Week 6; Measure Zero and Integration Math 2240, Spring ’24

Problem. 1.

- From the definition, show $\mathbb{R}^{n-1} \times \{0\}$ has measure zero.
- From the definition, show that if D is a cube in $\mathbb{R}^n$, then any face of D has measure zero.
- Show that if $A \subseteq \mathbb{R}^n$ is compact, and A has measure zero in $\mathbb{R}^n$, then given $\varepsilon > 0$ there exists a *finite* collection of cubes of total volume less than ε covering A.
- Let $f : [a, b] \to \mathbb{R}$ be continuous. From the definition, show that the *graph* of f has measure zero in $\mathbb{R}^2$.

Problem. 2. Transform the iterated integral $\int_0^1 \int_y^{y^{1/3}} \sin(x^2) dx dy$ into an integral over a subset of the plane.

1. Sketch this subset.
2. Compute the integral

Problem. 3. Let Q be a rectangle in $\mathbb{R}^n$, and $f : Q \to \mathbb{R}$. Assume f is integrable.

- Show that if $f(x) \geq 0$ for all $x \in Q$, then $\int_Q f \geq 0$
- Show that if $f(x) > 0$ for all $x \in Q$, then $\int_Q f > 0$

Problem. 4. We say a non-empty subset $A \subset \mathbb{R}^n$ is *discrete* if, for any $x \in A$, there exists an $\varepsilon > 0$ such that $B(x, \varepsilon) \cap A = \{x\}$. Prove or give a counterexample:

1. A discrete set has volume zero
2. A discrete set has measure zero
3. The closure of a discrete set has volume zero
4. The closure of a discrete set has measure zero

Problem. 5. Let $f \in C((0, 1))$ (i.e., a continuous function with domain $(0, 1)$). Suppose f has the property that, for any $g \in C_c((0, 1))$,

$$\int_{\mathbb{R}} fg = 0$$

where $C_c((0, 1)) = \{g \in C((0, 1)) : \text{support}(g) \subset (0, 1)\}$. (Alternatively support$(g) \subseteq [\varepsilon, 1 - \varepsilon]$ for some $\varepsilon > 0$). Does it follow that $f(x) = 0$ for all $x \in (0, 1)$? What if f is "only" integrable rather than continuous?

(extras)

Problem. 7. Does there exist an open set $A \subseteq (0, 1)$ such that ∂A *isn’t* measure zero?

Problem. 6. Does there exist a dense, open subset of $[0,1]$?

Problem. 7. We'll restrict our attention to $C([0,1])$, the (uniformly) continuous functions from $[0,1]$ to $\mathbb{R}$. Define

$$\|f\|_1 = \int_0^1 |f(x)|dx$$

- Show that $\|\cdot\|_1$ defines a norm on $C([0,1])$ (remember to check $\|f\|_1 = 0$ iff $f = 0$!)
- How does $\|f\|_1$ compare to $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$? (hard) Do they induce the same topology on $C([0,1])$?