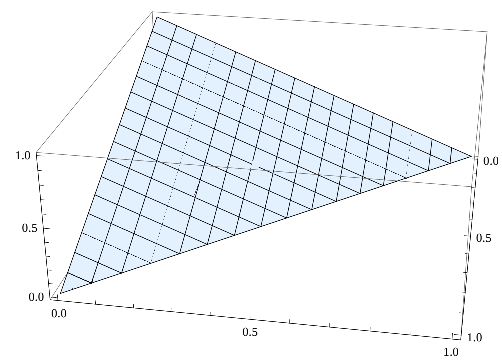
Tuesday, April 10 $\quad*\quad$ **Solutions** $\quad*\quad$ *Surface Parameterpalooza*

1. Let S be the portion of the plane $x+y+z=1$ which lies in the positive octant.

   (a) Draw a picture of S.

   Solution. The picture is shown below.



   (b) Find a parametrization $\mathbf{r}\colon D \to S$, being sure to clearly indicate the domain D. Check your answer with the instructor.

   Solution. One can use the parametrization $\mathbf{r}(u,v) = (u,\ v,\ 1-u-v)$ with the domain D given by $D = \{(u,v) : 0 \le u \le 1,\ 0 \le v \le 1-u\}$.

   (c) Use your answer in (b) to compute the area of S via an integral over D.

   Solution. Using the parametrization in (b), one gets

$$\mathbf{r}_u = (1,0,-1), \quad \mathbf{r}_v = (0,1,-1),$$

   so $\mathbf{r}_u \times \mathbf{r}_v = (1,1,1)$, and $\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{3}$. Hence the area of S is

$$\iint_D dS = \int_0^1 \int_0^{1-u} \|\mathbf{r}_u \times \mathbf{r}_v\|\, dv\, du = \frac{\sqrt{3}}{2}.$$

   (d) Check your answer in (c) using only things you learned in the first few weeks of this class.

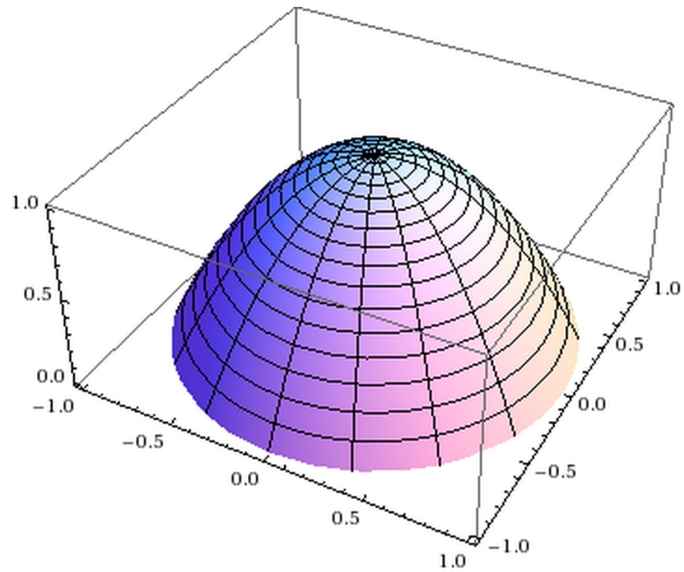
   Solution. The picture of S is a triangle with vertices $A=(1,0,0)$, $B=(0,1,0)$ and $C=(0,0,1)$. Thus $\overrightarrow{AB} = (-1,1,0)$ and $\overrightarrow{AC} = (-1,0,1)$, and the area is

$$\frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{\sqrt{3}}{2}.$$

2. Consider the surface S which is the part of $z + x^2 + y^2 = 1$ where $z \geq 0$.

   (a) Draw a picture of S.

   Solution. The picture is shown below.



   (b) Find a parametrization $\mathbf{r}\colon D \to S$. Check your answer with the instructor.

   Solution. One can use the parametrization $\mathbf{r}(r,\theta) = (r\cos\theta,\ r\sin\theta,\ 1-r^2)$ with the domain $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$.

3. Let S be the surface given by the following parametrization. Let $D = [-1,1] \times [0,2\pi]$ and define

$$\mathbf{r}(u,v) = (u\cos v, u\sin v, v).$$

   (a) Consider the vertical line segment $L = \{u = 0\}$ in D. Describe geometrically the image of L under $\mathbf{r}$.

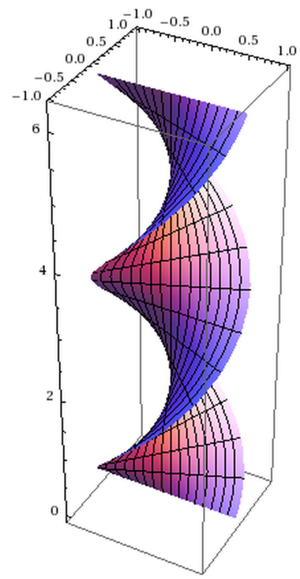
   Solution. The image of $u = 0$ under $\mathbf{r}$ is a line segment $(0,\ 0,\ v)$ where $0 \leq v \leq 2\pi$.

   (b) Repeat for the vertical segments where $u = -1$ and $u = 1$.

   Solution. When $u = 1$, the image $\mathbf{r}(1,v) = (\cos v,\ \sin v,\ v)$ is a helix with $0 \leq v \leq 2\pi$, and so is $u = -1$. Thus the images of $u = 1$ and $u = -1$ form the double helix.

   (c) Use your answers in (a) and (b) to make a sketch of S.

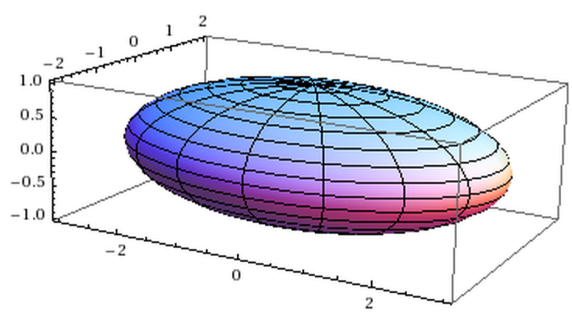
   Solution. The picture is shown below.



4. Consider the ellipsoid E given by $\dfrac{x^2}{9} + \dfrac{y^2}{4} + z^2 = 1$.

   (a) Draw a picture of E.

   Solution. The picture is shown below.



   (b) Find a parametrization of E. Hint: Find a transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ which takes the unit sphere S to E, and combine that with our existing parametrization of the plain sphere S.

   Solution. One can use the following parametrization

$$\mathbf{r}(\phi,\theta) = (3\sin\phi\cos\theta,\ 2\sin\phi\sin\theta,\ \cos\phi)$$

   with the domain $0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi$.