(Liam P., 2025-01-06)

1 Scale Decompositions for EQFT

The goal of this note is to clearly demonstrate two different scale decompositions of the Gaussian Free Field, which are qualitatively similar, that appear in recent works on Euclidean quantum field theory, see references. There's also lots of missing 2π's—please forgive me.

1.1 Brief motivating example: infinite-dimensional Gibbs measures

Formally, the ϕ^4_d measure on the torus $\mathbb{T}^d$ is a Gibbs measure

$$d\mu = \frac{1}{Z} e^{-H(\phi)} \prod_{x\in\mathbb{T}^d} d\phi_x \tag{1.1}$$

with

$$H(\phi) = \int_{\mathbb{T}^d} \phi^4 dx + \int_{\mathbb{T}^d} |\nabla\phi|^2 + \phi^2 dx$$

Of course, this doesn't make any sense. The worst offender of nonsense is the non-existent infinite-dimensional Lebesgue measure $\Pi_{x\in\mathbb{T}^d}\phi_x$, where a Lebesgue measure is somehow attached to each point on the torus. We can handle this by grouping some terms of H with this Lebesgue measure to form the Gaussian Free Field (GFF).

$$d\mu = \frac{1}{Z} e^{-V(\phi)} d\theta$$

where θ is the law of a GFF (to be described), and V has the left-over terms of H.

1.2 The Gaussian Free Field on $\mathbb{T}^3$

While $\prod_{x\in\mathbb{T}^3} d\phi_x$ makes no sense, we can group it with the "kinetic energy" and "mass" term $\int_{\mathbb{T}^3} |\nabla\phi|^2 + \phi^2 dx$ of the potential to form the Gaussian Free Field (GFF). Using the Fourier isometry, we see for ϕ nice enough,

$$\exp\left(-\int_{\mathbb{T}^3} |\nabla\phi|^2 + \phi^2\right) \prod_x d\phi_{x\in\mathbb{T}^3} \simeq \exp\left(-\sum_{n\in\mathbb{Z}^3} (1+|n|^2)\widehat{\phi}^2(n)\right) \prod_{n\in\mathbb{Z}^3} d\widehat{\phi}_n,$$

where $\simeq$ is used since the left-hand side doesn't make any sense. But the right-hand side is formally the law of a sum of countably many independent Gaussians, each with variance $(1+|n|^2)^{-1}$.

Definition 1.1. *A Gaussian Free Field X with mass m is a Gaussian process with covariance given formally by*

$$\mathbb{E}[X(x)X(y)] = (m^2 - \Delta)^{-1}(x-y). \tag{1.2}$$

Although defining something formally doesn't seem helpful, we see

$$X(x) = \sum_{n\in\mathbb{Z}^d} e^{i\langle n,x\rangle}(1+|n|^2)^{-\frac{1}{2}} g^n, \tag{1.3}$$

is a GFF with mass $m=1$, where $g^n = \frac{1}{\sqrt{2}}(g^n_1 + i g^n_2)$ are iid complex Gaussians with the constraint $\overline{g^n} = g^{-n}$. Indeed, since $\mathbb{E}[g^n g^m] = \delta_{m=-n}$,

$$\mathbb{E}[X(x)X(y)] = \sum_{n\in\mathbb{Z}^d} e^{i\langle n,x-y\rangle}(1+|n|^2)^{-1} = (1-\Delta)^{-1}(x-y)$$

Remark 1.2. In $d \geq 2$, X is almost surely not a function, so this really is a formal definition. To get a feel for why this is the case, we can return to the previous argument and plug into the Fourier expansion $x = y$. We then see $(m^2 - \Delta)^{-1}(0) = \infty$ in $d \geq 2$, so that $\mathbb{E}[X(x)^2] = \infty$ for all $x \in \mathbb{T}^3$. This indicates there will be some serious problems with "evaluating at a point."

However: (1.2) can be integrated against $f(x), g(y) \in L^2(\mathbb{T}^3)$ so X can be defined as a Gaussian process indexed by functions:

$$\mathbb{E}[\langle X, f\rangle \langle X, g\rangle] = \int_{\mathbb{T}^3 \times \mathbb{T}^3} f(x)(-\Delta + m^2)^{-1}(x-y)g(y)dxdy.$$

Because of this, $\langle X, f\rangle$ is sometimes called an "observable," since all we can do is probe X with a function, "observing" a Gaussian random variable $\langle X, f\rangle$ for each f.

2 Scale Decompositions

Recall one method of solution for Laplace's equation is to "run the heat equation forever." Written succinctly,

$$-\Delta^{-1} = \int_0^\infty e^{s\Delta} ds.$$

Heuristically, $-\Delta$ is a positive operator, and hence $s\Delta < 0$, so we expect $\lim_{s\to\infty} \Delta^{-1} e^{s\Delta} \to 0$.

But this equation tells more: we can stop the heat equation at any finite time to get an approximate inverse to Δ:

$$C_t = \int_0^t e^{s\Delta} ds.$$

We can even build Δ^{-1} by successively starting and stopping the heat equation,

$$\Delta^{-1} = C_0 + (C_1 - C_0) + (C_2 - C_1) + \cdots$$

One way to construct a GFF is to find a sequence of Gaussian processes, X_t, in some sense continuous in t, such that $\mathbb{E}[X_t(x)X_t(y)] \to (1-\Delta)^{-1}(x-y)$. If we can find a process for which $\mathbb{E}[X_t(x)X_t(y)] = C_t$, then we've accomplished our goal by the preceding discussion. But there are many different choices for C_t, and some could be more helpful than others depending on the problem. Either way, to accomplish this, we'll construct a process for Gaussian fields which is analogous to Brownian motion for Gaussian distributions.

2.1 Cylindrical BM

One way to construct white noise is to form the random Fourier series,

$$W_t = \sum_{n\in\mathbb{Z}} g^n e^{i\langle n,t\rangle}$$

where g^n are iid complex Gaussians as before. Brownian motion is obtained by "integrating white noise",

$$B_t = \left\langle W_\cdot, 1_{[0,t]}(\cdot)\right\rangle_{L^2(\mathbb{R})} = \sum_{n\in\mathbb{Z}} g^n \left\langle e^{i\langle n,\cdot\rangle}, 1_{[0,t]}(\cdot)\right\rangle_{L^2(\mathbb{R})}$$

Right now, $B_t \in \mathbb{R}$ for fixed t; we're going to look for a Gaussian process based on this construction with values in $H^{-\alpha}$.

To "evaluate" white noise at a particular time, we plug t into the Fourier series. If we make the Fourier series over $\mathbb{Z}^d$ instead of $\mathbb{Z}$, we would have a white noise in d dimensions. Re-labeling t to x, this gives "white noise in space:"

$$W(x) = \sum_{n\in\mathbb{Z}^d} g^n e^{i\langle n,x\rangle}.$$

Indeed, $\mathbb{E}[W(x)W(y)] = \sum_{n\in\mathbb{Z}^d} e^{i\langle n,x-y\rangle} = \delta(x-y)$, and $\{W_x\}_{x\in\mathbb{T}^d}$ is formally a random distribution on $\mathbb{T}^d$. To make this process time-dependent, we can turn the complex Gaussians g^n into complex Brownian motions B_t^n with the same constraint $\overline{B_t^n} = B_t^{-n}$,

$$Y_t(x) = \sum_{n\in\mathbb{Z}^d} B_t^n e^{i\langle n,x\rangle}.$$

Then $dY_t = \sum_n dB_t^n e^{i\langle n,x\rangle}$ plays the role of dB_t in standard SDEs; the spatial dimensions are thought of as fixed, so stochastic processes increments dY_t have values in a function space, and SDEs $dX_t = F_t dt + Q_t dY_t$ formally represent the integral equation

$$X_t = X_0 + \int_0^t F_s ds + \int_0^t Q_s dY_s \tag{2.1}$$

but where $\int dt$ yields a function or distribution rather than a number, and Q_t is a positive linear operator rather than a matrix or number.

Remark 2.1. One massive challenge introduced is, while $B_t \in \mathbb{R}$, Y_t is only a distribution, and a very rough one at that: for fixed time, its spatial covariance is $t\cdot\delta_{x-y} \in H^{-\frac{d}{2}-\epsilon}$. This makes the choice of spatial covariance Q_t in front of Y_t very important in applications. For example, while $Y_t \in H^{-\frac{d}{2}-\epsilon}$, if $Q_t^2 = \frac{d}{dt}C_t = e^{-t(1-\Delta)}$, then $Z_t = \int_0^t Q_t dY_t \in H^k$ for all k (its Fourier coefficients decay exponentially fast almost surely). This extra boost in regularity can make a big difference.

To demonstrate this point, suppose $F_t(X_t) = M_t X_t$ for some $M_t \in H^{\frac{d}{2}}(\mathbb{T}^d)$. If we call $Z_t = \int_0^t F_t(X_t)dt$, then we at least need $F_t(X_t) = M_t(Z_t + Y_t)$ to be well-defined. The very lowest requirement is that $M_t Y_t$ be well-defined, which can only hold when their regularities add to something positive. For $Q_s = 1$, we're out of luck: their regularities sum to $\frac{d}{2} - \frac{d}{2} - \epsilon < 0$. For $Q_t^2 = e^{-t(1-\Delta)}$, we have a chance. Obviously, the solutions to the two different SDEs won't be the same, but the point is that it's easier to get to the starting line with a covariance decomposition.

2.2 Heat kernel decomposition

As a different construction of the GFF, set $X_\infty = \int_0^\infty Q_t dY_t$ where $Q_t^2 = C_t$. Then X_∞ has covariance $\int_0^\infty Q_t^2 dt = (1-\Delta)^{-1}$, so X_∞ must be a GFF if it exists. Thus the construction and analytic properties of the GFF can be understood by studying (2.1). This way of constructing a GFF is qualitatively similar to performing a Fourier cutoff, and progressively restoring higher frequencies:

$$X_\infty(x) = \lim_{K\to\infty} \sum_{n\in\mathbb{Z}^d\cap[-K,K]^d} e^{i\langle n,x\rangle}(1+|n|^2)^{-\frac{1}{2}} g^n = \lim_{K\to\infty} \sum_{n\in\mathbb{Z}^d} \chi_K(n)(1+|n|^2)^{-\frac{1}{2}} e^{i\langle n,x\rangle} g^n$$

where $\chi_K(n) = 1_{[-K,K]^d}(n)$.

2.3 C_c^∞ decomposition

Observe that the heat kernel $e^{t\Delta}$ acts on Fourier coefficients through multiplication by $e^{-t|n|^2}$. This gives considerable room for regularity, as we only need $\sum_n (1+|n|^2)^k e^{-t|n|^2}|\widehat{f}(n)|^2$ to be summable for $e^{t\Delta}f$ to be in H^k. A natural generalization is to construct covariance decomposition through Fourier multipliers $\widehat{C}_t(n)$ such that $\widehat{C}_t(n) \to \frac{1}{(1+|n|^2)}$, C_t is differentiable, and $\dot{\widehat{C}}_t(n) \geq 0$ for all t . Then

$$C_t f(x) := \sum_n e^{i\langle n,x\rangle} \widehat{C}_t(n)\widehat{f}(n)$$

defines a heat kernel decomposition we can use in (2.1). $\widehat{C}_t(n) \geq 0$ ensures we can take a square root, setting

$$Q_t f = \sum_n e^{i\langle n,x\rangle} \hat{\dot{C}}_t(n)^{1/2} \widehat{f}(n).$$

For example, taking inspiration from the previous discussion on "restoring higher frequencies," we can take a compactly supported smooth function χ, set $\widehat{C}_t(n) = \chi^2(n/t)$, so that $\widehat{Q}_t(n) = \sqrt{\frac{d}{dt}\chi^2(n/t)} \geq 0$. Plugging this into (2.1) with $F_t = 0$,

$$X_t(x) = \left(\int_0^t Q_t dY_t\right)(x) = \sum_{n\in\mathbb{Z}^3} \frac{1}{(1+|n|^2)^{1/2}} e^{i\langle n,x\rangle} \int_0^t \sqrt{\frac{d}{dt}\chi_t^2(n)} dB_s^n.$$

Notice that $\mathbb{E}[X_t(x)X_t(y)] = \sum_n \frac{\chi_t(n)^2}{(1+|n|)^2} e^{i\langle n,x-y\rangle} \to (1-\Delta)^{-1}(x-y)$. Since the Fourier series is truncated, X_t is smooth for each t.

3 Scale decompositions for a different L^2 basis

Remember that we started with the potential V, and grouped the gradient and square term with the Lebesgue measure to form a GFF. If instead we started with a potential $V : L^2(\mathbb{R};\mathbb{C}) \to \mathbb{R}$

$$V(\phi) = \int_{\mathbb{R}} |\phi|^4 dx + \int_{\mathbb{R}} |\nabla\phi|^2 + |x|^2|\phi|^2 dx$$

the Fourier basis isn't as useful since we can only absorb the kinetic energy term into the GFF measure. But we can still run a similar argument to handle $|x|^2|\phi|^2$. First, we find a Gaussian field X with spatial covariance

$$\mathbb{E}[X(x)X(y)] = (|x|^2 - \Delta)^{-1}(x,y).$$

We can do this by emulating the construction of the GFF: since $|x|^2 - \Delta$ is very nice (compact resolvent, discrete spectrum), we have an orthogonal decomposition of eigenfunctions $\{H_n, iH_n\}_{n\in\mathbb{N}}$ which are orthogonal with respect to the measure $e^{-\frac{1}{2}x^2}dx$. Form the cylindrical Brownian motion as before by writing

$$Y_t(x) = \sum_{n\in\mathbb{N}} B_t^n H_n(x).$$

where B_t^n are iid complex Brownian motions (hidden by B_t^n are the two basis elements H_n and iH_n)

Then dY_t is a space-time white noise for the measure $d\gamma = e^{-\frac{1}{2}x^2}dx$, i.e.,

$$\mathbb{E}[\langle dY_t, f\rangle \overline{\langle dY_t, g\rangle}] = \langle f, g\rangle_{L^2(\gamma)}$$

we can write down SDEs,

$$dX_t = F_t dt + C_t dY_t.$$

Again, we can look for covariance decompositions that play nicely with the basis H_n. One option is to use the cutoff functions χ_t^2 again:

$$X_t = \int Q_t dY_t = \sum_{n\in\mathbb{N}} \frac{1}{\lambda_n^{1/2}} H_n(x) \int \sqrt{\frac{d}{dt}\chi_t^2(n)} dB_t^n$$

Then

$$\mathbb{E}[X_t(x)\overline{X_t(y)}] \to \sum_n \frac{1}{\lambda_n} H_n(x)\overline{H_n(y)}.$$

But this is exactly $(-|x|^2 - \Delta)^{-1}(x, y)$. Likewise, we could use the heat kernel decomposition

$$C_t = \int_0^t e^{s(-|x|^2+\Delta)} ds$$

where $e^{s(-|x|^2+\Delta)} f = \sum_n e^{-s\lambda_n} \langle f, H_n \rangle_{L^2(\gamma)} H_n$. Since $\lambda_n = \sqrt{2n+1}$, the multiplier sequence of $e^{t(-|x|^2+\Delta)}$ has decay $\lesssim e^{-t\sqrt{n}}$ with n.

So we've reduced the problem of understanding $d\mu = \frac{1}{Z} e^{-V(\phi)} d\phi$ to understanding $\mathbb{E}[e^{-\int_{\mathbb{R}} |X_\infty|^2}]$. As X_t is a function of a cylindrical Brownian motion, we still have the Boue-Dupuis formula to understand exponential functionals $\mathbb{E}[e^{-F(X_t)}]$, giving a starting point for defining the partition function of the system.

References

[1] N. Barashkov and M. Gubinelli. A variational method for Φ^4_3. *Duke Mathematical Journal*, 169(17), November 2020. Number: 17 arXiv:1805.10814 [math-ph].