# Elliptic PDE I

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5/21/2023

## 1 Three line summary

- Elliptic partial differential equations (PDE) are PDE with no time variable and whose leading order derivatives satisfify a positivity condition.
- Using Lax Milgram's theorem we can prove existence and uniqueness of weak (distributional) solutions if the transport term is zero.
- If the transport term is non-zero solutions still exist but they are no longer unique and are determined by the kernel of the homogeneous problem.

## 2 Why should I care?

Many problems arising in physics such as the Laplace and Poisson equation are elliptic PDE. Furthermore, the tools used to analyze them can be extrapolated to other settings such as elliptic PDE. The analysis also helps contextualize and provide motivation for theoretical tools such as Hilbert spaces, compact operators and Fredholm operators.

## 3 Notation

Given $\lambda \in \mathbb{C}$ we will write $\overline{\lambda}$ for the conjugate of $\lambda$. Given a subset $A$ of some topological space it is also common to write $\overline{A}$ for the closure of $A$. Though this is a slight abuse of notation we will do the same as the meaning will always be clear from context.

Given two topological vector spaces $X, Y$ we write $\mathcal{L}(X, Y)$ for the space of countinuous linear operators from $X$ to $Y$.
We will use the Einstein convention that indices when they are repeated are summed over. For example we will write

$$\nabla \cdot (a\nabla) = \sum_{i=1}^{n} \partial_i a_{ij} \partial_j = \partial_i a_{ij} \partial_j.$$

Furthermore, we will fix $U \subset \mathbb{R}^n$ to be an open **bounded** (we will see later why this is important) set in $\mathbb{R}^n$.
We will use Vinogradov notation.

# 4 Introduction

Welcome back to the second post on our series of PDE. In the first post of the series we dealt with the Fourier transform and it's application to defining spaces of weak derivatives and weak solutions to PDE. In this post we will consider an equation of the form

$$\mathcal{L}u = f; \quad u|_{\partial u} = 0. \tag{1}$$

Where $\mathcal{L}$ is some differential operator, $f : U \to \mathbb{R}$ is some known function and $u$ is the solution we want to find. As we shall soon see, the Sobolev spaces $H^s$ and the concept of weak solution will prove a natural setting for our analysis of elliptic PDE. We start right off with the definition

**Definition 1.** *Given $A : U \to \mathbb{R}^{d\times d}, b : U \to \mathbb{R}^d$ and $c : U \to \mathbb{R}$ we say that the differential operator*

$$\mathcal{L} := -\nabla \cdot A\nabla + \nabla \cdot b + c \tag{2}$$

*is* elliptic *if there exists $\lambda > 0$ such that*

$$\xi^T A(x)\xi \geq \lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \quad \forall x \in U. \tag{3}$$

There are some points to clear up. Firstly, if this is the first time you've encountered the ellipticity condition in (3) then it may seem a bit strange. Physically, speaking in a typical derivation of our PDE in (1), $u$ is the density of some substance and $A$ corresponds to a diffusion matrix. In this case (3)

says that flow occurs from the region of higher to lower density. Mathematically speaking (3) will provide the necessary bound we need to apply Lax Milgram's theorem (ref section?).
This said, we have not yet defined which function space our coefficients live in and what $\mathcal{L}$ acts on. This is always a tricky aspect. "How much can one get away with?".So as to not make our lives too complicated in what remains we will make the following assumption

**Assumption 1.** *We assume that $A_{ij}, b_i, c \in L^\infty(U)$ for all $i, j = 1, \dots, d$. Furthermore, $A$ is symmetric, that is $A_{ij} = A_{ji}$.*

T0 simplify the notation we will write for the bound on $a, b, c$

$$\|a\|_{L^\infty(U)} + \|b\|_{L^\infty(U)} + \|c\|_{L^\infty(U)} = M$$

The first assumption will make it easy to get bounds on $\mathcal{L}$ and the second will be necessary to apply Lax Milgram's theorem and the third will prove useful when we look at the spectral theory of $\mathcal{L}$ (blogpost on this coming). With these assumptions we have the following

**Proposition 1.** *The operator $\mathcal{L}$ defined in (2) defines for all $s \in \mathbb{R}$ a bounded linear operator*

$$\mathcal{L} : H_0^{s+2} \to H^s(U).$$

*Have to add some stuff on trace and Sobolev on bounded domain.*

*Proof.* We apply the usual trick of working first with a smooth functions $u$ that vanishes on the boundary. Then have that

$$\|\mathcal{L}u\|_{H_0^s(U)} = \left\| \langle \xi \rangle^s \widehat{\mathcal{L}f} \right\|_{L^2(U)} \lesssim M \left\| \langle \xi \rangle^{s+2} \widehat{u}(\xi) \right\|_{L^2(U)} = M \|u\|_{H^{s+2}(U)}.$$

Since $C_0^\infty(U)$ is dense in $H_0^k(U)$ for any $k \in \mathbb{R}$ we can extend $\mathcal{L}$ continuous to $H_0^{s+2}(U)$ by defining

$$\mathcal{L}u = \lim_{n\to\infty} \mathcal{L}u_n, \quad \forall u \in H_0^{s+2}(U).$$

Where $u_n \in C_0^\infty(U)$ is any sequence converging to $u$ in $H^{s+2}(U)$. $\square$

We now note that, by an integration by parts, if $u, v \in C_0^\infty(U)$ then

$$\int_U \mathcal{L}uv = \int_U a\nabla u \cdot \nabla v + \int_U b\nabla uv + \int_U cuv =: B(u,v).$$

It is clear that $B$ is bilinear in an algebraic sense. Furthermore from Cauchy Schwartz and the fact that

$$\|u\|_{H_1(U)} \sim \|u\|_{L^2(U)} + \|\nabla u\|_{L^2(U \to \mathbb{R}^d)}.$$

Shows that we have the bound

$$B(u,v) \lesssim M \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}. \tag{4}$$

This allows us as in the previous proposition to extend $B$ from $C_0^\infty(U)$ to a continuous bilinear operator on $H_0^1(U)$. We still have not mentioned what space $f$ should be in. We just saw that it makes sense to consider $u \in H_0^1(U)$. Since by Proposition 1 we have $\mathcal{L}u \in H^{-1}(U)$ and since we are looking for solutions to

$$\mathcal{L}u = f.$$

We see that we should impose $f \in H^{-1}(U)$. This can all be summarized as follows. holds

**Proposition 2.** *Given $f \in H^{-1}(U)$, solving equation (1) (under assumption 1) is equivalent to finding $u \in H_0^1(U)$ such that*

$$B(u,v) = (f,v), \quad \forall v \in H_0^1(U). \tag{5}$$

Where we recall the "duality notation"

$$(f,v) := f(v).$$

Labeling assumptions probably won't work. Check it. As a result we have reformulated our problem to something that looks very similar to the setup of Lax Milgram's theorem. In fact, if we suppose $b = 0$ and $c \geq 0$ we are done

**Theorem 1.** *Suppose $b = 0$ and $c \geq 0$. Then given $f \in H^{-1}(U)$ there exists a unique solution $u$ to (5) (and thus to (1)). Furthermore, the solution operator $\mathcal{L}^{-1}$ is a continuous operator $\mathcal{L}^{-1} : H^{-1}(U) \to H_0^1(U)$ with $\|\mathcal{L}^{-1}\| \lesssim_U \lambda^{-1}$.*

*Proof.* The continuity of $B$ was proved in (5). It remains to see that $B$ is coercive. This follows from the fact that for smooth $u$

$$B(u,u) = \int_U a\nabla u \cdot \nabla u + \int_U cu^2 \geq \lambda \left\|\nabla u\right\|_{L^2(U\to\mathbb{R}^d)} \gtrsim_U \left\|u\right\|_{H_0^1(U)}. \tag{6}$$

Where in first inequality we used the ellipticity assumption on $A$ and in the last inequality we used Poincaré's inequality. $\square$

Maybe best add $c \geq 0$ inn assumption here instead of 1 if we later add on $\alpha u$. Furthermore, by Rellich theorem $\mathcal{L}$ is compact and is self adjoint since $b$ is 0 so there is a countable basis of eigenvalues in $L^2(U)$. Furthermore they must me smooth by Prop 2 and Sobolev embedding.
In the previous result, we somewhat unsatisfyingly had to assume that $b$ was identically zero and had to impose the extra assumption $c \geq 0$. These extra assumptions can be done away with, but at the cost of modifying our initial problem by a correction term $\gamma$ so we can once more obtain a coercive operator $B_\gamma$

**Theorem 2** (Modified problem). *Given $f \in H^{-1}(U)$ there exists some constant $\nu \geq 0$ (depending on the coefficients) such that for all $\gamma \geq \nu$ there exists a unique solution $u$ to*

$$\mathcal{L}_\gamma u := \mathcal{L}u + \gamma u = f.$$

Furthermore $\mathcal{L}^{-1} : H^{-1}(U) \to H_0^1(U)$ is a continuous operator

*Proof.* Once more, the proof will go through the Lax-Milgram theorem, where now we work with the bilinear operator $B_\gamma$ associated to $\mathcal{L}_\gamma$

$$B_\gamma(u,v) := B(u,v) + \gamma(u,v).$$

The calculation procees in a similar fashion to (6), where now an additional application of Cauchy Schwartz to $\nabla uv = (\epsilon^{\frac{1}{2}}\nabla u)(\epsilon^{-\frac{1}{2}}v)$ shows that

$$\begin{aligned} B(u,u) = \int_U a\nabla u \cdot \nabla u + \int_U b \cdot \nabla uv + \int_U cu^2 \geq \lambda \left\|\nabla u\right\|_{L^2(U\to\mathbb{R}^d)} \\ - \left\|b\right\|_{L^\infty(U)} \left(\epsilon \left\|\nabla u\right\|_{L^2(U)} + \epsilon^{-1}\left\|u\right\|_{L^2(U)}\right) - \left\|c\right\|_{L^\infty(U)} \left\|u\right\|_{L^2(U)}. \end{aligned}$$

Taking $\epsilon$ small enough (smaller than $\frac{1}{2}\lambda \|b\|_{L^\infty(U)}^{-1}$ to be precise) and gathering up terms gives

$$B(u,u) \geq \frac{\lambda}{2} \|\nabla u\|_{L^2(U \to \mathbb{R}^d)} - \nu \|u\|_{L^2(U)}. \tag{7}$$

Where we defined $\nu = \|b\|_{L^\infty(U)} \epsilon^{-1} + \|c\|_{L^\infty(U)}$.The theorem now follows from the just proved (7) and Poincaré's inequality as for all $\gamma \geq \nu$

$$B_\gamma(u,u) = B(u,u) + \gamma \|u\|_{L^2(U)} \geq \frac{\lambda}{2} \|\nabla u\|_{L^2(U \to \mathbb{R}^d)} \gtrsim_U \|u\|_{H_0^1(U)}.$$

□

Now that we proved solutions for our modified problem $\mathcal{L}_\gamma$ it would be nice if we could somehow "unomodify". We reason as follows: we have that $u$ solves our original problem (1) if and only if

$$\mathcal{L}_\gamma u - \gamma u = f.$$

That is, moving $\gamma u$ over and taking the inverse of $\mathcal{L}_\gamma$, if and only if

$$\gamma^{-1} u - \mathcal{L}_\gamma^{-1} u = \mathcal{L}_\gamma^{-1} f.$$

By what we just proved in Proposition 2 the operator $\mathcal{L}_\gamma^{-1}$, and thus $\gamma\mathcal{L}_\gamma^{-1}$ is a continuous operator form $H^{-1}(U)$ to $H^1(U)$, as a result it is compact and we may apply the Fredholm alternative.
Might have to restrict $f$ to $L^2$ for some adjointness....

# A A brief recap of Hilbert spaces

In this blog post and in future ones Hilbert spaces will be a recurrent theme. In this appendix I present a brief rundown and some important results. I'm going to assume you know nothing (please don't feel insulted) and go over the main ideas and results.
There is a lot of theory to cover so the proofs will only be sketched. For more details see for example **??**. Firstly what is a Hilbert space? To answer that question let's start with what kind of spaces we want to model

**Example 1.** *The Euclidean space $\mathbb{R}^n$ together with the inner product*

$$x \cdot y := x_1 y_1 + \ldots x_n y_n.$$

**Example 2.** *The complex Euclidean space* $\mathbb{C}^n$ *together with the inner product*

$$\langle x, y \rangle := x_1\overline{y_1} + \ldots x_n\overline{y_n}.$$

You might be wondering why we took the conjugate in the definition of inner product above. That is simply because we want the inner product to define a norm by

$$\|x\|^2 := \langle x, x \rangle .$$

In the cases above note that this is the standard Euclidean norm

$$\|x\|^2 = \langle x, x \rangle = |x_1|^2 + \ldots + |x_n|^2 .$$

A Hilbert space is just this construction generalized to infinite dimensions. Firstly, the inner product is generalized as follows.

**Definition 2.** *Given a vector space* $V$ *over* $\mathbb{K}$ *where* $\mathbb{K} = \mathbb{R}$ *or* $K = \mathbb{C}$*, an inner product is a mapping*

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{K}; \quad (x, y) \to \langle x, y \rangle .$$

*Such that the following hold for all* $x, y, z \in V$ *and all* $\lambda \in \mathbb{K}$

1. *Linearity:* $\langle \lambda x + y, z \rangle = \lambda \langle x, z \rangle + \langle y, z \rangle$.
2. *Conjugate symmetry* $\langle x, y \rangle = \overline{\langle y, x \rangle}$.
3. *Positivity* $\langle x, x \rangle = 0$ *if and only if* $x = 0$.

**Definition 3.**

In what follows we will take always take the base field to be $\mathbb{K}$, that is, we will consider real and complex vector spaces as the theory is the same. In the real case the conjugate function just being the identity. Note that the first two properties also imply so called antilinearity of the second component as

$$\langle z, \lambda x + y \rangle = \overline{\langle \lambda x + y, z \rangle} = \overline{\lambda} \langle z, x \rangle + \lambda_2 \langle z, y \rangle .$$

A vector space with an inner product is called an inner product space. This inner product gives us a concept of geometry.

**Definition 4.** *Given an inner product space $V$ we define*

$$\|x\| := \sqrt{\langle x, x\rangle}.$$

*Furthermore, we say that $x, y$ are orthogonal if $\langle x, y\rangle = 0$.*

With this definition, Pythagoras's theorem follows directly, as if $x, y$ are orthogonal then by linearity of the inner product.

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + \|y\|^2 .$$

An important property of the inner product is the following

**Proposition 3** (Cauchy Schwartz)**.** *Let $V$ be an inner product space, then it holds that*

$$\langle x, y\rangle \leq \|x\| \, \|y\| .$$

In particular, this implies that the inner product is continuous jointly and in each component. An inner product space keeps to key properties of Euclidean space, namely the vector space structure and inner product. The third key property is that of completeness.

**Definition 5.** *A Hilbert space is a complete inner product space $(H, \langle\cdot,\cdot\rangle)$. Where completeness is respect to the inner product norm*

$$\|x\|^2 := \langle x, x\rangle .$$

Cool! Now we know what is a Hilbert space. Some examples that have come up in previous posts are the space of square integrable functions valued in a (another) Hilbert space $H$ with the inner product

$$\langle f, g\rangle_{L^2(\Omega\to H)} := \int_\Omega \langle f, g\rangle \, d\mu.$$

Another are the Sobolev spaces $H^s(U)$ with the inner product

$$\langle f, g\rangle_{H^s(U)} := \int_U \langle \omega\rangle^{2s} \widehat{f}(\xi)\overline{\widehat{g}(\xi)} d\xi.$$

(I apologize for using the same notation for the Japanese bracket and the inner product). One result that carries over from Euclidean spaces to Hilbert spaces is that of projection.

**Theorem 3** (Projection theorem)**.** *Let $F$ be a closed convex subset of a Hilbert space $H$. Then, given $h \in H$ there exists a unique $f \in F$ such that*

$$\min_{g \in F} \|h - g\| = \|h - f\| .$$

*Furthermore, if $F$ is a (vector) subspace, then it holds that $f$ is the unique element in $F$ such that $h - f$ is orthogonal to $F$. That is $f \in F$ is the only one verifying*

$$h - f \in F^{\perp} := \{g \in H : \langle g, F \rangle = \{0\}\} .$$

*Proof.* The idea of the proof is to consider a sequence $f_n$ that approaches the minimum distance

$$d(f_n, h) \to d(h, F).$$

The inner product comes in to show that $f_n$ is a Cauchy sequence. Since $H$ is complete it converges to some $f$. Then our assumptions on $F$ come in. Firstly, since $F$ is closed shows $f \in F$. Secondly, since $F$ is convex one can get uniqueness. The uniqueness is once more a consequence of the properties of the inner product. □

A consequence of the above theorem is that orthogonal complements in Hilbert spaces exist.

**Theorem 4** (Orthogonal complement)**.** *Let $V$ be a closed subspace of a vector space $H$. Then we have that*

$$\begin{aligned} V \oplus V^{\perp} &\longrightarrow H \\ (v_1, v_2) &\longmapsto v_1 + v_2. \end{aligned}$$

*Is a bijective isomorphism where we consider on $V \oplus V^{\perp}$ the inner product*

$$\langle (v_1, v_2), (v_1', v_2') \rangle = \langle v_1, v_1' \rangle + \langle v_2, v_2' \rangle .$$

*Proof.* Given $h \in H$ we know that there exists a unique closest point $v_1 \in V$ to $V$ (the projection of $h$ onto $v$). Furthermore, $v_2 := h - v \in V^{\perp}$ so we get the decomposition

$$h = v_1 + v_2.$$

This proves bijectivity, whereas the isomorphism property follows quickly from the orthogonality of $v_1, v_2$. □

An important corollary of this is the following

**Corollary 1.** *Let $V$ be a vector space, then*

$$\overline{V} = H \iff \overline{V}^{\perp} = \{0\} .$$

Note that the imposition that $V$ is closed in Theorem 4 is necessary. Finite dimensional subspaces are always closed, but infinite dimensional subspaces may not be. For example, consider $H = L^2(I)$ for some bounded $I$ and take $V$ to polynomials on $I$ . By the Stone-Weierstrass theorem $V$ is dense in $H$. That is $H = \overline{V}$, whereas

$$H \neq V \oplus V^{\perp} = V \oplus \{0\} .$$

Another property of Euclidean space that Hilbert spaces reproduce is that of existence of orthonormal basis

**Definition 6.** *Let $H$ be a Hilbert space and $I$ some index set. We say that $\mathcal{B} = \{\phi_\alpha\}_{\alpha\in I}$ is an orthonormal basis of $H$ if*

$$\langle \phi_\alpha, \phi_\beta \rangle = \delta_{\alpha,\beta}, \quad \forall \alpha, \beta \in I.$$

*And for every element $f \in H$ there exist $\lambda_\alpha \in \mathbb{C}$ such that*

$$f = \sum_{\alpha\in I} \lambda_\alpha \phi_\alpha. \tag{8}$$

Note that we impose no conditions on the Index set $I$ which may be countable or uncountable.

**Definition 7.** *Given an index set $I$ and a normed vector space $X$ we say that $x_\alpha$ is absolutely summable to $x$ and write*

$$x = \sum_{\alpha\in I} x_\alpha$$

*if given $\epsilon > 0$ there exists some finite subset $J_0 \in I$ such that for every $J$ containing $J_0$ it holds that*

$$\left\| x - \sum_{\alpha\notin J} x_\alpha \right\| < \epsilon.$$

If $I$ is countable, the definition says that $\sum_{\alpha\in I} x_\alpha$ converges to $x$ irregardless of the order in which we sum. In fact, the following shows that, even if we start with an uncountable sequence we will always end up back in this case.

**Proposition 4.** *Let $X$ be a normed space and $\{x_\alpha\}_{\alpha\in I} \subset X$ be absolutely summable to $x$, then only a countable number of the terms $x_\alpha$ are non-zero. Let us take the nonzero terms and relabel them $\{x_n\}_{n\in\mathbb{N}}$, then*

$$x = \sum_{n=0}^{\infty} x_n.$$

*Proof.* Take $J_n$ such that

$$\left\| x - \sum_{\alpha\notin J_n} x_\alpha \right\| < \frac{1}{n}.$$

Then $J := \cup_n J_n$ is countable (it is a countable union of countable sets) and a small reasoning shows that $J$ are the non-zero terms of $I$. $\square$

Now that we made sense of the sum over a potentially uncountable number of basis elements in (8), it remains to address the question of existence of orthonormal basis.

**Theorem 5** (Existence of orthonormal basis)**.** *Every Hilbert space has an orthonormal basis $\mathcal{B}$.*

*Proof.* The proof is formally identical to the proof that every vector space has a basis space . Let $\mathcal{A}$ be the collection of all orthonormal subsets of $H$. That is, $\mathcal{A}$ is comprised of sets of the form

$$\mathcal{S} = \left\{ \{\phi_\alpha\}_{\alpha\in J} : \langle \phi_\alpha, \phi_\beta\rangle = \delta_{\alpha,\beta}, \quad \forall \alpha,\beta\in J \right\}.$$

Such that $\phi_\alpha$ are orthonormal. Given an ordered chain $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots$ we have the bound

$$\mathcal{S}_0 \subset \mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots \subset \bigcup_{n=0}^{\infty} \mathcal{S}_n.$$

As a result by Zorn's lemma there exists a maximal element $\mathcal{B}$. If $\mathcal{B}$ is not complete (that is (8) doesn't hold), then there exists $f \in \overline{\mathcal{B}}^{\perp}$. By taking $f/\|f\|$ and forming $\mathcal{B}' := \mathcal{B} \cup \{f\,\|f\|\}$ we obtain that $\mathcal{B} \subsetneq \mathcal{B}' \in \mathcal{A}$. This contradicts the maximality of $\mathcal{B}$ and concludes the proof. $\square$

The next result is the natural generalization of Pythagoras's theorem to Hilbert spaces.

**Theorem 6** (Parseval)**.** *Let $H$ be a Hilbert space with orthonormal basis $\mathcal{B} = \{\phi_\alpha\}_{\alpha \in I}$. Then for every $f \in H$ it holds that*

$$f = \sum_{\alpha \in I} \langle f, \phi_\alpha \rangle \phi_\alpha; \quad \|f\|^2 = \sum_{\alpha \in I} \|\phi_\alpha\|^2 .$$

*Proof.* We have that by the orthonormality of $\phi_\alpha$ and the continuity of the inner product

$$\left\langle f - \sum_{\alpha \in I} \langle f, \phi_\alpha \rangle \phi_\alpha, \phi_\alpha \right\rangle = \langle f, \phi_\alpha \rangle - \langle f, \phi_\alpha \rangle = 0.$$

As a result, $f - \sum_{\alpha \in I} \langle f, \phi_\alpha \rangle \phi_\alpha$ is orthogonal to the closure of the span of $\mathcal{B}$, which by assumption is $H$. By Corollary 1 we conclude that

$$f - \sum_{\alpha \in I} \langle f, \phi_\alpha \rangle \phi_\alpha = 0.$$

This proves the first part of the theorem. The second follows by the first together with the orthonormality of $\phi_\alpha$. □

The above shows that, on fixing a basis, every Hilbert space can be identified with a space of square integrable sequences by the bijective isometry

$$H \to \ell^2(I); \quad f \to \{\langle f, \phi_\alpha \rangle\}_{\alpha \in I} .$$

In particular every Hilbert space with a countable basis is isometric to $\ell^2(\mathbb{N})$. However, the identification is not "canonical" as it depends on the bases chosen. The next example is all pervasive (in fact it has even invaded our blog)

**Example 3** (Plancherel)**.** *The Hilbert space of square integrable complex valued periodic functions $L^2(\mathbb{R}^d/\mathbb{Z}^d \to \mathbb{C})$ has orthonormal basis*

$$\{\phi_k(x)\}_{k \in \mathbb{Z}^d} := \left\{e^{2\pi i k \cdot x}\right\}_{k \in \mathbb{Z}^d} .$$

*Thus, every function $f \in L^2(\mathbb{R}^d/\mathbb{Z}^d)$ can be written as*

$$f(x) = \sum_{k \in \mathbb{Z}^d} \widehat{f}(k) e^{2\pi i k \cdot x}.$$

*Where* $\widehat{f} \in \ell^2(\mathbb{Z}^d)$ *is known as the Fourier transform of* $f$ *and defined by*

$$\widehat{f}(k) := \langle f, \phi_k \rangle = \int_{\mathbb{R}^d/\mathbb{Z}^d} f(x) e^{-2\pi i k \cdot x} \, \mathrm{d}x.$$

Another interesting property of Euclidean space is that every element of the dual $\ell : \mathbb{C}^n \to \mathbb{C}$ is represented by a vector in the space, that is

$$\ell(x) = \langle x, y_\ell \rangle, \quad \forall x \in \mathbb{C}^n.$$

Here one can calculate directly that $y_\ell$ is the conjugate of the "matrix" defined by $\ell$ as a linear function

$$y_\ell = (\overline{\ell(e_1)}, \ldots, \overline{\mathcal{L}(e_n)}).$$

Where $e_i$ is the standard orthonormal basis of $\mathbb{C}^n$. In Hilbert spaces, the same result holds for Hilbert spaces

**Theorem 7** (Riesz representation ). *Let* $H$ *be a Hilbert space, then given* $\ell \in H'$ *there exists a unique* $f_\ell \in H$ *such that*

$$\langle h, f_\ell \rangle, \quad \forall h \in H. \tag{9}$$

*Furthermore,* $\|f_\ell\| = \|\ell\|$.

*Proof.* Consider an orthonormal basis $\{\phi_\alpha\}_{\alpha \in I}$ for $H$. Then, just as in Euclidean space we have that

$$f_\ell = \sum_{\alpha \in I} \overline{\ell(\phi_\alpha)} \phi_\alpha.$$

The fact that $f_\ell$ verifies (9) is a direct application of the (anti)-linearity and continuity of the inner product. Uniqueness following from the fact that if $f_\ell, g_\ell$ both verify the equality then for all $h \in h$

$$\langle h, f_\ell - g_\ell \rangle = \ell(h) - \ell(h) = 0.$$

This can only occur if $f_\ell - g_\ell = 0$ (hint take $h = f_\ell - g_\ell$). To verify the norm we can use that, for all $h \in H$ with norm 1

$$\ell\left(\frac{f_\ell}{\|f_\ell\|}\right) = \left\langle \frac{f_\ell}{\|f_\ell\|}, f_\ell \right\rangle = \|f_\ell\|; \quad \ell(h) = \langle h, f_\ell \rangle \leq \|h\| \|f_\ell\| = \|f_\ell\|.$$

The equality shows that $\|\ell\| \geq \|f_\ell\|$ whereas the inequality shows the converse $\|\ell\| \leq \|f_\ell\|$, proving the theorem. $\square$

The previous theorem says that we have an antilinear isometry

$$\Phi_1 : H' \to H; \quad \ell \to f_\ell.$$

This allows us to identify $H$ with $H'$. In fact, the identification is canonical as it does not depend on the bases chosen. Yes we fixed a basis to prove it but the vector $f_\ell$ is unique independently of the basis. This allows us to make $H'$ into a Hilbert space in a canonical way

**Proposition 5.** *Let $H$ be a Hilbert space, then $H'$ is also a Hilbert space, with inner product given by*

$$\langle \ell_1, \ell_2 \rangle_{H'} = \langle f_{\ell_2}, f_{\ell_1} \rangle_H .$$

Where we had to "swap the order" of the representatives of $\ell_1, \ell_2$ due to the anti-linearity of the mapping $\ell \to f_\ell$. Since $H'$ is also a Hilbert space we can apply Riesz's theorem to $H'$ to show that $H''$ is also a Hilbert space and there exists a canonical antilinear isometry

$$\Phi_2 : H'' \to H; \varphi \to \ell_\varphi.$$

By construction, of $\Phi_1, \Phi_2$ it holds that

$$\Phi_2 \circ \Phi_1(f)(\ell) = \ell(f).$$

That is, $H$ is identified canonically with $H''$ and, the identification is such that

$$f(\ell) = \ell(f).$$

In other words

**Theorem 8.** *Every Hilbert space is reflexive.*

Hilbert spaces provide us a way to guarantee existence and uniqueness to a wide class of problems, an important tool is Lax-Milgram's theorem. We first need two definitions

**Definition 8.** *We say that a mapping*

$$B : V \times V \to \mathbb{K}.$$

*On a vector space $V$ is sesquilinear if $B$ is linear in the first component and antilinear in the second. That is, for all $x, y, z \in V$ and $\lambda \in \mathbb{K}$:*

$$B(\lambda x + y, z) = \lambda B(x, z) + B(y, z); \quad B(x, \lambda y + z) = \overline{\lambda} B(x, z) + B(y, z).$$

**Definition 9.** *Let $V$ be a normed vector space then we say that a sesquilinear form $B$ is $\alpha$ coercive if it is continuous and there exists a constant $\alpha > 0$ such that*

$$B(f,f) \geq c \|f\|^2 \quad \forall f \in H.$$

The coercivity condition essentially imposes that the bilinear form is not degenerate. As a particular example, a symmetric sesquilinear form is an inner product.

**Theorem 9** (Lax Milgram)**.** *Let $B, L$ be respectively an $\alpha$ coercive sesquilinear form and a linear form on a Hilbert space $H$. Then there exists an invertible linear operator $\mathcal{L} : H \to H$ and $f \in H$ such that*

$$B(v,u) = \langle v, \mathcal{L}u \rangle ; \quad L(v) = \langle v, f \rangle . \tag{10}$$

*As a result, equation*

$$B(v,u) = L(v) \quad \forall v \in H$$

*has a unique solution $u = \mathcal{L}^{-1} f$. Furthermore, the solution operator $\mathcal{L}^{-1}$ is continuous with*

$$\left\| \mathcal{L}^{-1} \right\| \leq \alpha^{-1}.$$

*Proof.* For each fixed $u \in H$, we have that $\ell_u := B(\cdot, u) \in H'$. As a result by Riesz's representation theorem (Theorem 7) there exists a unique $f_{\ell_u} \in H$ such that

$$B(v,u) = \ell_u(v) = \langle v, f_{\ell_u} \rangle . \tag{11}$$

Furthermore, it can be simply verified that the mapping $v \to f_{\ell_u}$ is linear in $u$. That is, there exists $\mathcal{L} : H \to H$ such that

$$\mathcal{L}u = f_{\ell_u} \quad \forall u \in H. \tag{12}$$

The existence of the representative $f \in H$ of $L$ is once more by Riesz's representation theorem. We now show that $\mathcal{L}$ verifies the desired properties. Firstly $\mathcal{L}$ is continuous as, given $v \in H$

$$\|\mathcal{L}u\|^2 = \langle \mathcal{L}u, \mathcal{L}u \rangle = B(\mathcal{L}u, u) \leq \|B\| \|u\| \|\mathcal{L}u\| .$$

So dividing by $\|\mathcal{L}u\|$ on either side shows that shows that $\|\mathcal{L}\| \leq \|B\|$. Now, $\mathcal{L}$ is injective as if $\mathcal{L}u = 0$ then

$$0 = \langle u, \mathcal{L}u \rangle = B(u,u) \geq \alpha \|u\|^2 .$$

For similar reasons $\mathcal{L}$ is surjective. Consider $v \in \mathbf{Im}(\mathcal{L})^{\perp}$, then it holds that

$$\langle u, \mathcal{L}u \rangle = B(u,u) \geq \alpha \|u\|^2 . \tag{13}$$

As a result, we deduce from the orthogonal complement theorem 4 that $\overline{\mathbf{Im}(\mathcal{L})} = 0$. Thus, if we show that $\mathcal{L}$ is closed we are done. From the estimate in (13) together with Cauchy Schwartz shows that for all $u \in H$

$$\|\mathcal{L}u\| \geq \|u\| .$$

As if $\mathcal{L}u_n \in \mathbf{Im}(\mathcal{L})$ is a Cauchy sequence then so must be $u_n$. By completeness of $\mathcal{L}$, the sequence $u_n$ converges to some $u \in H$ and we deduce, by continuity of $\mathcal{L}$, that $\mathcal{L}u_n \to \mathcal{L}u \in \mathbf{Im}(\mathcal{L})$. In consequence, $\mathcal{L}$ is invertible, finally to show the bound on $\mathcal{L}^{-1}$ let us write $u = \mathcal{L}^{-1} f$ then

$$\alpha \|u\|^2 \leq B(u,u) = \langle u, f \rangle \leq \|u\| \|f\| .$$

Dividing on either side by $\alpha \|u\|$ concludes the proof. □

If we had made the assumption that $B$ were anti-symmetric then the proof would have been simplified as $B$ would define an inner product $\langle \cdot, \cdot \rangle_B$. Applying Riesz's theorem to this inner product (as opposed to the original one) would transform our equation (**??**) into

$$\langle v, u \rangle_B = \langle v, f \rangle , \quad \forall v \in H.$$

That is, to solve (10) it would suffice to take $u = f$. In the case were $B$ is symmetric and real we can also find $u$ by solving a minimization problem

**Proposition 6** (Minimization formulation)**.** *Let $B : H \times H \to \mathbb{R}$ be a symmetric coercive bilinear operator on a real Hilbert space $H$. Then problem* (10) *is equivalent to minimizing*

$$J(u) := \frac{1}{2} B(u,u) - L(u).$$

*Proof.* To prove that a solution to (10) minimizes $J$ we can develop $J(u+v)$ and simplify it using $B(v,u) = L(v)$ to obtain

$$J(u+v) \geq J(u), \quad \forall v \in H.$$

To prove that a minimum of $J$ solves (10) one can show by taking limits as $\lambda \to 0$ in the expression

$$J(u) \leq J(u + \lambda(h - u)).$$

That for all $h$

$$L(h - u) \leq B(u, h - u).$$

Taking $h = u + v$ and $h = u - v$ where $v$ is any shows $-L(v) \leq -B(u, v)$ and $L(v) \leq B(u, v)$, which concludes the proof. $\square$

# B A little bit of operator theory

Finally we wrap up with some operator theory. This will be revisited in a more detail blog-post on spectral theory. For now we give the essentials.

**Definition 10.** *We say that an linear operator $K : X \to Y$ where $X, Y$ are two normed spaces is compact if $T(B)$ is relatively compact for all bounded $B \subset X$.*

The above will be abbreviated $K \in \mathcal{K}(X, Y)$. The reason we are interested in compact operators is that they are particularly simple. Note that every finite dimensional operator (that is operators whose image is finite dimensional) is compact by the Heine-Borel theorem in fact, a good way of thinking of compact operator is to see them as finite dimensional operators . Or more precisely, as the limit of them

**Proposition 7.** *Let $T \in \mathcal{K}(X, H)$ where $H$ is a separable Hilbert space . Then there exists a sequence of finite dimensional operators $T_n$ such that*

$$\lim_{n \to \infty} T_n = T.$$

*Proof.* Since $H$ is separable there exists a countable orthonormal basis $\{\phi_n\}_{n \in \mathbb{N}}$. One has that if we denote $T_n$ for the projection of $T$ onto the space generated by $\{\phi_1, \ldots, \phi_n\}$ then, by Parseval's Theorem pointwise convergence holds

$$T_n(x) \to T(x), \quad \forall x \in H.$$

Consider the unit ball $B_X$ in $X$. Then $T(B_X)$ is relatively compact. So $T(B_X)$ is totally bounded and given $\epsilon > 0$ we can form a finite $\epsilon$ net

$Tx_1, \ldots, Tx_m$ of $T(B(X)$. By pointwise convergence we can now take $n_0$ large enough so that for all $N \geq n_0$

$$\|T(x_j) - T_N(x_j)\|, \quad \forall j = 1, \ldots, m.$$

Now for any $x \in B_X$ we can find $x_j$ such that $\|Tx_j - Tx\| < \epsilon$. Using the triangle inequality

$$\|Tx - T_N x\| \leq \|Tx - T_N x_j\| + \|T_N x_j\| + \|T_N x_j - Tx_j\| < 3\epsilon.$$

This concludes the proof. □

An important property of compact operators is that they are preserved by continuous ones.

**Proposition 8.** *Let $K \in \mathcal{K}(X, Y), T \in L(Y, Z)$ with $Z$ complete, then $T \circ K$ is compact*

*Proof.* This is a consequence of the fact that in complete metric spaces relatively compact is equivalent to totally bounded. □

For us, an important example of compact operators will be the solution operator $\mathcal{L}^{-1}$ of a PDE. This is because of the following theorem

**Theorem 10** (Rellich-Kondrachov)**.** *Let $U \subset \mathbb{R}^n$ be a bounded open domain in $\mathbb{R}^n$ with smooth boundary. Then, given $s > 0$ it holds that the natural inclusion*

$$i : H^{s+\sigma}(U) \hookrightarrow H^s(U)$$

*is compact for all $\sigma > 0$.*

For the proof of a more general version see [1] page 334. It is important to observe the restriction that $U$ is bounded is necessary and the theorem no longer holds if $U$ is replaced by $\mathbb{R}^n$.

**Corollary 2** (Adding differentiability is compact)**.** *Let $U \subset \mathbb{R}^n$ be a bounded open domain in $\mathbb{R}^n$ with smooth boundary and $s, \sigma > 0$, then every continuous operator*

$$T : H^{s+\sigma}(U) \hookrightarrow H^s(U)$$

*is compact.*

*Proof.* We have that $T = T \circ i$ so we conclude by Rellich-Kondrachov's theorem **??** and the preservation of compact operators by continuous ones (Proposition 2). $\square$

Compact operators also have nice spectral theory. Let

**Theorem 11** (Spectral theorem)**.** *Let $T \in \mathcal{K}(H)$ be compact and self adjoint where $H$ is a separable Hilbert space. Then $T$ diagonalizes in an orthonormal basis. That is, there exists an orthonormal basis $\{e_n\}_{n\in\mathcal{N}}$ and $\lambda_n \in \mathbb{K}$ such that*

$$Tx = \left(\sum_{n=0}^{\infty} \lambda_n e_n \otimes e_n\right) x = \sum_{n=0}^{\infty} \lambda_n \langle x, e_n \rangle e_n.$$

This type of discrete representation of $T$ is very useful and links up with the theory of trace class and Hilbert-Schmidt operators which will become a mainstay of future posts. To end it all off we state without proof a theorem that will be useful in proving properties about the solution space to the solution of PDEs

**Theorem 12** (Fredholm alternative)**.** *Let $H$ be a Hilbert space and $K \in \mathcal{K}(H)$. Consider $T := Id - K$ then it holds that*

$$T \text{ is injective if and only if } Tis surjective$$

*Furthermore, it holds that*

- *a)* $\mathbf{ker}(T)$ *is finite dimensional.*
- *b)* $T$ *is closed.*
- *c)* $\mathbf{Im}(T) = \mathbf{ker}(T^*)^{\perp}$.
- *d)* $\dim(\mathbf{ker}(T)) = \dim(\mathbf{ker}(T^*))$

We delay the proof till another day, in the meantime, see [2] page 725.

# References

[1] M. Taylor, Partial differential equations II: Qualitative studies of linear equations, Vol. 116, Springer Science & Business Media, 2013.

[2] L. C. Evans, Partial differential equations, Vol. 19, American Mathematical Society, 2022.
URL https://math24.files.wordpress.com/2013/02/partial-differential-equations-by-evans.pdf