Question1:

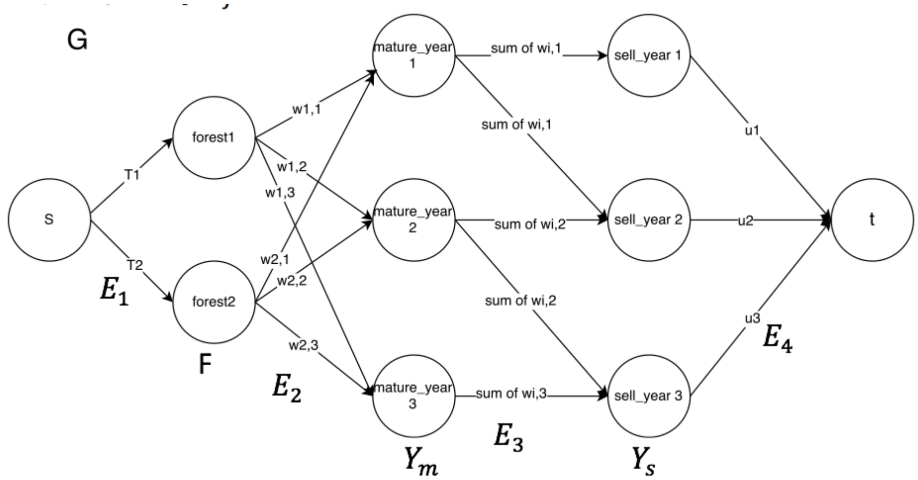
a). Formulate the task

- Create the digraph called G:
    1. G = (F ∪ Y_m ∪ Y_s ∪ {s, t} ∪ E)

    2. (k = total number of forests; Y = total number of years; i = the index from 1 to k; j = the index from j to Y)

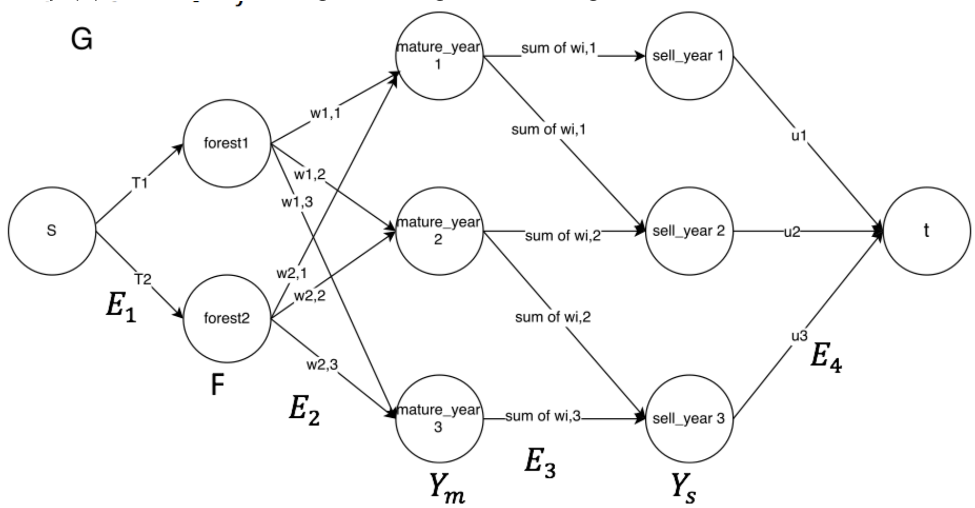
    3. Representations for the type of vertices:
        a. s = Source
        b. F = The forests which is {forest 1 to forest k} since k is the total number of forests
        c. Y_m = The year, which the trees will be matured from F (all the forests). The year would be declared as {mature_ year 1 to mature_ year Y} since Y is the total number of years.
        d. Y_s = The year, which the matured trees can be cut down and sell. The year would be declared as {sell_ year 1 to sell_ year Y} since Y is the total number of years.
        e. t = Sink

    4. Representations for the type of edges:
        - E_1 (the edge between s and F): The total trees in each forest
        Capacity in E_1: T_i or τ_i $(T_i$: $is\ tau\ (\tau_i)\ as\ denoted\ in\ the\ graph)$
        - E_2 (the edge between F and Y_m): The total amount of matured trees of each forest in each year
        Capacity in E_2: $w_{i,j}$
        - E_3 (the edge between Y_m and Y_s): The amount of matured-trees from the matured year to the available selling years, we can also says each tree matures in year j can only be sold in that year (j) or the δ_j - 1 years afterwards.
        Capacity in E_3: The sum of the $w_{i,j}$ of all the forests in each mature-year (Y_m), which goes to the available sell _year (Y_s).
        - E_4 (the edge between Y_s and t): The amount of the total sell-matured-trees from each year.
        Capacity in E_4: u_j



b). Prove the algorithm is correct:

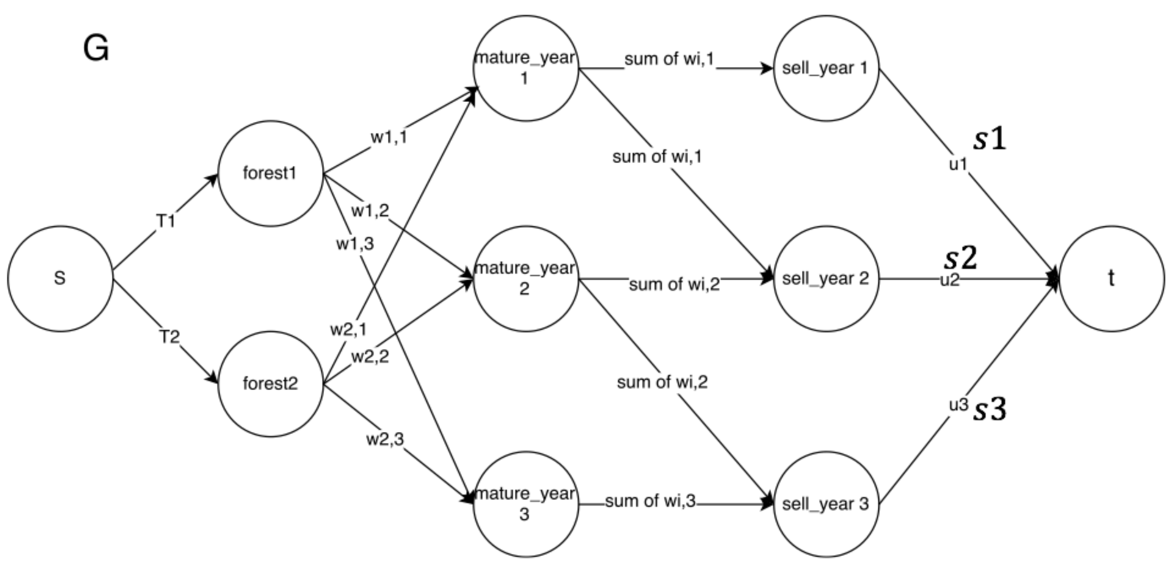
- Integrality: If all capacities are integers then every flow value f(e) and every residual capacities $cf(e)$ remains an integer throughout the algorithm.



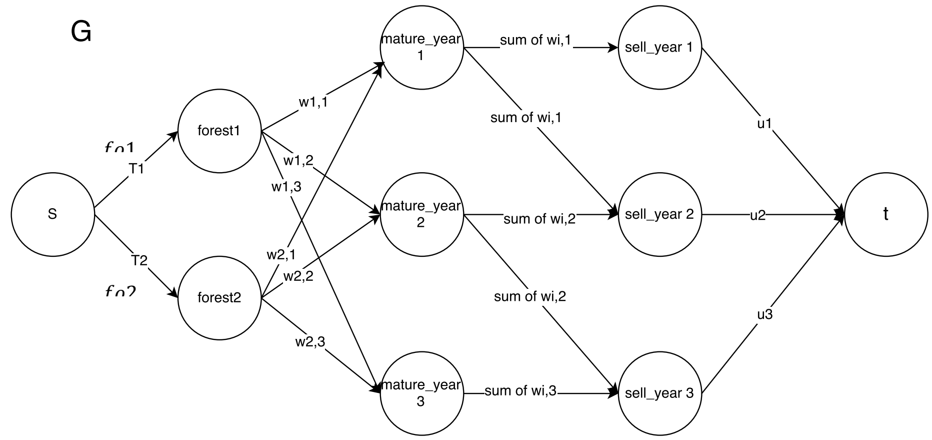
Proof (part I and part II):

- **Part I:**
  Purpose: If the maximum number of Christmas trees sold is f, we also have to prove that there exists a max flow f.

  1). The flow of the edges from sell_ year nodes to the sink (t) is represented as the schedule that sells f Christmas trees. We mark the flow from sell_ year 1 to t as s1, the flow from sell_ year 2 to t as s2 and the sell_ year 3 to t as s3. Since we have assumed the max schedule that sells the Christmas trees is f. Hence, the sum of the flows (s1 + s2 + s3) is f.



2). There are two edges going from the source to forest 1 and forest 2. The flow of the edges between source (s) to the forest nodes (F) is represented as the number of the trees (the mature_ trees can be sold) in each forest. We mark the flow between s to forest 1 as fo1 and the flow between s to forest 2 as fo2.

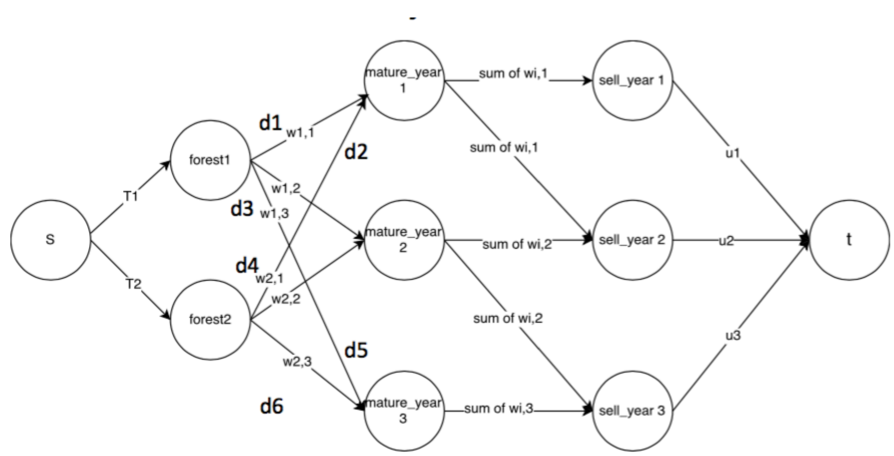


In the flows definition, the value of a flow is: $v(f) = \sum_{e \text{ } out \text{ } of \text{ } s} f(e)$.
Since we have assumed the max value of the schedule is f, therefore, the sum of the flow between s to the forests (fo1 + fo2) suppose to be f as well, which is the sum of the flows between sell_ years to t (s1 + s2 + s3).

“ fo1 + fo2 = s1 + s2 + s3 = max schedule f ”

Since the max schedule f is assumed, the capacity of the edge between s to forest nodes should be holds by the flow definition: $0 \leq f(e) \leq c(e)$. In our graph the definition should be $0 \leq fo1 \leq \tau 1 \text{ } and \text{ } 0 \leq fo2 \leq \tau 2$.

3). The flow of the edges between the forest nodes to the mature_ year nodes (Y_m) is represented as the number of forest i’s trees mature in year j. We mark the flow between forest 1 to mature_ year 1 as d1, forest2 to mature_ year 2 as d2, forest3 to mature_ year 3 as d3, forest4 to mature_ year 4 as d4, forest5 to mature_ year 5 as d5 and forest6 to mature_ year 6 as d6.



Since the max schedule is f, therefore:

$$0 \leq d1;\ d2;\ d3;\ d4;\ d5;\ d6 \leq w1{,}1;\ w1{,}2;\ w1{,}3;\ w2{,}1;\ w2{,}2;\ w2{,}3.$$

The capacity of the edge between forests to mature_ years holds.
We have to assume the max schedule is correct.
For each v ∈ $V - \{s, t\}$: $\sum_{e\ in\ to\ v} f(e) = \sum_{e\ out\ of\ v} f(e)$ in the flows conservation rule.

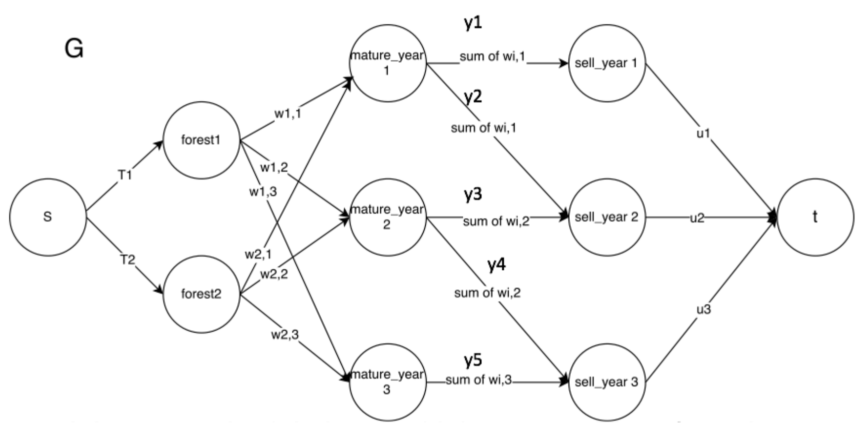
The conservation between flow fo and flow d holds, since:

$$\sum_{E1\ in\ to\ forest\ 1} fo(E1) = \sum_{E2\ out\ of\ forest1} d(E2)$$

and

$$\sum_{E1\ in\ to\ forest\ 2} fo(E1) = \sum_{E2\ out\ of\ forest2} d(E2)$$

4). The flow of the edge from the mature_ year nodes to the sell_ year nodes is represented as the number of the matured trees be allowed to cut down and sell in the year j or $\delta j - 1$ years afterwards. We mark the flow from mature_ year 1 to sell_ year 1 as $y1$, mature_ year1 to sell_ year 2 as y2, mature_ year 2 to sell_ year 1 as y3, mature_ year 2 to sell_ year 3 as y4 and mature_ year 3 to sell_ year 3 as y5.



We have assumed the max schedule is f and it is correct. Therefore, the conservation between flow d and flow y holds, since:

$$\sum_{E2\ in\ to\ mature\_year\ 1} d(E2) = \sum_{E3\ out\ of\ mature\_year1} y(E3)$$

and

$$\sum_{E2\ in\ to\ mature\_year\ 2} d(E2) = \sum_{E3\ out\ of\ mature\_year2} y(E3)$$

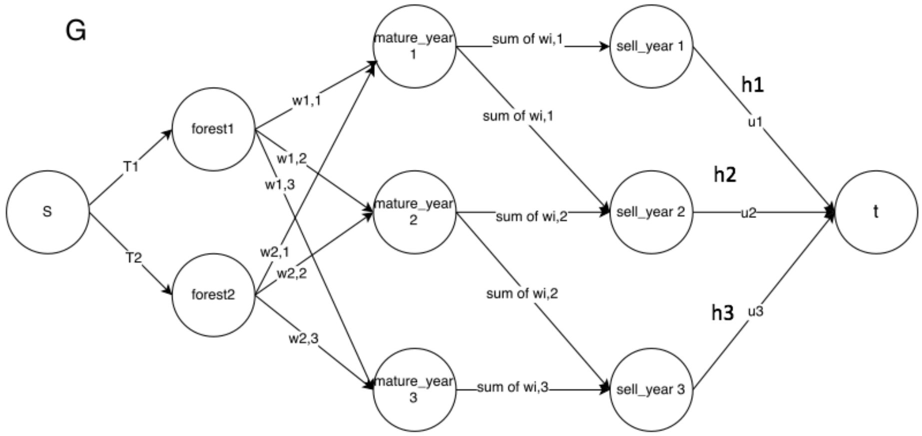
and

$$\sum_{E2\ in\ to\ mature\_year\ 3} d(E2) = \sum_{E3\ out\ of\ mature\_year3} y(E3)$$

Additionally, the capacity of the edge between mature_ year nodes to the sell_ year nodes would hold since:

$$0 \leq y1;\ y2; y3;\ y4;\ y5 \leq sum\ of\ wi, 1;\ sum\ of\ wi, 2;\ sum\ of\ wi, 3$$

5). The flow of the edges between sell_ year nodes to the sink (t) is represented as the number of the trees that be harvested in year j. we mark the flow of the edge from sell_ year 1 to t as $h1$, sell_ year 2 to t as $h2$ and sell_ year 3 to t as $h3$.



Since the assumption that max schedule is f. Hence the capacity of the edge between sell_ year nodes to t has to hold when:

$0 \leq sum\ of\ wi, 1; (sum\ of\ wi, 1 + sum\ of\ wi, 2); (sum\ of\ wi, 2 + \ sum\ of\ wi, 3) \leq$
$u1;\ u2;\ u3.$

In the flows conservation rule, the conservation between flow y and flow h should hold as well since:

$$\sum_{E3\ in\ to\ sell\_year\ 1} y(E3) = \sum_{E4\ out\ of\ sell\_year1} h(E4)$$
and
$$\sum_{E3\ in\ to\ sell\_year\ 2} y(E3) = \sum_{E4\ out\ of\ sell\_year2} h(E4)$$
and
$$\sum_{E3\ in\ to\ sell\_year\ 3} y(E3) = \sum_{E4\ out\ of\ sell\_year3} h(E4)$$

- **Part II:**

(each flow's meaning and the signs between the different type of set-nodes has been defined in part I and those flows would still be used for part II.)

Purpose: if assume f is the value of the max flow then we have to prove there exist a schedule that sells f Christmas trees as well.
Hence, the capacities ($\tau i, w_{i,j}\ , sum\ of\ w_{i,j}\ in\ each\ year\ and\ uj$) of the edge (E1, E2, E3, E4) should hold ($0 \leq f(e) \leq c(e)$).

The conservation of the s-t flow for each set of nodes (F, Y_s, Y_m) should hold as well
(v ∈ $V - \{s, t\}$: $\sum_{e\ in\ to\ v} f(e) = \sum_{e\ out\ of\ v} f(e)$)
Since there exists a max flow with value f, therefore, the sum of the flows of E4, which is $h1 + h2 + h3$ has to be f as well. Additionally, the sum of the flows of E1, which is $fo1 + fo2$ also has to be f since the definition of the flows: $v(f) = \sum_{e\ out\ of\ s} f(e)$

- **Conclusion for the proof:**

In part I we have proved that there exists a max flow with value f via holding the capacity for the edges and the conservation for the nodes since we assume the max schedule that sells f number of Christmas trees. In part II, we prove there exists a max schedule that sells f trees through assuming the f value of max flow exists. Hence, our algorithm is correct since we have proved the algorithm's correctness in both directions.

Question2:

a). Formulate the task

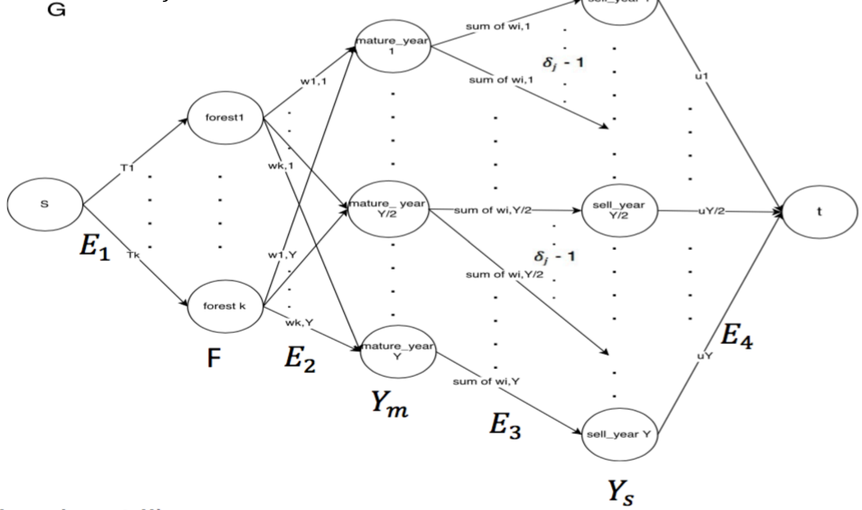
- Create the digraph called G:
    1. G = (F ∪ Y_m ∪ Y_s ∪ {s, t} ∪ E)

    2. (k = total number of forests; Y = total number of years; i = the index from 1 to k; j = the index from j to Y)

    3. Representations for the type of vertices:
        a. s = Source
        b. F = The forests which is {forest 1 to forest k} since k is the total number of forests
        c. Y_m = The year, which the trees will be matured from F (all the forests). The year would be declared as {mature_ year 1 to mature_ year Y} since Y is the total number of years.
        d. Y_s = The year, which the matured trees can be cut down and sell. The year would be declared as {sell_ year 1 to sell_ year Y} since Y is the total number of years.
        e. t = Sink

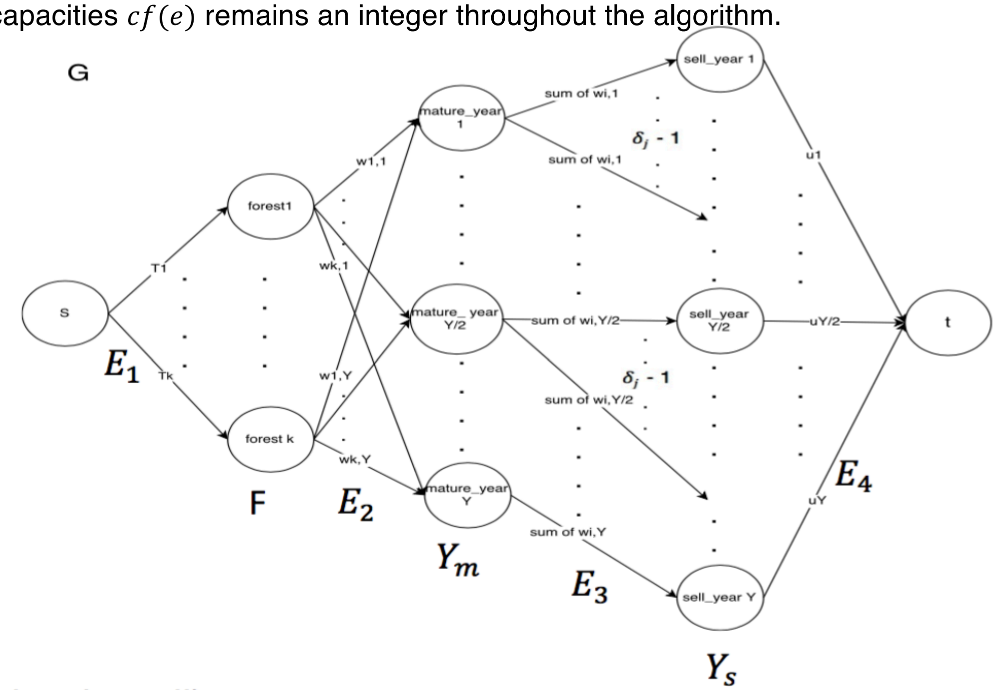
    4. Representations for the type of edges:
    (we make up the fake value of the capacities)
    - E_1 (the edge between s and F): The total trees in each forest
    Capacity in E_1: T_i
    - E_2 (the edge between F and Y_m): The total amount of matured trees of each forest in each year
    Capacity in E_2: $w_{i,j}$
    - E_3 (the edge between Y_m and Y_s): The amount of matured-trees from the matured year to the available selling years, we can also says each tree matures in year j can only be sold in that year (j) or the δ_j - 1 years afterwards.
    Capacity in E_3: The sum of the $w_{i,j}$ of all the forests in each mature-year (Y_m), which goes to the available sell _year (Y_s).
    - E_4 (the edge between Y_s and t): The amount of the total sell-matured-trees from each year
    Capacity in E_4: u_j

G



b). Prove the algorithm is correct:

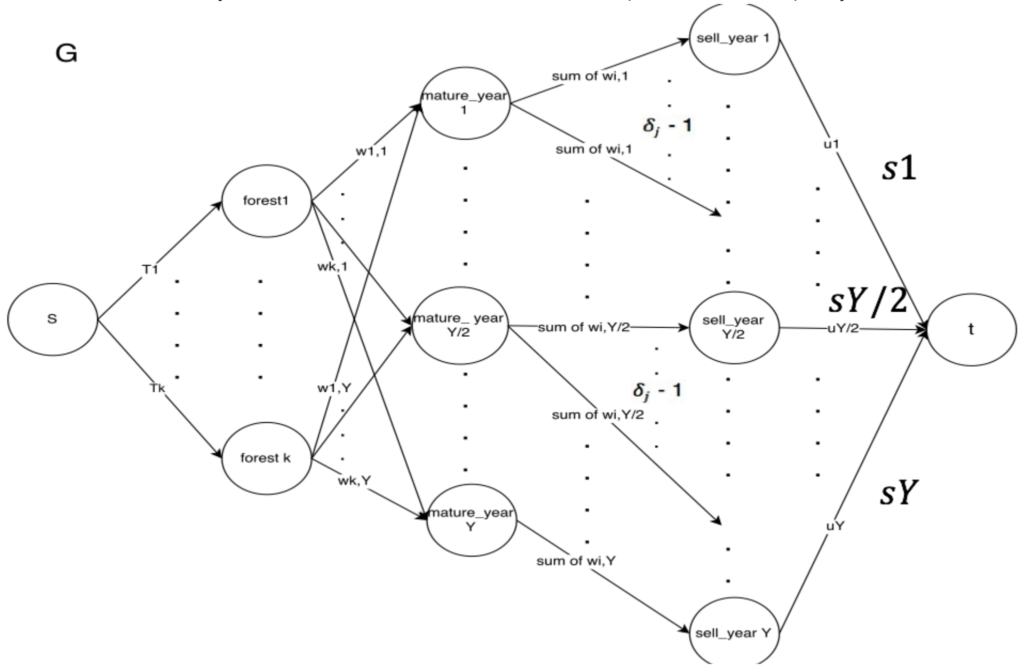
- Integrality: If all capacities are integers then every flow value f(e) and every residual capacities $cf(e)$ remains an integer throughout the algorithm.



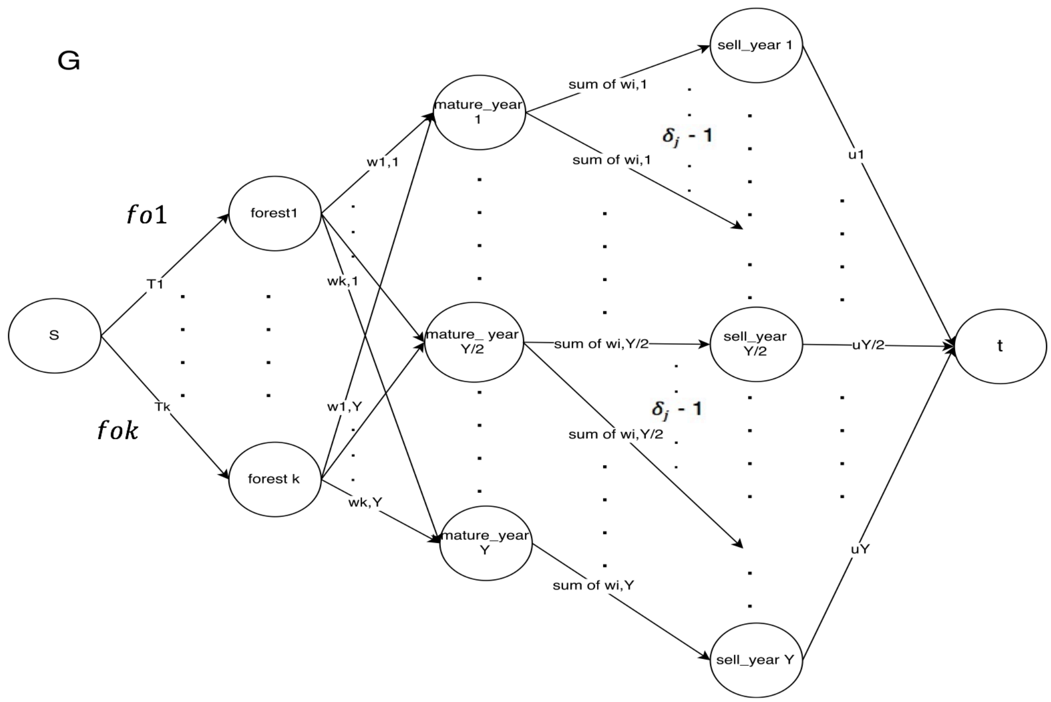
Proof (part I and part II):

- **Part I:**
  Purpose: If the maximum number of Christmas trees sold is f, we also have to prove that there exists a max flow f.

  1). The flow of the edges from sell_ year nodes to the sink (t) is represented as the schedule that sells f Christmas trees. We mark the flow from sell_ year 1 to t is s1 up to the flow from sell_ year Y to t is sY. Since we have assumed the max schedule that sells the Christmas trees is f. Hence, the sum of the flows (s1 + … + sY) is f.



2). There are two edges go from the source to forest 1 to forest k. The flow of the edges between source (s) to the forest nodes (F) is represented as the number of the trees (the mature_ trees can be sold) in each forest.
We mark the flow between s to forest 1 as fo1 up to the flow between s to forest k as fok.

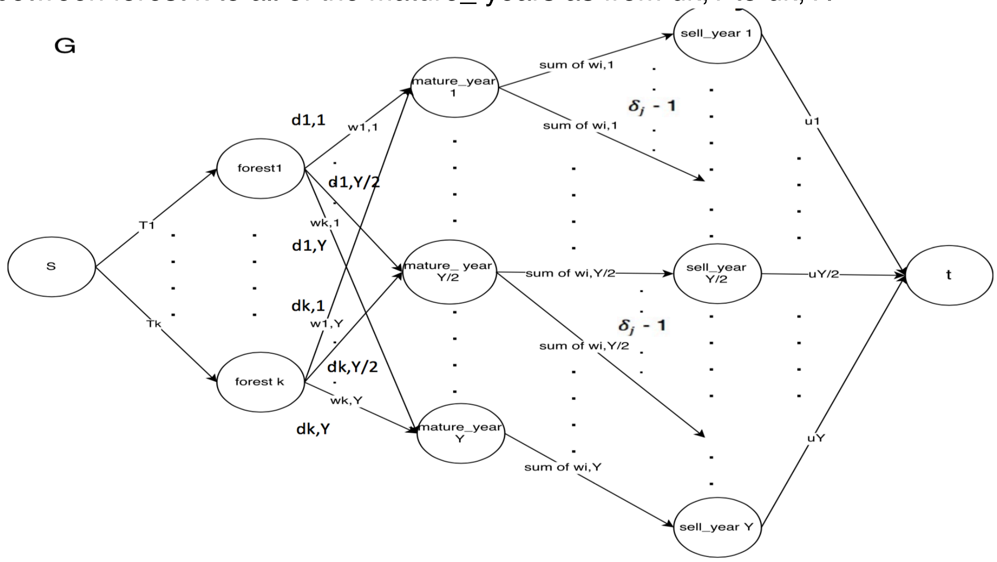


In the flows definition, the value of a flow is: $v(f) = \sum_{e\ out\ of\ s} f(e)$.
Since we have assumed the max value of the schedule is f, therefore, the sum of the flow between s to the forests (fo1 + … + fok) suppose to be f as well, which is the sum of the flows between sell_ years to t (s1 + … + sY).

“ fo1 + … + fok = s1 + … + sY = max schedule f ”

Since the max schedule f is assumed, the capacity of the edge between s to forest nodes should be holds by the flow definition: $0 \leq f(e) \leq c(e)$. In our graph the definition should be $0 \leq fo1 \leq \tau 1; to \ldots;\ 0 \leq fok \leq \tau k$.

3). The flow of the edges between the forest nodes to the mature_ year nodes (Y_m) is represented as the number of forest i's trees mature in year j. We mark the flow between forest 1 to all of the mature_ years from year1 to year Y as from d1,1 to d1,Y up to the flow between forest k to all of the mature_ years as from dk,1 to dk,Y.



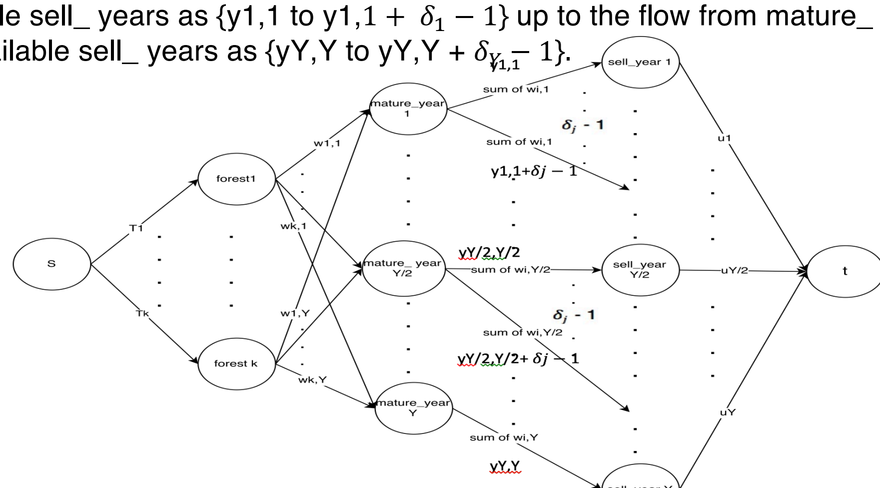
We have to assume the max schedule is correct.
Since the max schedule is f, therefore:

$$0 \leq d1{,}1, \ldots, d1,Y; \ldots;\ di, 1, \ldots, di, Y; \ldots;\ dk, 1, \ldots, dk, Y \leq w1{,}1, \ldots, wk, 1; \ldots;\ w1, j, \ldots, wk, j; \ldots;\ w1, Y, \ldots, wk, Y.$$

The capacity of the edge between forests to mature_ years holds.
For each v ∈ $V - \{s, t\}$: $\sum_{e\ in\ to\ v} f(e) = \sum_{e\ out\ of\ v} f(e)$ in the flows conservation rule.
The conservation between flow fo and flow d holds, since:

$$\sum_{E1\ in\ to\ forest\ i} fo(E1) = \sum_{E2\ out\ of\ foresti} d(E2)$$

4). The flow of the edge from the mature_ year nodes to the sell_ year nodes is represented as the number of the matured trees be allowed to cut down and sell in the year j or $\delta j - 1$ years afterwards. We mark the flow from mature_year1 to all the available sell_ years as {y1,1 to y1,1 + $\delta_1 - 1$} up to the flow from mature_ year Y to all the available sell_ years as {yY,Y to yY,Y + $\delta_Y - 1$}.



We have assumed the max schedule is f and it is correct. Therefore, the conservation between flow d and flow y holds, since:

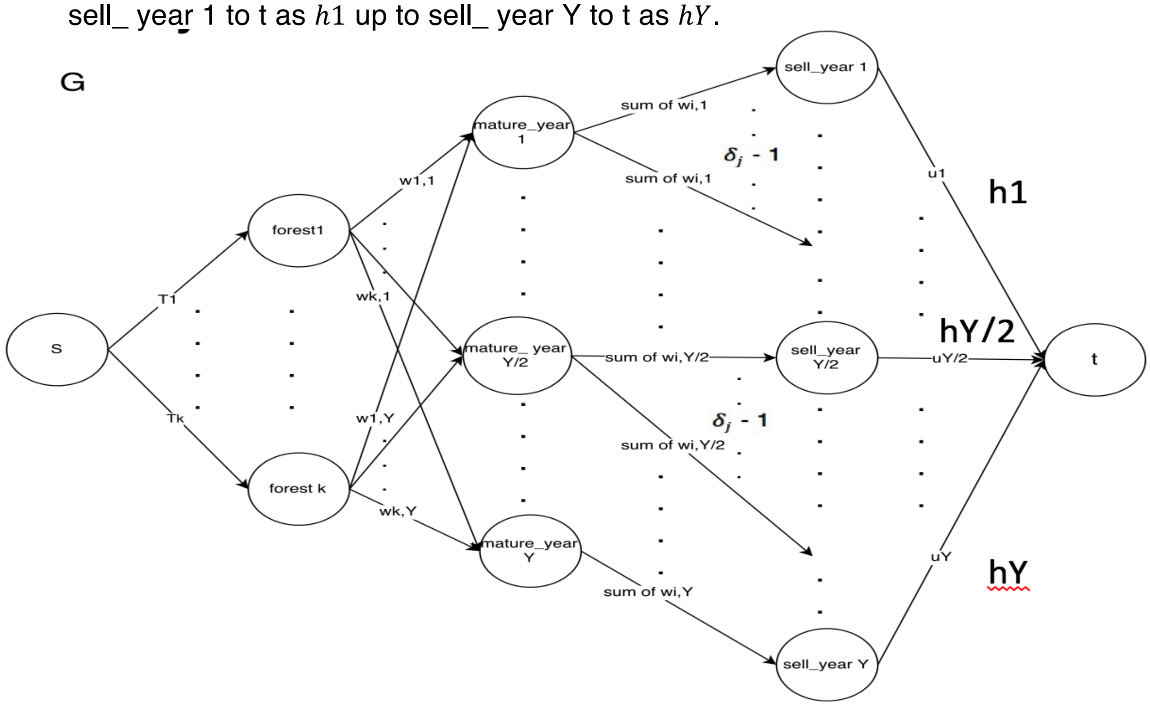
$$\textstyle\sum_{E2\ in\ to\ mature\_year\ j} d(E2) = \sum_{E3\ out\ of\ mature\_year j} y(E3)$$

Additionally, the capacity of the edge between mature_ year nodes to the sell_ year nodes would hold since:

$$0 \leq \{\mathrm{y1{,}1\ to\ y1{,}1} + \delta_1 - 1\}; \ldots; \{\mathrm{yj,j\ to\ yj,j} + \delta_\mathrm{j} - 1\}; \ldots; \{\mathrm{yY,Y\ to\ yY,Y} + \delta_Y - 1\} \leq \{sum\ of\ wi,1\ with\ \delta_j\ times\}; \ldots; \{sum\ of\ wi,j\ with\ \delta_j\ times\}; \ldots; \{sum\ of\ wi,Y\ with\ \delta_j\ times\}$$

5). The flow of the edges between sell_ year nodes to the sink (t) is represented as the number of the trees that be harvested in year j. we mark the flow of the edge from sell_ year 1 to t as $h1$ up to sell_ year Y to t as hY.

G



Since the assumption that max schedule is f. Hence the capacity of the edge between sell_ year nodes to t has to hold when:

$$0 \leq h1; \ldots; hY \leq u1; \ldots;\ uY.$$

In the flows conservation rule, the conservation between flow y and flow h should hold as well since:

$$\textstyle\sum_{E3\ in\ to\ sell\_year\ j} y(E3) = \sum_{E4\ out\ of\ sell\_year j} h(E4)$$

- **Part II:**

  (each flow's meaning and the signs between the different type of set-nodes has been defined in part I and those flows would still be used for part II.)

  Purpose: if assume f is the value of the max flow then we have to prove there exist a schedule that sells f Christmas trees as well.
  Hence, the capacities ($\tau i, w_{i,j}\ , sum\ of\ w_{i,j}\ in\ each\ year\ and\ uj$) of the edge (E1, E2, E3, E4) should hold ($0 \leq f(e) \leq c(e)$).

The conservation of the s-t flow for each set of nodes (F, Y_s, Y_m) should hold as well (v ∈ $V - \{s,t\}$: $\sum_{e\ in\ to\ v} f(e) = \sum_{e\ out\ of\ v} f(e)$)
Since there exists a max flow with value f, therefore, the sum of the flows of E4, which is $h1 + \cdots + hY$ has to be f as well. Additionally, the sum of the flows of E1, which is $fo1 + \ldots + fok$ also has to be f since the definition of the flows: $v(f) = \sum_{e\ out\ of\ s} f(e)$

- **Conclusion for the proof:**

In part I we have proved that there exists a max flow with value f via holding the capacity for the edges and the conservation for the nodes since we assume the max schedule that sells f number of Christmas trees. In part II, we prove there exists a max schedule that sells f trees through assuming the f value of max flow exists. Hence, our algorithm is correct since we have proved the algorithm's correctness in both directions.

c). Upper bound of the algorithm:

(The variables that have been defined in part a will be use for defining the upper bound of the time complexity for our algorithm).

We assume the max value of the flow is f, and m is the number of all the edges in the graph $num(E1 + E2 + E3 + E4)$.

- **The time complexity of Ford-Fulkerson O($\boldsymbol{num(E1 + E2 + E3 + E4)^2 log(f)}$):**

- **The time complexity of setting up the graph:**

1). We firstly start from setting up the edges connection between the source (s) and the set of forest nodes (F). Hence, we just need to run with one for- loop with the range of k forests. The upper bound time complexity of the connection between s and F would be O(k).

2). We set up the edges connection between the set of forests (F) and all of the mature_ years (Y_m). Since we have to set up each capacity ($w_{i,j}$) of the edge from each forest (1 to k) to all of the years (1 to Y). Hence, we have to do double for- loop with the outer range of k forests and the inner range of Y years. The upper bound time complexity of this operation would be O(kY).

3). We have to restore the capacity between the mature_ year (Y_m) and sell_ year (Y_s) since the capacity of the edge is the sum of the $w_{i,j}$ of all the forests in each mature-year (Y_m)), which goes to the available sell _year (Y_s). Therefore, we should use double for-loop with the outer range of Y years and the inner range of k forests. The upper bound time complexity for this set-up is O(kY).

4). We set up the edge connection between the mature_ years (Y_m) to the sell_ years(Y_s). Since the mature trees in each year sell to the available sell_ years depends on δ_j (each year j connects to sell_ year j to sell_ year $j + \delta_j - 1$), hence, we are using double for-loop with the outer range of Y years and the inner range of δ_j years. The time complexity for this operation should be O($\sum_{j=1}^{Y} \delta_j$) since each trees in the mature_ year can be cut down and sold out in year j and $\delta_j - 1$ years afterwards depends on the value of the δ_j

5). The last set up connection is between the sell_ years (Y_s) to the sink (t). This is done by simply using a for-loop with the range of Y years to collectively go to the sink. Hence, the upper bound time complexity of this part would be O(Y).

- **Overall:**

  We know {E1 + E2 + E3 + E4} set of edges include all the number of the edges in the graph, and we have declared the edge connection of each part in the graph. Hence, we convert from {E1 + E2 + E3 + E4} to {k + kY + $\sum_{j=1}^{Y} \delta_j$ + Y}. The Ford-Fulkerson's time complexity would be converted from O($num(E1 + E2 + E3 + E4)^2 log(f)$) to O($(\mathrm{k} + \mathrm{kY} + \sum_{j=1}^{Y} \delta_j + \mathrm{Y})^2 log(f)$) in our graph. ($f$ is the value of the maximum flow). The maximum value of δj can be infinite since the value can beyond Y years. Hence, we suppose to limit the δj's maximum value as Y years to prevent the algorithm might cause the extreme huge time complexity.

  Additionally, the max value of the flow would be the max schedule that trees sold, which is f. Since we know that the schedule that the trees sold would be the sum of the flows between the sell_ years and the sink from sell_ year 1 to sell_ year Y. Hence, the value max schedule would also be $\sum_{j=1}^{Y} u_j$. The f would also represent as $\sum_{j=1}^{Y} u_j$.

  The task is asking for the maximum flow through our set-up graph. Hence, we implement Ford- Fulkerson in our graph to find the value of the max flow. Additionally, we need to consider that the δj 's constraint, so the upper bound of the edge connection between mature_ year and sell_ year would be from $\sum_{j=1}^{Y} \delta_j$ to O(Y^2) (includes the part of Ford-Fulkerson's time complexity as well). As a result, the total time complexity of the whole operations is:

$$\mathrm{O}(\mathrm{k} + \mathrm{ckY} + \sum_{j=1}^{Y} \delta_j + \mathrm{Y}) + \mathrm{O}((\mathrm{k} + \mathrm{kY} + \sum_{j=1}^{Y} \delta_j + \mathrm{Y})^2 log(f))$$

Which equals to the upper bound of:

$$\mathrm{O}(\ (\mathrm{kY} + \mathrm{Y}^2) + (\mathrm{k} + \mathrm{kY} + \mathrm{Y}^2 + \mathrm{Y})^2 log(\sum_{j=1}^{Y} u_j)\ )$$