Lecture 4:
Divide & Conquer

General techniques in this course

- Greedy algorithms [Lecture 3]
- Divide & Conquer algorithms [today]
- Sweepline algorithms [28 Aug]
- Dynamic programming algorithms [4 and 11 Sep]
- Network flow algorithms [18 Sep and 9 Oct]

Divide-and-Conquer

- **Divide-and-conquer** [usually 3 parts]
  1. Divide: Break up problem into several parts.
  2. Conquer: Solve each part recursively.
  3. Combine solutions to sub-problems into overall solution.

- **Most common usage.**
  - Break up problem of size n into two equal parts of size ½n.
  - Solve two parts recursively.
  - Combine two solutions into overall solution in linear time.

Searching

Input: A sorted sequence S of n numbers $a_1, a_2, \ldots, a_n$, stored in an array A[1..n].

Question: Given a number x, is x in S?

0	1	3	4	5	7	10	13	15	18	19	23

Binary Search

- Compare x to the middle element of the array (A[n/2]).
- If A[n/2] = x then "Yes"
- Otherwise, if A[n/2] < x then recursively Search A[1…n/2-1].
- Otherwise, if A[n/2] > x then recursively Search A[n/2+1...n]

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Example: x=1 (non-integers are rounded up)

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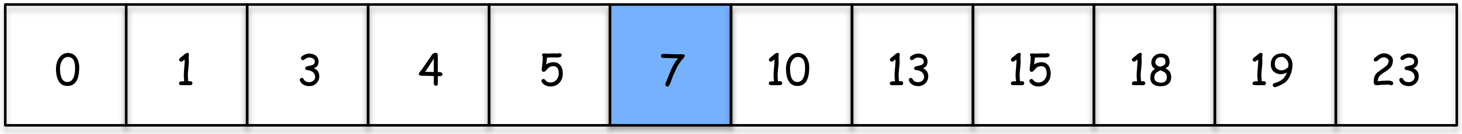

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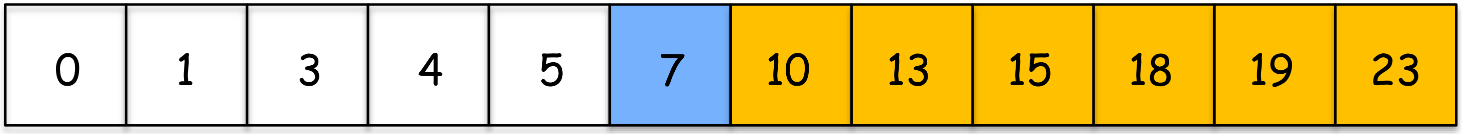

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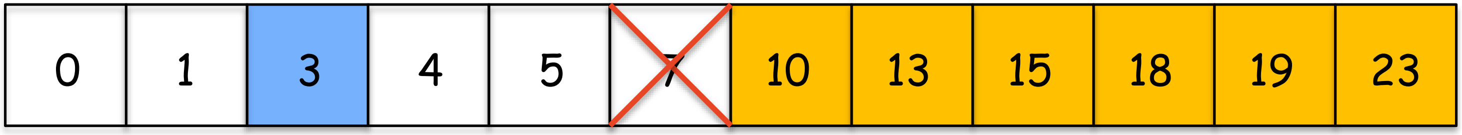

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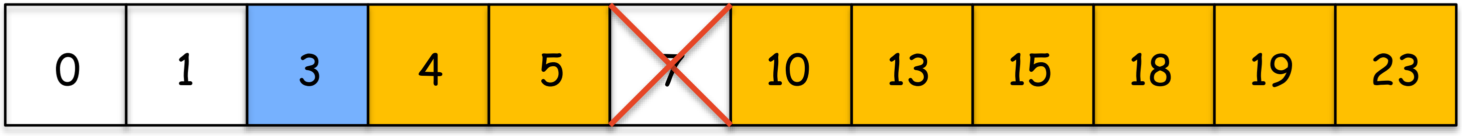

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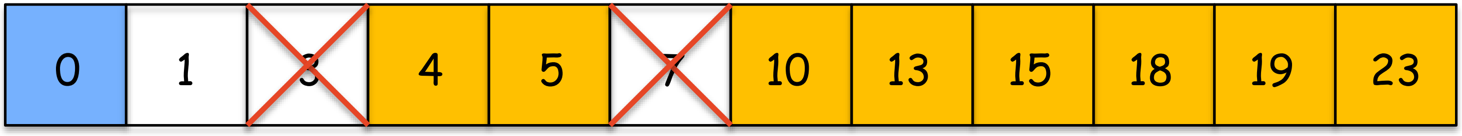

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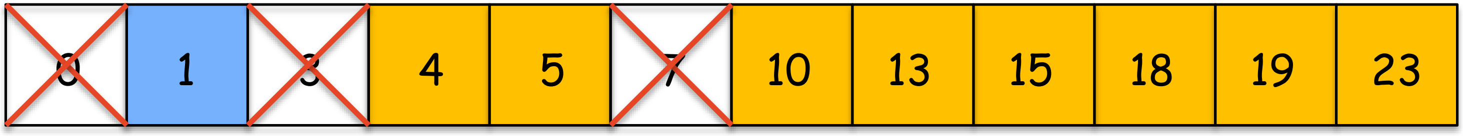

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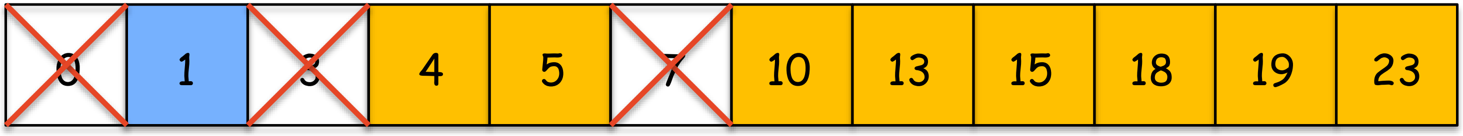

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Analysis: $T(n) = 1 + T(n/2)$

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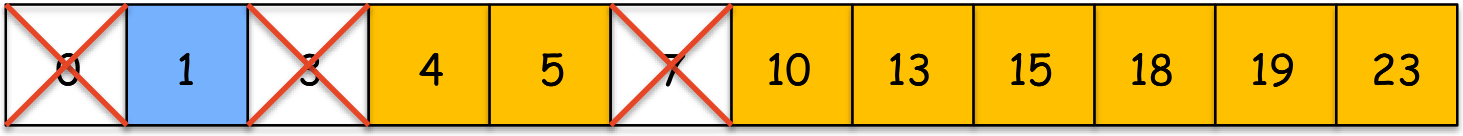
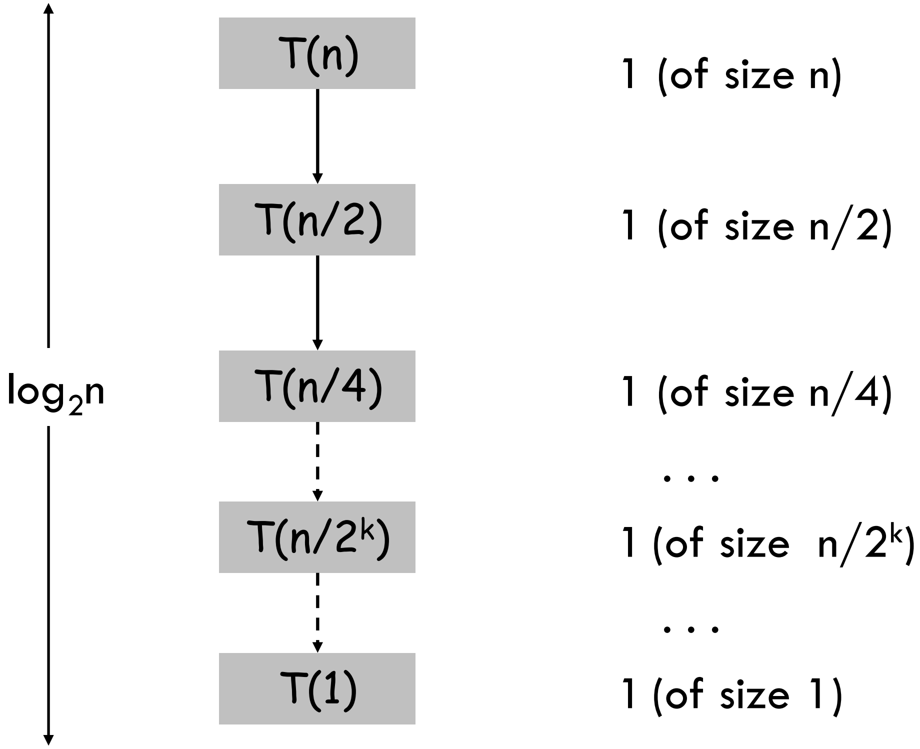


Analysis: $T(n) = 1 + T(n/2)$

Analyze recursion

$$T(n) = T(n/2)+O(1)$$



$\log_2 n$

$T(n)$ 1 (of size n)

$T(n/2)$ 1 (of size n/2)

$T(n/4)$ 1 (of size n/4)

. . .

$T(n/2^k)$ 1 (of size $n/2^k$)

. . .

$T(1)$ 1 (of size 1)

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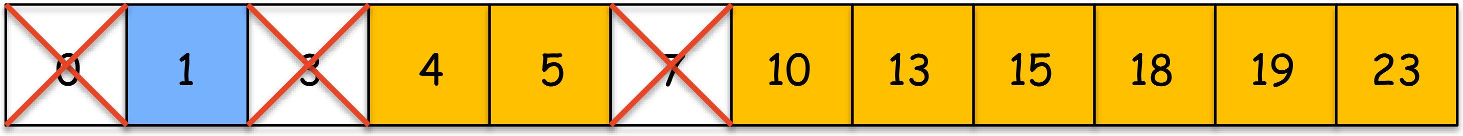


~~0~~	1	~~3~~	4	5	~~7~~	10	13	15	18	19	23

Analysis: $T(n) = 1 + T(n/2) = O(\log n)$

Sorting

- Sorting. Given n elements, rearrange in ascending order.

Obvious sorting applications.

  List files in a directory.
  Organize an MP3 library.
  List names in a phone book.
  Display Google PageRank results.

Problems become easier once sorted.

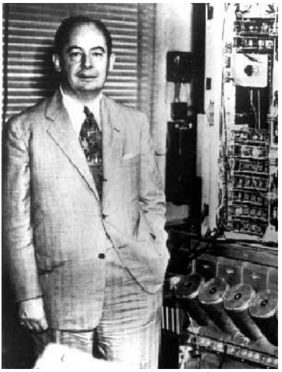
  Find the median.
  Find the closest pair.
  Binary search in a database.
  Identify statistical outliers.
  Find duplicates in a mailing list.

Non-obvious sorting applications.

  Data compression.
  Computer graphics.
  Interval scheduling.
  Computational biology.
  Minimum spanning tree.
  Supply chain management.
  Simulate a system of particles.
  Book recommendations on Amazon.
  Load balancing.
  ...

Mergesort

1. Divide array into two halves.
2. Conquer: Recursively sort each half.
3. Combine: Merge two halves to make sorted whol



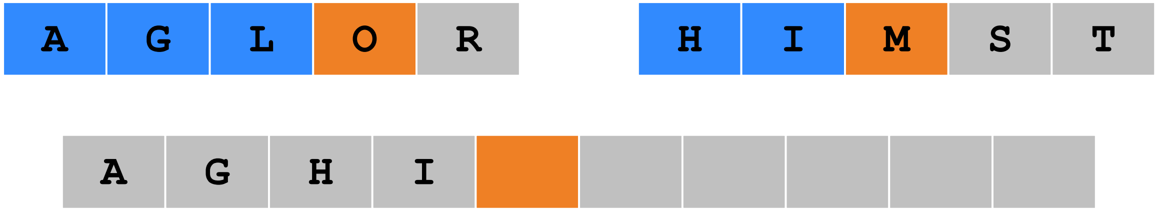
Jon von Neumann (1945)

A	L	G	O	R	I	T	H	M	S		
A	L	G	O	R	I	T	H	M	S	divide	O(1)
A	G	L	O	R	H	I	M	S	T	sort	2T(n/2)
A	G	H	I	L	M	O	R	S	T	merge	O(n)

Merging

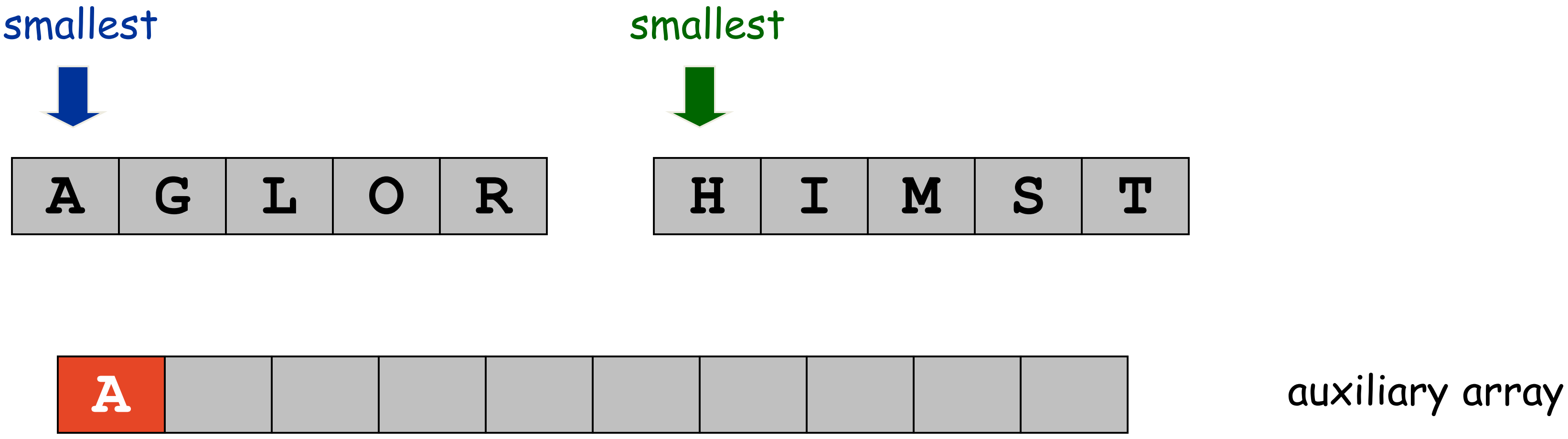
- Merging. Combine two pre-sorted lists into a sorted whole.

- How to merge efficiently?
  - Linear number of comparisons.
  - Use temporary array.



Merging

- Merge.
  - Keep track of smallest element in each sorted half.
  - Insert smallest of two elements into auxiliary array.
  - Repeat until done.



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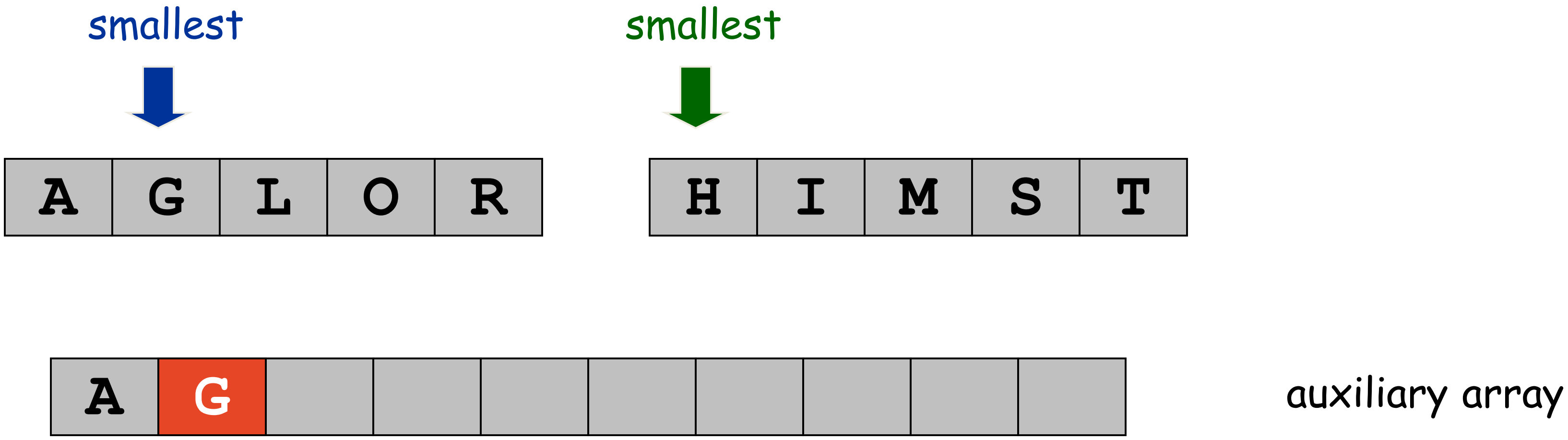

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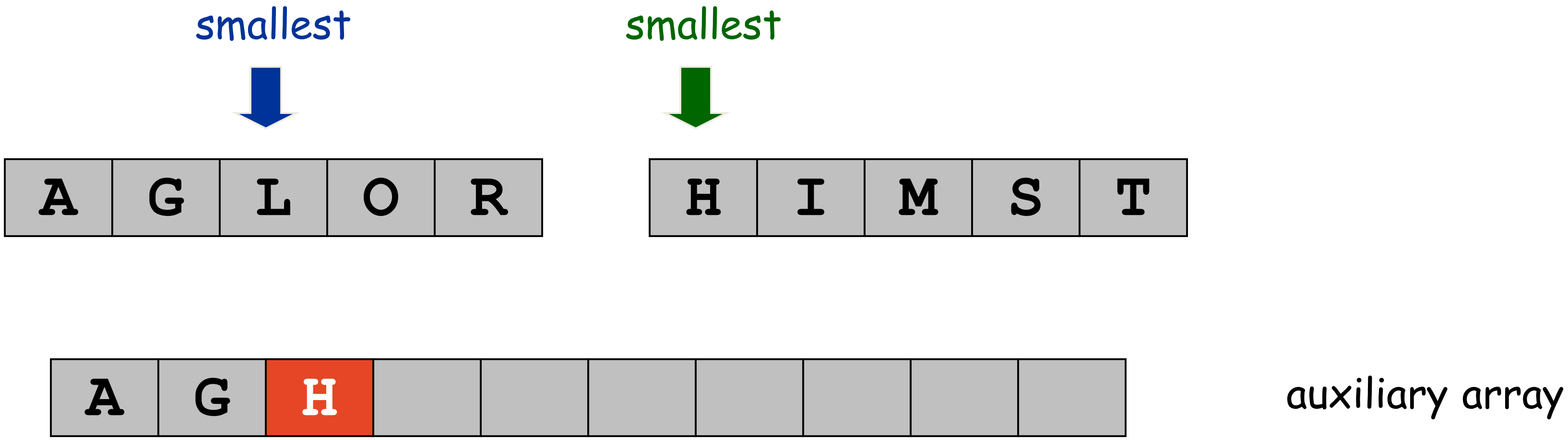

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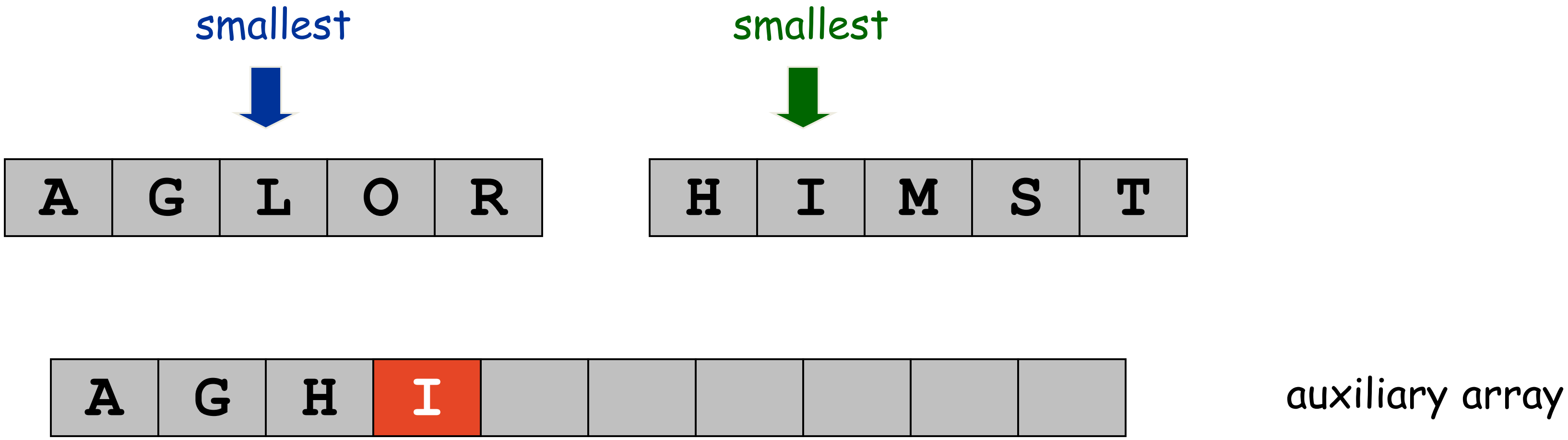

smallest

smallest

A	G	L	O	R

H	I	M	S	T

A	G	H	I						

auxiliary array

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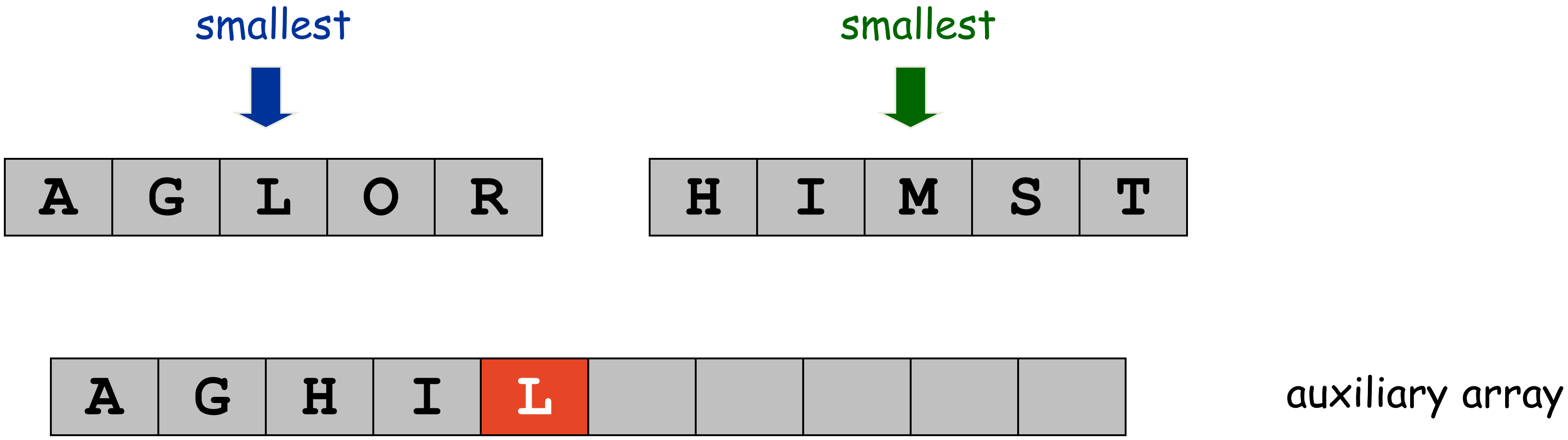

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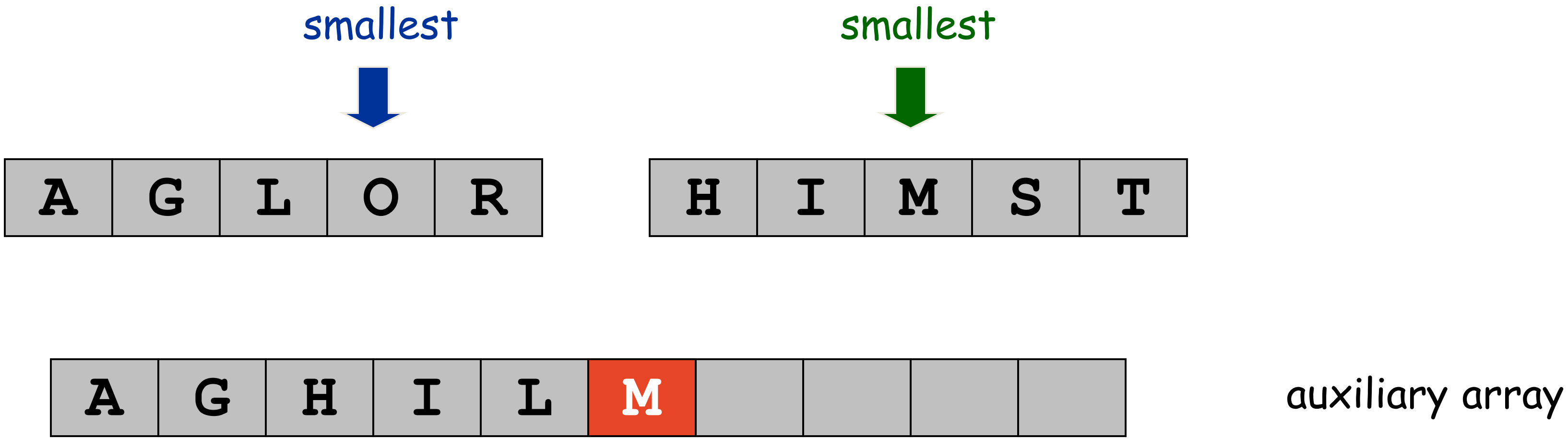

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A	G	H	I	L	M				

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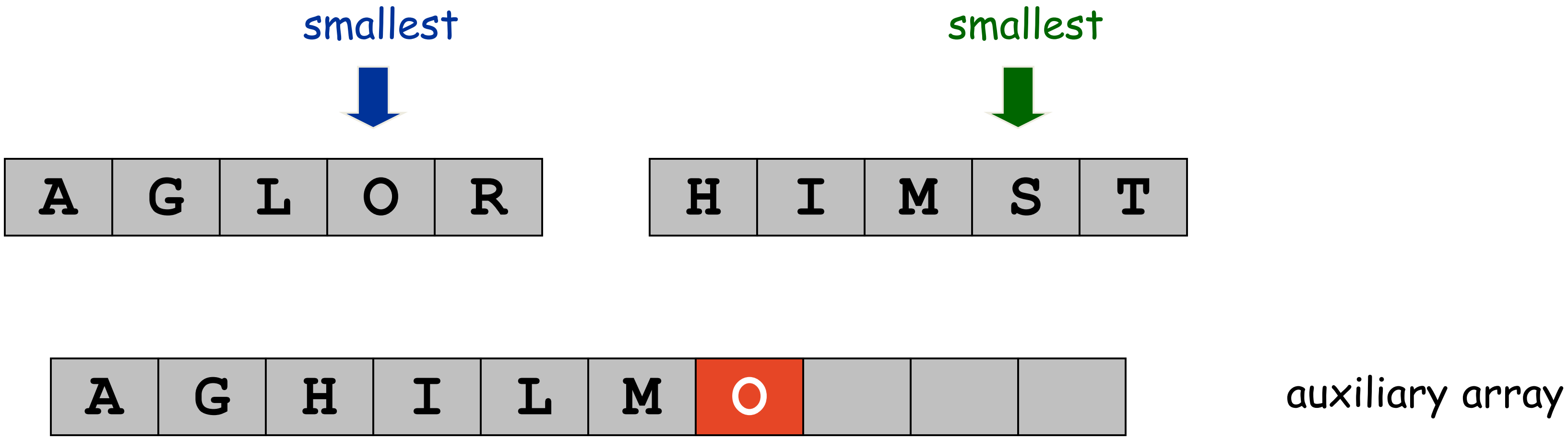

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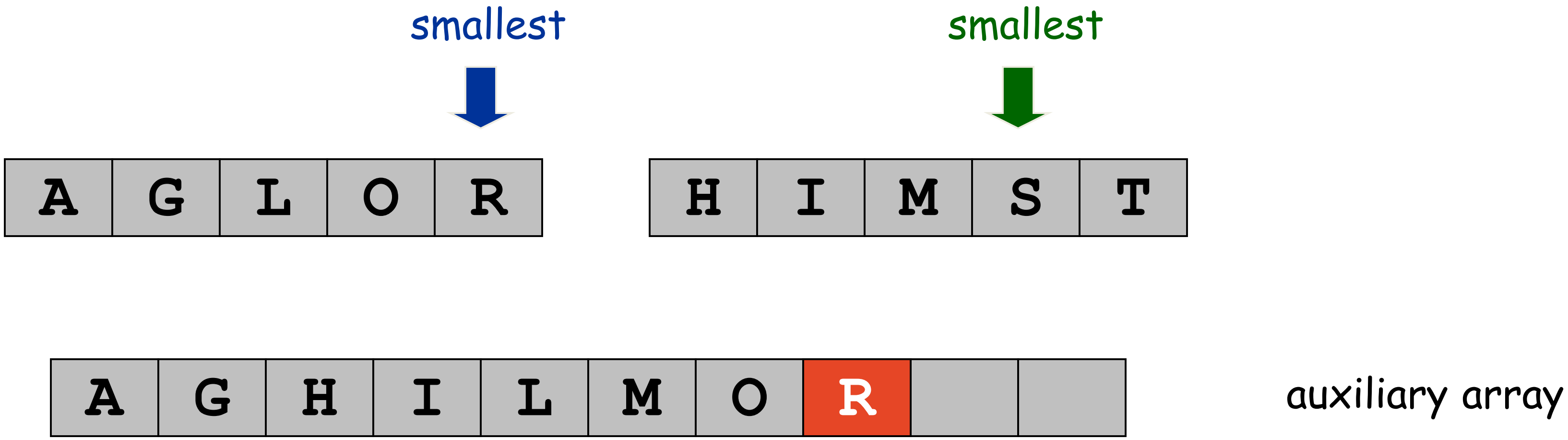

smallest

smallest

A	G	L	O	R

H	I	M	S	T

A	G	H	I	L	M	O	R		

auxiliary array

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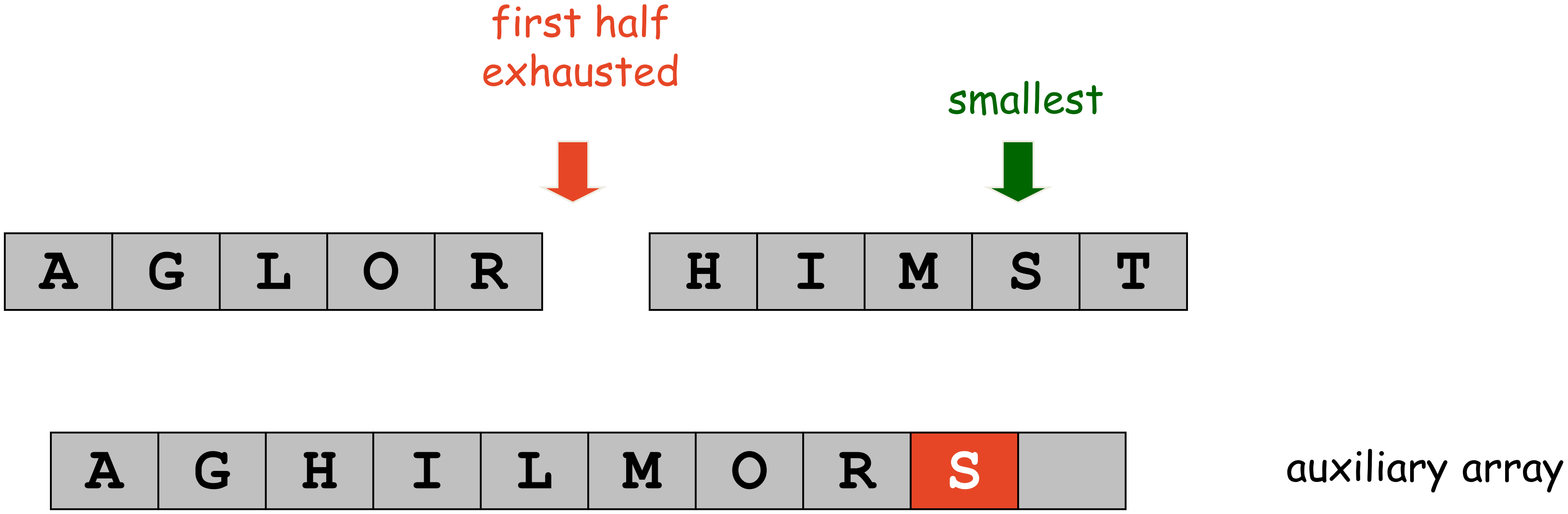

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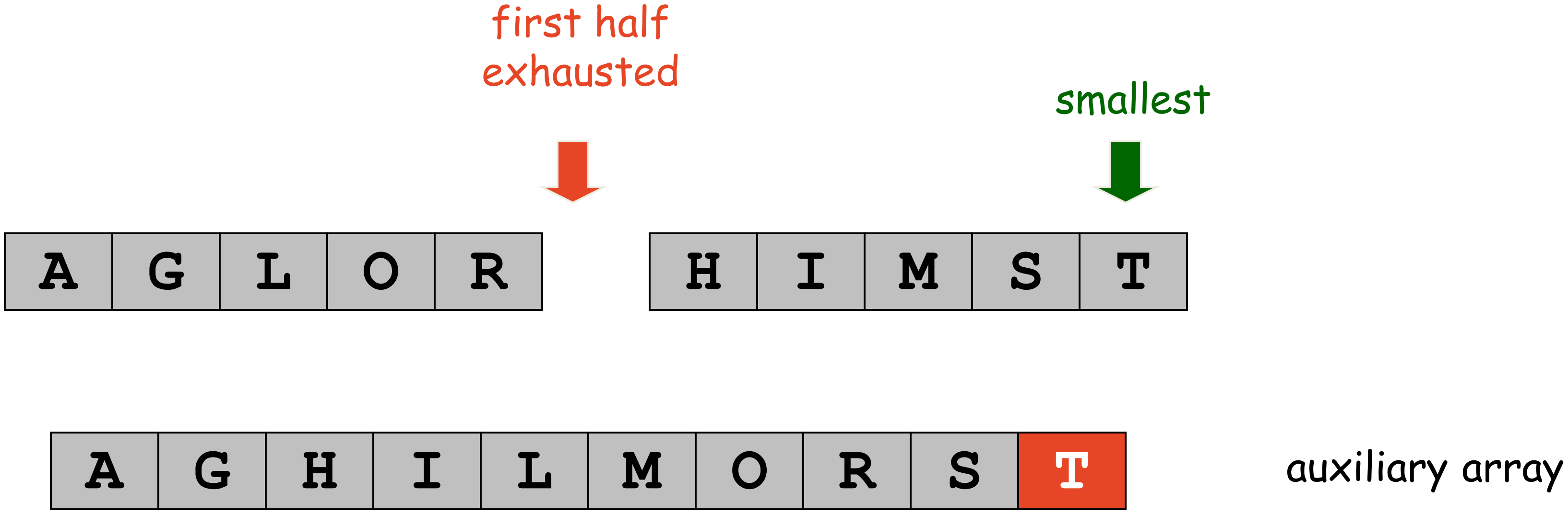

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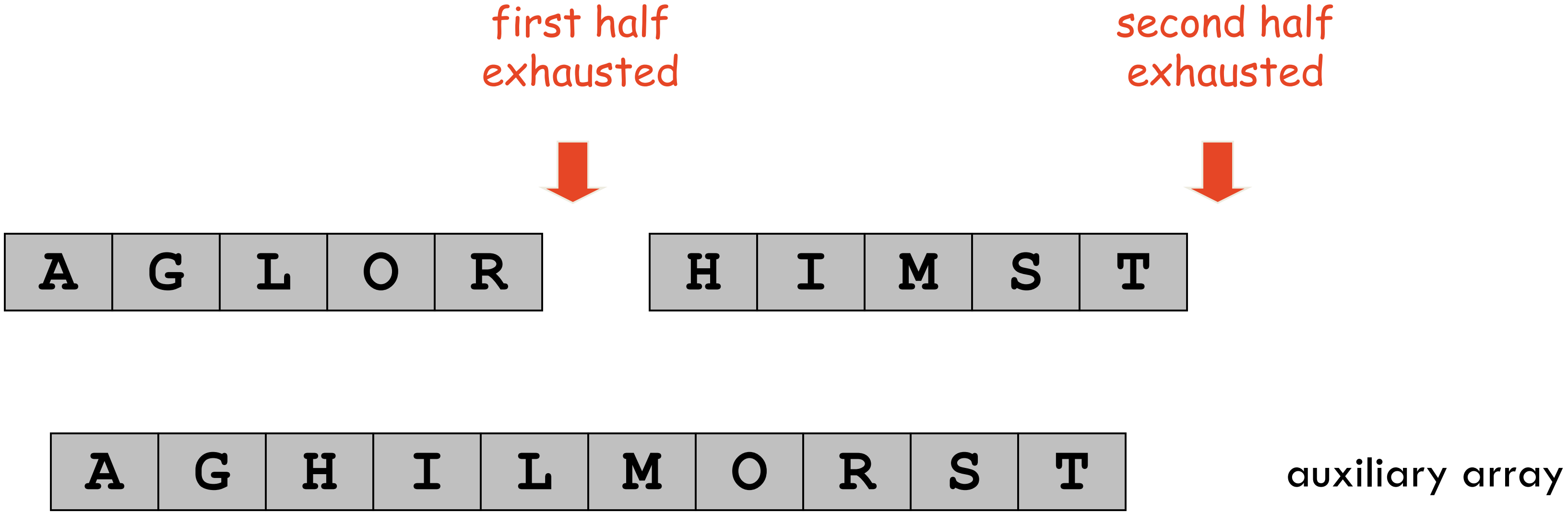

first half exhausted

second half exhausted

A	G	L	O	R

H	I	M	S	T

A	G	H	I	L	M	O	R	S	T

auxiliary array

Mergesort

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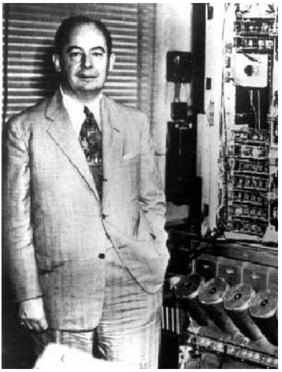

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A	L	G	O	R	I	T	H	M	S	divide	O(1)
A	G	L	O	R	H	I	M	S	T	sort	2T(n/2)
A	G	H	I	L	M	O	R	S	T	merge	O(n)

A Useful Recurrence Relation

- **Definition:** T(n) = number of comparisons to mergesort an input of size n.

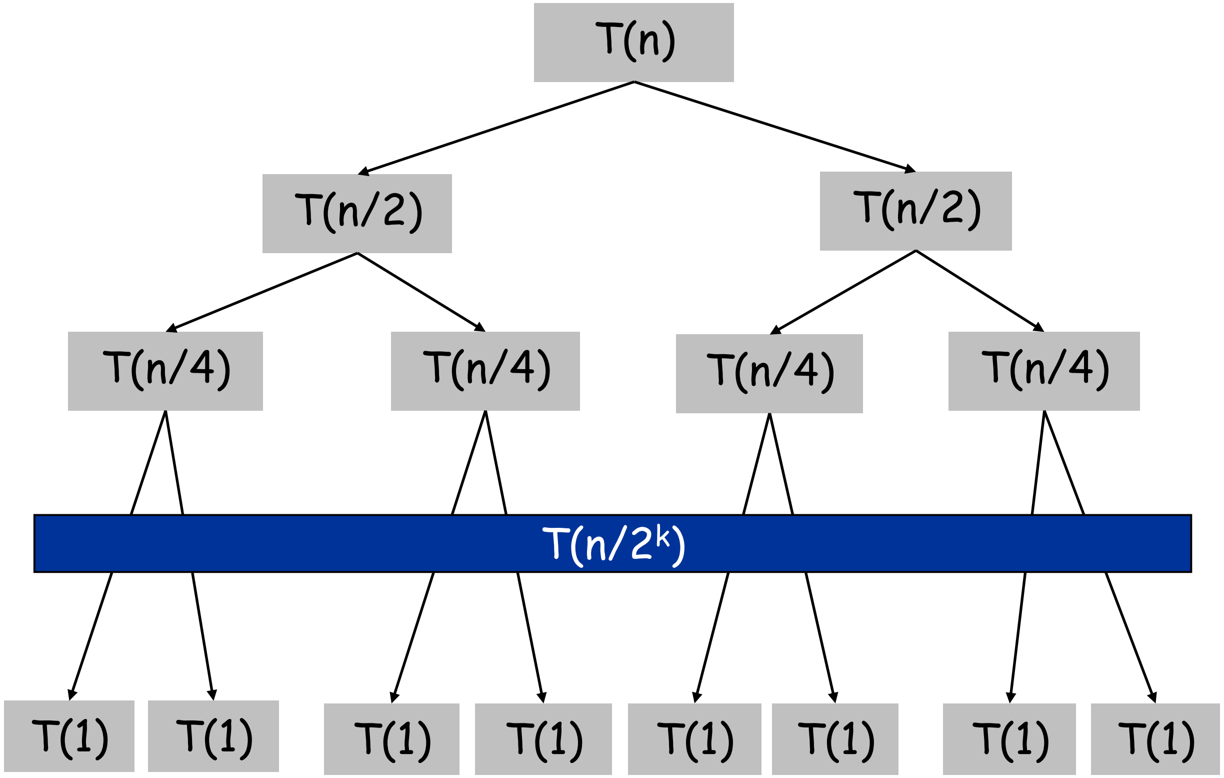
- **Mergesort recurrence.**

$$T(n) = \begin{cases} 0 & \text{if } n=1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + O(n) & \text{otherwise} \end{cases}$$

- **Solution:** T(n) = ?

Proof by unrolling

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{cn}_{\text{merging}} & \text{otherwise} \end{cases}$$



1 (of size n)

2 (of size n/2)

4 (of size n/4)

. . .

2^k (of size $n/2^k$)

. . .

n (of size 1)

Proof by unrolling

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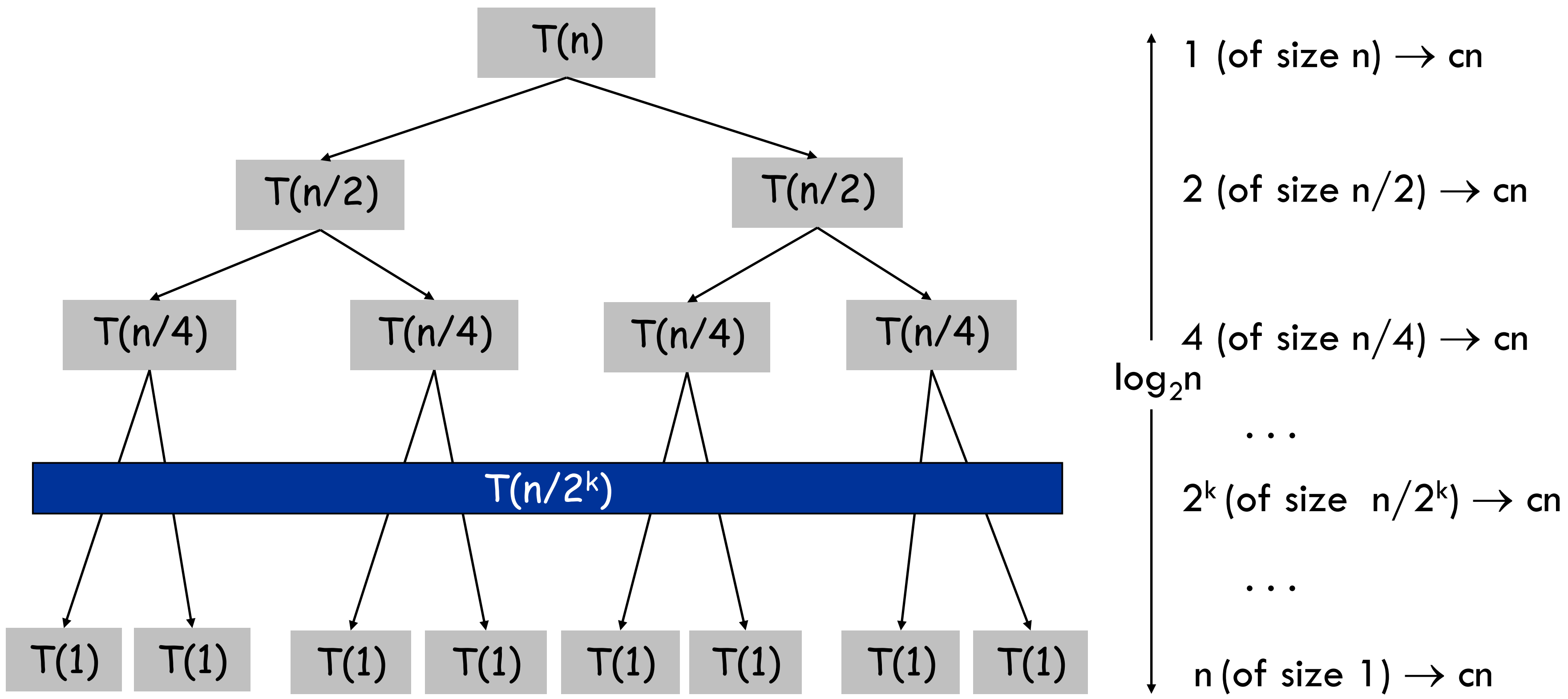

T(n)

T(n/2) T(n/2)

T(n/4) T(n/4) T(n/4) T(n/4)

$T(n/2^k)$

T(1) T(1) T(1) T(1) T(1) T(1) T(1) T(1)

$\log_2 n$

1 (of size n) → cn

2 (of size n/2) → cn

4 (of size n/4) → cn

. . .

2^k (of size $n/2^k$) → cn

. . .

n (of size 1) → cn

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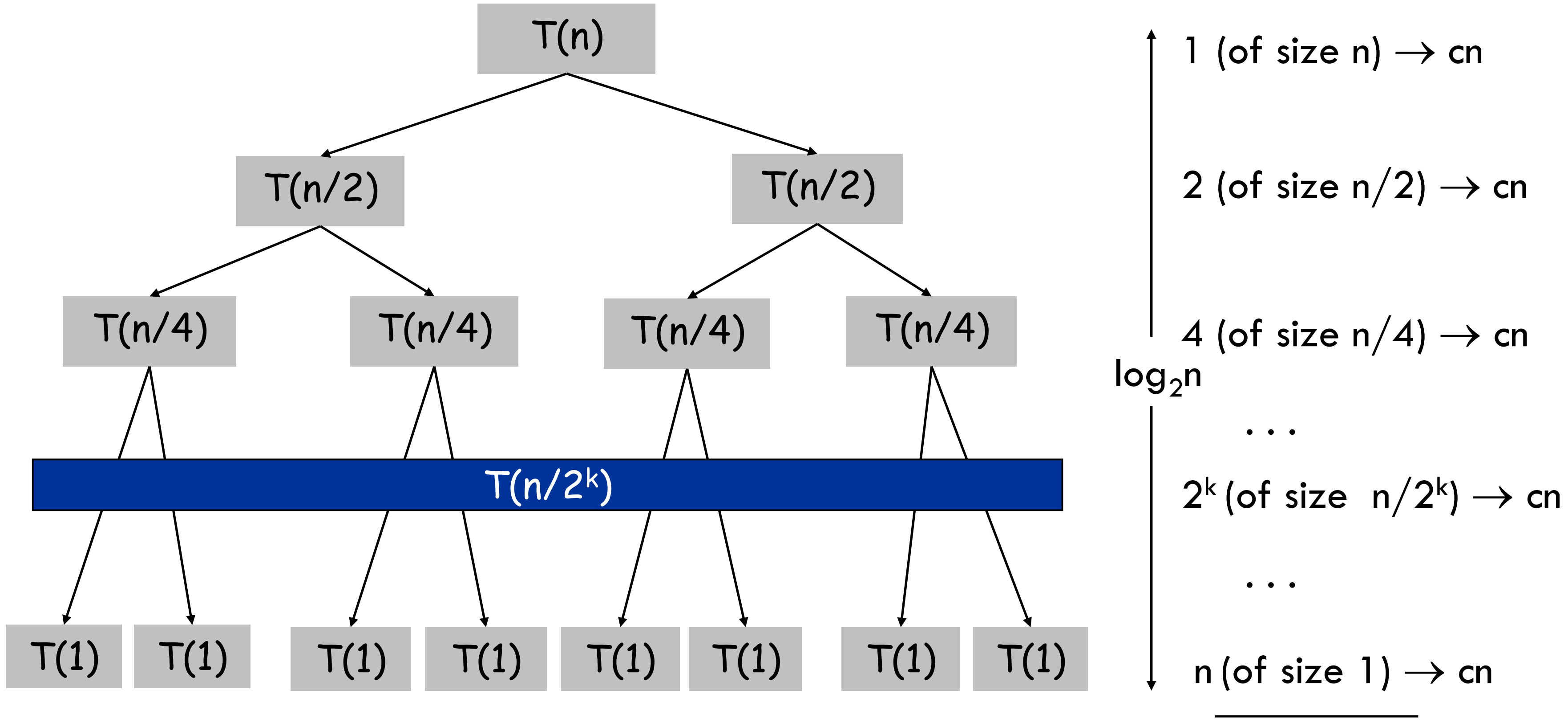

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2^k (of size $n/2^k$) → cn

. . .

n (of size 1) → cn

$cn \log_2 n$

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$$\mathrm{T}(n) \le \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{T(\lceil n/2 \rceil)}_{\text{solve left half}} + \underbrace{T(\lfloor n/2 \rfloor)}_{\text{solve right half}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

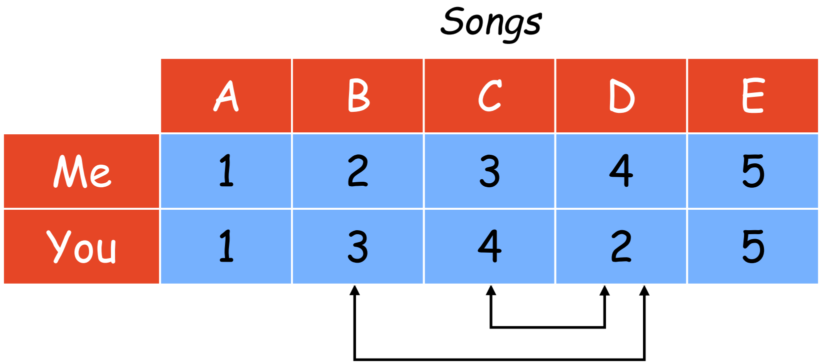
- **Solution:** $T(n) = O(n \log_2 n)$.

Counting Inversions

Counting Inversions

- Music site tries to match your song preferences with others.
  - You rank n songs.
  - Music site consults database to find people with **similar** tastes.

- Similarity metric: number of inversions between two rankings.
  - My rank: 1, 2, …, n.
  - Your rank: $a_1, a_2, \ldots, a_n$.
  - Songs i and k inverted if $i < k$, but $a_i > a_k$.



Songs

	A	B	C	D	E
Me	1	2	3	4	5
You	1	3	4	2	5

Inversions

3-2, 4-2

- Brute force: check all $\Theta(n^2)$ pairs i and k.

Applications

- Applications.
  - Voting theory.
  - Collaborative filtering.
  - Measuring the "sortedness" of an array.
  - Sensitivity analysis of Google's ranking function.
  - Rank aggregation for meta-searching on the Web.
  - Nonparametric statistics (e.g., Kendall's Tau distance).

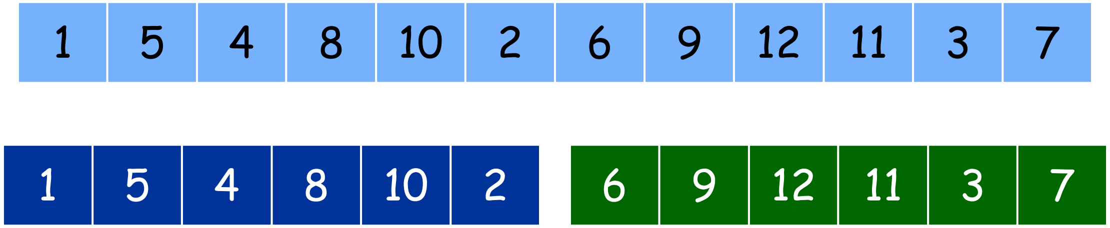
Counting Inversions: Divide-and-Conquer

- Divide-and-conquer.

1	5	4	8	10	2	6	9	12	11	3	7

Counting Inversions: Divide-and-Conquer

- Divide-and-conquer.
  - Divide: separate list into two pieces.



1	5	4	8	10	2	6	9	12	11	3	7

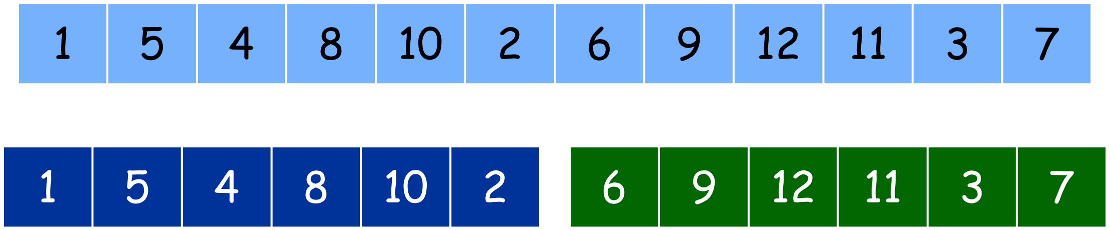
1	5	4	8	10	2

6	9	12	11	3	7

Divide: O(1).

Counting Inversions: Divide-and-Conquer

- Divide-and-conquer.
  - Divide: separate list into two pieces.
  - Conquer: recursively count inversions in each half.



1	5	4	8	10	2	6	9	12	11	3	7

Divide: O(1).

1	5	4	8	10	2

6	9	12	11	3	7

Conquer: 2T(n/2)

5 blue-blue inversions

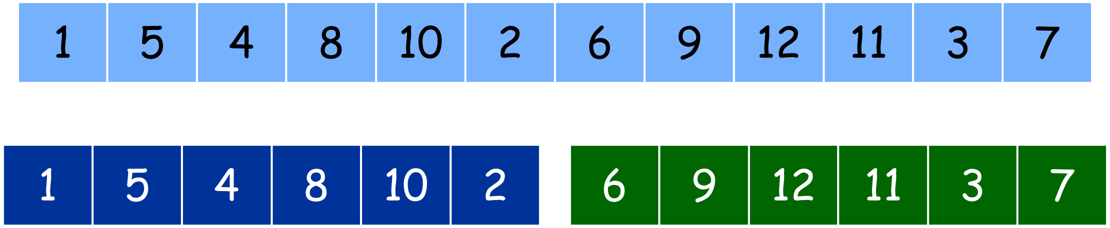
8 green-green inversions

5-4, 5-2, 4-2, 8-2, 10-2

6-3, 9-3, 9-7, 12-3, 12-7, 12-11, 11-3, 11-7

Counting Inversions: Divide-and-Conquer

- Divide-and-conquer.
  - Divide: separate list into two pieces.
  - Conquer: recursively count inversions in each half.
  - Combine: count inversions where a_i and a_j are in different halves, and return sum of three quantities.



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Divide: O(1).

1	5	4	8	10	2

6	9	12	11	3	7

5 blue-blue inversions

8 green-green inversions

Conquer: 2T(n/2)

9 blue-green inversions

5-3, 4-3, 8-6, 8-3, 8-7, 10-6, 10-9, 10-3, 10-7

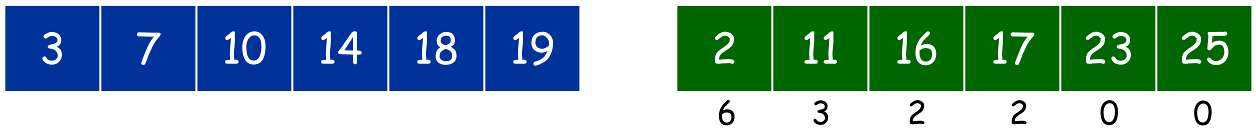
Combine: ???

Total = 5 + 8 + 9 = 22.

Counting Inversions: Combine

Combine: count blue-green inversions

- Assume each half is sorted.
- Count inversions where a_i and a_j are in different halves.
- Merge two sorted halves into sorted whole.



3	7	10	14	18	19

2	11	16	17	23	25
6	3	2	2	0	0

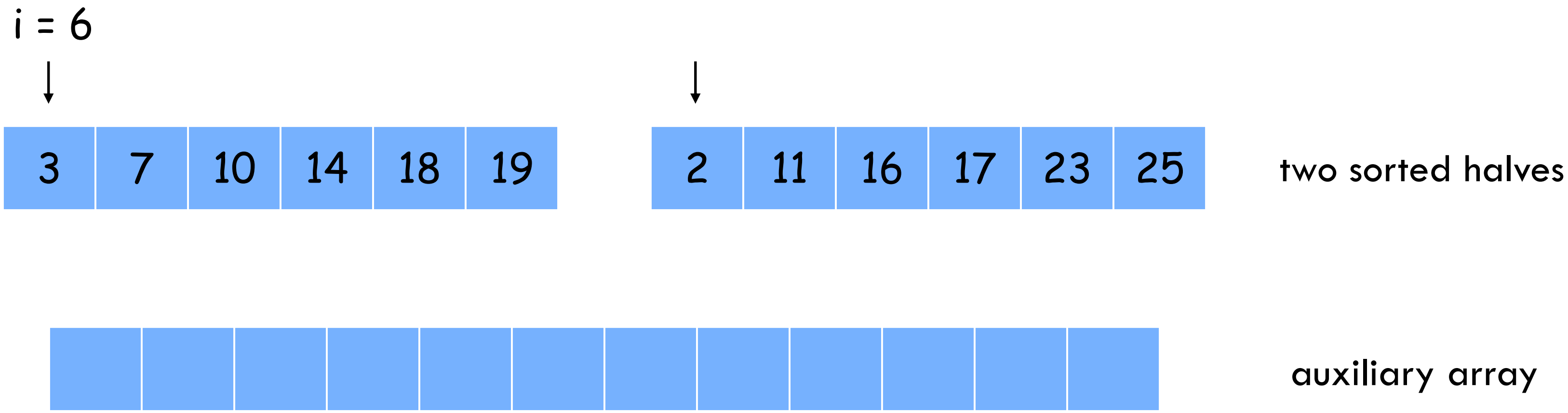
5 blue-blue inversions

8 green-green inversions

How many blue-green inversions?

Merge and Count

- Merge and count step.
  - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
  - Combine two sorted halves into sorted whole.



i = 6

3	7	10	14	18	19		2	11	16	17	23	25

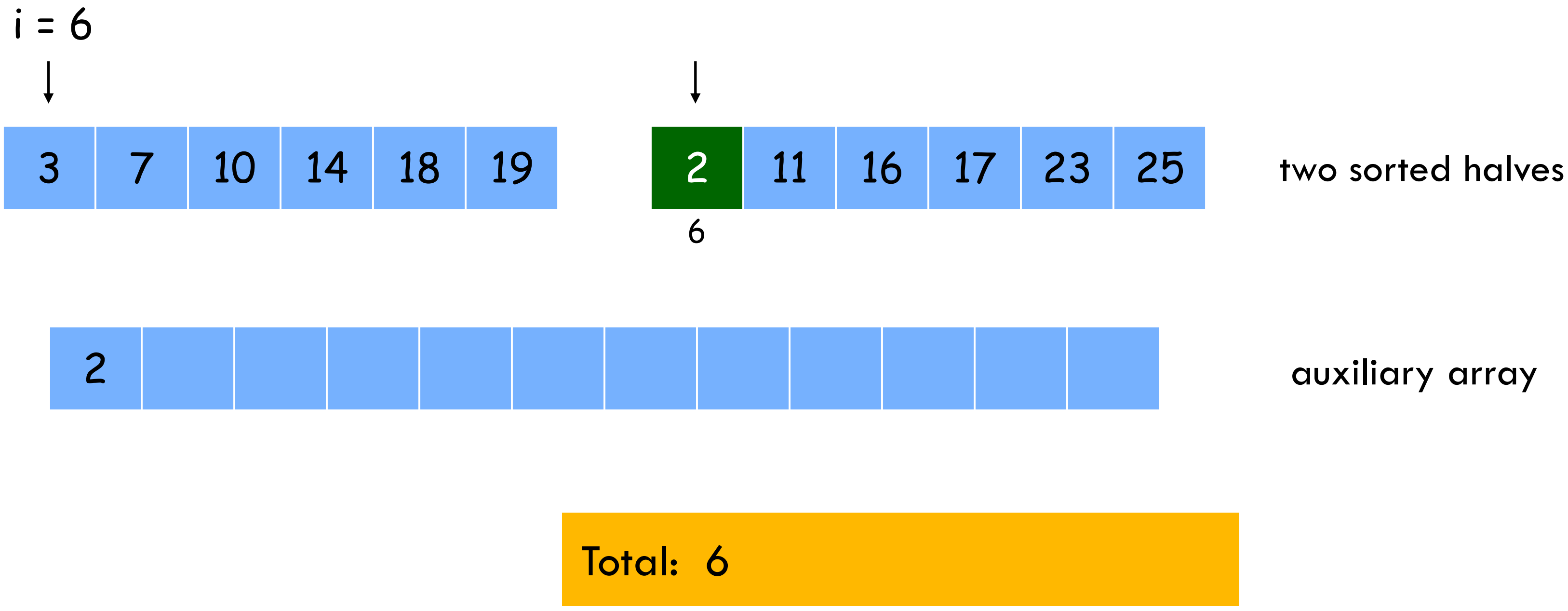
two sorted halves

auxiliary array

Total:

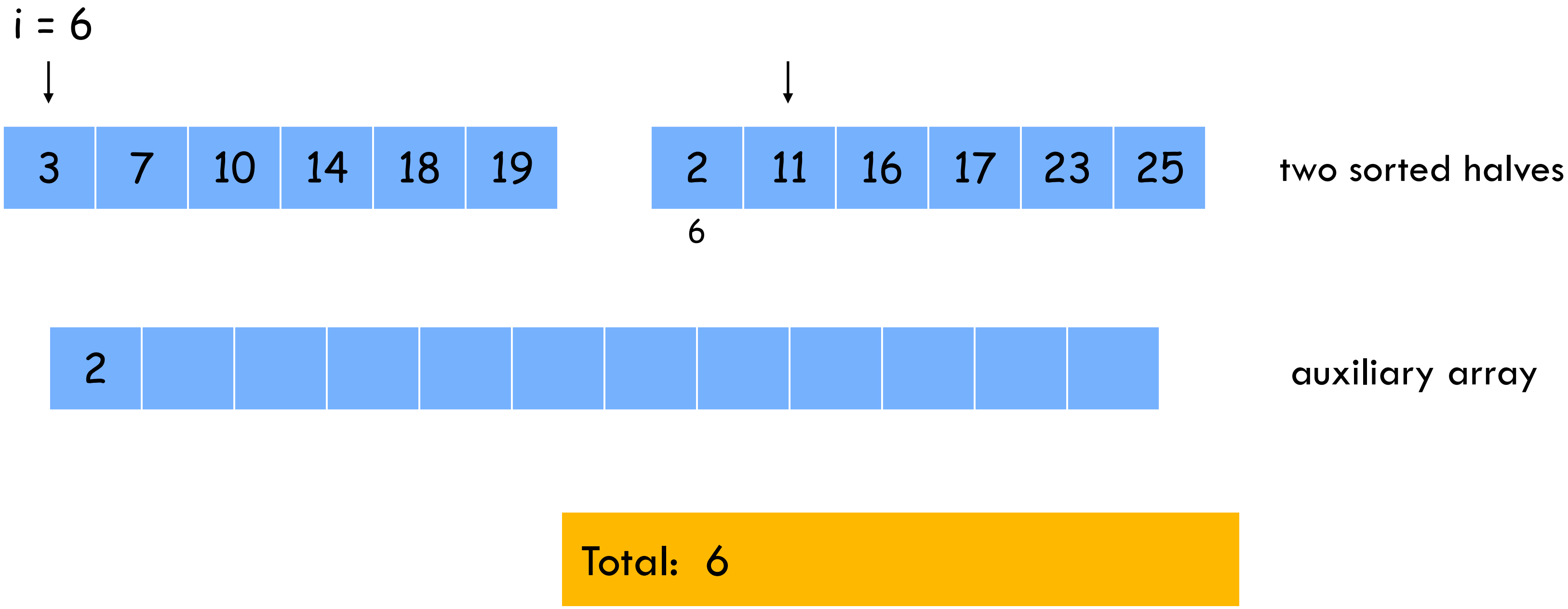
Merge and Count

- Merge and count step.
  - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
  - Combine two sorted halves into sorted whole.



Merge and Count

- Merge and count step.
  - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
  - Combine two sorted halves into sorted whole.



i = 6

3	7	10	14	18	19

2	11	16	17	23	25
6					

two sorted halves

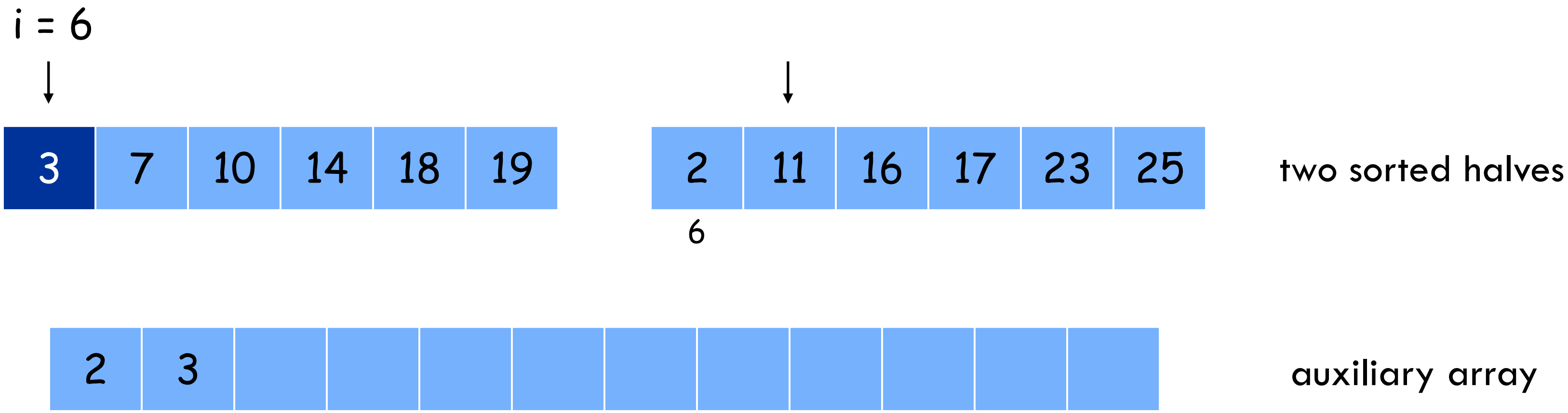
2											

auxiliary array

Total: 6

Merge and Count

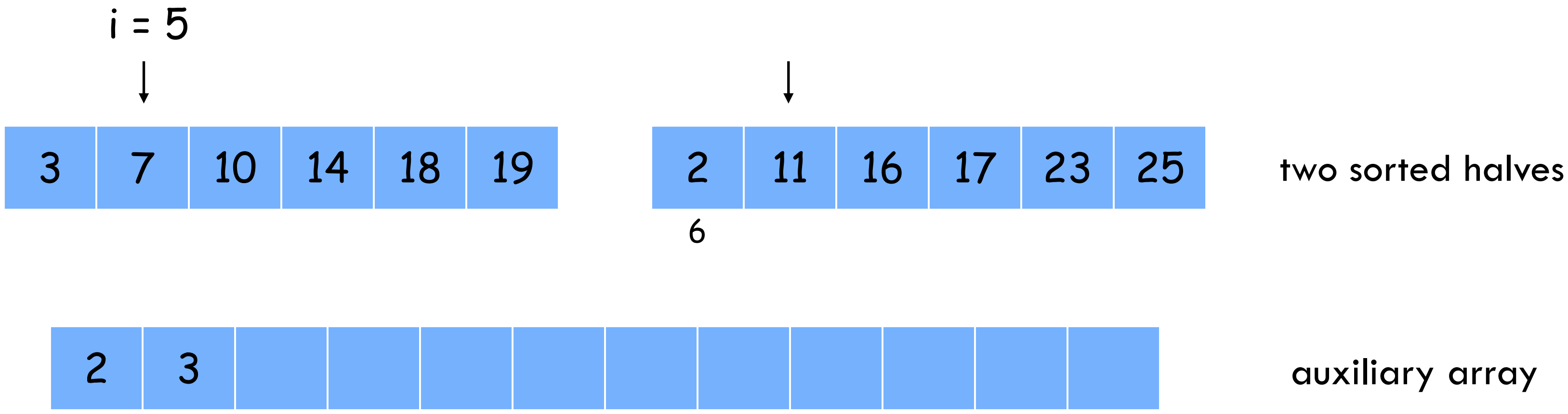
- Merge and count step.
  - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
  - Combine two sorted halves into sorted whole.



Total: 6

Merge and Count

- Merge and count step.
  - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
  - Combine two sorted halves into sorted whole.



i = 5

3	7	10	14	18	19

2	11	16	17	23	25
6					

two sorted halves

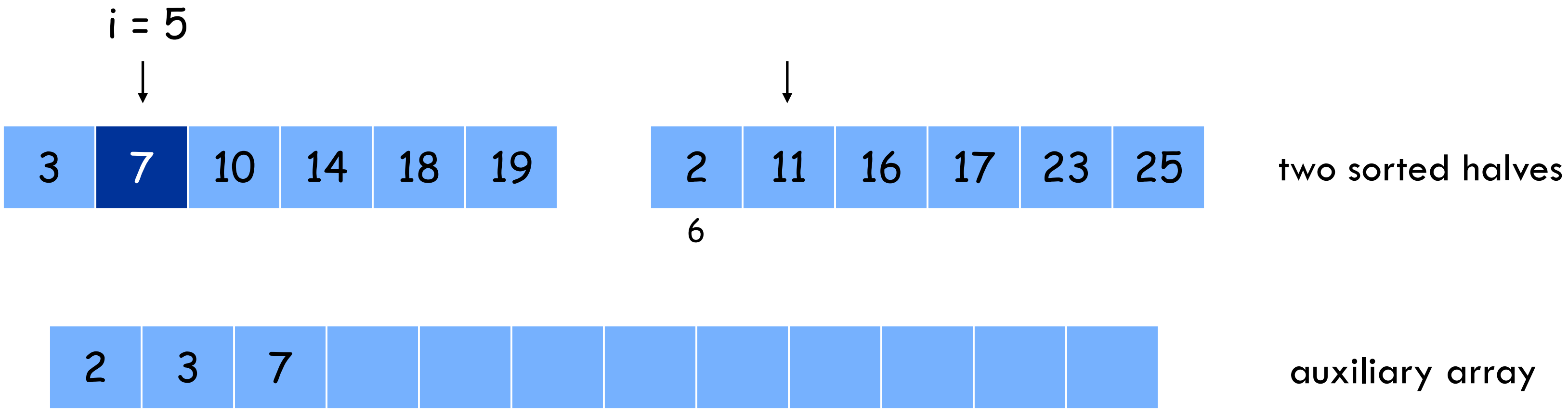
2	3										

auxiliary array

Total: 6

Merge and Count

- Merge and count step.
  - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
  - Combine two sorted halves into sorted whole.



i = 5

3	7	10	14	18	19	2	11	16	17	23	25
						6					

two sorted halves

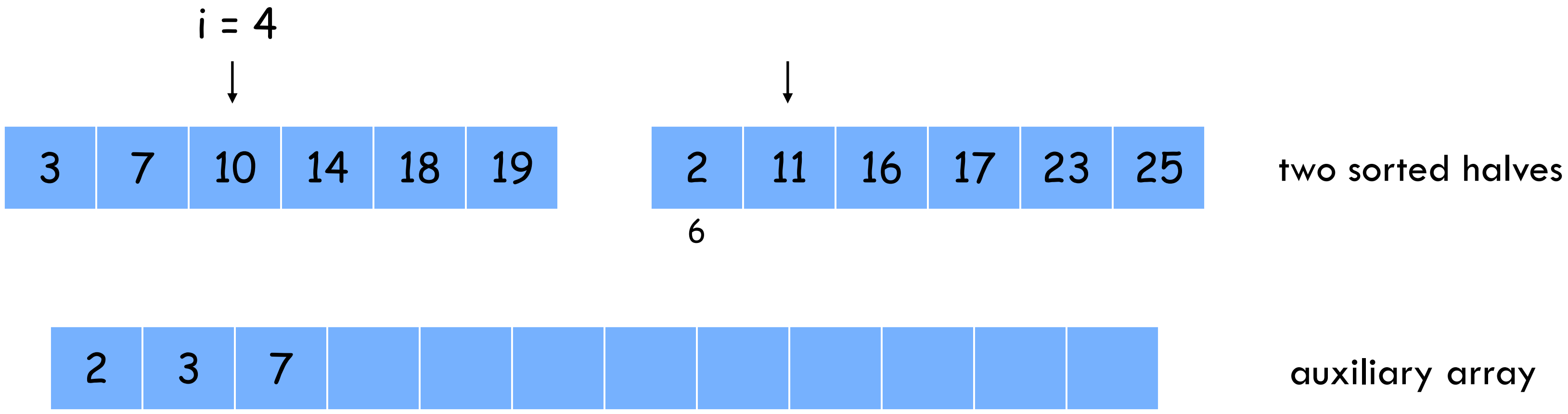
2	3	7									

auxiliary array

Total: 6

Merge and Count

- Merge and count step.
  - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
  - Combine two sorted halves into sorted whole.



i = 4

3	7	10	14	18	19

2	11	16	17	23	25
6					

two sorted halves

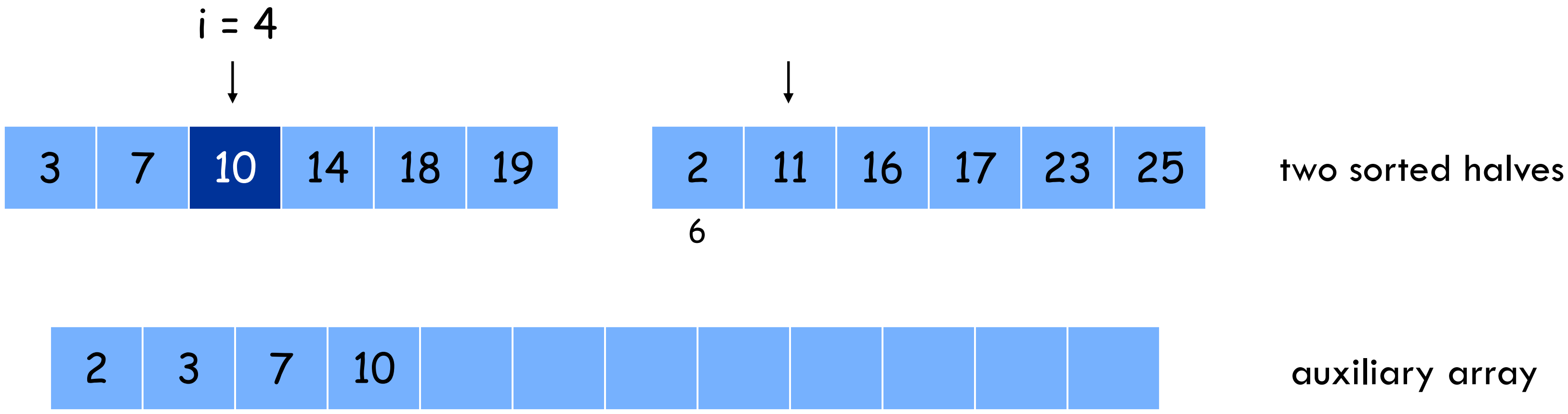
2	3	7									

auxiliary array

Total: 6

Merge and Count

- Merge and count step.
  - Given two sorted halves, count number of inversions where a_i and a_i are in different halves.
  - Combine two sorted halves into sorted whole.



i = 4

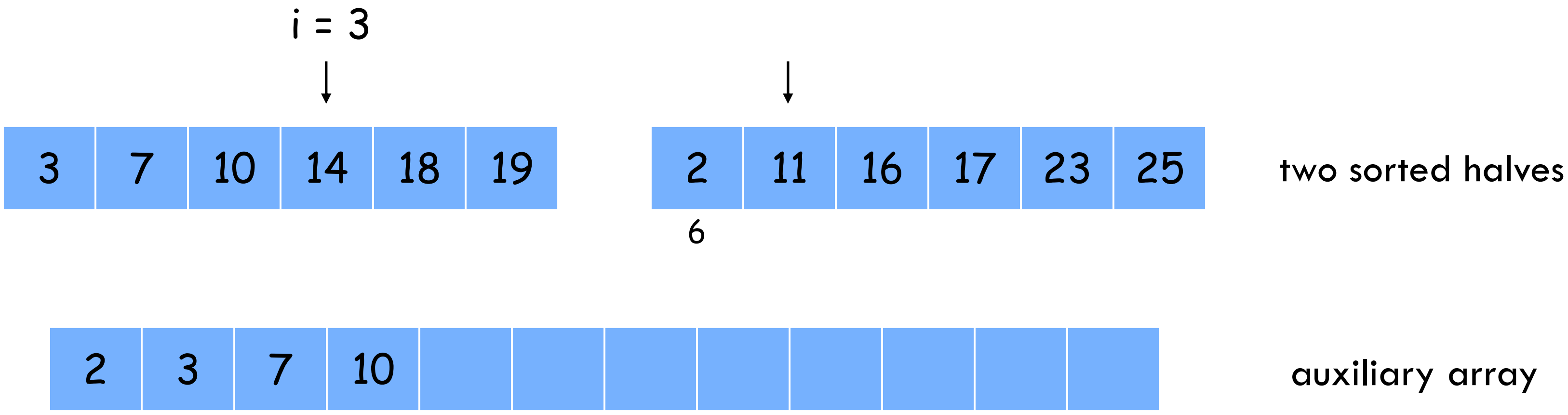
3	7	10	14	18	19	2	11	16	17	23	25	two sorted halves
						6						

2	3	7	10									auxiliary array

Total: 6

Merge and Count

- Merge and count step.
  - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
  - Combine two sorted halves into sorted whole.



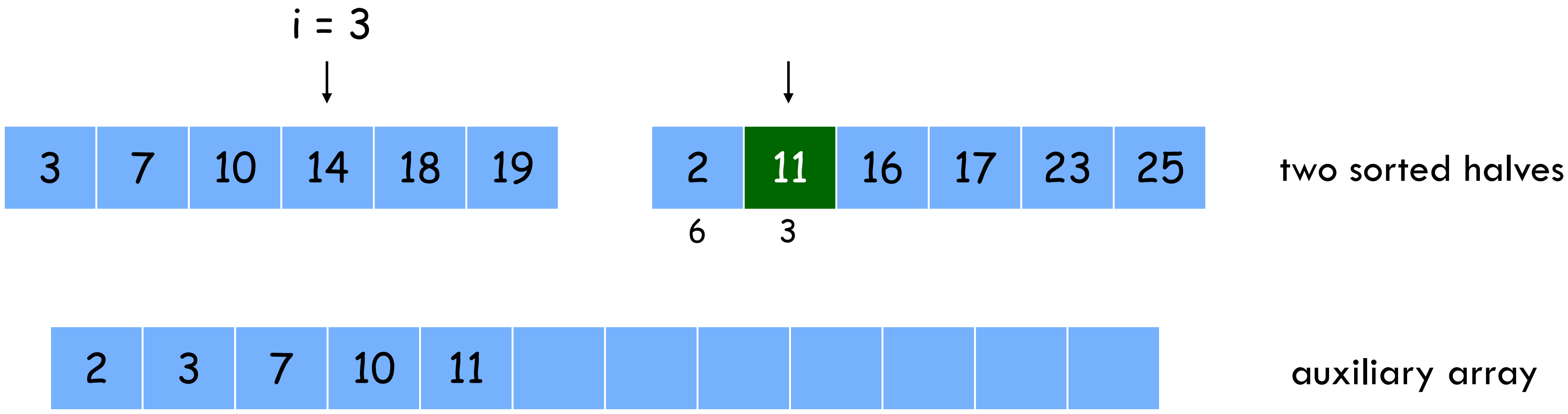
3	7	10	14	18	19		2	11	16	17	23	25
							6					

2	3	7	10								

Total: 6

Merge and Count

- Merge and count step.
  - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
  - Combine two sorted halves into sorted whole.



i = 3

3	7	10	14	18	19

2	11	16	17	23	25
6	3				

two sorted halves

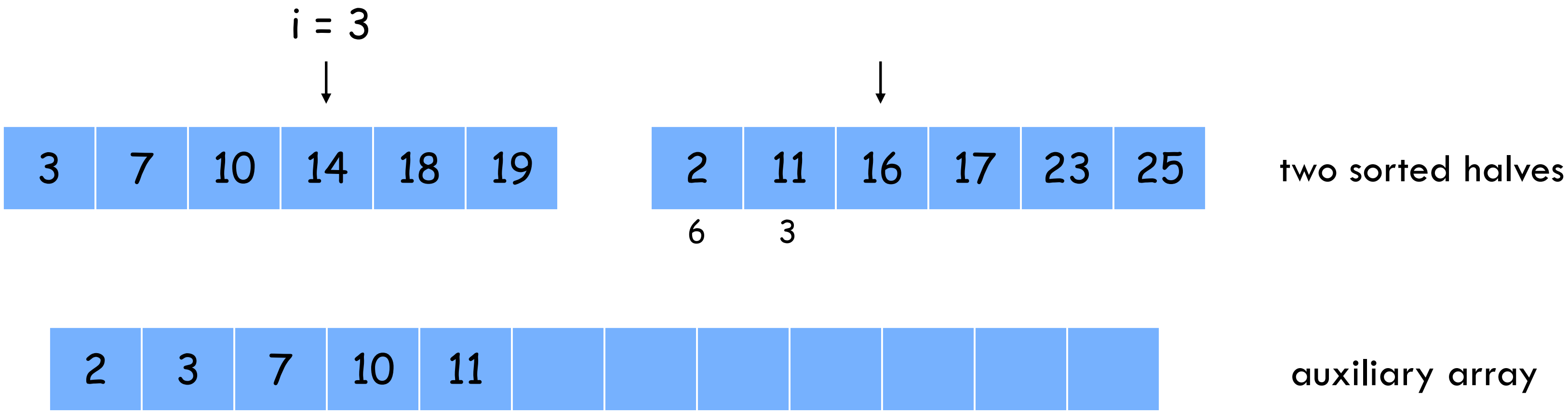
2	3	7	10	11							

auxiliary array

Total: 6 + 3

Merge and Count

- Merge and count step.
  - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
  - Combine two sorted halves into sorted whole.



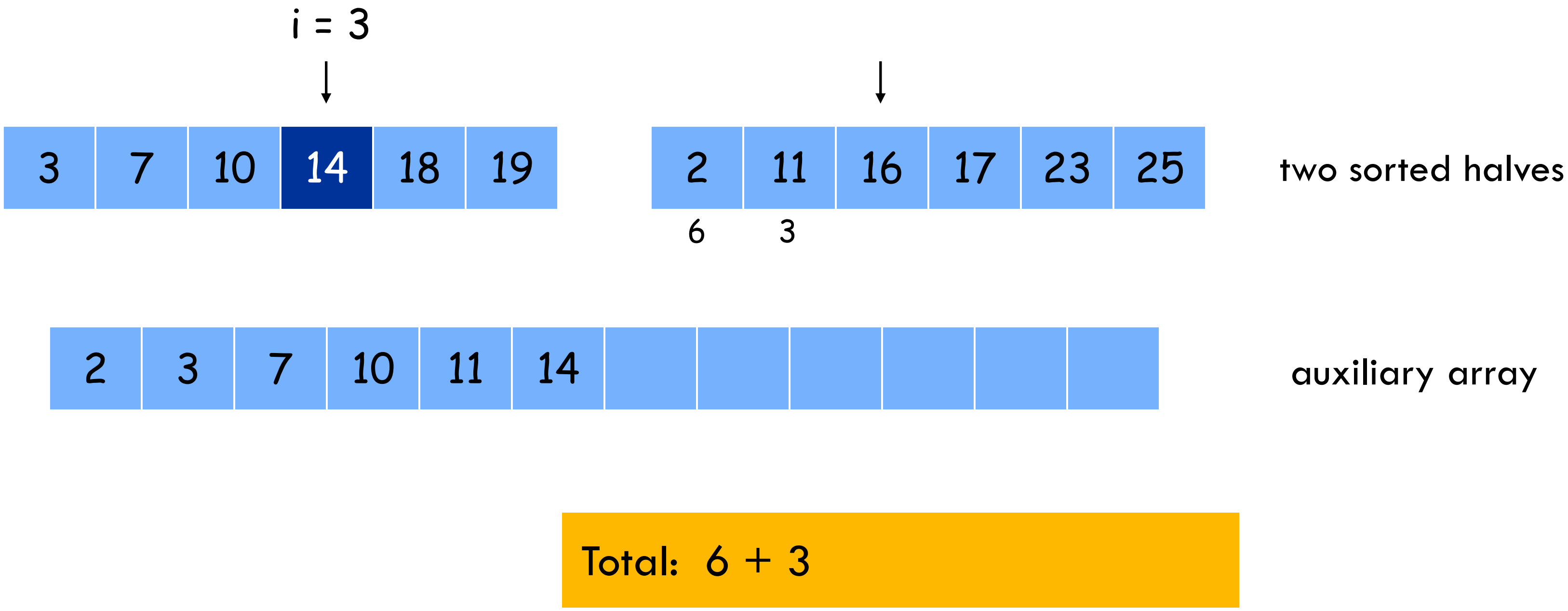
3	7	10	14	18	19	2	11	16	17	23	25	two sorted halves
						6	3					

2	3	7	10	11								auxiliary array

Total: 6 + 3

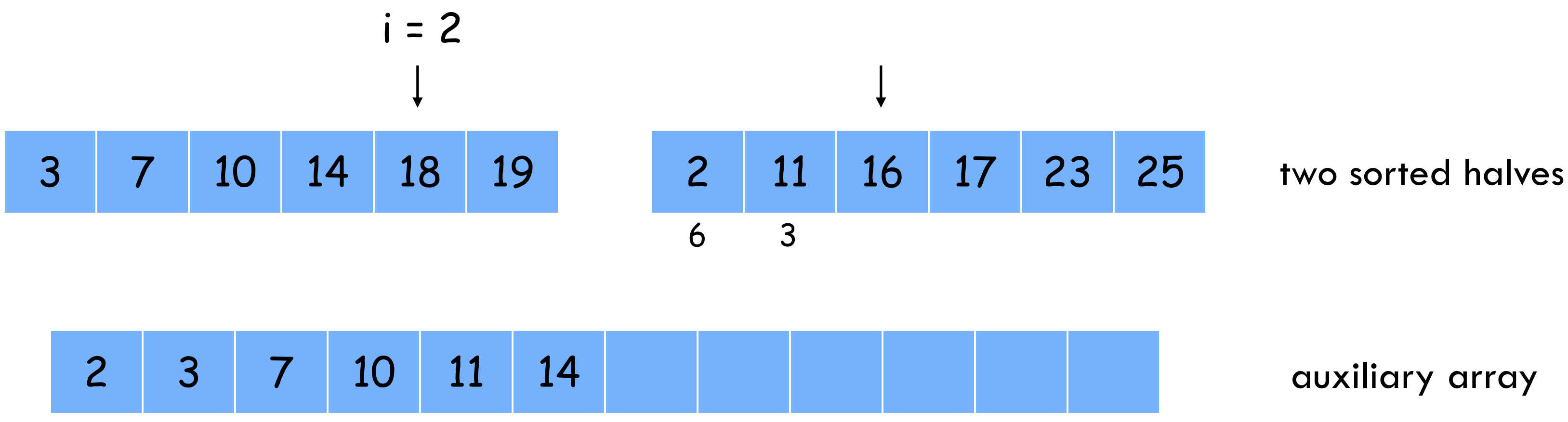
Merge and Count

- Merge and count step.
  - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
  - Combine two sorted halves into sorted whole.



Merge and Count

- Merge and count step.
  - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
  - Combine two sorted halves into sorted whole.



3	7	10	14	18	19	2	11	16	17	23	25
						6	3				

two sorted halves

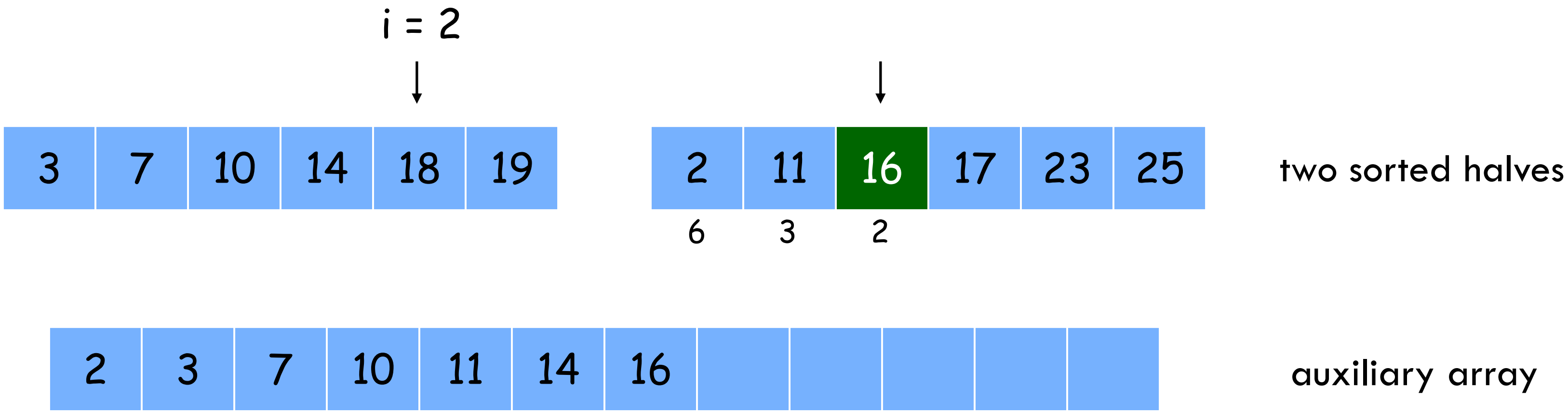
2	3	7	10	11	14						

auxiliary array

Total: 6 + 3

Merge and Count

- Merge and count step.
  - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
  - Combine two sorted halves into sorted whole.



i = 2

two sorted halves

3	7	10	14	18	19	2	11	16	17	23	25
						6	3	2			

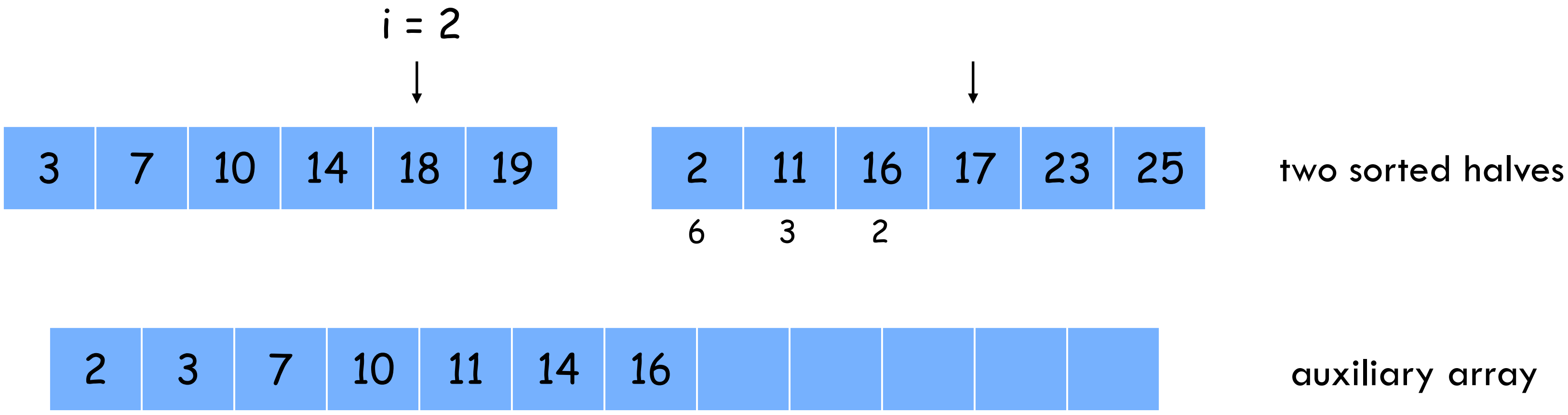
auxiliary array

2	3	7	10	11	14	16					

Total: 6 + 3 + 2

Merge and Count

- Merge and count step.
  - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
  - Combine two sorted halves into sorted whole.



i = 2

two sorted halves

3	7	10	14	18	19	2	11	16	17	23	25
						6	3	2			

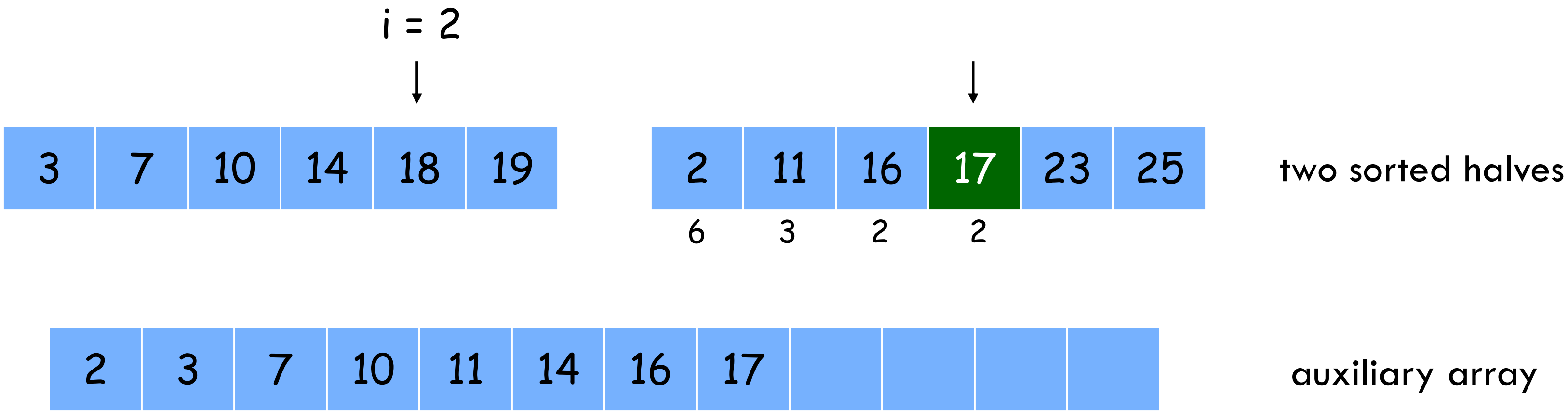
auxiliary array

2	3	7	10	11	14	16					

Total: 6 + 3 + 2

Merge and Count

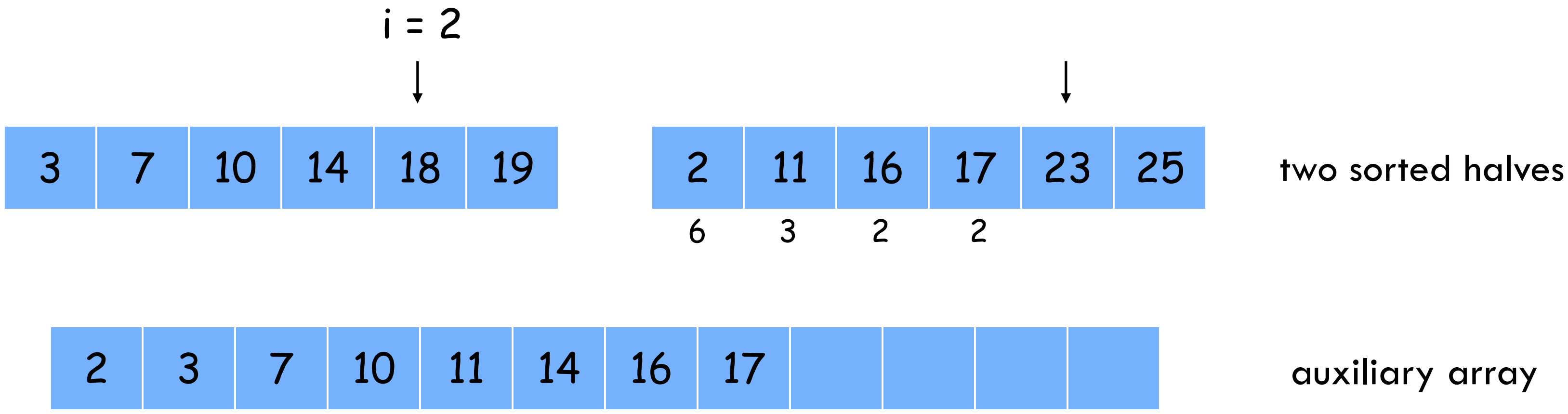
- Merge and count step.
  - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
  - Combine two sorted halves into sorted whole.



Total: 6 + 3 + 2 + 2

Merge and Count

- Merge and count step.
  - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
  - Combine two sorted halves into sorted whole.



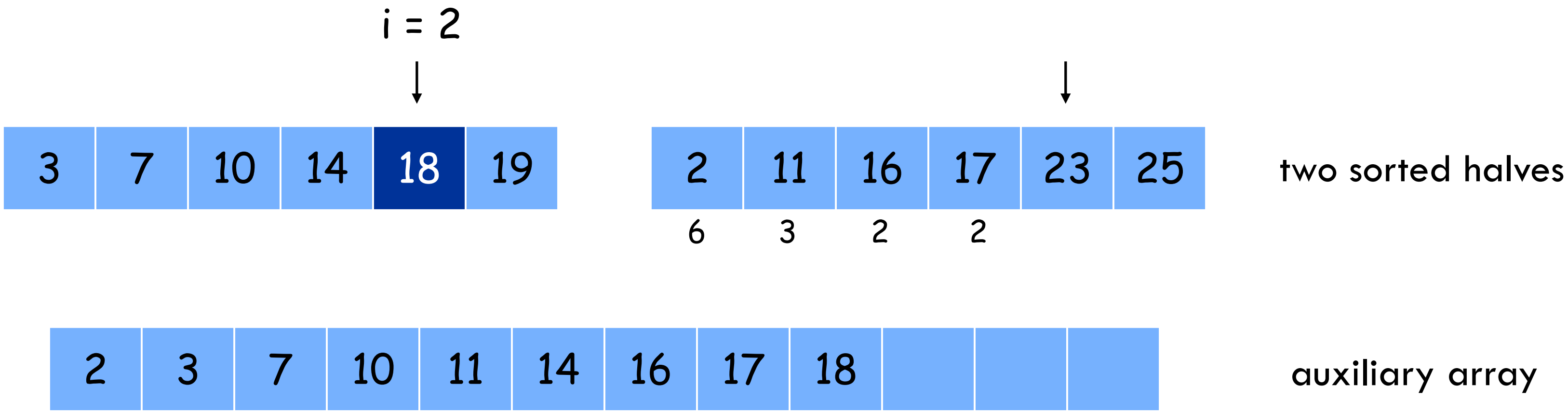
3	7	10	14	18	19	2	11	16	17	23	25	two sorted halves
						6	3	2	2			

2	3	7	10	11	14	16	17					auxiliary array

Total: $6 + 3 + 2 + 2$

Merge and Count

- Merge and count step.
  - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
  - Combine two sorted halves into sorted whole.



i = 2

two sorted halves

3	7	10	14	18	19

2	11	16	17	23	25
6	3	2	2		

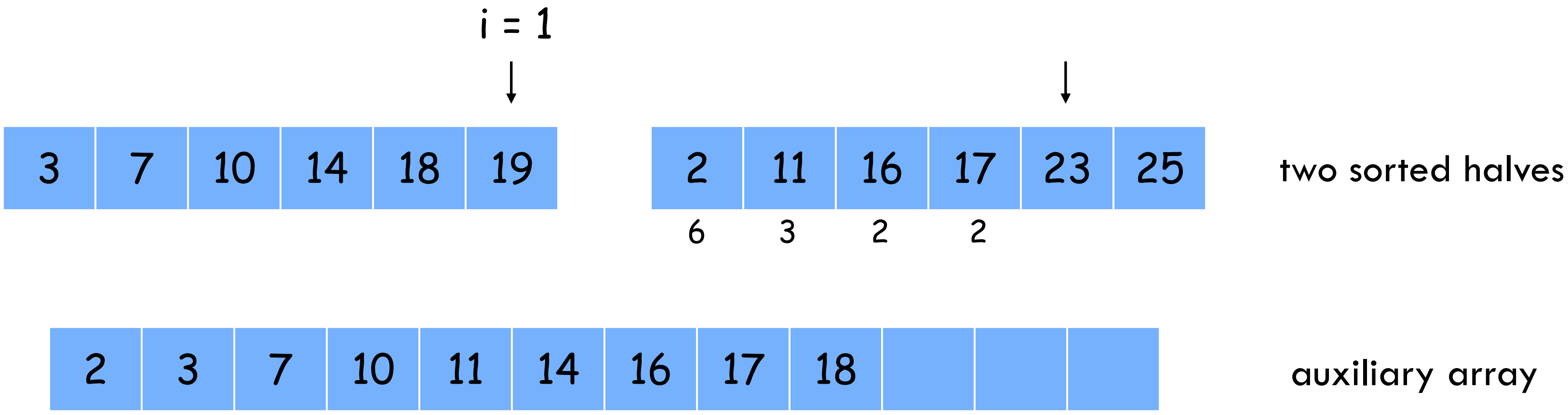
auxiliary array

2	3	7	10	11	14	16	17	18			

Total: 6 + 3 + 2 + 2

Merge and Count

- Merge and count step.
  - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
  - Combine two sorted halves into sorted whole.



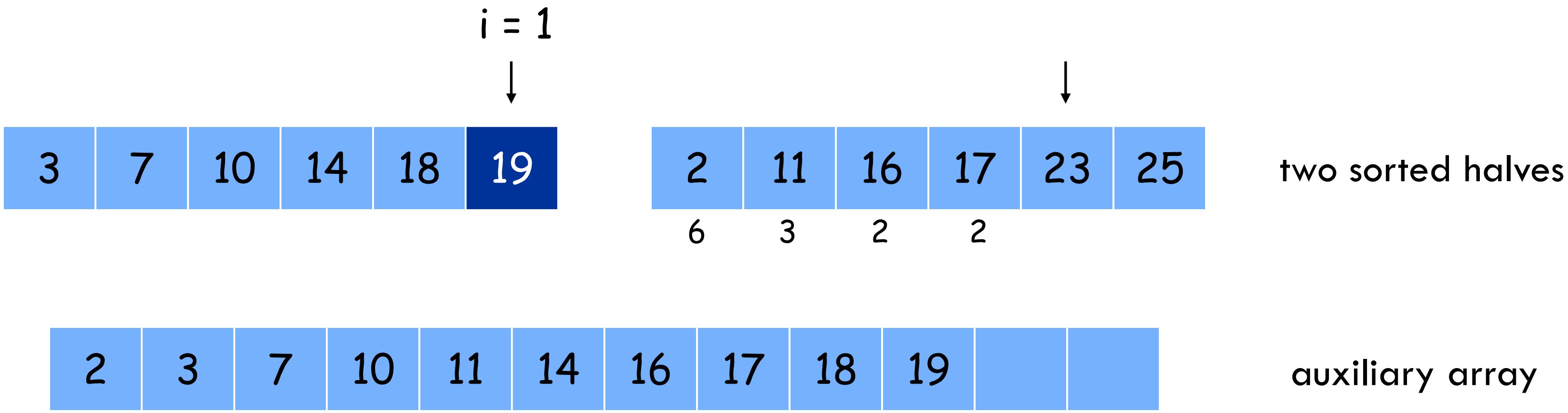
3	7	10	14	18	19	2	11	16	17	23	25	two sorted halves
						6	3	2	2			

2	3	7	10	11	14	16	17	18				auxiliary array

Total: 6 + 3 + 2 + 2

Merge and Count

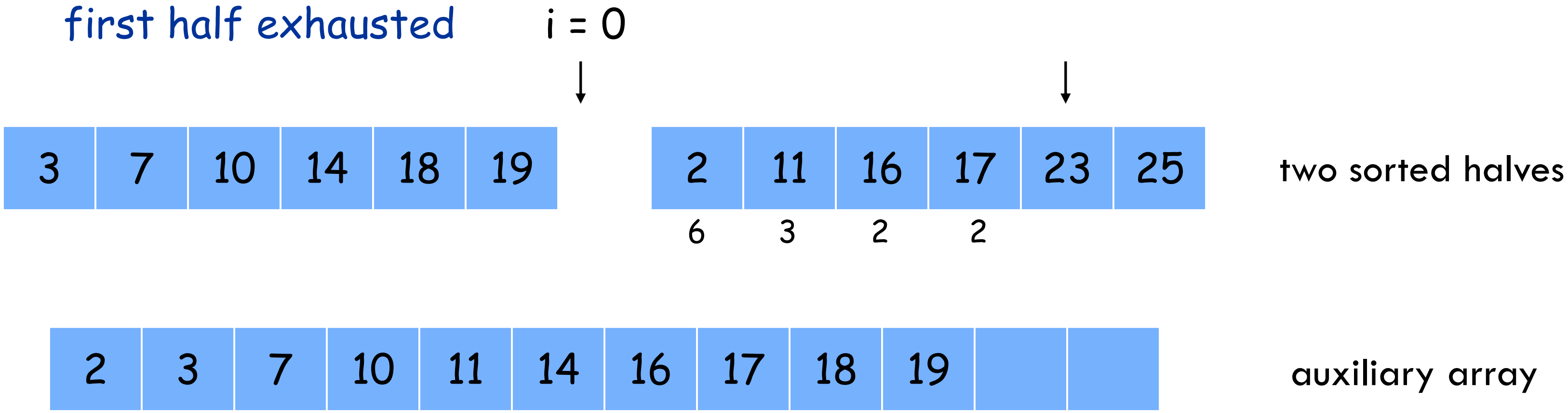
- Merge and count step.
  - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
  - Combine two sorted halves into sorted whole.



Total: 6 + 3 + 2 + 2

Merge and Count

- Merge and count step.
  - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
  - Combine two sorted halves into sorted whole.



first half exhausted

i = 0

3	7	10	14	18	19		2	11	16	17	23	25
							6	3	2	2		

two sorted halves

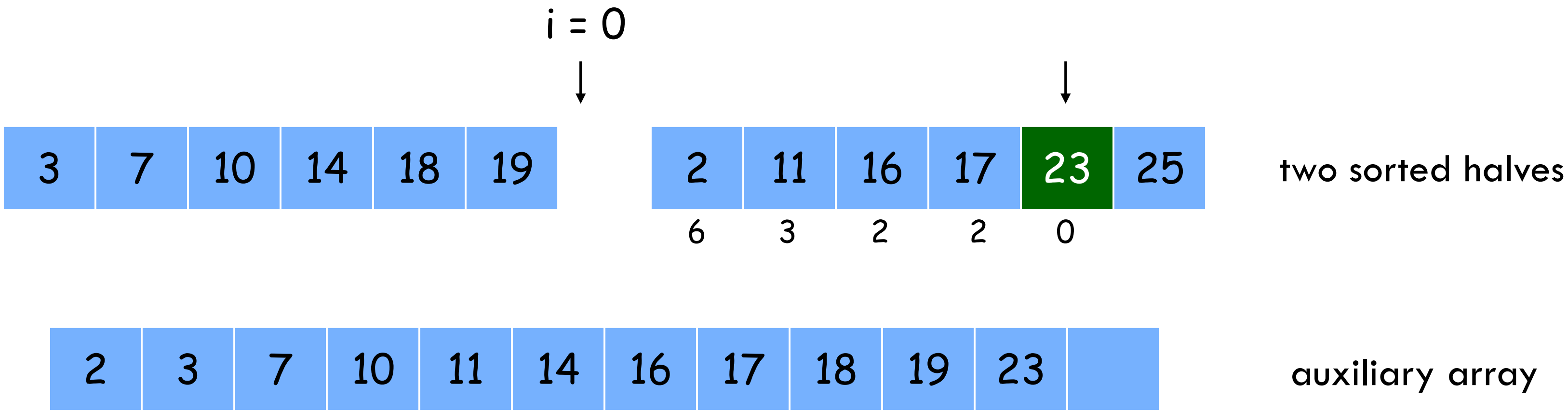
2	3	7	10	11	14	16	17	18	19		

auxiliary array

Total: 6 + 3 + 2 + 2

Merge and Count

- Merge and count step.
  - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
  - Combine two sorted halves into sorted whole.



3	7	10	14	18	19		2	11	16	17	23	25
							6	3	2	2	0	

two sorted halves

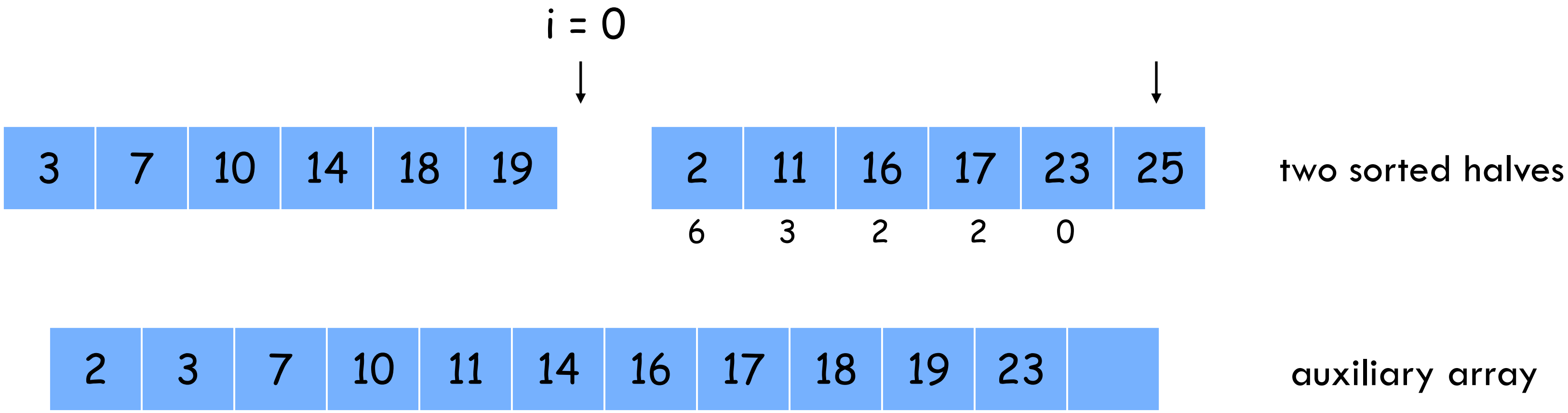
2	3	7	10	11	14	16	17	18	19	23	

auxiliary array

Total: 6 + 3 + 2 + 2 + 0

Merge and Count

- Merge and count step.
  - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
  - Combine two sorted halves into sorted whole.



two sorted halves

3	7	10	14	18	19	2	11	16	17	23	25
						6	3	2	2	0	

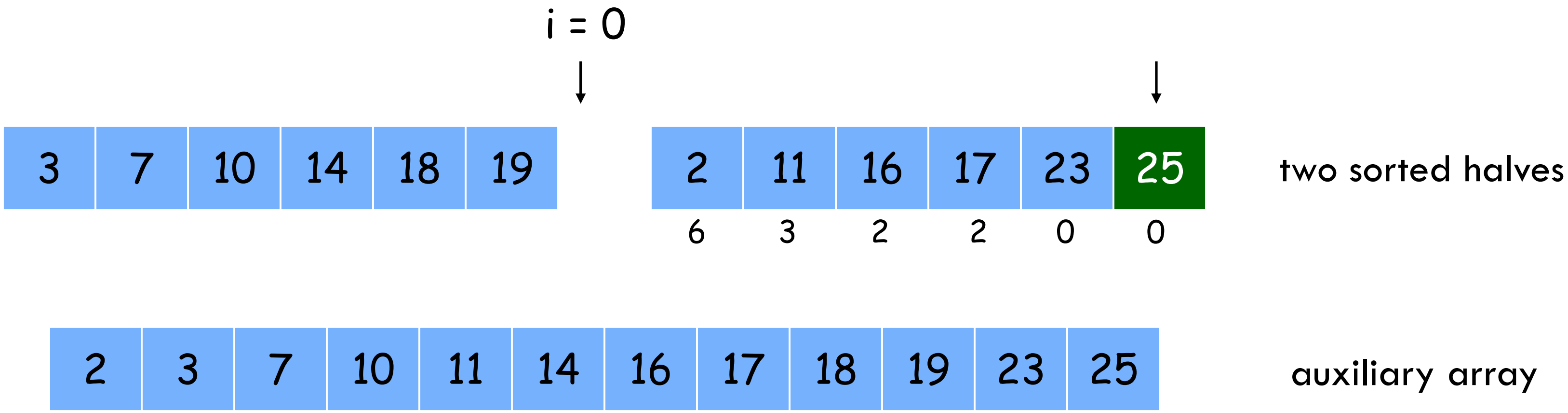
auxiliary array

2	3	7	10	11	14	16	17	18	19	23	

Total: 6 + 3 + 2 + 2 + 0

Merge and Count

- Merge and count step.
  - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
  - Combine two sorted halves into sorted whole.



3	7	10	14	18	19

2	11	16	17	23	25
6	3	2	2	0	0

two sorted halves

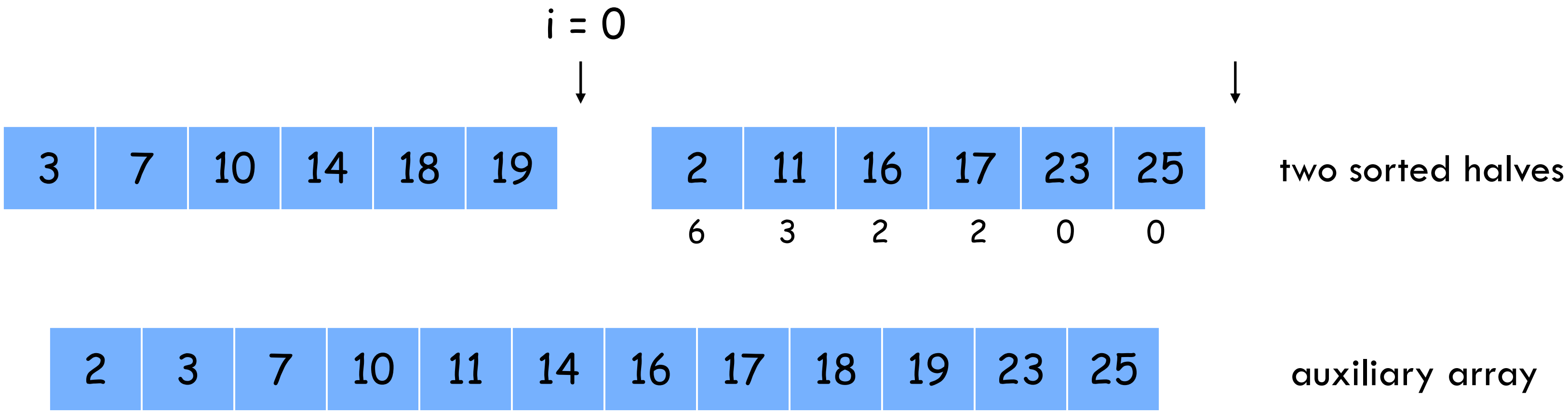
2	3	7	10	11	14	16	17	18	19	23	25

auxiliary array

Total: 6 + 3 + 2 + 2 + 0 + 0

Merge and Count

- Merge and count step.
  - Given two sorted halves, count number of inversions where a_i and a_j are in different halves.
  - Combine two sorted halves into sorted whole.



two sorted halves

3	7	10	14	18	19	2	11	16	17	23	25
						6	3	2	2	0	0

auxiliary array

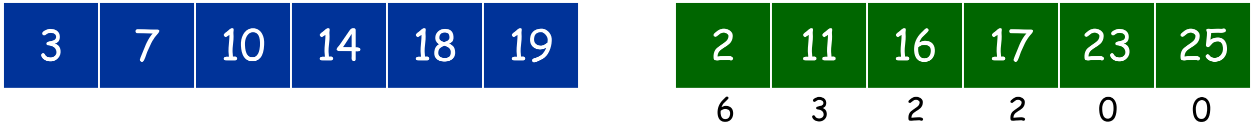
2	3	7	10	11	14	16	17	18	19	23	25

Total: $6 + 3 + 2 + 2 + 0 + 0 = 13$

Counting Inversions: Combine

Combine: count blue-green inversions

- Assume each half is sorted.
- Count inversions where a_i and a_j are in different halves.
- Merge two sorted halves into sorted whole.

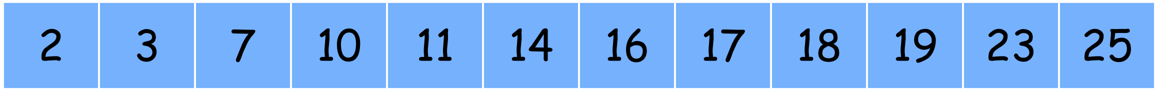


3	7	10	14	18	19

2	11	16	17	23	25
6	3	2	2	0	0

13 blue-green inversions: 6 + 3 + 2 + 2 + 0 + 0

Count: O(n)



2	3	7	10	11	14	16	17	18	19	23	25

Merge: O(n)

Time: $T(n) = 2T(n/2) + O(n) = O(n \log n)$

Counting Inversions: Implementation

- Pre-condition. [Merge-and-Count] A and B are sorted.
- Post-condition. [Sort-and-Count] L is sorted.

```
Sort-and-Count(L) {
   if list L has one element
      return 0 and the list L

   Divide the list into two halves A and B
   ($r_A$, A) ← Sort-and-Count(A)
   ($r_B$, B) ← Sort-and-Count(B)
   ($r_B$, L) ← Merge-and-Count(A, B)

   return r = $r_A$ + $r_B$ + r and the sorted list L
}
```

Closest Pair of Points

Closest Pair of Points

- **Closest pair.** Given n points in the plane, find a pair with smallest Euclidean distance between them.

- **Fundamental geometric primitive.**
  - Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
  - Special case of nearest neighbor, Euclidean MST, Voronoi.

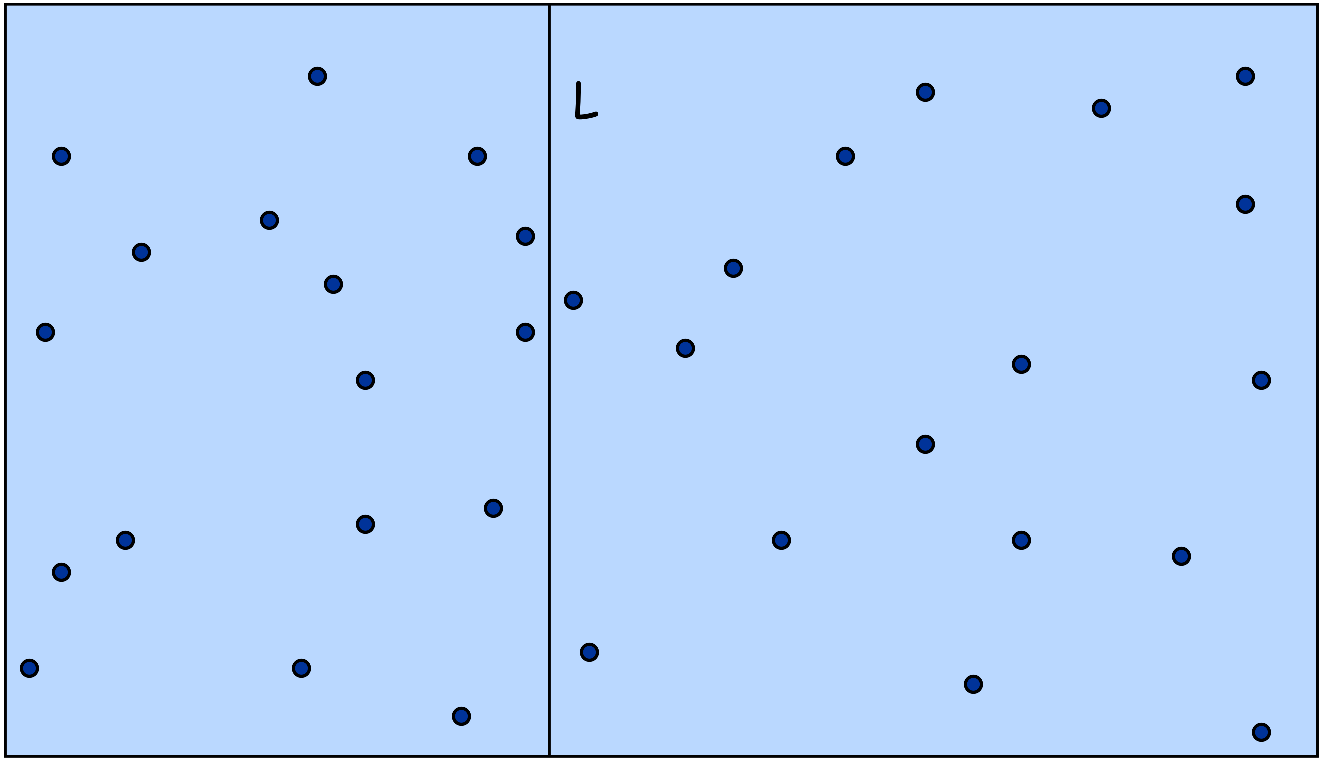
- **Brute force.** Check all pairs of points p and q with $\Theta(n^2)$ comparisons.

- **1-D version.** $O(n \log n)$ easy if points are on a line.

- **Assumption.** No two points have same x coordinate.

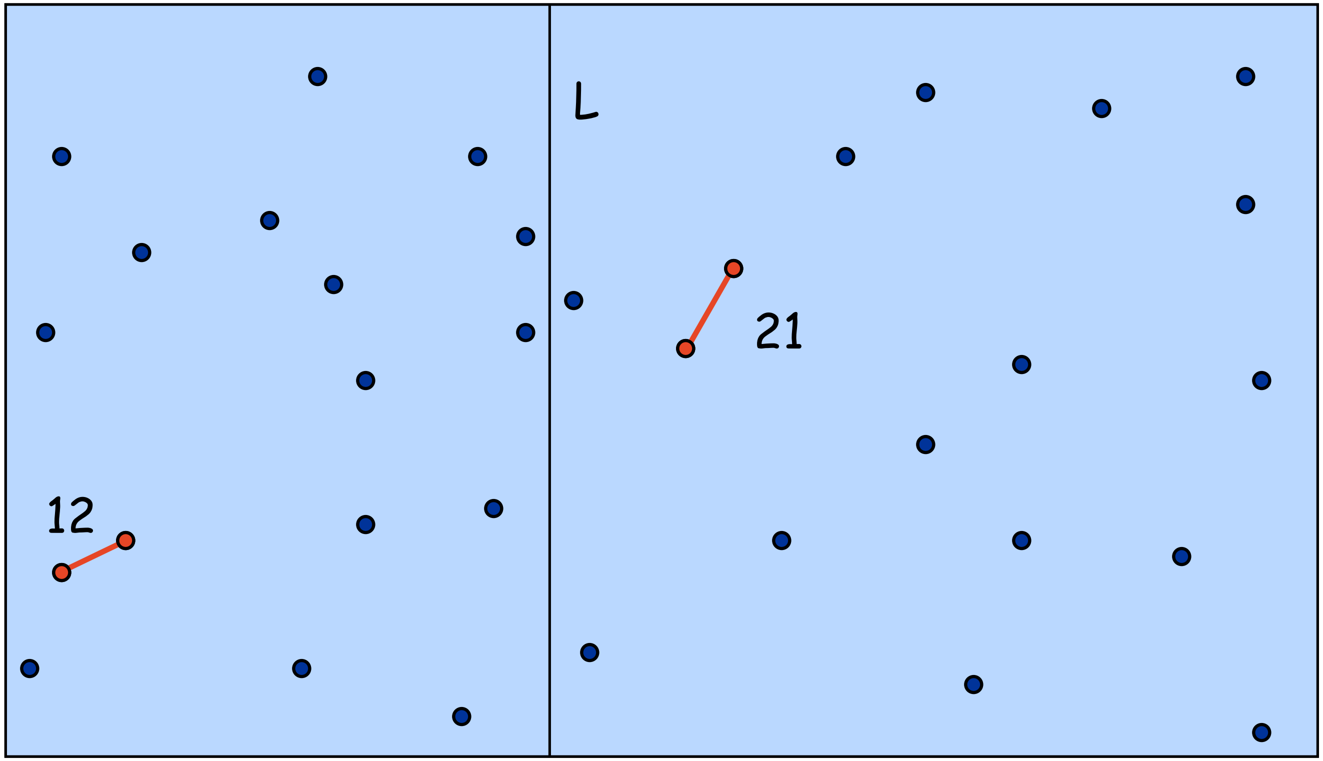
Closest Pair of Points

- **Algorithm.**
  - Divide: draw vertical line L so that roughly ½n points on each side.



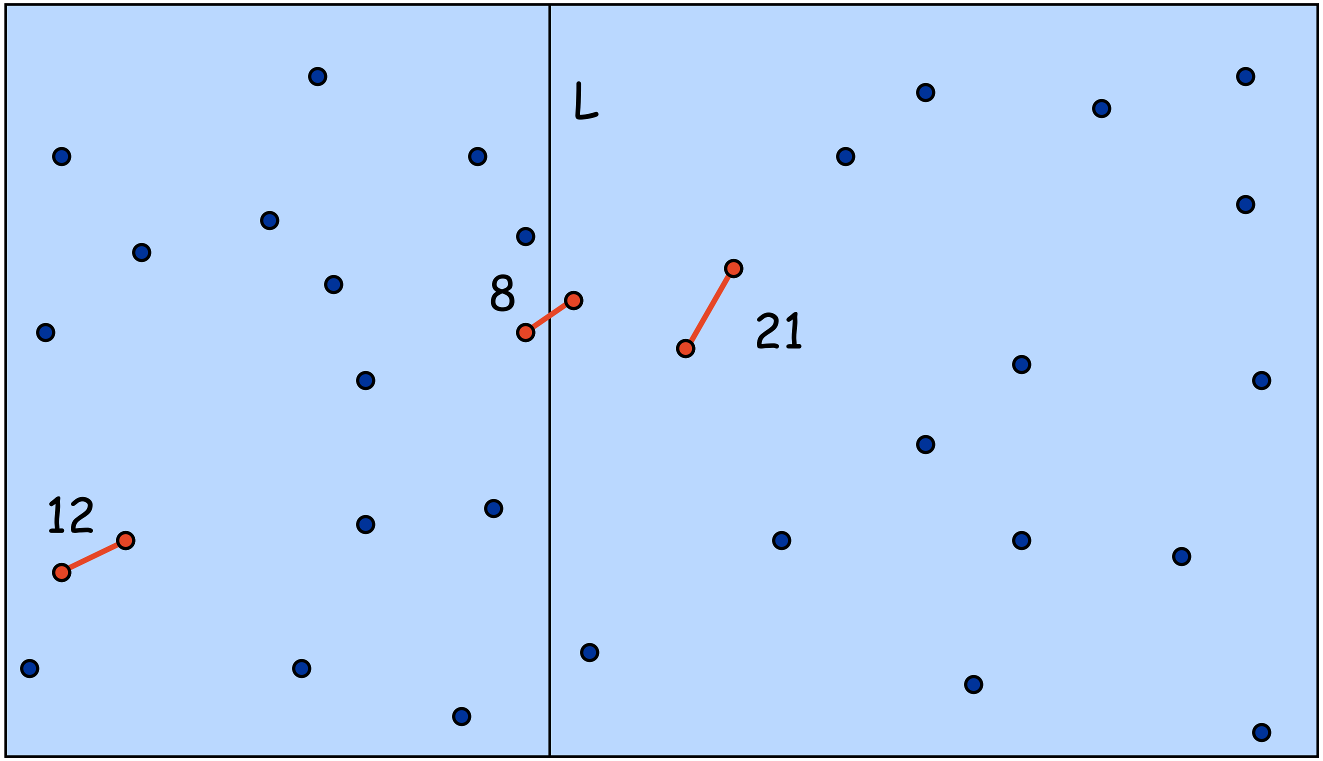
Closest Pair of Points

- **Algorithm.**
  - Divide: draw vertical line L so that roughly ½n points on each side.
  - Conquer: find closest pair in each side recursively.



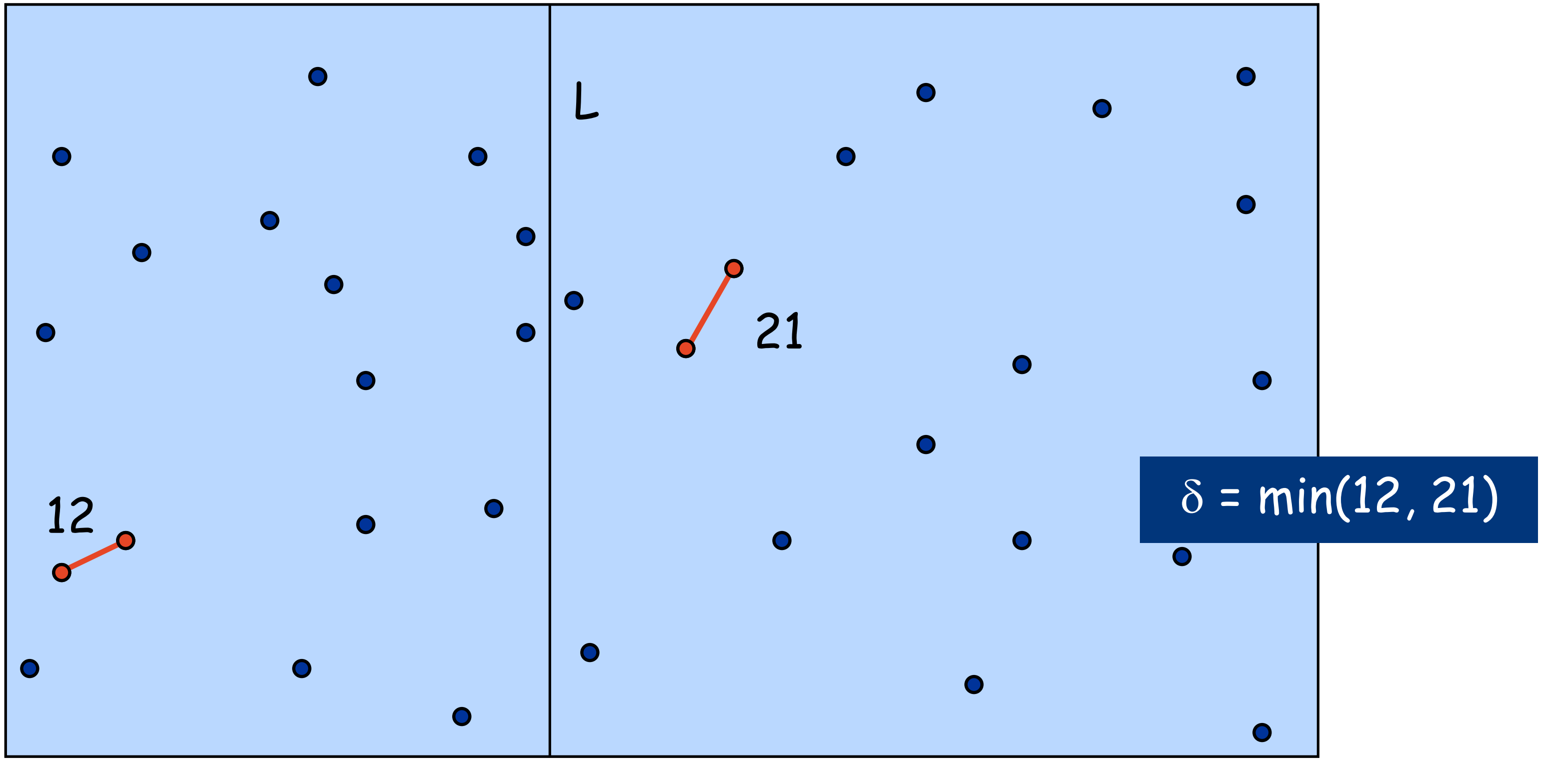
Closest Pair of Points

- **Algorithm.**
  - Divide: draw vertical line L so that roughly ½n points on each side.
  - Conquer: find closest pair in each side recursively.
  - Combine: find closest pair with one point in each side.
  - Return best of 3 solutions.



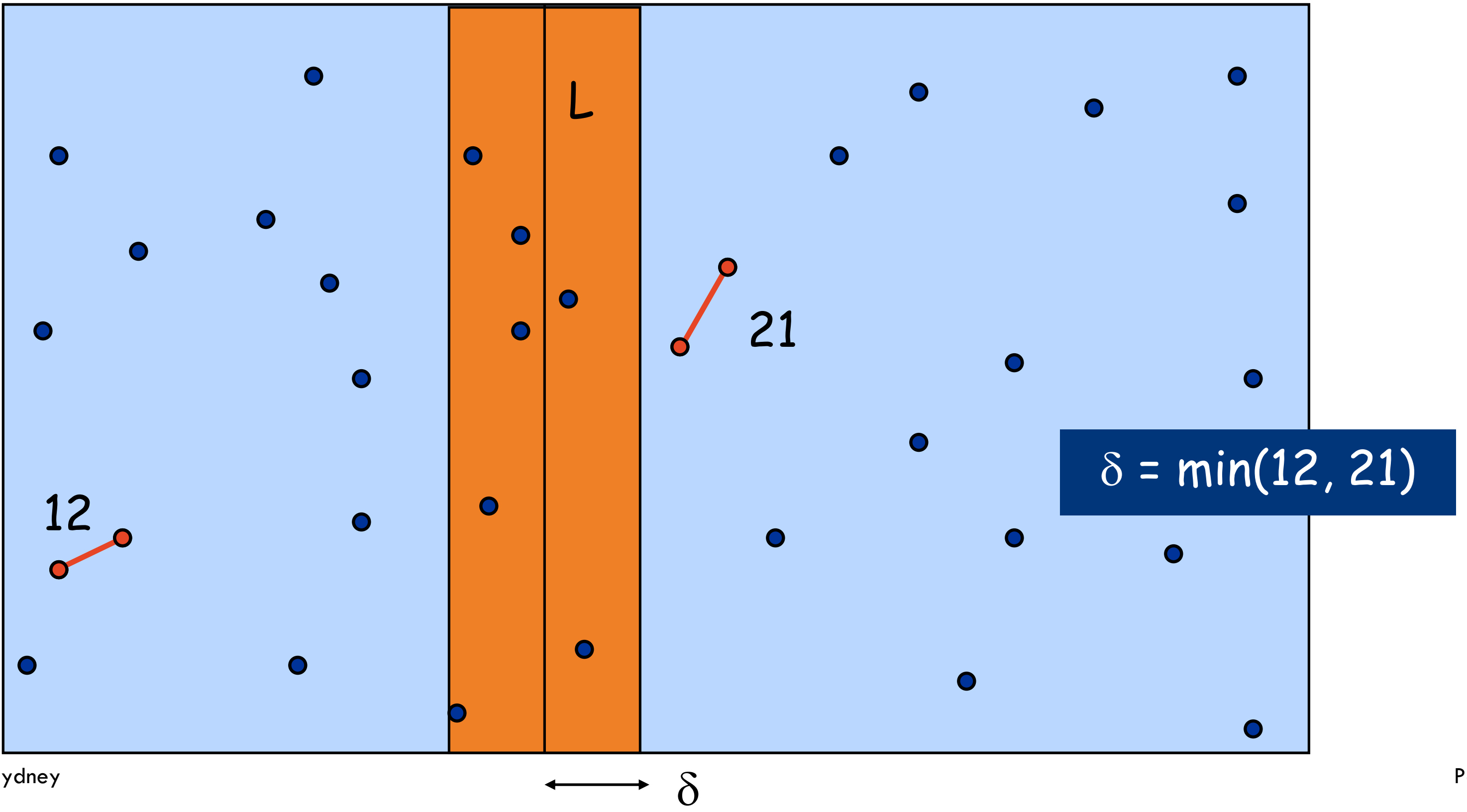
Closest Pair of Points

- Find closest pair with one point in each side, assuming that distance < δ.



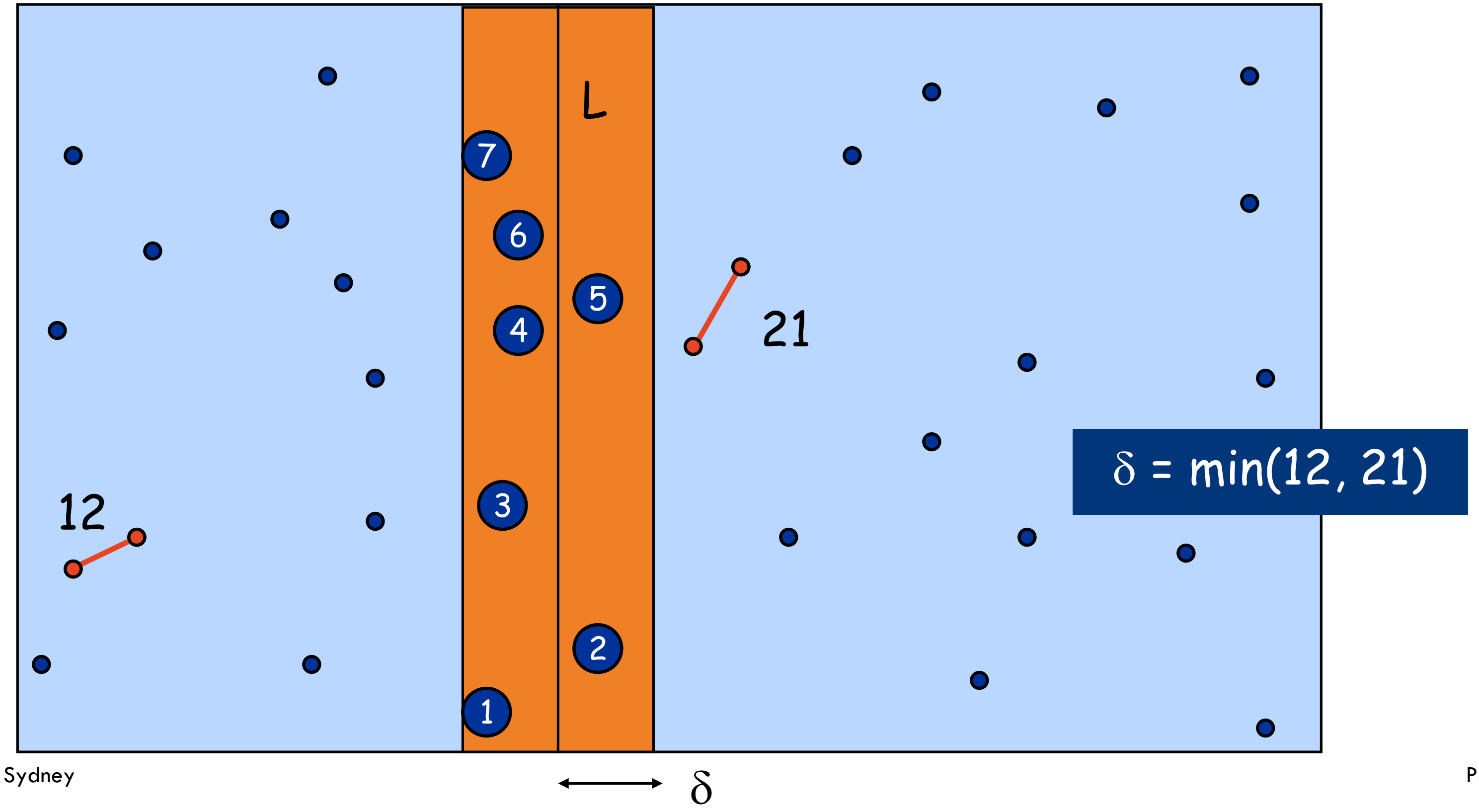
Closest Pair of Points

- Find closest pair with one point in each side, assuming that distance < δ.
  - **Observation:** only need to consider points within δ of line L.



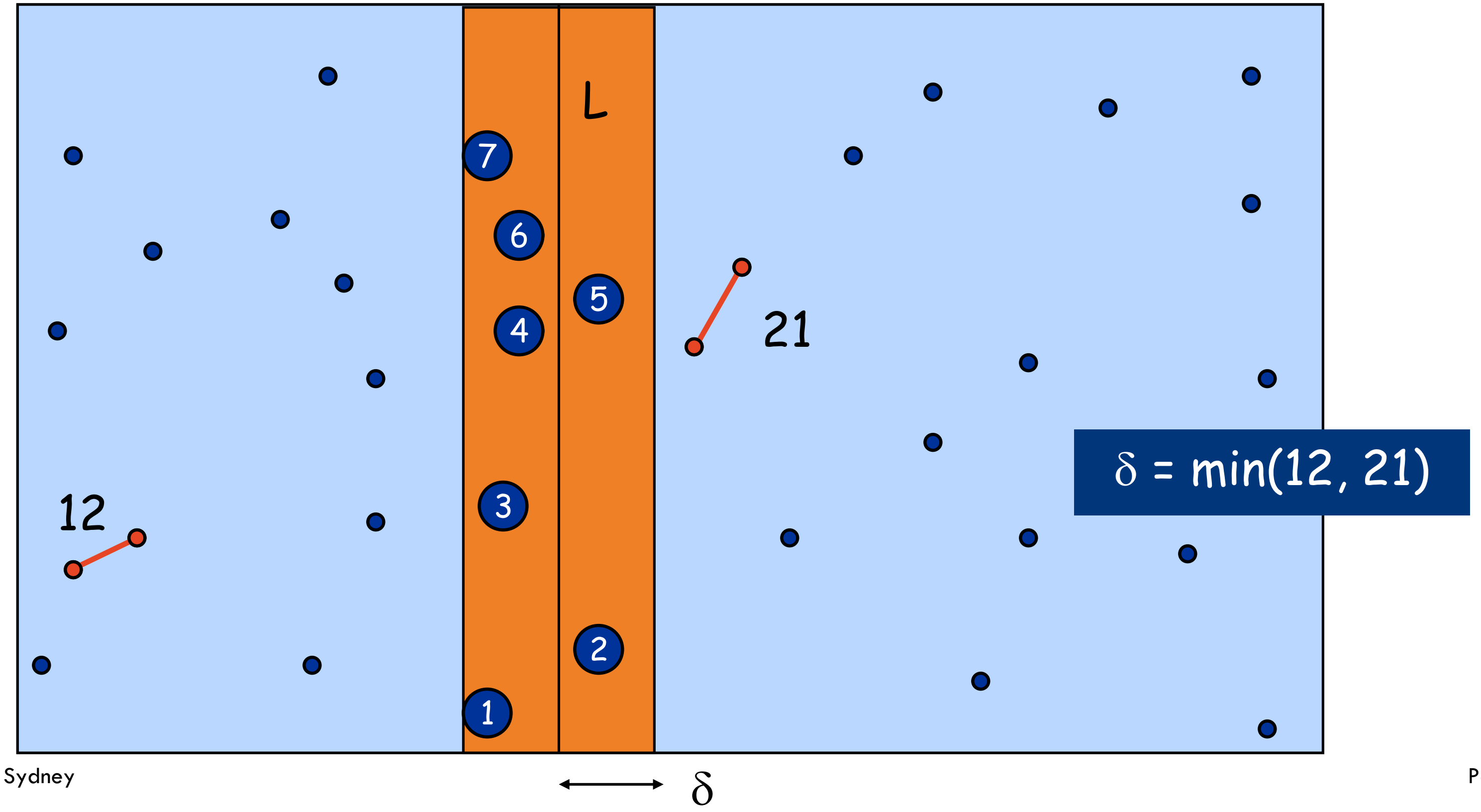
Closest Pair of Points

- Find closest pair with one point in each side, assuming that distance < δ.
  - **Observation:** only need to consider points within δ of line L.
  - Sort points in 2δ-strip by their y coordinate.



Closest Pair of Points

- Find closest pair with one point in each side, assuming that distance < δ.
  - **Observation:** only need to consider points within δ of line L.
  - Sort points in 2δ-strip by their y coordinate.
  - Only check distances of those within 11 positions in sorted list!



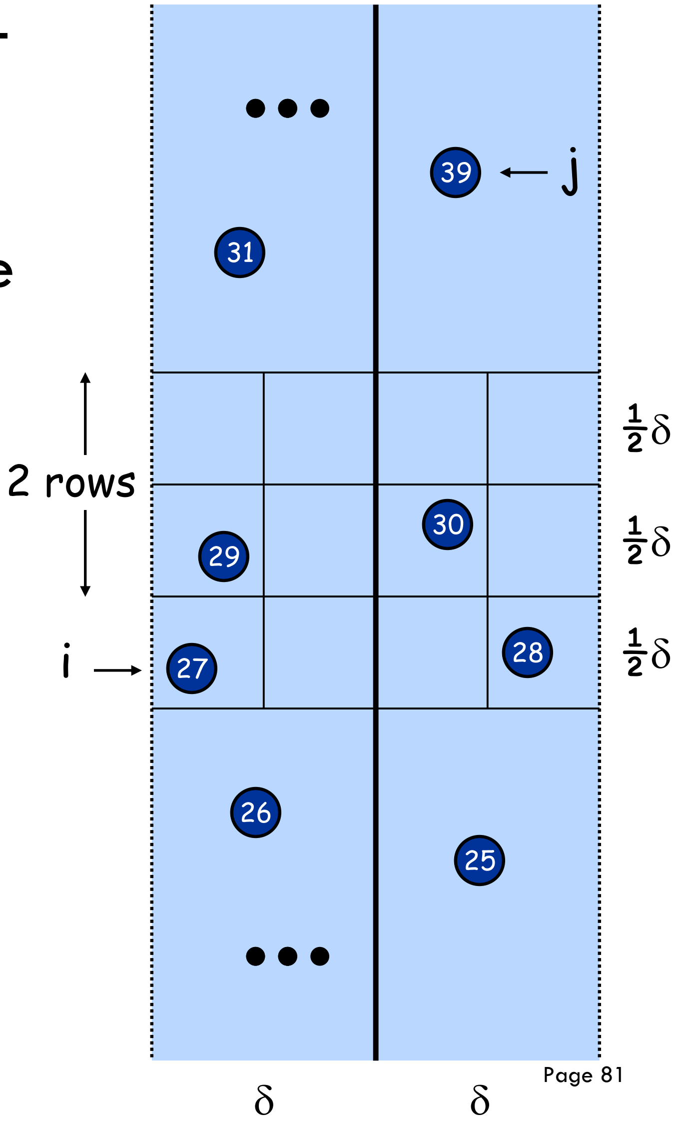
Closest Pair of Points

- **Definition:** Let s_i be the point in the 2δ-strip, with the i^{th} smallest y-coordinate.

- **Claim:** If $|i - j| \geq 12$, then the distance between s_i and s_j is at least δ.

- **Proof:**
  - No two points lie in same $\frac{1}{2}\delta$-by-$\frac{1}{2}\delta$ box.
  - Two points at least 2 rows apart have distance $\geq 2(\frac{1}{2}\delta)$. ▪

- **Fact:** Still true if we replace 12 with 7.



Closest Pair Algorithm

```
Closest-Pair($p_1$, …, $p_n$) {
    Compute separation line L such that half the points          O(n log n)
    are on one side and half on the other side.

    $\delta_1$ = Closest-Pair(left half)                         2T(n / 2)
    $\delta_2$ = Closest-Pair(right half)
    $\delta$  = min($\delta_1$, $\delta_2$)

    Delete all points further than $\delta$ from separation line L    O(n)

    Sort remaining points by y-coordinate.                        O(n log n)

    Scan points in y-order and compare distance between           O(n)
    each point and next 11 neighbors. If any of these
    distances is less than $\delta$, update $\delta$.

    return $\delta$.
}
```

Closest Pair of Points: Analysis

- **Running time**

$$T(n) \leq 2T(n/2) + O(n \log n) \Rightarrow T(n) = O(n \log^2 n)$$

- **Question:** Can we achieve O(n log n)?

- **Answer:** Yes. Don't sort points in strip from scratch each time.
  - Each recursive returns two lists: all points sorted by y coordinate, and all points sorted by x coordinate.
  - Sort by merging two pre-sorted lists.

$$T(n) \leq 2T(n/2) + O(n) \Rightarrow T(n) = O(n \log n)$$

Solving recursions

Unrolling and the Master method

Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

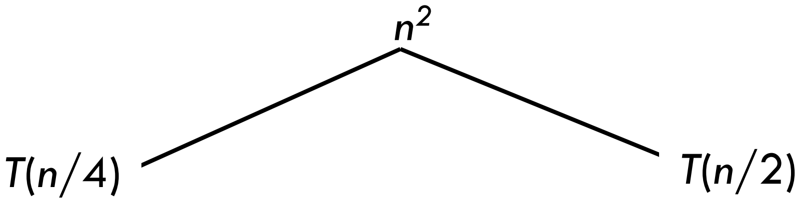
Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

$$T(n)$$

Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

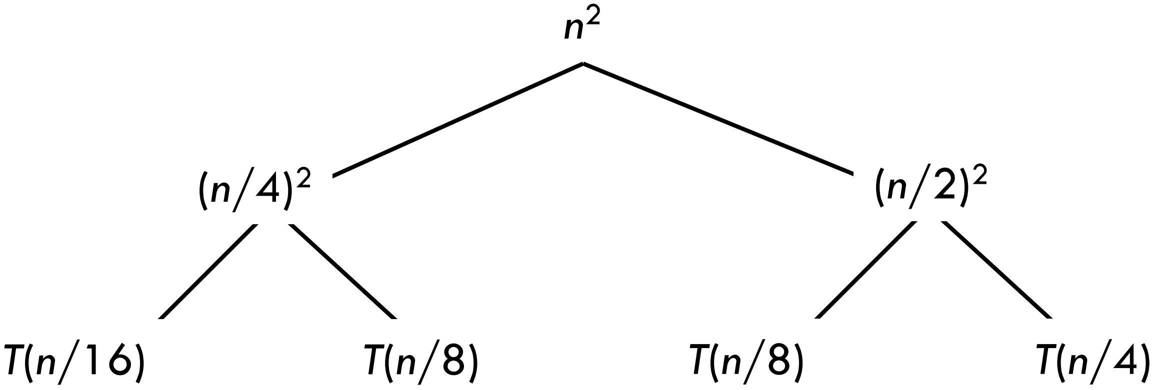


n^2

$T(n/4)$ $T(n/2)$

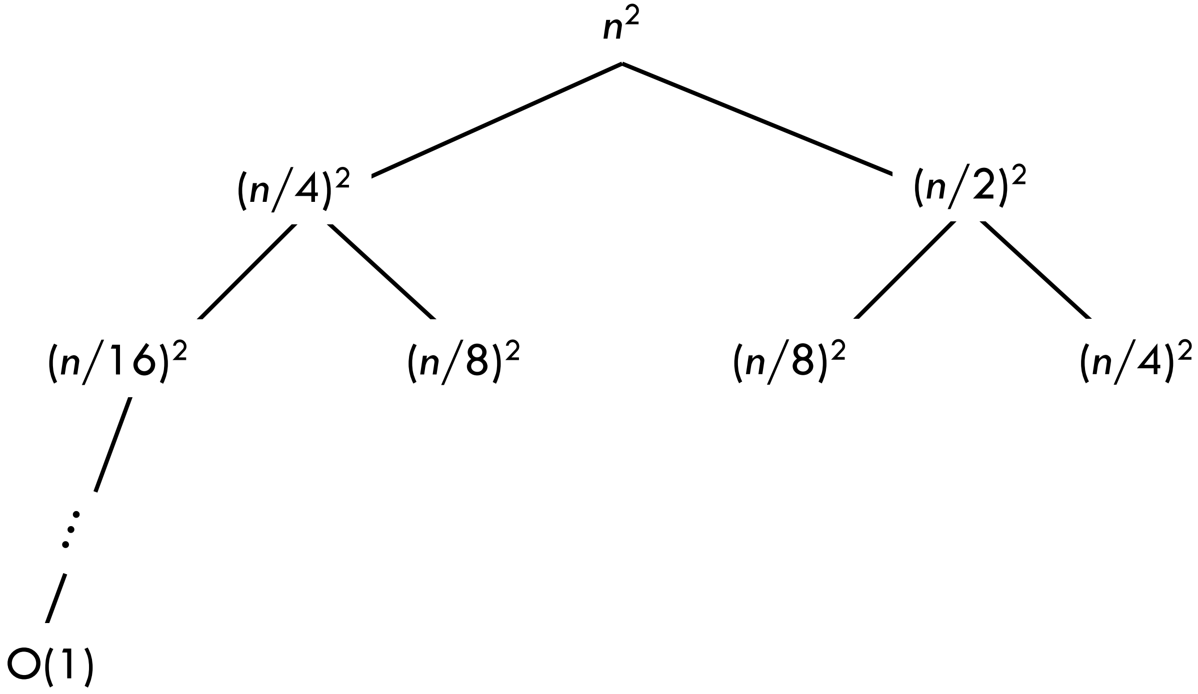
Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



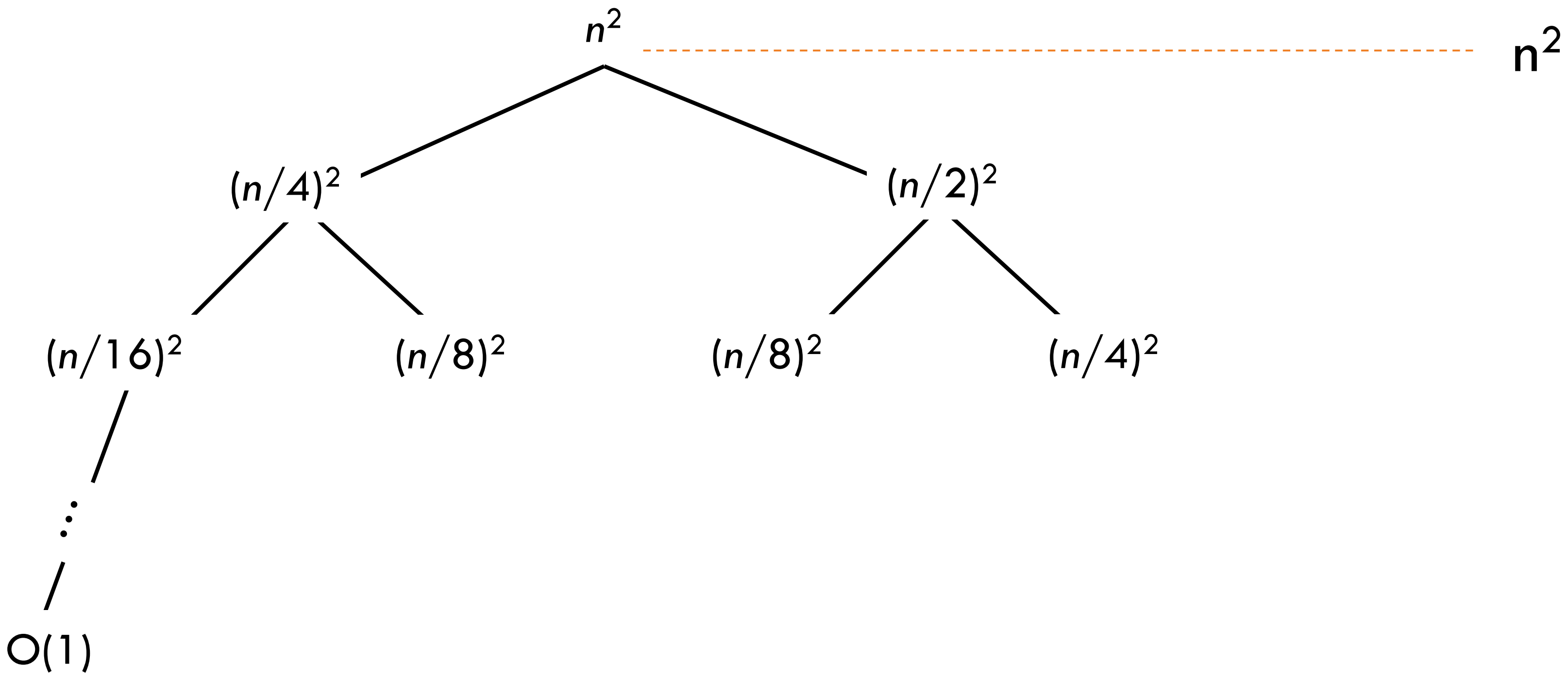
Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



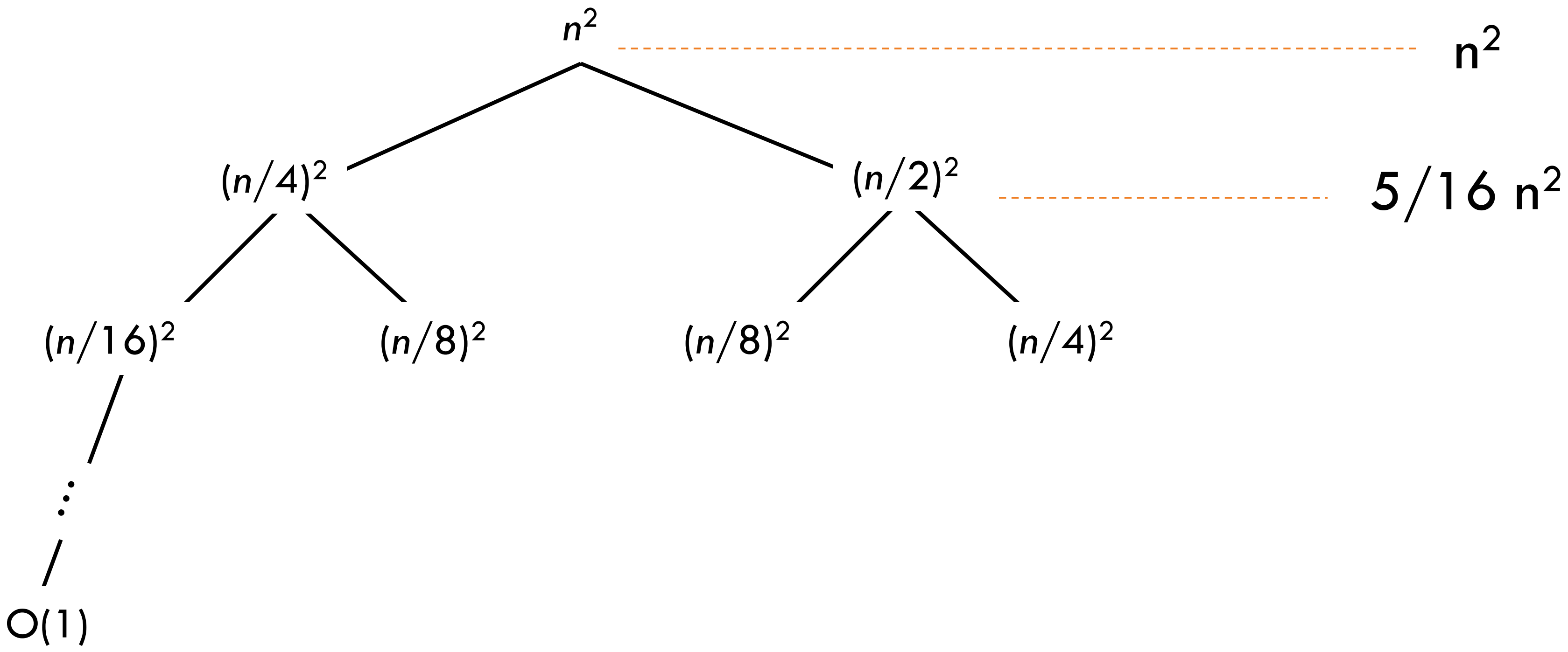
Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



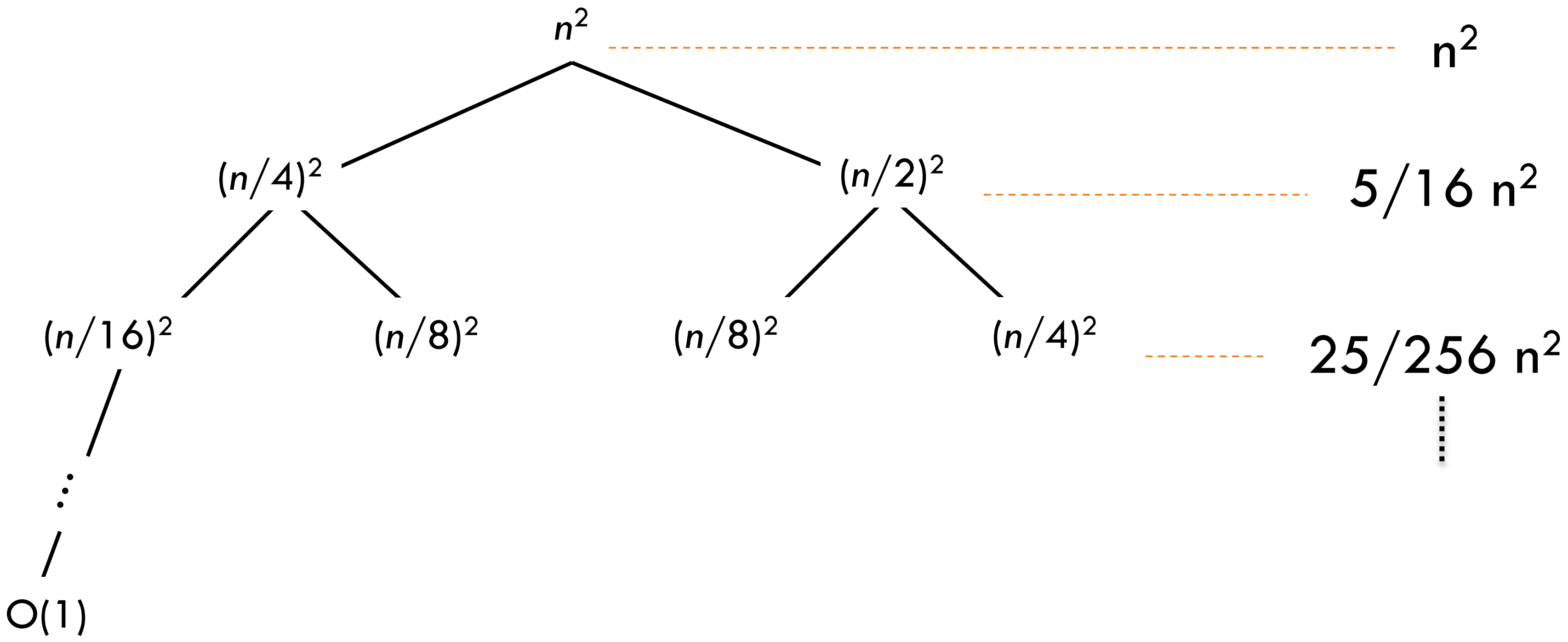
Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



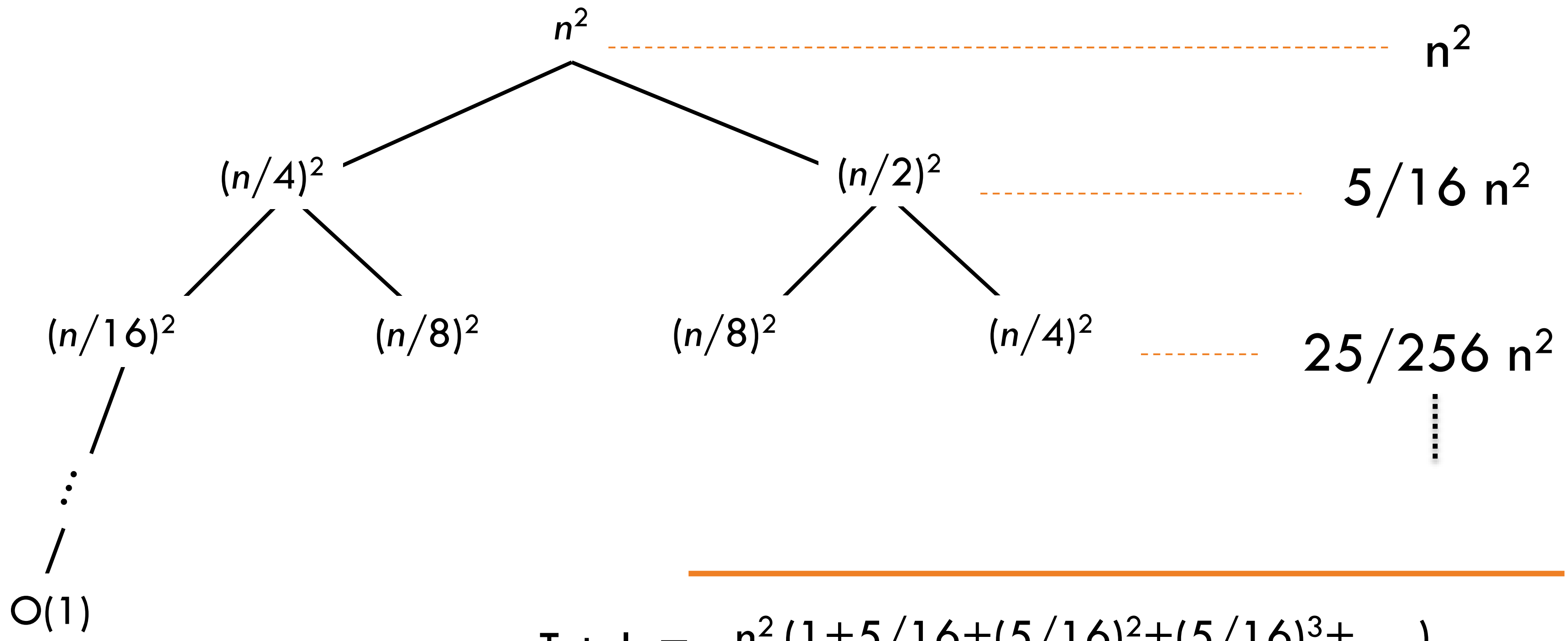
Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



$$\text{Total} = n^2\,(1+5/16+(5/16)^2+(5/16)^3+\ldots)$$

geometric series

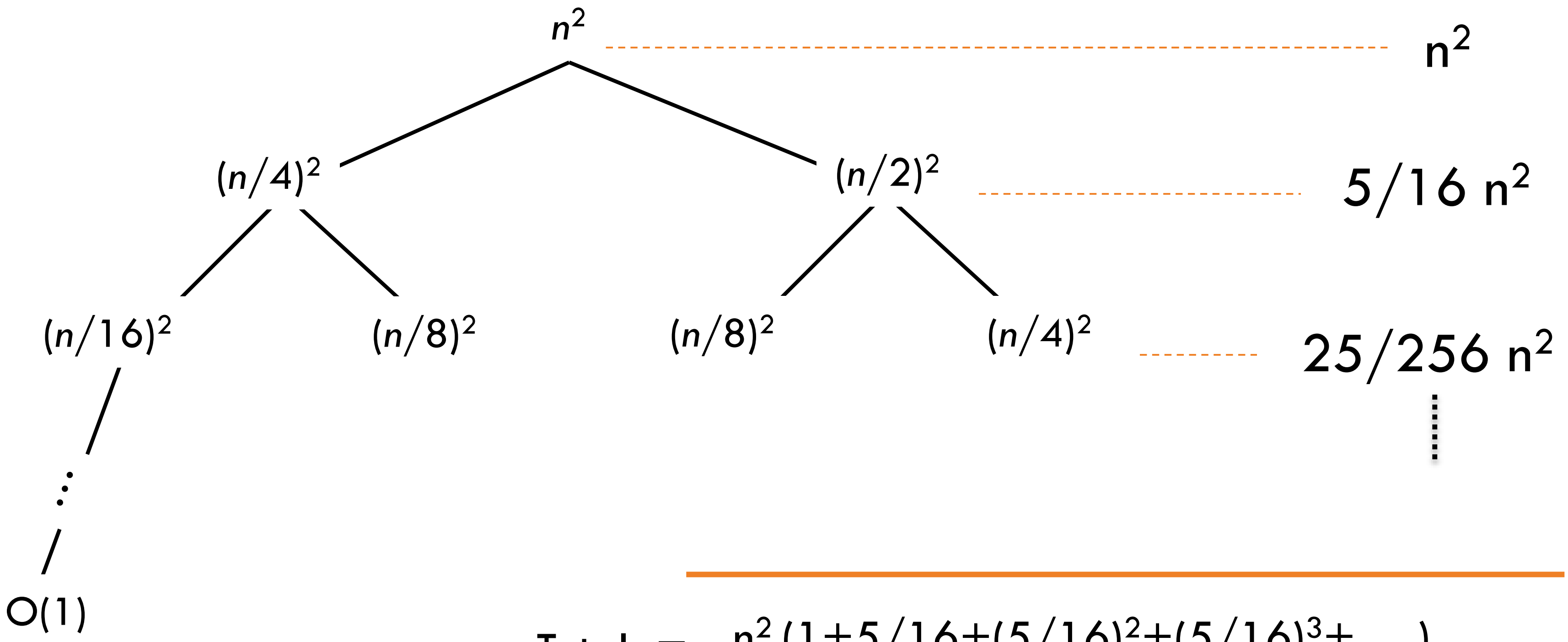
Appendix: geometric series

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$ for $x \neq 1$

$$1 + x + x^2 + \cdots = \frac{1}{1 - x}$$ for $|x| < 1$

Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



n^2

$(n/4)^2$ $(n/2)^2$

$(n/16)^2$ $(n/8)^2$ $(n/8)^2$ $(n/4)^2$

$O(1)$

n^2

$5/16\ n^2$

$25/256\ n^2$

Total $= n^2 (1+5/16+(5/16)^2+(5/16)^3+ \ldots)$

$= O(n^2)$ *geometric series*

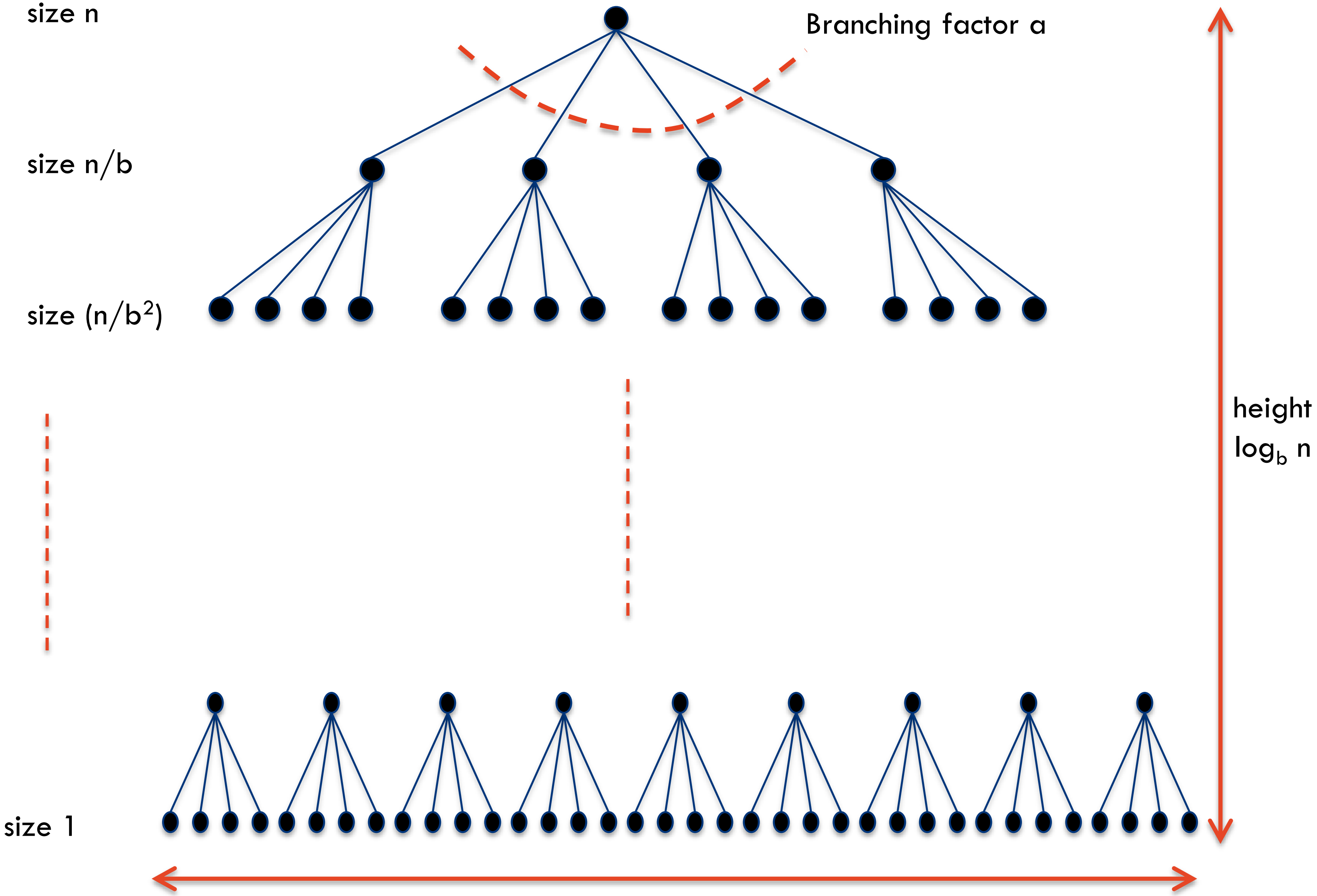
The master method

The master method applies to recurrences of the form

$$T(n) = a \cdot T(n/b) + f(n) \ ,$$

where $a \geq 1$, $b > 1$, and f is asymptotically positive.

$T(n) = a \cdot T(n/b) + f(n)$



size n

Branching factor a

size n/b

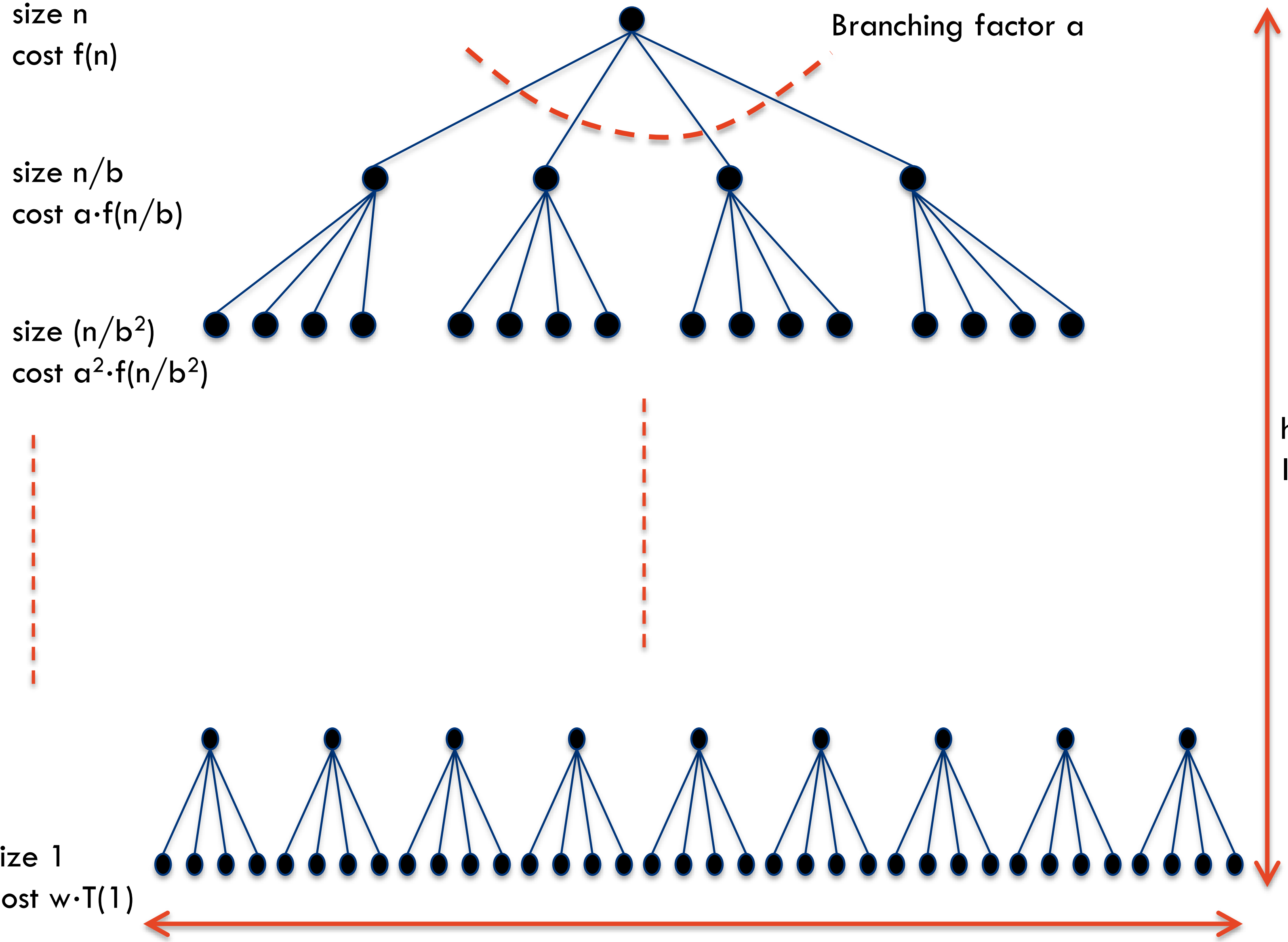
size (n/b^2)

height $\log_b n$

size 1

width $w = a^{\log_b n} = n^{\log_b a}$

$T(n) = a \cdot T(n/b) + f(n)$



size n
cost $f(n)$

Branching factor a

size n/b
cost $a \cdot f(n/b)$

size (n/b^2)
cost $a^2 \cdot f(n/b^2)$

height
$\log_b n$

size 1
cost $w \cdot T(1)$

width $w = a^{\log_b n} = n^{\log_b a}$

Three common cases

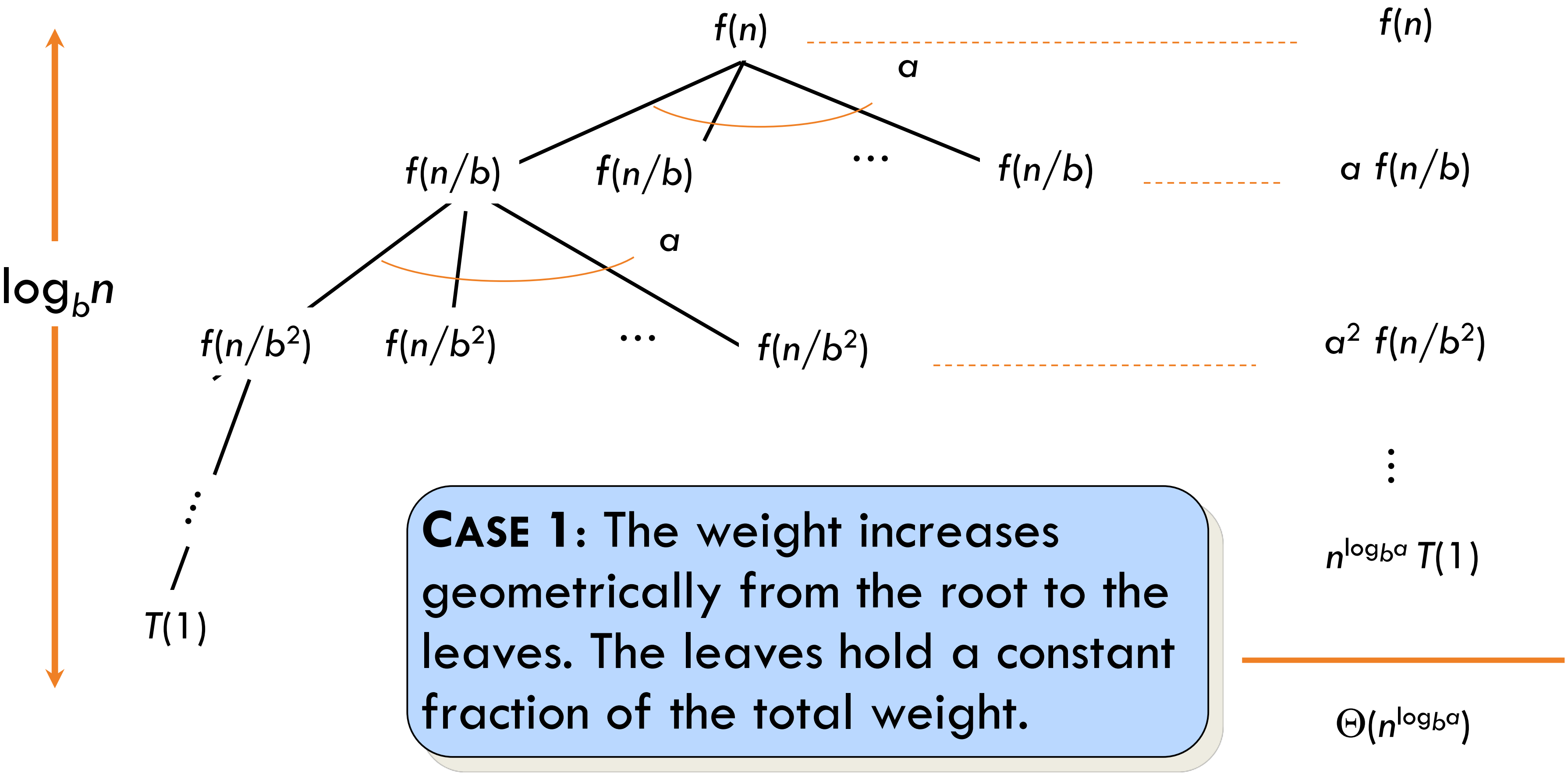
Compare $f(n)$ with $n^{\log_b a}$:

Case 1:

If $f(n) = O(n^{\log_b a - \varepsilon})$ for any constant $\varepsilon>0$ then $f(n)$ grows polynomially slower than $n^{\log_b a}$ (by an n^ε factor).

Solution: $T(n) = \Theta(n^{\log_b a})$.

Idea of master theorem: Case 1



CASE 1: The weight increases geometrically from the root to the leaves. The leaves hold a constant fraction of the total weight.

Three common cases [Compare f(n) with $n^{\log_b a - \varepsilon}$]

Case 1: Example

$T(n) = 8T(n/2) + 10n^2$

$\Rightarrow$ $a = 8$, $b=2$ and $f(n)=10n^2$

$f(n) = 10n^2$

$n^{\log_b a - \varepsilon} = n^{\log_2 8 - \varepsilon} = O(n^{3 - \varepsilon})$ for $\varepsilon=1>0$.

$\Rightarrow f(n) = O(n^{\log_b a - \varepsilon}) \Rightarrow$ Case 1 holds.

Solution: $T(n) = \Theta(n^{\log_b a}) = \Theta(n^3)$

Three common cases (cont'd) [Compare f(n) with $n^{\log_b a - \varepsilon}$]

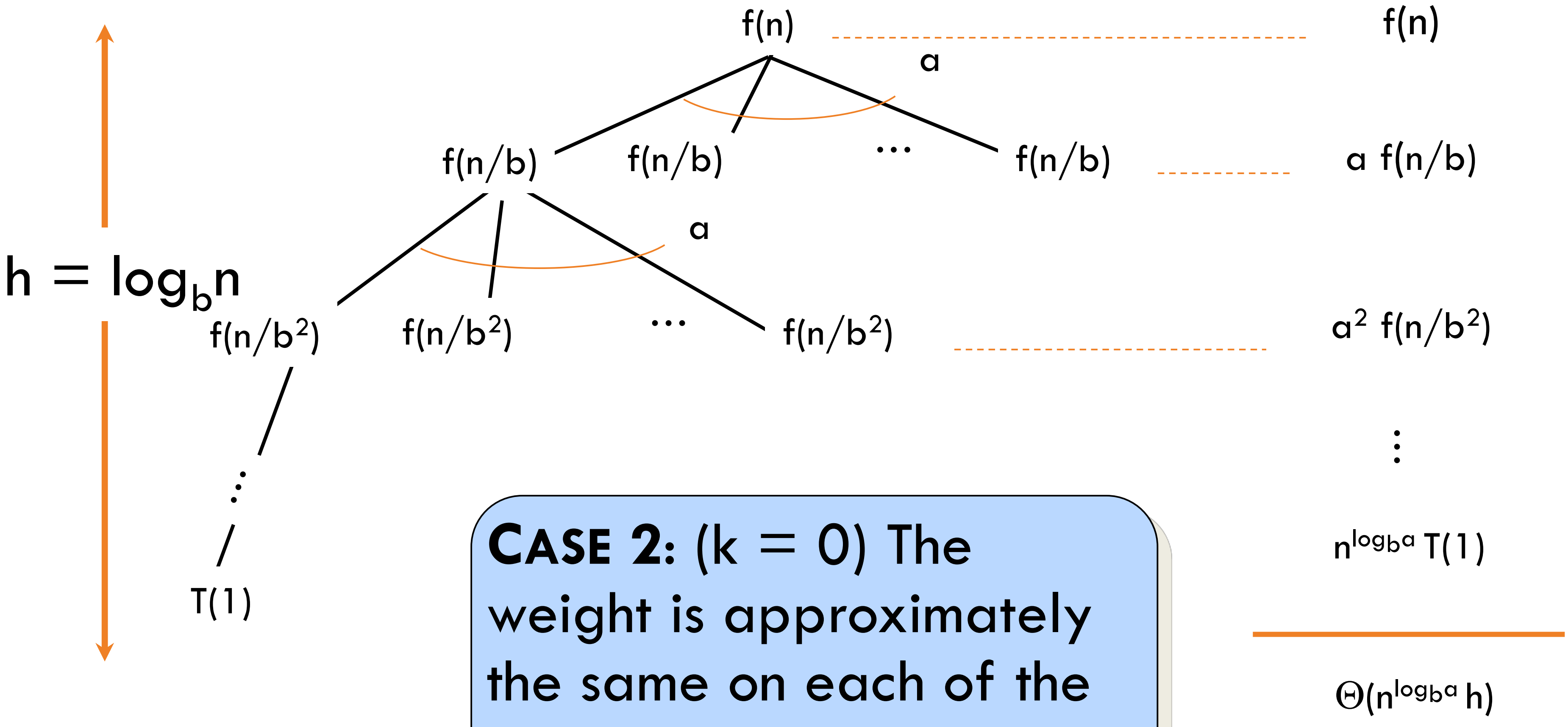
Case 2:

If $f(n) = \Theta(n^{\log_b a} \log^k n)$ for some constant $k \geq 0$ then f(n) and $n^{\log_b a}$ grow at similar rates.

Solution: $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$.

Idea of master theorem

Recursion tree:



CASE 2: ($k = 0$) The weight is approximately the same on each of the $\log_b n$ levels.

Three common cases [Compare f(n) with $n^{\log_b a - \varepsilon}$]

Case 2: Example

$T(n) = 2T(n/2) + n \log n$

$\Rightarrow$ a= 2, b=2 and f(n)=n log n

$f(n) = n \log n = \Theta(n^{\log_b a} \log^k n) = \Theta(n \log n)$ for k=1

$\Rightarrow$ Case 2 holds.

Solution: $T(n) = \Theta(n^{\log_b a} \log^{k+1} n) = \Theta(n \log^2 n)$

Three common cases (cont.) [Compare f(n) with $n^{\log_b a - \varepsilon}$]

Case 3:

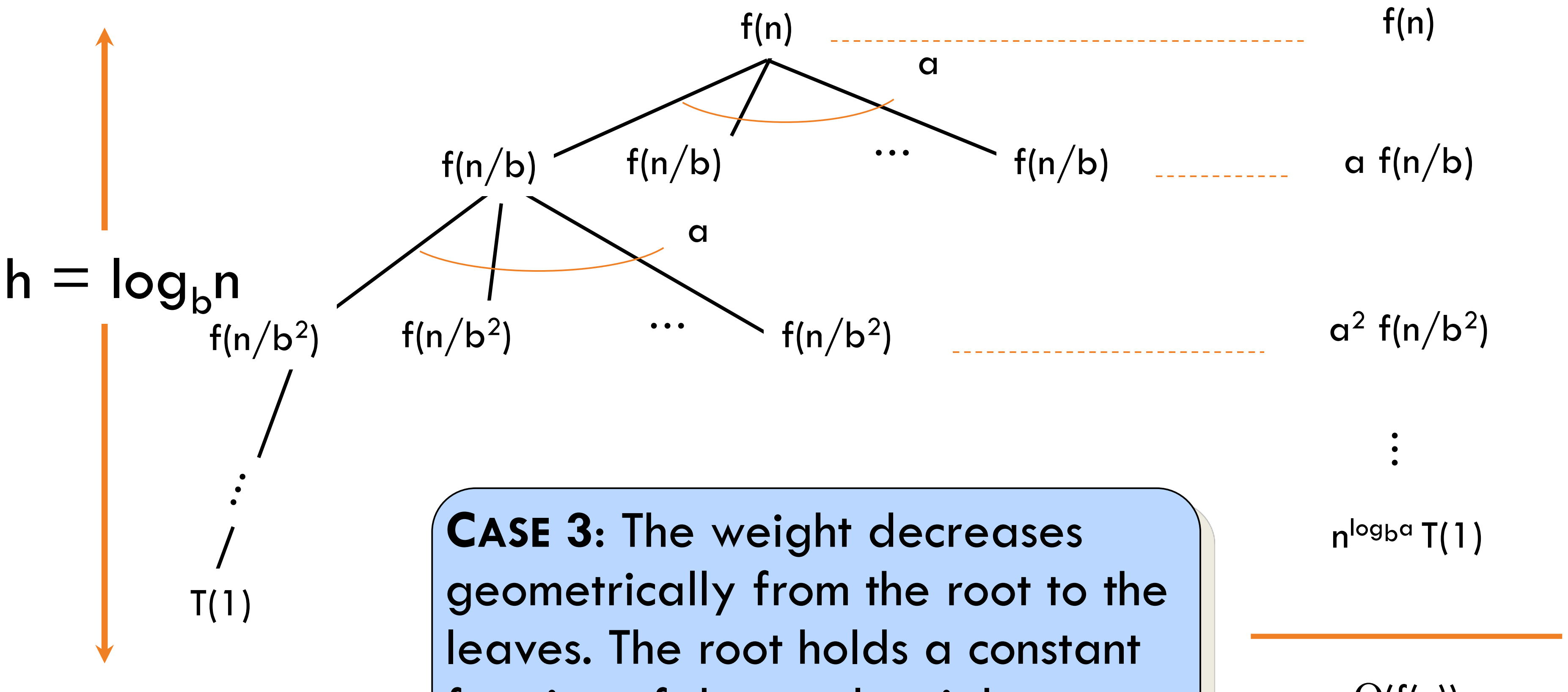
If $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$.

- f(n) grows polynomially faster than $n^{\log_b a}$ (by an n^{ε} factor), and
- f(n) satisfies the **regularity condition** that $a \cdot f(n/b) \leq c \cdot f(n)$ for some constant $c < 1$.

Solution: $T(n) = \Theta(f(n))$.

Idea of master theorem

Recursion tree:



CASE 3: The weight decreases geometrically from the root to the leaves. The root holds a constant fraction of the total weight.

Three common cases [Compare f(n) with $n^{\log_b a - \varepsilon}$]

$T(n) = 4T(n/2) + n^3$

$a = 4, b = 2 \quad \Rightarrow \quad n^{\log_b a} = n^2$ and $f(n) = n^3$.

CASE 3: $f(n) = \Omega(n^{2 + \varepsilon})$ for $\varepsilon = 1$

and

$4(cn/2)^3 \leq cn^3$ (reg. cond.) for $c = 1/2$.

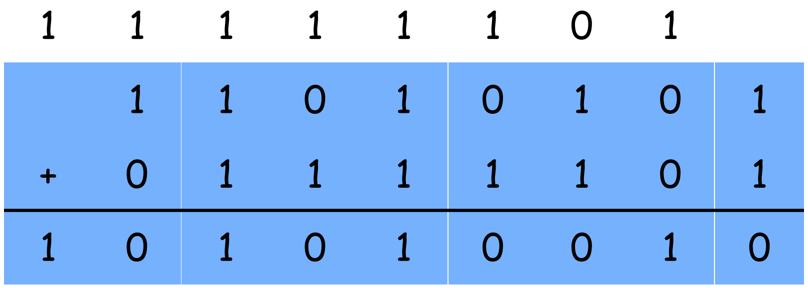
Solution: $T(n) = \Theta(n^3)$

Integer Multiplication

Integer Arithmetic

- **Add.** Given two n-digit integers a and b, compute a + b.
  - O(n) bit operations.

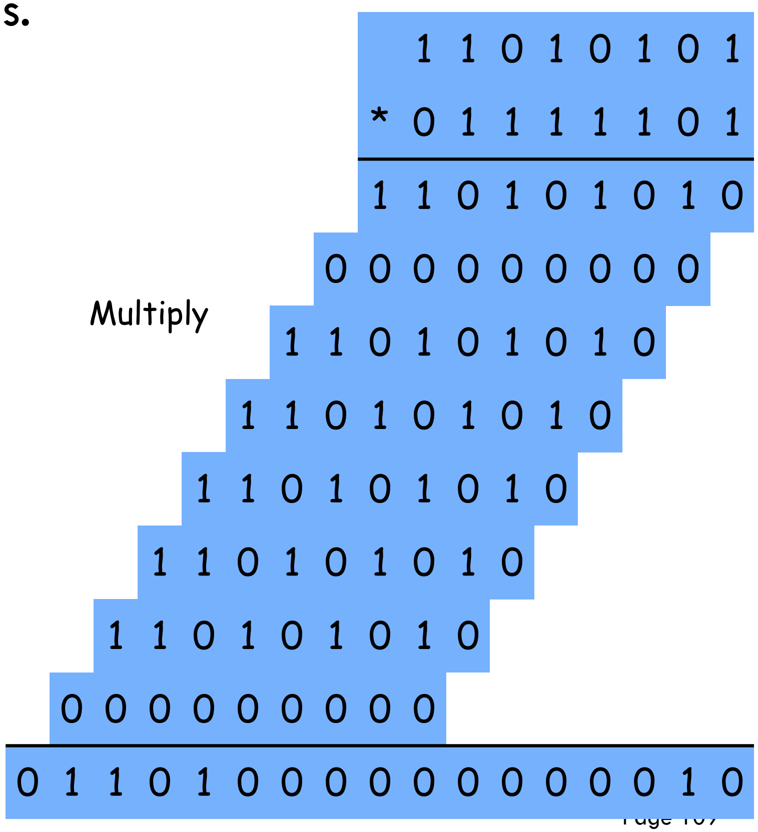
- **Multiply.** Given two n-digit integers a and b, compute a × b.
  - Brute force solution: $\Theta(n^2)$ bit operations.



1	1	1	1	1	1	0	1	
	1	1	0	1	0	1	0	1
+	0	1	1	1	1	1	0	1
1	0	1	0	1	0	0	1	0

Add

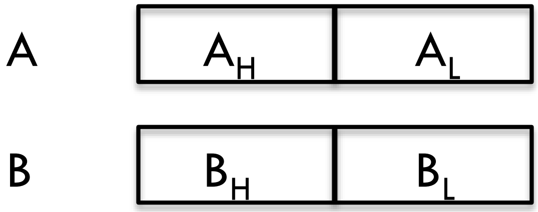
Multiply



									1	1	0	1	0	1	0	1
								*	0	1	1	1	1	1	0	1
								1	1	0	1	0	1	0	1	0
							0	0	0	0	0	0	0	0	0	
						1	1	0	1	0	1	0	1	0		
					1	1	0	1	0	1	0	1	0			
				1	1	0	1	0	1	0	1	0				
			1	1	0	1	0	1	0	1	0					
		1	1	0	1	0	1	0	1	0						
	0	0	0	0	0	0	0	0	0							
0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	1	0

Divide-and-Conquer Multiplication: Warmup

- To multiply two n-digit integers:
  - Given two numbers A and B with n bits each.
  - Partition the n bits into the n/2 “high” bits and the n/2 “low” bits.

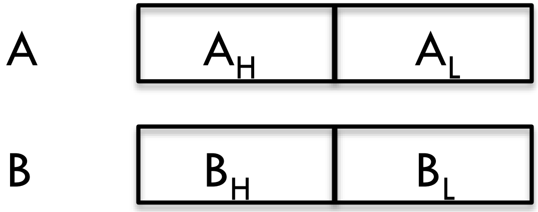


$$A = A_H \cdot 2^{n/2} + A_L$$

$$B = B_H \cdot 2^{n/2} + B_L$$

Divide-and-Conquer Multiplication: Warmup

- To multiply two n-digit integers:
  - Given two numbers A and B with n bits each.
  - Partition the n bits into the n/2 “high” bits and the n/2 “low” bits.



$$A = A_H \cdot 2^{n/2} + A_L$$

$$B = B_H \cdot 2^{n/2} + B_L$$

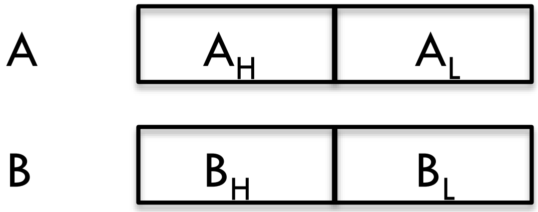
$$\rightarrow \; A \cdot B = (A_H \cdot 2^{n/2} + A_L) \cdot (B_H \cdot 2^{n/2} + B_L)$$

$$= A_H\, B_H \cdot 2^n + A_H\, B_L \cdot 2^{n/2} + A_L\, B_H \cdot 2^{n/2} + A_L\, B_L$$

- 4 multiplications of n/2-bit numbers: A_HB_H, A_HB_L, A_LB_H, A_LB_L , additions and shifts. Multiplications by powers of 2 are just shifts.

Divide-and-Conquer Multiplication: Warmup

- To multiply two n-digit integers:
  - Given two numbers A and B with n bits each.
  - Partition the n bits into the n/2 “high” bits and the n/2 “low” bits.



$A = A_H \cdot 2^{n/2} + A_L$

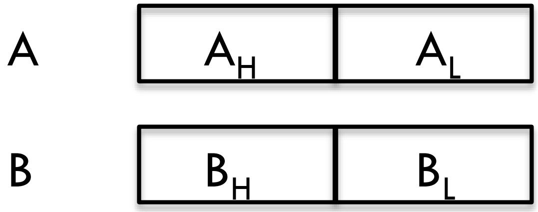
$B = B_H \cdot 2^{n/2} + B_L$

$$\rightarrow A \cdot B = (A_H \cdot 2^{n/2} + A_L) \cdot (B_H \cdot 2^{n/2} + B_L)$$
$$= A_H B_H \cdot 2^n + A_H B_L \cdot 2^{n/2} + A_L B_H \cdot 2^{n/2} + A_L B_L$$

$$\mathrm{T}(n) = \underbrace{4T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, shift}} \Rightarrow \mathrm{T}(n) = \Theta(n^2)$$

Karatsuba Multiplication [1960]

– Multiply two n-digit integers


A A_H A_L

B B_H B_L

$A = A_H \cdot 2^{n/2} + A_L$

$B = B_H \cdot 2^{n/2} + B_L$

Observation:

$$A \cdot B = A_H B_H \cdot 2^n + [(A_H + A_L) \cdot (B_H + B_L) - A_H B_H - A_L B_L] \cdot 2^{n/2} + A_L B_L$$

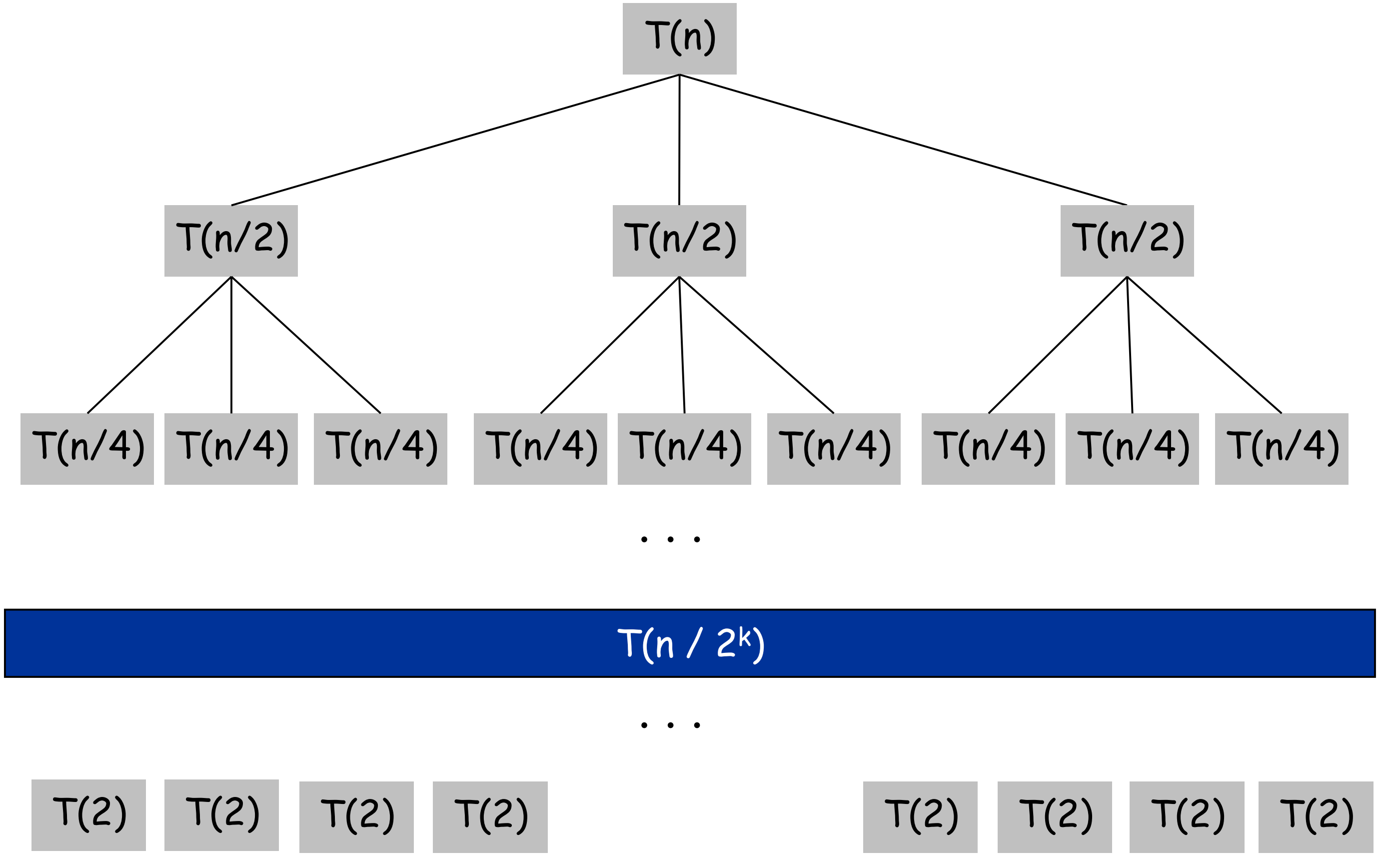
Theorem: [Karatsuba-Ofman, 1962]

3 multiplications of n/2-bit numbers + additions, subtractions and shifts.

Karatsuba: Recursion Tree

$$T(n) = \begin{cases} 0 & \text{if } n=1 \\ 3T(n/2) + n & \text{otherwise} \end{cases}$$

$$T(n) = \sum_{k=0}^{\log_2 n} n \left(\tfrac{3}{2}\right)^k = \frac{\left(\tfrac{3}{2}\right)^{1+\log_2 n} - 1}{\tfrac{3}{2} - 1} = 3n^{\log_2 3} - 2 = O(n^{1.59})$$



T(n) — $1 \cdot (n)$

T(n/2) T(n/2) T(n/2) — $3 \cdot (n/2)$

T(n/4) T(n/4) T(n/4) T(n/4) T(n/4) T(n/4) T(n/4) T(n/4) T(n/4) — $9 \cdot (n/4)$

. . . — . . .

$T(n / 2^k)$ — $3^k \cdot (n / 2^k)$

. . . — . . .

T(2) T(2) T(2) T(2) T(2) T(2) T(2) T(2) — $n^{\log3\ 2} \cdot (2)$

Summary: Divide-and-Conquer

- **Divide-and-conquer.**
  - Break up problem into several parts.
  - Solve each part recursively.
  - Combine solutions to sub-problems into overall solution.

- **Master theorem**

- **Problems**
  - Merge Sort
  - Closest pair
  - Multiplication

This weeks quiz is all about solving recursions!