Elliptic PDE I

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1 Three line summary

- Elliptic partial differential equations (PDE) are PDE with no time variable and whose leading order derivatives satisfy a positivity condition.
- Using Lax Milgram's theorem we can prove existence and uniqueness of weak (distributional) solutions if the transport term is zero.
- If the transport term is non-zero solutions still exist but they are no longer unique and are determined by the kernel of the homogeneous problem.

2 Why should I care?

Many problems arising in physics such as the Laplace and Poisson equation are elliptic PDE. Furthermore, the tools used to analyze them can be extrapolated to other settings such as elliptic PDE. The analysis also helps contextualize and provide motivation for theoretical tools such as Hilbert spaces, compact operators and Fredholm operators.

3 Notation

Given $\alpha \in \mathbb{C}$ we will write $\overline{\alpha}$ for the conjugate of α. Given a subset S of some topological space it is also common to write $\overline{S}$ for the closure of S. Though this is a slight abuse of notation we will do the same as the meaning will always be clear from context.

Given two topological vector spaces X, Y we write $\mathcal{L}(X, Y)$ for the space of continuous linear operators from X to Y.

We will use the Einstein convention that indices when they are repeated are summed over. For example we will write

$$\nabla \cdot (\mathbf{A}\nabla) = \sum_{i=1}^{n} \partial_i a_{ij} \partial_j = \partial_i A_{ij} \partial_j.$$

Furthermore, we will fix $U \subset \mathbb{R}^n$ to be an open **bounded** (this will be necessary to apply the Poincaré inequality) set in $\mathbb{R}^n$ with **no conditions** on the regularity of ∂U. If we need to impose regularity on the boundary we will write Ω instead of U.

We will use Vinogradov notation.

4 Introduction

Welcome back to the second post on our series of PDE. In the first post of the series we dealt with the Fourier transform and it's application to defining spaces of weak derivatives and weak solutions to PDE. In this post we will consider an equation of the form

$$\mathcal{L}u = f; \quad u|_{\partial U} = 0. \tag{1}$$

Where $\mathcal{L}$ is some differential operator, $f : U \to \mathbb{R}$ is some known function and u is the solution we want to find. We will need the following condition on $\mathbf{A}$ to prove *well-posedness* of (1).

Definition 4.1. *We say that an equation is* well-posed *if*

1. *It has a solution.*
2. *The solution is unique.*
3. *The solution depends continuously on the data.*

Definition 4.2. *Given* $\mathbf{A} : U \to \mathbb{R}^{d\times d}, \mathbf{b} : U \to \mathbb{R}^d$ *and* $c : U \to \mathbb{R}$ *we say that the differential operator*

$$\mathcal{L}u := -\nabla \cdot (\mathbf{A}\nabla u) + \nabla \cdot (\mathbf{b}u) + c \tag{2}$$

is elliptic *if there exists* $\alpha > 0$ *such that*

$$\xi^T \mathbf{A}(x)\xi \geq \alpha |\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \quad \forall x \in U. \tag{3}$$

There are some points to clear up. Firstly, if this is the first time you've encountered the ellipticity condition in (3) then it may seem a bit strange. Physically speaking, in a typical derivation of our PDE in (1), u is the density of some substance and $\mathbf{A}$ corresponds to a diffusion matrix. Due to the ellipticity condition (3) says that flow occurs from the region of higher to lower density. Mathematically speaking (3) will provide the necessary bound we need to apply Lax Milgram's theorem. We have not yet defined which function space our coefficients live in and what $\mathcal{L}$ acts on. It would be natural to assume that we need for $\mathbf{A}, \mathbf{b}$ to be differentiable. However, the following will suffice.

Assumption 1. We assume that $A_{ij}, b_i, c \in L^\infty(U)$ for all $i, j = 1, \ldots, d$. Furthermore, A is symmetric, that is $A_{ij} = A_{ji}$.

To simplify the notation we will write for the bound on $\mathbf{A}, \mathbf{b}, c$

$$\|\mathbf{A}\|_{L^\infty(U)} + \|\mathbf{b}\|_{L^\infty(U)} + \|c\|_{L^\infty(U)} = M$$

The first assumption will make it easy to get bounds on $\mathcal{L}$ and the second will be necessary to apply Lax Milgram's theorem and the third will prove useful when we look at the spectral theory of $\mathcal{L}$. Now, to make sense of (1) we need to define what we mean by a solution. Here the theory of Sobolev Spaces and the Fourier transform prove crucial. We will work with the following space

Definition 4.3 (Negative Sobolev space)**.** *Given* $k \in \mathbb{N}$ *we define*

$$H^{-k}(U) := H^k_0(U)'$$

For more details on why this notation is used see Theorem B.1 in the appendix.

Exercise 1. Suppose $A_{ij} \in C^{s+1}(\overline{U}), b_i, c \in C^s(\overline{U})$ for some $s \in \mathbb{R}$. Then, $\mathcal{L}$ defines a bounded linear operator

$$\mathcal{L} : W_0^{s+2,p}(U) \to W^{s,p}(U).$$

Hint. Use the chain rule to write $\mathcal{L}u$ in the form

$$\mathcal{L}u = \sum_{|\alpha|\le 2} g_\alpha D^\alpha, \quad g_\alpha \in C^s(U).$$

Using the chain rule again, to show that, for $k = \lfloor s \rfloor$ and $\gamma = s - k$

$$\sum_{|\alpha|\le k} \mathcal{L}u = \sum_{|\alpha|\le k+2} h_\alpha D^\alpha u, \quad h_\alpha \in C^\gamma(U),$$

and conclude by the definition of $W^{s,p}(U)$.

In the particular case $s = 1$, we have that $\mathcal{L}$ maps $H_0^1(U)$ to its dual $H^{-1}(U)$. This allows us to define the weak formulation of (1) and study its well-posedness using Lax Milgram's theorem. We will do this in the next section.

5 Weak solutions and well posedness

We can make sense of the equation $\mathcal{L}u = f$ in a distributional (weak) sense as follows. By an integration by parts, if $u, v \in C_0^\infty(U)$ then

$$\int_U \mathcal{L}uv = \int_U \mathbf{A}\nabla u \cdot \nabla v + \int_U \mathbf{b} \cdot (\nabla u)v + \int_U cuv =: B(u,v).$$

It is clear that B is bilinear in an algebraic sense. Furthermore from Cauchy Schwartz and the fact that $\|u\|_{H^1(U)} \sim \|u\|_{L^2(U)} + \|\nabla u\|_{L^2(U\to\mathbb{R}^d)}$ we have the bound

$$B(u,v) \lesssim M \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)} . \tag{4}$$

This allows us as to extend by density B from $C_0^\infty(U)$ to a continuous bilinear operator on $H_0^1(U)$. In which case, by Exercise 1 we can consider $f \in H^{-1}(U)$. This gives the following definition.

Definition 5.1 (Weak formulation). *Given $f \in H^{-1}(U)$, we say that $u \in H_0^1(U)$ solves (1) if*

$$B(u,v) = (v,f), \quad \forall v \in H_0^1(U). \tag{5}$$

We recall the "duality notation" $(v,f) := f(v)$. We have now reformulated our problem to something that looks very similar to the setup of Lax Milgram's theorem. In fact, if we suppose $\mathbf{b} = 0$ and $c \ge 0$ we are done.

Theorem 5.2. *Suppose $\mathbf{b} = 0$ and $c \ge 0$. Then, equation (1) is well posed. That is,*

$$\mathcal{L} : H_0^1(U) \xrightarrow{\sim} H^{-1}(U),$$

is a homeomorphism. Furthermore, $\left\|\mathcal{L}^{-1}\right\| \lesssim_U \alpha^{-1}$.

Proof. The continuity of B was proved in (5). It remains to see that B is coercive. This follows from the fact that for smooth u

$$B(u,u) = \int_U \mathbf{A}\nabla u \cdot \nabla u + \int_U cu^2 \ge \alpha \|\nabla u\|_{L^2(U\to\mathbb{R}^d)} \gtrsim_U \|u\|_{H_0^1(U)} . \tag{6}$$

Where in first inequality we used the ellipticity assumption on $\mathbf{A}$ and in the last inequality we used Poincaré's inequality. The result now follows from Lax Milgram's theorem. $\square$

Furthermore, by Rellich's theorem $\mathcal{L}$ is compact and is self adjoint since $\mathbf{b}$ is 0 so there is a countable basis of eigenvalues in $L^2(U)$. Furthermore they must me smooth by Prop 2 and Sobolev embedding.

In the previous result, we somewhat unsatisfyingly had to assume that $\mathbf{b}$ was identically zero and had to impose the extra assumption $c \geq 0$. These extra assumptions can be done away with, but at the cost of modifying our initial problem by a correction term γ so we can once more obtain a coercive operator B_γ

Theorem 5.3 (Modified problem). *There exists some constant $\nu \geq 0$ (depending on the coefficients) such that for all $\gamma \geq \nu$ the operator $\mathcal{L}_\gamma := \mathcal{L} + \gamma \mathbf{I}$ defines a homeomorphism*

$$\mathcal{L}_\gamma : H_0^1(U) \xrightarrow{\sim} H^{-1}(U).$$

That is, the problem $\mathcal{L}u + \gamma u = f$ is well posed for all $\gamma \geq \nu$.

Proof. Once more, the proof will go through the Lax-Milgram theorem, where now we work with the bilinear operator B_γ associated to $\mathcal{L}_\gamma$

$$B_\gamma(u, v) := B(u, v) + \gamma(u, v).$$

The calculation proceeds in a similar fashion to (6), where now an additional application of Cauchy's inequality $ab \leq a^2/2 + b^2/2$ $\nabla uv = (\epsilon^{\frac{1}{2}} \nabla u)(\epsilon^{-\frac{1}{2}} v)$ shows that

$$\begin{aligned} B(u,u) = \int_U (\mathbf{A}\nabla u) \cdot \nabla u + \int_U \mathbf{b} \cdot (\nabla u) u + \int_U c u^2 \geq \alpha \|\nabla u\|_{L^2(U \to \mathbb{R}^d)} \\ - \frac{1}{2} \|\mathbf{b}\|_{L^\infty(U)} \left(\epsilon \|\nabla u\|_{L^2(U)} + \epsilon^{-1} \|u\|_{L^2(U)} \right) - \|c\|_{L^\infty(U)} \|u\|_{L^2(U)} . \end{aligned}$$

Taking ϵ small enough (smaller than $\alpha \|\mathbf{b}\|^{-1}_{L^\infty(U)}$ to be precise) and gathering up terms gives

$$B(u,u) \geq \frac{\alpha}{2} \|\nabla u\|_{L^2(U \to \mathbb{R}^d)} - \nu \|u\|_{L^2(U)} . \tag{7}$$

Where we defined $\nu = \|\mathbf{b}\|_{L^\infty(U)} \epsilon^{-1} + \|c\|_{L^\infty(U)}$.The theorem now follows from the just proved (7) and Poincaré's inequality as for all $\gamma \geq \nu$

$$B_\gamma(u,u) = B(u,u) + \gamma \|u\|_{L^2(U)} \geq \frac{\alpha}{2} \|\nabla u\|_{L^2(U \to \mathbb{R}^d)} \gtrsim_U \|u\|_{H_0^1(U)} .$$

□

We now consider $\mathcal{L}u = \lambda u + f$, which is a small generalization of our original problem (1). We have that,

$$\mathcal{L}u = \lambda u + f \iff \mathcal{L}_\gamma u = (\gamma + \lambda) u + f.$$

If we introduce the notation $\mu := (\gamma + \lambda)$ and rename $v = \mu u + f$ we obtain that the above is equivalent to

$$(\boldsymbol{I} - \mu \mathcal{L}_\gamma^{-1}) v = f.$$

By Rellich-Kondrachov, we know that the inclusion $i : H^1(U) \hookrightarrow L^2(U)$ is compact. As a result, by Theorem 5.2, we deduce that $\mathcal{L}_\gamma^{-1} : L^2(U) \to L^2(U)$, which we are now viewing as an operator on $L^2(U)$, is compact. More precisely $K := i \circ \mathcal{L}_\gamma^{-1}\big|_{L^2(U)}$ is compact and the previous reasoning show that, given $f \in L^2(U)$, and $u \in H_0^1(U)$

$$\mathcal{L}u + \lambda u = f \iff Tu := (\mathbf{I} - \mu K) u = f. \tag{8}$$

Which is exactly the form the Fredholm alternative takes. We can now state the following result.

Theorem 5.4. *Let $\mathcal{L}$ verify Assumption 1, let $\lambda > 0, f \in L^2(U)$ be any and consider the problems*

$$\begin{cases} \mathcal{L}u = \lambda u + f \\ u \in H_0^1(U) \end{cases} \qquad (9) \qquad \begin{cases} \mathcal{L}u = \lambda u \\ u \in H_0^1(U) \end{cases} \qquad (10).$$

1. *Equation (9) is well posed if and only if (10) has no non-zero solutions ($\lambda \notin \sigma(\mathcal{L})$).*
2. *The spectrum $\sigma(\mathcal{L})$ is discrete. If $\sigma(\mathcal{L}) = \{\lambda_n\}_{n=1}^{\infty}$ is infinite, then $\lambda_n \to +\infty$.*
3. *The dimension of the following spaces is equal*

$$N := \left\{u \in H_0^1(U) : \mathcal{L}u = \lambda u\right\}, \quad N^* := \left\{f \in L^2(U) : \mathcal{L}^* f = \lambda f\right\},$$

4. *Equation, (9) has a solution if and only if $f \in (N^*)^{\perp}$ (equivalently $\langle w, f\rangle = 0$ for all $w \in N^*$).*

Proof. Given $f \in L^2(U)$ Consider the following two problems

$$\begin{cases} Tv = f \\ v \in L^2(U) \end{cases} \qquad (11) \qquad \begin{cases} Tv = 0 \\ v \in L^2(U) \end{cases} \qquad (12).$$

The reasoning in (8) showed that a solution u to (9) gives a solution to (11) via the transformation $v = \mu u + f$. The converse is not clear, as given $v \in L^2(U)$ the inverse transformation $u = \mu^{-1}(v - f)$ may not return a function in $H_0^1(U)$. However, if v solves (11), then u verifies

$$Tv = v - \mu K v = \mu u + f - \mu K v = f.$$

Cancelling out the f and dividing by μ we obtain that

$$u = Kv.$$

By Theorem 5.2 we know that $Kv = \mathcal{L}_{\gamma}^{-1} v \in H_0^1(U)$ for all $v \in L^2(U)$. As a result, u solves (9) and we have that (11) has a solution if and only if (9) has a solution. Taking $f = 0$ we also obtain that u solves (10) if and only v solves (12). In conclusion,

$$(9) \text{ is w.p} \iff (11) \text{ is w.p} \iff \ker(T) = 0 \iff \ker(\mathcal{L} + \lambda) = 0,$$

where the second equivalence is the Fredholm alternative.

To see the second part, note that (12) has non zero solutions if and only if $\mu^{-1} \in \sigma(K)$. Since K is compact, $\sigma(K)$ is discrete and if it is infinite then its eigenvalues, which we denote by $\left\{\mu_n^{-1}\right\}_{n=1}^{\infty}$ go to 0. The claim follows by the correspondence $\lambda = \mu - \gamma$.

For the final part, we note that we have already proved that $\ker(T) = N$. Additionally,

$$T^* = (\mathbf{I} - \mu K^*) = \mathbf{I} - \mu(\mathcal{L}^* + \gamma)^{-1},$$

from where

$$\ker(T^*) = \ker(\mathcal{L}^* - \lambda \mathbf{I}) = N^*.$$

Applying the Fredholm alternative once more concludes the proof. □

Corollary 5.5. *Equation (1) is well posed unless the homogeneous problem $\mathcal{L}u = 0$ has a non null solution (that is, $\ker(\mathcal{L}) \neq 0$). Furthermore, $\ker(\mathcal{L})$ and $\ker(\mathcal{L}^*)$ have the same dimension. And (1) will have a solution if and only if f is orthogonal to the kernel of $\mathcal{L}^*$.*

Exercise 2. In Theorem 5.4 we used that, for γ large enough, $K = \mathcal{L}_\gamma^{-1}$ is compact. However, $\mathcal{L}_\gamma^{-1}$ is invertible with inverse $\mathcal{L}_\gamma$. As a result $\mathbf{I} = \mathcal{L}_\gamma \circ \mathcal{L}_\gamma^{-1}$ is compact. How is this possible?

Hint. In fact, $\mathcal{L}_\gamma^{-1}$ is only invertible as an operator from $H^{-1}(U) \to H_0^1(U)$. However, it is not invertible as an operator from $K : L^2(U) \to L^2(U)$. Given $f \in L^2(U)$ it is not generally possible to find an $u \in L^2(U)$ such that $\mathcal{L}_\gamma u = f$.

Exercise 3. Where does the proof of Theorem 5.4 break down if we replace U with $\mathbb{R}^d$?

Hint. Can you apply Rellich-Kondrachov to unbounded domains? What is the spectrum of the Laplacian on $\mathbb{R}^d$?

6 Higher regularity

We saw in Theorem 5.4 that given bounded coefficients $\mathbf{A}, \mathbf{b}, c$and $f \in L^2(U)$ the solution to (1) is in $H_0^1(U)$. However, since we are differentiating twice, we can expect that the solution is in fact in $H^2(U)$. This is indeed if we assume some additional regularity on the coefficients. The idea is to use difference quotients to approximate the derivatives.

$$D_j^h u := \frac{u(x + he_j) - u(x)}{h}, \quad e_j = (0, \dots, \overset{(j)}{1}, \dots, 0).$$

The following lemma shows that, if we are able to bound these, then we can obtain the desired regularity. We show the proof for unbounded domains.

Lemma 6.1 (Difference quotients and regularity)**.** *Let $p \in (1, +\infty)$, then the following hold.*

1. *If $u \in L^p(\mathbb{R}^d)$ and for all h sufficiently small $\left\| D_j^h u \right\|_{L^p(\mathbb{R}^d)} \le C$. Then $u \in W^{1,p}(\mathbb{R}^d)$.*
2. *If $u \in W^{1,p}(\mathbb{R}^d)$. Then, for all h sufficiently small $\left\| D_j^h u \right\|_{L^p(\mathbb{R}^d)} \le C \left\| \nabla u \right\|_{L^p(\mathbb{R}^d)}$.*

Proof. The idea is to use the fundamental theorem of calculus and the density of smooth functions in $L^p(\mathbb{R}^d)$ to bound the difference quotients. Since $L^p(\mathbb{R}^d)$ is reflexive, this will give a subsequence that converges weakly to some $v \in L^p(\mathbb{R}^d)$. We will show that $v = \nabla u$ almost everywhere. Suppose first that u is smooth. Then,

□

Theorem 6.2 (Higher regularity unbounded domain)**.** *Suppose that $\mathbf{A} \in C^1(\mathbb{R}^d)$ and that $\mathbf{b}, \in L^\infty(\mathbb{R}^d \to \mathbb{R}^d), c \in L^\infty(\mathbb{R}^d)$. Then, if u solves $\mathcal{L}u = f$, it holds that $u \in H^2(U)$.*

Proof. The idea is to use difference quotients to approximate the second derivative. We will use the following notation □

Theorem 6.3. *Suppose that $\mathbf{A} \in C^1(U)$ and that $\mathbf{A}$ is uniformly elliptic. Then, the solution to (1) is in $H^2_{\text{loc}}(U)$.*

Proof. □

For bounded domains the proof is similar, however one has to be careful as the difference quotients may not be well defined at the boundary. As a result, it is necessary to work locally and use bump functions. This makes the proofs technically a bit messier, though the idea is the same. The proof of the following results can all be found in [1] pages (326-344).

A Fractional Sobolev spaces

In this section we discuss the following:

1. There are three ways to define fractional Sobolev spaces. Two of them, denoted by $W^{s,p}, B^{s,p}$, are defined by using the analogous to the definition of Hölder spaces and coincide when s is not an integer. The second, denoted by $H^{s,p}$, is defined by using the Fourier transform and coincides with $W^{s,p}$ for integer order regularity. Here s denotes the (fractional) regularity and p the integrability.

2. All these spaces coincide with H^s when $p = 2$ is.

3. The dual of $H^{s,p}(\mathbb{R}^d)$ is $H^{-s,p'}(\mathbb{R}^d)$. For integer order regularity and on smooth domains, the dual of $H^{k,p}(\Omega)$ is $H^{-k,p'}(\Omega)$ and can equivalently be defined by differentiating k times functions in $L^p(\mathbb{R}^d)$. The analogous result holds for $W^{k,p}(\Omega)$ and $B^{k,p}(\Omega)$.

4. Using these fractional estimates, one can obtain finer regularity results such as in the trace theorem.

A.1 Fractional Sobolev spaces: two definitions

The definitions developed in the next three subsections can be found in [2] page 222.

A.1.1 Soboelv-Slodeckij spaces

Definition A.1 (Sobolev-Slodeckij spaces). *Let $s = k + \gamma$ where $k \in \mathbb{N}_0$, and $\gamma \in [0, 1)$. Then, given $p \in [1, \infty)$ and $\Omega \subset \mathbb{R}^n$ be an arbitrary open set. Write $k = \lfloor s \rfloor, \gamma = k - s$. We define*

$$W^{s,p}(U) := \left\{ u \in W^{k,p}(U) : \|u\|_{W^{s,p}(U)} < \infty \right\},$$

where

$$\|u\|_{W^{s,p}(U)} := \left(\|u\|^p_{W^{k,p}(U)} + \sum_{|\alpha|=k} \int_U \int_U \frac{|D^\alpha u(x+y) - D^\alpha u(x)|^p}{|y|^{n+\gamma p}} \,\mathrm{d}x \,\mathrm{d}y \right)^{\frac{1}{p}}. \tag{13}$$

We will later define $W^{s,p}(U)$ also for negative s (see Definition A.10). We observe that the above definition coincides with our usual definition of Sobolev space when $s = k \in \mathbb{N}_0$ and mimics that of the Hölder spaces, with the addition that we now require integrability. The factor $|x - y|^{n+\gamma p}$ is chosen so that the integral is scale invariant.

Exercise 4. Show that $W^{s,p}(U)$ is a Banach space.

Hint. To show that $|\cdot|_{s,p}$ is a norm apply Minkowski's inequality to u and to $f_u(x, y) := (u(x) - u(y))/(x - y)^{n/p+s}$. Given a Cauchy sequence show that, since $L^p(U)$ is complete, $u_n \to u$ in $L^p(U)$ and that $f_{u_n} \to f_u$ in $L^p(U \times U)$ to conclude that $u_n \to u$ in $W^{s,p}(U)$.

Though the Sobolev-Slodeckij spaces can be defined for any open set U, they are most useful when $U = \mathbb{R}^d$ or U is bounded open and Lipschitz (that is U is of class $C^{0,1}$). This is because of the following result.

Proposition A.2 (Inclusion ordered by regularity). *Let $\Omega \subset \mathbb{R}^d$ be bounded open and Lipschitz. Then, for $p \in [1, \infty)$ and $0 < s < s'$ it holds that*

$$W^{s',p}(\Omega) \hookrightarrow W^{s,p}(\Omega), \quad W^{s',p}(\mathbb{R}^d) \hookrightarrow W^{s,p}(\mathbb{R}^d).$$

The proof can be found in [3] page 10. The regularity of the domain is necessary to be able to extend functions in $W^{1,p}(\Omega)$ to $W^{1,p}(\mathbb{R}^d)$. The result is not true otherwise and an example is given in this same reference.

A.1.2 Bessel potential spaces

We now give a second definition of fractional Sobolev spaces through the Fourier transform.

Definition A.3 (Bessel potential spaces on $\mathbb{R}^d$)**.** *Let $s > 0$ and $p \in [1, \infty)$. Define for $u \in \mathcal{S}'(\mathbb{R}^d)$*

$$\Lambda^s u := \mathcal{F}^{-1} \left(\langle \xi \rangle^s \widehat{u}(\xi) \right).$$

Then, we define the Bessel potential space

$$H^{s,p}(\mathbb{R}^d) := \left\{ u \in \mathcal{S}'(\mathbb{R}^d) : \Lambda^s u \in L^p(\mathbb{R}^d) \right\},$$

and give it the norm

$$\|u\|_{H^{s,p}(\mathbb{R}^d)} := \|\Lambda^s u\|_{L^p(\mathbb{R}^d)}.$$

We also define the space $H_0^{s,p}(\mathbb{R}^d)$ as the closure of $C_c^\infty(\mathbb{R}^d)$ in $H^{s,p}(\mathbb{R}^d)$.

In the definition above, is motivated by the case $p = 2$. As we saw when we studied Sobolev spaces through the Fourier transform, $u \in H^k(\mathbb{R}^d)$ if and only if $\Lambda^k u \in L^2(\mathbb{R}^d)$. That is, $H^{k,2}(\mathbb{R}^d) = H^k(\mathbb{R}^d)$. The natural generalization of this fact gives Definition A.3.

Exercise 5. Show that $\Lambda^s \Lambda^r = \Lambda^{s+r}$. Use this to show that

$$\Lambda^r : H^{r+s,p}(\mathbb{R}^d) \xrightarrow{\sim} H^{s,p}(\mathbb{R}^d),$$

is an invertible isomorphism.

Hint. Use that $\langle \xi \rangle^s \langle \xi \rangle^r = \langle \xi \rangle^{s+r}$ and show that the inverse of Λ^r is Λ^{-r}.

We now extend this to general domains

Definition A.4 (Bessel potential spaces on U)**.** *Let $U \subset \mathbb{R}^d$ be an arbitrary open set. We define,*

$$H^{s,p}(U) := \left\{ u \in \mathcal{D}'(U) : \text{ there exists } v \in H^{s,p}(\mathbb{R}^d) \text{ such that } v|_U = u \right\}.$$

And give it the norm

$$\|u\|_{H^{s,p}(U)} := \inf \left\{ \|v\|_{H^{s,p}(\mathbb{R}^d)} : v|_U = u \right\}.$$

The restriction above is in the sense of distributions. That is, we define $u = v|_U$ by

$$(\phi, u) := (\phi, v), \quad \forall \phi \in C_c^\infty(U).$$

It would be tempting to define $\|u\|_{H^{s,p}(U)} := \|\Lambda^s v\|_{L^p(U)}$. However, since the Fourier transform, and thus Λ^s, is a nonlocal operator, the norm would depend on the extension v of u to $\mathbb{R}^d$. So the norm would be ill-defined.

A.1.3 Besov spaces

Definition A.5 (Besov spaces)**.** *Let $s = k_- + \gamma$ where $k^- \in \mathbb{N}_0$, and $\gamma \in (0,1]$. Then, given $p \in [1,\infty)$ and $\Omega \subset \mathbb{R}^n$ be an arbitrary open set we define*

$$B^{s,p}(U) := \left\{u \in W^{k^-,p}(U) : \|u\|_{B^{s,p}(U)} < \infty\right\},$$

where

$$\|u\|_{B^{s,p}(U)} := \left(\|u\|^p_{W^{k^-,p}(U)} + \sum_{|\alpha|=k} \int_U \int_U \frac{|D^\alpha u(x+y) - D^\alpha u(x)|^p}{|y|^{n+\gamma p}} \,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{p}}.$$

The above definition is extremely similar in form to that of the Sobolev-Slodeckij spaces A.1. In fact, it is equivalent for $s \notin \mathbb{N}$. The difference is that in the definition fo Besov spaces A.5 we require that $\gamma > 0$. As a result, always $k^- < s$. We have chosen to indicate this fact by the index "$-$" on k_-. An equivalent definition is possible which extends the above to negative values of s

Definition A.6 (Besov spaces, negative s)**.** *Let $s \in \mathbb{R}$ and choose any $\sigma \notin \mathbb{N}_0$ with $\sigma > 0$. Then, given $p \in [1,\infty)$ we define*

$$\|u\|_{B^{s,p}(\Omega)} = \left\|\Lambda^{s-\sigma} u\right\|_{W^\sigma_p(\Omega)}.$$

The requirement $\sigma > 0$ is necessary as in general $B^{s,p}(\mathbb{R}^d) \neq H^{s,p}(\mathbb{R}^d)$ This can then be extended to general open sets U in the same way as for the Bessel potential spaces,

Definition A.7 (Besov spaces on U)**.** *Let $U \subset \mathbb{R}^d$ be an arbitrary open set. We define,*

$$B^{s,p}(U) := \left\{u \in \mathcal{D}'(U) : \text{ there exists } v \in B^{s,p}(\mathbb{R}^d) \text{ such that } v|_U = u\right\},$$

and give it the norm

$$\|u\|_{B^{s,p}(U)} := \inf\left\{\|v\|_{B^{s,p}(\mathbb{R}^d)} : v|_U = u\right\}.$$

Observation 1. Different authors use different notation for these spaces. For example, in [4], the notation $W^{s,p} := B^{s,p}$ is used. With this notation, one has that, for $p \neq 2$,

$$H^{k,p} \neq B^{k,p} = W^{k,p}.$$

Whereas, with our notation, as we will later see, $H^{k,p} = W^{k,p}$. Other notations which can be found are the notation $B^{s,p} = \Lambda^p_s$ and $H^{s,p} = \mathcal{L}^p_s$. See [5] and [6].

A.1.4 Interpolation

Both the Sobolev-Slobodeckij and Bessel potential spaces can be viewed as a way to fill the gaps between integer valued Sobolev spaces.

Proposition A.8 (Interpolation)**.** *Let $s_0 \neq s_1 \in \mathbb{R}, p \in (1,\infty)$, $0 < \theta < 1$ and*

$$s = s_0(1-\theta) + s_1\theta, \quad p = p_0(1-\theta) + p_1\theta.$$

Then, given $\Omega \subset \mathbb{R}^d$ open with uniformly Lipschitz boundary it holds that

$$B^{s,p} = [B^{s_0,p}(\Omega), B^{s_1,p}(\Omega)]_\theta, \quad H^{s,p}(\Omega) = [H^{s_0,p_0}(\Omega), H^{s_1,p_1}(\Omega)]_\theta,$$

where $[X,Y]_\theta$ denotes the complex interpolation space.

The result can be found in [4] page 45 for $\Omega = \mathbb{R}^d$. The general results follows by extension. See [7] page 424. In particular, if we write $k := \lfloor s \rfloor$ and $\gamma := s - k$, then

$$H^{s,p}(\Omega) = \left[H^{k,p}(\Omega), H^{k+1,p}(\Omega)\right]_\gamma = \left[L^p(\Omega), H^{k+1,p}(\Omega\right]_{s/(k+1)}.$$

A.2 Relationship between the definitions

The following result shows the inclusions between $W^{s,p}, H^{s,p}, B^{s,p}$ and can be found in [2] page 224 and in [5] page 155.

Theorem A.9. *Let $\Omega \subset \mathbb{R}^d$ be open with uniformly Lipschitz boundary and $s \in [0, \infty)$. Then,*

$$\begin{aligned} H^{s+\epsilon,p}(\Omega) \subset B^{s,p}(\Omega) \subset H^{s,p}(\Omega) \quad \forall p \in (1, 2] \\ B^{s+\epsilon,p}(\Omega) \subset H^{s,p}(\Omega) \subset B^{s,p}(\Omega) \quad \forall p \in [2, \infty), \end{aligned}$$

where the above inclusions are continuous and dense. Furthermore,

$$W^{s,p}(\Omega) = \begin{cases} H^{s,p}(\Omega) & \text{if } s \in \mathbb{N}_0 \\ B^{s,p}(\Omega) & \text{if } s \notin \mathbb{N}_0 \end{cases}. \tag{14}$$

In consequence, for $p = 2$,

$$H^{s,2}(\Omega) = W^{s,2}(\Omega) = B^{s,2}(\Omega). \tag{15}$$

The equality in (14) shows that, as long as we understand the behaviour of $H^{s,p}(\Omega)$ and $B^{s,p}(\Omega)$ we can completely determine that of $W^{s,p}(\Omega)$. It also justifies the following extension of $W^{s,p}(\Omega)$ to negative regularity.

Definition A.10 (Slodeckij space negative s)**.** *Let $\Omega \subset \mathbb{R}^d$ be open with uniformly Lipschitz boundary. Then, given $p \in [1, \infty)$ and any $s \in \mathbb{R}$ we define*

$$W^{s,p}(\Omega) = \begin{cases} H^{s,p}(\Omega) & \text{if } s \in \mathbb{N}_0 \\ B^{s,p}(\Omega) & \text{if } s \notin \mathbb{N}_0 \end{cases}.$$

The equality for $p = 2$ in (15) justifies that, for sufficiently regular domains, all three spaces are written $H^s(\Omega)$. We will prove the left hand side of this equivalence in Exercise 6. For $p \neq 2$ the inclusions are in general strict. An example is constructed in [5] page 161 exercise 6.8.

Exercise 6 (Equivalence of fractional spaces)**.** Show that

$$H^{s,2}(\mathbb{R}^d) = W^{s,2}(\mathbb{R}^d).$$

Hint. We want to show that the norms are equivalent. That is, that

$$\|u\|_{B^{s,2}(\mathbb{R}^d)} \sim \|u\|_{H^{s,2}(\mathbb{R}^d)}.$$

We already know this is the case when s is an integer so it suffices to show that the norms are equivalent for $s = \gamma \in (0, 1)$. That is, that

$$|u|_{s,2}^2 \sim \int_{\mathbb{R}^d} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 d\xi$$

By multiple changes of variable and Plancherel's theorem we have that

$$\begin{aligned} |u|_{\gamma,2}^2 &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x+y) - u(y)|^2}{|x|^{d+2\gamma}} \,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^d} \frac{\|\mathcal{F}(u(x+\cdot) - u)\|^2}{|x|^{d+2\gamma}} \,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|e^{-2\pi i x\cdot\xi} - 1|^2}{|x|^{d+2\gamma}} |\widehat{u}(\xi)|^2 \,\mathrm{d}x\,\mathrm{d}\xi = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{1 - \cos(2\pi\xi \cdot x)}{|x|^{d+2\gamma}} \,\mathrm{d}x \right) |\widehat{u}(\xi)|^2 \,\mathrm{d}\xi. \end{aligned}$$

To treat the inner integral we note that it is rotationally invariant and so, by rotating, to the first axis and later changing variable $x \to x/|\xi|$ we get

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{1-\cos(2\pi\xi\cdot x)}{|x|^{d+2\gamma}}\,\mathrm{d}x &= \int_{\mathbb{R}^d} \frac{1-\cos(2\pi\,|\xi|\,x_1)}{|x|^{d+2\gamma}}\,\mathrm{d}x \\ &= |\xi|^{2\gamma}\int_{\mathbb{R}^d} \frac{1-\cos(2\pi x_1)}{|x|^{d+2\gamma}}\,\mathrm{d}x \sim |\xi|^{2\gamma}\,.\end{aligned}$$

The last integral is finite as, since $d+2\gamma > d$, the tails $|\xi| \to \infty$ are controlled, and since $1-\cos(2\pi x_1) \sim x_1^2 \leq |x|^2$ the integrand has order $-d+2(1-\gamma) > -d$ for $|\xi| \sim 0$. That said, substituting this back into the previous expression gives the desired result.

Exercise 7. Use the previous exercise 6 to show that, if Ω is a open set with uniformly Lipschitz boundary, then

$$H^{s,2}(\Omega) = W^{s,2}(\Omega).$$

Hint. By definition A.3 choose a sequence $v_n \in H^{s,2}(\mathbb{R}^d)$ such that $\|v_n\|_{H^{s,2}(\mathbb{R}^d)} \to \|u\|_{H^{s,2}(\Omega)}$ in $H^s(\Omega)$. Then,

$$\|u\|_{H^{s,2}(\Omega)} = \lim_{n\to\infty} \|v_n\|_{H^{2,2}(\mathbb{R}^d)} \sim \lim_{n\to\infty} \|v_n\|_{B^{s,2}(\mathbb{R}^d)} \geq \|u\|_{B^{s,2}(\Omega)}\,.$$

To obtain the reverse inequality use the existence of a continuous extension operator $E : W^{s,2}(\Omega) \to W^{s,2}(\mathbb{R}^d)$ (see [3] page 33) to obtain

$$\|u\|_{W^{s,2}(\Omega)} \sim \|Eu\|_{W^{s,2}(\mathbb{R}^d)} \geq \|u\|_{H^{s,2}(\mathbb{R}^d)}\,.$$

The above suggests that integrals appearing in the definition of the slodeckij spaces A.1 correspond to differentiating a fractional amount of times. This indeed is the case

Definition A.11. *Given $\gamma \in [0,+\infty)$ and $u \in \mathcal{S}(\mathbb{R}^d)$ we define the fractional Laplacian as*

$$(-\Delta)^\gamma u(x) := \mathcal{F}^{-1}(|2\pi\xi|^{2\gamma}\,\widehat{u}(\xi)).$$

Proposition A.12. *For $\gamma \in (0,1)$ and $u \in H^s(\mathbb{R}^d)$ it holds that*

$$(-\Delta)^\gamma u(x) = C\int_{\mathbb{R}^d} \frac{u(x)-u(x+y)}{|y|^{d+2\gamma}}\,\mathrm{d}y,$$

where C is a constant that depends on d,γ.

Proof. The above equality may seem odd at first if we compare with the integral in A.1 where a square appears in the numerator which gives us our 2 in the 2γ. However, it is justified by the fact that, by the change of variables $y \to -y$,

$$\int_{\mathbb{R}^d} \frac{u(x)-u(x+y)}{|y|^{d+2\gamma}}\,\mathrm{d}y = \int_{\mathbb{R}^d} \frac{u(x)-u(x-y)}{|y|^{d+2\gamma}}\,\mathrm{d}y.$$

So we can get the *second* order difference in the numerator by adding the two integrals.

$$\int_{\mathbb{R}^d} \frac{u(y)-u(x+y)}{|y|^{d+2\gamma}}\,\mathrm{d}y = -\frac{1}{2}\int_{\mathbb{R}^d} \frac{u(x+y)-2u(x)+u(x-y)}{|y|^{d+2\gamma}}\,\mathrm{d}y. \tag{16}$$

That said, we must show that

$$|\xi|^{2\gamma}\,\widehat{u}(\xi) \sim \mathcal{F}\left(\int_{\mathbb{R}^d} \frac{u(x)-u(x+y)}{|y|^{d+2\gamma}}\,\mathrm{d}y\right)$$

Using (16) and proceeding as in exercise A.9 gives

$$\mathcal{F}\left(\int_{\mathbb{R}^d} \frac{u(x)-u(x+y)}{|y|^{d+2\gamma}}\,\mathrm{d}y\right) = -\frac{1}{2}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} \frac{e^{-2\pi i y\cdot\xi}-2+e^{2\pi i y\cdot\xi}}{|y|^{d+2\gamma}}\,\mathrm{d}y\right)\widehat{u}(\xi)\,\mathrm{d}\xi$$

$$= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} \frac{1-\cos(2\pi y\cdot\xi)}{|y|^{d+2\gamma}}\,\mathrm{d}y\right)\widehat{u}(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}^d} \frac{1-\cos(2\pi y_1)}{|y|^{d+2\gamma}}\,\mathrm{d}y \int_{\mathbb{R}^d} |\xi|^{2\gamma}\,\widehat{u}(\xi)\,\mathrm{d}\xi$$

$$\sim |\xi|^{2\gamma}\,\widehat{u}(\xi)\,\mathrm{d}\xi.$$

This completes the proof, and shows that the explicit expression for C is

$$C = \frac{1}{(2\pi)^{2\gamma}}\int_{\mathbb{R}^d} \frac{1-\cos(2\pi y_1)}{|y|^{d+2\gamma}}\,\mathrm{d}y.$$

$\square$

B Dual of Sobolev spaces and correspondence with negative regularity

Negative orders of regularity correspond to the dual of Sobolev spaces. This is best seen in the integer case, where the following result holds (see [1] pages (326-344) for the case $p = 2$).

Theorem B.1. *For all $k \in \mathbb{Z}$ and $p \in [1,\infty)$ it holds that*

$$H_0^{k,p}(\Omega)' = H^{-k,p'}(\Omega), \quad W_0^{k,p}(\Omega)' = W^{-k,p'}(\Omega).$$

The first equality will be discussed in the next subsection and is most easily proves when $\Omega = \mathbb{R}^d$, in which case one can use the correspond $\Lambda^s : H^{r,p}(\mathbb{R}^d) \xrightarrow{\sim} H^{r-s,p}(\mathbb{R}^d)$ together with the reflexivity of $L^p(\mathbb{R}^d)$. The second equality is a direct consequence of the integer order equality $W^{k,p}(\Omega) = H^{k,p}(\Omega)$ of Theorem A.9. For fractional order regularities, we have the following result which can be found in [2] page 228.

Theorem B.2. *The spaces $W^{s,p}(\Omega), H^{s,p}(\Omega), B^{s,p}(\Omega)$ are reflexive Banach spaces with duals*

$$W^{s,p}(\Omega)' = W_{\overline{\Omega}}^{-s,p'}(\mathbb{R}^d), \quad H^{s,p}(\Omega)' = H_{\overline{\Omega}}^{-s,p'}(\mathbb{R}^d), \quad B^{s,p}(\Omega)' = B_{\overline{\Omega}}^{-s,p'}(\mathbb{R}^d).$$

where p' is the conjugate exponent of p and given a space of distributions X on $\mathbb{R}^d$ we define $X_{\overline{\Omega}}$ as the space of distributions on $\mathbb{R}^d$ which are supported in $\overline{\Omega}$. In particular, for $\Omega = \mathbb{R}^d$,

$$W^{s,p}(\mathbb{R}^d)' = W^{-s,p'}(\mathbb{R}^d), \quad H^{s,p}(\mathbb{R}^d)' = H^{-s,p'}(\mathbb{R}^d), \quad B^{s,p}(\mathbb{R}^d)' = B^{-s,p'}(\mathbb{R}^d).$$

We have already seen that $W^{s,p}(\Omega)$ and $H^{s,p}(\Omega)$ are Banach spaces, one can similarly show that $B^{s,p}(\Omega)$ is a Banach space. Furthermore, the spaces are all reflexive for smooth domains.

Observation 2. Some authors define given $s > 0$

$$W^{-s,p'}(\Omega)' := W^{s,p}(\Omega)'.$$

See for example [6]. This is equivalent to our definition when $\Omega = \mathbb{R}^d$ or when $s \in k$. However, in other cases, the two definitions are not equivalent.

B.1 The dual of $H^{s,p}(\mathbb{R}^d)$ and $B^{s,p}(\mathbb{R}^d)$

For some motivation we start by considering the case $\Omega = \mathbb{R}^d$. In this case, since the closure of $C_c^\infty(\mathbb{R}^d)$ in $H^{s,p}(\mathbb{R}^d)$ (which by definition is $H_0^{s,p}(\mathbb{R}^d)$) is itself $H^{s,p}(\mathbb{R}^d)$, we have that $H_0^{s,p}(\mathbb{R}^d) = H^{s,p}(\mathbb{R}^d)$.

Exercise 8 (Dual identification)**.** Prove the identification $H^{-s,p'}(\mathbb{R}^d) = H^{s,p}(\mathbb{R}^d)'$.

Hint. Consider the mapping $H_0^{-s,p'}(\mathbb{R}^d) \to H_0^{s,p}(\mathbb{R}^d)'$ given by $f \mapsto \ell_f$ where

$$\ell_f(u) := \int_{\mathbb{R}^d} (\Lambda^s u)(\Lambda^{-s} f).$$

Show that this mapping is well defined and continuous. To see that it is invertible, show that, by duality, given $\ell \in H^{s,p}(\mathbb{R}^d)'$ and $u \in H^{s,p}(\mathbb{R}^d)$, it holds that

$$(u, \ell) = (\Lambda^s u, \Lambda^{-s} \ell).$$

Since $\Lambda^s u \in L^p(\mathbb{R}^d)$ we deduce that $\Lambda^{-s}\ell \in L^p(\mathbb{R}^d)'$ and so by the Riesz representation theorem there exists $f_\ell \in L^{p'}(\mathbb{R}^d)$ such that $\Lambda^{-s}\ell = \langle \cdot, f_\ell \rangle$. Show that the mapping

$$H^{s,p}(\mathbb{R}^d)' \longrightarrow H^{-s,p'}(\mathbb{R}^d); \quad \ell = \langle \cdot, \Lambda^s f_\ell \rangle \to \Lambda^s f_\ell,$$

is the inverse of the previous one.

Exercise 9. We also know that, since $H^s(\mathbb{R}^d)$ is a Hilbert space, so by the Riesz representation theorem we have the identification $H^s(\mathbb{R}^d) = H^s(\mathbb{R}^d)'$. So by the previous exercise $H^{-s}(\mathbb{R}^d) = H^s(\mathbb{R}^d)$ How is this possible?

Hint. It does **not** hold that $H^{-s}(\mathbb{R}^d) = H^s(\mathbb{R}^d)$. The problem occurs when considering too many identifications at once, as we are identifying duals using different inner products. By following the mappings we obtain a bijective isomorphism

$$\begin{aligned} &H^s(\mathbb{R}^d) \to H^s(\mathbb{R}^d)' \to H^{-s}(\mathbb{R}^d) \\ &u \longmapsto \langle \cdot, u \rangle_{H^s(\mathbb{R}^d)} = \langle \cdot, \Lambda^{2s} u \rangle \mapsto \Lambda^{2s} u. \end{aligned}$$

However, the isomorphism is Δ^{2s}, which is hardly the identity mapping.

For another example where confusion with these kind of identifications can arise see remark 3 on page 136 of [8].

B.2 The dual of $H_0^{s,p}(\Omega)$

Given an extension domain Ω and $s \in \mathbb{R}$, one can define extension and restriction operators,

$$E : H^{s,p}(\Omega) \to H^{s,p}(\mathbb{R}^d), \quad \rho : H^{s,p}(\mathbb{R}^d) \to H^{s,p}(\Omega),$$

which verify $\rho \circ E = \boldsymbol{I}_{H^s(\Omega)}$. As a result, restriction is surjective and we can factor $H^{s,p}(\Omega)$ as

$$H^{s,p}(\Omega) \simeq H^{s,p}(\mathbb{R}^d)/H^s_{\Omega^c}(\mathbb{R}^d), \tag{17}$$

where given a closed set $K \subset \mathbb{R}^d$ we define

$$H_K^{s,p}(\mathbb{R}^d) := \left\{ u \in H^{s,p}(\mathbb{R}^d) : \operatorname{supp}(u) \subset K \right\},$$

where the support is to be understood in the sense of distributions Now, given a Banach space X and a closed subspace $Y \hookrightarrow X$, elements of X' can be restricted to Y, obtaining functionals in Y'. The kernel of this restriction is $Y^\circ := \{\ell \in X' : Y \subset \ker(\ell)\}$. Since, by the Hahn Banach theorem, the restriction is surjective, we obtain the factorization

$$Y' \simeq X'/Y^\circ. \tag{18}$$

Applying this to $Y = H_0^{k,p}(\Omega) \hookrightarrow H^{k,p}(\mathbb{R}^d) = X$ we obtain the result of Theorem B.1.

$$H_0^{k,p}(\Omega)' \simeq H^{k,p}(\mathbb{R}^d)'/H^{k,p}_{\Omega^c}(\mathbb{R}^d)' \simeq H^{-k,p'}(\mathbb{R}^d)/H^{-k,p'}_{\Omega^c}(\mathbb{R}^d) \simeq H^{-k,p'}(\Omega),$$

where the second equality is by Exercise 8 and the third by (17). This shows that the dual of $H_0^{k,p}(\Omega)$ is $H^{-k,p'}(\Omega)$. By also using the integer order equivalence of Theorem A.9 we obtain Theorem B.1.

As a final note, if our domain has a boundary, $H_0^k(\Omega)'$ and $H^k(\Omega)'$ are not equal. Rather,

$$H^{k,p}(\Omega)' \simeq H^{-k,p'}_{\overline{\Omega}}(\mathbb{R}^d), \quad H^{-k,p'}(\Omega) \simeq H^{-k,p'}(\mathbb{R}^d)/H^{-k,p'}_{\Omega^c}(\mathbb{R}^d).$$

See [9] Section 4 for more details.

C Representation theorems

We know that we can identify the spaces $H^{s,p}(\mathbb{R}^d)$ and $B^{s,p}(\mathbb{R}^d)$ with the lower order spaces $H^{s-r,p}(\mathbb{R}^d)$ and $B^{s-r,p}(\mathbb{R}^d)$ by application of Λ^r. That is, by differentiating r times. That is, spaces of lower order regularity are obtained by differentiating functions with higher regularity. We show how to extend this idea to smooth domains in some particular cases.

Theorem C.1 (Representation of $W_0^{k,p}(\Omega)'$)**.** *Let $\Omega \subset \mathbb{R}^d$ be be open with uniformly Lipschitz boundary and let $k \in \mathbb{N}$ and $p \in [1, \infty)$. Then, every element in $W^{-k,p'}(\Omega) = W^{k,p}(\Omega)'$ is the unique extension of a distribution of the form*

$$\sum_{1 \le |\alpha| \le k} D^\alpha u_\alpha \in \mathcal{D}'(\Omega), \quad \textit{where } u_\alpha \in L^{p'}(\Omega).$$

Proof. Define the mapping

$$\begin{aligned} T : W^{k,p}(\Omega) &\longrightarrow L^p(\Omega \to \mathbb{R}^n) \\ u &\longmapsto (D^\alpha u)_{1 \le |\alpha| \le k}. \end{aligned}$$

Where the notation just says that we send u to the vector formed by all its derivatives. By our definition of the norm on $W^{k,p}(\Omega)$, we have that T is an isometry and in particular continuously invertible on its image. Denote the image of T by $X := \mathrm{Im}(T)$. Given $\ell \in W^{-k,p'}(\Omega)$ we define

$$\ell_0 : X \to \mathbb{R}, \quad \ell_0(\mathbf{w}) := \ell(T^{-1}\mathbf{w}), \quad \forall \mathbf{w} \in X.$$

By Hahn Banach's theorem we can extend ℓ_0 from X to a functional $\ell_1 \in L^p(\Omega \to \mathbb{R}^n)'$ and by the Riesz representation theorem we have that there exists a unique $\mathbf{f} = (f_\alpha)_{1 \le |\alpha| \le k} \in L^{p'}(\Omega \to \mathbb{R}^n)$ such that

$$\ell_1(\mathbf{w}) = \int_\Omega \mathbf{w} \cdot \mathbf{h}, \quad \forall \mathbf{w} \in L^p(\Omega \to \mathbb{R}^n).$$

By construction, it holds that, for all $v \in W^{k,p}(\Omega)$

$$\ell(u) = \ell_0(Tv) = \int_\Omega Tv \cdot \mathbf{f} = \sum_{1 \le |\alpha| \le k} \int_\Omega f_\alpha D^\alpha v.$$

In particular, this holds for $v \in \mathcal{D}(\Omega)$ and if we set $u_\alpha := (-1)^\alpha h_\alpha$ we obtain that for all $v \in \mathcal{D}(\Omega)$

$$\ell(v) = \left(v, \sum_{1 \le |\alpha| \le k} D^\alpha u_\alpha \right) =: \omega(v) \tag{19}$$

(we recall the notation (v, ω) for the duality pairing). By definitions of the norm on $W^{k,p}(\Omega)$ and Cauchy Schwartz, we have that ω is continuous with respect to the norm on $W^{k,p}(\Omega)$ and so we may extend it uniquely to the closure of $\mathcal{D}(\Omega)$ in $W^{k,p}(\Omega)$ which is $W_0^{k,p}(\Omega)$. By (19) the extension is necessarily ω. This completes the proof. □

The above theorem shows that $W^{-k,p'}(\Omega)$ can be equivalently formed by differentiating k times functions in $L^{p'}(\Omega)$. The proof also sheds some light as to why $W^{-s,p'}(\Omega)$ is the dual of $W_0^{k,p}(\Omega)$ and not the dual of $W^{k,p}(\Omega)$. The reason is that the elements of $W_0^{k,p}(\Omega)$ are the ones that can be extended to distributions in $\mathcal{D}'(\Omega)$ and so are the ones that we can integrate against. Finally, though the extension from $\mathcal{D}'(\Omega)$ to $W^{-s,p}(\Omega)$ is unique the functions u_α will not be, for example if $|\alpha| > 0$ it is possible to add a constant to u_α and still obtain the same result.

Exercise 10. Show that for $s > 0$ and $p \in [1, \infty)$ every element in $H^{-s,p'}(\Omega)$ can be written in the form $w|_{\partial\Omega}$, where

$$w = \sum_{0 \le |\alpha| \le k} \Lambda^\gamma D^\alpha u_\alpha \in \mathcal{D}'(\mathbb{R}^d), \quad \text{where } u_\alpha \in L^{p'}(\mathbb{R}^d),$$

where $k = \lfloor s \rfloor$ and $\gamma = s - k$.

Hint. Use that $\Lambda^\gamma : H^{s,p}(\mathbb{R}^d) \to H^{k,p}(\mathbb{R}^d)$ is an isomorphism and the just proved theorem C.1 together with the integer equivalence in Theorem A.9 to show that

$$H^{s,p}(\mathbb{R}^d)' = \left\{ \sum_{0 \le |\alpha| \le k} \Lambda^\gamma D^\alpha u_\alpha \in \mathcal{D}'(\mathbb{R}^d), \quad \text{where } u_\alpha \in L^{p'}(\mathbb{R}^d) \right\}.$$

Now conclude by the definition of $H^{-s,p'}(\Omega)$ for open domains A.4.

The above results extend to Besov spaces, see [2] page 227. This gives,

Theorem C.2. *Let $k \in \mathbb{N}_0, \gamma \in [0,1)\theta \in (0,1)$ and $p \in [1, \infty)$. Then,*

$$B^{\theta-k,p}(\Omega) = \left\{ \sum_{0 \le |\alpha| \le k} D^\alpha u_\alpha \in \mathcal{D}'(\Omega), \quad \textit{where } u_\alpha \in B^{\theta,p}(\Omega) \right\}$$

$$H^{\gamma-k,p}(\Omega) = \left\{ \sum_{0 \le |\alpha| \le k} D^\alpha u_\alpha \in \mathcal{D}'(\Omega), \quad \textit{where } u_\alpha \in H^{\gamma,p}(\Omega) \right\}$$

$$setminus W^{\gamma-k,p}(\Omega) = \left\{ \sum_{0 \le |\alpha| \le k} D^\alpha u_\alpha \in \mathcal{D}'(\Omega), \quad \textit{where } u_\alpha \in W^{\gamma,p}(\Omega) \right\}.$$

C.1 Some applications

The following theorem can be found in [2] page 228 and serves as a generalization of the trace theorem for fractional Sobolev spaces.

Theorem C.3 (Fractional trace theorem). *Let $\Omega \subset \mathbb{R}^d$ be an open set with uniformly Lipschitz boundary. Then, for all $s > 1/p$, the trace operator* Tr *can be extended from $C_c^\infty(\mathbb{R}^d)$ to a bounded operator*

$$\mathrm{Tr} : H^{s,p}(\Omega) \to B^{s-1/p,p}(\partial\Omega), \quad \mathrm{Tr} : B^{s,p}(\Omega) \to B^{s-1/p,p}(\partial\Omega).$$

In particular, as we have respective equalities for $W^{s,p}(\Omega)$ with $H^{s,p}(\Omega)$ and $W^{s,p}(\Omega)$ for respectively non-integer s, we can also extend

$$\mathrm{Tr} : W^{s,p}(\Omega) \to B^{s-1/p,p}(\partial\Omega).$$

One can also obtain improved Sobolev embeddings for fractional Sobolev spaces. For example, see [2] page 224

Theorem C.4. *Given $s > n/p + \gamma$, where $t > 0$, the space $W_p^s\left(\mathbb{R}^f\right)$ is continuously embedded in $C_b^\gamma\left(\mathbb{R}^d\right)$. This embedding also holds for $s = n/p + t$ provided that γ is noninteger. In particular, $W_p^s(\Omega)$ is continuously embedded in $C_b(\Omega)$ for $s > n/p$.*

Theorem C.5 (Rellich Kondratov). *Let $\Omega \subset \mathbb{R}^d$ be an open set with uniformly Lipschitz boundary. Suppose $1 < p \leq q < \infty$ and $-\infty < t \leq s < \infty$ satisfy $s - \frac{n}{p} \geq t - \frac{n}{q}$. Then,*

$$W^{s,p}(\Omega) \hookrightarrow W^{t,q}\left(\overline{\Omega}\right)$$

In particular, $W^{s,\bar{p}}(\Omega) \hookrightarrow W^{t,p}(\Omega)$. Furthermore, if Ω is bounded, the inclusion is compact.

Observation 3. Edit extension theorem for uniformly Lipschitz domains and then whenever C^k boundary is invoked $C^{0,1}$ can be used. The above extension result can also be proved when Ω is open with uniformly Lipschitz boundary. In practice, this just means that $\partial\Omega$ and of class $C^{0,1}$ (Lipschitz continuous). See [7] pages 423-430 for the details.

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