The Ornstein-Uhlenbeck Semigroup

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date

1 Three line summary

- There is a natural extension of the Laplacian to the Wiener space.
- The generator of the Laplacian is the Ornstein-Uhlenbeck semigroup.
- The Ornstein-Uhlenbeck semigroup in finite dimensions is the generator of the Ornstein-Uhlenbeck process, from which it derives its name.

2 The Laplacian of a random variable

First, we give some finite-dimensional motivation. Suppose that $f \in C_c^\infty(\mathbb{R}^d \to \mathbb{R}^d)$ and $g \in C_c^\infty(\mathbb{R}^d)$. Then an integration by parts shows that the adjoint of the gradient in $L^2(\mathbb{R}^d)$ is minus the divergence. That is,

$$\int_{\mathbb{R}^d} f(x) \cdot \nabla g(x) dx = -\int_{\mathbb{R}^d} \nabla \cdot f(x) \nabla g(x) dx.$$

Then, we define the Laplacian as minus the adjoint of the gradient ∇ composed with the gradient

$$\Delta := -\nabla^* \circ \nabla.$$

Which gives the familiar

$$\Delta_{\mathbb{R}^d} = \nabla \cdot \nabla = \partial_1^2 + \dots \partial_d^2$$

.

Of course, this is all well and good when the domain of f, g is a finite-dimensional space. Otherwise, there is no Lebesgue measure. We now move to what is our base case in our series of blog posts and consider a probability space $(\Omega, \mathbb{P}, \mathcal{F}_t)$ where $\mathcal{F}_t$ is generated by a Wiener process W_t. Then, as we have seen previously (link) the Skorohod integral δ is the adjoint of the Malliavin derivative D so we would like to define

$$\Delta := -\delta \circ D.$$

On what kind of random variables can we define this? Well let us take $X = \sum_{n=0}^{\infty} I_n(f_n)$ with a rapidly decaying chaos expansion, then

$$\Delta X = -\delta(DX) = -\delta \left(\sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)) \right) = -\sum_{n=1}^{\infty} n I_n(f_n).$$

All we require for this expression to make sense is that- the right-hand side is in $L^2(\Omega)$. That is, by Ito's n-th isometry (link), that

$$\sum_{n=0}^{\infty} n^2 \|f_n\|_{L^2(I_n)} < \infty.$$

Is this a space we've dealt with before? Well if we recall our old spaces $\mathbb{D}^{k,p}$ (link). Then we have that

$$\int_{I^2} \|D_{t,s}X\|^2_{L^2(\Omega)} dsdt = \int_{I^2} \left\| \sum_{n=2}^{\infty} n(n-1) I_{n-2}(f_n(\cdot, s, t)) \right\|^2_{L^2(\Omega)}$$

$$= \int_{I^2} \sum_{n=2}^{\infty} n^2(n-1)^2(n-2)! \|f_n(\cdot, s, t)\|^2_{L^2(I_{n-2})} = \sum_{n=2}^{\infty} n(n-1)n! \|f_n(\cdot, s, t)\|^2_{L^2(I_n)}.$$

Where analogous calculations go through if we have more derivatives to get the terms $n(n-1)\cdots(n-(k-1))$. This shows that

$$\mathbb{D}^{k,p} := \left\{ X \in L^2(\Omega) : \quad \|X\|_{\mathbb{D}^{k,2}} = \sum_{n=0}^{\infty} n^k n! \|f_n\|_{L^p(I_n)} < \infty \right\}.$$

Thus, the domain of Δ is exactly $\mathbb{D}^{2,2}$. This is quite pleasing as, as we have observed earlier, the spaces $\mathbb{D}^{k,p}$ mimic the Sobolev spaces $W^{k,p}$, when $p = 2$ this resemblance is quite strong as we have that the norm on $H^k := W^{k,2}$ is

$$\|f\|_{H^k \mathbb{R}^d)} = \int_{\mathbb{R}^d} \langle \xi \rangle^k \hat{f}(\xi)^2 d\xi.$$

Which is formally equal to the one just derived for $\mathbb{D}^{k,2}$. It is very interesting to observe that, directly from the definition, we obtain a basis of eigenvalues of Δ. Let us define

$$H_n := \{X \in L^2(\Omega) : X = I_n(f_n), \quad \text{for some } f_n \in L^2_S(I^n)\}.$$

That is, H_n are the random variables that only have the n-th term in their chaos expansion to be non-zero. Then by the chaos expansion theorem, we know that

$$L^2(\Omega) = \overline{\oplus_{n=0}^{\infty} H_n}.$$

And by construction of the Laplacian, $\Delta e_n = n e_n$ for every $e_n \in H_n$. In fact, by the uniqueness of the chaos expansion, the elements of H_n for some $n \in \mathbb{N}$ are the unique eigenvectors of Δ.

3 The Ornstein-Uhlenbeck semigroup

As it turns out, Δ defines a semigroup

Definition 1. *The* Ornstein-Uhlenbeck semigroup *is the family of operators* $\Phi(t) : L^2(\Omega) \to L^2(\Omega)$

$$\Phi(t)X := \sum_{n=0}^{\infty} e^{-nt} I_n(f_n), \quad \forall t \in I.$$

The term e^{-nt} is quite reminiscent of the semigroup for the heat equation

$$e^{t\Delta}u_0 := \int_{\mathbb{R}^d} e^{-4\pi^2\xi^2 t}\widehat{u_0}(\xi)d\xi,$$

and will cause an analogous smoothing effect by making the terms in the chaos expansion to decrease faster. To see that Φ defines a semigroup first note that, by the linearity of the iterated integrals,

$$\Phi(t)X := \sum_{n=0}^{\infty} I_n(e^{-nt} f_n).$$

So as a result

$$\Phi(t+s)X = \sum_{n=0}^{\infty} e^{-nt} I_n(e^{-ns} f_n) = \sum_{n=0}^{\infty} \Phi(t)\Phi(s)X.$$

Which shows that $\Phi(t+s) = \Phi(t) \circ \Phi(s)$. Finally, note that

$$\frac{\Phi(t)X - X}{t} = \sum_{n=0}^{\infty} \left(\frac{e^{-nt} - 1}{nt} \right) nI_n(f_n) \to -\sum_{n=0}^{\infty} nI_n(f_n) = \Delta X \in L^2(\Omega).$$

Where the commutation under the integral sign (with the counting measure) is justified as $(e^{-nt} - 1)/(nt)$ is uniformly bounded in n. There's an explicit formula for $\Phi(t)$.

Proposition 1 (Mehler's formula)**.** *Let $(\Omega, \mathcal{F}_t, \gamma)$ be the Wiener space, then*

$$\Phi(t)X(\omega) = \int_\Omega X\left(e^{-t}\omega + \sqrt{1 - e^{-2t}}\eta\right) \gamma(\eta) \in L^2(\Omega).$$

The proof is technical and can be found in Nualart's book [1] on page 74. Let us try to understand the formula and also the reason for the name of the semigroup. We consider as at the beginning of this post the finite-dimensional case but now with some Gaussian measure μ

$$\mu(A) := \int_A e^{-\frac{\|x\|^2}{2}} dx.$$

Then, integration by parts shows that

$$\begin{aligned}\int_{\mathbb{R}^d} f(x) \cdot \nabla g(x) d\mu(x) &= -\int_{\mathbb{R}^d} \nabla \cdot \left(e^{-\frac{\|x\|^2}{2}} f(x)\right) \nabla g(x) dx \\ &= \int_{\mathbb{R}^d} (x \cdot f(x) - \nabla \cdot f(x)) d\mu(x).\end{aligned}$$

That is, the adjoint of the gradient in $L^2(\mathbb{R}^d, \mu)$ is $x \cdot -\nabla \cdot$. Notice that we get the extra term that corresponds to multiplication by $x \cdot$.As a result, the Laplacian on $L^2(\mathbb{R}^d, \mu)$ is given by

$$\Delta_\mu = \nabla \cdot \nabla - x \cdot \nabla.$$

Furthermore, by Itô's formula, Δ_μ is the generator of the SDE

$$dX(t) = -X(t)dt + \sqrt{2}dW(t)$$

Let us write X_x for the solution to the above SDE with initial data $x \in \mathbb{R}^d$ That is, if we define

$$P_t X(x) := E[\varphi(X(t))],$$

then

$$\partial_t P_t X(x) = \Delta_\mu P_t X(x).$$

The process X that solves the SDE above is known as the *Ornstein-Uhlenbeck process* and, by the theory of linear SDEs, is given by

$$X_x(t) = e^{-t}x + \sqrt{2}\int_0^t e^{s-t}dW(t).$$

Since

$$\sqrt{2}\int_0^t e^{s-t}dW(t) \sim \sqrt{2}e^{-t}\mathcal{N}\left(0, \|e^{\cdot}\|^2_{L^2([0,t])}\right) = \sqrt{1-e^{-2t}}\mathcal{N}(0,1)$$

We deduce that for each fixed t we can find a measure $\gamma \sim \mathcal{N}(0,1)$ with

$$X(t) = e^{-t}x + \sqrt{1-e^{-2t}}\gamma.$$

We then get that

$$P_t\varphi(x) = \mathbb{E}\left[\varphi\left(e^{-t}x + \sqrt{1-e^{-2t}}\gamma\right)\right]$$

And by taking $\varphi = Id$ we recover Mehler's formula. This correspondence is expanded on in chapter 7 of Hairer's notes [2].

References

[1] D. Nualart, E. Nualart, Introduction to Malliavin calculus, Vol. 9, Cambridge University Press, 2018.
URL https://books.google.co.uk/books?hl=zh-CN&lr=&id=l_1uDwAAQBAJ&oi=fnd&pg=PR11&dq=nualart+introduction+malliavin&ots=_JuMhMkTMt&sig=Tx5y0Ou4kMNs73jLtMEs-kyXAuU&redir_esc=y#v=onepage&q=nualart%20introduction%20malliavin&f=false

[2] M. Hairer, An introduction to stochastic pdes, arXiv preprint arXiv:0907.4178 (2009).