$\left|T_{LPn}\left(\frac{j\omega_{SB}}{\omega_{PB}}\right)\right| = |T_{LPn}(j\Omega_n)| = T_{SB} = \frac{1}{\sqrt{1+\varepsilon^2\,\Omega_n^{2N}}} \quad . \qquad (2\text{-}10)$$

This equation is useful for determining the order required to satisfy a given filter specification. Often, the filter specification is given in terms of dB. In this case, Eq. (2-10) is rewritten as

$$20\log_{10}(T_{SB}) = T_{SB}\,(\mathrm{dB}) = -10\log_{10}\left(1+\varepsilon^2\,\Omega_n^{2N}\right) \quad . \qquad (2\text{-}11)$$

Example 2-2 - Determining the Order of A Butterworth Filter Approximation

Assume that a normalized, low-pass filter is specified as T_{BP} = -3dB, T_{SB} = -20 dB, and Ω_n = 1.5. Find the smallest integer value of N of the Butterworth filter approximation which will satisfy this specification.

Solution

T_{BP} = -3dB corresponds to T_{BP} = 0.707 which implies that ε = 1. Thus, substituting ε = 1 and Ω_n = 1.5 into Eq. (2-11) gives

$$T_{SB}\,(\mathrm{dB}) = -\,10\log_{10}(1+1.5^{2N}) \quad .$$

Substituting values of N into this equation gives T_{SB} = -7.83 dB for N = 2, -10.93 dB for N = 3, -14.25 dB for N = 4, -17.68 dB for N = 5, and -21.16 dB for N = 6. Thus, N must be 6 or greater to meet the filter specification.

Once, the order of the Butterworth filter approximation is known, we must next find the normalized root locations. Of course, all zeros are at infinity because the realization is low-pass. The poles depend on the value of ε. For ε = 1, the pole locations are on a unit circle. The normalized poles are designated as $p_{kn} = \sigma_{kn} + j\omega_{kn}$ and are given as

$$\sigma_{kn} = -\sin\left(\frac{(2k-1)\pi}{2N}\right) \quad \text{and} \quad \omega_{kn} = \cos\left(\frac{(2k-1)\pi}{2N}\right), \; k = 1,2,3,\cdots,N \, . \qquad (2\text{-}12)$$

To help illustrate this formula, the poles for fifth-order, Butterworth filter approximation have been evaluated and are given in Table 2-2. Figure 2-7 shows the pole locations for the fifth-order, Butterworth filter approximation. It can be shown that these poles are angularly spaced by an amount of π/N (36° for N = 5).

k	$\sigma_{kn} = -\sin\left(\frac{(2k-1)\pi}{2N}\right)$	$\omega_{kn} = \cos\left(\frac{(2k-1)\pi}{2N}\right)$
1	-0.3090 rps	0.9511 rps
2	-0.8090 rps	0.5878 rps
3	-1.0000 rps	0.0000 rps
4	-0.8090 rps	-0.5878 rps
5	-0.3090 rps	-0.9511 rps

Table 2-2 - Normalized pole locations for a fifth-order, Butterworth filter approximation.

Cascade Realization of Butterworth Filter Approximations

It is important to realize that while the Butterworth filter approximation is monotonic, the individual pole pairs are not. For example, let us write the normalized transfer function of the fifth-order example of Table 2-2 into the product of two, second-order terms and one, first-order terms. The filter transfer function is written as

$$T_{LPn}(s_n) = \left(\frac{p_{3n}}{s_n+p_{3n}}\right)\left(\frac{p_{1n}p_{5n}}{(s_n+p_{1n})(s_n+p_{5n})}\right)\left(\frac{p_{2n}p_{4n}}{(s_n+p_{2n})(s_n+p_{4n})}\right) \qquad (2\text{-}13)$$

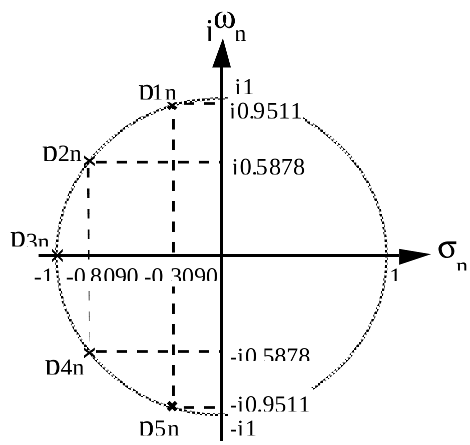

Figure 2-7 - Example of the normalized pole locations for a fifth-order, normalized Butterworth filter approximation.

where we have grouped the complex-conjugate poles into second-order terms. Substituting the values of Table 2-2 into Eq. (2-13) gives

$$T_{LPn}(s_n) = T_1(s_n)T_2(s_n)T_3(s_n) = \left(\frac{1}{s_n+1}\right)\left(\frac{1}{s_n^2+0.6180s_n+1}\right)\left(\frac{1}{s_n^2+1.6180s_n+1}\right) \quad . \tag{2-14}$$

The contributions of the first-order term, $T_1(s_n)$, and the two second-order terms, $T_2(s_n)$ and $T_3(s_n)$, can be illustrated by plotting each one separately and then taking the products of all three. Fig. 2-8 shows the result. Interestingly enough, we see that the magnitude of $T_2(s_n)$ has a peak that is about 1.7 times the gain of the fifth-order filter at low frequencies. If we plotted Fig. 2-8 with the vertical scale in dB, we could identify the Q by comparing the results with the normalized second-order, magnitude responses of Fig. 1-6a. Consequently, all filter approximations that are made up from first-order and/or second-order products do not necessarily have the properties of the filter approximation until all the terms are multiplied (added on a dB scale).

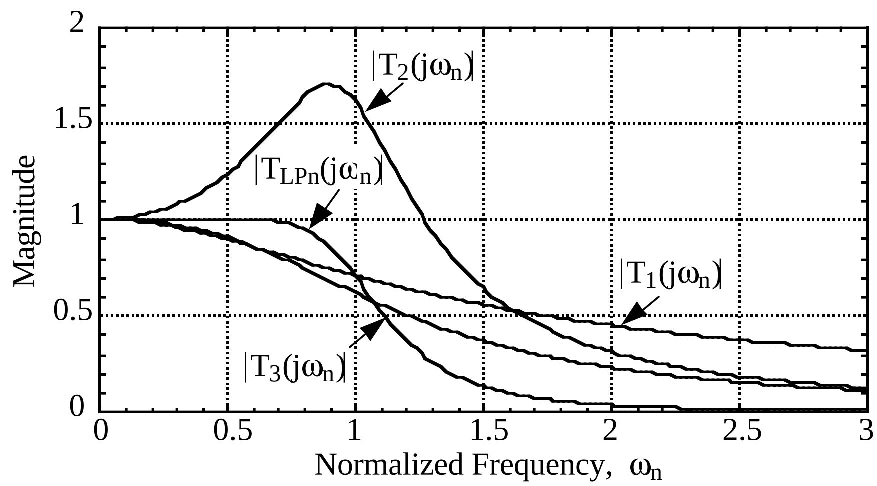

Figure 2-8 - Individual magnitude contributions of a fifth-order, Butterworth filter approximation.

Now we see how the Butterworth filter approximation can be realized by the cascading of second-order stages and, at most, one-first order stage. The design procedure is stated as follows:

1.) From T_{PB}, T_{SB}, and Ω_n (or A_{PB}, A_{SB}, and Ω_n) determine the required order of the Butterworth filter approximation using Eq. (2-10) or Eq. (2-11).

2.) From Eq. (2-12) find the normalized poles of the approximation.

3.) Group the complex-conjugate poles into second-order realizations. For odd-order realizations there will be one first-order term of the form $1/(s_n+1)$.

4.) Realize each of the second-order terms using the active filters of Sec. 1. Realize the first-order section (if any) by the first-order low-pass circuits of Sec. 4.2.

5.) Cascade the realizations in the order from input to output of the lowest-Q stage first (first-order stages generally should be first).

6.) Denormalize to the desired passband frequency and denormalize the impedances if desired.

Step 5 covers an aspect we have not considered and that is the order of the stages. The principle behind the ordering suggested in step 5 is to prevent one stage from being overdriven. Consider the fifth-order filter of Fig. 2-8 to illustrate this principle. If the frequency applied to the filter is at the passband (ω_n = 1), the gain of $T_2(s_n)$ is about 1.6 while the gain of $T_1(s_n)$ is 0.707 and $T_3(s_n)$ is 0.6. If the order were T_2, T_1, and T_3 and the amplitude of the sinusoid at ω_n = 1 was 1V, the output of the first stage (T_2) would be 1.6V, the output of the second stage (T_1) would be 1.13V, and finally the output of the third stage (T_3) would be 0.707. If the input signal is too large, or the power supply voltages too small, the first and possibly the second stages may saturate or clip. However, if we put the stages in the order of T_1, T_3, and T_2 or T_3, T_1 and T_2, saturation will not occur and one can achieve maximum signal amplitude. Let us illustrate the cascade design approach with an example.

Example 2-3 - Design of a Fifth-Order, Low-Pass Butterworth Filter

Design a cascade, active filter realization for a Butterworth filter approximation to the filter specifications of A_{PB} = 3dB, A_{SB} = 30 dB, f_{PB} = 1 kHz, and f_{SB} = 2 kHz. Give a schematic and component values for the realization using the negative feedback, second-order, low-pass active filter of Fig. 1-14 and any first-order stage that may be necessary.

Solution

First we must convert the specifications to T_{PB} = - 3dB, T_{SB} = -30dB, and Ω_n = f_{SB}/f_{PB} = 2.0. T_{PB} = -3 dB means ε = 1. Trying different values of N in Eq. (2-11) shows that for N = 5 that T_{SB} = -30.1 dB. Thus, a fifth-order Butterworth approximation barely satisfies the requirement. We might be smart to go to a sixth-order realization to obtain a margin of safety but we shall stay with N = 5 for this example. N = 5 allows us to take advantage of the previous results given above. Let us do the design stage-by-stage.

Stage 1: Stage 1 is simply a first-order stage. We will use Fig. 4.2-6c with R_{11} = R_{21} = 1Ω and C_{21} = 1F where the second subscript stands for the stage number.

Stage 2: The transfer function for the second stage is

$$T_2(s_n) = \frac{1}{s_n^2+0.6181s_n+1} \quad .$$

From Eq. (1-3) we see that ω_o = 1rps and Q = 1.6181. Using the design equations of Eqs. (1-41) through (1-45) give C_{52} = 1 F, $C_{42} = 4(1.6181)^2(1+1)C$ = 20.95 F, R_{12} = 1/[(2)(1)(1)(1.6181)(1)] = 0.3090 Ω, R_{22} = 1/[(2)(1)(1.6181)(1)] = 0.3090 Ω, and R_{32} = 1/[(2)(1)(1.6181)(1+1)] = 0.1545 Ω.

Stage 3: The transfer function for the third stage is

$$T_3(s_n) = \frac{1}{s_n^2+1.6180s_n+1} \quad .$$

From Eq. (1-3) we see that ω_o = 1rps and Q = 0.6181. Using the design equations of Eqs. (1-41) through (1-45) give C_{53} = 1 F, $C_{43} = 4(0.6181)^2(1+1)C$ = 3.056 F, R_{13} = 1/[(2)(1)(1)(0.6181)(1)] = 0.8090 Ω, R_{23} = 1/[(2)(1)(0.6181)(1)] = 0.8090 Ω, and R_{33} = 1/[(2)(1)(0.6181)(1+1)] = 0.4045 Ω.

Next, we frequency denormalize the filter realization by $\omega_{PB} = 2\pi x10^3$. At the same time we will impedance denormalize by 10^5 (arbitrarily chosen). The resulting values are shown on the realization of Fig. 2-9 and are achieved using the bottom row of Table 2-1. Note, that we have placed stage 1 first, stage 3 second, and stage 2 last. This filter realization will meet the specifications given and will permit maximum signal amplitude.

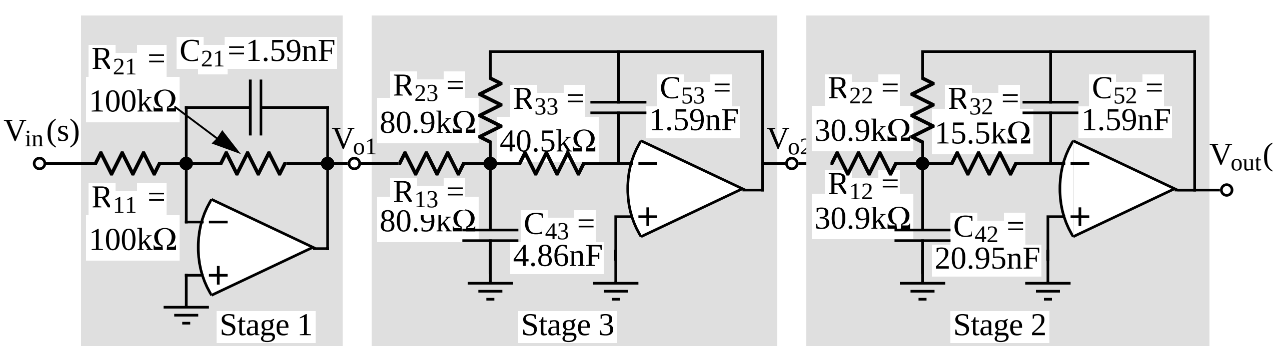

Figure 2-9 - A denormalized, fifth-order, active filter realization of a low-pass , Butterworth filter approximation.

Normalized, Low-Pass, Chebyshev Filter Approximation

A second useful filter approximation to the normalized low-pass filter is called a Chebyshev† filter approximation. The Chebyshev low-pass filter approximation has equal-ripples in the passband and then is monotonic outside of the passband. The equal-ripple in the passband allows the Chebyshev filter approximation to fall off more quickly than the Butterworth filter approximation of the same order. This increased rolloff occurs only for frequencies just above ω_{PB}. As the frequency becomes large, filter approximations of the same order will have the same rate of decrease in the magnitude response. The magnitude of the normalized, Chebyshev, low-pass, filter approximation can be expressed as

$$|T_{LPn}(j\omega_n)| = \frac{1}{\sqrt{1 + \varepsilon^2 \cos^2[N\cos^{-1}(\omega_n)]}} \quad , \quad \omega_n \le 1 \tag{2-15}$$

and

$$|T_{LPn}(j\omega_n)| = \frac{1}{\sqrt{1 + \varepsilon^2 \cosh^2[N\cosh^{-1}(\omega_n)]}} \quad , \quad \omega_n > 1 \tag{2-16}$$

where N is the order of the filter approximation and ε is defined in Fig. 2-10. Fig. 2-10 shows the magnitude response of the Chebyshev filter approximation for $\varepsilon = 0.5088$.

The values of ε are normally chosen so that the ripple width is between 0.1dB (ε = 0.0233) and 1 dB ($\varepsilon = 0.5088$). We can show that the Chebyshev is has a smaller transition region by considering the order necessary to satisfy the partial specification of $T_{SB} = 0.1$ and $\Omega_n = 1.5$. We see from Fig. 2-10 that N = 4 will easily satisfy this requirement. We also note that $T_{PB} = 0.8913$ which is better than 0.7071 of the Butterworth filter approximation. Thus, we see that ε determines the width of the passband ripple and is given as

$$|T_{LP}(\omega_{PB})| = |T_{LPn}(1)| = T_{PB} = \frac{1}{\sqrt{1+\varepsilon^2}} \quad . \tag{2-17}$$

† The Chebyshev filter approximation was first used to study the construction of steam engines as described by P.L. Cheybshev in the paper "Thé orie des mé canismes connus sous le nom de parallelogrammes," *Oeuvres*, vol. 1, St. Petersburg, 1899.

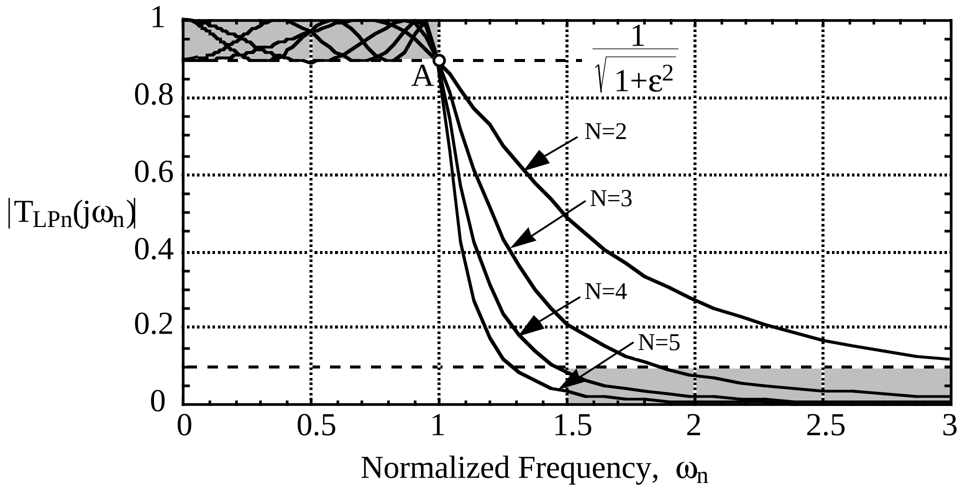

Figure 2-10 - Magnitude response of a normalized Chebyshev low-pass filter approximation for various orders of N and for $\varepsilon = 0.5088$.

The magnitude of the Chebyshev filter approximation at ω_{SB} can be expressed from Eq. (2-16) as

$$\left|T_{LPn}\left(\frac{\omega_{SB}}{\omega_{PB}}\right)\right| = |T_{LPn}(\Omega_n)| = T_{SB} = \frac{1}{\sqrt{1 + \varepsilon^2 \cosh^2[N\cosh^{-1}(\Omega_n)]}} \quad . \quad (2\text{-}18)$$

If the specifications are in terms of decibels, then Eq. (2-19) is more convenient in the form

$$20 \log_{10}(T_{SB}) = T_{SB}\,(\mathrm{dB}) = -10 \log_{10}[\,1 + \varepsilon^2 \cosh^2[N\cosh^{-1}(\Omega_n)]\,] \quad . \quad (2\text{-}19)$$

Example 2-4 - Determining the Order of A Chebyshev Filter Approximaton

Repeat Ex. 2-2 for the Chebyshev filter approximation.

Solution

In Ex. 2-2, $\varepsilon = 1$ which means the ripple width is 3 dB or $T_{PB} = 0.707$. Now we substitute $\varepsilon = 1$ into Eq. (2-19) and find the value of N which satisfies $T_{SB} = -20$dB. For N = 2, we get $T_{SB} = -11.22$ dB. For N =3, we get $T_{SB} = -19.14$ dB. Finally, for N = 4, we get $T_{SB} = -27.43$ dB. Thus N = 4 must be used although N = 3 almost satisfies the specifications.

The normalized pole locations, p_{kn}, of the low-pass, normalized Chebyshev filter approximation can be found from the following formula

$$p_{kn} = \sigma_{kn} + j\omega_{kn} = -\sin\left(\frac{(2k-1)\pi}{2N}\right)\sinh\left(\frac{1}{N}\sinh^{-1}\frac{1}{\varepsilon}\right) + j\cos\left(\frac{(2k-1)\pi}{2N}\right)\cosh\left(\frac{1}{N}\sinh^{-1}\frac{1}{\varepsilon}\right)$$

,

$$k = 1, 2, 3, \cdots, N\ . \tag{2-20}$$

To illustrate this formula, the poles for a fifth-order, normalized, Chebyshev filter have been evaluated are are given in Table 2-3 for the case where T_{PB} = -1dB. It can be shown that these poles lie on an ellipse centered about the origin of the complex frequency plane. The pole locations for Table 2-3 are illustrated on Fig. 2-11.

k	$\sigma_{kn} = -\sin\left(\frac{(2k-1)\pi}{2N}\right)\sinh\left(\frac{1}{N}\sinh^{-1}\frac{1}{\varepsilon}\right)$	$\omega_{kn} = \cos\left(\frac{(2k-1)\pi}{2N}\right)\cosh\left(\frac{1}{N}\sinh^{-1}\frac{1}{\varepsilon}\right)$
1	-0.0895 rps	0.9901 rps
2	-0.2342 rps	0.6119 rps
3	-0.2895 rps	0.0000 rps
4	-0.2342 rps	-0.6119 rps
5	-0.0895 rps	-0.9901 rps

Table 2-3 - Normalized pole locations for a fifth-order, Chebyshev filter approximation for ε = 0.5088.

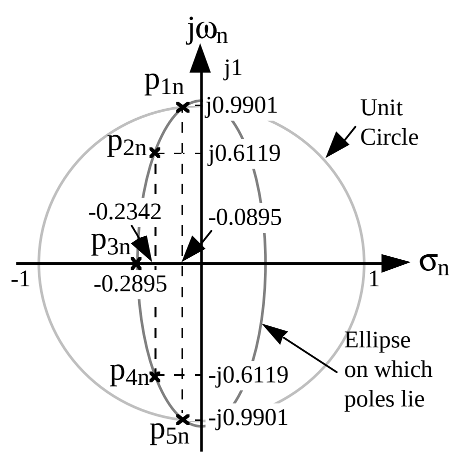

Figure 2-11 - Location of the normalized poles for a fifth-order, normalized Chebyschev filter approximation for ε = 0.5088.

The normalized transfer function of the Chebyshev filter approximation can be written as the product of second-order terms and one, first-order term if the order is odd. For the fifth-order, Chebysheve filter approximation, the filter transfer function is written as

$$T_{LPn}(s_n) = \left(\frac{p_{3n}}{s_n+p_{3n}}\right)\left(\frac{p_{1n}p_{5n}}{(s_n+p_{1n})(s_n+p_{5n})}\right)\left(\frac{p_{2n}p_{4n}}{(s_n+p_{2n})(s_n+p_{4n})}\right) \qquad (2\text{-}21)$$

where we have grouped the complex-conjugate poles into second-order terms. Substituting the values of Table 2-3 into Eq. (2-21) gives

$$T_{LPn}(s_n) = T_1(s_n)T_2(s_n)T_3(s_n) =$$

$$\left(\frac{0.2895}{s_n+0.2895}\right)\left(\frac{0.9883}{s_n^2+0.1789s_n+0.9883}\right)\left(\frac{0.4239}{s_n^2+0.4684s_n+0.4239}\right) . \qquad (2\text{-}22)$$

As before, each product does not necessarily have the characteristic of a Chebychev filter. However, when all the products are multiplied together, the result is a Chebyschev filter approximation.

The design procedure for designing a cascaded, Chebyshev filter approximation using active filters is stated as follows:

1.) From T_{PB}, T_{SB}, and Ω_n (or A_{PB}, A_{SB}, and Ω_n) determine the required order of the Chebyshev filter approximation using Eq. (2-18) or Eq. (2-19).

2.) From Eq. (2-20) find the normalized poles of the approximation.

3.) Group the complex-conjugate poles into second-order realizations. For odd-order realizations there will be one first-order term of the form $1/(s_n+1)$.

4.) Realize each of the second-order terms using the active filters of Sec. 1. Realize the first-order section (if any) by the first-order low-pass circuits of Sec. 4.2.

5.) Cascade the realizations in the order from input to output of the lowest-Q stage first (first-order stages generally should be first).

6.) Denormalize to the desired passband frequency and denormalize the impedances if desired.

The following example will illustrate the application of this design procedure.

Example 2-5 - Design of a Fifth-Order, Low-Pass Chebyshev Filter

Design a cascade, active filter realization for a Chebyshev filter approximation to the filter specifications of A_{PB} = 1dB, A_{SB} = 45 dB, f_{PB} = 1 kHz, and f_{SB} = 2 kHz. Give a schematic for the realization using the negative feedback, second-order, low-pass active filter of Fig. 1-14 and any first-order state that may be necessary. Show all values of the components.

Solution

First we must convert the specifications to T_{PB} = - 1dB, T_{SB} = -45dB, and Ω_n = f_{SB}/f_{PB} = 2.0. T_{PB} = -1 dB means ε = 0.5088. Trying different values of N in Eq. (2-20) shows that for N = 5 that T_{SB} = -45.3 dB. Thus, a fifth-order Chebyshev approximation barely satisfies the requirement. Again, we might be smart to go to a sixth-order realization to achieve a margin of safety but we shall stay with N = 5 for this example. N = 5 allows us to take advantage of the previous results given above. Let us do the design stage-by-stage.

Stage 1: Stage 1 is simply a first-order stage. Let us use Fig. 4.2-6c and choose $R_{11} = R_{21} = 1\Omega$. Therefore, from Eq. (4.2-17) we get $C_{21} = \dfrac{1}{0.2895R_{21}} = 3.454$ F where the second subscript stands for the stage number.

Stage 2: The transfer function for the second stage is

$$T_2(s_n) = \frac{0.9883}{s_n^2+0.1789s_n+0.9883} .$$

From Eq. (1-3) we see that $\omega_o = \sqrt{0.9883} = 0.9941$ rps and $Q = Q_2 = \dfrac{\sqrt{0.9883}}{0.1789} = 5.557$. Using the design equations of Eqs. (1-41) through (1-45) give C_{52} = 1 F, $C_{42} = 4(5.557)^2(1+1)C = 247.04$ F, $R_{12} = \dfrac{1}{(2)(1)(0.9941)(5.557)(1)} = 0.09051\ \Omega$, $R_{22} = \dfrac{1}{(2)(0.9941)(5.557)(1)} = 0.09051\ \Omega$, and $R_{32} = \dfrac{1}{(2)(0.9941)(5.557)(1+1)} = 0.04525\ \Omega$.

Stage 3: The transfer function for the second stage is

$$T_3(s_n) = \frac{0.4239}{s_n^2+0.4684s_n+0.4239} \quad .$$

From Eq. (1-3) we see that $\omega_o = \sqrt{0.4239} = 0.6522$ rps and $Q_3 = \frac{\sqrt{0.4239}}{0.4684} = 1.390$. Using the design equations of Eqs. (1-41) through (1-45) give $C_{53} = 1$ F, $C_{43} = 4(1.390)^2(1+1)C = 15.457$ F, $R_{13} = \frac{1}{(2)(1)(0.6522)(1.390)(1)} = 0.5515\ \Omega$, $R_{23} = \frac{1}{(2)(0.6522)(1.390)(1)} = 0.5515\ \Omega$, and $R_{33} = \frac{1}{(2)(0.6522)(1.390)(1+1)} = 0.2758\ \Omega$.

Next, we frequency denormalize the filter realization by $\omega_{PB} = 2\pi x10^3$. At the same time we will impedance denormalize by 10^5 (arbitrarily chosen). The resulting values are shown on the realization of Fig. 2-12 and are achieved using the bottom row of Table 2-1. Note, that we have placed stage 1 first, stage 3 second, and stage 2 last. This filter realization will meet the specifications given and will permit maximum signal amplitude.

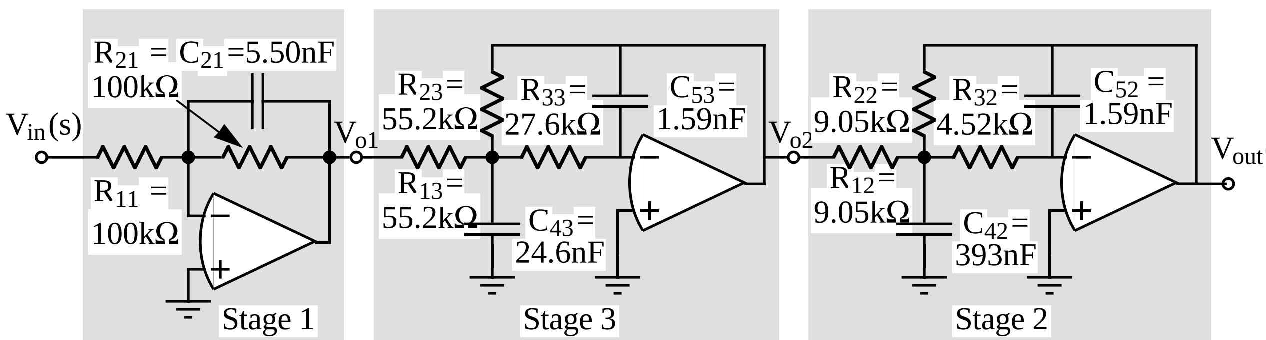

Figure 2-12 - A denormalized, fifth-order, active filter realization of a low-pass, Chebyshev filter approximation.

In comparing the results of Examples 2-3 and 2-5, we see that the realizations are identical and it is the value of the components which distinguishes the Butterworth approximation from the Chebyshev approximation. However, we should note that for the same ω_{PB} and ω_{SB}, the Chebyshev approximation has a 1dB passband ripple compared to a 3dB passband "ripple" for the Butterworth approximation. More importantly, the stopband attenuation of the Chebyshev approximation is greater than 45 dB as compared to

30 dB for the Butterworth. In fact, we could easily satisfy the requirements of Ex. 2-3 with a fourth-order Chebyshev approximation which would eliminate stage 1 of Fig. 2-12 (obviously, the components of stages 3 and 2 will change values).

High-Pass Active Filters

One of the useful principles of active filter design is that all design is based on the normalized, low-pass filter approximation. Therefore, all we have to do is to transform the normalized, low-pass filter design to a high-pass, bandpass, or bandstop design. If we let s_{ln} be the normalized, low-pass frequency variable, the *normalized, low-pass to normalized, high-pass transformation* is defined as

$$s_{ln} = \frac{1}{s_{hn}} \tag{2-23}$$

where s_{hn} is the normalized, high-pass frequency variable (normally the subscripts h and l are not used when the meaning is understood). We have seen from the previous work that a general form of the normalized, low-pass transfer function is

$$T_{LPn}(s_{ln}) = \frac{p_{1ln}p_{2ln}p_{3ln}\cdots p_{Nln}}{(s_{ln}+p_{1ln})(s_{ln}+p_{2ln})(s_{ln}+p_{3ln})\cdots(s_{ln}+p_{Nln})} \tag{2-24}$$

where p_{kln} is the kth normalized, low-pass pole. If we apply the normalized, low-pass to high-pass transformation to Eq. (2-24) we get

$$T_{HPn}(s_{hn}) = \frac{p_{1ln}p_{2ln}p_{3ln}\cdots p_{Nln}}{\left(\frac{1}{s_{hn}}+p_{1ln}\right)\left(\frac{1}{s_{hn}}+p_{2ln}\right)\left(\frac{1}{s_{hn}}+p_{3ln}\right)\cdots\left(\frac{1}{s_{hn}}+p_{Nln}\right)}$$

$$= \frac{s_{hn}^N}{\left(s_{hn}+\frac{1}{p_{1ln}}\right)\left(s_{hn}+\frac{1}{p_{2ln}}\right)\left(s_{hn}+\frac{1}{p_{3ln}}\right)\cdots\left(s_{hn}+\frac{1}{p_{Nln}}\right)}$$

$$= \frac{s_{hn}^N}{(s_{hn}+p_{1hn})(s_{hn}+p_{2hn})(s_{hn}+p_{3hn})\cdots(s_{hn}+p_{Nhn})} \tag{2-25}$$

where p_{khn} is the kth normalized high-pass pole.

We see that the normalized, low-pass to high-pass transformation inverts the pole locations and causes all of the zeros to appear at the origin of the complex frequency plane. Therefore, the design procedure is essentially the same as for the normalized, low-pass filters except the poles are inverted and we associate with each pole a zero at the origin. The order of the high-pass filter is determined by translating its specifications to an equivalent low-pass filter. The general cascade design procedure for a high-pass filter is as follows:

1.) Start with the high-pass specification in the form of Fig. 2-3b (or the equivalent in terms of attenuation). Normalize the frequency by dividing by ω_{PB}. Therefore, the normalized passband frequency is 1 rps and the normalized bandstop frequency is

$$\frac{1}{\Omega_n} = \Omega_{hn} = \frac{\omega_{SB}}{\omega_{PB}} \quad . \tag{2-26}$$

Note that Ω_{hn} will always be less than unity.

2.) From T_{PB}, T_{SB}, and Ω_n (or A_{PB}, A_{SB}, and Ω_n) determine the required order of the filter approximation using the proper equations for the selected approximation.

3.) Find the normalized, low-pass poles of the approximation.

4.) Invert the normalized, low-pass poles (p_{kln}) to get the normalized, high-pass filter poles (p_{khn}).

5.) Group the complex-conjugate poles of the normalized, high-pass filter into second-order realizations including 2 zeros at the origin. For odd-order realizations there will be one first-order high-pass product.

6.) Realize each of the second-order terms using the high-pass, first- or second-order active filters of Chapter 1.

7.) Cascade the realizations in the order from input to output of the lowest pole-Q stage first (first-order stages generally should be first).

8.) Denormalize to the desired passband frequency and denormalize the impedances.

The following example will illustrate the application of this design procedure.

Example 2-6 - Design of a Butterworth, High-Pass Filter

Design a high-pass filter having a -3dB ripple bandwidth above 1 kHz and a gain of less than -35 dB below 500 Hz using the Butterworth approximation. Use the positive feedback active filter realization and give a complete circuit and all component values. What is the value of the filter gain as frequency approaches infinity?

Solution

From the specification, we know that T_{PB} = -3 dB and T_{SB} = -35 dB. Eq. (2-26) gives us an $\Omega_n = 2$ ($\Omega_{hn} = 0.5$). $\varepsilon = 1$ because T_{PB} = -3 dB. Therefore, we use Eq. (2-11) to find that N = 6 will give T_{SB} = -36.12 dB which is the lowest, integer value of N which meets the specifications.

Next, we evaluate the normalized, low-pass poles from Eq. (2-12) as

$$p_{1ln}, p_{6ln} = -0.2588 \pm j\,0.9659$$

$$p_{2ln}, p_{5ln} = -0.7071 \pm j\,0.7071$$

and

$$p_{3ln}, p_{4ln} = -0.9659 \pm j\,0.2588$$

where the first subscript on the poles corresponds to k in Eq. (2-12). Inverting the normalized, low-pass poles gives the normalized, high-pass poles which are

$$p_{1hn}, p_{6hn} = -0.2588 \mp j\,0.9659$$

$$p_{2hn}, p_{5hn} = -0.7071 \mp j\,0.7071$$

and

$$p_{3hn}, p_{4hn} = -0.9659 \mp j\,0.2588 \ .$$

We note the inversion of the Butterworth poles simply changes the sign of the imaginary part of the pole.

The next step is to group the poles in second-order products, since there are no first-order products. This result gives the following normalized, high-pass transfer function.

$$T_{HPn}(s_{hn}) = T_1(s_{hn})T_2(s_{hn})T_3(s_{hn})$$

$$= \left(\frac{s_{hn}^2}{(s_{hn}+p_{1hn})(s_{hn}+p_{6hn})}\right)\left(\frac{s_{hn}^2}{(s_{hn}+p_{2hn})(s_{hn}+p_{5hn})}\right)\left(\frac{s_{hn}^2}{(s_{hn}+p_{3hn})(s_{hn}+p_{4hn})}\right)$$

$$= \left(\frac{s_{hn}^2}{s_{hn}^2+0.5176s_{hn}+1}\right)\left(\frac{s_{hn}^2}{s_{hn}^2+1.4141s_{hn}+1}\right)\left(\frac{s_{hn}^2}{s_{hn}^2+1.9318s_{hn}+1}\right) .$$

Now we are in a position to do the stage-by-stage design. We will use Fig. 1-21a and the design equations indicated on this figure for all three, second-order stages.

Stage 1: Let us select $C = C_{11} = C_{31} = 1$ F. The resistors become $R = R_{21} = R_{41} = 1$ Ω. If we select $R_{A1} = 1\Omega$, then $R_{B1} = 2 - (1/1.9320) = 1.4823\ \Omega$.

Stage 2: Let us select $C = C_{12} = C_{32} = 1$F. Therefore the resistors are $R = R_{22} = R_{42} = 1\ \Omega$. If $R_{A2} = 1\ \Omega$, then $R_{B2} = 2 - 1.4141 = 0.5858\ \Omega$.

Stage 3: Let us select $C = C_{13} = C_{33} = 1$F. Therefore the resistors are $R = R_{23} = R_{43} = 1\ \Omega$. If $R_{A3} = 1\ \Omega$, then $R_{B3} = 2 - 1.9318 = 0.0682\ \Omega$.

Finally, we denormalize the components by $\omega_{PB} = 2\pi x 10^3$. At the same time we will arbitrarily denormalize the impedance by 10^5. Fig. 2-13 shows the resulting, 6th-order, unnormalized, high-pass active filter realization using the Butterworth approximation. Note that the stages are cascaded in the order of T_3, T_2, and T_1 for maximum signal swing. The gain of the filter at high frequencies is 1.0682x1.589x2.480 = 4.201.

The above procedure is general and can be used to design a cascaded, active filter realization of a high-pass filter. The problems give several other examples of designing higher-order, high-pass filters (see PR2-8 and PR2-9).

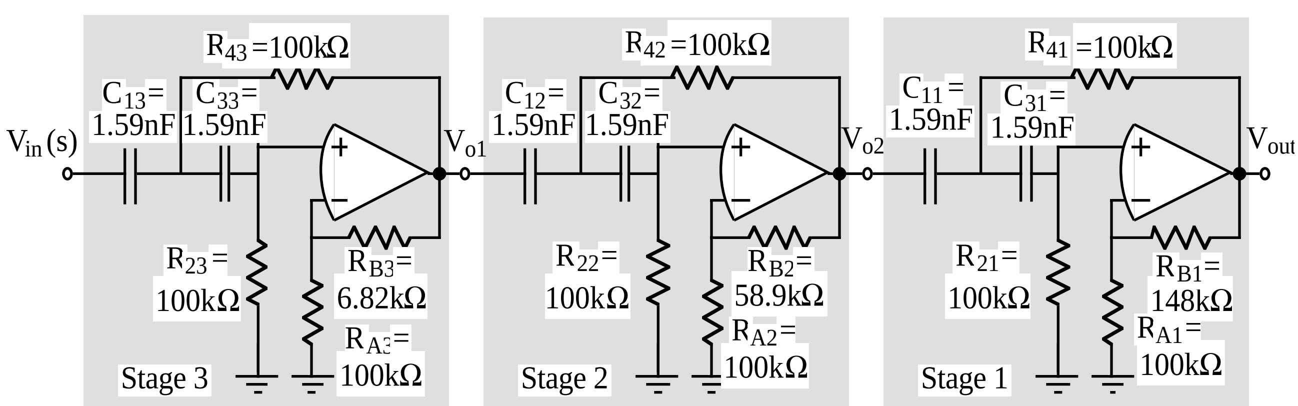

Figure 2-13 - Denormalized, sixth-order, active filter realization of Ex. 2-6.

Bandpass Active Filters

We can continue to use our knowledge of low-pass filters to develop the other types of filter responses indicated in Fig. 2-3. We will now develop bandpass filters which are based on the normalized, low-pass filter. First, we define the width of the passband and the width of the stopband of the bandpass filter as

$$BW = \omega_{BP2} - \omega_{PB1} \tag{2-27}$$

and

$$SW = \omega_{SB2} - \omega_{SB1} \ , \tag{2-28}$$

respectively. Our study here only pertains to a certain category of bandpass filters. This category is one where the passband and stopband are geometrically centered about a frequency, ω_r, which is called *the geometric center frequency* of the bandpass filter. The geometric center frequency of the bandpass filter is defined as

$$\omega_r = \sqrt{\omega_{PB1}\omega_{PB2}} = \sqrt{\omega_{SB2}\omega_{SB1}} \ . \tag{2-29}$$

The geometrically centered bandpass filter can be developed from the normalized low-pass filter by the use of a frequency transformation. If s_b is the bandpass complex frequency variable, then we define a *normalized low-pass to unnormalized bandpass transformation* as

$$s_{ln} = \frac{1}{BW}\left(\frac{s_b^2 + \omega_r^2}{s_b}\right) = \frac{1}{BW}\left(s_b + \frac{\omega_r^2}{s_b}\right). \tag{2-30}$$

A *normalized low-pass to normalized bandpass transformation* is achieved by dividing the bandpass variable, s_b, by the geometric center frequency, ω_r, to get

$$s_{ln} = \left(\frac{\omega_r}{BW}\right)\left(\frac{s_b}{\omega_r} + \frac{1}{(s_b/\omega_r)}\right) = \left(\frac{\omega_r}{BW}\right)\left(s_{bn} + \frac{1}{s_{bn}}\right) \tag{2-31}$$

where

$$s_{bn} = \frac{s_b}{\omega_r} \ . \tag{2-32}$$

We can multiply Eq. (2-31) by BW/ω_r and define yet a further normalization of the low-pass, complex frequency variable as

$$s_{ln}' = \left(\frac{BW}{\omega_r}\right) s_{ln} = \Omega_b s_{ln} = \Omega_b\left(\frac{s_l}{\omega_{PB}}\right) = \left(s_{bn} + \frac{1}{s_{bn}}\right) \tag{2-33}$$

where Ω_b is a bandpass normalization of the low-pass frequency variable given as

$$\Omega_b = \frac{BW}{\omega_r} \quad . \tag{2-34}$$

We will call the normalization of Eq. (2-33) a *bandpass normalization* of the low-pass complex frequency variable.

In order to be able to use this transformation, we need to solve for s_{bn} in terms of s_{ln}' . From Eq. (2-33) we get the following quadratic equation.

$$s_{bn}^2 - s_{ln}' \; s_{bn} + 1 = 0 \quad . \tag{2-35}$$

Solving for s_{bn} from Eq. (2-34) gives

$$s_{bn} = \left(\frac{s_{ln}'}{2}\right) \pm \sqrt{\left(\frac{s_{ln}'}{2}\right)^2 - 1} \quad . \tag{2-36}$$

Figure 2-14 shows how transformation of Eq. (2-30) is used to create an unnormalized bandpass filter from an unnormalized low-pass filter. We must remember that the low-pass filter magnitude includes negative frequencies as indicated by the area enclosed by dashed lines to the left of the vertical axis of Fig. 2-14a. The low-pass filter has been amplitude normalized so that the passband gain is unity. Fig. 2-14b shows the normalization of the frequency by ω_{PBl}. Next, the low-pass to bandpass transformation of Eq. (2-31) is applied to get the normalized, band-pass magnitude in Fig. 2-14c. Finally, the bandpass filter is frequency denormalized to get the frequency unnormalized bandpass magnitude response of Fig. 2-14d. The stopbands of the bandpass filter were not included for purposes of simplicity but can be developed in the same manner.

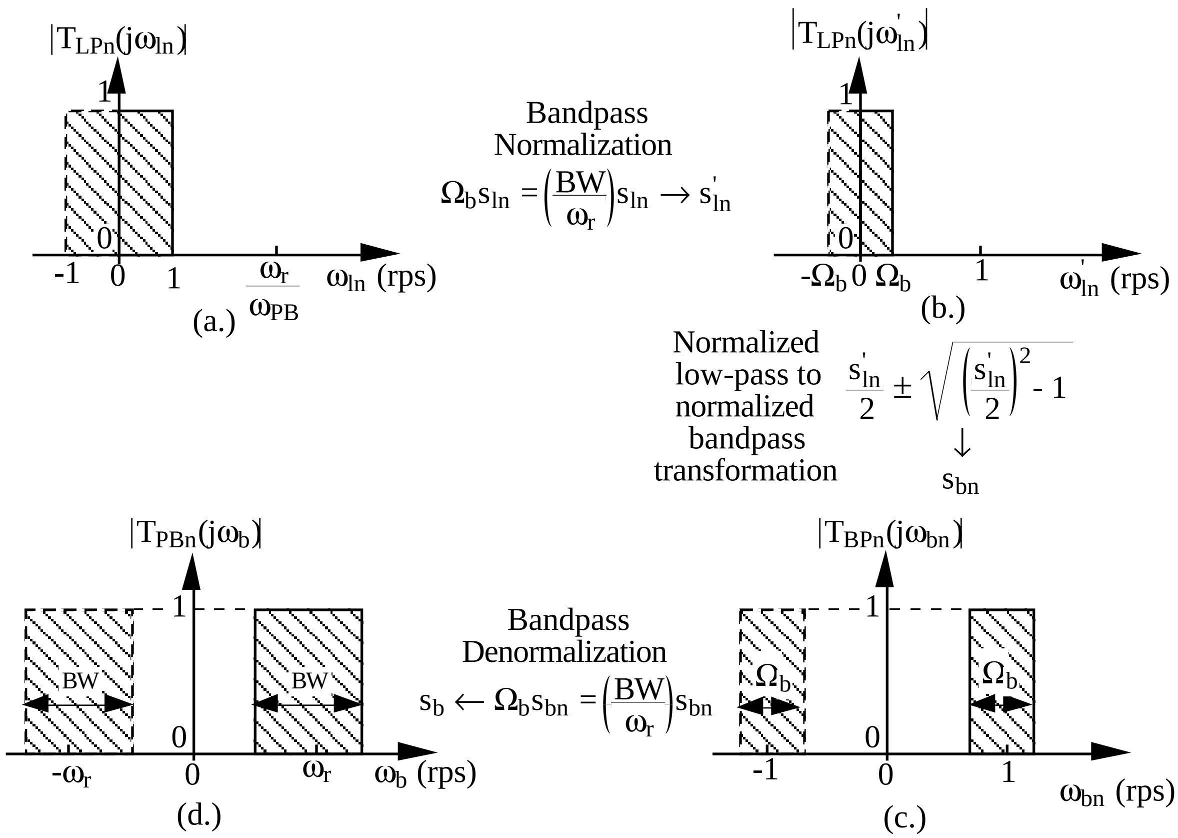

Figure 2-14 - Illustration of the development of a bandpass filter from a low-pass filter. (a.) Ideal normalized, low-pass filter. (b.) Normalization of (a.) for bandpass transformation. (c.) Application of low-pass to bandpass transformation. (d.) Denormalized bandpass filter.

Once the normalized, low-pass poles, p'_{kln} , are known, then the normalized bandpass poles can be found from

$$p_{kbn} = \frac{p'_{kln}}{2} \pm \sqrt{\left(\frac{p'_{kln}}{2}\right)^2 - 1} \quad . \tag{2-37}$$

which is written from Eq. (2-36). For each pole of the low-pass filter, two poles result for the bandpass filter. Consequently, the order of complexity based on poles is 2N for the bandpass filter. If the low-pass pole is on the negative real axis, the two bandpass poles are complex conjugates. However, if the low-pass pole is complex, two bandpass poles result from this pole and two bandpass poles result from its conjugate. Fig. 2-15 shows how the complex conjugate low-pass poles contribute to a pair of complex conjugate bandpass poles. p^* is the designation for the conjugate of p. This figure shows that both poles of the complex conjugate pair must be transformed in order to identify the resulting two pairs of complex conjugate poles.

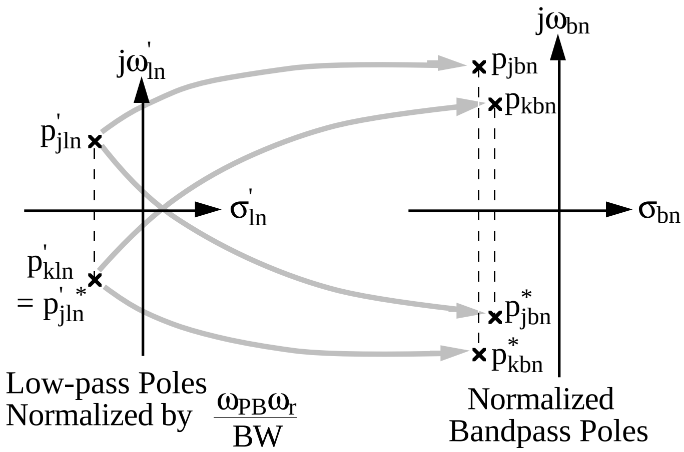

Figure 2-15 - Illustration of how the normalized, low-pass, complex conjugate poles are transformed into two normalized, bandpass, complex conjugate poles.

It can also be shown that the low-pass to bandpass transformation takes each zero at infinity and transforms to a zero at the origin and a zero at infinity. After the low-pass to bandpass transformation is applied to N-th order low-pass filter, there will be N complex conjugate poles, N zeros at the origin, and N zeros at infinity. We can group the poles and zeros into second-order products having the following form

$$T_k(s_{bn}) = \frac{K_k s_{bn}}{(s_{bn} + p_{kbn})(s_{bn} + p_{jbn}^*)} = \frac{K_k s_{bn}}{(s_{bn}+\sigma_{kbn}+j\omega_{kbn})(s_{bn}+\sigma_{kbn}-j\omega_{kbn})}$$

$$= \frac{K_k s_{bn}}{s_{bn}^2+(2\sigma_{kbn})s_{bn}+(\sigma_{bn}^2+\omega_{kbn}^2)} = \frac{T_k(\omega_{kon})\left(\frac{\omega_{kon}}{Q_k}\right)s_{bn}}{s_{bn}^2 + \left(\frac{\omega_{kon}}{Q_k}\right)s_{bn} + \omega_{kon}^2} \tag{2-38}$$

where j and k corresponds to the jth and kth low-pass poles which are a complex conjugate pair, K_k is a gain constant, and

$$\omega_{kon} = \sqrt{\sigma_{kbn}^2+\omega_{kbn}^2} \tag{2-39}$$

and

$$Q_k = \frac{\sqrt{\sigma_{bn}^2+\omega_{kbn}^2}}{2\sigma_{bn}} \quad . \tag{2-40}$$

Normally, the gain of $T_k(\omega_{kon})$ is unity.

The order of the bandpass filter is determined by translating its specifications to an equivalent low-pass filter. The ratio of the stop bandwidth to the pass bandwidth for the bandpass filter is defined as

$$\Omega_n = \frac{SW}{BW} = \frac{\omega_{SB2} - \omega_{SB1}}{\omega_{BP2} - \omega_{PB1}} \quad . \tag{2-41}$$

The general cascade design procedure for a bandpass filter follows:

1.) Start with the bandpass specification in the form of Fig. 2-3c (or the equivalent in terms of attenuation). Normalize the frequency by dividing by ω_r . Therefore, the normalized geometric center frequency is 1 rps and the normalized bandwidth Ω_b.

2.) From T_{PB}, T_{SB}, and Ω_n (or A_{PB}, A_{SB}, and Ω_n) determine the required order of the normalized, low-pass filter approximation using the proper equations for the selected approximation.

3.) Find the normalized, low-pass poles of the approximation.

4.) Frequency scale (normalize) the normalized, low-pass poles by $\Omega_b = (BW/\omega_r)$.

5.) Find the normalized poles of the bandpass filter by inserting each normalized low-pass pole into Eq. (2-37).

6.) Group the complex conjugate poles in to the form of Eq. (2-38).

7.) Realize each complex conjugate pole pair by a second-order, bandpass active filter.

8.) Cascade the realizations in the order from input to output of the lowest pole-Q stage first.

9.) Denormalize to the desired passband frequency and denormalize the impedances if desired using Eq. (2-32) and Table 2-1.

The following example will illustrate the application of this design procedure.

Example 2-7 - Design of a Butterworth, BandPass Filter

Design a bandpass, Butterworth filter having a -3dB ripple bandwidth of 200 Hz geometrically centered at 1 kHz and a stopband of 1 kHz with an attenuation of 40 dB or greater, geometrically centered at 1 kHz. Use the Tow-Thomas active filter realization and give a complete circuit and all component values. The gain at 1 kHz is to be unity.

Solution

From the specifications, we know that T_{PB} = -3 dB and T_{SB} = -40 dB. Eq. (2-39) gives a value of Ω_n = 1000/200 = 5. ε = 1 because T_{PB} = -3 dB. Therefore, we use Eq. (2-11) to find that N = 3 will give T_{SB} = -41.94 dB which is the lowest, integer value of N which meets the specifications.

Next, we evaluate the normalized, low-pass poles from Eq. (2-12) as

$$p_{1ln},\ p_{3ln} = -0.5000 \pm j0.8660$$

and

$$p_{2ln} = -1.0000\ .$$

where the first subscript on the poles corresponds to k in Eq. (2-12). Normalizing these poles by the bandpass normalization of Ω_b = 200/1000 = 0.2 gives

$$p'_{1ln},\ p'_{3ln} = -0.1000 \pm j\,0.1732$$

and

$$p'_{2ln} = -0.2000\ .$$

Each one of the p'_{kln} will contribute a second-order term of the form given in Eq. (2-37). The normalized bandpass poles are found by using Eq. (2-36) which results in 6 poles given as follows. For p'_{1ln} = -0.1000 + j0.1732 we get

$$p_{1bn},\ p_{2bn} = -0.0543 + j1.0891,\ -0.0457 - j0.9159.$$

For p'_{3ln} = -0.1000 - j0.1732 we get

$$p_{3bn},\ p_{4bn} = -0.0457 + j0.9159,\ -0.543 - j\,1.0891.$$

For p'_{2ln} = -0.2000 we get

$$p_{5bn},\ p_{6bn} = -0.1000 \pm j\,0.9950.$$

Fig. 2-16 shows the normalized low-pass pole locations, p_{kln}, the bandpass normalized, low-pass pole locations, p'_{kln}, and the normalized bandpass poles, p_{kbn}. Note that the bandpass poles have very high pole-Qs if BW < ω_r.

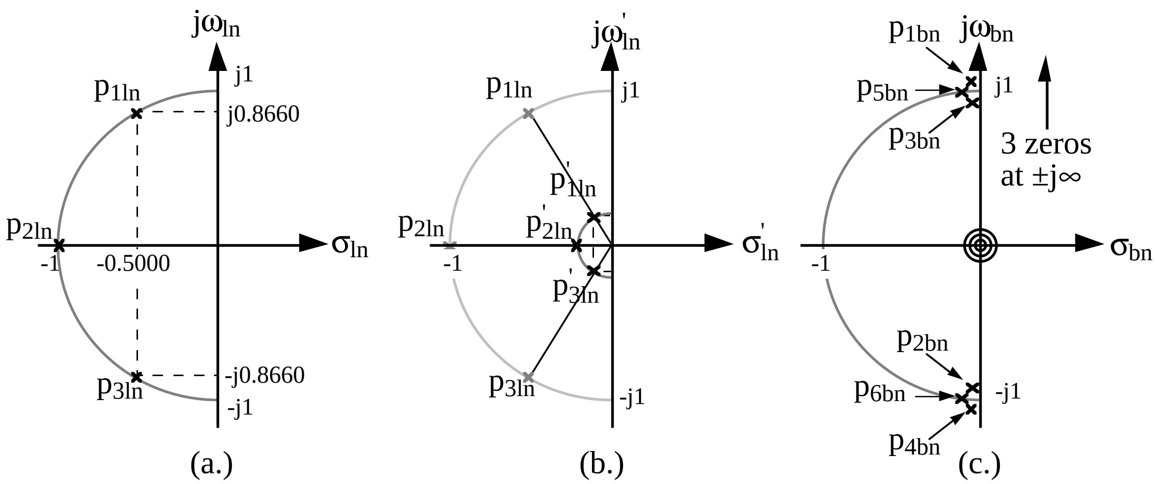

Figure 2-16 - Pole locations for Ex. 2-7. (a.) Normalized low-pass poles. (b.) Bandpass normalized low-pass poles. (c.) Normalized bandpass poles.

Grouping the complex conjugate bandpass poles gives the following second-order transfer functions.

$$T_1(s_{bn}) = \frac{K_1 s_{bn}}{(s+p_{1bn})(s+p_{4bn})} = \frac{K_1 s_{bn}}{(s_{bn}+0.0543+j1.0891)(s_{bn}+0.0543-j1.0891)}$$

$$= \frac{\left(\frac{1.0904}{10.0410}\right)s_{bn}}{s_{bn}^2+\left(\frac{1.0904}{10.0410}\right)s_{bn}+1.0904^2} \quad .$$

$$T_2(s_{bn}) = \frac{K_2 s_{bn}}{(s+p_{2bn})(s+p_{3bn})} = \frac{K_2 s_{bn}}{(s_{bn}+0.0457+j0.9159)(s_{bn}+0.0457-j0.9159)}$$

$$= \frac{\left(\frac{0.9170}{10.0333}\right)s_{bn}}{s_{bn}^2+\left(\frac{0.9170}{10.0333}\right)s_{bn}+0.9159^2} \quad .$$

and

$$T_3(s_{bn}) = \frac{K_3 s_{bn}}{(s+p_{5bn})(s+p_{6bn})} = \frac{K_3 s_{bn}}{(s_{bn}+0.1000+j0.9950)(s_{bn}+0.1000-j0.9950)}$$

$$= \frac{\left(\frac{1.0000}{5.0000}\right)s_{bn}}{s_{bn}^2+\left(\frac{1.0000}{5.0000}\right)s_{bn}+1.0000^2} \quad .$$

Now we are in a position to do the stage-by-stage design. We will use Fig. 1-22 and the design equations of Eqs. (1-60), (1-61), and (1-73) for all three, second-order stages.

Stage 1: Let us select $C = C_{11} = C_{21} = 1$ F. From Eq. (1-60), the resistors R_{21} and R_{31} are designed as $R = R_{21} = R_{31} = 1/(\omega_{1on}C) = 1/(1.09043)(1) = 0.9171\ \Omega$. Eq. (1-61) gives $R_{41} = QR = (10.0410)(0.9171) = 9.2083\ \Omega$. Finally, Eq. (1-73) gives $R_{11} = R_{41} = 9.2083\ \Omega$.

Stage 2: Let us select $C = C_{12} = C_{22} = 1$ F. From Eq. (1-60), the resistors R_{22} and R_{32} are designed as $R = R_{22} = R_{32} = 1/(\omega_{2on}C) = 1/(0.9159)(1) = 1.0918\ \Omega$. Eq. (1-61) gives $R_{42} = QR = (10.0333)(1.0918) = 10.9546\ \Omega$. Finally, Eq. (1-73) gives $R_{12} = R_{42} = 10.9546\ \Omega$.

Stage 3: Let us select $C = C_{13} = C_{23} = 1$ F. From Eq. (1-60), the resistors R_{23} and R_{33} are designed as $R = R_{23} = R_{33} = 1/(\omega_{3on}C) = 1/(1)(1) = 1.0000\ \Omega$. Eq. (1-61) gives $R_{41} = QR = (5.0000)(1.0000) = 5.0000\ \Omega$. Finally, Eq. (1-73) gives $R_{13} = R_{43} = 5.0000\ \Omega$.

Finally, we denormalize the components by $\omega_r = 2\pi x10^3$. At the same time we will arbitrarily denormalize the impedance by 10^4. Fig. 2-17 shows the resulting, sixth-order, unnormalized, bandpass active filter realization using the Butterworth approximation. Note that the stages are cascaded in the order of T_3, T_2, and T_1 for maximum signal swing.

The bandpass design procedure illustrated in Ex. 2-7 is general and can be used to design a cascaded, active filter realization of a bandpass filter whose bandwidth is geometrically centered around a frequency, ω_r. The problems give several other examples of designing higher-order, high-pass filters (see PR2-11 and PR2-12).

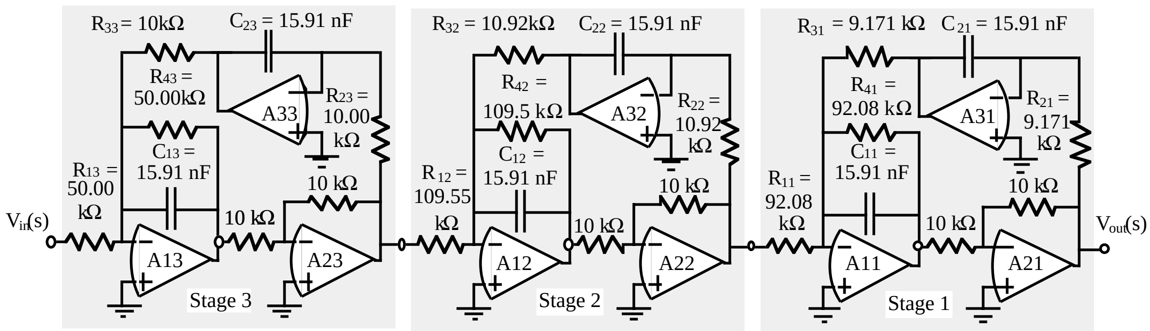

Figure 2-17 - Sixth-order, Butterworth filter realization for Ex. 2-7.

Active Filters with Finite Complex Conjugate Zeros

Some filter approximations which we have not studied use finite complex conjugate zeros as well as complex conjugate poles. Typically, these zeros are on the $j\omega$ axis although they may be either in the left-half or right-half complex frequency plane. The advantage of having complex conjugate zeros is that these zeros may be placed in the stopband to make the attenuation of the filter approximation greater for a given transition region. Fig. 2-18 shows an approximation called elliptic filter approximation. The elliptic filter has equal ripple bands in both the passband and the stopband. The number of poles must be equal to or greater than the number of zeros so that the magnitude of the low-pass filter rolls off to zero as frequency approaches infinity.

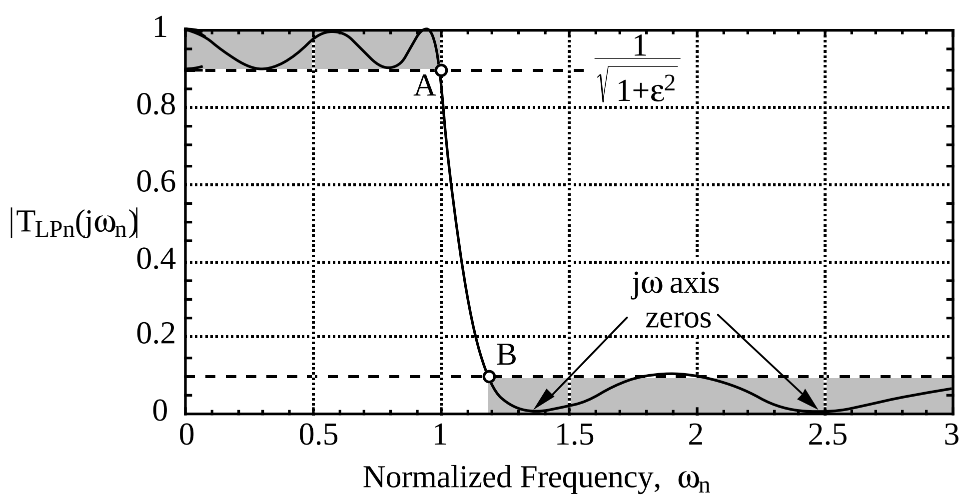

Figure 2-18 - Magnitude response of a fifth-order, elliptic filter approximation.

The elliptic filter approximation has the steepest possible roll off in the transition region of any type of filter approximation. This steep roll off is due to the presence of complex conjugate zeros on the jω axis just outside of the passband. The realization of filters containing jω axis zeros is exactly the same as been previously demonstrated in this section. The filter design starts from the normalized, low-pass structure which will contain complex conjugate zeros. The realization uses the cascade of first- and second-order active filters. However, this time, the active filters must be capable of realizing the complex conjugate zeros. The biquad, second-order active filter discussed in the last section is useful for this purpose. See the problems for further details on the design of filters with complex conjugate zeros (see PA2-2 and PA2-3).

Summary

The emphasis of this section has been on the design of higher-order filters using the cascade of first- and second-order active filter realizations. The normalized low-pass filter approximation is the starting point of all filter designs. The normalized low-pass filter has a passband from 0 to 1 rps. A filter is completely specified by four quantities: 1.) passband region, 2.) ripple of the passband region, 3.) stopband region, and 4.) the ripple of the stopband region. We use the word "ripple" even for those filter approximations which are monotonic.

We have examined two popular filter approximations. They are the Butterworth and the Chebyshev approximations. An approximation is a transfer function in the complex frequency variable which will satisfy the filter specification. The order of the approximation is a measure of the complexity of a realization. It is generally preferable to keep the order as small as possible.

There are four types of filters that have been considered. These types are the low-pass, high-pass, bandpass, and bandstop. The normalized low-pass filter is the starting point for the design of these different types of filters. Frequency transformations are used to take the low-pass filter approximation to the other types of filters. It is important to remember if the frequency transformations are used for the bandpass filters (and bandstop filters) that the filter passband and stopband must be geometrically related to a center frequency, ω_r.

The realization of higher-order filters discussed in this section uses the cascade of first- and second-order active filters. The design of each cascaded active filter uses a set of design equations which permit the designer to find the value of the components of the filter in terms of the first- or second-order transfer function. The design of these active filters is usually done for the normalized approximation and then a frequency denormalization and an impedance denormalization (which is generally arbitrary) is used to achieve the actual filter specifications and to get practical component values.

A word of caution is in order concerning the filters that have been discussed in this section. The design techniques introduced work well until the frequency of the filter begins to become larger than about 10 kHz. At this point, the frequency response of the op amp can no longer be ignored. Some of the realizations are more susceptible than others to the influence of the op amp frequency response. There are methods which permit active filters to be extended to 100 kHz and above but they are beyond the scope of this chapter. It has been observed that the influence of the op amp frequency response on the filter performance increases as the Q of the pole becomes higher.