

# Fractional Sobolev spaces

Liam Llamazares

February 23, 2024

## 1 Three line summary

1. There are three ways to define Sobolev spaces with fractional regularity  $s$  and integrability  $p$ 
  - a) The spaces  $W^{s,p}(\Omega), B^{s,p}(\Omega)$  are defined by using the analogous to the definition of Hölder spaces. Both spaces are equal when the regularity  $s$  is not an integer.
  - b) The space  $H^{s,p}(\Omega)$  is defined by using the Fourier transform and coincides with  $W^{s,p}(\Omega)$  for integer  $s$ .
  - c) All these spaces coincide with  $H^s(\Omega)$  when  $p = 2$ .
2. There is a natural correspondence between negative regularity and the dual. Additionally, negative regularity can be obtained by differentiating functions with higher regularity.
3. Using these fractional spaces, one can obtain finer regularity results such as in the trace theorem or Sobolev embeddings.

## 2 Introduction

In previous posts, we covered the theory of Sobolev spaces  $W^{k,p}(\Omega)$  where  $k$  is an integer. In the case  $k = 2$  and when  $\Omega = \mathbb{R}^d$  we saw that this space coincided with  $H^k(\mathbb{R}^d)$ . Furthermore, we had also seen how to define  $H^s(\mathbb{R}^d)$  when  $s$  was any real number. This motivates the following two questions.

1. How can we define  $H^s(\Omega)$  when  $\Omega$  is not  $\mathbb{R}^d$  and  $s$  is not an integer ?
2. Is it possible to extend such a definition to other orders of integrability  $p$ ?

In this post, we aim to answer these questions, and show that both of these questions can be answered in the affirmative. The first point can be resolved by restricting functions in  $H^s(\mathbb{R}^d)$  to  $\Omega$ . The second point is trickier and, in fact, like any good trick question, has multiple answers. Three, to be precise. This leads to the theory of *Bessel* spaces, *Sobolev-Slobodeckij* spaces and *Besov* spaces

$$H^{s,p}(\Omega), W^{s,p}(\Omega), B^{s,p}(\Omega).$$

We will cover the basic properties of these spaces as well as their relationship to each other. We will see how these spaces can be used to obtain finer regularity results, such as in the trace theorem or Sobolev embeddings. The material in this post is based on [1], [2], [3] and [4]. The material can be quite technical, and there are multiple 800 plus page books on the subject, so in many cases, we will mainly state the main results, providing references for the proofs, as well as proving some of the more tractable results.

## 2.1 Preliminaries

In this post, we make frequent use of the fact that (as [shown](#) in a previous post) functions in  $L^p(U)$  can be identified as elements of the larger space of distributions  $\mathcal{D}'(\Omega) := C_c^\infty(U)'$ . This is done by identifying a function  $f \in L^p(U)$  with the linear functional  $\phi \mapsto \int_U f \phi \, dx$  and allows us to extend by duality operators that are defined on  $C_c^\infty(U)$  to  $L^p(U)$ . For example, given  $u \in L^p(U) \in \mathcal{D}'(\Omega)$  we can define both its Fourier transform and  $\alpha$ -th derivative to be the distributions defined by

$$(\varphi, \mathcal{F}u) := (\mathcal{F}^{-1}\varphi, u), \quad (\varphi, D^\alpha u) := (-1)^{|\alpha|} (D^\alpha \varphi, u), \quad \forall \varphi \in C_c^\infty(U).$$

The above definition is justified by the fact that if it turns out that  $u$  is smooth and integrable enough after all, this coincides with the usual definitions of  $\mathcal{F}, D^\alpha$ .

In terms of notation, we will always denote  $U$  by an open subset of  $\mathbb{R}^d$  whereas  $\Omega \subset \mathbb{R}^d$  will be open with a smooth enough boundary.

**Definition 2.1.** Let  $X(U)$  be a topological space of functions for each open  $U \subset \mathbb{R}^d$ . We say that  $\Omega$  is an extension domain for  $X$  if there exists a continuous operator

$$E : X(\Omega) \rightarrow X(\mathbb{R}^d)$$

such that  $Eu(x) = u(x)$  for almost all  $x \in \Omega$ .

## 3 Fractional Sobolev spaces: three definitions

The definitions developed in the next three subsections can be found in [\[1\]](#) page 222.

### 3.1 Sobolev-Slobodeckij spaces

**Definition 3.1** (Sobolev-Slobodeckij spaces). Let  $s = k + \gamma$  where  $k \in \mathbb{N}_0$ , and  $\gamma \in [0, 1)$ . Then, given  $p \in [1, \infty)$  and  $\Omega \subset \mathbb{R}^n$  be an arbitrary open set. Write  $k = \lfloor s \rfloor, \gamma = s - k$ . We define

$$W^{s,p}(U) := \left\{ u \in W^{k,p}(U) : \|u\|_{W^{s,p}(U)} < \infty \right\},$$

where

$$\|u\|_{W^{s,p}(U)} := \left( \|u\|_{W^{k,p}(U)}^p + \sum_{|\alpha|=k} \int_U \int_U \frac{|D^\alpha u(x+y) - D^\alpha u(x)|^p}{|y|^{n+\gamma p}} \, dx \, dy \right)^{\frac{1}{p}}. \quad (1)$$

We will later define  $W^{s,p}(U)$  also for negative  $s$  (see [Definition 4.2](#)). We observe that the above definition coincides with our usual definition of Sobolev space when  $s = k \in \mathbb{N}_0$  and mimics that of the Hölder spaces, with the addition that we now require integrability. The factor  $|x - y|^{n+\gamma p}$  is chosen so that the integral is scale invariant.

**Exercise 1.** Show that  $W^{s,p}(U)$  is a Banach space.

**Hint.** To show that  $\|\cdot\|_{W^{s,p}(U)}$  is a norm apply Minkowski's inequality to  $u$  and to  $f_u(x, y) := (D^\alpha u(x+y) - D^\alpha u(x))/|y|^{n/p+s}$ . Given a Cauchy sequence show that, since  $L^p(U)$  is complete,  $u_n \rightarrow u$  in  $L^p(U)$  and that  $f_{u_n} \rightarrow f_u$  in  $L^p(U \times U)$  to conclude that  $u_n \rightarrow u$  in  $W^{s,p}(U)$ .

Though the Sobolev-Slobodeckij spaces can be defined for any open set  $U$ , they are most useful when  $U = \mathbb{R}^d$  or  $U$  is regular enough. Otherwise, basic properties such as the following break down

**Proposition 3.2** (Inclusion ordered by regularity). *Let  $\Omega$  be an extension domain for  $W^{1,p}$ . Then, for  $p \in [1, \infty)$  and  $0 < s < s'$  it holds that*

$$W^{s',p}(\Omega) \hookrightarrow W^{s,p}(\Omega), \quad W^{s',p}(\mathbb{R}^d) \hookrightarrow W^{s,p}(\mathbb{R}^d).$$

The proof can be found in [2] page 10. The regularity of the domain is necessary to be able to extend functions in  $W^{1,p}(\Omega)$  to  $W^{1,p}(\mathbb{R}^d)$ . The result is not true otherwise, and an example is given in this same reference.

### 3.2 Bessel potential spaces

We now give a second definition of fractional Sobolev spaces through the Fourier transform.

**Definition 3.3** (Bessel potential spaces on  $\mathbb{R}^d$ ). *Let  $s > 0$  and  $p \in [1, \infty)$ . Define for  $u \in \mathcal{S}'(\mathbb{R}^d)$*

$$\Lambda^s u := \mathcal{F}^{-1}(\langle \xi \rangle^s \widehat{u}(\xi)).$$

*Then, we define the Bessel potential space*

$$H^{s,p}(\mathbb{R}^d) := \left\{ u \in \mathcal{S}'(\mathbb{R}^d) : \Lambda^s u \in L^p(\mathbb{R}^d) \right\},$$

*and give it the norm*

$$\|u\|_{H^{s,p}(\mathbb{R}^d)} := \|\Lambda^s u\|_{L^p(\mathbb{R}^d)}.$$

*We also define the space  $H_0^{s,p}(\mathbb{R}^d)$  as the closure of  $C_c^\infty(\mathbb{R}^d)$  in  $H^{s,p}(\mathbb{R}^d)$ .*

The definition above is motivated by the case  $p = 2$ . As we saw when we studied Sobolev spaces through the Fourier transform,  $u \in H^k(\mathbb{R}^d)$  if and only if  $\Lambda^k u \in L^2(\mathbb{R}^d)$ . That is,  $H^{k,2}(\mathbb{R}^d) = H^k(\mathbb{R}^d)$ . The natural generalization of this fact gives Definition 3.3.

**Exercise 2.** Show that  $\Lambda^s \Lambda^r = \Lambda^{s+r}$ . Use this to show that the following is an invertible isomorphism

$$\Lambda^r : H^{r+s,p}(\mathbb{R}^d) \xrightarrow{\sim} H^{s,p}(\mathbb{R}^d).$$

**Hint.** Use that  $\langle \xi \rangle^s \langle \xi \rangle^r = \langle \xi \rangle^{s+r}$  and show that the inverse of  $\Lambda^r$  is  $\Lambda^{-r}$ .

We now extend this to general domains

**Definition 3.4** (Bessel potential spaces on  $U$ ). *Let  $U \subset \mathbb{R}^d$  be an arbitrary open set. We define,*

$$H^{s,p}(U) := \left\{ u \in \mathcal{D}'(U) : \text{there exists } v \in H^{s,p}(\mathbb{R}^d) \text{ such that } v|_U = u \right\}.$$

*And give it the norm*

$$\|u\|_{H^{s,p}(U)} := \inf \left\{ \|v\|_{H^{s,p}(\mathbb{R}^d)} : v|_U = u \right\}.$$

The restriction above is in the sense of distributions. That is, we define  $u = v|_U$  by

$$(\phi, u) := (\phi, v), \quad \forall \phi \in C_c^\infty(U).$$

It would be tempting to define  $\|u\|_{H^{s,p}(U)} := \|\Lambda^s v\|_{L^p(U)}$ . However, since the Fourier transform, and thus  $\Lambda^s$ , is a nonlocal operator, the norm would depend on the extension  $v$  of  $u$  to  $\mathbb{R}^d$ . So, the norm would be ill-defined.

### 3.3 Besov spaces

**Definition 3.5** (Besov spaces). *Let  $s = k_- + \gamma$  where  $k_- \in \mathbb{N}_0$ , and  $\gamma \in (0, 1]$ . Then, given  $p \in [1, \infty)$  and  $\Omega \subset \mathbb{R}^n$  be an arbitrary open set we define*

$$B^{s,p}(U) := \left\{ u \in W^{k_-,p}(U) : \|u\|_{B^{s,p}(U)} < \infty \right\},$$

where

$$\|u\|_{B^{s,p}(U)} := \left( \|u\|_{W^{k_-,p}(U)}^p + \sum_{|\alpha|=k} \int_U \int_U \frac{|D^\alpha u(x+y) - D^\alpha u(x)|^p}{|y|^{n+\gamma p}} dx dy \right)^{\frac{1}{p}}.$$

The above definition is extremely similar in form to that of the Sobolev-Slobodeckij spaces 3.1. In fact, it is equivalent when  $s \notin \mathbb{N}$ . The difference is that in the definition of Besov spaces 3.5, we require that  $\gamma > 0$ . As a result, always  $k_- < s$ . We have chosen to indicate this fact by the index “-” on  $k_-$ . An equivalent definition is possible which extends the above to negative values of  $s$

**Definition 3.6** (Besov spaces, negative  $s$ ). *Let  $s \in \mathbb{R}$  and choose any  $\sigma \notin \mathbb{N}_0$  with  $\sigma > 0$ . Then, given  $p \in [1, \infty)$  we define*

$$\|u\|_{B^{s,p}(\Omega)} = \|\Lambda^{s-\sigma} u\|_{W_p^\sigma(\Omega)}.$$

The requirement  $\sigma > 0$  is necessary as in general  $B^{s,p}(\mathbb{R}^d) \neq H^{s,p}(\mathbb{R}^d)$ .

**Exercise 3.** Show that  $\Lambda^r$  defines an invertible isomorphism

$$\Lambda^r : B^{s,p}(\mathbb{R}^d) \xrightarrow{\sim} H^{s-r,p}(\mathbb{R}^d).$$

**Hint.** Apply definition 3.6 and use that  $W^{\sigma,p}(\Omega) = B^{\sigma,p}$  for non-integer  $\sigma$ . Finally,  $\Lambda^r$  has inverse  $\Lambda^{-r}$ .

The definition of  $B^{s,p}(\mathbb{R}^d)$  can then be extended to general open sets  $U$  in the same way as for the Bessel potential spaces,

**Definition 3.7** (Besov spaces on  $U$ ). *Let  $U \subset \mathbb{R}^d$  be an arbitrary open set. We define,*

$$B^{s,p}(U) := \left\{ u \in \mathcal{D}'(U) : \text{there exists } v \in B^{s,p}(\mathbb{R}^d) \text{ such that } v|_U = u \right\},$$

and give it the norm

$$\|u\|_{B^{s,p}(U)} := \inf \left\{ \|v\|_{B^{s,p}(\mathbb{R}^d)} : v|_U = u \right\}.$$

**Observation 1.** Different authors use different notations for these spaces. For example, in [3], the notation  $W^{s,p} := B^{s,p}$  is used. With this notation, one has that, for  $p \neq 2$ ,

$$H^{k,p} \neq B^{k,p} = W^{k,p}.$$

Whereas, with our notation, as we will later see,  $H^{k,p} = W^{k,p}$ . Other notations which can be found are the notation  $B^{s,p} = \Lambda_s^p$  and  $H^{s,p} = \mathcal{L}_s^p$ . See [4] and [5].

### 3.3.1 Interpolation

Both the Sobolev-Slobodeckij and Bessel potential spaces can be viewed as a way to fill the gaps between integer-valued Sobolev spaces.

**Proposition 3.8** (Interpolation ). *Let  $s_0 \neq s_1 \in \mathbb{R}, p \in (1, \infty), 0 < \theta < 1$  and*

$$s = s_0(1 - \theta) + s_1\theta, \quad p = p_0(1 - \theta) + p_1\theta.$$

*Then, given  $\Omega \subset \mathbb{R}^d$  open with uniformly Lipschitz boundary it holds that*

$$B^{s,p} = [B^{s_0,p}(\Omega), B^{s_1,p}(\Omega)]_\theta, \quad H^{s,p}(\Omega) = [H^{s_0,p_0}(\Omega), H^{s_1,p_1}(\Omega)]_\theta,$$

*where  $[X, Y]_\theta$  denotes the complex interpolation space.*

The result can be found in [3] page 45 for  $\Omega = \mathbb{R}^d$ . The general result follows by extension. See [6] page 424. In particular, if we write  $k := \lfloor s \rfloor$  and  $\gamma := s - k$ , then

$$H^{s,p}(\Omega) = [H^{k,p}(\Omega), H^{k+1,p}(\Omega)]_\gamma = [L^p(\Omega), H^{k+1,p}(\Omega)]_{s/(k+1)}.$$

## 4 Relationship between the definitions

The following result shows the inclusions between  $W^{s,p}, H^{s,p}, B^{s,p}$  and can be found in [1] page 224 and in [4] page 155.

**Theorem 4.1.** *Let  $\Omega \subset \mathbb{R}^d$  be open with uniformly Lipschitz boundary and  $s \in [0, \infty)$ . Then,*

$$\begin{aligned} H^{s+\epsilon,p}(\Omega) &\subset B^{s,p}(\Omega) \subset H^{s,p}(\Omega) \quad \forall p \in (1, 2] \\ B^{s+\epsilon,p}(\Omega) &\subset H^{s,p}(\Omega) \subset B^{s,p}(\Omega) \quad \forall p \in [2, \infty), \end{aligned}$$

*where the above inclusions are continuous and dense. Furthermore,*

$$W^{s,p}(\Omega) = \begin{cases} H^{s,p}(\Omega) & \text{if } s \in \mathbb{N}_0 \\ B^{s,p}(\Omega) & \text{if } s \notin \mathbb{N}_0 \end{cases}. \quad (2)$$

*In consequence, for  $p = 2$ ,*

$$H^{s,2}(\Omega) = W^{s,2}(\Omega) = B^{s,2}(\Omega). \quad (3)$$

The equality in (2) shows that, as long as we understand the behaviour of  $H^{s,p}(\Omega)$  and  $B^{s,p}(\Omega)$ , we can completely determine that of  $W^{s,p}(\Omega)$ . It also justifies the following extension of  $W^{s,p}(\Omega)$  to negative regularity.

**Definition 4.2** (Slobodeckij space negative  $s$ ). *Let  $\Omega \subset \mathbb{R}^d$  be open with uniformly Lipschitz boundary. Then, given  $p \in [1, \infty)$  and any  $s \in \mathbb{R}$  we define*

$$W^{s,p}(\Omega) = \begin{cases} H^{s,p}(\Omega) & \text{if } s \in \mathbb{N}_0 \\ B^{s,p}(\Omega) & \text{if } s \notin \mathbb{N}_0 \end{cases}.$$

The equality for  $p = 2$  in (3) justifies that, for sufficiently regular domains, all three spaces are written  $H^s(\Omega)$ . We will prove the left-hand side of this equivalence in Exercise 4. For  $p \neq 2$ , the inclusions are, in general, strict. An example is constructed in [4] page 161 exercise 6.8.

**Exercise 4** (Equivalence of fractional spaces). Show that

$$H^{s,2}(\mathbb{R}^d) = W^{s,2}(\mathbb{R}^d).$$

**Hint.** We want to show that the norms are equivalent. That is, that

$$\|u\|_{B^{s,2}(\mathbb{R}^d)} \sim \|u\|_{H^{s,2}(\mathbb{R}^d)}.$$

We already know this is the case when  $s$  is an integer, so it suffices to show that the norms are equivalent for  $s = \gamma \in (0, 1)$ . That is, that

$$|u|_{s,2}^2 \sim \int_{\mathbb{R}^d} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 d\xi$$

By multiple changes of variable and Plancherel's theorem, we have that

$$\begin{aligned} |u|_{\gamma,2}^2 &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x+y) - u(y)|^2}{|x|^{d+2\gamma}} dx dy = \int_{\mathbb{R}^d} \frac{\|\mathcal{F}(u(x+\cdot) - u)\|^2}{|x|^{d+2\gamma}} dx \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|e^{-2\pi i x \cdot \xi} - 1|^2}{|x|^{d+2\gamma}} |\widehat{u}(\xi)|^2 dx d\xi = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \frac{1 - \cos(2\pi \xi \cdot x)}{|x|^{d+2\gamma}} dx \right) |\widehat{u}(\xi)|^2 d\xi. \end{aligned}$$

To treat the inner integral, we note that it is rotationally invariant, and so, by rotating to the first axis and later changing variable  $x \rightarrow x/|\xi|$ , we get

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{1 - \cos(2\pi \xi \cdot x)}{|x|^{d+2\gamma}} dx &= \int_{\mathbb{R}^d} \frac{1 - \cos(2\pi |\xi| x_1)}{|x|^{d+2\gamma}} dx \\ &= |\xi|^{2\gamma} \int_{\mathbb{R}^d} \frac{1 - \cos(2\pi x_1)}{|x|^{d+2\gamma}} dx \sim |\xi|^{2\gamma}. \end{aligned}$$

The last integral is finite as, since  $d + 2\gamma > d$ , the tails  $|\xi| \rightarrow \infty$  are controlled, and since  $1 - \cos(2\pi x_1) \sim x_1^2 \leq |x|^2$  the integrand has order  $-d + 2(1 - \gamma) > -d$  for  $|\xi| \sim 0$ . That said, substituting this back into the previous expression gives the desired result.

**Exercise 5.** Use the previous exercise 4 to show that if  $\Omega$  is an open set with uniformly Lipschitz boundary, then

$$H^{s,2}(\Omega) = W^{s,2}(\Omega).$$

**Hint.** By definition 3.3 choose a sequence  $v_n \in H^{s,2}(\mathbb{R}^d)$  such that  $\|v_n\|_{H^{s,2}(\mathbb{R}^d)} \rightarrow \|u\|_{H^{s,2}(\Omega)}$  in  $H^s(\Omega)$ . Then,

$$\|u\|_{H^{s,2}(\Omega)} = \lim_{n \rightarrow \infty} \|v_n\|_{H^{s,2}(\mathbb{R}^d)} \sim \lim_{n \rightarrow \infty} \|v_n\|_{B^{s,2}(\mathbb{R}^d)} \geq \|u\|_{B^{s,2}(\Omega)}.$$

To obtain the reverse inequality, use the existence of a continuous extension operator  $E : W^{s,2}(\Omega) \rightarrow W^{s,2}(\mathbb{R}^d)$  (see [2] page 33) to obtain

$$\|u\|_{W^{s,2}(\Omega)} \sim \|Eu\|_{W^{s,2}(\mathbb{R}^d)} \geq \|u\|_{H^{s,2}(\mathbb{R}^d)}.$$

The above suggests that, for  $p = 2$ , the integrals appearing in the definition of the Slobodeckij spaces 3.1 correspond to differentiating a fractional amount of times. This indeed is the case

**Definition 4.3.** Given  $\gamma \in [0, +\infty)$  and  $u \in \mathcal{S}(\mathbb{R}^d)$  we define the fractional Laplacian as

$$(-\Delta)^\gamma u(x) := \mathcal{F}^{-1}(|2\pi \xi|^{2\gamma} \widehat{u}(\xi)).$$

**Proposition 4.4.** For  $\gamma \in (0, 1)$  and  $u \in H^s(\mathbb{R}^d)$  it holds that

$$(-\Delta)^\gamma u(x) = C \int_{\mathbb{R}^d} \frac{u(x) - u(x+y)}{|y|^{d+2\gamma}} dy,$$

where  $C$  is a constant that depends on  $d, \gamma$ .

*Proof.* The above equality may seem odd at first if we compare it with the integral in 3.1 where a square appears in the numerator, which gives us our 2 in the  $2\gamma$ . However, it is justified by the fact that, by the change of variables  $y \rightarrow -y$ ,

$$\int_{\mathbb{R}^d} \frac{u(x) - u(x+y)}{|y|^{d+2\gamma}} dy = \int_{\mathbb{R}^d} \frac{u(x) - u(x-y)}{|y|^{d+2\gamma}} dy.$$

So, we can get the *second* order difference in the numerator by adding the two integrals.

$$\int_{\mathbb{R}^d} \frac{u(y) - u(x+y)}{|y|^{d+2\gamma}} dy = -\frac{1}{2} \int_{\mathbb{R}^d} \frac{u(x+y) - 2u(x) + u(x-y)}{|y|^{d+2\gamma}} dy. \quad (4)$$

That said, we must show that

$$|\xi|^{2\gamma} \widehat{u}(\xi) \sim \mathcal{F} \left( \int_{\mathbb{R}^d} \frac{u(x) - u(x+y)}{|y|^{d+2\gamma}} dy \right)$$

Using (4) and proceeding as in exercise 4.1 gives

$$\begin{aligned} \mathcal{F} \left( \int_{\mathbb{R}^d} \frac{u(x) - u(x+y)}{|y|^{d+2\gamma}} dy \right) &= -\frac{1}{2} \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \frac{e^{-2\pi i y \cdot \xi} - 2 + e^{2\pi i y \cdot \xi}}{|y|^{d+2\gamma}} dy \right) \widehat{u}(\xi) d\xi \\ &= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \frac{1 - \cos(2\pi y \cdot \xi)}{|y|^{d+2\gamma}} dy \right) \widehat{u}(\xi) d\xi = \int_{\mathbb{R}^d} \frac{1 - \cos(2\pi y_1)}{|y|^{d+2\gamma}} dy \int_{\mathbb{R}^d} |\xi|^{2\gamma} \widehat{u}(\xi) d\xi \\ &\sim |\xi|^{2\gamma} \widehat{u}(\xi) d\xi. \end{aligned}$$

This completes the proof and shows that the explicit expression for  $C$  is

$$C = \frac{1}{(2\pi)^{2\gamma}} \int_{\mathbb{R}^d} \frac{1 - \cos(2\pi y_1)}{|y|^{d+2\gamma}} dy.$$

□

## 5 Dual of Sobolev spaces and correspondence with negative regularity

Negative orders of regularity correspond to the dual of Sobolev spaces. This is best seen in the integer case, where the following result holds (see [7] pages (326-344) for the case  $p = 2$ ).

**Theorem 5.1.** For all  $k \in \mathbb{Z}$  and  $p \in [1, \infty)$  it holds that

$$H_0^{k,p}(\Omega)' = H^{-k,p'}(\Omega), \quad W_0^{k,p}(\Omega)' = W^{-k,p'}(\Omega).$$

The first equality will be discussed in the next subsection and is most easily proven when  $\Omega = \mathbb{R}^d$ , in which case one can use the corresponding  $\Lambda^s : H^{r,p}(\mathbb{R}^d) \xrightarrow{\sim} H^{r-s,p}(\mathbb{R}^d)$  together with the reflexivity of  $L^p(\mathbb{R}^d)$ . The second equality is a direct consequence of the integer order equality  $W^{k,p}(\Omega) = H^{k,p}(\Omega)$  of Theorem 4.1. For fractional order regularities, we have the following result, which can be found in [1] page 228.

**Theorem 5.2.** *The spaces  $W^{s,p}(\Omega)$ ,  $H^{s,p}(\Omega)$ ,  $B^{s,p}(\Omega)$  are reflexive Banach spaces with duals*

$$W^{s,p}(\Omega)' = W_{\overline{\Omega}}^{-s,p'}(\mathbb{R}^d), \quad H^{s,p}(\Omega)' = H_{\overline{\Omega}}^{-s,p'}(\mathbb{R}^d), \quad B^{s,p}(\Omega)' = B_{\overline{\Omega}}^{-s,p'}(\mathbb{R}^d).$$

where  $p'$  is the conjugate exponent of  $p$  and given a space of distributions  $X$  on  $\mathbb{R}^d$  we define  $X_{\overline{\Omega}}$  as the space of distributions on  $\mathbb{R}^d$  which are supported in  $\overline{\Omega}$ . In particular, for  $\Omega = \mathbb{R}^d$ ,

$$W^{s,p}(\mathbb{R}^d)' = W^{-s,p'}(\mathbb{R}^d), \quad H^{s,p}(\mathbb{R}^d)' = H^{-s,p'}(\mathbb{R}^d), \quad B^{s,p}(\mathbb{R}^d)' = B^{-s,p'}(\mathbb{R}^d).$$

We have already seen that  $W^{s,p}(\Omega)$  and  $H^{s,p}(\Omega)$  are Banach spaces, one can similarly show that  $B^{s,p}(\Omega)$  is a Banach space. Furthermore, the spaces are all reflexive for smooth domains.

**Observation 2.** Some authors define given  $s > 0$

$$W^{-s,p'}(\Omega)' := W^{s,p}(\Omega)'. \quad (5)$$

See, for example, [5]. The definition in (5) is equivalent to our definition when  $\Omega = \mathbb{R}^d$  or when  $s \in k$ . However, in other cases, the two definitions are not equivalent.

### 5.1 The dual of $H^{s,p}(\mathbb{R}^d)$ and $B^{s,p}(\mathbb{R}^d)$

For some motivation, we start by considering the case  $\Omega = \mathbb{R}^d$ . In this case, since the closure of  $C_c^\infty(\mathbb{R}^d)$  in  $H^{s,p}(\mathbb{R}^d)$  (which by definition is  $H_0^{s,p}(\mathbb{R}^d)$ ) is itself  $H^{s,p}(\mathbb{R}^d)$ , we have that  $H_0^{s,p}(\mathbb{R}^d) = H^{s,p}(\mathbb{R}^d)$ .

**Exercise 6** (Dual identification). Prove the identification  $H^{-s,p'}(\mathbb{R}^d) = H^{s,p}(\mathbb{R}^d)'$ .

**Hint.** Consider the mapping  $H_0^{-s,p'}(\mathbb{R}^d) \rightarrow H_0^{s,p}(\mathbb{R}^d)'$  given by  $f \mapsto \ell_f$  where

$$\ell_f(u) := \int_{\mathbb{R}^d} (\Lambda^s u)(\Lambda^{-s} f).$$

Show that this mapping is well-defined and continuous. To see that it is invertible, show that, by duality, given  $\ell \in H^{s,p}(\mathbb{R}^d)'$  and  $u \in H^{s,p}(\mathbb{R}^d)$ , it holds that

$$(u, \ell) = (\Lambda^s u, \Lambda^{-s} \ell).$$

Since  $\Lambda^s u \in L^p(\mathbb{R}^d)$  we deduce that  $\Lambda^{-s} \ell \in L^p(\mathbb{R}^d)'$  and so by the Riesz representation theorem there exists  $f_\ell \in L^{p'}(\mathbb{R}^d)$  such that  $\Lambda^{-s} \ell = \langle \cdot, f_\ell \rangle$ . Show that the inverse of the previous mapping is

$$H^{s,p}(\mathbb{R}^d)' \longrightarrow H^{-s,p'}(\mathbb{R}^d); \quad \ell = \langle \cdot, \Lambda^s f_\ell \rangle \rightarrow \Lambda^s f_\ell.$$

**Exercise 7.** We also know that since  $H^s(\mathbb{R}^d)$  is a Hilbert space, so by the Riesz representation theorem, we have the identification  $H^s(\mathbb{R}^d) = H^s(\mathbb{R}^d)'$ . So by the previous exercise  $H^{-s}(\mathbb{R}^d) = H^s(\mathbb{R}^d)$  How is this possible?

**Hint.** It does **not** hold that  $H^{-s}(\mathbb{R}^d) = H^s(\mathbb{R}^d)$ . The problem occurs when considering too many identifications at once, as we are identifying duals using different inner products. By following the mappings, we obtain a bijective isomorphism

$$\begin{aligned} H^s(\mathbb{R}^d) &\rightarrow H^s(\mathbb{R}^d)' \rightarrow H^{-s}(\mathbb{R}^d) \\ u &\mapsto \langle \cdot, u \rangle_{H^s(\mathbb{R}^d)} = \langle \cdot, \Lambda^{2s} u \rangle \mapsto \Lambda^{2s} u. \end{aligned}$$

However, the isomorphism is  $\Delta^{2s}$ , which is hardly the identity mapping.

For another example where confusion with this kind of identification can arise, see remark 3 on page 136 of [8].



## 5.2 The dual of $H_0^{s,p}(\Omega)$

Given an extension domain  $\Omega$  and  $s \in \mathbb{R}$ , one can define extension and restriction operators,

$$E : H^{s,p}(\Omega) \rightarrow H^{s,p}(\mathbb{R}^d), \quad \rho : H^{s,p}(\mathbb{R}^d) \rightarrow H^{s,p}(\Omega),$$

which verify  $\rho \circ E = \mathbf{I}_{H^s(\Omega)}$ . As a result, the restriction is subjective, and we can factor  $H^{s,p}(\Omega)$  as

$$H^{s,p}(\Omega) \simeq H^{s,p}(\mathbb{R}^d) / H_{\Omega^c}^s(\mathbb{R}^d), \quad (6)$$

where given a closed set  $K \subset \mathbb{R}^d$  we define

$$H_K^{s,p}(\mathbb{R}^d) := \left\{ u \in H^{s,p}(\mathbb{R}^d) : \text{supp}(u) \subset K \right\},$$

where the support is to be understood in the sense of distributions. Now, given a Banach space  $X$  and a closed subspace  $Y \hookrightarrow X$ , elements of  $X'$  can be restricted to  $Y$ , obtaining functionals in  $Y'$ . The kernel of this restriction is  $Y^\circ := \{\ell \in X' : Y \subset \ker(\ell)\}$ . Since, by the Hahn Banach theorem, the restriction is surjective, we obtain the factorization

$$Y' \simeq X' / Y^\circ. \quad (7)$$

Applying this to  $Y = H_0^{k,p}(\Omega) \hookrightarrow H^{k,p}(\mathbb{R}^d) = X$  we obtain the result of Theorem 5.1.

$$H_0^{k,p}(\Omega)' \simeq H^{k,p}(\mathbb{R}^d)' / H_{\Omega^c}^{k,p}(\mathbb{R}^d)' \simeq H^{-k,p'}(\mathbb{R}^d) / H_{\Omega^c}^{-k,p'}(\mathbb{R}^d) \simeq H^{-k,p'}(\Omega),$$

where the second equality is by Exercise 6 and the third by (6). This shows that the dual of  $H_0^{k,p}(\Omega)$  is  $H^{-k,p'}(\Omega)$ . By also using the integer order equivalence of Theorem 4.1, we obtain Theorem 5.1.

As a final note, if our domain has a boundary,  $H_0^k(\Omega)'$  and  $H^k(\Omega)'$  are not equal. Rather,

$$H^{k,p}(\Omega)' \simeq H_{\Omega}^{-k,p'}(\mathbb{R}^d), \quad H^{-k,p'}(\Omega) \simeq H^{-k,p'}(\mathbb{R}^d) / H_{\Omega^c}^{-k,p'}(\mathbb{R}^d).$$

See [9] Section 4 for more details.

## 6 Representation theorems

We know that we can identify the spaces  $H^{s,p}(\mathbb{R}^d)$  and  $B^{s,p}(\mathbb{R}^d)$  with the lower order spaces  $H^{s-r,p}(\mathbb{R}^d)$  and  $B^{s-r,p}(\mathbb{R}^d)$  by application of  $\Lambda^r$ . That is, by differentiating  $r$  times. That is, spaces of lower-order regularity are obtained by differentiating functions with higher regularity. We show how to extend this idea to smooth domains in some particular cases.

**Theorem 6.1** (Representation of  $W_0^{k,p}(\Omega)'$ ). *Let  $\Omega \subset \mathbb{R}^d$  be open with uniformly Lipschitz boundary and let  $k \in \mathbb{N}$  and  $p \in [1, \infty)$ . Then, every element in  $W^{-k,p'}(\Omega) = W^{k,p}(\Omega)'$  is the unique extension of a distribution of the form*

$$\sum_{1 \leq |\alpha| \leq k} D^\alpha u_\alpha \in \mathcal{D}'(\Omega), \quad \text{where } u_\alpha \in L^{p'}(\Omega).$$

*Proof.* Define the mapping

$$\begin{aligned} T : W^{k,p}(\Omega) &\longrightarrow L^p(\Omega \rightarrow \mathbb{R}^n) \\ u &\longmapsto (D^\alpha u)_{1 \leq |\alpha| \leq k}. \end{aligned}$$

Where the notation says that we send  $u$  to the vector formed by all its derivatives. By our definition of the norm on  $W^{k,p}(\Omega)$ , we have that  $T$  is an isometry and, in particular, continuously invertible on its image. Denote the image of  $T$  by  $X := \text{Im}(T)$ . Given  $\ell \in W^{-k,p'}(\Omega)$  we define

$$\ell_0 : X \rightarrow \mathbb{R}, \quad \ell_0(\mathbf{w}) := \ell(T^{-1}\mathbf{w}), \quad \forall \mathbf{w} \in X.$$

By Hahn Banach's theorem, we can extend  $\ell_0$  from  $X$  to a functional  $\ell_1 \in L^p(\Omega \rightarrow \mathbb{R}^n)'$  and by the Riesz representation theorem, we have that there exists a unique  $\mathbf{f} = (f_\alpha)_{1 \leq |\alpha| \leq k} \in L^{p'}(\Omega \rightarrow \mathbb{R}^n)$  such that

$$\ell_1(\mathbf{w}) = \int_{\Omega} \mathbf{w} \cdot \mathbf{h}, \quad \forall \mathbf{w} \in L^p(\Omega \rightarrow \mathbb{R}^n).$$

By construction, it holds that, for all  $v \in W^{k,p}(\Omega)$

$$\ell(u) = \ell_0(Tv) = \int_{\Omega} Tv \cdot \mathbf{f} = \sum_{1 \leq |\alpha| \leq k} \int_{\Omega} f_\alpha D^\alpha v.$$

In particular, this holds for  $v \in \mathcal{D}(\Omega)$  and if we set  $u_\alpha := (-1)^\alpha h_\alpha$  we obtain that for all  $v \in \mathcal{D}(\Omega)$

$$\ell(v) = \left( v, \sum_{1 \leq |\alpha| \leq k} D^\alpha u_\alpha \right) =: \omega(v) \quad (8)$$

(we recall the notation  $(v, \omega)$  for the duality pairing). By definitions of the norm on  $W^{k,p}(\Omega)$  and Cauchy Schwartz, we have that  $\omega$  is continuous with respect to the norm on  $W^{k,p}(\Omega)$  and so we may extend it uniquely to the closure of  $\mathcal{D}(\Omega)$  in  $W^{k,p}(\Omega)$  which is  $W_0^{k,p}(\Omega)$ . By (8), the extension is necessarily  $\omega$ . This completes the proof.  $\square$

The above theorem shows that  $W^{-k,p'}(\Omega)$  can be equivalently formed by differentiating  $k$  times functions in  $L^{p'}(\Omega)$ . The proof also sheds some light as to why  $W^{-s,p'}(\Omega)$  is the dual of  $W_0^{k,p}(\Omega)$  and not the dual of  $W^{k,p}(\Omega)$ . The reason is that the elements of  $W_0^{k,p}(\Omega)$  are the ones that can be extended to distributions in  $\mathcal{D}'(\Omega)$  and so are the ones that we can integrate against. Finally, though the extension from  $\mathcal{D}'(\Omega)$  to  $W^{-s,p}(\Omega)$  is unique, the functions  $u_\alpha$  will not be, for example, if  $|\alpha| > 0$  it is possible to add a constant to  $u_\alpha$  and still obtain the same result.

**Exercise 8.** Show that for  $s > 0$  and  $p \in [1, \infty)$  every element in  $H^{-s,p'}(\Omega)$  can be written in the form  $w|_{\partial\Omega}$ , where

$$w = \sum_{0 \leq |\alpha| \leq k} \Lambda^\gamma D^\alpha u_\alpha \in \mathcal{D}'(\mathbb{R}^d), \quad \text{where } u_\alpha \in L^{p'}(\mathbb{R}^d),$$

where  $k = \lfloor s \rfloor$  and  $\gamma = s - k$ .

**Hint.** Use that  $\Lambda^\gamma : H^{s,p}(\mathbb{R}^d) \rightarrow H^{k,p}(\mathbb{R}^d)$  is an isomorphism and the just proved theorem 6.1 together with the integer equivalence in Theorem 4.1 to show that

$$H^{s,p}(\mathbb{R}^d)' = \left\{ \sum_{0 \leq |\alpha| \leq k} \Lambda^\gamma D^\alpha u_\alpha \in \mathcal{D}'(\mathbb{R}^d), \quad \text{where } u_\alpha \in L^{p'}(\mathbb{R}^d) \right\}.$$

Now conclude by the definition of  $H^{-s,p'}(\Omega)$  for open domains 3.4.

The above results extend to Besov spaces; see [1] page 227. This gives,

**Theorem 6.2.** Let  $k \in \mathbb{N}_0, \gamma \in [0, 1), \theta \in (0, 1)$  and  $p \in [1, \infty)$ . Then,

$$\begin{aligned} B^{\theta-k,p}(\Omega) &= \left\{ \sum_{0 \leq |\alpha| \leq k} D^\alpha u_\alpha \in \mathcal{D}'(\Omega), \quad \text{where } u_\alpha \in B^{\theta,p}(\Omega) \right\} \\ H^{\gamma-k,p}(\Omega) &= \left\{ \sum_{0 \leq |\alpha| \leq k} D^\alpha u_\alpha \in \mathcal{D}'(\Omega), \quad \text{where } u_\alpha \in H^{\gamma,p}(\Omega) \right\} \\ \text{setminus } W^{\gamma-k,p}(\Omega) &= \left\{ \sum_{0 \leq |\alpha| \leq k} D^\alpha u_\alpha \in \mathcal{D}'(\Omega), \quad \text{where } u_\alpha \in W^{\gamma,p}(\Omega) \right\}. \end{aligned}$$

## 6.1 Some applications

The following theorem can be found in [1] page 228 and [10] page 350 and serves as a generalization of the trace theorem for fractional Sobolev spaces.

**Theorem 6.3** (Fractional trace theorem). Let  $\Omega \subset \mathbb{R}^d$  be open with  $C^{0,1}$  boundary. Then, for all  $p \in (1, \infty), s \in (1/p, \infty)$ , the trace operator  $\text{Tr}$  can be extended from  $C(\overline{\Omega})$  to a bounded operator

$$\text{Tr} : H^{s,p}(\Omega) \rightarrow B^{s-1/p,p}(\partial\Omega), \quad \text{Tr} : B^{s,p}(\Omega) \rightarrow B^{s-1/p,p}(\partial\Omega).$$

In particular, as we have respective equalities for  $W^{s,p}(\Omega)$  with  $H^{s,p}(\Omega)$  and  $B^{s,p}(\Omega)$  for respectively non-integer  $s$ , we can also extend

$$\text{Tr} : W^{s,p}(\Omega) \rightarrow B^{s-1/p,p}(\partial\Omega).$$

In terms of extensions, the following completely characterizes the set of extension domains  $\Omega$  for  $W^{s,p}(\Omega)$ . See [6] page 313.

**Theorem 6.4** (Plump sets are extension domains). Let  $\Omega \subseteq \mathbb{R}^N$  be an open connected set,  $p \in [1, +\infty]$ , and  $\gamma \in (0, 1)$ . Then  $\Omega$  is an extension domain for  $W^{\gamma,p}(\Omega)$  if and only if there exists a constant  $C > 0$  such that

$$\lambda^d(B(x, r) \cap \Omega) \geq Cr^d$$

for all  $x \in \Omega$  and all  $0 < r \leq 1$ . Where  $\lambda^d$  is the Lebesgue measure on  $\mathbb{R}^d$ .

For higher orders of regularity, the following is sufficient: see [10] page 454 and [6] page 424.

**Theorem 6.5.** Let  $\Omega \subset \mathbb{R}^d$  be open with uniformly Lipschitz boundary and consider  $p \in [1, \infty), s \in [1, \infty)$ . Then,  $\Omega$  is an extension domain for  $W^{s,p}$ .

One can also obtain improved Sobolev embeddings for fractional Sobolev spaces. Here, the analogous of the exponent  $p_k^*$  is

**Definition 6.6.** Given  $p \in [1, \infty)$  and  $s \in (0, 1)$ , with  $sp < d$ , we define the Sobolev critical exponent  $p_s^*$  by

$$\frac{1}{p_s^*} := \frac{1}{p} - \frac{s}{d}.$$

The natural extension of the Sobolev embedding theorem to the fractional case is the following. See [10] page 262 for the following result.

**Theorem 6.7** (fractional Sobolev-Gagliardo-Nirenberg). *Given  $s \in (0, 1)$ ,  $p \in [1, \infty)$  and an extension domain  $\Omega$  for  $W^{s,p}$ , it holds that*

$$\|u\|_{L^{p_s^*}(\Omega)} \lesssim |u|_{W^{s,p}(\Omega)}, \quad \forall s < d/p$$

*In particular, by interpolation, for all  $q \in [p, p_s^*]$ .*

$$\|u\|_{L^q(\Omega)} \lesssim \|u\|_{W^{s,p}(\Omega)}, \quad \forall s < d/p.$$

The critical case  $s = d/p$  corresponding to  $p_s^* = \infty$  is now (see [10] .page 265)

**Theorem 6.8.** *Given  $s \in (0, 1)$ ,  $q \in [p, \infty)$  and an extension domain  $\Omega$  for  $W^{s,p}$ , it holds that*

$$\|u\|_{L^q(\mathbb{R}^d)} \lesssim \|u\|_{W^{s,p}(\Omega)}, \quad s = d/p.$$

**Exercise 9.** Using Theorem 6.7 prove Theorem 6.8

**Hint.** We are in the subcritical case for  $r < d/p = s$ . Extend  $u$  to  $\mathbb{R}^d$  to form  $\tilde{u}$ . Then, by Proposition 3.2, we have

$$\|u\|_{L^{p_r^*}(\Omega)} \leq \|u\|_{L^{p_r^*}(\mathbb{R}^d)} \lesssim \|u\|_{W^{r,p}(\mathbb{R}^d)} \leq \|u\|_{W^{s,p}(\mathbb{R}^d)} \lesssim \|u\|_{W^{s,p}(\Omega)}.$$

Conclude by finding  $r$  such that  $p_r^* = q$ .

The supercritical case  $s > d/p$  can be found in [1] page 224 and [10] page 275.

**Theorem 6.9** (Morrey's fractional embedding). *Let  $\Omega$  be an extension domain for  $W^{s,p}$ . Then, given  $k \in \mathbb{N}_0$ ,  $\gamma \in [0, 1)$  we have a continuous embedding*

$$W_p^s(\Omega) \hookrightarrow C^{k,\gamma}(\Omega), \quad \forall s > \frac{d}{p} + k + \gamma.$$

*This embedding also holds for  $s = d/p + k + \gamma$  provided that  $\gamma$  is non-integer.*

The extension to higher orders of regularity can be found in Section 11.4 of [10].

## References

- [1] M. S. Agranovich, Sobolev spaces, their generalizations and elliptic problems in smooth and Lipschitz domains, Springer, 2015.  
URL <https://link.springer.com/book/10.1007/978-3-319-14648-5>
- [2] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional sobolev spaces, Bulletin des sciences mathématiques 136 (5) (2012) 521–573.  
URL <https://www.sciencedirect.com/science/article/pii/S0007449711001254>
- [3] H. Triebel, Theory of Function Spaces II, Vol. 84, Birkhäuser, 1992.  
URL <https://link.springer.com/book/10.1007/978-3-0346-0419-2>
- [4] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton university press, 1970.
- [5] U. Biccari, M. Warma, E. Zuazua, Local regularity for fractional heat equations, Recent Advances in PDEs: Analysis, Numerics and Control: In Honor of Prof. Fernández-Cara's 60th Birthday (2018) 233–249.

- [6] G. Leoni, A first course in Sobolev spaces, American Mathematical Soc., 2017.  
URL [https://www.google.co.uk/books/edition/A\\_First\\_Course\\_in\\_Sobolev\\_Spaces/qoA8DwAAQBAJ?hl=en&gbpv=0](https://www.google.co.uk/books/edition/A_First_Course_in_Sobolev_Spaces/qoA8DwAAQBAJ?hl=en&gbpv=0)
- [7] L. C. Evans, Partial differential equations, Vol. 19, American Mathematical Society, 2022.  
URL <https://math24.files.wordpress.com/2013/02/partial-differential-equations-by-evans.pdf>
- [8] H. Brezis, H. Brézis, Functional analysis, Sobolev spaces and partial differential equations, Vol. 2, Springer, 2011.
- [9] M. Taylor, Partial differential equations II: Qualitative studies of linear equations, Vol. 116, Springer Science & Business Media, 2013.
- [10] G. Leoni, A first course in fractional Sobolev spaces, Vol. 229, American Mathematical Society, 2023.  
URL [https://www.google.co.uk/books/edition/A\\_First\\_Course\\_in\\_Fractional\\_Sobolev\\_Spa/lh2\\_EAAAQBAJ?hl=en&gbpv=1&dq=giovanni+leoni+fractional+sobolev&pg=PP1&printsec=frontcover](https://www.google.co.uk/books/edition/A_First_Course_in_Fractional_Sobolev_Spa/lh2_EAAAQBAJ?hl=en&gbpv=1&dq=giovanni+leoni+fractional+sobolev&pg=PP1&printsec=frontcover)