The Malliavin derivative, part 2

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1 The Malliavin derivative as a Fréchet derivative

Let $C_0([0,T])$ be the Banach space of all continuous functions $f\colon [0,T] \to \mathbb{R}$ such that $f(0) = 0$. On this space, we can associate a special Borel probability measure μ such that $W_t(\omega) := \omega(t)$ is a Brownian motion. Given a random variable on $C_0([0,T])$, i.e. a μ-measurable function $X\colon C_0([0,T]) \to \mathbb{R}$, we want to know how the value $X(\omega)$ changes upon perturbing the path ω by a small quantity $\gamma \in C_0([0,T])$. This can be described by the Fréchet derivative $\nabla X(\omega)$, which is a bounded linear map $C_0([0,T]) \to \mathbb{R}$, i.e. a member of the dual space $C_0([0,T])^*$, giving the best linear approximation to the difference $X(\omega+\gamma) - X(\omega)$. Formally, ∇X satisfies

$$X(\omega+\gamma) = X(\omega) + \langle \nabla X(\omega), \gamma \rangle + \mathrm{o}(\|\gamma\|_{C_0([0,T])}).$$

If X has a Fréchet derivative $\nabla X\colon C_0([0,T]) \to C_0([0,T])^*$, we say it is *Fréchet differentiable*.

Within the Banach space $C_0([0,T])$ lies a Hilbert space H of distinguished elements. This is the space of paths of the form

$$\gamma(t) = \int_0^t \psi(s)\, \mathrm{d}s$$

for some $\psi \in L^2([0,T])$. In other words, it is the space of $W^{1,2}$ functions on $[0,T]$ starting at 0. Its inner product is given by

$$(\gamma_1, \gamma_2)_H = (\dot{\gamma}_1, \dot{\gamma}_2)_{L^2([0,T])}.$$

H is continuously imbedded in C_0 by the theory of Sobolev spaces. We call H the *Cameron-Martin space*. It acts in some sense as the heart of $C_0([0,T])$ with the probability measure μ, with its elements having better analytical properties compared to a general element in $C_0([0,T])$. Ideally, we could restrict μ to this space and only work here, but unfortunately μ is not a measure on H (in particular, it is not σ-additive), forcing us to work in a larger Banach space.

Returning to our Fréchet differentiable random variable X, given some path $\omega \in C_0([0,T])$, we consider the restriction of its derivative at ω to H, namely $\nabla X(\omega)|_H$. Since H is continously imbedded in C_0, this restriction is an element of H^*. Then, as H is a Hilbert space, the dual space H^* is isomorphic to H through its inner product, so there exists some $DX(\omega) \in L^2([0,T])$ such that

$$\left\langle \nabla X(\omega), \int_0^\cdot \psi\, \mathrm{d}t \right\rangle = \left(\int_0^\cdot D_t X(\omega)\, \mathrm{d}t, \int_0^\cdot \psi\, \mathrm{d}t \right)_H = \int_0^T D_t X(\omega) \psi(t)\, \mathrm{d}t.$$

This object DX is precisely the Malliavin derivative of X.

2 The Skorokhod integral and the Malliavin derivative

Given a stochastic process $(X_t)_{t\in[0,T]} \in L^2([0,T] \times \Omega, \mathrm{d}t \otimes \mathbb{P})$ such that X_t is $\mathscr{F}_T$-measurable for all $t \in [0,T]$, let

$$X_t = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t))$$

be its chaos expansion for some $f_n \in L^2([0,T]^{n+1})$ symmetric in the first n variables. Recall we say X is *Skorokhod integrable*, and define its Skorokhod integral by

$$\int_0^T X_t\,\delta W_t := \sum_{n=0}^{\infty} I_{n+1}(f_{n,S}),$$

whenever this sum converges in $L^2(\Omega)$. The following result is fundamental.

1 Theorem. *The Skorokhod integral and Malliavin derivative are* adjoint *in the following sense:*

Let $(X_t)_{t\in[0,T]}$ be a Skorokhod-integrable. Let $Y \in \mathbb{D}^{1,2}$ be a Malliavin differentiable random variable. Then

$$\left(Y, \int_0^T X_t\,\delta W_t\right)_{L^2(\Omega)} = (DY, X)_{L^2([0,T]\times\Omega)}.$$

More concretely,

$$\mathbb{E}\left[Y \int_0^T X_t\,\delta W_t\right] = \mathbb{E}\left[\int_0^T D_t Y X_t\,\mathrm{d}t\right].$$

Proof. As usual, we apply the definitions in terms of the chaos expansions. Let

$$X_t = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t))$$

be the chaos expansion of X, and

$$Y = \sum_{n=0}^{\infty} I_n(g_n)$$

the chaos expansion of Y. Then

$$\begin{aligned}
\mathbb{E}\left[Y \int_0^T X_t\,\delta W_t\right] &= \mathbb{E}\left[\sum_{n=0}^{\infty} I_n(g_n) \sum_{m=0}^{\infty} I_{m+1}(f_{m,S})\right] \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \mathbb{E}[I_n(g_n) I_{m+1}(f_{m,S})] \\
&= \sum_{n=0}^{\infty} n!(g_n, f_{n-1,S})_{L^2([0,T]^n)},
\end{aligned}$$

and on the other side,

$$\begin{aligned}
\mathbb{E}\left[\int_0^T D_t Y X_t\,\mathrm{d}t\right] &= \int_0^T \mathbb{E}\left[\sum_{n=1}^{\infty} n I_{n-1}(g_n(\cdot,t)) \sum_{m=0}^{\infty} I_m(f_m(\cdot,t))\right]\mathrm{d}t \\
&= \int_0^T \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} n\mathbb{E}[I_{n-1}(g_n(\cdot,t)) I_m(f_m(\cdot,t))]\,\mathrm{d}t \\
&= \sum_{n=0}^{\infty} n(n-1)! \int_0^T (g_n(\cdot,t), f_{n-1}(\cdot,t))_{L^2([0,T]^{n-1})}\,\mathrm{d}t \\
&= \sum_{n=0}^{\infty} n!(g_n, f_{n-1})_{L^2([0,T]^n)}.
\end{aligned}$$

Finally, by definition of the symmetrization,

$$\begin{aligned}(g_n, f_{n-1,S})_{L^2([0,T]^n)} &= \int_{[0,T]^n} g_n(t_1,\dots,t_n)\frac{1}{n}\sum_{k=1}^{n} f(t_1,\dots,t_{k-1},t_n,t_{k+1},\dots,t_{n-1},t_k)\,\mathrm{d}t_1\cdots\mathrm{d}t_n \\ &= \frac{1}{n}\sum_{k=0}^{n}\int_{[0,T]^n} g_n(t_1,\dots,t_n)f_{n-1}(t_1,\dots,t_n)\,\mathrm{d}t_1\cdots\mathrm{d}t_n \\ &= (g_n, f_{n-1})_{L^2([0,T]^n)},\end{aligned}$$

where we change variables $t_k \mapsto t_n, t_n \mapsto t_k$, use the property that g_n is symmetric, and apply Fubini's theorem. □

2 Remark. The symbol δ is often used for a divergence-like operator in Hodge theory. The analogy with our case is that in the Hodge situation, δ is defined via a duality formula which looks like $\langle d\alpha, \beta\rangle = \langle \alpha, \delta\beta\rangle$, where d is the exterior derivative on differential forms. Indeed, even in vector calculus, the negative of the divergence is in some sense adjoint to the gradient:

$$\int_\Omega \operatorname{div} f\phi\,\mathrm{d}x = -\int_\Omega f\cdot\nabla\phi\,\mathrm{d}x$$

whenever ϕ has zero boundary. So, in a sense, the Skorokhod integral is just a divergence operator.

Using this, we can immediately prove the following:

3 Corollary. *Let $(X^n)_{n\in\mathbb{N}}$ be a sequence of Skorokhod-integrable stochastic processes. Suppose there exist $X \in L^2([0,T]\times\Omega)$ and $Y \in L^2(\Omega)$ such that $X^N \to X$ in $L^2([0,T]\times\Omega)$, and $\delta X^N \to Y$ in $L^2(\Omega)$. Then X is Skorokhod integrable, and $\delta X^N \to \delta X$.*

Proof. Recall that Skorokhod integrability of X can be expressed in terms of convergence of the series

$$\sum_{n=0}^{\infty}(n+1)!\|f_{n,S}\|^2_{L^2([0,T]^{n+1})},$$

where $f_n(\cdot,t)$ are the components of the chaos expansion of X. Since $X^N \to X$ strongly in L^2, and each X^N is Skorokohod integrable, the components of their chaos expansions must satisfy the above condition, and we can take limits.

Let $Z \in \mathbb{D}^{1,2}$. Then by adjointness,

$$(Z, \delta X^N)_{L^2(\Omega)} = (DZ, X^N)_{L^2([0,T]\times\Omega)}.$$

Taking limits on both sides and using adjointness on the limiting objects gives us

$$(Z,Y)_{L^2(\Omega)} = (DZ,X)_{L^2([0,T]\times\Omega)} = (Z,\delta X)_{L^2(\Omega)}.$$

Then, since $\mathbb{D}^{1,2}$ is dense in $L^2(\Omega)$, we see that $Y = \delta X$ a.s., as required. □

4 Remark. Perhaps a more intuitive way to say the Skorokhod integral is "closable" in the book's words is that it is sequentially continuous as a map $D(\delta) \subseteq L^2([0,T]\times\Omega) \to L^2(\Omega)$ with respect to the strong L^2 topology in its domain and weak L^2 topology in its codomain.

5 Theorem. *Let $X \in L^2([0,T]\times\Omega)$ be a Skorokhod integrable random process, and let $Y \in \mathbb{D}^{1,2}$ be such that FX is also Skorokhod integrable. Then*

$$Y \int_0^T X_t \,\delta W_t = \int_0^T Y X_t \,\delta W_t + \int_0^T D_t Y X_t \,\mathrm{d}t$$

almost surely.

Proof. Suppose Y has finite chaos expansion, and choose some $Z \in \mathbb{D}^{1,2}$ also with finite chaos expansion. Then by adjointness and the product rule,

$$\begin{aligned}
\mathbb{E}\left[Z \int_0^T Y X_t \,\delta W_t\right] &= \mathbb{E}\left[\int_0^T D_t Z Y X_t \,\mathrm{d}t\right] \\
&= \mathbb{E}\left[\int_0^T (D_t(YZ) - Z D_t Y) X_t \,\mathrm{d}t\right] \\
&= \mathbb{E}\left[YZ \int_0^T X_t \,\delta W_t\right] - \mathbb{E}\left[Z \int_0^T D_t Y X_t \,\mathrm{d}t\right].
\end{aligned}$$

Since the set of all test functions $Z \in \mathbb{D}^{1,2}$ with finite chaos expansion is dense in $L^2(\Omega)$, we conclude the result for Y with finite chaos expansion. For general Y, we approximate. □

6 Remark. A similar formula crops up in vector calculus, namely the following:

$$\operatorname{div}(fX) = \nabla f \cdot X + f \operatorname{div} X,$$

where f is a scalar function and X a vector field. Again, in the above theorem, the Malliavin derivative takes the place of the gradient, the Skorokhod integral take the place of the divergence, and the usual inner product on $\mathbb{R}^n$ (the dot product) is replaced with the $L^2([0,T])$ inner product. There is a sign difference owing to the fact the adjointness in the Malliavin case does not induce a sign change, unlike in the vector calculus case (see the remark above).

7 Theorem. *Let $X \in L^2([0,T]\times\Omega)$ be a stochastic process such that for all $s \in [0,T]$, X_s is in $\mathbb{D}^{1,2}$, DX_s is Skorokhod integrable, and*

$$\int_0^T DX_s \,\delta W_s \in L^2([0,T]\times\Omega).$$

Then δX lies in $\mathbb{D}^{1,2}$, and

$$D_t(\delta X) = \int_0^T D_t X_s \,\delta W_s + X_t.$$

8 Remark. The technical constraints in the theorem above are an unfortunate consequence of the fact the Skorokhod and Malliavin operators δ and D are not defined on the full space $L^2([0,T]\times\Omega)$ and $L^2(\Omega)$ respectively - we have to ensure an operator throws us to the right spot before we can consider applying the other one.

Note that this theorem is simply an expression of the Malliavin derivative and Skorokhod integral's failure to commute, with the error simply being the identity on $L^2(\Omega\times[0,1])$. That is,

$$D\delta = \delta D + \mathrm{id}.$$

This contrasts with our vector calculus analogy, where the divergence and gradient most certainly commute (assuming enough regularity).