# Predator Prey Systems

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## 1 Introduction

In the early 20th century, Vito Volterra, an Italian mathematician and physicist, developed equations to predict changing marine life populations. Around the same time, an American Mathematician and Chemist, Alfred Lotka, created strikingly similar equations to represent concentration changes in chemical reactions, completely independent of Volterra's work. As such, this model was dubbed the "Lotka-Volterra Equations".

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## 2 Exponential Growth

Lotka-Volterra Interactions:

$$\frac{dx}{dt} = \alpha x - \beta xy$$
$$\frac{dy}{dt} = -\gamma y + \delta xy$$

### 2.1 Physical Realization of the Model

- $\alpha, \beta, \gamma, \delta$ : positive real constants
- $\alpha x$ : Exponential increase of prey population through reproduction.
- $-\beta xy$ : Decrease in prey population proportional to the number of interactions between predator and prey.
- $-\gamma y$ : Exponential decrease of predator population through natural causes as it is assumed that predator cannot kill predators.
- $\delta xy$ : Increase in predator population proportional to the number of interactions between predator and prey.

### 2.2 Assumptions

This model requires a few simplifying assumptions that may not be the most "realistic", but still give a good indicator of the dynamics of each population.

- $\beta \neq \delta$ : the rate at which the predator population grows is **not necessarily** equal to the rate at which it kills prey.
- $\alpha >> \gamma$ : The rate at which the prey reproduce is significantly higher than that of predators.
- The environment does not affect the interaction parameters, the relationship between predators and prey is invariant.
- In the absence of predators, the prey population will grow without bound as they have an infinite food supply.

## 2.3 Steady State Analysis

To find our steady states, we need to find the points where $\frac{dx}{dt} = \frac{dy}{dt} = 0$. As such,

$$\frac{dx}{dt} = \alpha x - \beta xy = 0$$

$$\alpha x = \beta xy$$

$$\boxed{y = \frac{\alpha}{\beta}}$$

$$\frac{dy}{dt} = -\gamma y + \delta xy = 0$$

$$\gamma y = \delta xy$$

$$\boxed{x = \frac{\gamma}{\delta}}$$

Thus, a steady state is located at

$$(x, y) = (\frac{\gamma}{\delta}, \frac{\alpha}{\beta})$$

Additionally, there is the trivial case when

$$(x, y) = (0, 0)$$

As there are 0 predators and 0 prey at this point, this is where the species have reached extinction.

To determine the stability of our steady states we will take the Jacobian of our model and analyze the eigenvalues.

$$\mathbf{A^J}_{(x,y)} = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{bmatrix} = \begin{bmatrix} \alpha - \beta y & \text{-}\beta x \\ \delta y & \text{-}\gamma + \delta x \end{bmatrix}$$

First steady state:

$$\mathbf{A^J}_{(0,0)} = \begin{bmatrix} \alpha & 0 \\ 0 & \text{-}\gamma \end{bmatrix}$$

$$(\alpha - \lambda_1)(-\gamma - \lambda_2) = 0$$

$$\boxed{\lambda_1 = \alpha} \boxed{\lambda_2 = -\gamma}$$

Since the Eigenvalues of the Jacobian are real and of opposite sign, we get a saddle point, which is an unstable equilibrium point. This means our populations of both predator and prey can approach infinitesimally close to 0 but will not collapse, and the regenerative cycle we see in the model will persist.

Second Steady State:

$$\mathbf{A^J}_{(\frac{\gamma}{\delta}, \frac{\alpha}{\beta})} = \begin{bmatrix} 0 & \frac{-\beta\gamma}{\delta} \\ \frac{\delta\alpha}{\beta} & 0 \end{bmatrix}$$

$$\lambda^2 + (\frac{\delta\alpha}{\beta} * \frac{\beta\gamma}{\delta}) = 0$$

$$\lambda^2 + \alpha\gamma = 0$$

$$\lambda = \pm\sqrt{-\alpha c}$$

$$\boxed{\lambda_1 = 0 + i\sqrt{\alpha\gamma}} \boxed{\lambda_2 = 0 - i\sqrt{\alpha\gamma}}$$

Since $Re(\lambda_i) = 0$, Taylor series approximation is not capable of determining stability of this particular steady state. However, it is seen that the eigenvalues are complex conjugates. Equilibrium points with this property are known as "Centers". The trajectories form closed orbits and tend neither towards nor away from the equilibrium point. We could shift the coordinates such that our equilibrium point $(x, y) = (\frac{\gamma}{\delta}, \frac{\alpha}{\beta})$, is at the new origin. $(x*, y*) = (x - \frac{\gamma}{\delta}, y - \frac{\alpha}{\beta}) = (0, 0)$. As such, we can prove that this point is stable in the sense of Lyapunov.

However, it is sufficient to observe the graph in Section 3.3, noting at each nullcline - horizontal and vertical - there are only 2 solutions: the minimum and maximum of both predator and prey populations. This also shows that this equilibrium point is stable.

This Stability Analysis shows that, under the exponential growth model, this system will maintain regular, cyclic interactions between predator and prey populations unless at least one of the populations is completely eradicated.

Additionally, we can notice the non-trivial steady state solution is composed of only the interaction parameters. Thus the steady state is independent of the initial predator and prey population sizes.

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# 3 Logistic Growth

Modified Lotka-Volterra Interactions:

$$\frac{dx}{dt} = \frac{\alpha x(K - x)}{K} - \beta xy$$

$$\frac{dy}{dt} = -\gamma y + \delta xy$$

## 3.1 Physical Realization of the Model

- Similar to Exponential Model
  - $\alpha, \beta, \gamma, \delta$ : positive real constants
  - $-\beta xy$ : Decrease in prey population proportional to the number of interactions between predator and prey.
  - $-\gamma y$ : Exponential decrease of predator population through natural causes as it is assumed that predator cannot kill predators.
  - $\delta xy$ : Increase in predator population proportional to the number of interactions between predator and prey.
- $\frac{\alpha x(K-x)}{K}$ : Increase of prey population through reproduction is no longer unbounded and is thereby limited by some carrying capacity, K.

## 3.2 Steady State Analysis

Likewise with our Exponential Model, we need to find the points where $\frac{dx}{dt} = \frac{dy}{dt} = 0$.

$$\frac{dx}{dt} = \frac{\alpha x(K - x)}{K} - \beta xy = 0$$

$$\frac{\alpha x(K - x)}{K} = \beta xy$$

$$\boxed{y = \frac{\alpha}{\beta K}(K - x)}$$

$$\frac{dy}{dt} = -\gamma y + \delta xy = 0$$

$$\gamma y = \delta xy$$

$$\boxed{x = \frac{\gamma}{\delta}}$$

Thus, a steady state is located at

$$(x, y) = (\frac{\gamma}{\delta}, \frac{\alpha}{\beta K}(K - x)) = (\frac{\gamma}{\delta}, \frac{\alpha}{\beta K}(K - \frac{\gamma}{\delta}))$$

Again, there is also the trivial case of

$$(x, y) = (0, 0)$$

where the species have reached extinction.

Stability of our steady states.

$$\mathbf{A^J}_{(x,y)} = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\alpha(K-2x)}{K} - \beta y & -\beta x \\ \delta y & -\gamma + \delta x \end{bmatrix}$$

First steady state:

$$\mathbf{A^J}_{(0,0)} = \begin{bmatrix} \alpha & 0 \\ 0 & -\gamma \end{bmatrix}$$

$$(\alpha - \lambda_1)(-\gamma - \lambda_2) = 0$$

$$\boxed{\lambda_1 = \alpha} \boxed{\lambda_2 = -\gamma}$$

Identical to the Exponential Growth Model, the Eigenvalues of the Jacobian are real and of opposite sign, and thus we get a saddle point - an unstable equilibrium point.

Second Steady State:

$$\mathbf{A^J}_{(\frac{\gamma}{\delta}, \frac{\alpha}{\beta K}(K - \frac{\gamma}{\delta}))} = \begin{bmatrix} \frac{\alpha(K-2(\frac{\gamma}{\delta}))}{K} - \beta(\frac{\alpha}{\beta K}(K - \frac{\gamma}{\delta})) & \frac{-\beta\gamma}{\delta} \\ \delta(\frac{\alpha}{\beta K}(K - \frac{\gamma}{\delta})) & 0 \end{bmatrix}$$

This row reduces to

$$= \begin{bmatrix} \delta(\frac{\alpha}{\beta K}(K - \frac{\gamma}{\delta})) & 0 \\ 0 & \frac{-\beta\gamma}{\delta} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\delta\alpha}{\beta}(1 - \frac{\gamma}{\delta K}) & 0 \\ 0 & \frac{-\beta\gamma}{\delta} \end{bmatrix}$$

Getting the Eigenvalues:

$$(\frac{\delta\alpha}{\beta}(1 - \frac{\gamma}{\delta K})) - \lambda_1) * (\frac{\beta\gamma}{\delta} - \lambda_2) = 0$$

$$\boxed{\lambda_1 = \frac{\delta\alpha}{\beta}(1 - \frac{\gamma}{\delta K})} \boxed{\lambda_2 = \frac{\beta\gamma}{\delta}}$$

Note: $\alpha, \beta, \gamma, \delta, K$ are all positive constants.

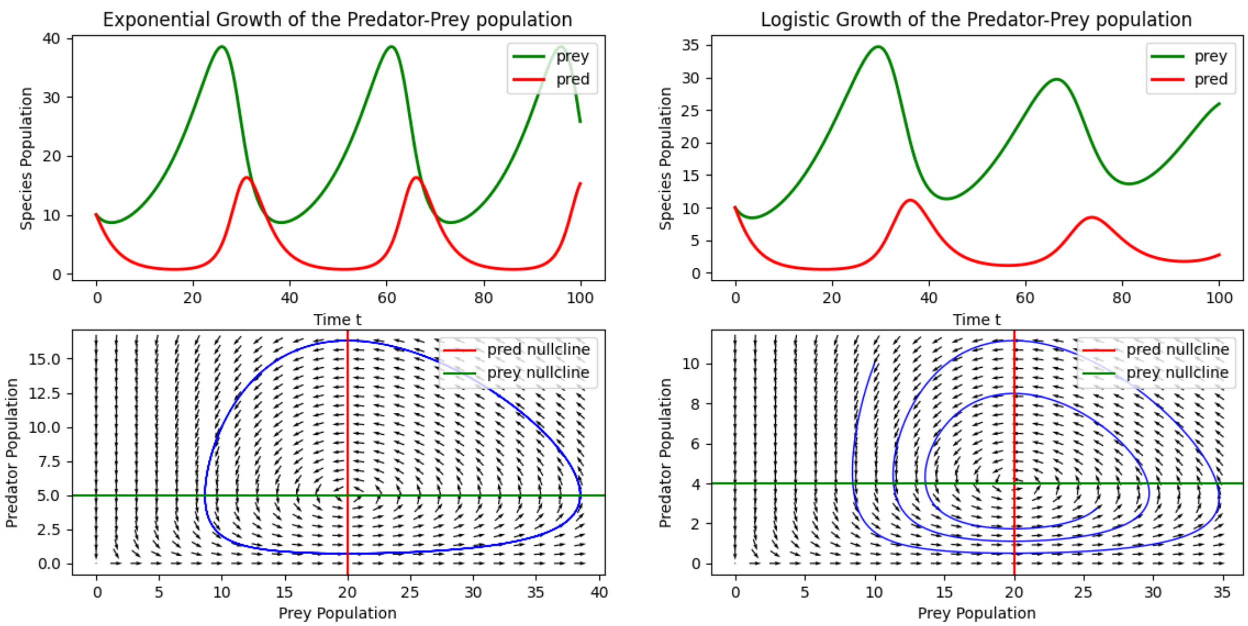
This immediately shows that $\lambda_2$ is always positive. Also, it is assumed that the capacity K is much larger than parameters $\alpha, \beta, \gamma$, and $\delta$. It holds that $1 - \frac{c}{dK}$ is positive and thus so is $\lambda_1$.

As at least one of the eigenvalues are greater than 0, this steady state is asymptotically unstable.

## 3.3 Plotting

Parameters:

- $\alpha = 0.1, \beta = \delta = 0.02, \gamma = 0.4$, Predator initial: 10, Prey initial: 10, K = 100
- This scenario shows when $\beta = \delta$. Thus for each interaction between predator and prey, there is an equal exchange; prey is killed, predator is born.

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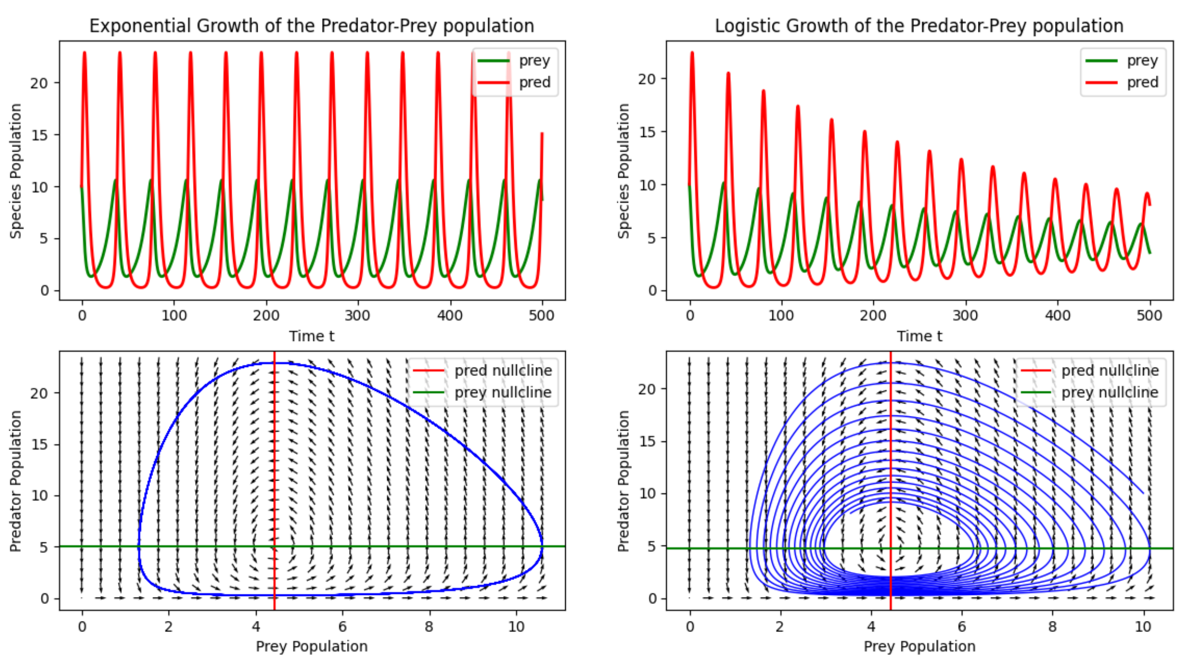
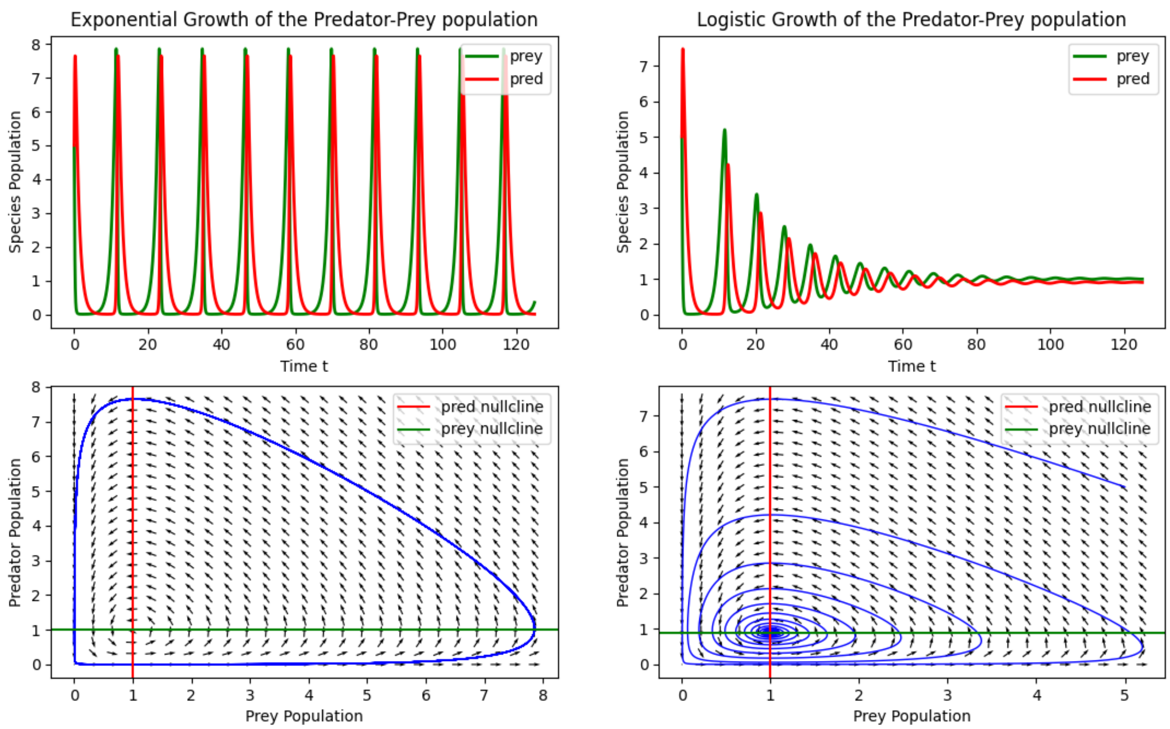
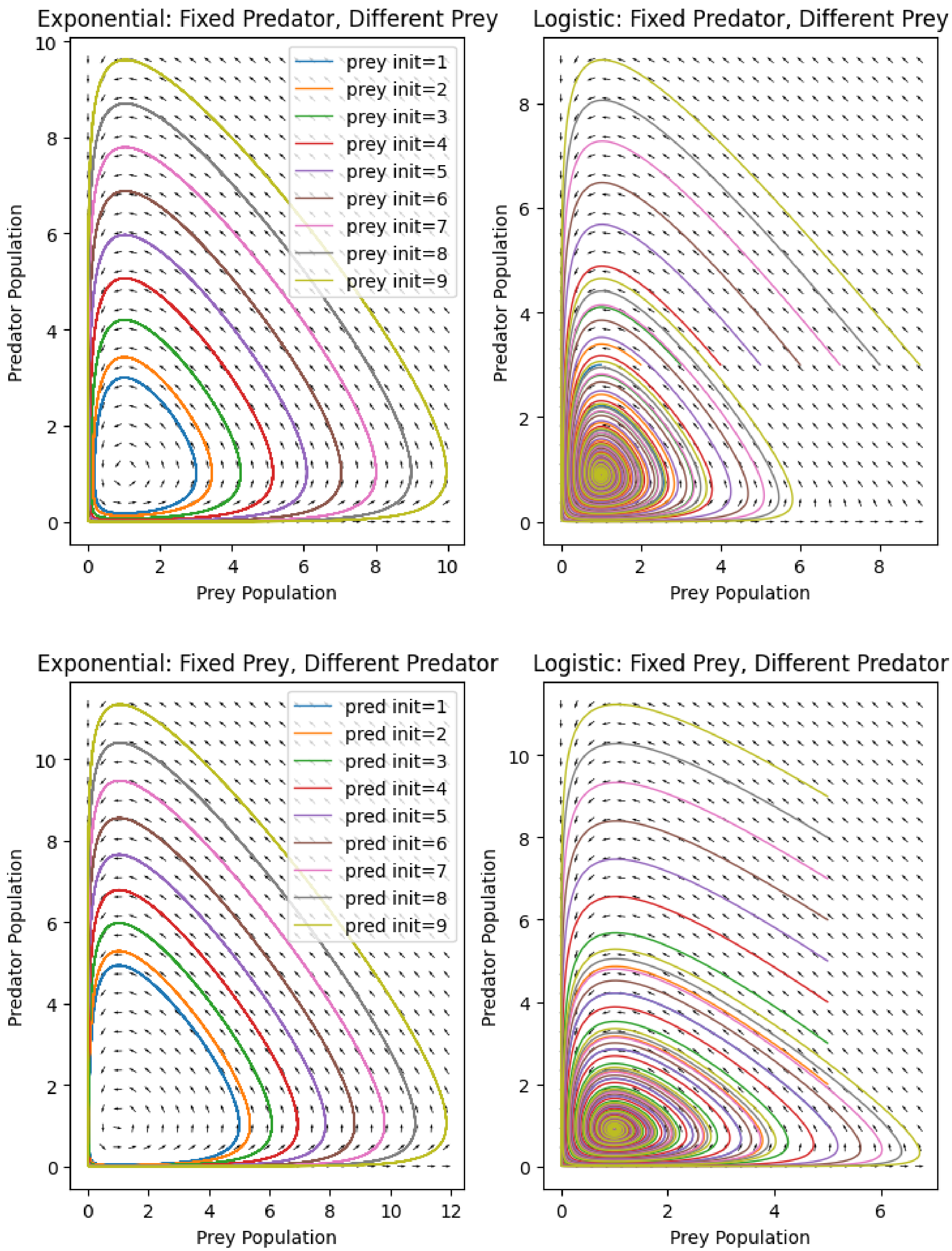
- $\alpha = 0.1, \beta = 0.02, \delta = 0.09, \gamma = 0.4$, Predator initial: 8, Prey initial: 10, K = 100
- This scenario depicts a more resourceful predator; there is a higher rate of predator reproduction for each iteration between the species.



- $\alpha = \beta = \delta = \gamma = 1$, Predator initial: 5, Prey initial: 5, K = 10
- This run shows the typical behavior of the model without much interference from the interacting parameters. All conflicts between species are 1-to-1 trade-offs, and all populations grow and diminish at the same rate. This is to show the cyclic and tapering nature of the exponential and logistic models, respectively.




Exponential: Fixed Predator, Different Prey
Logistic: Fixed Predator, Different Prey
Exponential: Fixed Prey, Different Predator
Logistic: Fixed Prey, Different Predator
prey init=1
prey init=2
prey init=3
prey init=4
prey init=5
prey init=6
prey init=7
prey init=8
prey init=9
pred init=1
pred init=2
pred init=3
pred init=4
pred init=5
pred init=6
pred init=7
pred init=8
pred init=9
Predator Population
Prey Population

# 4 Conclusion

The Lotka-Volterra equations is a revolutionary model in ecology and population dynamics. Having even the slightest perturbation in the parameters can lead to wildly different changes to the system. While many assumptions have to be made to make the model more reasonable on the computation side, these equations are strikingly accurate to nature. Additionally, much research has been done on this model and different terms and parameters are added and changed in the model - This system has a very modular nature to it and one can simplify or complexify as much as needed.

# 5 References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- Berryman, A. (1992) "The Origins and Evolution of Predator-Prey Theory"
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