Fractional and negative order Sobolev spaces

Liam Llamazares

February 22, 2024

1 Three line summary

1. There are three main ways to define fractional Sobolev spaces. Two of them, denoted by $W^{s,p}, B^{s,p}$, are defined by using the analogous to the definition of Hölder spaces and coincide when s is not an integer. The second, denoted by $H^{s,p}$, is defined by using the Fourier transform and coincides with $W^{s,p}$ for integer order regularity. Here s denotes the (fractional) regularity and p the integrability.

2. All these spaces coincide with H^s when $p = 2$ is.

3. The dual of $H^{s,p}(\mathbb{R}^d)$ is $H^{-s,p'}(\mathbb{R}^d)$. For integer order regularity and on smooth domains, the dual of $H^{k,p}(\Omega)$ is $H^{-k,p'}(\Omega)$ and can equivalently be defined by differentiating k times functions in $L^p(\mathbb{R}^d)$. The analogous result holds for $W^{k,p}(\Omega)$ and $B^{k,p}(\Omega)$.

4. Using these fractional estimates, one can obtain finer regularity results such as in the trace theorem.

1.1 Fractional Sobolev spaces: two definitions

The definitions developed in the next three subsections can be found in [**?**] page 222.

1.1.1 Soboelv-Slodeckij spaces

Definition 1.1 (Sobolev-Slodeckij spaces). *Let $s = k + \gamma$ where $k \in \mathbb{N}_0$, and $\gamma \in [0, 1)$. Then, given $p \in [1, \infty)$ and $\Omega \subset \mathbb{R}^n$ be an arbitrary open set. Write $k = \lfloor s \rfloor , \gamma = k - s$. We define*

$$W^{s,p}(U) := \Big\{ u \in W^{k,p}(U) : \|u\|_{W^{s,p}(U)} < \infty \Big\{ ,$$

where

$$\|u\|_{W^{s,p}(U)} := \left(\|u\|^p_{W^{k,p}(U)} + \sum_{|\alpha|=k} \int_U \int_U \frac{|D^\alpha u(x+y) - D^\alpha u(x)|^p}{|y|^{n+\gamma p}} \,\mathrm{d}x\,\mathrm{d}y \right)^{\frac{1}{p}} . \tag{1}$$

We will later define $W^{s,p}(U)$ also for negative s (see Definition 1.10). We observe that the above definition coincides with our usual definition of Sobolev space when $s = k \in \mathbb{N}_0$ and mimics that of the Hölder spaces, with the addition that we now require integrability. The factor $|x - y|^{n+\gamma p}$ is chosen so that the integral is scale invariant.

Exercise 1. Show that $W^{s,p}(U)$ is a Banach space.

Hint. To show that $|\cdot|_{s,p}$ is a norm apply Minkowski's inequality to u and to $f_u(x,y) := (u(x) - u(y))/(x-y)^{n/p+s}$. Given a Cauchy sequence show that, since $L^p(U)$ is complete, $u_n \to u$ in $L^p(U)$ and that $f_{u_n} \to f_u$ in $L^p(U \times U)$ to conclude that $u_n \to u$ in $W^{s,p}(U)$.

Though the Sobolev-Slodeckij spaces can be defined for any open set U, they are most useful when $U = \mathbb{R}^d$ or U is bounded open and Lipschitz (that is U is of class $C^{0,1}$). This is because of the following result.

Proposition 1.2 (Inclusion ordered by regularity). *Let $\Omega \subset \mathbb{R}^d$ be bounded open and Lipschitz. Then, for $p \in [1,\infty)$ and $0 < s < s'$ it holds that*

$$W^{s',p}(\Omega) \hookrightarrow W^{s,p}(\Omega), \quad W^{s',p}(\mathbb{R}^d) \hookrightarrow W^{s,p}(\mathbb{R}^d).$$

The proof can be found in [**?**] page 10. The regularity of the domain is necessary to be able to extend functions in $W^{1,p}(\Omega)$ to $W^{1,p}(\mathbb{R}^d)$. The result is not true otherwise and an example is given in this same reference.

1.1.2 Bessel potential spaces

We now give a second definition of fractional Sobolev spaces through the Fourier transform.

Definition 1.3 (Bessel potential spaces on $\mathbb{R}^d$). *Let $s > 0$ and $p \in [1,\infty)$. Define for $u \in \mathcal{S}'(\mathbb{R}^d)$*

$$\Lambda^s u := \mathcal{F}^{-1}\left(\langle\xi\rangle^s \,\widehat{u}(\xi)\right).$$

Then, we define the Bessel potential space

$$H^{s,p}(\mathbb{R}^d) := \left\{u \in \mathcal{S}'(\mathbb{R}^d) : \Lambda^s u \in L^p(\mathbb{R}^d)\right\},$$

and give it the norm

$$\|u\|_{H^{s,p}(\mathbb{R}^d)} := \|\Lambda^s u\|_{L^p(\mathbb{R}^d)}.$$

We also define the space $H_0^{s,p}(\mathbb{R}^d)$ as the closure of $C_c^\infty(\mathbb{R}^d)$ in $H^{s,p}(\mathbb{R}^d)$.

In the definition above, is motivated by the case $p = 2$. As we saw when we studied Sobolev spaces through the Fourier transform, $u \in H^k(\mathbb{R}^d)$ if and only if $\Lambda^k u \in L^2(\mathbb{R}^d)$. That is, $H^{k,2}(\mathbb{R}^d) = H^k(\mathbb{R}^d)$. The natural generalization of this fact gives Definition 1.3.

Exercise 2. Show that $\Lambda^s\Lambda^r = \Lambda^{s+r}$. Use this to show that

$$\Lambda^r : H^{r+s,p}(\mathbb{R}^d) \xrightarrow{\sim} H^{s,p}(\mathbb{R}^d),$$

is an invertible isomorphism.

Hint. Use that $\langle\xi\rangle^s \langle\xi\rangle^r = \langle\xi\rangle^{s+r}$ and show that the inverse of Λ^r is Λ^{-r}.

We now extend this to general domains

Definition 1.4 (Bessel potential spaces on U). *Let $U \subset \mathbb{R}^d$ be an arbitrary open set. We define,*

$$H^{s,p}(U) := \left\{u \in \mathcal{D}'(U) : \text{ there exists } v \in H^{s,p}(\mathbb{R}^d) \text{ such that } v|U = u\right\}.$$

And give it the norm

$$\|u\|_{H^{s,p}(U)} := \inf\left\{\|v\|_{H^{s,p}(\mathbb{R}^d)} : v|U = u\right\{.$$

The restriction above is in the sense of distributions. That is, we define $u = v|U$ by

$$(\phi, u) := (\phi, v), \quad \forall\phi \in C_c^\infty(U).$$

It would be tempting to define $\|u\|_{H^{s,p}(U)} := \|\Lambda^s v\|_{L^p(U)}$. However, since the Fourier transform, and thus Λ^s, is a nonlocal operator, the norm would depend on the extension v of u to $\mathbb{R}^d$. So the norm would be ill-defined.

1.1.3 Besov spaces

Definition 1.5 (Besov spaces)**.** *Let $s = k_- + \gamma$ where $k^- \in \mathbb{N}_0$, and $\gamma \in (0,1]$. Then, given $p \in [1,\infty)$ and $\Omega \subset \mathbb{R}^n$ be an arbitrary open set we define*

$$B^{s,p}(U) := \left\{ u \in W^{k^-,p}(U) : \|u\|_{B^{s,p}(U)} < \infty \right\},$$

where

$$\|u\|_{B^{s,p}(U)} := \left(\|u\|^p_{W^{k^-,p}(U)} + \sum_{|\alpha|=k} \int_U \int_U \frac{|D^\alpha u(x+y) - D^\alpha u(x)|^p}{|y|^{n+\gamma p}} \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{p}}.$$

The above definition is extremely similar in form to that of the Sobolev-Slodeckij spaces 1.1. In fact, it is equivalent for $s \notin \mathbb{N}$. The difference is that in the definition fo Besov spaces 1.5 we require that $\gamma > 0$. As a result, always $k^- < s$. We have chosen to indicate this fact by the index "$-$" on k_-. An equivalent definition is possible which extends the above to negative values of s

Definition 1.6 (Besov spaces, negative s)**.** *Let $s \in \mathbb{R}$ and choose any $\sigma \notin \mathbb{N}_0$ with $\sigma > 0$. Then, given $p \in [1,\infty)$ we define*

$$\|u\|_{B^{s,p}(\Omega)} = \left\| \Lambda^{s-\sigma} u \right\|_{W^\sigma_p(\Omega)}.$$

The requirement $\sigma > 0$ is necessary as in general $B^{s,p}(\mathbb{R}^d) \neq H^{s,p}(\mathbb{R}^d)$ This can then be extended to general open sets U in the same way as for the Bessel potential spaces,

Definition 1.7 (Besov spaces on U)**.** *Let $U \subset \mathbb{R}^d$ be an arbitrary open set. We define,*

$$B^{s,p}(U) := \left\{ u \in \mathcal{D}'(U) : \text{ there exists } v \in B^{s,p}(\mathbb{R}^d) \text{ such that } v|U = u \right\},$$

and give it the norm

$$\|u\|_{B^{s,p}(U)} := \inf \left\{ \|v\|_{B^{s,p}(\mathbb{R}^d)} : v|U = u \right\}.$$

Observation 1. Different authors use different notation for these spaces. For example, in [?], the notation $W^{s,p} := B^{s,p}$ is used. With this notation, one has that, for $p \neq 2$,

$$H^{k,p} \neq B^{k,p} = W^{k,p}.$$

Whereas, with our notation, as we will later see, $H^{k,p} = W^{k,p}$. Other notations which can be found are the notation $B^{s,p} = \Lambda^p_s$ and $H^{s,p} = \mathcal{L}^p_s$. See [?] and [?].

1.1.4 Interpolation

Both the Sobolev-Slobodeckij and Bessel potential spaces can be viewed as a way to fill the gaps between integer valued Sobolev spaces.

Proposition 1.8 (Interpolation)**.** *Let $s_0 \neq s_1 \in \mathbb{R}, p \in (1,\infty)$, $0 < \theta < 1$ and*

$$s = s_0(1-\theta) + s_1\theta, \quad p = p_0(1-\theta) + p_1\theta.$$

Then, given $\Omega \subset \mathbb{R}^d$ open with uniformly Lipschitz boundary it holds that

$$B^{s,p} = [B^{s_0,p}(\Omega), B^{s_1,p}(\Omega)]_\theta, \quad H^{s,p}(\Omega) = [H^{s_0,p_0}(\Omega), H^{s_1,p_1}(\Omega)]_\theta,$$

where $[X,Y]_\theta$ denotes the complex interpolation space.

The result can be found in [?] page 45 for $\Omega = \mathbb{R}^d$. The general results follows by extension. See [?] page 424. In particular, if we write $k := \lfloor s \rfloor$ and $\gamma := s - k$, then

$$H^{s,p}(\Omega) = \left[H^{k,p}(\Omega), H^{k+1,p}(\Omega) \right]_\gamma = \left[L^p(\Omega), H^{k+1,p}(\Omega \right]_{s/(k+1)}.$$

1.2 Relationship between the definitions

The following result shows the inclusions between $W^{s,p}, H^{s,p}, B^{s,p}$ and can be found in [?] page 224 and in [?] page 155.

Theorem 1.9. *Let $\Omega \subset \mathbb{R}^d$ be open with uniformly Lipschitz boundary and $s \in [0, \infty)$. Then,*

$$\begin{aligned} H^{s+\epsilon,p}(\Omega) \subset B^{s,p}(\Omega) \subset H^{s,p}(\Omega) \quad \forall p \in (1,2] \\ B^{s+\epsilon,p}(\Omega) \subset H^{s,p}(\Omega) \subset B^{s,p}(\Omega) \quad \forall p \in [2,\infty), \end{aligned}$$

where the above inclusions are continuous and dense. Furthermore,

$$W^{s,p}(\Omega) = \begin{cases} H^{s,p}(\Omega) & \text{if } s \in \mathbb{N}_0 \\ B^{s,p}(\Omega) & \text{if } s \notin \mathbb{N}_0 \end{cases}. \tag{2}$$

In consequence, for $p = 2$,

$$H^{s,2}(\Omega) = W^{s,2}(\Omega) = B^{s,2}(\Omega). \tag{3}$$

The equality in (2) shows that, as long as we understand the behaviour of $H^{s,p}(\Omega)$ and $B^{s,p}(\Omega)$ we can completely determine that of $W^{s,p}(\Omega)$. It also justifies the following extension of $W^{s,p}(\Omega)$ to negative regularity.

Definition 1.10 (Slodeckij space negative s)**.** *Let $\Omega \subset \mathbb{R}^d$ be open with uniformly Lipschitz boundary. Then, given $p \in [1, \infty)$ and any $s \in \mathbb{R}$ we define*

$$W^{s,p}(\Omega) = \begin{cases} H^{s,p}(\Omega) & \text{if } s \in \mathbb{N}_0 \\ B^{s,p}(\Omega) & \text{if } s \notin \mathbb{N}_0 \end{cases}.$$

The equality for $p = 2$ in (3) justifies that, for sufficiently regular domains, all three spaces are written $H^s(\Omega)$. We will prove the left hand side of this equivalence in Exercise 3. For $p \neq 2$ the inclusions are in general strict. An example is constructed in [?] page 161 exercise 6.8.

Exercise 3 (Equivalence of fractional spaces)**.** Show that

$$H^{s,2}(\mathbb{R}^d) = W^{s,2}(\mathbb{R}^d).$$

Hint. We want to show that the norms are equivalent. That is, that

$$\|u\|_{B^{s,2}(\mathbb{R}^d)} \sim \|u\|_{H^{s,2}(\mathbb{R}^d)}.$$

We already know this is the case when s is an integer so it suffices to show that the norms are equivalent for $s = \gamma \in (0, 1)$. That is, that

$$|u|_{s,2}^2 \sim \int_{\mathbb{R}^d} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 d\xi$$

By multiple changes of variable and Plancherel's theorem we have that

$$\begin{aligned} |u|_{\gamma,2}^2 &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x+y) - u(y)|^2}{|x|^{d+2\gamma}} \,\mathrm{d}x \,\mathrm{d}y = \int_{\mathbb{R}^d} \frac{\|\mathcal{F}(u(x+\cdot) - u)\|^2}{|x|^{d+2\gamma}} \,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|e^{-2\pi i x\cdot\xi} - 1|^2}{|x|^{d+2\gamma}} |\widehat{u}(\xi)|^2 \,\mathrm{d}x \,\mathrm{d}\xi = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{1 - \cos(2\pi\xi \cdot x)}{|x|^{d+2\gamma}} \,\mathrm{d}x \right) |\widehat{u}(\xi)|^2 \,\mathrm{d}\xi. \end{aligned}$$

To treat the inner integral we note that it is rotationally invariant and so, by rotating, to the first axis and later changing variable $x \to x/|\xi|$ we get

$$\int_{\mathbb{R}^d} \frac{1-\cos(2\pi\xi\cdot x)}{|x|^{d+2\gamma}}\,\mathrm{d}x = \int_{\mathbb{R}^d} \frac{1-\cos(2\pi|\xi|x_1)}{|x|^{d+2\gamma}}\,\mathrm{d}x$$
$$= |\xi|^{2\gamma}\int_{\mathbb{R}^d} \frac{1-\cos(2\pi x_1)}{|x|^{d+2\gamma}}\,\mathrm{d}x \sim |\xi|^{2\gamma}.$$

The last integral is finite as, since $d+2\gamma > d$, the tails $|\xi| \to \infty$ are controlled, and since $1-\cos(2\pi x_1) \sim x_1^2 \leq |x|^2$ the integrand has order $-d+2(1-\gamma) > -d$ for $|\xi| \sim 0$. That said, substituting this back into the previous expression gives the desired result.

Exercise 4. Use the previous exercise 3 to show that, if Ω is a open set with uniformly Lipschitz boundary, then

$$H^{s,2}(\Omega) = W^{s,2}(\Omega).$$

Hint. By definition 1.3 choose a sequence $v_n \in H^{s,2}(\mathbb{R}^d)$ such that $\|v_n\|_{H^{s,2}(\mathbb{R}^d)} \to \|u\|_{H^{s,2}(\Omega)}$ in $H^s(\Omega)$. Then,

$$\|u\|_{H^{s,2}(\Omega)} = \lim_{n\to\infty}\|v_n\|_{H^{2,2}(\mathbb{R}^d)} \sim \lim_{n\to\infty}\|v_n\|_{B^{s,2}(\mathbb{R}^d)} \geq \|u\|_{B^{s,2}(\Omega)}.$$

To obtain the reverse inequality use the existence of a continuous extension operator $E : W^{s,2}(\Omega) \to W^{s,2}(\mathbb{R}^d)$ (see [?] page 33) to obtain

$$\|u\|_{W^{s,2}(\Omega)} \sim \|Eu\|_{W^{s,2}(\mathbb{R}^d)} \geq \|u\|_{H^{s,2}(\mathbb{R}^d)}.$$

The above suggests that integrals appearing in the definition of the slodeckij spaces 1.1 correspond to differentiating a fractional amount of times. This indeed is the case

Definition 1.11. *Given $\gamma \in [0,+\infty)$ and $u \in \mathcal{S}(\mathbb{R}^d)$ we define the fractional Laplacian as*

$$(-\Delta)^\gamma u(x) := \mathcal{F}^{-1}(|2\pi\xi|^{2\gamma}\,\widehat{u}(\xi)).$$

Proposition 1.12. *For $\gamma \in (0,1)$ and $u \in H^s(\mathbb{R}^d)$ it holds that*

$$(-\Delta)^\gamma u(x) = C\int_{\mathbb{R}^d} \frac{u(x)-u(x+y)}{|y|^{d+2\gamma}}\,\mathrm{d}y,$$

where C is a constant that depends on d, γ.

Proof. The above equality may seem odd at first if we compare with the integral in 1.1 where a square appears in the numerator which gives us our 2 in the 2γ. However, it is justified by the fact that, by the change of variables $y \to -y$,

$$\int_{\mathbb{R}^d} \frac{u(x)-u(x+y)}{|y|^{d+2\gamma}}\,\mathrm{d}y = \int_{\mathbb{R}^d} \frac{u(x)-u(x-y)}{|y|^{d+2\gamma}}\,\mathrm{d}y.$$

So we can get the *second* order difference in the numerator by adding the two integrals.

$$\int_{\mathbb{R}^d} \frac{u(y)-u(x+y)}{|y|^{d+2\gamma}}\,\mathrm{d}y = -\frac{1}{2}\int_{\mathbb{R}^d} \frac{u(x+y)-2u(x)+u(x-y)}{|y|^{d+2\gamma}}\,\mathrm{d}y. \tag{4}$$

That said, we must show that

$$|\xi|^{2\gamma}\,\widehat{u}(\xi) \sim \mathcal{F}\left(\int_{\mathbb{R}^d} \frac{u(x)-u(x+y)}{|y|^{d+2\gamma}}\,\mathrm{d}y\right)$$

Using (4) and proceeding as in exercise 1.9 gives

$$\begin{aligned}
&\mathcal{F}\left(\int_{\mathbb{R}^d} \frac{u(x)-u(x+y)}{|y|^{d+2\gamma}}\,\mathrm{d}y\right) = -\frac{1}{2}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} \frac{e^{-2\pi i y\cdot\xi}-2+e^{2\pi i y\cdot\xi}}{|y|^{d+2\gamma}}\,\mathrm{d}y\right)\widehat{u}(\xi)\,\mathrm{d}\xi\\
&= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} \frac{1-\cos(2\pi y\cdot\xi)}{|y|^{d+2\gamma}}\,\mathrm{d}y\right)\widehat{u}(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}^d} \frac{1-\cos(2\pi y_1)}{|y|^{d+2\gamma}}\,\mathrm{d}y \int_{\mathbb{R}^d} |\xi|^{2\gamma}\,\widehat{u}(\xi)\,\mathrm{d}\xi\\
&\sim |\xi|^{2\gamma}\,\widehat{u}(\xi)\,\mathrm{d}\xi.
\end{aligned}$$

This completes the proof, and shows that the explicit expression for C is

$$C = \frac{1}{(2\pi)^{2\gamma}}\int_{\mathbb{R}^d} \frac{1-\cos(2\pi y_1)}{|y|^{d+2\gamma}}\,\mathrm{d}y.$$

□

2 Dual of Sobolev spaces and correspondence with negative regularity

Negative orders of regularity correspond to the dual of Sobolev spaces. This is best seen in the integer case, where the following result holds (see [?] pages (326-344) for the case $p = 2$).

Theorem 2.1. *For all $k \in \mathbb{Z}$ and $p \in [1,\infty)$ it holds that*

$$H_0^{k,p}(\Omega)' = H^{-k,p'}(\Omega), \quad W_0^{k,p}(\Omega)' = W^{-k,p'}(\Omega).$$

The first equality will be discussed in the next subsection and is most easily proves when $\Omega = \mathbb{R}^d$, in which case one can use the correspond $\Lambda^s : H^{r,p}(\mathbb{R}^d) \xrightarrow{\sim} H^{r-s,p}(\mathbb{R}^d)$ together with the reflexivity of $L^p(\mathbb{R}^d)$. The second equality is a direct consequence of the integer order equality $W^{k,p}(\Omega) = H^{k,p}(\Omega)$ of Theorem 1.9. For fractional order regularities, we have the following result which can be found in [?] page 228.

Theorem 2.2. *The spaces $W^{s,p}(\Omega), H^{s,p}(\Omega), B^{s,p}(\Omega)$ are reflexive Banach spaces with duals*

$$W^{s,p}(\Omega)' = W_{\overline{\Omega}}^{-s,p'}(\mathbb{R}^d), \quad H^{s,p}(\Omega)' = H_{\overline{\Omega}}^{-s,p'}(\mathbb{R}^d), \quad B^{s,p}(\Omega)' = B_{\overline{\Omega}}^{-s,p'}(\mathbb{R}^d).$$

where p' is the conjugate exponent of p and given a space of distributions X on $\mathbb{R}^d$ we define $X_{\overline{\Omega}}$ as the space of distributions on $\mathbb{R}^d$ which are supported in $\overline{\Omega}$. In particular, for $\Omega = \mathbb{R}^d$,

$$W^{s,p}(\mathbb{R}^d)' = W^{-s,p'}(\mathbb{R}^d), \quad H^{s,p}(\mathbb{R}^d)' = H^{-s,p'}(\mathbb{R}^d), \quad B^{s,p}(\mathbb{R}^d)' = B^{-s,p'}(\mathbb{R}^d).$$

We have already seen that $W^{s,p}(\Omega)$ and $H^{s,p}(\Omega)$ are Banach spaces, one can similarly show that $B^{s,p}(\Omega)$ is a Banach space. Furthermore, the spaces are all reflexive for smooth domains.

Observation 2. Some authors define given $s > 0$

$$W^{-s,p'}(\Omega)' := W^{s,p}(\Omega)'.$$

See for example [?]. This is equivalent to our definition when $\Omega = \mathbb{R}^d$ or when $s \in k$. However, in other cases, the two definitions are not equivalent.

2.1 The dual of $H^{s,p}(\mathbb{R}^d)$ and $B^{s,p}(\mathbb{R}^d)$

For some motivation we start by considering the case $\Omega = \mathbb{R}^d$. In this case, since the closure of $C_c^\infty(\mathbb{R}^d)$ in $H^{s,p}(\mathbb{R}^d)$ (which by definition is $H_0^{s,p}(\mathbb{R}^d)$) is itself $H^{s,p}(\mathbb{R}^d)$, we have that $H_0^{s,p}(\mathbb{R}^d) = H^{s,p}(\mathbb{R}^d)$.

Exercise 5 (Dual identification)**.** Prove the identification $H^{-s,p'}(\mathbb{R}^d) = H^{s,p}(\mathbb{R}^d)'$.

Hint. Consider the mapping $H_0^{-s,p'}(\mathbb{R}^d) \to H_0^{s,p}(\mathbb{R}^d)'$ given by $f \mapsto \ell_f$ where

$$\ell_f(u) := \int_{\mathbb{R}^d} (\Lambda^s u)(\Lambda^{-s} f).$$

Show that this mapping is well defined and continuous. To see that it is invertible, show that, by duality, given $\ell \in H^{s,p}(\mathbb{R}^d)'$ and $u \in H^{s,p}(\mathbb{R}^d)$, it holds that

$$(u, \ell) = (\Lambda^s u, \Lambda^{-s} \ell).$$

Since $\Lambda^s u \in L^p(\mathbb{R}^d)$ we deduce that $\Lambda^{-s}\ell \in L^p(\mathbb{R}^d)'$ and so by the Riesz representation theorem there exists $f_\ell \in L^{p'}(\mathbb{R}^d)$ such that $\Lambda^{-s}\ell = \langle \cdot, f_\ell \rangle$. Show that the mapping

$$H^{s,p}(\mathbb{R}^d)' \longrightarrow H^{-s,p'}(\mathbb{R}^d); \quad \ell = \langle \cdot, \Lambda^s f_\ell \rangle \to \Lambda^s f_\ell,$$

is the inverse of the previous one.

Exercise 6. We also know that, since $H^s(\mathbb{R}^d)$ is a Hilbert space, so by the Riesz representation theorem we have the identification $H^s(\mathbb{R}^d) = H^s(\mathbb{R}^d)'$. So by the previous exercise $H^{-s}(\mathbb{R}^d) = H^s(\mathbb{R}^d)$ How is this possible?

Hint. It does **not** hold that $H^{-s}(\mathbb{R}^d) = H^s(\mathbb{R}^d)$. The problem occurs when considering too many identifications at once, as we are identifying duals using different inner products. By following the mappings we obtain a bijective isomorphism

$$\begin{aligned} &H^s(\mathbb{R}^d) \to H^s(\mathbb{R}^d)' \to H^{-s}(\mathbb{R}^d) \\ &u \longmapsto \langle \cdot, u \rangle_{H^s(\mathbb{R}^d)} = \langle \cdot, \Lambda^{2s} u \rangle \mapsto \Lambda^{2s} u. \end{aligned}$$

However, the isomorphism is Δ^{2s}, which is hardly the identity mapping.

For another example where confusion with these kind of identifications can arise see remark 3 on page 136 of [**?**].

2.2 The dual of $H_0^{s,p}(\Omega)$

Given an extension domain Ω and $s \in \mathbb{R}$, one can define extension and restriction operators,

$$E : H^{s,p}(\Omega) \to H^{s,p}(\mathbb{R}^d), \quad \rho : H^{s,p}(\mathbb{R}^d) \to H^{s,p}(\Omega),$$

which verify $\rho \circ E = \boldsymbol{I}_{H^s(\Omega)}$. As a result, restriction is surjective and we can factor $H^{s,p}(\Omega)$ as

$$H^{s,p}(\Omega) \simeq H^{s,p}(\mathbb{R}^d)/H^s_{\Omega^c}(\mathbb{R}^d), \tag{5}$$

where given a closed set $K \subset \mathbb{R}^d$ we define

$$H_K^{s,p}(\mathbb{R}^d) := \Big\{ u \in H^{s,p}(\mathbb{R}^d) : \operatorname{supp}(u) \subset K \Big\{ ,$$

where the support is to be understood in the sense of distributions Now, given a Banach space X and a closed subspace $Y \hookrightarrow X$, elements of X' can be restricted to Y, obtaining functionals in Y'. The kernel of this restriction is $Y^\circ := \{\ell \in X' : Y \subset \ker(\ell)\{$. Since, by the Hahn Banach theorem, the restriction is surjective, we obtain the factorization

$$Y' \simeq X'/Y^\circ. \tag{6}$$

Applying this to $Y = H_0^{k,p}(\Omega) \hookrightarrow H^{k,p}(\mathbb{R}^d) = X$ we obtain the result of Theorem 2.1.

$$H_0^{k,p}(\Omega)' \simeq H^{k,p}(\mathbb{R}^d)'/H_{\Omega^c}^{k,p}(\mathbb{R}^d)' \simeq H^{-k,p'}(\mathbb{R}^d)/H_{\Omega^c}^{-k,p'}(\mathbb{R}^d) \simeq H^{-k,p'}(\Omega),$$

where the second equality is by Exercise 5 and the third by (5). This shows that the dual of $H_0^{k,p}(\Omega)$ is $H^{-k,p'}(\Omega)$. By also using the integer order equivalence of Theorem 1.9 we obtain Theorem 2.1.

As a final note, if our domain has a boundary, $H_0^k(\Omega)'$ and $H^k(\Omega)'$ are not equal. Rather,

$$H^{k,p}(\Omega)' \simeq H_{\overline{\Omega}}^{-k,p'}(\mathbb{R}^d), \quad H^{-k,p'}(\Omega) \simeq H^{-k,p'}(\mathbb{R}^d)/H_{\Omega^c}^{-k,p'}(\mathbb{R}^d).$$

See [?] Section 4 for more details.

3 Representation theorems

We know that we can identify the spaces $H^{s,p}(\mathbb{R}^d)$ and $B^{s,p}(\mathbb{R}^d)$ with the lower order spaces $H^{s-r,p}(\mathbb{R}^d)$ and $B^{s-r,p}(\mathbb{R}^d)$ by application of Λ^r. That is, by differentiating r times. That is, spaces of lower order regularity are obtained by differentiating functions with higher regularity. We show how to extend this idea to smooth domains in some particular cases.

Theorem 3.1 (Representation of $W_0^{k,p}(\Omega)'$)**.** *Let $\Omega \subset \mathbb{R}^d$ be be open with uniformly Lipschitz boundary and let $k \in \mathbb{N}$ and $p \in [1, \infty)$. Then, every element in $W^{-k,p'}(\Omega) = W^{k,p}(\Omega)'$ is the unique extension of a distribution of the form*

$$\sum_{1 \leq |\alpha| \leq k} D^\alpha u_\alpha \in \mathcal{D}'(\Omega), \quad \textit{where } u_\alpha \in L^{p'}(\Omega).$$

Proof. Define the mapping

$$\begin{aligned} T : W^{k,p}(\Omega) &\longrightarrow L^p(\Omega \to \mathbb{R}^n) \\ u &\longmapsto (D^\alpha u)_{1 \leq |\alpha| \leq k}. \end{aligned}$$

Where the notation just says that we send u to the vector formed by all its derivatives. By our definition of the norm on $W^{k,p}(\Omega)$, we have that T is an isometry and in particular continuously invertible on its image. Denote the image of T by $X := \operatorname{Im}(T)$. Given $\ell \in W^{-k,p'}(\Omega)$ we define

$$\ell_0 : X \to \mathbb{R}, \quad \ell_0(\mathbf{w}) := \ell(T^{-1}\mathbf{w}), \quad \forall \mathbf{w} \in X.$$

By Hahn Banach's theorem we can extend ℓ_0 from X to a functional $\ell_1 \in L^p(\Omega \to \mathbb{R}^n)'$ and by the Riesz representation theorem we have that there exists a unique $\mathbf{f} = (f_\alpha)_{1 \leq |\alpha| \leq k} \in L^{p'}(\Omega \to \mathbb{R}^n)$ such that

$$\ell_1(\mathbf{w}) = \int_\Omega \mathbf{w} \cdot \mathbf{h}, \quad \forall \mathbf{w} \in L^p(\Omega \to \mathbb{R}^n).$$

By construction, it holds that, for all $v \in W^{k,p}(\Omega)$

$$\ell(u) = \ell_0(Tv) = \int_\Omega Tv \cdot \mathbf{f} = \sum_{1 \leq |\alpha| \leq k} \int_\Omega f_\alpha D^\alpha v.$$

In particular, this holds for $v \in \mathcal{D}(\Omega)$ and if we set $u_\alpha := (-1)^\alpha h_\alpha$ we obtain that for all $v \in \mathcal{D}(\Omega)$

$$\ell(v) = \left(v, \sum_{1 \le |\alpha| \le k} D^\alpha u_\alpha \right) =: \omega(v) \tag{7}$$

(we recall the notation (v, ω) for the duality pairing). By definitions of the norm on $W^{k,p}(\Omega)$ and Cauchy Schwartz, we have that ω is continuous with respect to the norm on $W^{k,p}(\Omega)$ and so we may extend it uniquely to the closure of $\mathcal{D}(\Omega)$ in $W^{k,p}(\Omega)$ which is $W_0^{k,p}(\Omega)$. By (7) the extension is necessarily ω. This completes the proof. $\square$

The above theorem shows that $W^{-k,p'}(\Omega)$ can be equivalently formed by differentiating k times functions in $L^{p'}(\Omega)$. The proof also sheds some light as to why $W^{-s,p'}(\Omega)$ is the dual of $W_0^{k,p}(\Omega)$ and not the dual of $W^{k,p}(\Omega)$. The reason is that the elements of $W_0^{k,p}(\Omega)$ are the ones that can be extended to distributions in $\mathcal{D}'(\Omega)$ and so are the ones that we can integrate against. Finally, though the extension from $\mathcal{D}'(\Omega)$ to $W^{-s,p}(\Omega)$ is unique the functions u_α will not be, for example if $|\alpha| > 0$ it is possible to add a constant to u_α and still obtain the same result.

Exercise 7. Show that for $s > 0$ and $p \in [1, \infty)$ every element in $H^{-s,p'}(\Omega)$ can be written in the form $w|\,\partial\Omega$, where

$$w = \sum_{0 \le |\alpha| \le k} \Lambda^\gamma D^\alpha u_\alpha \in \mathcal{D}'(\mathbb{R}^d), \quad \text{where } u_\alpha \in L^{p'}(\mathbb{R}^d),$$

where $k = \lfloor s \rfloor$ and $\gamma = s - k$.

Hint. Use that $\Lambda^\gamma : H^{s,p}(\mathbb{R}^d) \to H^{k,p}(\mathbb{R}^d)$ is an isomorphism and the just proved theorem 3.1 together with the integer equivalence in Theorem 1.9 to show that

$$H^{s,p}(\mathbb{R}^d)' = \left\{ \sum_{0 \le |\alpha| \le k} \Lambda^\gamma D^\alpha u_\alpha \in \mathcal{D}'(\mathbb{R}^d), \quad \text{where } u_\alpha \in L^{p'}(\mathbb{R}^d) \right\}.$$

Now conclude by the definition of $H^{-s,p'}(\Omega)$ for open domains 1.4.

The above results extend to Besov spaces, see [?] page 227. This gives,

Theorem 3.2. *Let $k \in \mathbb{N}_0, \gamma \in [0,1)\theta \in (0,1)$ and $p \in [1, \infty)$. Then,*

$$B^{\theta-k,p}(\Omega) = \left\{ \sum_{0 \le |\alpha| \le k} D^\alpha u_\alpha \in \mathcal{D}'(\Omega), \quad \textit{where } u_\alpha \in B^{\theta,p}(\Omega) \right\}$$

$$H^{\gamma-k,p}(\Omega) = \left\{ \sum_{0 \le |\alpha| \le k} D^\alpha u_\alpha \in \mathcal{D}'(\Omega), \quad \textit{where } u_\alpha \in H^{\gamma,p}(\Omega) \right\}$$

$$setminus W^{\gamma-k,p}(\Omega) = \left\{ \sum_{0 \le |\alpha| \le k} D^\alpha u_\alpha \in \mathcal{D}'(\Omega), \quad \textit{where } u_\alpha \in W^{\gamma,p}(\Omega) \right\}.$$

3.1 Some applications

The following theorem can be found in [?] page 228 and serves as a generalization of the trace theorem for fractional Sobolev spaces.

Theorem 3.3 (Fractional trace theorem)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open set with uniformly Lipschitz boundary. Then, for all $s > 1/p$, the trace operator* Tr *can be extended from $C_c^\infty(\mathbb{R}^d)$ to a bounded operator*

$$\mathrm{Tr} : H^{s,p}(\Omega) \to B^{s-1/p,p}(\partial\Omega), \quad \mathrm{Tr} : B^{s,p}(\Omega) \to B^{s-1/p,p}(\partial\Omega).$$

In particular, as we have respective equalities for $W^{s,p}(\Omega)$ with $H^{s,p}(\Omega)$ and $W^{s,p}(\Omega)$ for respectively non-integer s, we can also extend

$$\mathrm{Tr} : W^{s,p}(\Omega) \to B^{s-1/p,p}(\partial\Omega).$$

One can also obtain improved Sobolev embeddings for fractional Sobolev spaces. For example, see [**?**] page 224

Theorem 3.4. *Given $s > n/p + \gamma$, where $t > 0$, the space $W_p^s\left(\mathbb{R}^f\right)$ is continuously embedded in $C_b^\gamma\left(\mathbb{R}^d\right)$. This embedding also holds for $s = n/p + t$ provided that γ is noninteger. In particular, $W_p^s(\Omega)$ is continuously embedded in $C_b(\Omega)$ for $s > n/p$.*

Theorem 3.5 (Rellich Kondratov)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open set with uniformly Lipschitz boundary. Suppose $1 < p \leq q < \infty$ and $-\infty < t \leq s < \infty$ satisfy $s - \frac{n}{p} \geq t - \frac{n}{q}$. Then,*

$$W^{s,p}(\Omega) \hookrightarrow W^{t,q}\left(\overline{\Omega}\right)$$

In particular, $W^{s,\bar{p}}(\Omega) \hookrightarrow W^{t,p}(\Omega)$. Furthermore, if Ω is bounded, the inclusion is compact.

Observation 3. Edit extension theorem for uniformly Lipschitz domains and then whenever C^k boundary is invoked $C^{0,1}$ can be used. The above extension result can also be proved when Ω is open with uniformly Lipschitz boundary. In practice, this just means that $\partial\Omega$ and of class $C^{0,1}$ (Lipschitz continuous). See [**?**] pages 423-430 for the details.