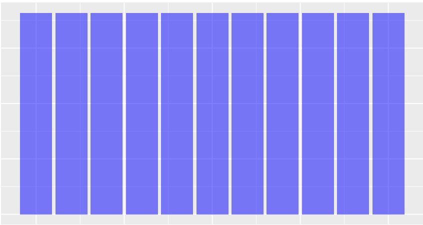
Probability Distributions

(Discrete) Uniform

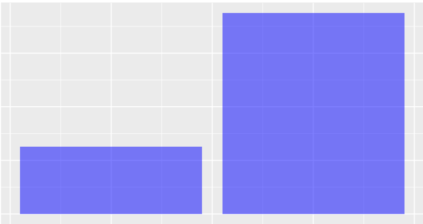
Each integer value in a specified range is assigned the same value.



Parameters	lower bound $a \in \mathbb{Z}$, upper bound $b \in \mathbb{Z}$ s.t. $a < b$
Possible values	$\{a, a+1, \ldots, b-1, b\}$
Notation	$U(a, b)$
Probability function	$p(x) = \frac{1}{b-a+1}$
Expectation	$\frac{a+b}{2}$
Variance	$\frac{(b-a)(b-a+2)}{12}$
MGF	$M(t) = \frac{1}{b-a+1} \sum_{k=a}^{b} e^{kt}$
Kernel	1

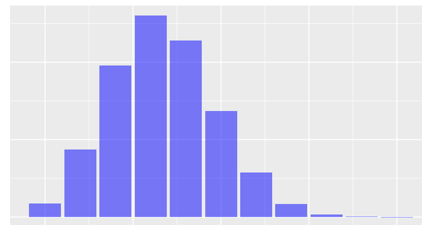
Bernoulli

Represents the success of a single experiment as a binary outcome.



Parameters	probability of success $0 \leq p \leq 1$
Possible values	$\{0, 1\}$
Notation	$\text{Bern}\,(p)$
Probability function	$p(x) = p^x (1-p)^{1-x}$
Expectation	p
Variance	$p(1-p)$
MGF	$M(t) = 1 + p\left(e^t - 1\right)$
Kernel	$p^x (1-p)^{-x}$

Binomial



Represents the number of successes in a fixed number of independent and repeated trials of the same Bernoulli experiment.

Parameters	number of trials $n \in \mathbb{N}$, probability of success on a single trial $0 \leq p \leq 1$
Possible values	$\{0, 1, 2, \ldots, n\}$
Notation	$\text{Bin}\,(n, p)$
Probability function	$p(x) = \binom{n}{x} p^x (1-p)^{n-x}$
Expectation	np
Variance	$np(1-p)$
MGF	$M(t) = \left[1 + p\left(e^t - 1\right)\right]^n$
Kernel	$\binom{n}{x} p^x (1-p)^{-x}$

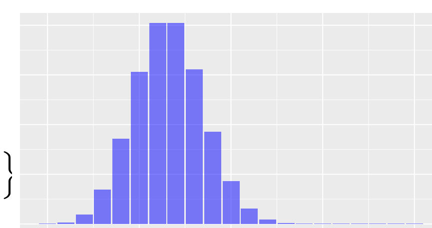
Related Distributions

- If $X \sim \text{Bin}\,(n, p)$, then

$$X = \sum_{i=1}^{n} X_i,$$

  where the $X_i \sim \text{Bern}\,(p)$ independently.

Hypergeometric

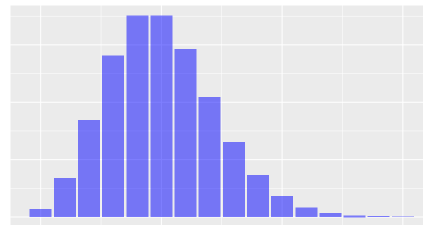


Represents the number of when drawing a fixed number of samples from a population containing a known number of successes.

Possible values	$\Big\{\max\{0, n - (n - (N - k))\}, \ldots, \min\{n, k\}\Big\}$
Parameters	size of population $N \in \mathbb{N}$, number of successes in population $k \in \mathbb{N}$, number of samples drawn $n \in \mathbb{N}$
Notation	$\text{Hypergeometric}\,(N, k, n)$
Probability function	$p(x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$
Expectation	$\frac{nk}{N}$
Variance	$n\left(\frac{k}{N}\right)\left(\frac{N-k}{N}\right)\left(\frac{N-n}{N-1}\right)$
MGF	No useful expression
Kernel	$\binom{k}{x}\binom{N-k}{n-x}$

Poisson

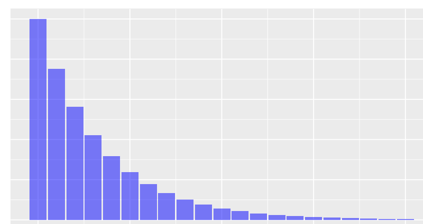
Represents the number of events occuring in a fixed interval.



Possible values	$\{0, 1, 2, \ldots\}$
Parameters	average number of events in interval $\lambda > 0$,
Notation	$\text{Po}\,(\lambda)$
Probability function	$p(x) = \frac{e^{-\lambda}\lambda^x}{x!}$
Expectation	λ
Variance	λ
MGF	$M(t) = e^{\lambda\left(e^t - 1\right)}$
Kernel	$\frac{\lambda^x}{x!}$

Geometric

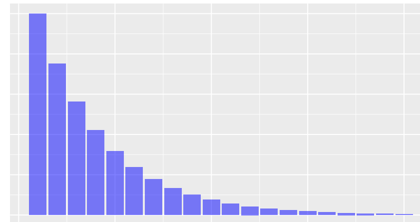
Represents the number of failed Bernoulli trials preceeding the first success.



Parameters	probability of success on a single trial $0 \leq p \leq 1$,
Possible values	$\{0, 1, 2, \ldots\}$
Notation	$\text{Geom}\,(p)$
Probability function	$p(x) = (1-p)^x p$
Expectation	$\frac{1-p}{p}$
Variance	$\frac{1-p}{p^2}$
MGF	$M(t) = \frac{p}{1-(1-p)e^t}$ for $t < -\ln(1-p)$
Kernel	$(1-p)^x$

Geometric (alternative)

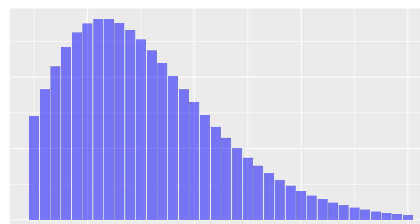
Represents the position of the first success in a sequence of Bernoulli trials.



Parameters	probability of success on a single trial $0 \leq p \leq 1$,
Possible values	$\{1, 2, \ldots\}$
Notation	$\text{Geom}\,(p)$
Probability function	$p(x) = (1-p)^{x-1}p$
Expectation	$\frac{1}{p}$
Variance	$\frac{1-p}{p^2}$
MGF	$M(t) = \frac{pe^t}{1-(1-p)e^t}$ for $t < -\ln(1-p)$
Kernel	$(1-p)^x$

Negative Binomial

Represents the number of Bernoulli trials preceeding the rth success.



Parameters	number of desired successes $r \in \mathbb{N}$, probability of success on a single trial $0 \leq p \leq 1$,
Possible values	$\{r, r+1, \ldots\}$
Notation	$\text{NB}\,(r, p)$
Probability function	$p(x) = \binom{x-1}{r-1}(1-p)^{x-r}p^r$
Expectation	$\frac{r}{p}$
Variance	$\frac{r(1-p)}{p^2}$
MGF	$M(t) = \left[\frac{pe^t}{1-(1-p)e^t}\right]^r$ for $t < -\ln p$
Kernel	$\binom{x-1}{r-1}(1-p)^x$

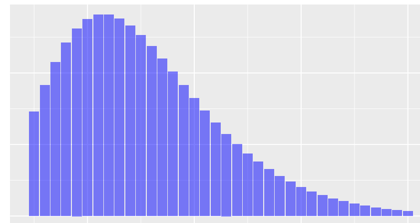
Related Distributions

- If $X \sim \text{NB}\,(r, p)$, then

$$X = \sum_{i=1}^{n} X_i,$$

  where the $X_i \sim \text{Geom}\,(p)$ independently.

Negative Binomial (Alternative)



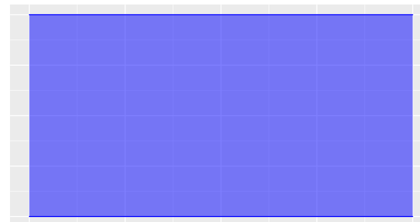
Represents the number of failures preceeding the rth success in a sequence of Bernoulli trials.

Parameters	number of desired successes $r \in \mathbb{N}$, probability of success on a single trial $0 \leq p \leq 1$,
Possible values	$\{0, 1, 2, \ldots\}$
Notation	$\mathrm{NB}\,(r, p)$
Probability function	$p(x) = \binom{x+r-1}{x}(1-p)^x p^r$
Expectation	$\frac{r(1-p)}{p}$
Variance	$\frac{r(1-p)}{p^2}$
MGF	$M(t) = \left[\frac{p}{1-(1-p)e^t}\right]^r$ for $t < -\ln p$
Kernel	$\binom{x+r-1}{x}(1-p)^x$

Continuous

(Continuous) Uniform

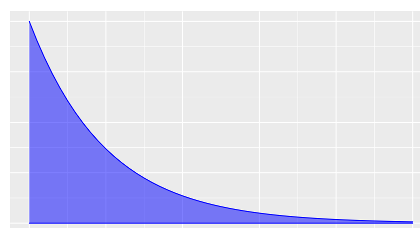
Each value in a specified interval has the same probability density.



Parameters	lower bound $a \in \mathbb{R}$, upper bound $b \in \mathbb{R}$ s.t. $a < b$
Possible values	(a, b)
Notation	$\mathrm{U}(a, b)$
Probability density function	$f(x) = \frac{1}{b-a}$
Cumulative distribution function	$F(x) = \begin{cases} 0, & \text{if } x \leq a, \\ \frac{x-a}{b-a}, & \text{if } a < x < b, \\ 1, & \text{if } b \leq x. \end{cases}$
Expectation	$\frac{a+b}{2}$
Variance	$\frac{(b-a)^2}{12}$
MGF	$M(t) = \begin{cases} 1, & \text{if } t = 0, \\ \frac{e^{tb}-e^{ta}}{t(b-a)}, & \text{if } t \neq 0. \end{cases}$
Kernel	1

Exponential

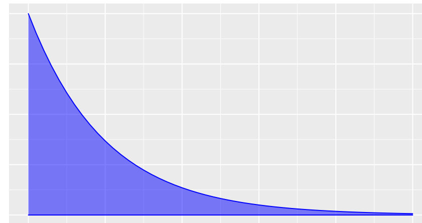
Represents the waiting time until an event occurs.



Parameters	rate of events $\lambda > 0$
Possible values	$(0, \infty)$
Notation	$\mathrm{Exp}(\lambda)$
Probability density function	$f(x) = \lambda e^{-\lambda x}$
Cumulative distribution function	$F(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1 - e^{-\lambda x}, & \text{if } x > 0. \end{cases}$
Expectation	$\frac{1}{\lambda}$
Variance	$\frac{1}{\lambda^2}$
MGF	$M(t) = \frac{\lambda}{\lambda - t}$ for $t < \lambda$
Kernel	$e^{-\lambda x}$

Exponential (Alternative)

Characterised by average time until the first event $\beta = 1/\lambda$.



Parameters	average time until event $\beta > 0$
Possible values	$(0, \infty)$
Notation	$\text{Exp}\,(\beta)$
Probability density function	$f(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}}$
Cumulative distribution function	$F(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1 - e^{-\frac{x}{\beta}}, & \text{if } x > 0. \end{cases}$
Expectation	β
Variance	β^2
MGF	$M(t) = \frac{1}{1-\beta t}$ for $t < \frac{1}{\beta}$
Kernel	$e^{-\frac{x}{\beta}}$

Gamma

Generalisation of the exponential distribution. For integer shape parameter, represents the waiting time until the rth event.

Parameters	shape $r > 0$ scale $\lambda > 0$
Possible values	$(0, \infty)$
Notation	$\text{Gamma}\,(a, b)$
Probability density function	$f(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}$
Cumulative distribution function	no simple expression
Expectation	$\frac{r}{\lambda}$
Variance	$\frac{r}{\lambda^2}$
MGF	$M(t) = \left(\frac{\lambda}{\lambda - t}\right)^r$ for $t < \lambda$
Kernel	$x^{r-1} e^{-\lambda x}$

Properties

- If $X \sim \text{Gamma}\,(r, \lambda)$, then for any $k > 0$, $kX \sim \text{Gamma}\left(r, \frac{\lambda}{k}\right)$

Related Distributions

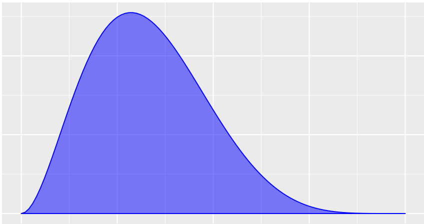
- If $r \in \mathbb{N}$ and $X \sim \text{Gamma}\,(r, \lambda)$, then

$$X = \sum_{i=1}^{n} X_i,$$

  where $X_i \sim \text{Exp}\,(\lambda)$ independently.
- If $X \sim \chi^2_r$, then $X \sim \text{Gamma}\left(\frac{r}{2}, \frac{1}{2}\right)$.

Beta

Represents a random variable which is positive and bounded. Useful for modelling proportions and percentages.



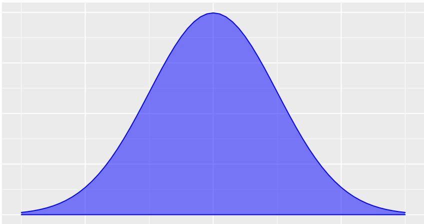
Parameters	shape $\alpha > 0$ shape $\beta > 0$
Possible values	$(0, 1)$
Notation	$\text{Beta}\,(a, b)$
Probability density function	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$
Cumulative distribution function	no simple expression
Expectation	$\frac{\alpha}{\alpha+\beta}$
Variance	$\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$
MGF	exists but no simple expression
Kernel	$x^{\alpha-1}(1-x)^{\beta-1}$

Related Distributions

- If $X \sim \text{Gamma}\,(\alpha, \lambda)$ and $Y \sim \text{Gamma}\,(\beta, \lambda)$ independently, then $\frac{X}{X+Y} \sim \text{Beta}\,(\alpha, \beta)$.
- The Beta distribution can be generalised to any bounded interval $[a, b]$, with properties

Normal

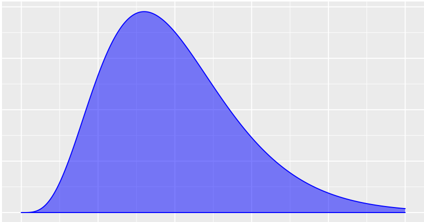
The classic bell-shaped curve.



Parameters	mean $\mu \in \mathbb{R}$ variance $\sigma^2 > 0$
Possible values	$\mathbb{R}$
Notation	$\mathcal{N}\left(\mu, \sigma^2\right)$
Probability density function	$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
Cumulative distribution function	no simple expression
Expectation	μ
Variance	σ^2
MGF	$e^{\mu t + \frac{\sigma^2 t^2}{2}}$
Kernel	$e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

χ^2

The sum of the squares of n independent standard normal random variables.



Parameters	degrees of freedom $n \in \mathbb{N}$
Possible values	$(0, \infty)$ if $n = 1$; $[0, \infty)$ otherwise
Notation	χ^2_n
Probability density function	$f(x) = \frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)} x^{\frac{n}{2}-1} e^{-\frac{x}{2}}$
Cumulative distribution function	$F(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ \frac{1}{\Gamma\left(\frac{n}{2}\right)} \int_0^{x/2} t^{\frac{n}{2}-1} e^{-\frac{t}{2}} \, \mathrm{d}t. & \text{if } x > 0 \end{cases}$
Expectation	n
Variance	$2n$
MGF	$(1-2t)^{-\frac{n}{2}}$
Kernel	$x^{\frac{n}{2}-1} e^{-\frac{x}{2}}$

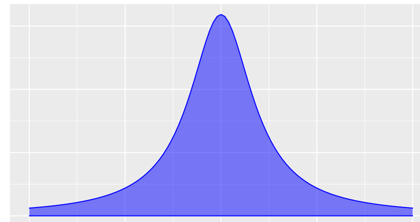
Related Distributions

- If $X \sim \chi^2_n$, then
$$X = \sum_{i=1}^{n} Z_i^2,$$
where $Z_i \sim \mathcal{N}(0, 1)$ independently.
- If $X \sim \chi^2_r$, then $X \sim \text{Gamma}\left(\frac{r}{2}, \frac{1}{2}\right)$.
- If $Y \sim F_{\nu_1,\nu_2}$, then $\lim_{\nu_2 \to \infty} \nu_1 Y \sim \chi^2_{\nu_1}$.

Cauchy

The ratio of two independent, identically distributed and central normal random variables.

The sum of the squares of n independent standard normal random variables.



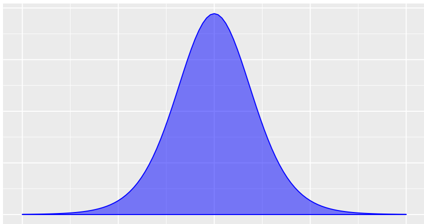
Parameters	location $\mu \in \mathbb{R}$ scale $\sigma > 0$
Possible values	$\mathbb{R}$
Notation	$\text{Cauchy}\,(\mu, \sigma)$
Probability density function	$f(x) = \frac{1}{\pi\sigma\left(1+\frac{2(x-\mu)}{\sigma}\right)}$
Cumulative distribution function	$F(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{x-\mu}{\sigma}\right), & \text{if } x > 0 \end{cases}$
Expectation	does not exist
Variance	does not exist
MGF	does not exist
Kernel	$\sigma + 2(x-\mu)$

Related Distributions

- If $Z_1, Z_2 \sim \mathcal{N}\,(0,1)$ independently, then $Z_1/Z_2 \sim \text{Cauchy}\,(0,1)$.
- If $X \sim \text{Cauchy}\,(\mu,\sigma)$ then $X \sim t_n\,(\mu,\sigma)$.

t

The ratio of a standard normal random variable and χ^2 random variable.



Parameters	location $\mu \in \mathbb{R}$ scale $\sigma > 0$ degrees of freedom $\nu \in \mathbb{N}$
Possible values	$(0, \infty)$ if $n = 1$; $[0, \infty)$ otherwise
Notation	$t_n\left(\mu, \sigma^2\right)$
Probability density function	$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sigma\sqrt{\pi\nu}}\left[1 + \frac{1}{\nu}\left(\frac{x-\mu}{\sigma}\right)^2\right]^{-\frac{\nu+1}{2}}$
Cumulative distribution function	no simple expression
Expectation	μ for $\nu \geq 2$
Variance	$\frac{\sigma^2\nu}{\nu-2}$ for $\nu \leq 3$
MGF	does not exist
Kernel	$\left[1 + \frac{1}{\nu}\left(\frac{x-\mu}{\sigma}\right)^2\right]^{-\frac{\nu+1}{2}}$

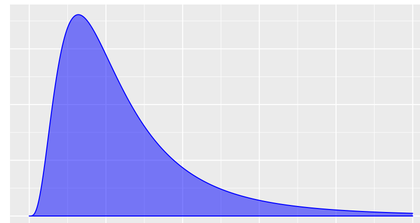
Related Distributions

- If $Z \sim \mathcal{N}(0, 1)$ and $W \sim \chi^2_\nu$ independently, then

$$\mu + \sigma\frac{Z}{\sqrt{\frac{W}{\nu}}} \sim t_\nu\left(\mu, \sigma^2\right).$$

F

The ratio of the sample variance of two normal random variables.



Parameters	degrees of freedom $\nu_1 \in \mathbb{N}$ degrees of freedom $\nu_2 \in \mathbb{N}$
Possible values	$(0, \infty)$
Notation	$F_n \mu, \sigma^2$
Probability density function	$f(x) = \frac{\Gamma\left(\frac{\nu_1 \nu_2}{2}\right) \nu_1^{\nu_1/2} \nu_2^{\nu_2} x^{\nu_1/2-1}}{\Gamma\left(\frac{\nu_1}{2}\right) \Gamma \nu_2 2 (\nu_1 x + \nu_2)^{\frac{\nu_1+\nu_2}{2}}}$
Cumulative distribution function	no simple expression
Expectation	$\frac{\nu_2}{\nu_2 - 2}$ for $\nu \geq 2$
Variance	$\frac{2\nu_2^2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-2)^2(\nu_2-4)}$ for $\nu \geq 4$
MGF	does not exist
Kernel	$\frac{x^{\nu_1/2-1}}{(\nu_1 x + \nu_2)^{\frac{\nu_1+\nu_2}{2}}}$

Related Distributions

- If $U \sim \chi^2_{\nu_1}$ and $V \sim \chi^2_{\nu_2}$ independently, then

$$\frac{U/\nu_1}{V/\nu_2} \sim F_{\nu_1, \nu_2}.$$

- If $X_1 \sim \text{Gamma}\,(\alpha_1, \beta_1)$ and $X_2 \sim \text{Gamma}\,(\alpha_2, \beta_2)$ independently, then

$$\frac{\alpha_1 \beta_1 X_1}{\alpha_2 \beta_2 X_2} \sim F_{2\alpha_1, 2\alpha_2}.$$

- If $X \sim \text{Beta}\left(\alpha/2, \beta/2\right)$ then

$$\frac{\beta X}{\alpha(1-X)} \sim F_{\alpha, \beta}.$$

- If $X \sim t_\nu$, then $X^2 \sim F_{1,\nu}$.

Multivariate

Discrete

Trinomial

A generalisation of the binomial distribution to 3 possible outcomes.

Parameters	number of trials $n \in \mathbb{N}$ probabilities of success $\pi_1, \pi_2 > 0$ such that $\pi_1 + \pi_2 < 1$
Possible values	$x_1, x_2 \in \{0, 1, \ldots, n\}$ such that $x_1 + x_2 \leq n$
Notation	$\text{Tri}\left(n, \pi_1, \pi_2\right)$
Probability function	$p(x_1, x_2) = \frac{n!}{x_1! x_2! (n - x_1 - x_2)!} \pi_1^{x_1} \pi_2^{x_2} (1 - \pi_1 - \pi_2)^{n - \pi_1 - \pi_2}$
Expectation	$\mathbb{E}\left[X_i\right] = n\pi_i$
Variance	$\text{var}\left(X_i\right) = n\pi_i(1 - \pi_i)$
Covariance	$\text{cov}\left(X_1, X_2\right) = -n\pi_1\pi_2$
Marginal	$X_1 \sim \text{Bin}\left(n, \pi_1\right)$ $X_2 \sim \text{Bin}\left(n, \pi_2\right)$
Conditional	$X_1 \mid X_2 = x_2 \sim \text{Bin}\left(n - x_2, \frac{\pi_1}{1 - \pi_2}\right)$ $X_2 \mid X_1 = x_1 \sim \text{Bin}\left(n - x_1, \frac{\pi_2}{1 - \pi_1}\right)$

Multinomial

Generalisation of the binomial distribution to r possible outcomes.

Parameters	number of trials $n \in \mathbb{N}$ probabilities of success $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_r$ such that $\pi_i \geq 0$ and $\sum_{i=1}^{r} \pi_i = n$
Possible values	$x_1, \ldots, x_r \in \{0, 1, \ldots, n\}$ such that $\sum_{i=1}^{r} x_i = n$
Notation	$\text{Multi}_r\left(n, \boldsymbol{\pi}\right)$
Probability function	$p(x_1, \ldots, x_r) = \frac{n!}{x_1! \cdots x_r!} \pi_1^{x_1} \cdots \pi_r^{x_r}$
Expectation	$\mathbb{E}\left[X_i\right] = n\pi_i$
Variance	$\text{var}\left(X_i\right) = n\pi_i(1 - \pi_i)$
Covariance	$\text{cov}\left(X_i, X_j\right) = -n\pi_i\pi_j$ for $i \neq j$

Continuous

Uniform on Unit Square

A bivariate extension of the univariate (continuous) uniform distribution.

Parameters	none
Possible values	$0 < x_1, x_2 < 1$
Probability density function	$f(x_1, x_2) = 1$

Uniform on Unit Circle

Another bivariate extension of the univariate (continuous) uniform distribution.

Parameters	none
Possible values	$-1 < x_1, x_2 < 1$ such that $x_1^2 + x_2^2 < 1$
Probability density function	$f(x_1, x_2) = \frac{1}{\pi}$

Dirichlet

Generalisation of the Beta distribution.

Parameters	$\alpha_1, \ldots, \alpha_{r+1} > 0$
Possible values	$x_1, \ldots, x_r > 0$ such that $\sum_{i=1}^{r} x_i < 1$
Notation	$\mathrm{Dir}\left(\boldsymbol{\alpha}\right)$
Probability density function	$f(x_1, \ldots, x_r) = \frac{\Gamma\left(\alpha_1 + \cdots + \alpha_{r+1}\right)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_{r+1})} \prod_{i=1}^{r} x_i^{\alpha_i - 1}$
Expectation	$\mathbb{E}\left[X_i\right] = \frac{\alpha_i}{\alpha_0}$
Variance	$\mathrm{var}\left(X_i\right) = \frac{\alpha_i(\alpha_0 - \alpha_i)}{\alpha_0^2(\alpha_0 + 1)}$
Covariance	$\mathrm{cov}\left(X_1, X_2\right) = -\frac{\alpha_i \alpha_j}{\alpha_0^2(\alpha_0 + 1)}$ where $\alpha_0 = \sum_{i=1}^{r+1} \alpha_i$
Marginal	$X_i \sim \mathrm{Beta}\left(\alpha_i, \alpha_0 - \alpha_i\right)$

Multivariate Normal

Generalisation of the normal distribution.

Parameters	mean $\boldsymbol{\mu} \in \mathbb{R}^r$ non-negative definite variance matrix Σ
Possible values	$\boldsymbol{x} \in \mathbb{R}^r$
Notation	$\mathcal{N}_r\left(\boldsymbol{\mu}, \Sigma\right)$
Probability density function	$p(\boldsymbol{x}) = (2\pi)^{-\frac{r}{2}} \lvert\Sigma\rvert^{-\frac{1}{2}}\, e^{-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})^T \Sigma^{-1} (\boldsymbol{x} - \boldsymbol{\mu})}$
Expectation	$\mathbb{E}\left[\boldsymbol{X}\right] = \boldsymbol{\mu}$
Variance	$\mathrm{var}\left(\boldsymbol{X}\right) = \Sigma$
MGF	$e^{\boldsymbol{t}^T \boldsymbol{\mu} + \frac{1}{2} \boldsymbol{t}^T \Sigma \boldsymbol{t}}$