Chaos in Seemingly Simple Optical Systems

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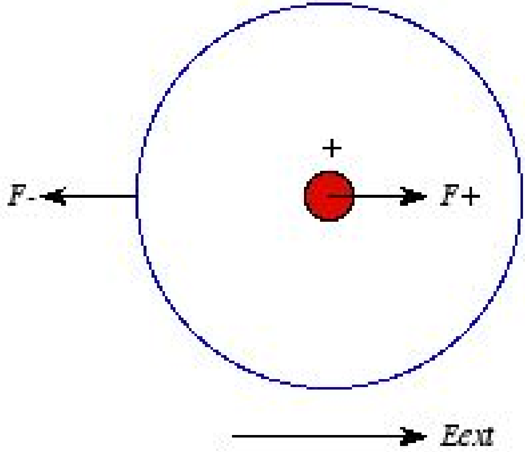
A Brief Intro to Nonlinear Optics

The Linear Polarization

- In linear optics: $\mathbf{P} \propto \mathbf{E}$
- Refractive index n a constant of a material for a given wavelength
- Most general, $\mathbf{P} = \epsilon_0 \chi^{(1)}(\mathbf{E})$, or $P^i = \epsilon_0 \chi_{ij} E^j$



The Nonlinear Polarization

- Nonlinear optics idea: Expand **P** in iterated products of **E**

$$\begin{aligned} \mathbf{P} &= \epsilon_0[\chi^{(1)}(\mathbf{E}) + \chi^{(2)}(\mathbf{E},\mathbf{E}) + \chi^{(3)}(\mathbf{E},\mathbf{E},\mathbf{E}) + \cdots] \\ &= \mathbf{P}^{(1)} + \mathbf{P}^{(2)} + \mathbf{P}^{(3)} + \cdots \end{aligned}$$

$$\text{or } P^i = \epsilon_0[\chi^{(1)}_{ij}E^j + \chi^{(2)}_{ijk}E^jE^k + \chi^{(3)}_{ijkl}E^jE^kE^l + \cdots]$$

- where $P^{(n)} \equiv \epsilon_0\chi^{(n)}(\mathbf{E},\mathbf{E},\cdots)$
- (assumes instantaneous response of medium to electric field)

Intensity-Dependent Refractive Index

Lets consider some third-order effects. For simplicity, we'll only work with magnitudes and say that $P^{(3)}(t) = \epsilon_0 \chi^{(3)} E^3(t) \propto E^3(t)$.

- Picking a monochromatic wave, $E(t) = \mathcal{E} \cos \omega t$
- Then, using the identity $\cos^3 \omega t = \frac{1}{4}\cos(3\omega t) + \frac{3}{4}\cos \omega t$:

$$P^{(3)}(t) = \frac{1}{4}\epsilon_0 \chi^{(3)} \mathcal{E}^3 \cos 3\omega t + \frac{3}{4}\epsilon_0 \chi^{(3)} \mathcal{E}^3 \cos \omega t$$

- Since $n(\omega) = \sqrt{1 + \chi_{eff}(\omega)}$ and $I(\omega) \propto |E(\omega)|^2$, this effects the index of refraction: $n(\omega) = n_0 + n_2 I(\omega)$ (See Boyd §6.2, 2008).

A Simple System

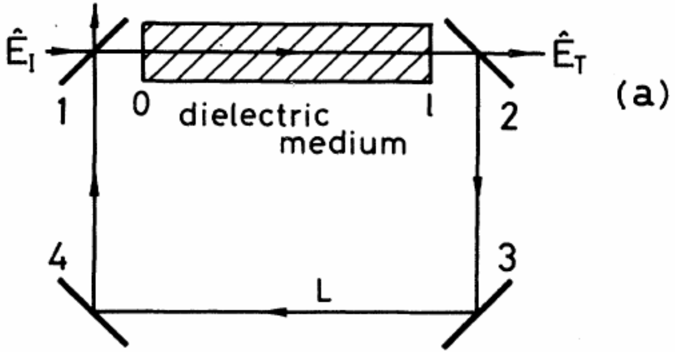


Figure: [3]

A Simple System (intuition, guesses)

- Linear case: constant phase shift $\phi_0 = \frac{2\pi L}{\lambda} n_0$
- Nonlinear case: variable phase shift $\phi(\omega) = \frac{2\pi L}{\lambda} n(\omega)$
- Third-order effects: $n(\omega) = n_0 + n_2 I(\omega)$
- Equation of motion guess: $\dot{E}(t) \propto E(t) \exp(i\phi)$

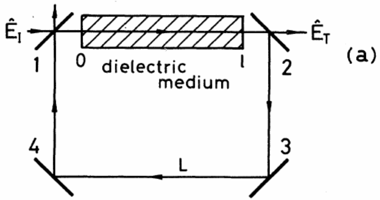


Figure: [3]

Analysis

The Driving Equations (complex amplitude)

$$\dot{E}(t) = A + BE(t - t_R)\exp\{i[\phi(t) - \phi_0]\}, \qquad (1)$$

$$\gamma^{-1}\dot{\phi}(t) = -\phi(t) + \mathrm{sgn}(n_2)|E(t - t_R)|^2. \qquad (2)$$

[3]

- $E(t) \equiv$ dimensionless (complex) field amplitude at top-left corner of ring
- $\phi(t) \equiv$ time-dependent phase shift of the electric field
- $A \propto \sqrt{(2\pi/\lambda)n_2}E_{in}$, $B \propto \sqrt{(2\pi/\lambda)n_2}\hat{E}(t, 0)$ a dissipation parameter
- $t_R \equiv$ time delay due to propagation of light through cavity
- $\phi_0 \equiv (2\pi/\lambda)n_0$

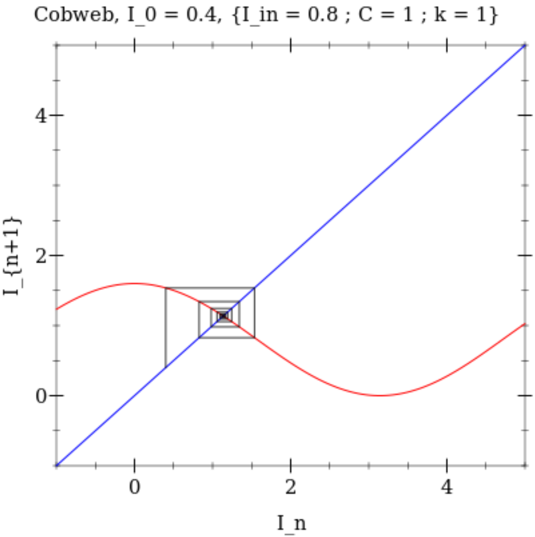
The Driving Equations (intensity map)

$$I_{n+1} = I_{in}\left(1 + C\cos\left(\frac{2\pi L}{\lambda}(n_0 + n_2 I_n)\right)\right) \tag{3}$$

[2]

- $I_n \equiv$ Intensity of electric field at start of n^{th} round trip
- $I_{in} \equiv$ Intensity of input electric field
- $n_0 \equiv$ linear index of refraction, $n_2 \equiv$ second-order index of refraction
- $L \equiv$ length of cavity, $\lambda \equiv$ wavelength of monochromatic input light

Non-Chaotic Regime



Figure

The Road to Chaos

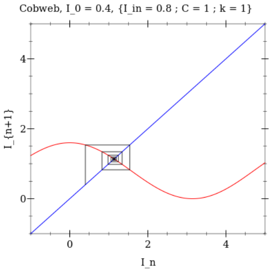


Figure: $\lambda \approx -0.3205$, $f'(x_i) \approx 0.72575$
$\lambda \approx -0.10966$, $f'(x_i) \approx 0.64491$

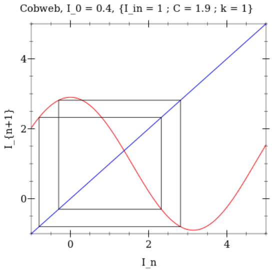
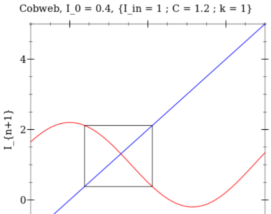
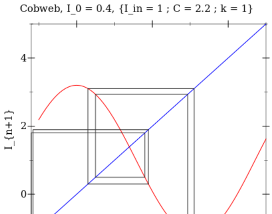


Figure: $\lambda \approx -0.39309$, $f'(x_i) \approx 0.45558$
$\lambda \approx -0.08662$, $f'(x_i) \approx 0.50001$





Chaos

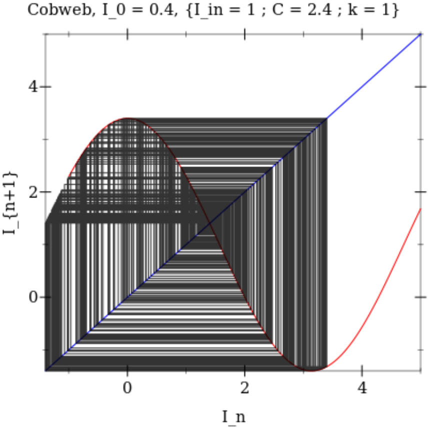
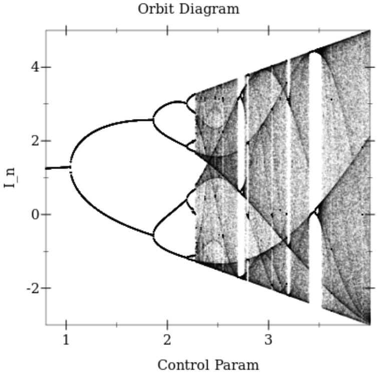
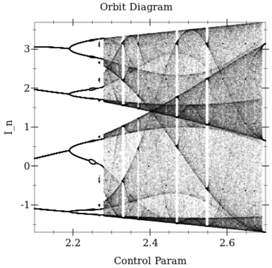
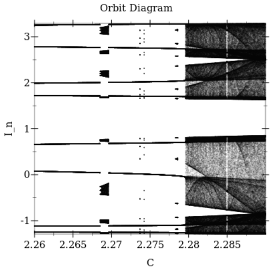


Figure: $\lambda \approx 0.3937816$, $f'(x_i) \approx$ really big.. (10^8)

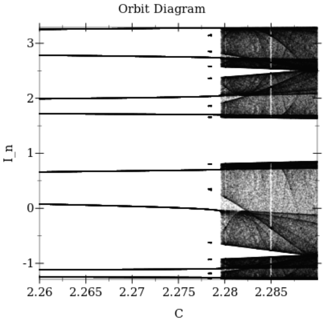
Orbit Diagram

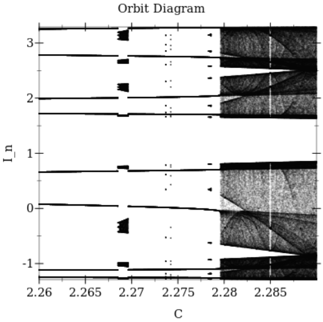




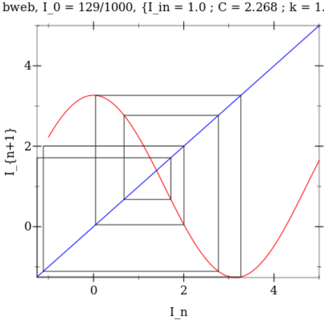


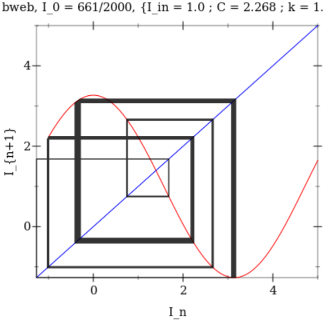
Two Different Orbit Diagrams





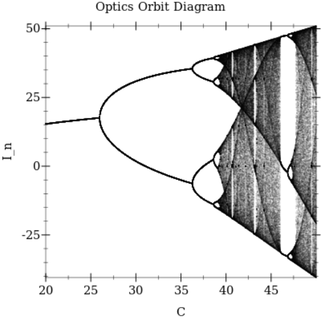
Two Different Stable Orbits

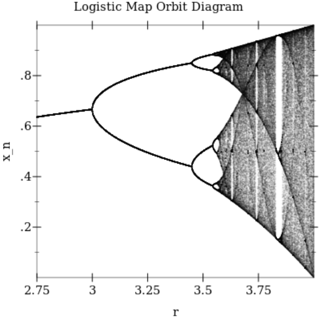




Logistic Map Orbit Comparison

When n_2 is small, the above map reduces to $I_{n+1} \approx I_{in}(1 + C(1 - \frac{1}{2}x^2))$, a quadratic form reminiscent of the logistic map $x_{n+1} = rx_n(1 - x_n)$:





Periodic Harmony in the Chaos

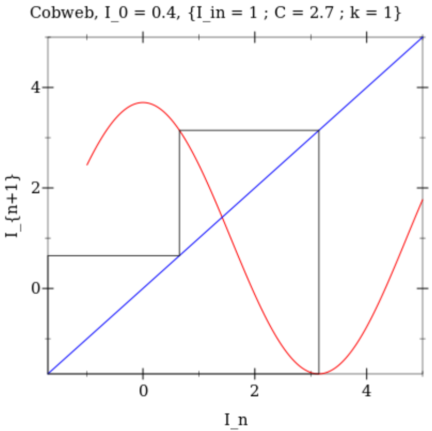


Figure: $\lambda \approx -0.9956$, $f'(x_i) \approx 0.05$

Spooky Ghosts

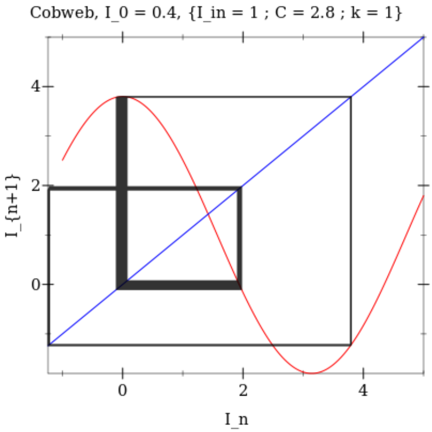
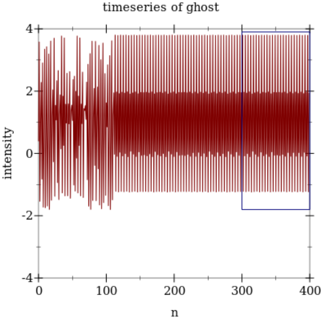
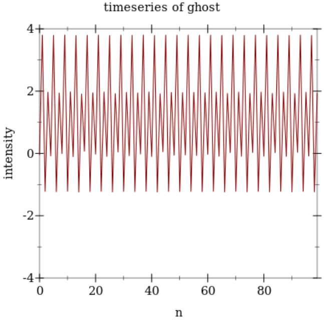


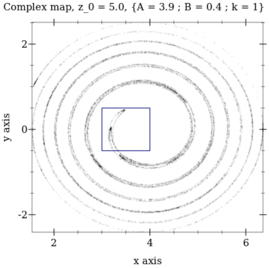
Figure: $\lambda \approx -0.9997$, $f'(x_i) \approx 0$

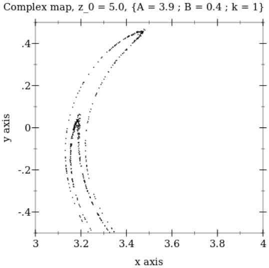
Spooky Ghosts





The Strange Attractor





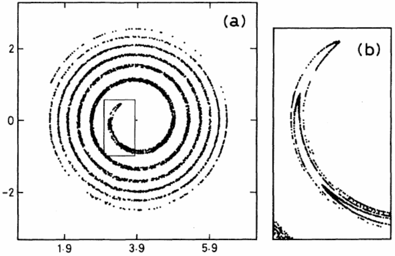
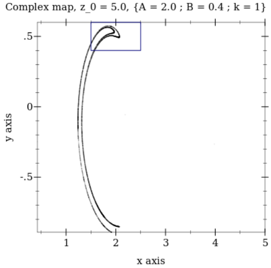
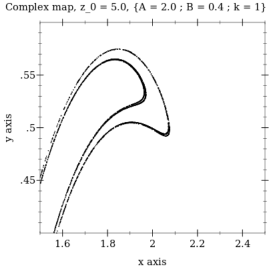
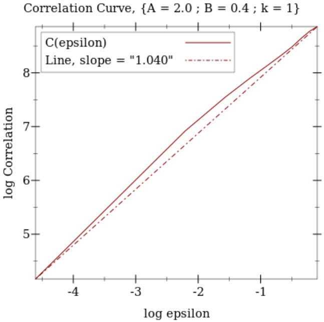


FIG. 2. (a) Plot of 5000 successive points of a series E_n on the complex E plane. The parameter values chosen are $B = 0.4$, $A = 3.9$, and $\varphi_0 = 0$. (b) Enlargement of the rectangular region of (a).

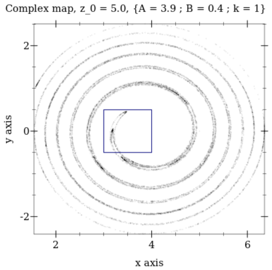
Fractal Dimension

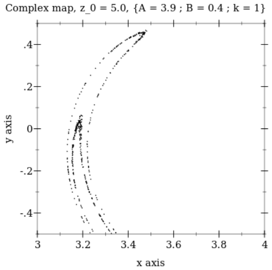


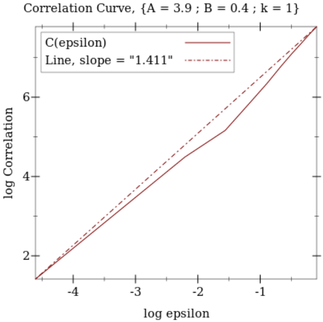




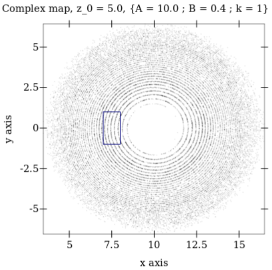
Fractal Dimension

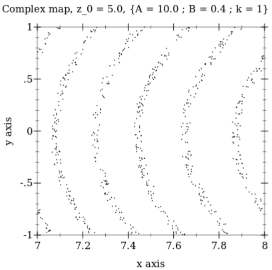


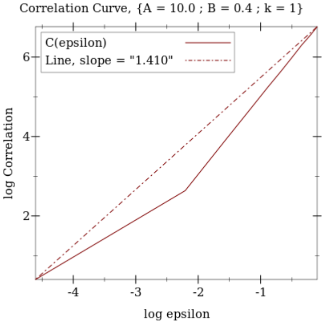




Fractal Dimension







References

References

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Polarization image.
http://teacher.pas.rochester.edu/PHY217/LectureNotes/Chapter4/LectureNotesChapter4.html.
Accessed: 2022-05-02.

Robert G. Harrison and Dhruba J. Biswas.
Chaos in light.
Nature, 321(6068):394–401, May 1986.
Number: 6068 Publisher: Nature Publishing Group.

K. Ikeda, H. Daido, and O. Akimoto.
Optical Turbulence: Chaotic Behavior of Transmitted Light from a Ring Cavity.
Physical Review Letters, 45(9):709–712, September 1980.
Publisher: American Physical Society.