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1 Scale Decompositions for EQFT

The goal of this note is to clearly demonstrate the similarities between two different scale decompositions of the Gaussian Free Field that appear in recent works on Euclidean quantum field theory (see references). There's also lots of missing 2π's—please forgive me.

Brief motivating example: infinite-dimensional Gibbs measures

Formally, the ϕ^4_d measure on the torus $\mathbb{T}^d$ is a Gibbs measure

$$d\mu = \frac{1}{Z} e^{-H(\phi)} \prod_{x \in \mathbb{T}^d} d\phi_x \tag{1.1}$$

with

$$H(\phi) = \int_{\mathbb{T}^d} \phi^4 dx + \int_{\mathbb{T}^d} |\nabla \phi|^2 + \phi^2 dx$$

Of course, this doesn't make any sense. The worst offender is the non-existent infinite-dimensional Lebesgue measure $\Pi_{x\in\mathbb{T}^d}\phi_x$, where a Lebesgue measure is somehow attached to each point on the torus. To handle this problem, we can group the "Lebesgue" measure with the the quadratic terms of H, leaving $V(\phi) = \int_{\mathbb{T}^d} \phi^4 dx$, and

$$d\mu = \frac{1}{Z} e^{-V(\phi)} d\theta(\phi)$$

where θ is the law of a GFF (to be described), and (formally) $Z = \int e^{-V(\phi)} d\theta(\phi)$.

The point: a very convenient way to construct the measure (1.1) is to write down explicitly a GFF with $\mathrm{Law}(X) = \theta$ and then try to understand expectations with respect to X tilted by e^{-V},

$$\int f d\mu = \mathbb{E}[f(X)e^{-V(X)}].$$

A (very effective) recent approach to compute expectations under this tilted measure is to represent these expectations as minimizers of a related stochastic control problem (first implemented in [1], followed by [3] [4] [13] and many more). For example, we can formally compute the normalization constant $Z = \mathbb{E}[e^{-V(X)}]$ by a formula of Boue and Dupuis [10]

$$-\log \mathbb{E}[e^{-V(X_\infty)}] = \inf_v \mathbb{E}[V(X_\infty + \int_0^\infty Q_s v_s ds) + \frac{1}{2}\int_0^\infty \|v_s\|_{L^2} ds] \tag{1.2}$$

where X_t is a scale decomposition of X by $X_t = \int_0^t Q_s dY_s$ (to be described; think of $X_t = \int_0^t \sigma_s dB_s$ in $\mathbb{R}^d$), and v_t is a stochastic process with values in $H^1(\mathbb{T}^d)$, called a "control", that is adapted to X_t. Modifications of this control problem can be used to compute $\mathbb{E}[f(X)e^{-V(X)}]$ for nice f (see [3], [2], [13] for this approach).

Independently, scale decompositions were used to explicitly construct the measure μ; this is dubbed the "Polchinski flow" approach and has advantages and disadvantages when compared with the control approach (although a comparison isn't explicitly detailed in the literature, and the two approaches aren't mutually disjoint, see [4]). The first use of the Polchinski approach was found through obtaining log-Sobolev constants of the sine-Gordon field with $\beta < 6\pi$ [5] and for ϕ^4_2, ϕ^4_3 by slightly different means but still with the Polchinski flow [8]. Further applications followed in [7] [9] [6] [4] and probably more.

Hopefully this gives an indication that scale decompositions X_t of the GFF X have become an important tool in Euclidean quantum field theory, which justifies the brief explanation given by this note.

1.1 The Gaussian Free Field on $\mathbb{T}^3$

As promised, let's try to group together the quadratic terms $\int_{\mathbb{T}^3} |\nabla\phi|^2 + \phi^2 dx$ with $\prod_{x\in\mathbb{T}^3} d\phi_x$. Using the Fourier isometry, we see for ϕ nice enough,

$$\exp\left(-\int_{\mathbb{T}^3} |\nabla\phi|^2 + \phi^2\right) \prod_x d\phi_{x\in\mathbb{T}^3} \simeq \exp\left(-\sum_{n\in\mathbb{Z}^3} (1+|n|^2)\widehat{\phi}^2(n)\right) \prod_{n\in\mathbb{Z}^3} d\widehat{\phi}_n,$$

where $\simeq$ is used since the left-hand side doesn't make any sense. But the right-hand side is formally the law of a sum of countably many independent Gaussians, each with variance $(1+|n|^2)^{-1}$.

Definition 1.1. *A Gaussian Free Field X with mass m on the torus $\mathbb{T}^d$ is a Gaussian process with covariance given formally by*

$$\mathbb{E}[X(x)X(y)] = (m^2 - \Delta)^{-1}(x-y). \tag{1.3}$$

Defining something formally doesn't seem helpful, but notice that

$$X(x) = \sum_{n\in\mathbb{Z}^d} e^{i\langle n,x\rangle}(1+|n|^2)^{-\frac{1}{2}} g^n, \tag{1.4}$$

where $g^n = \frac{1}{\sqrt{2}}(g_1^n + ig_2^n)$ are iid complex Gaussians with the constraint $\overline{g^n} = g^{-n}$, is a GFF with mass $m=1$. Indeed, since $\mathbb{E}[g^n g^m] = \delta_{m=-n}$,

$$\mathbb{E}[X(x)X(y)] = \sum_{n\in\mathbb{Z}^d} e^{i\langle n,x-y\rangle}(1+|n|^2)^{-1} = (1-\Delta)^{-1}(x-y)$$

Remark 1.2. In $d\geq 2$, X is almost surely not a function, so this really is a formal definition. To get a feel for why this is the case, we can return to the previous argument and plug into the Fourier expansion $x=y$. We then see $(m^2-\Delta)^{-1}(0) = \infty$ in $d\geq 2$, so that $\mathbb{E}[X(x)^2] = \infty$ for all $x\in\mathbb{T}^d$. This indicates there will be some serious problems with "evaluating at a point."

2 Scale Decompositions

Definition 2.1. *A scale decomposition of the GFF with mass m is a Gaussian stochastic process X_t such that $X_\infty = X$, $\mathbb{E}[X_t(x)X_t(y)] = C_t(x,y)$ is a positive-definite kernel, continuous in time, and increasing to $(m^2-\Delta)^{-1}(x-y)$ in the sense of positive linear operators.*

The name of the game is to explicitly construct these scale decompositions. We can take a hint from finite-dimensional Brownian motion; if X_t satisfies

$$X_t = \int_0^t \sigma_s dB_s$$

where σ_s is a positive definite matrix for each s, then $\{X_t(i)\}_{i=1}^n$ is a Gaussian vector with covariance $E[X_t(i)X_t(j)] = \int_0^t \sigma_s(i,j)^2 ds$ by the Itô isometry.

2.1 Cylindrical Brownian Motion

One way to construct white noise is to form the random Fourier series,

$$W_t = \sum_{n\in\mathbb{Z}} g^n e^{i\langle n,t\rangle}$$

where g^n are iid complex Gaussians as before. Brownian motion is obtained by "integrating white noise",

$$B_t = \left\langle W_\cdot, 1_{[0,t]}(\cdot)\right\rangle_{L^2(\mathbb{R})} = \sum_{n\in\mathbb{Z}} g^n \left\langle e^{i\langle n,\cdot\rangle}, 1_{[0,t]}(\cdot)\right\rangle_{L^2(\mathbb{R})}$$

Right now, $B_t \in \mathbb{R}$ for fixed t; we want a Gaussian process based on this construction with values in $H^\alpha(\mathbb{T}^d)$ for some $\alpha \in \mathbb{R}$.

To (formally) evaluate white noise at a particular time, we plug t into the Fourier series. If we make the Fourier series over $\mathbb{Z}^d$ instead of $\mathbb{Z}$, we would have a white noise in d dimensions. Re-labeling t to x, this gives white noise in space:

$$W(x) = \sum_{n \in \mathbb{Z}^d} g^n e^{i\langle n,x\rangle}.$$

Indeed, $\mathbb{E}[W(x)W(y)] = \sum_{n\in\mathbb{Z}^d} e^{i\langle n,x-y\rangle} = \delta(x-y)$, making W a random distribution on $\mathbb{T}^d$. To make this process time-dependent, we can turn the complex Gaussians g^n into complex Brownian motions B_t^n with the same constraint $\overline{B_t^n} = B_t^{-n}$.

$$Y_t(x) = \sum_{n \in \mathbb{Z}^d} B_t^n e^{i\langle n,x\rangle}.$$

Definition 2.2. *Y_t as given above is called a cylindrical Brownian motion over $\{e^{i\langle n,x\rangle}\}_{n\in\mathbb{Z}^d}$.*

From here, $dY_t = \sum_n dB_t^n e^{i\langle n,x\rangle}$ plays the role of dB_t in standard SDEs; the spatial dimensions are thought of as fixed, so stochastic processes increments dY_t have values in a function space, and SDEs $dX_t = F_t dt + Q_t dY_t$ formally represent the integral equation

$$X_t = X_0 + \int_0^t F_s ds + \int_0^t Q_s dY_s \tag{2.1}$$

but where $\int dt$ yields a function or distribution rather than a number, and Q_t is a positive linear operator rather than a matrix or number.

Remark 2.3. One massive challenge is introduced: while standard Brownian motion B_t has values in $\mathbb{R}$, the cylindrical Brownian motion Y_t is distribution-valued: for fixed time, its spatial covariance is $t \cdot \delta_{x-y} \in H^{-\frac{d}{2}-\epsilon}(\mathbb{T}^d)$. This makes the choice of spatial covariance Q_t in front of Y_t very important in applications. For example, while $Y_t \in H^{-\frac{d}{2}-\epsilon}(\mathbb{T}^d)$, if $Q_t^2 = e^{-t(1-\Delta)}$, then $Z_t = \int_0^t Q_t dY_t \in H^k(\mathbb{T}^d)$ for all k (its Fourier coefficients decay exponentially fast almost surely). This extra boost in regularity can make a big difference.

Example 2.4. To demonstrate the previous remark, suppose $F_t(X_t) = MX_t$ for some $M \in H^{\frac{d}{2}}(\mathbb{T}^d)$. If we call $Z_t = \int_0^t F_t(X_t)dt = \int_0^t M(Z_s + Y_s)ds$, for (2.1) to be well-posed, we need MY_t be well-defined; this can only happen when their regularities add to something positive. For $Q_s = 1$, we're out of luck: almost surely, their regularities sum to $\frac{d}{2} - \frac{d}{2} - \epsilon < 0$. For $Q_t^2 = e^{-t(1-\Delta)}$, we have a chance. Obviously, the solutions to the two different SDEs won't be the same, but the point is that it's easier to get to the starting line with a covariance decomposition.

Remark 2.5. One can also can use a basis free definition of a cylindrical Brownian motion by asking for a mean-zero Gaussian stochastic process with the covariance $\mathbb{E}[Y_t(x)Y_s(y)]$ of Y given above. From this definition there are many ways to construct a cylindrical Brownian motion. The essential property of $e^{i\langle n,x\rangle}$ used above was that it forms an $L^2(\mathbb{T}^d)$ orthonormal basis. Any other $L^2(\mathbb{T}^d)$ basis would have done the job, but one should choose a basis that plays nicely with $(\Delta - m^2)^{-1}$ or whichever covariance you have on hand (see section 3 for an example).

2.2 Heat kernel decomposition

Proposition 2.6. *If a Gaussian stochastic process X_t has covariance kernel*

$$C_t = \int_0^t e^{s(\Delta - m^2)} ds$$

then X_t is a scale decomposition for the GFF with mass m, called the heat kernel scale decomposition.

Proof. We can write

$$C_t(x) = \int_0^t e^{s(\Delta - m^2)}(x)ds = \int_0^t \sum_{n\in\mathbb{Z}^d} e^{-t(|n|^2+1)} e^{inx} ds$$

Then

$$\sup_{x\in\mathbb{T}^d} |C_{t+\epsilon}(x) - C_{t-\epsilon}(x)| = |\int_{t-\epsilon}^{t+\epsilon} \sum_{n\in\mathbb{Z}^d} e^{-s(|n|^2+1)} ds| \leq \sum_{n\in\mathbb{Z}^d} \frac{1}{|n|^2+1} e^{-s(|n|^2+1)} \left| e^{\epsilon(|n|^2+1)} - e^{-\epsilon(|n|^2+1)} \right|$$

Everything on the right-hand side is summable and positive, and each term of the sum goes to zero as $\epsilon \to 0$, so $|C_t - C_s|_\infty \to 0$ as $|t-s| \to 0$. Given $f \in L^2(\mathbb{T}^d)$, we then have

$$\|C_t f - C_s f\|_{H^1(\mathbb{T}^d)} \leq |C_t - C_s|_\infty \|f\|_{H^1(\mathbb{T}^d)} \to 0$$

□

2.3 C_c^∞ decomposition

The essential property of the heat kernel $e^{t\Delta}$ is that it acts on Fourier coefficients through multiplication by $e^{-t|n|^2}$; from this we see $e^{t\Delta} f$ has exponentially decaying Fourier coefficients, so it is smooth for any $t > 0$. A natural generalization is to construct a covariance decomposition through Fourier multipliers $\widehat{C}_t(n)$ such that $\widehat{C}_t(n) \to \frac{1}{(1+|n|^2)}$, C_t is differentiable, and $\widehat{\dot{C}}_t(n) \geq 0$ for all t. This would make

$$C_t f(x) := \sum_n e^{i\langle n,x\rangle} \widehat{C}_t(n) \widehat{f}(n)$$

a covariance decomposition to be used in a scale decomposition; the condition $\widehat{C}_t(n) \geq 0$ ensures we can take a square root, so setting

$$Q_t f = \sum_n e^{i\langle n,x\rangle} \widehat{\dot{C}}_t(n)^{1/2} \widehat{f}(n)$$

we can explicitly write

$$\int_0^t Q_s dY_s = \sum_{n\in\mathbb{Z}^d} e^{i\langle n,x\rangle} \int_0^t \widehat{\dot{C}}_s(n)^{1/2} dB_s^n$$

Definition 2.7. *Let $\chi(x)$ be a smooth bump function supported on $B(0,1)$ with $\chi(0) = 0$. Let $\widehat{C_t}(n) = \chi^2(n/t)$ so that $\widehat{Q}_t(n) = \sqrt{\frac{d}{dt}\chi^2(n/t)}$. A Gaussian stochastic process with covariance kernel*

$$C_t(x) = \int_0^t Q_s^2 ds = \sum_n \frac{1}{m^2 + |n|^2} e^{i\langle n,x\rangle} \widehat{C}_t(n)$$

is called a $C_c^\infty(\mathbb{T}^d)$ scale decomposition of the GFF with mass m.

Proposition 2.8. *With Q_t as in the previous definition, the process*

$$X_t = \int_0^t Q_t dY_t$$

is a C_c^∞ scale decomposition.

Proof. Since $\chi^2(n/t) \geq 0$ and $\frac{1}{1+|n|^2}\chi^2(n/t) \uparrow \frac{1}{1+|n|^2}$ for each n, this follows from the Itô isometry. □

3 Scale decompositions for a different L^2 basis

Remember that we started with the potential H in (1.1), and grouped the gradient and square term with the Lebesgue measure to form a GFF. Let's consider a different energy H, this time defined on $L^2(\mathbb{R})$ with values in $\mathbb{C}$ rather than $L^2(\mathbb{T}^d)$ with values in $\mathbb{R}$,

$$H(\phi) = \int_{\mathbb{R}} |\phi(x)|^4 dx + \int_{\mathbb{R}} |\nabla\phi(x)|^2 + |x|^2|\phi(x)|^2 dx.$$

Our trick of absorbing the square terms into a Fourier decomposition isn't going to work: since the functions ϕ are defined over $\mathbb{R}$, the Fourier series turns into the Fourier transform, and we no longer have a simple decomposition by sums of iid Gaussians.

The problem is that the Fourier isometry doesn't respect the operator $\Delta - |x|^2$. We should instead look for a basis of $L^2(\mathbb{R})$ composed of eigenfuctions of $\Delta - |x|^2$. In general, this is a fool's errand, but $\Delta - |x|^2$ is very nice (compact resolvent, discrete spectrum), so we immediately have an orthogonal decomposition of $L^2(\mathbb{R})$ by eigenfunctions $\{H_n, iH_n\}$ (these are the Hermite polynomials, multiplied by $e^{-\frac{x^2}{2}}$, which makes them a basis with respect to the Lebesgue measure instead of the Gaussian measure, see [11] [12]). Immediately we can write down a GFF-like field,

$$X(x) = \sum_{n\in\mathbb{N}} \frac{1}{\lambda_n} g^n H_n(x)$$

where g^n are iid complex standard Gaussians. This time, since $\lambda_n = \sqrt{2n+1}$,

$$\mathbb{E}[\|X\|_2^2] = \sum_{n\in\mathbb{N}} \frac{1}{\lambda_n^2} \sim \sum_{n=1}^{\infty} \frac{1}{n}$$

which is a logarithmic divergence. So even in dimension $d = 1$ we need to renormalize small-scale divergences (for the standard GFF, this only happened when $d \geq 2$).

Anyway, from the $\{H_n, iH_n\}$ basis we can construct the cylindrical Brownian motion as before,

$$Y_t(x) = \sum_{n\in\mathbb{N}} B_t^n H_n(x).$$

where B_t^n are iid complex Brownian motions (hidden by B_t^n are the two basis elements H_n and iH_n). Then dY_t is again a space-time white noise,

$$\mathbb{E}[\langle dY_t, f\rangle \overline{\langle dY_t, g\rangle}] = \langle f, g\rangle_{L^2(\mathbb{R})}$$

so we can write down SDEs,

$$dX_t = F_t dt + C_t dY_t.$$

Now we can look for covariance decompositions that play nicely with the basis H_n. One option is to use the C_c^∞ bump functions χ_t^2:

$$X_t = \int Q_t dY_t = \sum_{n\in\mathbb{N}} \frac{1}{\lambda_n^{1/2}} H_n(x) \int \sqrt{\frac{d}{dt}\chi_t^2(n)}.dB_t^n$$

Then

$$\mathbb{E}[X_t(x)\overline{X_t(y)}] \to \sum_n \frac{1}{\lambda_n} H_n(x)\overline{H_n(y)},$$

which is precisely $(|x|^2 - \Delta)^{-1}(x, y)$. We could also use the heat kernel decomposition

$$C_t = \int_0^t e^{s(-|x|^2+\Delta)} ds$$

where $e^{s(-|x|^2+\Delta)} f = \sum_n e^{-s\lambda_n} \langle f, H_n\rangle_{L^2(\mathbb{R})} H_n$.

4 Brief Takeaway

So we've reduced the problem of understanding $d\mu = \frac{1}{Z}e^{-H(\phi)}d\phi$ to understanding $\mathbb{E}[e^{-V(X_\infty)}]$. As X_t is a function of a cylindrical Brownian motion, we still have the Boue-Dupuis formula (1.2) to understand exponential functionals $\mathbb{E}[e^{-F(X_t)}]$, giving a starting point for defining the partition function of the system. Alternatively, we could try the Polchinski flow approach. Either way, we're in a good spot to start doing work.

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