SPDE Notes based on Lototsky

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1 Notation

Stochastic basis, i.e. filtered probability space with the usual conditions $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{F}_t$ and where we take as our index set some interval $I \subset \mathbb{R}_+$.

2 Main points

- White noise takes $f \in L^2(\mathbb{R}^d)$ as an input and returns the stochastic integral of the function (that is, a normal random variable). This is called a generalized function.
- This works in the same way both for time and space time deterministic functions.
- A White noise gives a Gaussian noise and vice versa.

3 Introduction

On such a space there exists a countable family of independent Gaussian processes G_k as a result we can construct explicitly a Brownian motion as

$$W(t) = \sum_{n=0}^{\infty} \left(\int_0^t e_n(s) ds \right) G_n. \tag{1}$$

Where e_n is orthonormal basis of $L^2(I)$. The fact that this is a Wiener process follows from the fact that the limit of Gaussian processes (in distribution and thus also in $L^2(\Omega)$) is Gaussian, that by Parseval's identity

$$\mathbb{E}[W(t)W(s)] = \left\langle 1_{[0,t]}, 1_{[0,s]} \right\rangle_{L^2(\mathbb{R}_+)} = t \wedge s.$$

And the independence also follows by Parseval. This construction can be extended to $\mathbb{R}^d$ (where d by contain both time and space) by considering e_n to be orthonormal basis in $L^2(\mathbb{R}^d)$ and setting for $u \in \mathbb{R}^d$

$$W(u) = \sum_{n=0}^{\infty} \left(\int_{[0,u]} e_n(v)dv \right) G_n.$$

The above is often called the *Brownian sheet*. A term by term differentiation in (1) suggests that we should have

$$\dot{W}(t) = \sum_{n=0}^{\infty} e_n(t) G_n.$$

However, this is problematic as, even if pointwise evaluation of f_n is well defined, the series may well diverge. For this reason it will be necessary to view $\dot{W}$ as a *generalized process*.

Definition 1. *Let $A \in \mathcal{B}(\mathbb{R}^d)$ we define* white noise $\dot{W}$ *on $L^2(A)$ as*

$$\dot{W}(f) := \sum_{n=0}^{\infty} \langle f, e_n \rangle_{L^2(A)} G_n; \quad f \in L^2(A)$$

We note that $\dot{W}(f) \sim \mathcal{N}\Big(0, \|f\|_{L^2(A)}\Big)$. To motivate this definition we note that if we take $G_n = \int_{\mathbb{R}} e_n(t) dW(t)$ (they're Gaussian because they're the integral of a deterministic function which is defined as a L^2 limit of Gaussian variables) and define $f_n = \langle f, e_n \rangle_{L^2(A)}$ then

$$\int_A f(u)dW(u) = \sum_{n=0}^{\infty} f_n \int_A e_n(u)dW(u) = \sum_{n=0}^{\infty} f_n G_n = \dot{W}(f).$$

Observation 1. *The relationship between white noise and the previously defined Gaussian noise dZ is that of $\dot{W}$ is a white noise on $\mathbb{R}^d$ then*

$$\dot{W}(B) := \dot{W}(1_B); \quad dZ(f) := \int_{\mathbb{R}^d} f dZ$$

This gives validity to us saying that Gaussian noise is just white noise and justifies the notation

$$\int_{\mathbb{R}^d} f(u)dW(u) := \dot{W}(f).$$

In particular we have the familiar properties

$$\left\langle \dot{W}(f), \dot{W}(g) \right\rangle_{L^2(\Omega)} = \int_A f(u)g(u)du.$$

Observation 2. *The above construction already includes time. Given a sequence of independent Wiener processes W_n one may consider time explicitly by setting*

$$W(t,x) := \sum_{n=0}^{\infty} \left(\int_{[0,x]} e_n(y)dy \right) W_n(s); \quad \dot{W}(t,x) := \sum_{n=0}^{\infty} \int_{\mathbb{R}_+} f_n(s)dW_n(s).$$

Where $f_n(s) := \langle f(s,\cdot), e_n(\cdot) \rangle_{L^2(\mathbb{R}^d)}$. Note that this is equivalent to the previous definition in the case $W_n(t) = \sum_{k=0}^{\infty} \left(\int_0^t v_k(s)ds \right) G_{k,n}$ for some orthonormal basis $v_k \in L^2(I)$ and independent Gaussian variables $G_{k,n}$, as the product of orthonormal basis on $L^2(A)$ and $L^2(I)$ is an orthonormal basis on $L^2(I \times A)$.

4 Reproducing Kernel Hilbert Space

Definition from [**?**] page 105.

Definition 2. *A generalized random field over a topological vector space V is a collection of random variables $\{X(u)\}_{u \in V}$ such that the following hold*

1. Linearity*: $X(u + av) = X(u) + aX(v)$ for all $u, v \in V$ and $a \in \mathbb{R}$.*
2. Continuity*: If $u_n \to u \in V$ then also $X(u_n) \xrightarrow{\mathbb{P}} X(u)$.*

References

Equivalent conditions for trace class operators are:

- $\sqrt{|T|}$ is a Hilbert Schmidt operator.

- For some orthonormal basis $\sum_{n=0}^{\infty} |\langle Te_n, e_n \rangle| <$
  $infty$.
- T is compact and $\sum_{n=0}^{\infty} \lambda_n < \infty$ where λ_n are the eigenvalues of T.
- There exists $\lambda_n \in \mathbb{R}, \ell_n \in H^*$ and $y_n \in H$ bounded sequences such that

$$T(x) = \sum_{n=0}^{\infty} \lambda_n \ell_n(x) y_n \quad \forall x \in H.$$