Local Boundedness — De Giorgi and Moser

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Setup

In the following we consider a weak sub-solution $u \in H_0^1(B_1)$ of the PDE

$$Lu + cu \le f$$

where

$$Lu = \sum_{i,j} D_i(a_{ij}D_j).$$

In other words, for any non-negative $\phi \in H_0^1(B_1)$,

$$\int a_{ij}D_iuD_j\phi + cu\phi \le \int f\phi \tag{1}$$

In the following, we assume $\{a_{ij}(x)\}_{ij}$ is bounded and satisfies an ellipticity condition,

$$a_{ij}(x)\xi_i\xi_j \ge \lambda|\xi|^2 > 0,$$

and $c, f \in L^q$ for $q > \frac{n}{2}$. We also take Λ so that

$$|a_{ij}|_\infty + \|c\|_q \le \Lambda.$$

Remark. The requirement that $q > \frac{n}{2}$ is a bare-minimum to guarantee the integrals in 1 are finite. Indeed, by Hölder,

$$\|cu\phi\| \le \|c\|_q\|u\phi\|_{1-\frac{1}{q}} \le \|f\|_q\|u\|_{2-\frac{2}{q}}\|\phi\|_{2-\frac{2}{q}}$$

If we only know $u, \phi \in H^1$, we need $2(1-\frac{1}{q}) < \frac{2n}{n-2}$ to make use of the Sobolev inequality, or $q > \frac{n}{2}$.

Theorem 1 (De Giorgi [1], Nash [4], Moser [3]). With the above conditions, assume u satisfies the above weak form. Write u^+ as the positive part of u. Then for some constant C depending only on n, λ, Λ, p,

$$\sup_{B_\theta} u^+ \le C(n,\lambda,\Lambda,p)\left\{\frac{1}{(1-\theta)^{n/p}}\|u\|_{L^p(B_1)} + \|f\|_q\right\}.$$

This note will sketch the proof of this classical result for the case $p = 2$, $\theta = 1/2$ using Moser's and De Giorgi's methods. For the extension to all $p \in (0,\infty)$ and $\theta \in (0,1)$, and for many more (important and instructive) details, see section 4 of [2].

Moser's Approach

The Homogeneous Case

Let's simplify things and assume $f = 0$ and $c = 0$,

$$\int a_{ij}D_iuD_j\phi \le 0 \tag{2}$$

This was the original equation studied in the papers referenced above.

Suppose $u \geq 0$; otherwise replace u with its positive part. Plugging in $\phi = \eta^2 u$ and applying both boundedness and ellipticity of a_{ij}, we obtain the estimate

$$\int |D(u\eta)|^2 \leq C\left\{\int |D\eta|^2 u^2\right\}.$$

With $0 < r < R < 1$, choose η supported on B_R with $\eta \equiv 1$ on B_r. Then by the Sobolev inequality,

$$\|u\|_{L^{2\chi}(B_r)} \leq C\frac{1}{R-r}\|u\|_{L^2(B_R)}$$

where $\chi = \frac{n}{n-2} > 1$ so that $2\chi = 2^*$. We have successfully bounded a higher L^p norm of u by its L^2 norm on a larger set, at the cost of a factor $\frac{1}{R-r}$ and a smaller domain, a price we'll gladly pay. If the same were true for this higher power of u, we could iterate, $u \to u^\chi \to u^{\chi^i}$ and obtain a chain of estimates leading back to $\|u\|_{L^2(B_R)}$. As $\chi^i \to \infty$, this gives control over the L^∞ norm of u, albeit on a smaller set than the initial ball B_R. So long as the decrements of the radius, $R - r$, decrease fast enough, we can get an estimate of $\sup_{x\in B_{\frac{1}{2}}} u = \lim_{i\to\infty} \|u\|_{L^{2\chi^i}(B_{r_i})}$ in terms of $\|u\|_{L^2(B_R)}$ with $R = 1$.

Lets see the details. Notice that since $x \mapsto x^\chi$ is a convex function when $\chi > 1$, and has positive derivative when $x > 0$, u^χ *is also a sub-solution to 2*. A-fortiori, we iterate the bound in the previous paragraph; set $r_i = \frac{1}{2} + \frac{1}{2^{i+1}}$. Then

$$\|u\|_{L^{2\chi^i}(B_{r_i})} \leq C^{\frac{1}{\chi^i}} 2^{-\frac{i}{\chi^i}} \|u\|_{L^{2\chi^{i-1}}B(r_{i-1})}$$

Repeatedly applying this bound to the left-hand side,

$$\|u\|_{L^{2\chi^i}(B_{r_i})} \leq C^{\sum_{j\leq i}\frac{1}{\chi^j}} 2^{-\sum_{j\leq i}\frac{j}{\chi^j}} \|u\|_{L^2B(r_0)}$$

Letting $i \to \infty$ on both sides gives the result: the left-hand side becomes $\|u\|_{L^\infty(B_{1/2})}$, while the constants appearing on the right have convergent sums as exponents:

$$\sup_{B_{1/2}} u \leq C\|u\|_{L^2(B_1)}$$

The General Case

We return to the general case of $c, f \in L^q(B_1)$ for $q > \frac{n}{2}$,

$$\int a_{ij}D_i u D_j\phi + cu\phi \leq \int f\phi. \tag{3}$$

With care, Moser's approach will work once again. Set $\tilde{u} = u^+ + k$ with $k \geq 0$ to be determined in order to handle $\|f\|_q$. Plug in $\phi = \eta^2\tilde{u}$, again applying ellipticity, to obtain

$$\int |D(\tilde{u}\eta)^2| \leq C\left\{\int |D\eta|^2\tilde{u}^2 + \int |c|\eta^2\tilde{u}^2 + \int f\eta^2\tilde{u}\right\}$$

Note that $\tilde{u} \geq k$, so $f\tilde{u} \geq \frac{f}{k}\tilde{u}^2$. Choosing $k = \|f\|_q$, we can group the $|c|$ and $|f|$ terms, apply Holder's inequality and use the condition that $q > n/2$ to obtain

$$\int |D(\tilde{u}\eta)^2| \leq C\left\{\int |D\eta|^2\tilde{u}^2 + \int \eta^2\tilde{u}^2\right\}$$

Choosing η in the same way as above and applying the Sobolev inequality

$$\|\tilde{u}\|_{L^{2\chi}(B_r)} \leq C\frac{1}{R-r}\|\tilde{u}\|_{L^2(B_R)}$$

This is once again the starting point of the Moser iteration scheme. But there is a problem: we don't know whether $\tilde{u}^\chi$ satisfies the same type of estimate as $\tilde{u}$. Before, convexity saved us. But now we need to check by hand. Luckily, $x \mapsto x^\chi$ is a nice enough function for things to work; we can still apply Holder's inequality to products involving u^χ. For the details, see section 4.2 of [2]. At the end of the day, we found the right bound,

$$\sup_{B_{1/2}} u \le \sup_{B_{1/2}} \tilde{u} \le C\|\tilde{u}\|_{L^2(B_1)} \le C\left\{\|u\|_{L^2(B_1)} + \|f\|_q\right\}.$$

De Giorgi's Approach

Starting from 3, set $\phi = \eta^2 v$ where $v = (u-k)^+$, with k to be chosen later (but *not* for the purpose of removing $\|f\|_q$). Our goal is to show, for k large enough,

$$\int_{B_{1/2}} ((u-k)^+)^2 = 0$$

It immediately follows that

$$\sup_{B_{1/2}} u^+ \le k$$

Hopefully, k is of the form in the theorem (it will be, don't worry). This is a decidedly different method from Moser's approach: instead of using the Sobolev inequality to directly lower-bound $\int |D(\eta\tilde{u})|^2$, in this approach, we want the L^2 of v on the *left-hand side* rather than the Sobolev norm. To make progress, apply Holder's inequality:

$$\begin{aligned}\|v\eta\|_2 &\le \|v\eta\|_{2^*}^2 |\{v\eta > 0\}|^{1-\frac{2}{2^*}} \\ &\le \|D(v\eta)\|_2^2 |\{v\eta > 0\}|^{\frac{2}{n}}\end{aligned}$$

This $\frac{2}{n}$ will be crucial later. If we apply the usual ellipticity conditions to 3 with our chosen ϕ, we find

$$\int D(v\eta)^2 \le C\left\{\int |D\eta|^2 v^2 + \int |c|uv\eta^2 + \int |f|v\eta^2\right\}$$

To make progress, we apply Holder's inequality to the c and f terms. For example,

$$\begin{aligned}\int |f|v\eta^2 &\le \|f\|_q \|\eta v\|_{2^*} |\{v\eta > 0\}|^{1-\frac{1}{2^*}-\frac{1}{q}} \\ &\le \|f\|_q \|D(\eta v)\|_2 |\{v\eta > 0\}|^{\frac{1}{2}+\frac{1}{n}-\frac{1}{q}} \\ &\le \frac{1}{2\varepsilon}\|f\|_q |\{v\eta > 0\}|^{1+\frac{2}{n}-\frac{2}{q}} + \frac{\varepsilon}{2}\|D(\eta v)\|_2^2\end{aligned}$$

We can ε as small as we like to absorb the second term into our constant C. Since $q > n/2$, we can replace the measure term by

$$|\{v\eta > 0\}|^{1+(\frac{2}{n}-\frac{1}{q})-\frac{1}{q}} = C(n,q)|\{v\eta > 0\}|^{1-\frac{1}{q}}$$

Doing the same for the c term and applying the above inequality to $\int |D(\eta v)|^2$, we obtain the start of an iteration scheme different from Moser's.

Define $A(k,r) = \{u > k\} \cap B_r$. Using the same η function as in the Moser section

$$\|v\|_{L^2(B_r)}^2 \le C\left\{\frac{1}{(R-r)^2}|A(k,R)|^\varepsilon \|v\|_{L^2(B_R)} + (k + \|f\|_q)^2 |A(k,R)|^{1+\varepsilon}\right\}$$

where $\varepsilon = \frac{2}{n} - \frac{1}{q} > 0$. Without this ε of room, there is no hope of iteration. Remember that $v = (u-k)^+$. In order to iterate, when going from right to left, we need to decrease the size of the domain and *increase* the cutoff to some $h > k$. We can then chain estimates to end with $\|(u-h)^+\|_{L^2(B_{1/2})}$ on the left-hand side.

To see this in action, we need bounds on $|A(k,R)|$. We follow [2] closely. By Markov's inequality, $|A(k,R)| \leq \frac{1}{k}\|u^+\|_{L^2}$, so the above inequality holds for $k_0 = C\|u^+\|_{L^2}$ with C large enough. Now note that $A(k,r) \subset A(k,R)$, and if $h > k$, then $A(k,r) \supset A(h,r)$. We can apply Markov's inequality along with these inclusions to obtain

$$|A(k,r)| \leq \frac{1}{(h-k)^2}\int_{A(k,R)}(u-k)^2$$

Set $r_i = \frac{1}{2} + \frac{1}{2^{i+1}}$ and $k_i = k_0 + k(1 - \frac{1}{2^i})$. Writing $\phi(k,r) = \|(u^+ - k)^+\|_{L^2(B_r)}$, we have a chain of inequalities,

$$\phi(k_i, r_i) \leq C2^i\phi(k_{i-1}, r_{i-1})^{1+\varepsilon}$$

We have the freedom to choose our end-point, k, as large as we like. In particular, we can make it so that, for some constant $\gamma > 1$,

$$\phi(k_i, r_i) \leq \frac{\phi(k_0, r_0)}{\gamma^i} \tag{4}$$

From this we can show $\Phi(k_\infty, r_\infty) = \|(u-k_\infty)^+\|^2_{L^2(B_{1/2})} = 0$. Since $k_\infty = k$, whichever value of k we choose to make 4 hold will yield our desired bound. For details, once again see [2]. The important point is that the $1+\varepsilon$ power of the right-hand side allows the inequality chain to accumulate powers of $\phi(k_0, r_0)$. If this balances with the accumulating powers of 2^i and C, the argument goes through.

References

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