(Liam P., 2025-01-06, modified 2025-02-01)

1 Scale Decompositions for EQFT

The goal of this note is to clearly demonstrate the similarities between two different scale decompositions of the Gaussian Free Field that appear in recent works on Euclidean quantum field theory (see references). There's also lots of missing 2π's—please forgive me.

1.1 Brief motivating example: infinite-dimensional Gibbs measures

Formally, the ϕ^4_d measure on the torus $\mathbb{T}^d$ is a Gibbs measure

$$d\mu = \frac{1}{Z} e^{-H(\phi)} \prod_{x \in \mathbb{T}^d} d\phi_x \tag{1.1}$$

with

$$H(\phi) = \int_{\mathbb{T}^d} \phi^4 dx + \int_{\mathbb{T}^d} |\nabla \phi|^2 + \phi^2 dx$$

Of course, this doesn't make any sense. The worst offender is the non-existent infinite-dimensional Lebesgue measure $\Pi_{x\in\mathbb{T}^d}\phi_x$, where a Lebesgue measure is somehow attached to each point on the torus. Solving this problem is well-understood (albeit, not yet by me...); we can group the quadratic terms of H with this Lebesgue measure to form the Gaussian Free Field (GFF).

$$d\mu = \frac{1}{Z} e^{-V(\phi)} d\theta(\phi)$$

where θ is the law of a GFF (to be described), and V has the left-over terms of H.

The point: a very convenient way to construct the measure (1.1) is to write down explicitly the GFF $X \sim \theta$ and then understand expectations of the tilted measure,

$$\mathbb{E}_\mu[f] = \mathbb{E}[f(X) e^{-V(X)}].$$

A (very effective) recent approach to compute expectations under this tilted measure is to represent these expectations as minimizers of a related stochastic control problem (first implemented in [1], followed by [3] [4] [10] and many more). For example, we can formally compute the normalization constant $Z = \mathbb{E}[e^{-V(X)}]$ by a formula of Boue and Dupuis [9]

$$-\log \mathbb{E}[e^{-V(X_\infty)}] = \inf_v \mathbb{E}[V(X_\infty + \int_0^\infty Q_s v_s ds) + \frac{1}{2}\int_0^\infty \|v_s\|_{L^2} ds] \tag{1.2}$$

where X_t is a scale decomposition of X by $X_t = \int_0^t Q_s dY_s$ (to be described, think of $X_t = \int_0^t \sigma_s dB_s$ in $\mathbb{R}^d$), and v_t is a stochastic process with values in $H^1(\mathbb{T}^d)$, called a "control", that is adapted to X_t. Modifications of this control problem can be used to compute $\mathbb{E}[f(X)e^{-V(X)}]$ for nice f (see [3], [2], [10] for this approach).

Independently, scale decompositions were used to explicitly construct the measure μ; this is approach is dubbed the "Polchinski flow" approach (see [5] [7] [8] [6] [4]).

Hopefully this gives an indication that scale decompositions X_t of the GFF X have become an important tool in Euclidean quantum field theory, which justifies the brief explanation given by this note.

1.2 The Gaussian Free Field on $\mathbb{T}^3$

Let's try to group together the quadratic terms $\int_{\mathbb{T}^3} |\nabla\phi|^2 + \phi^2 dx$ ("Kinetic energy" and "mass") with $\prod_{x\in\mathbb{T}^3} d\phi_x$. Using the Fourier isometry, we see for ϕ nice enough,

$$\exp\left(-\int_{\mathbb{T}^3} |\nabla\phi|^2 + \phi^2\right) \prod_x d\phi_{x\in\mathbb{T}^3} \simeq \exp\left(-\sum_{n\in\mathbb{Z}^3} (1+|n|^2)\widehat{\phi}^2(n)\right) \prod_{n\in\mathbb{Z}^3} d\widehat{\phi}_n,$$

where $\simeq$ is used since the left-hand side doesn't make any sense. But the right-hand side is formally the law of a sum of countably many independent Gaussians, each with variance $(1+|n|^2)^{-1}$.

Definition 1.1. *A Gaussian Free Field X with mass m on the torus $\mathbb{T}^d$ is a Gaussian process with covariance given formally by*

$$\mathbb{E}[X(x)X(y)] = (m^2 - \Delta)^{-1}(x-y). \tag{1.3}$$

Defining something formally doesn't seem helpful, but notice that

$$X(x) = \sum_{n\in\mathbb{Z}^d} e^{i\langle n,x\rangle}(1+|n|^2)^{-\frac{1}{2}} g^n, \tag{1.4}$$

where $g^n = \frac{1}{\sqrt{2}}(g_1^n + i g_2^n)$ are iid complex Gaussians with the constraint $\overline{g^n} = g^{-n}$, is a GFF with mass $m = 1$. Indeed, since $\mathbb{E}[g^n g^m] = \delta_{m=-n}$,

$$\mathbb{E}[X(x)X(y)] = \sum_{n\in\mathbb{Z}^d} e^{i\langle n,x-y\rangle}(1+|n|^2)^{-1} = (1-\Delta)^{-1}(x-y)$$

Remark 1.2. In $d \geq 2$, X is almost surely not a function, so this really is a formal definition. To get a feel for why this is the case, we can return to the previous argument and plug into the Fourier expansion $x = y$. We then see $(m^2-\Delta)^{-1}(0) = \infty$ in $d \geq 2$, so that $\mathbb{E}[X(x)^2] = \infty$ for all $x \in \mathbb{T}^d$. This indicates there will be some serious problems with "evaluating at a point."

However: (1.3) can be integrated against $f(x), g(y) \in L^2(\mathbb{T}^d)$ so X can be defined as a Gaussian process indexed by functions:

$$\mathbb{E}[\langle X,f\rangle \langle X,g\rangle] = \int_{\mathbb{T}^d\times\mathbb{T}^d} f(x)(-\Delta+m^2)^{-1}(x-y)g(y)dxdy.$$

With this interpretation of X, f is sometimes called an "observable," since all we can do is probe X with a function f , "observing" a Gaussian random variable $\langle X, f\rangle$.

2 Scale Decompositions

Definition 2.1. *A scale decomposition of the GFF with mass m is a Gaussian stochastic process X_t such that $X_\infty = X$, $\mathbb{E}[X_t(x)X_t(y)] = C_t(x,y)$ is a positive-definite kernel, continuous in time, and increasing to $(m^2-\Delta)^{-1}(x-y)$ in the sense of positive linear operators.*

The name of the game is to explicitly construct these scale decompositions. We can take a hint from finite-dimensional Brownian motion; if X_t satisfies

$$X_t = \int_0^t Q_s dB_s$$

where Q_s is a positive definite matrix for each s, then $\{X_t(i)\}_{i=1}^n$ is a Gaussian vector with covariance $E[X_t(i)X_t(j)] = \int_0^t Q_s(i,j)^2 ds$ by the Itô isometry.

2.1 Cylindrical Brownian Motion

One way to construct white noise is to form the random Fourier series,

$$W_t = \sum_{n\in\mathbb{Z}} g^n e^{i\langle n,t\rangle}$$

where g^n are iid complex Gaussians as before. Brownian motion is obtained by "integrating white noise",

$$B_t = \left\langle W_\cdot, 1_{[0,t]}(\cdot)\right\rangle_{L^2(\mathbb{R})} = \sum_{n\in\mathbb{Z}} g^n \left\langle e^{i\langle n,\cdot\rangle}, 1_{[0,t]}(\cdot)\right\rangle_{L^2(\mathbb{R})}$$

Right now, $B_t \in \mathbb{R}$ for fixed t; we're going to look for a Gaussian process based on this construction with values in $H^{-\alpha}$.

To (formally) evaluate white noise at a particular time, we plug t into the Fourier series. If we make the Fourier series over $\mathbb{Z}^d$ instead of $\mathbb{Z}$, we would have a white noise in d dimensions. Re-labeling t to x, this gives "white noise in space:"

$$W(x) = \sum_{n \in \mathbb{Z}^d} g^n e^{i\langle n,x\rangle}.$$

Indeed, $\mathbb{E}[W(x)W(y)] = \sum_{n\in\mathbb{Z}^d} e^{i\langle n,x-y\rangle} = \delta(x-y)$, and $\{W_x\}_{x\in\mathbb{T}^d}$ is formally a random distribution on $\mathbb{T}^d$. To make this process time-dependent, we can turn the complex Gaussians g^n into complex Brownian motions B_t^n with the same constraint $\overline{B_t^n} = B_t^{-n}$,

$$Y_t(x) = \sum_{n\in\mathbb{Z}^d} B_t^n e^{i\langle n,x\rangle}.$$

Then $dY_t = \sum_n dB_t^n e^{i\langle n,x\rangle}$ plays the role of dB_t in standard SDEs; the spatial dimensions are thought of as fixed, so stochastic processes increments dY_t have values in a function space, and SDEs $dX_t = F_t dt + Q_t dY_t$ formally represent the integral equation

$$X_t = X_0 + \int_0^t F_s ds + \int_0^t Q_s dY_s \tag{2.1}$$

but where $\int dt$ yields a function or distribution rather than a number, and Q_t is a positive linear operator rather than a matrix or number.

Remark 2.2. One massive challenge introduced is, while $B_t \in \mathbb{R}$, Y_t is only a distribution, and a very rough one at that: for fixed time, its spatial covariance is $t \cdot \delta_{x-y} \in H^{-\frac{d}{2}-\epsilon}$. This makes the choice of spatial covariance Q_t in front of Y_t very important in applications. For example, while $Y_t \in H^{-\frac{d}{2}-\epsilon}$, if $Q_t^2 = \frac{d}{dt}C_t = e^{-t(1-\Delta)}$, then $Z_t = \int_0^t Q_t dY_t \in H^k$ for all k (its Fourier coefficients decay exponentially fast almost surely). This extra boost in regularity can make a big difference.

Example 2.3. To demonstrate the previous remark, suppose $F_t(X_t) = MX_t$ for some $M \in H^{\frac{d}{2}}(\mathbb{T}^d)$. If we call $Z_t = \int_0^t F_t(X_t)dt = \int_0^t M(Z_s + Y_s)ds$, for (2.1) to be well-posed, we need MY_t be well-defined; this can only happen when their regularities add to something positive. For $Q_s = 1$, we're out of luck: almost surely, their regularities sum to $\frac{d}{2} - \frac{d}{2} - \epsilon < 0$. For $Q_t^2 = e^{-t(1-\Delta)}$, we have a chance. Obviously, the solutions to the two different SDEs won't be the same, but the point is that it's easier to get to the starting line with a covariance decomposition.

2.2 Heat kernel decomposition

As a different construction of the GFF, set $X_\infty = \int_0^\infty Q_t dY_t$ where $Q_t^2 = C_t = \int_0^t e^{s(\Delta - m^2)}ds$. Then X_∞ has covariance $\int_0^\infty Q_t^2 dt = (1-\Delta)^{-1}$, so X_∞ must be a GFF if it exists. Thus the construction and analytic properties of the GFF can be understood by studying (2.1). This way of constructing a GFF is qualitatively similar to performing a Fourier cutoff, and progressively restoring higher frequencies:

$$X_\infty(x) = \lim_{K\to\infty} \sum_{n\in\mathbb{Z}^d\cap[-K,K]^d} e^{i\langle n,x\rangle}(1+|n|^2)^{-\frac{1}{2}} g^n = \lim_{K\to\infty}\sum_{n\in\mathbb{Z}^d} \chi_K(n)(1+|n|^2)^{-\frac{1}{2}} e^{i\langle n,x\rangle} g^n$$

where $\chi_K(n) = 1_{[-K,K]^d}(n)$.

Proposition 2.4. *If a Gaussian stochastic process X_t has covariance kernel*

$$C_t = \int_0^t e^{s(\Delta - m^2)} ds$$

then X_t is a scale decomposition for the GFF with mass m.

Proof. We can write

$$C_t(x) = \int_0^t e^{s(\Delta - m^2)}(x)ds = \int_0^t \sum_{n\in\mathbb{Z}^d} e^{-t(|n|^2+1)} e^{inx} ds$$

Then

$$\sup_{x\in\mathbb{T}^d} |C_{t+\epsilon}(x) - C_{t-\epsilon}(x)| = |\int_{t-\epsilon}^{t+\epsilon} \sum_{n\in\mathbb{Z}^d} e^{-s(|n|^2+1)} ds| \le \sum_{n\in\mathbb{Z}^d} \frac{1}{|n|^2+1} e^{-s(|n|^2+1)} \left| e^{\epsilon(|n|^2+1)} - e^{-\epsilon(|n|^2+1)} \right|$$

Everything on the right-hand side is summable and positive, and each term of the sum goes to zero as $\epsilon \to 0$, so $|C_t - C_s|_\infty \to 0$ as $|t-s| \to 0$. Given $f \in L^2(\mathbb{T}^d)$, we then have

$$\|C_t f - C_s f\|_{H^1(\mathbb{T}^d)} \le |C_t - C_s|_\infty \|f\|_{H^1(\mathbb{T}^d)} \to 0$$

□

2.3 C_c^∞ decomposition

The essential property of the heat kernel $e^{t\Delta}$ is that it acts on Fourier coefficients through multiplication by $e^{-t|n|^2}$; from this we see $e^{t\Delta}f$ has exponentially decaying Fourier coefficients, so it is smooth for any $t > 0$. A natural generalization is to construct a covariance decomposition through Fourier multipliers $\widehat{C}_t(n)$ such that $\widehat{C}_t(n) \to \frac{1}{(1+|n|^2)}$, C_t is differentiable, and $\widehat{\dot{C}}_t(n) \ge 0$ for all t. This would make

$$C_t f(x) := \sum_n e^{i\langle n,x\rangle} \widehat{C}_t(n) \widehat{f}(n)$$

a covariance decomposition to be used in a scale decomposition; the condition $\widehat{C}_t(n) \ge 0$ ensures we can take a square root, so setting

$$Q_t f = \sum_n e^{i\langle n,x\rangle} \widehat{\dot{C}}_t(n)^{1/2} \widehat{f}(n)$$

we can explicitly write

$$\int_0^t Q_s dY_s = \sum_{n\in\mathbb{Z}^d} e^{i\langle n,x\rangle} \int_0^t \widehat{\dot{C}}_s(n)^{1/2} dB_s^n$$

Definition 2.5. *Let $\chi(x)$ be a smooth bump function supported on $B(0,1)$ with $\chi(0) = 0$. Let $\widehat{C_t}(n) = \chi^2(n/t)$ so that $\widehat{Q}_t(n) = \sqrt{\frac{d}{dt}\chi^2(n/t)}$. A Gaussian stochastic process with covariance kernel*

$$C_t(x) = \int_0^t Q_s^2 ds = \sum_n \frac{1}{m^2+|n|^2} e^{i\langle n,x\rangle} \widehat{C}_t(n)$$

is called a $C_c^\infty(\mathbb{T}^d)$ scale decomposition of the GFF with mass m.

Proposition 2.6. *With Q_t as in the previous definition, the process*

$$X_t = \int_0^t Q_t dY_t$$

is a C_c^∞ scale decomposition.

Proof. Since $\chi^2(n/t) \ge 0$ and $\frac{1}{1+|n|^2}\chi^2(n/t) \uparrow \frac{1}{1+|n|^2}$ for each n, this follows from the Itô isometry. □

3 Scale decompositions for a different L^2 basis

Remember that we started with the potential H in (1.1), and grouped the gradient and square term with the Lebesgue measure to form a GFF. If instead we started with a different potential

$$H(\phi) = \int_{\mathbb{R}} |\phi|^4 dx + \int_{\mathbb{R}} |\nabla \phi|^2 + |x|^2 |\phi|^2 dx$$

the Fourier basis isn't as useful since we can only absorb the kinetic energy term into the GFF measure. But we can still run a similar argument to handle $|x|^2|\phi|^2$: first, we find a Gaussian field X with spatial covariance

$$\mathbb{E}[X(x)X(y)] = (|x|^2 - \Delta)^{-1}(x, y).$$

We can do this by emulating the construction of the GFF; since $|x|^2 - \Delta$ is very nice (compact resolvent, discrete spectrum), we have an orthogonal decomposition of eigenfunctions $\{H_n, iH_n\}_{n\in\mathbb{N}}$ which are orthogonal with respect to the measure $e^{-\frac{1}{2}x^2}dx$. Form the cylindrical Brownian motion as before by writing

$$Y_t(x) = \sum_{n\in\mathbb{N}} B_t^n H_n(x).$$

where B_t^n are iid complex Brownian motions (hidden by B_t^n are the two basis elements H_n and iH_n). Then dY_t is a space-time white noise for the measure $d\gamma = e^{-\frac{1}{2}x^2}dx$, i.e.,

$$\mathbb{E}[\langle dY_t, f\rangle \overline{\langle dY_t, g\rangle}] = \langle f, g\rangle_{L^2(\gamma)}$$

so we can write down SDEs,

$$dX_t = F_t dt + C_t dY_t.$$

Again, we can look for covariance decompositions that play nicely with the basis H_n. One option is to once again use the C_c^∞ bump functions χ_t^2:

$$X_t = \int Q_t dY_t = \sum_{n\in\mathbb{N}} \frac{1}{\lambda_n^{1/2}} H_n(x) \int \sqrt{\frac{d}{dt}\chi_t^2(n)} dB_t^n$$

Then

$$\mathbb{E}[X_t(x)\overline{X_t(y)}] \to \sum_n \frac{1}{\lambda_n} H_n(x)\overline{H_n(y)},$$

but this is exactly $(-|x|^2 - \Delta)^{-1}(x, y)$. Likewise, we could use the heat kernel decomposition

$$C_t = \int_0^t e^{s(-|x|^2+\Delta)} ds$$

where $e^{s(-|x|^2+\Delta)} f = \sum_n e^{-s\lambda_n} \langle f, H_n\rangle_{L^2(\gamma)} H_n$. Since $\lambda_n = \sqrt{2n+1}$, the multiplier sequence of $e^{t(-|x|^2+\Delta)}$ has decay $\lesssim e^{-t\sqrt{n}}$ with n, which isn't nearly as fast as the $\sim e^{-t|n|^2}$ decay of $e^{t\Delta}$.

4 Brief Takeaway

So we've reduced the problem of understanding $d\mu = \frac{1}{Z}e^{-H(\phi)}d\phi$ to understanding $\mathbb{E}[e^{-V(X_\infty)}]$. As X_t is a function of a cylindrical Brownian motion, we still have the Boue-Dupuis formula (1.2) to understand exponential functionals $\mathbb{E}[e^{-F(X_t)}]$, giving a starting point for defining the partition function of the system. Alternatively, we could try the Polchinski flow approach. Either way, we're in a good spot to start doing work.

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