The Skorohod Integral

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1 Three line summary

- By fixing t, one can obtain a chaos expansion for (possibly non-adapted) square integrable stochastic processes $X(t)$.
- The Itô integral of an Itô integrable process $X(t)$ has a chaos expansion.
- This chaos expansion can converge even when $X(t)$ is not adapted to the filtration $\mathcal{F}_t$ (and thus not Itô integrable). This allows us to extend the Itô integral to non-adapted processes.

2 Why should I care?

If you want to define the Malliavin derivative you need the Skorohod integral.

3 Notation

The same as in the previous post [1] on the chaos expansion. We will also write $\mathbb{L}^2(I \times \Omega)$ for the space of Itô integrable functions (this was defined in [2]).

4 The chaos expansion of an Itô integral

Our goal in this post is to construct the Skorohod integral. This serves as a generalization of the Itô integral and the starting point for the definition of the Malliavin derivative. How is this done? Let us first consider a (not-necessarily adapted) stochastic process X such that $X_t \in L^2(\Omega, \mathcal{F}_\infty)$ for each $t \in I$. Then we know that by the chaos expansion proved in the previous post, for each t there exists $f_{n,t} \in L^2(S_n)$ such that

$$X = \sum_{n=0}^{\infty} I_n(f_{n,t}).$$

Let us write $f_n(\cdot, t) := f_{n,t}$. Note that we are now considering f_n as a function of $n+1$ variables instead of n. In particular, we will be able to consider expressions like $I_{n+1}(f_n)$ later on. The first thing we do is study what the adaptedness of X means in term of the functions f_n appearing in its chaos expansion.

Lemma 1. *Let $X(t) \in L^2(\Omega, \mathcal{F}_\infty)$ for each $t \in I$, then X is adapted iff*

$$f_n(t_1, \dots, t_n, t) = 0, \quad \forall t \le \max_{i=1,\dots,n} t_i.$$

Proof. Firstly, we note that a stochastic process X is adapted iff

$$X_t = \mathbb{E}_{\mathcal{F}_t}[X(t)] \quad \forall t \in I.$$

Since the Itô integral is a martingale, we obtain that, by commuting the sum and using the uniqueness of the chaos expansion this is equivalent to requiring that, for all t

$$\begin{aligned}
&I_n(f_n(\cdot, t)) \\
&\quad = n! \mathbb{E}_{\mathcal{F}_t}\left[\int_I \left(\int_0^{t_n} \cdots \int_0^{t_2} f(t_1 \dots t_n, t) dW(t_1) \cdots dW(t_{n-1})\right) dW(t_n)\right] \\
&\quad = n! \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} f(t_1 \dots t_n, t) dW(t_1) \cdots dW(t_{n-1}) dW(t_n) \\
&\quad = I_n(f_n(\cdot, t) 1_{\max_{t_i \le t}})
\end{aligned}$$

Where the commutation of the sum and the integral is justified by the $L^2(\Omega)$ convergence of the chaos expansion ($L^1(\Omega)$ convergence would have been enough). □

In particular, we obtain that, since f_n is already symmetric in its first n-coordinates, its symmetrization verifies that

$$f_{n,S}(t_1, \dots, t_n, t_{n+1}) = \frac{1}{n+1} f_n(t_1, \dots \hat{t_j}, \dots, t_{n+1}, t_j), \quad \text{where } j = \arg\max_i t_i.$$

Using this relationship we can directly calculate the Itô integral of a stochastic process to obtain that.

Theorem 1. *Let $X \in \mathbb{L}^2(I \times \Omega)$ then the Itô integral of X is*

$$\int_I X(t)dW(t) = \sum_{n=0}^{\infty} I_{n+1}(f_{n,S}).$$

Proof. This is a direct calculation using the previous result as

$$\begin{aligned}
\int_I X(t)dW(t) &= \sum_{n=0}^{\infty} \int_I I_n(f_{n,t})dW(t) \\
&= \sum_{n=0}^{\infty} n! \int_I \int_{S_n} f_{n,t}(t_1, \dots, t_n)dW(t_1)\dots dW(t_n)dW(t) \\
&= \sum_{n=0}^{\infty} (n+1)! \int_I \int_{S_n} f_{n,S}(t_1, \dots, t_n, t)dW(t_1)\dots dW(t_n)dW(t) \\
&= \sum_{n=0}^{\infty} (n+1)! J_{n+1}(f_{n,S}) = \sum_{n=0}^{\infty} I_{n+1}(f_{n,S}).
\end{aligned}$$

□

5 The Skorohod integral

The last term appearing in the equality is what we will call the Skorohod integral.

Definition 1. *Let $X(t) \in L^2(\Omega, \mathcal{F}_\infty)$ be a stochastic process such that*

$$\delta(X) := \int_I X(t)\delta W(t) := \sum_{n=0}^{\infty} I_{n+1}(f_{n,S}) \in L^2(\Omega).$$

Then we will say that X has Skorohod integral $\delta(W)$ and write $X \in dom(\delta)$.

As we saw in the previous theorem the Skorohod integral is equal to the Itô integral for all stochastic processes in $\mathbb{L}^2(I \times \Omega)$. However, it may also be defined for non-adapted stochastic processes. In fact, by using the orthogonality of the iterated integrals (what we called *Itô's n-th isometry* in the last post [1], we deduce the following).

Proposition 1. *A stochastic process $X(t) \in L^2(\Omega, \mathcal{F}_\infty)$ has a Skorohod integral iff*

$$\sum_{n=0}^{\infty}(n+1)!\|f_{n,S}\|_{L^2([0,T]^n)} < \infty.$$

Proof. By Itô's n-th isometry we have that

$$\|\delta(X)\|_{L^2(\Omega)} = \sum_{n=0}^{\infty}\|I_{n+1}(f_{n,S})\| = \sum_{n=0}^{\infty}(n+1)!\|f_{n,S}\|_{L^2(I^{n+1})}.$$

□

Of course, a priori the above condition is not that easy to check for a given function as it involves calculating the chaos expansion for the given process X. In some cases however it is possible, to consider for example the stochastic process defined by $X(t) = W(T)$ on the interval $I = [0,T]$. Then we have that

$$X(t) = \int_0^T dW(t) = I_1(1).$$

Thus, for all $t \in I$ we have that

$$f_1 = 1; \quad f_n = 0 \quad \forall n \in \mathbb{N} \setminus \{1\}.$$

So $X \in dom(\delta)$ with

$$\delta(X) = I_2(1) = \int_0^T \int_0^t dW(t_1)dW(t) = \int_0^T W(t)dW(t) = W^2(T) - T.$$

Note however that the Itô integral of $W(T)$ is undefined as it is not $\mathcal{F}_t$ adapted. Since the Skorohod integral of 1 is equal to $W(T)$, the above example shows how one cannot simply "pull out constants in t" in the sense that, if G is a random variable independent of t and $X(t) = G \cdot u(t)$, then

$$\int_I X(t)\delta W(t) = \int_I G \cdot u(t)\delta W(t) \neq G \int_I u(t)dW(t).$$

Though this may seem unintuitive, it is a consequence of the fact that, even though f_i may not depend on t, the terms

$$g(t) := \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} f_i dW(t_1) \cdots dW(t_{n-1}) dW(t_n).$$

Can depend on t. Despite this, the Skorohod integral still maintains some of the natural properties we associate with integration.

Proposition 2. *Let $X(t), Y(t) \in dom(\delta)$, $\lambda \in \mathbb{R}$. Then it holds that*

- *$X(t) + \lambda Y(t) \in dom(\delta)$ with $\delta(X + \lambda Y) = \delta(X) + \lambda\delta(Y)$.*
- *$\mathbb{E}[\delta(X)] = 0$.*
- *$X \cdot 1_A \in dom(\delta)$ for any measurable subset $A \subset I$. Furthermore, if $A \cup B = I$ then*

$$\int_A X(t)\delta(t) + \int_B X(t)\delta W(t) := \delta(X \cdot 1_A) + \delta(X \cdot 1_B) = \delta(X).$$

Proof. The first property is a consequence of the chaos expansion's linearity (which is itself a consequence of the linearity of iterated Itô integration). The second is due to the expectation of the Itô integral being 0. The final property is a consequence of the fact that the chaos expansion of $X \cdot 1_A$ is

$$X \cdot 1_A = \sum_{n=0}^{\infty} I_n([f_n 1_A]_S).$$

Which shown by the equivalent characterization of Skorohod functions that $X \cdot 1_A \in dom(\delta)$. The final property is a consequence of the previously proved linearity. □

We now know what the Skorohod expansion is, how to characterize it, and its main properties, in the next post we will construct the Malliavin derivative as its adjoint.

References

[1] L. Llamazares, The chaos expansion (2022).
URL https://liamllamazares.github.io/2022-05-26-Malliavin-Calculus-1/

[2] L. Llamazares, The ito integral (2022).
URL https://liamllamazares.github.io/2022-05-26-The-Ito-integral/