Gelfand's theorems

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1 Three line summary

- The spectrum can be thought of heuristically as the frequencies that represent an element of a Banach algebra.
- In a unital Banach algebra every element has non-empty spectrum.
- A commutative Banach algebra A can be thought of as a subgroup of continuous functions.

2 Why should I care?

We introduce the basis of spectral theory which is useful to ...

3 Preliminary definitions

Here I include all the needed definitions. Many of them will be familiar if the reader has an algebra background and can be skimmed over.

Definition 1. *An* algebra A *is a vector space together with a multiplication which is bilinear and associative. That is,*

$$(a+b)c = ac + bc; \quad a(b+c) = ab + ac; \quad a(bc) = (ab)c \forall a, b, c \in A.$$

Note that in general we do not require that the product is commutative. Though this will be a common requirement later on. We will take A for the remainder to be a vectro space over $\mathbb{C}$

Definition 2. *Given a norm* $\|\cdot\|$ *on algebra* A *we say that* $(A, \|\|)$ *is a* normed algebra *if the norm is* submultiplicative*. That is,*

$$\|ab\| \leq \|a\|\|b\|.$$

Definition 3. *We say that a normed algebra* $(A, \|\cdot\|)$ *is a* Banach algebra *if it is complete. We say that it is* unital *if there exists an identity* 1 *for the multiplication*

$$a1 = 1a, \quad \forall a \in A. \tag{1}$$

and we say that it is commutative *if*

$$ab = ba, \quad \forall a, b \in A.$$

Definition 4. *Given* $a \in A$ *we say that* $\lambda \in \mathbb{C}$ *is in the* spectrum *of* a *if* $\lambda 1 - a$ *is* ***not*** *invertible. We write* $\sigma(a) \subset \mathbb{C}$ *for the set of such* λ*.*

Definition 5. *Given two algebras* A, B *we say that* $\varphi : A \to B$ *is a* homomorphism *if it is linear and respects the product. That is,*

$$\varphi(a + \lambda b) = \varphi(a) + \lambda\varphi(b); \quad \varphi(ab) = \varphi(a)\varphi(b), \quad \forall a, b \in A.$$

If A, B *are unital and* $\varphi(1) = 1$ *we say that* φ *is* unital*. If* φ *has an inverse* φ *and* φ *and* φ^{-1} *are continuous we say that it is a* homeomorphism

Note that any morphism φ between unital algebras conserves invertability. However that, even if A, B are unital it may be the case that $\varphi(1)$ is not (consider for example the zero morphism).

Definition 6. *We say that* φ *is a character of* A *if* $\varphi : A \to \mathbb{C}$ *is a homomorphism.*

Definition 7. *Given an algebra* A *we say that* $B \subset A$ *is a sub-algebra if it is subspace of* A *closed under multiplication. That is* B *itself is an algebra.*

Definition 8. *Given an algebra* A*, we say that* $I \subset A$ *is an* ideal *of* A *if* I *is stable under multiplication with elements is* A*. That is*

$$AI := \{ab : a \in A, \quad b \in I\} \subset I; \quad IA \subset I.$$

We say that I *is* maximal *if given any other ideal* J *such that* $I \subset J$ *it holds that* $J = A$*.*

We recall that, by Zorn's Lemma every algebra has a maximal ideal.

Definition 9. *Given a closed ideal I of an algebra A we define the quotient algebra A/I to be the set of equivalence classes under the relation*

$$a \sim b \iff a - b \in I.$$

With the product $\overline{a}\overline{b} := \overline{ab}$ and the norm

$$\|\overline{a}\| := \inf_{b \in I} \|a + b\|.$$

We note that it is necessary for I to be closed so that if $\|\overline{a}\| = 0$ then $a \in I$, that is $\overline{a} = 0$.

4 Introduction

5 The big theorems

We now state the main theorems and outline the proof

Theorem 1. *Given a unital Banach algebra A it holds that*

1. *The set of invertible elements in A is open.*
2. *Taking the inverse is smooth.*
3. *$\sigma(a)$ is a closed subset of $\subset B(0, \|a\|)$*

Proof. All of these facts can be proved via the Von-Neumman series for the inverse

$$(1 - a)^{-1} = \sum_{n=0}^{\infty} a^n, \quad \forall a \in B(0, 1).$$

□

Theorem 2 (Gelfand's theorem)**.** *Given a unital Banach algebra A, $\sigma(a) \neq \emptyset$ for all $a \in A$.*

Proof. Suppose not, then $(\lambda 1 - a)^{-1}$ exists for all $\lambda \in \mathbb{C}$. Let us consider the function $f : \mathbb{C} \to \mathbb{C}$

$$f(\lambda) := (\lambda 1 - a)^{-1}. \tag{2}$$

Since taking inverse is smooth, f is smooth and thus bounded on $B(0, 2\|a\|)$. Furthermore, using Neumann's series for the inverse we obtain that

$$\|f(\lambda)\| = \left|\lambda^{-1}\right| \|\sum_{n=0}^{\infty}(a/\lambda)^n\| \leq |\lambda|^{-1}\sum_{n=0}^{\infty}(\|a\|/\lambda)^n \leq \frac{1}{|\lambda|} \leq \frac{1}{2\|a\|}.$$

As a result f is a bounded smooth function and $\ell(f(\lambda))$ is a bounded entire function for any $\ell \in A^*$. By Liouville's theorem $\ell(f(\lambda))$ is constant. Since this holds for all $\ell \in A^*$ we deduce that $f(\lambda)$ is constant. This leads to a contradiction by taking $\lambda = 1, \lambda = 0$ as we get

$$I - a = a. \tag{3}$$

Citing (2) and (3) and referencing [1]. □

We now discuss Gelfand's representation theorem. First some lemmas

Lemma 1 (**Codimension of a maximal ideal is 1**)**.** *Let I be a maximal ideal in a commutative unital Banach Algebra A, then $A \simeq I + \mathbb{C} \cdot 1$.*

Proof. We must show that any $a \in A$ can be written $b + \lambda$ where $b \in I$. Given $\overline{a} \in A/I$ there exists by Gelfand's theorem $\lambda \in \mathbb{C}$ such that $\overline{\lambda - a}$ is not invertible. Since A/I is a field we deduce that $\overline{\lambda - a} = 0$, that is there exists $b \in I$ such that $b + \lambda = a$ as desired. □

Proposition 1. *Let A be a commutative Banach algebra.There is a bijective correspondence*

$$\begin{aligned} \Omega(A) :\simeq \{\textit{Maximal ideals } I \subset A\}. \\ \varphi \longmapsto \textit{ker}(\varphi). \end{aligned}$$

Proof. The mapping is well defined as $a - \varphi(a) \in \ker(\varphi)$ for all $a \in A$ so $\ker(\varphi)$ is maximal. The mapping is surjective by the previous lemma as given a maximal ideal I we have that $A = I \oplus \mathbb{C}$ and we can define $\varphi((a, \lambda)) := \lambda$. The mapping is injective as if $\ker\varphi_1 = \ker\varphi_2$ then $\varphi_1(a - \varphi_2(a)) = 0$ for all $a \in A$. □

Proposition 2. *Let A be a commutative Banach algebra. Then $\|\varphi\| = 1$ for every $\varphi \in A$.*

Proof. We have that $\varphi(a) \subset \sigma(a) \subset B(0, \|a\|)$ as a result $\|\varphi\| \leq 1$. On the other hand $\varphi(1)^2 = 1$ proves that $\varphi(1) = 1$ (we recall that characters by definition are non-zero). This shows that $\|\varphi\| \geq 1$ and concludes the proof. $\square$

Theorem 3. *Given a commutative unital Banach algebra A it holds that*

$$\sigma(a) = \{\varphi(a) : \varphi \in \Omega(A)\}.$$

If A is not unital then

$$\sigma(a) = \{\varphi(a) : \varphi \in \Omega(A)\} \cup \{0\}.$$

Proof. To prove the direct inclusion consider $\lambda \notin \sigma(a)$, then we may take by Zorn's lemma a proper maximal ideal I with $\lambda - a \in I$. By the previous proposition we can take $I = \ker\varphi$ for some character φ, from which we deduce that $\varphi(\lambda - a) = \lambda - \varphi(a) = 0$.
The reverse inclusion is immediate from the fact that $\varphi(a - \varphi(a)) = 0$ so $a - \varphi(a)$ is not invertible for any $a \in A$.
Suppose now A is not unital, then we unitize it by considering $\tilde{A}$ as previously. The characters of $\tilde{A}$ are $\Omega(A) \cup \{\tau_0\}$ where $\tau_0((a, \lambda)) := \lambda$. Applying the just prove result to $\tilde{A}$ shows that, since $\tau_0(a, 0) = 0$

$$\sigma(a) := \sigma(\tilde{a}) = \{\varphi(a) : \varphi \in \Omega(A)\} \cup \{0\}.$$

$\square$

We now consider A^* with the weak* topology and $\Omega(A)$ included within it. We recall that A^* is Hausdorff and thus so must be $\Omega(A)$.

Theorem 4. *Let A be an Abelian Banach algebra. Then $\Omega(A)$ is locally compact. Furthermore, if A is also unital then $\Omega(A)$ is compact.*

Proof. By the previous theorem we know that $\Omega(A) \subset B(0, 1)$ whihc is compact with the weak star topology (Banach Aloglu's theorem). Now, if $\varphi_n(a) \to \varphi(a)$ for all $a \in A$ then it holds that, by a simple argument φ is also a morphism. Thus φ will be a character if and only if $\varphi \neq 0$. For this reason it is necessary to adjoin 0 if A is not unital. This shows that

$\Omega(A) \cup \{0\}$ is closed in $B(0,1)$ and is thus compact. In consequence $\Omega(A)$ is locally compact.
Suppose now that A is unital, then we have seen than $\|\varphi\| = 1$ for all $\varphi \in \Omega(A)$. The previous argument shows that $\Omega(A)$ is closed (we can never reach 0) and as a result $\Omega(A)$ is compact. □

We can now identify A with a sub-algebra of continuous functions on $\Omega(A)$ via the mapping $a \to \widehat{a}$ where we define $\widehat{a}(\varphi) := \varphi(a)$. Let us denote the image of this mapping by $\widehat{A}$, then we obtain the following theorem

Theorem 5 (**Gelfand's represntation theorem**)**.** *Let A be a commutative Banach algebra, then the mapping*

$$A \to \widehat{A} \subset C_0(\Omega(A)); \quad a \to \widehat{a}.$$

is a norm decreasing injective homomorphism with $\|\widehat{a}\| = r(a)$. Furhtermore, if a is unital then $\sigma(a) = \widehat{a}(\Omega(A))$ and otherwise $\sigma(a) = \widehat{a}(\Omega(A)) \cup \{0\}$.

Proof. To show that $\widehat{a}$ vanishes at infinity note that the set

$$\{\varphi : \widehat{a}(\varphi) \geq \epsilon\} = \widehat{a}^{-1}([\epsilon, \|a\|]).$$

Which is a closed subset in the compact $B(0, a)$ and thus compact. □

We note however that the mapping $a \to \widehat{a}$ is in general neither injective (this means the identification isn't prefect) not surjective.

Example 1 (Continuous Fourier Transform)**.** *Consider the Banach Algebra $A = L^1(\mathbb{R}^d)$ with multiplication given by the convolution*

$$f * g(x) := \int_{\mathbb{R}^d} f(y)g(x-y)\,\mathrm{d}y.$$

Then the Gelfand transform can be identified with the Fourier transform and

$$\widehat{f}(\Omega) = \left\{ \widehat{f}(\omega) := \int_{\mathbb{R}^d} f(x)e^{-2\pi i\omega\cdot x}\,\mathrm{d}x : \omega \in \mathbb{R}^d \right\}.$$

Proof. We know that the dual of $L^1(\mathbb{R}^d)$ is $L^\infty(\mathbb{R}^d)$. In particular every character is of the form

$$\varphi_g(f) = \int_{\mathbb{R}^d} f\overline{g}.$$

The condition that φ_g preserves multiplication becomes by a change of variable

$$g(x+y)=g(x)g(y), \quad \forall x,y\in\mathbb{R}^d.$$

The only $g\in L^\infty(\mathbb{R}^d)$ verifying the above are of the form $g(x)=e^{2\pi i\omega\cdot x}$ for some $\omega\in\mathbb{R}^d$. That is, the character group is

$$\Omega(L^1(\mathbb{R}^d))\simeq\{e^{2\pi i\omega\cdot}:\omega\in\mathbb{R}^d\}.$$

This conclues the proof (we use the abuse of notation $\widehat{f}(\omega):=\widehat{f}(\varphi_{e^{2\pi i\omega\cdot}})$). □

Example 2 (Discrete Fourier Transform). *Consider the Banach Algebra* $A=\ell^\infty(\mathbb{Z}^d)$ *with multiplication given by the convolution*

$$g*h(k):=\sum_{k\in\mathbb{Z}^d}g(j)h(k-j).$$

Then the Gelfand transform can be identified with the (inverse) Fourier transform and

$$\widehat{g}(\Omega)=\left\{\check{g}(k):=\sum_{k\in\mathbb{Z}^d}g(x)e^{2\pi ik\cdot x}\,\mathrm{d}x:k\in\mathbb{Z}^d\right\}.$$

Furthermore

$$g(k)=\int_0^1\check{g}(k)e^{2\pi i(j-k)\cdot x}\,\mathrm{d}x.$$

Proof. Let $e_1,\dots e_d\in A$ be defined by

$$e_n(k):=\delta_{nk_1}, \quad \forall n=1,\dots d, k\in\mathbb{Z}^d.$$

Then we have that A is generated by $e_1,\dots e_n$ with

$$g(k)=\sum_k g(k)e_1^{k_1}\dots e_d^{k_d}.$$

Any character φ is thus determined by where it sends the e_n and they take the form

$$\Omega(A)=\left\{\varphi_z(g):=\sum_{k\in\mathbb{Z}^d}g(k)z^k:z\in\mathbb{T}^d\right\}\simeq\mathbb{T}^d.$$

Noting that $\mathbb{T}^d = \left\{e^{2\pi i x} : x \in \mathbb{R}^d\right\}$ and defining

$$\check{g}(x) := \varphi_{e^{2\pi i x \cdot}}(g) = \sum_{k\in\mathbb{Z}^d} g(k)e^{2\pi i k\cdot x}.$$

Gives the first part of the proof and using the orthonormality trick

$$\int_0^1 \check{g}(x)e^{-2\pi i k\cdot x}\,\mathrm{d}x = \sum_{j\in\mathbb{Z}^d} g(j)\int_0^1 e^{2\pi i(j-k)\cdot x}\,\mathrm{d}x = g(k).$$

Gives the second. Where in the above we applied the dominated convergence theorem. $\square$

References

[1] G. J. Murphy, C*-algebras and operator theory, Academic press, 2014.

[2] R. J. Adler, J. E. Taylor, et al., Random fields and geometry, Vol. 80, Springer, 2007.
URL https://www.dcs.warwick.ac.uk/~feng/papers/Random_Fields_and_Geometry_%5BAdler%5D.pdf