Mathematical Notes

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Groups

Group Axioms

A group G is a set G with a binary operation denoted by $\bullet$ which satisfies the following axioms:

1. **Identity**

   $\forall$ a $\in$ G, $\exists$ e $\in$ G, s.t. e $\bullet$ a = a $\bullet$ e = a

2. **Inverse**

   $\forall$ a $\in$ G, $\exists$ a^{-1} $\in$ G, s.t. a^{-1} $\bullet$ a = a $\bullet$ a^{-1} = e

3. **Associativity**

   $\forall$ a, b, c $\in$ G, a $\bullet$ (b $\bullet$ c) = (a $\bullet$ b) $\bullet$ c

4. **Closure**

   $\forall$ a, b $\in$ G, (a $\bullet$ b) $\in$ G

Order

The order of a group G is denoted $\|$ G $\|$ and gives the number of elements in G.
The order of an element g $\in$ G, denoted by n, is the smallest integer n such that g^n = e. It is trivial to see that ord(e) = 1 and if an element t $\in$ G is self inverse, ord(t) = 2.

Proof that the order of g & g^{-1} are the same

Let n_1 be s.t. g^{n_1} = e and n_1 is the smallest such integer.

$$\begin{aligned}
&(g^{-1}g) &&= e \\
\Longrightarrow\ &(g^{-1}g)^{n_1} &&= e^{n_1} = e \\
\Longrightarrow\ &(g^{-1})^{n_1}g^{n_1} &&= e \\
\Longrightarrow\ &(g^{-1})^{n_1} &&= e
\end{aligned}$$

Here n_1 is a multiple of the smallest such integer for g^{-1}. Similarly we can define the smallest integer n_2 which satisfies $(g^{-1})^{n_2} = e$. By symmetry we get n_1 and n_2 being multiples of each other. $\therefore n_1 = n_2$

Exercises

1. Proof of the uniqueness of the identity and the inverse.

2. Prove that $\forall$ a, b $\in G$ $ord(ba) = ord(ab)$.

Types of Groups

Abelian Groups

A group G is abelian if it follows the group axioms and also has a commutativity axiom. Where $\forall$ a, b $\in G$ $a \bullet b = b \bullet a$.

Cyclic Group

For a cyclic group G of order n, all emements of the group can be generated from one element of the group, e.g. X.

$$G = \{I, X, X^2, X^3, ..., X^{n-1}\}$$

A cyclic group is abelian and all elements of G, apart from I have order n, the same as the ord(G) - the number of elements in group G.

Subgroup

A group H is a subgroup of G, denoted by $H \leq G$, **iff** the set H is a subset of the set G, denoted by $H \subseteq G$, and follows the group axioms. This can be summarised by the following, where the group H is a non empty set with a binary operator $\bullet$:

$$\forall a, b \in H, (ab^{-1}) \in H$$

The logic follows as such:

$$\begin{aligned}
& a \in H \\
& aa^{-1} \in H \implies e \in H \text{ (By substitution of b for a- identity axiom)} \\
& ea^{-1} \in H \implies a^{-1} \in H \text{(Inverse Axiom)} \\
& b \in H \qquad \implies b^{-1} \in H \therefore a(b^{-1})^{-1} \in H \implies ab \in H \text{(Closure)}
\end{aligned}$$

List of Common DEs

First Order ODEs

Separable:

$\frac{dy}{dx} = f(x)g(y)$

If you can't do this, just give up.

Exact:

$A(x, y)dx + B(x, y)dy = 0$ and $\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$

This follows from the symmetry of mixed partial derivatives.
To find the function U which satisfies $\frac{\partial U}{\partial x} = A$ and $\frac{\partial U}{\partial y} = B$ we can take the union of the generalised functions solved by integrating. Remember due to the partial derivatives we get some functions of x or y instead of constants of integration.

Inexact:

$A(x, y)dx + B(x, y)dy = 0$ and $\frac{\partial A}{\partial y} \neq \frac{\partial B}{\partial x}$