Group: ______________ ______________ ______________ Name: Solutions.

Math 231 A. Fall, 2015. Worksheet 14. 10/29/15

1. In this problem we find an approximation for $\int_0^1 x \arctan x\, dx$.

a) For each function, write down the **first four terms** of the power series. State the radius of convergence as $|x| < R$.

- $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$ when $|x| < 1$. (geom.)

- $\frac{1}{1+x^2} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ when $|-x^2| < 1$ so $|x| < 1$.

- $\arctan x = \int \frac{1}{1+x^2}\, dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\, dx = \sum_{n=0}^{\infty} (-1)^n \int x^{2n}\, dx = \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \right) + C$

$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

and since $\arctan(0) = 0$, $C = 0$, so $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ when $|x| < 1$ (integrating doesn't change R.o.C.)

- $x \arctan x = x \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2n+1}$ when $|x| < 1$ (mult. by x doesn't change R. o. C).

$x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \frac{x^8}{7} + \cdots$

b) Write down a series for $\int_0^1 x \arctan x\, dx$.

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2n+1}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int_0^1 x^{2n+2}\, dx = \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{2n+1} \cdot \frac{1}{2n+3} \right) = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots$$

c) Write down enough terms to approximate $\int_0^1 x \arctan x\, dx$ to within $\frac{1}{100}$, and show that you are correct. (Hint: your series should be alternating.) $b_n = \frac{1}{(2n+1)(2n+3)} < \frac{1}{100}$ when $n = 5$, since $\frac{1}{11\cdot 13} < \frac{1}{100}$, so $S_4 =$

$\frac{1}{3} - \frac{1}{15} + \frac{1}{35} - \frac{1}{63} + \frac{1}{99}$ will approximate $\int_0^1 x \arctan x\, dx$ to within $\frac{1}{100}$.

2. Given $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, compute the power series for

- $e^{x^2} = 1 + (x^2) + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$

- $\int e^{x^2}\, dx = \int \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}\, dx = \sum_{n=0}^{\infty} \frac{1}{n!} \int x^{2n}\, dx = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!\,(2n+1)} + C$

Is the series for $\int_0^1 e^{x^2}\, dx$ an alternating series? No, it's

$\sum_{n=0}^{\infty} \frac{1}{n!\,(2n+1)}$, not alternating.

3. Augustin-Jean Fresnel (1788-1827) was an engineer, mathematician and the French commissioner of lighthouses. He is famous for his work in optics and for developing the Fresnel lens. Originally developed for lighthouses, Fresnel lenses are still used today in many consumer items including computer and overhead projectors. The integral

$$\int_0^1 \frac{\sin(x)}{x}dx$$

occurs in Fresnel's theory of diffraction, and is known as a Fresnel integral.

$\leftarrow \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

(a) Use the power series $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ to evaluate the Fresnel integral as an infinite series.

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}$$

$$\int \frac{\sin(x)}{x}dx = C + x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!} + C$$

$$\int_0^1 \frac{\sin(x)}{x}dx = 1 - \frac{1}{3\cdot 3!} + \frac{1}{5\cdot 5!} - \frac{1}{7\cdot 7!} + \cdots$$

(b) Estimate the Fresnel integral to within 10^{-3}.

Since: $\frac{1}{7\cdot 7!} < \frac{1}{1000}$, by the alternating series error estimate, the partial sum

$1 - \frac{1}{3\cdot 3!} + \frac{1}{5\cdot 5!}$ will estimate $\int_0^1 \frac{\sin(x)}{x}dx$ to within $\frac{1}{1000}$

4. Find a power series representation for each of the following functions. Use summation notation, and give the radius of convergence as $|x| < R$.

a) $$\frac{x}{(1+2x)^2} = \frac{-1}{2}x\left(\frac{-2}{(1+2x)^2}\right) = \frac{-1}{2}x\left(\frac{d}{dx}\frac{1}{1+2x}\right) = \frac{-1}{2}x\left(\frac{d}{dx}\sum_{n=0}^{\infty}(-2x)^n\right)$$

if $|-2x| < 1$ so $|x| < \frac{1}{2}$.

$$= \frac{-1}{2}x\sum_{n=0}^{\infty}(-2)^n \frac{d}{dx}x^n = \frac{-1}{2}x\sum_{n=1}^{\infty}(-1)^n 2^n \cdot n \cdot x^{n-1}$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1} 2^{n-1} n x^n$$

b) $$\ln(1+3x^2) = \int \frac{6x}{1+3x^2}dx = \int 6x \cdot \frac{1}{1-(-3x^2)}dx$$

$$= \int 6x \sum_{n=0}^{\infty}(-3x^2)^n dx = \int \sum_{n=0}^{\infty} 6x(-1)^n 3^n x^{2n} dx = \sum_{n=0}^{\infty}(-1)^n \cdot 2 \cdot 3^{n+1}\int x^{2n+1}dx$$

$|-3x^2| < 1 \Rightarrow |x^2| < \frac{1}{3} \Rightarrow |x| < \frac{1}{\sqrt{3}}$.

$$= \sum_{n=0}^{\infty}\frac{(-1)^n \cdot 2 \cdot 3^{n+1} x^{2n+2}}{2n+2} + C$$

and since $\ln(1+3x^2) = 0$ when $x = 0$ and the $\leftarrow$ series $= 0$ when $x = 0$ we have $0 = 0 + C$ so $C = 0$.