Elliptic PDE I

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1 Three point summary

- Elliptic partial differential equations (PDE) are PDE with no time variable and whose leading order derivatives satisfy a positivity condition.
- Using Lax Milgram's theorem, we can prove the existence and uniqueness of weak (distributional) solutions if the reaction term dominates the transport term. Using the Fredholm alternative, we can characterize the spectrum of the elliptic operator and the existence of solutions.
- Under suitable smoothness assumptions on the coefficients and domain, the solution map of the PDE adds two derivatives to the input function. This improved regularity allows us to recover classical solutions if the coefficients are smooth enough.

2 Why should I care?

Many problems arising in physics, such as the Laplace and Poisson equation, are elliptic PDE. Furthermore, the tools used to analyze them can be extrapolated to other settings, such as parabolic PDE. The analysis also helps contextualize and provide motivation for theoretical tools such as Hilbert spaces, compact operators and Fredholm operators.

3 Notation

We will use the Einstein convention that indices, when they are repeated, are summed over. For example we will write

$$\nabla \cdot (\boldsymbol{A}\nabla) = \sum_{i=1}^{n} \partial_i a_{ij} \partial_j = \partial_i A_{ij} \partial_j.$$

We will use Vinogradov notation $f \lesssim g$ to mean that there exists a constant $C > 0$ such that $f \leq Cg$. If we want to emphasize that the constant depends on a parameter α, we will write $f \lesssim_\alpha g$.

We fix $U \subset \mathbb{R}^n$ to be an open subset of $\mathbb{R}^n$ with **no conditions** on the regularity of ∂U. If we need to impose regularity on the boundary, we will write Ω instead of U.

4 Introduction

Welcome back to the second post on our series of PDE. In posts 1, 2, 3, 5 of the series we built up the theoretical framework necessary to define Sobolev spaces, spaces of weakly differentiable functions to which we could extend the concept of differentiation. In post 4 of the series, we gave a physical derivation of partial differential equations (both parabolic and elliptic) that justify why we are interested in such equations. We are now going to use the previous theory to study these equations.

5 The problem: Mathematical framework

We consider the following problem: given a bounded open set $U \subset \mathbb{R}^n$ and some coefficients $\boldsymbol{A}, \boldsymbol{b}, c$, we want to solve the following elliptic PDE

$$\mathcal{L}u := -\nabla \cdot (\boldsymbol{A}\nabla u) + \nabla \cdot (\boldsymbol{b}u) + cu = f; \quad u|_{\partial U} = 0, \tag{1}$$

where $f : U \to \mathbb{R}$ is some known function, u is the solution we want to find.

We recall from post 4 that physically; we can interpret u as the density of some substance, $\boldsymbol{A}$ as a diffusion matrix, $\boldsymbol{b}$ as a transport vector, c as a reaction coefficient and f as the source term. For the mathematical theory, we will need to assume that $\mathcal{L}$ is *elliptic*.

Definition 5.1. *Given $\boldsymbol{A} : U \to \mathbb{R}^{d\times d}, \boldsymbol{b} : U \to \mathbb{R}^d$ and $c : U \to \mathbb{R}$ we say that the operator*

$$\mathcal{L}u := -\nabla \cdot (\boldsymbol{A}\nabla u) + \nabla \cdot (\boldsymbol{b}u) + c \tag{2}$$

is elliptic *if there exists $\alpha > 0$ such that*

$$\xi^T \boldsymbol{A}(x)\xi \geq \alpha |\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \quad \forall x \in U. \tag{3}$$

We also say that $\boldsymbol{A}$ is elliptic.

There are some points to clear up. Firstly, if this is the first time you've encountered the ellipticity condition in (3), then it may seem a bit strange. With the previous physical interpretation, the ellipticity condition (3) says that diffusion occurs from the region of higher to lower density. Mathematically speaking, (3) will prove necessary to apply Lax Milgram's theorem and obtain regularity estimates on u.

When developing the mathematical theory of any equation, the first step to establish is whether the equation is *well-posed*.

Definition 5.2. *We say that an equation is* well-posed *if*

1. *It has a solution.*
2. *The solution is unique.*
3. *The solution depends continuously on the data.*

The above definition is due to Hadamard and is the standard definition of well-posedness in the context of PDE. The three properties above make the problem nice to work with and may be familiar from the basic theory of ODE. However, not all problems are well-posed. Ill-posedness often arises when one works with inverse problems, such as the backward heat equation, where one tries to recover the initial heat distribution from the final one.

The well-posedness of any given PDE is highly contingent on the space considered. In our case, we still need to define which function space our coefficients $\boldsymbol{A}, \boldsymbol{b}, c$ live in and what space $\mathcal{L}$ acts on. It would be natural to assume that we need $\boldsymbol{A}$ and $\boldsymbol{b}$ to be differentiable. However, the following will suffice.

Assumption 1. We assume that $A_{ij}, b_i, c \in L^\infty(U)$ for all $i, j = 1, \ldots, d$. Furthermore, $\boldsymbol{A}$ is symmetric ($A_{ij} = A_{ji}$) and elliptic.

In the future, i, j will always run from 1 to d, where d is the dimension of the space.

Observation 1. We lose no generality by assuming that $\boldsymbol{A}$ is symmetric as $\partial_{ij} u = \partial_{ji} u$. If $\boldsymbol{A}$ is not symmetric, we can replace $\boldsymbol{A}$ by $(\boldsymbol{A} + \boldsymbol{A}^T)/2$ and equation (1) will remain unchanged.

The first part of Assumption 1 will make it easy to get bounds on $\mathcal{L}$, and the second part will prove useful when we look at the spectral theory of $\mathcal{L}$. Now, to make sense of (1), we need to define what we mean by a solution. Here, the theory of Sobolev Spaces and the Fourier transform prove crucial. We will work with the following space.

Definition 5.3 (Negative Sobolev space). *Given $k \in \mathbb{N}$ we define*

$$H^{-k}(U) := H_0^k(U)'$$

For more details on why we denote the dual using negative exponents, see the relevant section in the previous post on fractional Sobolev spaces. We recall also that every element in this space can be written as the sum of derivatives up to order k of a function in $L^2(U)$.

Exercise 1. Suppose $A_{ij}, b_i, c \in L^\infty(U)$. Then, $\mathcal{L}$ defines a bounded linear operator

$$\mathcal{L} : H_0^1(U) \to H^{-1}(U).$$

Hint. By definition of the weak derivative, show that given $v \in C_c^\infty(U)$,

$$(v, \mathcal{L}u) = \int_U \boldsymbol{A}\nabla v \cdot \nabla u + \int_U \boldsymbol{b} \cdot \nabla v u + \int_U cvu.$$

Use this to conclude that,

$$|(v, \mathcal{L}u)| \lesssim \|v\|_{H^1(U)} \|u\|_{H^1(U)} .$$

So, $\mathcal{L}u \in H^{-1}(U)$ is well defined and $\mathcal{L}$ is bounded. Extend by density to $H_0^1(U)$.

Exercise 1 allows us to define the weak formulation of (1) and study its well-posedness using Lax Milgram's theorem. We will do this in the next section.

6 Weak solutions and well-posedness

By Exercise 1, we can make sense of the equation $\mathcal{L}u = f$ in a distributional (weak) sense as long as $f \in H^{-1}(U)$.

Definition 6.1 (Weak formulation). *Given $f \in H^{-1}(U)$, we say that $u \in H_0^1(U)$ solves (1) if*

$$B(u, v) := (v, \mathcal{L}u) = \int_U \boldsymbol{A}\nabla u \cdot \nabla v + \int_U \boldsymbol{b} \cdot (\nabla u) v + \int_U cuv = (v, f), \quad \forall v \in H_0^1(U). \tag{4}$$

In (4) we used the "duality notation" $(v, f) := f(v)$ for $f \in X, v \in X'$ (here $X = H_0^1(U)$). We have now reformulated our problem to something that looks very similar to the setup of Lax Milgram's theorem. We can now prove the well-posedness of (1) under certain conditions.

Theorem 6.2. *Let $U \subset \mathbb{R}^d$ be an arbitrary open set. Suppose Assumption 1 holds and let $\boldsymbol{b} = 0$. Then, equation (1) is well-posed, and we have the homeomorphism*

$$\mathcal{L} : H_0^1(U) \xrightarrow{\sim} H^{-1}(U).$$

Furthermore, $\left\|\mathcal{L}^{-1}\right\| \lesssim_U \alpha^{-1}$. The above also holds if $c \geq 0$ and U is bounded.

Proof. The continuity of B is a consequence of Exercise 1. It remains to see that B is coercive. For smooth $u \in C_c^\infty(U)$ we have that

$$B(u,u) = \int_U \boldsymbol{A}\nabla u \cdot \nabla u + \int_U cu^2 \geq \alpha \left\|\nabla u\right\|_{L^2(U\to\mathbb{R}^d)} + \int_U cu^2 \gtrsim \left\|u\right\|_{H_0^1(U)}. \tag{5}$$

Where in the first inequality, we used the ellipticity assumption on $\boldsymbol{A}$, and in the last inequality, we used Poincaré's inequality if U is bounded. The result now follows from Lax Milgram's theorem. $\square$

Theorem 6.2 is an example of the advantages of working with a weak formulation instead of classical solutions. The weak formulation allows us not only to make sense of our equation (1) for a wider class of coefficients but also provides a natural framework to study the well-posedness of (1).

Exercise 2. Show that, under the conditions of Theorem 6.2, if U is bounded, there is a countable basis of eigenfunctions for $\mathcal{L}$.

Hint. By Rellich's theorem $\mathcal{L}^{-1} : L^2(U) \to L^2(U)$ is compact and, since $\boldsymbol{b}$ is 0, $\mathcal{L}$ is also is self adjoint. As a result, so there is a countable basis of eigenvectors in $L^2(U)$.

In Theorem 6.2, we somewhat unsatisfyingly had to impose the extra assumption that $\boldsymbol{b}$ was identically zero and that $c > 0$. These extra assumptions can be done away with but at the cost of modifying our initial problem by a correction term γ so we can obtain a coercive operator B_γ.

Theorem 6.3 (Modified problem)**.** *Let $U \subset \mathbb{R}^d$ be any open set and let Assumption 1 hold. Then, there exists some constant $\nu \geq 0$ (depending on the coefficients) such that for all $\gamma \geq \nu$ the operator $\mathcal{L}_\gamma := \mathcal{L} + \gamma\boldsymbol{I}$ is positive definite and defines a homeomorphism*

$$\mathcal{L}_\gamma : H_0^1(U) \xrightarrow{\sim} H^{-1}(U).$$

That is, the problem $\mathcal{L}u + \gamma u = f$ is well-posed for all $\gamma > \nu$.

Proof. Once more, the proof will go through the Lax-Milgram theorem, where now we work with the bilinear operator B_γ associated with $\mathcal{L}_\gamma$

$$B_\gamma(u,v) := (u, \mathcal{L}_\gamma u) = B(u,v) + \gamma(u,v).$$

The calculation proceeds similarly to (5), where now an additional application of Cauchy's inequality

$$ab \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2 \tag{6}$$

to $a = \nabla u$ and $b = v$, shows that

$$\begin{aligned} B(u,u) = \int_U (\boldsymbol{A}\nabla u)\cdot\nabla u + \int_U \boldsymbol{b}\cdot(\nabla u)u + \int_U cu^2 \geq \alpha \left\|\nabla u\right\|_{L^2(U\to\mathbb{R}^d)} \\ - \frac{1}{2}\left\|\boldsymbol{b}\right\|_{L^\infty(U)}\left(\varepsilon\left\|\nabla u\right\|_{L^2(U)} + \varepsilon^{-1}\left\|u\right\|_{L^2(U)}\right) - \left\|c\right\|_{L^\infty(U)}\left\|u\right\|_{L^2(U)}. \end{aligned}$$

Taking ε small enough (smaller than $\alpha \|\boldsymbol{b}\|_{L^\infty(U)}^{-1}$ to be precise) and gathering up terms gives

$$B(u,u) \geq \frac{\alpha}{2} \|\nabla u\|_{L^2(U\to\mathbb{R}^d)} - \nu \|u\|_{L^2(U)} . \tag{7}$$

Where we defined $\nu = \|\boldsymbol{b}\|_{L^\infty(U)} \varepsilon^{-1} + \|c\|_{L^\infty(U)}$. The theorem follows from (7) as for all $\gamma > \nu$

$$B_\gamma(u,u) = B(u,u) + \gamma \|u\|_{L^2(U)} \geq \frac{\alpha}{2} \|\nabla u\|_{L^2(U\to\mathbb{R}^d)} + (\gamma - \nu) \|u\|_{L^2(U)} \gtrsim \|u\|_{H_0^1(U)} . \tag{8}$$

Equation (8) also shows that $\mathcal{L}_\gamma$ is positive definite and the proof is complete. □

We now consider $\mathcal{L}u = \lambda u + f$, which is a small generalization of our original problem (1). Take $\gamma > |\lambda|$ large enough as in Theorem 6.3. We have that,

$$\mathcal{L}u = \lambda u + f \iff \mathcal{L}_\gamma u = (\gamma + \lambda)u + f. \tag{9}$$

If we write $\mu := (\gamma + \lambda)$ and rename $v := \mu u + f$ we obtain that (9) is equivalent to

$$(\boldsymbol{I} - \mu \mathcal{L}_\gamma^{-1})v = f, \tag{10}$$

where $\boldsymbol{I}$ is the identity operator. Suppose now that U is bounded, then, by Rellich's theorem, we know that the following inclusion is compact

$$i : H^1(U) \hookrightarrow L^2(U).$$

As a result, by Theorem 6.2, we deduce that $\mathcal{L}_\gamma^{-1} : L^2(U) \to L^2(U)$, which we are now viewing as an operator on $L^2(U)$, is compact. More precisely,

$$K := i \circ \mathcal{L}_\gamma^{-1}\big|_{L^2(U)}$$

is compact and the reasoning in (9), (10) shows that, given $f \in L^2(U)$, and $u \in H_0^1(U)$

$$\mathcal{L}u = \lambda u + f \iff Tv := (\boldsymbol{I} - \mu K)v = f. \tag{11}$$

Equation (11) is exactly the form the Fredholm alternative takes and justifies the following.

Theorem 6.4. *Let $U \subset \mathbb{R}^d$ be bounded, let $\mathcal{L}$ verify Assumption 1, let $\lambda \in \mathbb{R}, f \in L^2(U)$ be any and consider the problems*

$$\mathcal{L}u = \lambda u + f \quad \textit{and} \quad u \in H_0^1(U) \tag{12}$$
$$\mathcal{L}u = \lambda u \quad \textit{and} \quad u \in H_0^1(U) \tag{13}$$

1. *Equation (12) is well-posed if and only if (13) has no non-zero solutions ($\lambda \notin \sigma(\mathcal{L})$).*
2. *The spectrum $\sigma(\mathcal{L})$ is discrete. If $\sigma(\mathcal{L}) = \{\lambda_n\}_{n=1}^\infty$ is infinite, then $\lambda_n \to +\infty$.*
3. *The dimensions of the following spaces are equal*

$$N := \left\{u \in H_0^1(U) : \mathcal{L}u = \lambda u\right\}, \quad N^* := \left\{f \in L^2(U) : \mathcal{L}^* f = \lambda f\right\},$$

4. *Equation, (12) has a solution if and only if $f \in (N^*)^\perp$ (equivalently $\langle w, f\rangle = 0$ for all $w \in N^*$).*

Proof. Given $f \in L^2(U)$ and $\lambda \in \mathbb{R}$ as before, we consider $\gamma > |\lambda|$ large and define $\mathcal{L}_\gamma = \mathcal{L} + \gamma \boldsymbol{I}$, $K := i \circ \mathcal{L}_\gamma^{-1}\big|_{L^2(U)}$, $\mu = \gamma + \lambda$ and $T = (\boldsymbol{I} - \mu K)$, where $i : H^1(U) \hookrightarrow L^2(U)$ is the inclusion. Consider the following two problems

$$Tv = f \quad \text{and} \quad v \in L^2(U), \tag{14}$$
$$Tv = 0 \quad \text{and} \quad v \in L^2(U). \tag{15}$$

The reasoning in (11) showed that a solution u to (12) gives a solution to (14) via the transformation $v = \mu u + f$. The converse needs to be clarified, as given $v \in L^2(U)$, the inverse transformation $u = \mu^{-1}(v - f)$ may not return a function in $H_0^1(U)$. However, if v solves (14), then u verifies

$$Tv = v - \mu K v = \mu u + f - \mu K v = f.$$

Cancelling out the f and dividing by μ we obtain that

$$u = Kv.$$

By Theorem 6.2 we know that $Kv = \mathcal{L}_\gamma^{-1} v \in H_0^1(U)$ for all $v \in L^2(U)$. As a result, u solves (12), and by the transformation $v \leftrightarrow u$ problem (14) has a solution if and only if (12) has a solution. Taking $f = 0$, we also obtain that u solves (13) if and only v solves (15). In conclusion,

$$(12) \text{ is w.p} \iff (14) \text{ is w.p} \iff \ker(T) = 0 \iff \ker(\mathcal{L} - \lambda \boldsymbol{I}) = 0,$$

where the second equivalence is due to the Fredholm alternative, and the third can be verified by an algebraic manipulation.

To see the second point, note that, by definition of T, equation (15) has non-zero solutions if and only if $\mu^{-1} \in \sigma(K)$. Since K is compact, $\sigma(K)$ is discrete and if $\sigma(K)$ is infinite, then its eigenvalues, which we denote by $\{\mu_n^{-1}\}_{n=1}^\infty$, go to 0. Furthermore, since by Theorem 6.3 K is positive definite, $\mu_n > 0$ and the claim follows by the correspondence $\lambda_n = \mu_n - \gamma$.

For the third and fourth points, we use that, as we have already proved, $\ker(T) = N$. Additionally,

$$T^* = (\boldsymbol{I} - \mu K^*) = \boldsymbol{I} - \mu(\mathcal{L}^* + \gamma)^{-1},$$

from where

$$\ker(T^*) = \ker(\mathcal{L}^* - \lambda \boldsymbol{I}) = N^*.$$

Applying the Fredholm alternative concludes the proof. □

Setting $\lambda = 0$ in Theorem 6.4, we recover our original problem and obtain the following corollary.

Corollary 6.5. *Equation* (1) *is well-posed unless the homogeneous problem $\mathcal{L}u = 0$ has a non-zero solution (that is,* $\ker(\mathcal{L}) \neq 0$*). Furthermore,* $\ker(\mathcal{L})$ *and* $\ker(\mathcal{L}^*)$ *have the same dimension. And* (1) *will have a solution if and only if f is orthogonal to the kernel of $\mathcal{L}^*$.*

In particular, to study the *existence* of solutions to (1), it is enough to study the *uniqueness* of solutions to (1)!

Exercise 3. In Theorem 6.4 we used that, for γ large enough, $K = \mathcal{L}_\gamma^{-1}$ is compact. However, $\mathcal{L}_\gamma^{-1}$ is invertible with inverse $\mathcal{L}_\gamma$. As a result $\boldsymbol{I} = \mathcal{L}_\gamma \circ \mathcal{L}_\gamma^{-1}$ is compact. How is this possible?

Hint. In fact, $\mathcal{L}_\gamma^{-1}$ is only invertible as an operator from $H^{-1}(U) \to H_0^1(U)$. However, it is not invertible as an operator from $K : L^2(U) \to L^2(U)$. Given $f \in L^2(U)$, it is not generally possible to find an $u \in L^2(U)$ such that $\mathcal{L}_\gamma u = f$.

Exercise 4. Where does the proof of Theorem 6.4 break down if we replace U with $\mathbb{R}^d$?

Hint. Can you apply Rellich's theorem to unbounded domains? What is the spectrum of the Laplacian on $\mathbb{R}^d$?

Exercise 5. Show using Theorem 6.4 that equation (14) (the generalization of (1)) is well-posed saved for at most a discrete set of λ.

Hint. Combine the first and second points of Theorem 6.4.

Exercise 6. Show the necessity of point 4 in Theorem 6.4 using only linear algebra.

Hint. Suppose $\mathcal{L}u = \lambda u + f$ and $w \in N^*$. Then,

$$\langle w, f \rangle = \langle w, \mathcal{L}u - \lambda u \rangle = \langle w, \mathcal{L}u \rangle - \lambda \langle w, u \rangle = \langle \mathcal{L}^* w, u \rangle - \lambda \langle w, u \rangle = 0.$$

7 Higher regularity

We have so far seen that, under the previous assumptions, solutions to (1) are in $H_0^1(U)$. However, analogously to the classical setting, we may expect that u is two degrees of regularity smoother than f. That is that $u \in H^2(U)$. This improved regularity is true, but only with the caveat that the domain U is sufficiently regular. Counterexamples with non-smooth domains exist. See [1].

We will also see how, for smoother coefficients, we can iterate to obtain a higher regularity of u. As a result, when the coefficients are smooth, u will be as well, and we will obtain a classical solution to (1).

7.1 Finite differences

In our study of regularity, we will make use of the difference quotients. Given a function $u \in L^p(\mathbb{R}^d)$, we define the difference quotients in the j-th direction as

$$D_j^h u := \frac{u(x + he_j) - u(x)}{h}, \quad e_j = (0, \dots, \overset{(j)}{1}, \dots, 0).$$

If u is differentiable, then $D_j^h u \to \partial_j u$ as $h \to 0$. The following lemma shows that, on $\mathbb{R}^d$, the difference quotients of u are bounded if and only if u is weakly differentiable.

Lemma 7.1 (Difference quotients and regularity). *Let $p \in (1, +\infty)$, and $C > 0$ be some constant. Then, the following hold.*

1. *If $u \in L^p(\mathbb{R}^d)$ and for all h sufficiently small $\left\| D_j^h u \right\|_{L^p(\mathbb{R}^d)} \le C$. Then $u \in W^{1,p}(\mathbb{R}^d)$.*

2. *If $u \in W^{1,p}(\mathbb{R}^d)$. Then, $\left\| D_j^h u \right\|_{L^p(\mathbb{R}^d)} \le \|\partial_j u\|_{L^p(\mathbb{R}^d)}$.*

Proof. We begin by proving the first point. Since $L^p(\mathbb{R}^d)$ is reflexive, every bounded sequence in $L^p(\mathbb{R}^d)$ has a weakly convergent subsequence. Thus, we can find h_n and $v \in L^p(\mathbb{R}^d)$ such that $D_j^{h_n} u \rightharpoonup v$ weakly in $L^p(\mathbb{R}^d)$. We want to show that $v = \partial_j u$. To this aim, let $\varphi \in C_c^\infty(\mathbb{R}^d)$. Then,

$$\begin{aligned} \int_{\mathbb{R}^d} v\varphi &= \lim_{n\to\infty} \int_{\mathbb{R}^d} D_j^{h_n} u\varphi = \lim_{n\to\infty} \int_{\mathbb{R}^d} u(x) \frac{\varphi(x - h_n e_j) - \varphi(x)}{h_n} \,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} u(x) \lim_{n\to\infty} -D_j^{-h_n} \varphi \,\mathrm{d}x = -\int_{\mathbb{R}^d} u\partial_j\varphi, \end{aligned} \tag{16}$$

where in the first equality, we used the weak convergence of $D_j^{h_n}u$ to v; in the second, we separated the integral in two and used the change of variable $x \to x - h_n e_j$ on the first of the integrals (from now on we will call this "discrete integration by parts"). The final equality follows from the smoothness of φ. Since $\varphi \in C_c^\infty(\mathbb{R}^d)$ was arbitrary, we have that $v = \partial_j u$ almost everywhere. Since $u \in L^p(\mathbb{R}^d)$, this shows that $u \in W^{1,p}(\mathbb{R}^d)$.

For the second point, suppose that u is smooth; then, by the fundamental theorem of calculus, we have that

$$D_j^h u(x) = \int_0^1 \partial_j u(x + the_j)\,\mathrm{d}t.$$

Taking norms and using Minkowski's integral inequality we obtain

$$\left\|D_j^h u\right\|_{L^p(\mathbb{R}^d)} \le \int_0^1 \|\partial_j u(\cdot + the_j)\|_{L^p(\mathbb{R}^d)}\,\mathrm{d}t = \int_0^1 \|\partial_j u\|_{L^p(\mathbb{R}^d)}\,\mathrm{d}t = \|\partial_j u\|_{L^p(\mathbb{R}^d)},$$

where in the second equality, we used the change of variables $x \to x - the_j$. We conclude by using the density of smooth functions in $W^{1,p}(\mathbb{R}^d)$ to take limits in the above inequality. □

The result can be extended to arbitrary open subsets $U \subset \mathbb{R}^d$. In this case, one can only obtain local regularity as the translation $u(x + he_j)$ is not well defined on the whole of U.

Lemma 7.2 (Difference quotients and local regularity). *Let $p \in (1, +\infty)$, $C > 0$ be a constant and $V \Subset U$ open. Then, the following hold.*

1. *If $u \in L^p(U)$ and for all h sufficiently small $\left\|D_j^h u\right\|_{L^p(V)} \le C$. Then, $u \in W^{1,p}(V)$.*
2. *If $u \in W^{1,p}(U)$. Then, $\left\|D_j^h u\right\|_{L^p(V)} \le \|\partial_j u\|_{L^p(U)}$ for all $h < \mathrm{d}(V, \partial U)$.*

Exercise 7. Prove Lemma 7.2.

Hint. Adapt the proof of Lemma 7.1. Take $\varphi \in C_c^\infty(V)$ for the first point. For the second point, use the local density of smooth functions in $W^{1,p}(U)$.

7.2 Regularity on $\mathbb{R}^d$

By using second-order finite difference, we now show that the solution to (1) is in $H^2(\mathbb{R}^d)$ if we impose additionally that $\boldsymbol{A}$ is continuously differentiable.

Theorem 7.3 (Improved regularity on $\mathbb{R}^d$). *Suppose that $A_{ij} \in C^1(\mathbb{R}^d)$ is elliptic and that $b_i \in L^\infty(\mathbb{R}^d), c \in L^\infty(\mathbb{R}^d)$. Then, if $u \in H^1(\mathbb{R}^d)$ solves $\mathcal{L}u = f$, it holds that $u \in H^2(\mathbb{R}^d)$ with*

$$\|u\|_{H^2(\mathbb{R}^d)} \lesssim_{\boldsymbol{A},\boldsymbol{b},c} \|f\|_{L^2(\mathbb{R}^d)} + \|u\|_{L^2(\mathbb{R}^d)}.$$

Proof. The idea is to use difference quotients to approximate the second derivative of u

$$v := -D_k^{-h} D_k^h u = \frac{u(x + he_k) - 2u(x) + u(x - he_k)}{h^2}.$$

Since $v \in H^1(U)$, we can substitute v into the weak formulation (4), do a discrete integration by parts and use Cauchy's inequality to show that $\left\|D_k^h \nabla u\right\|_{L^2(\mathbb{R}^d)}$ is bounded. Using Lemma 7.1, we will then conclude that $u \in H^2(\mathbb{R}^d)$ and finish off the proof. We now put this plan into action. From (4), we have that

$$\int_{\mathbb{R}^d} \boldsymbol{A}\nabla u \cdot \nabla v = \int_{\mathbb{R}^d} (f - \boldsymbol{b} \cdot \nabla u - cu)v. \tag{17}$$

Applying a discrete integration by parts to the left-hand side of (17) as in (16), we obtain

$$\int_{\mathbb{R}^d} \boldsymbol{A}\nabla u\cdot\nabla v = \int_{\mathbb{R}^d} D_k^h(\boldsymbol{A}\nabla u)\cdot(D_k^h\nabla u) = \int_{\mathbb{R}^d} \boldsymbol{A}^h D_k^h\nabla u\cdot D_k^h\nabla u + \int_{\mathbb{R}^d}(D_k^h\boldsymbol{A})\nabla u\cdot D_k^h\nabla u,$$

where in the last equality, we used the notation $\boldsymbol{A}^h(x) := \boldsymbol{A}(x+h)$ and the product rule for difference quotients (this can be checked by basic algebra). Using the ellipticity of $\boldsymbol{A}$ and Cauchy's inequality (6) to put ε on the higher order negative term $D_k^h\nabla u$ we obtain

$$\int_{\mathbb{R}^d} \boldsymbol{A}\nabla u\cdot\nabla v \geq \alpha\left\|D_k^h\nabla u\right\|_{L^2(\mathbb{R}^d)}^2 - \frac{C}{\varepsilon}\|\nabla u\|_{L^2(\mathbb{R}^d)}^2 - \varepsilon\left\|D_k^h\nabla u\right\|_{L^2(\mathbb{R}^d)}^2, \tag{18}$$

where we used that, since $\boldsymbol{A}\in C^1(\mathbb{R}^d)$, the term $D_k^h\boldsymbol{A}$ is bounded. Setting $\varepsilon=\alpha/3$ in (18) we obtain that

$$\int_{\mathbb{R}^d} \boldsymbol{A}\nabla u\cdot\nabla v \geq \frac{2\alpha}{3}\left\|D_k^h\nabla u\right\|_{L^2(\mathbb{R}^d)}^2 - \frac{3C}{\alpha}\|\nabla u\|_{L^2(\mathbb{R}^d)}^2. \tag{19}$$

We now estimate the right-hand side of (17). We have that, once more by Cauchy's inequality,

$$\int_{\mathbb{R}^d}(f-\boldsymbol{b}\cdot\nabla u - cu)v \leq \frac{C}{\varepsilon}\left(\|f\|_{L^2(\mathbb{R}^d)} + \|\nabla u\|_{L^2(\mathbb{R}^d)}^2 + \|u\|_{L^2(\mathbb{R}^d)}\right) + \varepsilon\left\|D_k^h\nabla u\right\|_{L^2(\mathbb{R}^d)}^2.$$

Once more, setting $\varepsilon=\alpha/3$ gives

$$\int_{\mathbb{R}^d}(f-\boldsymbol{b}\cdot\nabla u - cu)v \leq \frac{3C}{\alpha}\left(\|f\|_{L^2(\mathbb{R}^d)} + \|\nabla u\|_{L^2(\mathbb{R}^d)}^2 + \|u\|_{L^2(\mathbb{R}^d)}\right) + \frac{\alpha}{3}\left\|D_k^h\nabla u\right\|_{L^2(\mathbb{R}^d)}^2. \tag{20}$$

Using (19) and (20) in (17) shows that

$$\left\|D_k^h\nabla u\right\|_{L^2(\mathbb{R}^d)}^2 \leq \frac{\widetilde{C}}{\alpha^2}\left(\|f\|_{L^2(\mathbb{R}^d)} + \|\nabla u\|_{L^2(\mathbb{R}^d)}^2 + \|u\|_{L^2(\mathbb{R}^d)}\right). \tag{21}$$

Equation (21) is almost the desired result save the presence of $\|\nabla u\|_{L^2(\mathbb{R}^d)}$ on the right-hand side. However, by setting $v=u$ in (17) and once more using Cauchy's inequality, we obtain that

$$\|\nabla u\|_{L^2(\mathbb{R}^d)}^2 \lesssim_{\boldsymbol{A},\boldsymbol{b},c} \|f\|_{L^2(\mathbb{R}^d)} + \|u\|_{L^2(\mathbb{R}^d)}^2. \tag{22}$$

Combining (21) and (22) gives the bound

$$\left\|D_k^h\nabla u\right\|_{L^2(\mathbb{R}^d)}^2 \lesssim_{\boldsymbol{A},\boldsymbol{b},c} \|f\|_{L^2(\mathbb{R}^d)} + \|u\|_{L^2(\mathbb{R}^d)}^2.$$

Applying the first point of Lemma 7.1 concludes the proof. □

By induction, we can obtain higher-order regularity. For notational convenience, we write

$$X^k := H^k(\mathbb{R}^d)\cap W^{k,\infty}(\mathbb{R}^d).$$

This space corresponds to functions that are k times weakly differentiable with bounded and square-integrable derivatives up to order k.

Theorem 7.4 (Higher regularity on $\mathbb{R}^d$)**.** *Suppose that $\mathcal{L}$ is elliptic and that its coefficients verify*

$$A_{ij}\in C^1(\mathbb{R}^d)\cap X^{k+1}, \quad b_i, c\in X^k, \quad f\in H^k(\mathbb{R}^d).$$

Then, if $u\in H^1(\mathbb{R}^d)$ solves $\mathcal{L}u=f$, it holds that $u\in H^{k+2}(\mathbb{R}^d)$ with

$$\|u\|_{H^{k+2}(\mathbb{R}^d)} \lesssim_{\boldsymbol{A},\boldsymbol{b},c} \|f\|_{H^k(\mathbb{R}^d)} + \|u\|_{L^2(\mathbb{R}^d)}.$$

Proof. The theorem holds for $k = 0$ by Theorem 7.3. Suppose by hypothesis of induction that the theorem holds up to order k. Let

$$A_{ij} \in C^1(\mathbb{R}^d) \cap X^{k+2}, \quad b_i, c \in X^{k+1}, \quad f \in H^{k+1}(\mathbb{R}^d). \tag{23}$$

Then, by the induction hypothesis $u \in H^{k+2}(\mathbb{R}^d)$ with

$$\|u\|_{H^{k+2}(U)} \lesssim_{\boldsymbol{A},\boldsymbol{b},c} \|f\|_{H^k(\mathbb{R}^d)} + \|u\|_{L^2(\mathbb{R}^d)} \,. \tag{24}$$

Consider a multi-index α with $|\alpha| = k+1$ and $\widetilde{v} \in C_c^\infty(\mathbb{R}^d)$ Then, substituting $v := (-1)^{|\alpha|} D^\alpha \widetilde{v}$ in the weak formulation (4) we obtain by integrating by parts that

$$\int_{\mathbb{R}^d} D^\alpha(\boldsymbol{A}\nabla u)\cdot\nabla\widetilde{v} + \int_{\mathbb{R}^d} D^\alpha(\boldsymbol{b}\nabla u)\cdot\nabla\widetilde{v} + \int_{\mathbb{R}^d} D^\alpha(cu)\widetilde{v} = \int_{\mathbb{R}^d} D^\alpha f\widetilde{v}.$$

Let us write $\widetilde{u} := D^\alpha u$. Applying the chain rule repeatedly and keeping only the derivatives of order $k+3$ of u on the left-hand side to obtain

$$B(\widetilde{u},\widetilde{v}) = \int_{\mathbb{R}^d} \boldsymbol{A}\nabla D^\alpha u\cdot\nabla\widetilde{v} + \int_{\mathbb{R}^d} \boldsymbol{b}\nabla D^\alpha u\cdot\nabla\widetilde{v} + \int_{\mathbb{R}^d} cD^\alpha u\widetilde{v} = \int_{\mathbb{R}^d} \widetilde{f}\widetilde{v} = (\widetilde{f},\widetilde{v}), \tag{25}$$

where $\widetilde{f}$ involves only $D^\alpha f$ as well as sums and products of derivatives up to order $k+2$ of $u, \boldsymbol{A}$ and up to order $k+1$ of $\boldsymbol{b}$ and c. As a result, by the conditions on the coefficients in (23) and the induction hypothesis $u \in H^{k+2}(\mathbb{R}^d)$, we have that $\widetilde{f} \in L^2(\mathbb{R}^d)$ with

$$\|\widetilde{f}\|_{L^2(\mathbb{R}^d)} \lesssim_{\boldsymbol{A},\boldsymbol{b},c} \|f\|_{H^{k+1}(\mathbb{R}^d)} + \|u\|_{L^2(\mathbb{R}^d)} \,. \tag{26}$$

By equation (25), $\widetilde{u}$ is a solution to (1) and applying (24) and (26) shows that $\widetilde{u} \in H^2(\mathbb{R}^d)$ with

$$\|\widetilde{u}\|_{H^2(\mathbb{R}^d)} \lesssim_{\boldsymbol{A},\boldsymbol{b},c} \|\widetilde{f}\|_{L^2(\mathbb{R}^d)} + \|\widetilde{u}\|_{L^2(\mathbb{R}^d)} \lesssim_{\boldsymbol{A},\boldsymbol{b},c} \|f\|_{H^{k+1}(\mathbb{R}^d)} + \|u\|_{L^2(\mathbb{R}^d)} \,.$$

Since α was any coefficient of order $k+1$, we deduce that $u \in H^{k+3}(\mathbb{R}^d)$ with

$$\|u\|_{H^{k+3}(\mathbb{R}^d)} \lesssim_{\boldsymbol{A},\boldsymbol{b},c} \|f\|_{H^{k+1}(\mathbb{R}^d)} + \|u\|_{L^2(\mathbb{R}^d)} \,.$$

The equation above is the hypothesis of induction for $k+1$, and the proof is complete. □

Iterating the above theorem, we obtain that if the coefficients of $\mathcal{L}$ are smooth, then the solution to (1) is smooth as well. And u is a classical solution to (1).

Theorem 7.5 (Infinite regularity on $\mathbb{R}^d$)**.** *Let $A_{ij}, b_i, c \in C^\infty(\mathbb{R}^d)$ with $\boldsymbol{A}$ elliptic. Then, if $u \in H^1(U)$ solves $\mathcal{L}u = f$, it holds that $u \in C^\infty(\mathbb{R}^d)$*

Proof. By Theorem 7.7, we have that $u \in H^k(\mathbb{R}^d)$ for all $k \in \mathbb{N}$. By Sobolev embeddings we have that $u \in C^\infty(\mathbb{R}^d)$. □

At first sight, it may seem as if the above results can be extended to solutions of (1) on $U \subsetneq \mathbb{R}^d$ with the following reasoning. However, there is a mistake in the reasoning. Can you spot it?

Exercise 8. The following argument is **false**. Show the flaw in the reasoning.

Let $U \subset \mathbb{R}^d$ be any open subset. Suppose that $A_{ij} \in C^1(\overline{U})$ is elliptic and that $b_i, c \in L^\infty(U)$. Let $u \in H_0^1(U)$ solve $\mathcal{L}u = f$. The extension $\widetilde{u}$ to $\mathbb{R}^d$ by zero of u is in $H^1(\mathbb{R}^d)$. The coefficients b, c can likewise be extended by 0 to functions $\widetilde{\boldsymbol{b}}, \widetilde{c} \in L^\infty(\mathbb{R}^d)$. Likewise for f to $\widetilde{f} \in L^2(\mathbb{R}^d)$ and by Assumption, $\boldsymbol{A}$ is the restriction to U of some function $\widetilde{A} \in C^1(\mathbb{R}^d)$. We have that

$$\widetilde{\mathcal{L}}\widetilde{u} := -\nabla\cdot(\widetilde{A}\nabla\widetilde{u}) + \widetilde{b}\cdot\nabla\widetilde{u} + \widetilde{c}\widetilde{u} = \widetilde{f}. \tag{27}$$

As a result by Theorem 7.3 it holds that $\widetilde{u} \in H^2(\mathbb{R}^d)$ with

$$\|u\|_{H^2(U)} = \|\widetilde{u}\|_{H^2(\mathbb{R}^d)} \lesssim_{\boldsymbol{A},\boldsymbol{b},c} \|f\|_{L^2(\mathbb{R}^d)} + \|u\|_{L^2(\mathbb{R}^d)} \,.$$

Hint. Are you sure that $\widetilde{u}$ solves (27)? Consider for example the case $\boldsymbol{A} = \boldsymbol{I}, \boldsymbol{b} = c = 0$. For $\widetilde{u}$ to solve (27) it is necessary that for all $\varphi \in C_c^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \nabla \widetilde{u} \cdot \nabla \varphi = \int_{\mathbb{R}^d} \widetilde{f} \varphi.$$

That is, that

$$\int_U \nabla u \cdot \nabla \varphi = \int_U f \varphi, \quad \forall \varphi \in C^\infty(\mathbb{R}^d).$$

Whereas we only know that u solves (4). That is,

$$\int_U \nabla u \cdot \nabla \varphi = \int_U f \varphi, \quad \forall \varphi \in C_c^\infty(U).$$

This equality does not imply the previous one. The problem is that extension by zero does not respect the second derivative of functions in $H_0^1(\mathbb{R}^d)$. For example, if $u \in H^2(U) \cap H_0^1(U)$ we do not necessarily have that $\widetilde{u}$ is in $H^2(\mathbb{R}^d)$. Consider for example $U = (-1, 1)$ and $u(x) = 1 - x^2$. Then, u solves our equation (1) with $f = 2$. However, $\widetilde{u}$ is not in $H^2(\mathbb{R})$ and given $\varphi \in C_c^\infty(\mathbb{R})$

$$\int_{\mathbb{R}} \widetilde{u}' \varphi' = -2 \int_{-1}^1 x \varphi' = -2(\varphi(1) - \varphi(-1)) + 2 \int_{-1}^1 \varphi \neq 2 \int_{-1}^1 \varphi = \int_{\mathbb{R}} \widetilde{f} \varphi.$$

7.3 Interior regularity

We have just seen that a direct extension of Theorem 7.3 to unbounded domains is not possible using an extension by zero. However, by adapting the proof of Theorem 7.3, one can prove the analogous result.

In this case, however, one has to be careful as the difference quotients may not be well defined at the boundary. As a result, it is necessary to work locally and use a bump function. This makes the proofs a bit messier, though the idea is the same. We sketch the proof, which can also be found in [2] page 326.

Theorem 7.6 (Improved interior regularity). *Let $u \in H^1(U)$ be a solution to $\mathcal{L}u = f$ where $f \in L^2(U)$, $\boldsymbol{A} \in C^1(U)$ is elliptic and $v_i, c \in L^\infty(U)$. Then, $u \in H^2_{\mathrm{loc}}(U)$ and*

$$\|u\|_{H^2_{\mathrm{loc}}(U)} \lesssim_{\boldsymbol{A},\boldsymbol{b},c} \|f\|_{L^2(U)} + \|u\|_{L^2(U)}.$$

Note that we do not require u to be in $H_0^1(U)$.

Proof. Let $V \Subset W \Subset U$ be open and let η be a bump function supported on W and identically equal to 1 on V. Write

$$v = -D_k^h \eta^2 D_k^h u.$$

Proceeding as in the proof of Theorem 7.6, we obtain that

$$\int_V \left| D_k^h \nabla u \right|^2 dx \leq \int_U \eta^2 \left| D_k^h D u \right|^2 dx \lesssim C \int_U f^2 + u^2 + |\nabla u|^2.$$

Applying the first point of Lemma 7.2 we obtain that $u \in H^2_{\mathrm{loc}}(U)$ with

$$\|u\|_{H^2(U)} \lesssim_{\boldsymbol{A},\boldsymbol{b},c} \|f\|_{L^2(U)} + \|u\|_{H^1(U)}. \tag{28}$$

Analogously, we also obtain by setting $v = \eta^2 u$ that

$$\int_V |\nabla u|^2 \leq \int_U \eta^2 |\nabla u|^2 \lesssim \|f\|_{L^2(U)} + \|u\|_{H^1(U)}. \tag{29}$$

Combining (28) and (29), we obtain the desired result. ☐

As for $\mathbb{R}^d$, we can obtain higher-order regularity by induction. As before, we now write

$$X^k(U) := H^k(U) \cap W^{k,\infty}(U).$$

In the case that U is bounded then $X^k(U) = W^{k,\infty}(U)$.

Theorem 7.7 (Improved interior regularity). *Suppose that $\mathcal{L}$ is elliptic and that its coefficients verify*

$$A_{ij} \in C^1(U) \cap X^{k+1}(U), \quad b_i, c \in X^k(U), \quad f \in H^k(U).$$

Then, if $u \in H^1(U)$ solves $\mathcal{L}u = f$, it holds that $u \in H^{k+2}_{\text{loc}}(U)$ with

$$\|u\|_{H^{k+2}_{\text{loc}}(U)} \lesssim_{\boldsymbol{A},\boldsymbol{b},c} \|f\|_{H^k(U)} + \|u\|_{L^2(U)} .$$

Proof. The theorem holds for $k = 0$ by Theorem 7.6. Suppose by hypothesis of induction that the theorem holds up to order k. Let

$$A_{ij} \in C^1(U) \cap X^{k+2}(U), \quad b_i, c \in X^{k+1}(U), \quad f \in H^{k+1}(U). \tag{30}$$

Then, by the induction hypothesis $u \in H^{k+2}_{\text{loc}}(U)$ with

$$\|u\|_{H^{k+2}_{\text{loc}}(U)} \lesssim_{\boldsymbol{A},\boldsymbol{b},c} \|f\|_{H^k(U)} + \|u\|_{L^2(U)} . \tag{31}$$

Let $V \Subset U$ be open, consider a multi-index α with $|\alpha| = k+1$ and $\widetilde{v} \in C^\infty_c(V)$. Then, substituting $v := (-1)^{|\alpha|} D^\alpha \widetilde{v}$ in the weak formulation (4) we obtain by integrating by parts that

$$\int_V D^\alpha(\boldsymbol{A}\nabla u) \cdot \nabla\widetilde{v} + \int_V D^\alpha(\boldsymbol{b}\nabla u) \cdot \nabla\widetilde{v} + \int_V (D^\alpha cu)\widetilde{v} = \int_V D^\alpha f \widetilde{v}.$$

Let us write $\widetilde{u} := D^\alpha u$. Applying the chain rule repeatedly and keeping only the derivatives of order $k+2$ of u on the left-hand side to obtain

$$B(\widetilde{u}, \widetilde{v}) = \int_V \boldsymbol{A}\nabla D^\alpha u \cdot \nabla\widetilde{v} + \int_V \boldsymbol{b}\nabla D^\alpha u \cdot \nabla\widetilde{v} + \int_V cD^\alpha u\widetilde{v} = \int_V \widetilde{f}\widetilde{v} = (\widetilde{f}, \widetilde{v}), \tag{32}$$

where $\widetilde{f}$ involves only $D^\alpha f$ as well as sums and products of derivatives up to order $k+2$ of u, $\boldsymbol{A}$ and up to order $k+1$ of $\boldsymbol{b}$ and c. As a result, by the conditions on the coefficients in (30) and the induction hypothesis $u \in H^{k+2}_{\text{loc}}(U)$, we have that $\widetilde{f} \in L^2(V)$ with

$$\|\widetilde{f}\|_{L^2(V)} \lesssim_{\boldsymbol{A},\boldsymbol{b},c} \|f\|_{H^{k+1}(U)} + \|u\|_{L^2(U)} . \tag{33}$$

By equation (32), $\widetilde{u}$ is a solution to (1) on V and applying (31) and (33) shows that $\widetilde{u} \in H^2_{\text{loc}}(V)$ with

$$\|\widetilde{u}\|_{H^2_{\text{loc}}(V)} \lesssim_{\boldsymbol{A},\boldsymbol{b},c} \|\tilde{f}\|_{L^2(V)} + \|\widetilde{u}\|_{L^2(V)} \lesssim_{\boldsymbol{A},\boldsymbol{b},c} \|f\|_{H^{k+1}(U)} + \|u\|_{L^2(U)} .$$

Since α was any coefficient of order $k+1$, we deduce that $u \in H^{k+3}(W)$ with

$$\|u\|_{H^{k+3}_{\text{loc}}(V)} \lesssim_{\boldsymbol{A},\boldsymbol{b},c} \|f\|_{H^{k+1}(U)} + \|u\|_{L^2(U)} .$$

Since $V \Subset U$ is any, we deduce that

$$\|u\|_{H^{k+3}_{\text{loc}}(U)} \lesssim_{\boldsymbol{A},\boldsymbol{b},c} \|f\|_{H^{k+1}(U)} + \|u\|_{L^2(U)} .$$

The above is the hypothesis of induction for $k+1$ and completes the proof. $\square$

Using Sobolev embeddings we obtain once more infinite regularity for smooth coefficients.

Theorem 7.8 (Infinite interior regularity). *Let $A_{ij}, b_i, c \in C^\infty(U)$ with $\boldsymbol{A}$ elliptic. Then, if $u \in H^1(U)$ solves $\mathcal{L}u = f$, it holds that $u \in C^\infty_{\text{loc}}(U)$.*

Proof. By Theorem 7.7, we have that $u \in H^k(U)$ for all $k \in \mathbb{N}$. By Sobolev embeddings we have that $u \in C^\infty_{\text{loc}}(U)$. $\square$

7.4 Regularity at the boundary

Regularity at the boundary can also be obtained; however, in this case, it is necessary to impose the boundary condition $u|_{\partial\Omega} = 0$. We can then work on bounded smooth domains by reasoning first on open sets of the form $B(0,r) \cap \mathbb{R}^d_+$ and then using a partition of unity and smooth change of coordinates to translate these results back to Ω. The details can be found in [2] pages $334 - 343$. We summarize the main results, which are analogous to the interior regularity results of Theorems 7.6, 7.7 and 7.8.

Theorem 7.9 (Lower regularity). *Let $\Omega \subset \mathbb{R}^d$ be bounded of class C^2. Let $A_{ij} \in C^1(\overline{\Omega})$ be elliptic and $b_i, c \in L^\infty(\Omega)$. Let $u \in H^1_0(\Omega)$ be a weak solution to (1). Then, $u \in H^2(\Omega)$ with*

$$\|u\|_{H^2(\Omega)} \lesssim_{\boldsymbol{A},\boldsymbol{b},c,\Omega} \|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} .$$

Theorem 7.10 (Higher regularity). *Let $\Omega \subset \mathbb{R}^d$ be bounded of class C^{k+2}. Let $A_{ij} \in C^1(\overline{\Omega}) \cap W^{k,\infty}(\Omega)$ be elliptic and $b_i, c \in H^k(\Omega) \cap W^{k,\infty}(\Omega)$. Let $u \in H^1_0(\Omega)$ be a weak solution to (1). Then, $u \in H^{k+2}(\Omega)$ with*

$$\|u\|_{H^{k+2}(\Omega)} \lesssim_{\boldsymbol{A},\boldsymbol{b},c,\Omega} \|f\|_{H^k(\Omega)} + \|u\|_{L^2(\Omega)} .$$

Theorem 7.11 (Infinite regularity). *Let $\Omega \subset \mathbb{R}^d$ be bounded of class C^∞. Let $A_{ij}, b_i, c \in C^\infty(\overline{\Omega})$ with $\boldsymbol{A}$ elliptic. Let $u \in H^1_0(\Omega)$ be a weak solution to (1). Then, $u \in C^\infty(\overline{\Omega})$.*

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