

Malliavin Calculus

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Based on the book by el pringado [1].

1 Wiener Itô chaos expansion

1.1 Three line summary

- Square integrable *deterministic* functions of multiple variables can be repeatedly Itô integrated to get a square integrable *random* function.
- Repeated integration satisfies an *isometry*.
- Every square integrable function can be uniquely written as the *sum* of repeated integrals.

1.2 Why this is important

I have no idea yet what this links to or why it's important. **To be filled out.**

1.3 Iterated integrals

We will write \mathcal{F}_t for the completion of the natural filtration generated by $W(t)$, on some measure space Ω and consider an interval $I = [0, T]$ or $I = [0, +\infty)$. We also recall the notation

$$\mathcal{M}_t^2 := \{f \in L^2([0, t] \times \Omega) \text{ that are continuous martingales}\}.$$

We will work with the spaces $L^2(I^n)$, $L^2_S(I^n)$, $L^2(S_n)$ of respectively square integrable functions, symmetric square integrable functions and square integrable functions on

$$S_n := \{0 \leq t_0 \leq \dots \leq t_n \leq T\}.$$

Since these functions don't involve a random variable we call them *deterministic*. Given $f \in L^2(I^n)$ we denote it's symmetrization by

$$f_S := \frac{1}{n!} \sum_{\sigma} f(t_{\sigma_1}, \dots, t_{\sigma_n}).$$

Where σ is summed over all the permutations of $\{1, \dots, n\}$. Note that if f is already symmetric then $f = f_S$. Given a *deterministic* function we can transform it into a *random* one by taking it's Itô integral. By integrating multiple times we obtain the following definition

Definition 1. Let $f \in L^2(S_n)$, then we define the n-fold Itô integral as

$$J_n(f) := \int_I \dots \int_0^{t_3} \int_0^{t_2} f(t_1, \dots, t_n) dW(t_1) dW(t_2) \dots dW(t_n).$$

The definition requires some care to verify to make sense of and to make sure it is well defined. For example note that the variables appearing in the upper bounds of the integration domain are the same as those appearing as integration variables. To better understand the definition, first note that, by the properties of the Itô integral, for each $t_2 \leq \dots \leq t_n$,

$$F_2(t_2, \dots, t_n) := \int_0^{t_2} f(t_1, \dots, t_n) dW(t_1) \in \mathcal{M}_{t_2}^2.$$

Where $F_2(\cdot, t_3, \dots, t_n) \in \mathcal{M}_{t_3}^2$ is viewed as a function of t_2 . Due to the above inclusion $F_2(\cdot, t_3, \dots, t_n)$ is Itô integrable and

$$F_3(t_3, \dots, t_n) := \int_0^{t_3} F_2(t_2, \dots, t_n) dW(t_2) \in \mathcal{M}_{t_3}^2.$$

Where the above is viewed as a function of t_3 . Iterating this process we get that $F_n \in \mathcal{M}_T^2$ and

$$J_n(f) = \int_I F_n(t_n) dW(t_n) \in L^2(\Omega).$$

To notice the subtleties involved, we note that one example of an integral that would be ill-defined however is if for $f \in L^2(I^n)$ we defined

$$\textcolor{red}{J}_n(f) := \int_I \cdots \int_I \int_I f(t_1, \dots, t_n) dW(t_1) dW(t_2) \dots dW(t_n).$$

In this case we have that the first integral is \mathcal{F}_T adapted and not \mathcal{F}_{t_2} adapted so we cannot continue integrating! As a result we instead give the following definition for symmetric functions.

Definition 2. Let $f \in L^2_S(I^n)$, then we define

$$I_n(f) := \int_I \cdots \int_I \int_I f(t_1, \dots, t_n) dW(t_1) dW(t_2) \dots dW(t_n) := n! J_n(f).$$

Where it is important to note that the second equality is by definition and where the rescaling factor is motivated by the fact that, by counting permutations, if I is finite then $\mu(I^n) = n! \mu(S_n)$.

Proposition 1 (Itô's n-th isometry). Let $f, g \in L^2(S^n)$ then

$$\langle J_n(f), J_m(g) \rangle_{L^2(\Omega)} = \langle f, g \rangle_{L^2(S_n)} \delta_{nm}$$

Proof. The proof is an application of Itô's (product) isometry to move the expectation into the iterated integrals. If the number of integrals are unequal ($n \neq m$) you get the expectation of a Itô integral which is zero. Otherwise you just get the (deterministic) inner product. \square

As a result we also get by a calculation that if $f, g \in L^2(I^n)$ then

$$\langle I_n(f), I_m(g) \rangle_{L^2(\Omega)} = n! \langle f, g \rangle_{L^2(I^n)} \delta_{nm}.$$

1.4 The chaos expansion

Lemma 1 (Itô's representation theorem). Let $\xi \in L^2(\Omega, \mathcal{F}_\infty)$, then there exists a unique $X(t) \in \mathbb{L}^2(I)$ such that

$$\xi = \mathbb{E}[\xi] + \int_I X(t) dW(t).$$

Furthermore it holds that $\|X\|_{L^2(I \times \Omega)} \leq \|\xi\|_{L^2(\Omega)}.$

Theorem 1 (Chaos expansion). *Let $\xi \in L^2(\Omega, \mathcal{F}_\infty)$, then there exists a unique sequence of functions $f_n \in L^2(S_n), g_n \in L^2(I^n)$ such that*

$$\xi = \sum_{n=0}^{\infty} f_n; \quad \xi = \sum_{n=0}^{\infty} g_n.$$

Proof. The proof is a bit technical but we sketch the main idea which is to iteratively apply Itô's representation theorem to get

$$\begin{aligned} \xi &= \mathbb{E}[\xi] + \int_I X_1(t) dW(t_1) = \mathbb{E}[\xi] + \int_I \mathbb{E}[X_1(t_1)] dW(t_1) \\ &+ \int_I \int_0^{t_2} X_2(t_1, t_2) dW(t_1) dW(t_2) = \dots = \sum_{n=0}^N J_n(g_n) + \int_{S_{N+1}} X_{N+1} dW^{\otimes(N+1)}. \end{aligned}$$

Where $g_0 = \mathcal{E}[\xi], g_1(t_1) = \mathcal{E}[X_1](t_1)$ and so on. By an iteration we deduce that if we write ϕ_{N+1} for the last integral term, then

$$\mathbb{E}[\phi_{N+1}] \leq \dots \leq \mathbb{E}[\xi^2].$$

Furthermore, by Itô's n -th isometry we have that the terms in the above sum are orthogonal so

$$\|\xi\|_{L^2(\Omega)}^2 = \sum_{n=0}^N \|J_n(f_n)\|_{L^2(\Omega)}^2 + \|\phi_{N+1}\|_{L^2(\Omega)}^2.$$

Since the last term is bounded, we obtain a bound on the $L^2(\Omega)$ norm of the sum uniform in n so the sum converges. As a result so does ϕ_N to some ϕ_∞ . It suffices to see that $\phi_\infty = 0$. This can be proved using that, by Itô's n -th isometry, ϕ_{N+1} is orthogonal to $J_n(h)$ for any $h \in L^2(\Omega)$ and $n \leq N$. As a result the limit ϕ_∞ is orthogonal to $J_n(h)$ for all n and by a density argument (there are a lot of these functions $J_n(h)$!) we obtain $\phi_\infty = 0$. This proves the first sum of the theorem.

It remains to prove the second part, but this follows by extending the f_n by 0 on $I^n \setminus S_n$ and taking g_n to be the symmetrization of this extension. \square

References

- [1] G. D. Nunno, B. Øksendal, F. Proske, Malliavin calculus for Lévy processes with applications to finance, Springer, 2008.