Group: ____________ ____________ ____________ Name: solutions

Math 231 A. Fall 2015. Worksheet 4. 9/3/15

1. Fill in the table.

Expression	Substitution	dx	Identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$dx = a\cos\theta d\theta$	$a^2 - a^2\sin^2\theta = a^2\cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$dx = a\sec^2\theta d\theta$	$a^2 + a^2\tan^2\theta = a^2\sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$dx = a\sec\theta\tan\theta d\theta$	$a^2\sec^2\theta - a^2 = a^2\tan^2\theta$

2. Evaluate the integrals using trigonometric substitution. State the necessary restriction on the angle θ.

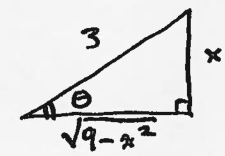
(a) $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx$ $\quad\begin{bmatrix} x = 3\sin\theta \\ dx = 3\cos\theta d\theta \end{bmatrix} = \displaystyle\int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}} \cdot 3\cos\theta d\theta$

$$= \int \frac{9\sin^2\theta \cdot 3\cos\theta}{\sqrt{9\cos^2\theta}}\,d\theta = \int \frac{27\sin^2\theta\cos\theta}{3\cos\theta}\,d\theta = \int 9\sin^2\theta\,d\theta$$

$$= 9\int \frac{1}{2}(1-\cos(2\theta))\,d\theta = \frac{9}{2}\left(\int d\theta - \int \cos(2\theta)\,d\theta\right)$$

$$= \frac{9\theta}{2} - \frac{9}{4}\sin(2\theta) + C = \frac{9\theta}{2} - \frac{9}{2}\sin\theta\cos\theta + C$$

$$= \frac{9}{2}\arcsin\left(\frac{x}{3}\right) - \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C = \boxed{\frac{9}{2}\arcsin\left(\frac{x}{3}\right) - \frac{1}{2}x\sqrt{9-x^2} + C}$$



$\theta = \arcsin\left(\frac{x}{3}\right)$

$\sin(\theta) = \frac{x}{3}$

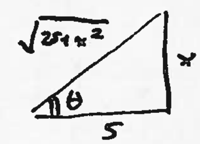
$\cos(\theta) = \frac{\sqrt{9-x^2}}{3}$

$\boxed{-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}}$

(b) $\displaystyle\int \frac{1}{\sqrt{25+x^2}}\,dx$ $\quad\begin{bmatrix} x = 5\tan\theta \\ dx = 5\sec^2\theta d\theta \end{bmatrix} = \displaystyle\int \frac{5\sec^2\theta}{\sqrt{25+25\tan^2\theta}}\,d\theta = \int \frac{5\sec^2\theta}{\sqrt{25\sec^2\theta}}\,d\theta$

$$= \int \frac{5\sec^2\theta}{5\sec\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$$ [see: WS #3]

$$= \boxed{\ln\left|\frac{\sqrt{25+x^2}}{5} + \frac{x}{5}\right| + C}$$



$\tan\theta = \frac{x}{5}$

$\sec\theta = \frac{\sqrt{25+x^2}}{5}$

$\boxed{-\frac{\pi}{2} < \theta < \frac{\pi}{2}}$

$$u = \sqrt{x^2-9}, \quad du = \frac{x}{\sqrt{x^2-9}}\,dx, \quad x^2 = u^2+9$$

3. Evaluate $\displaystyle\int \frac{x^3}{\sqrt{x^2-9}}\,dx$.

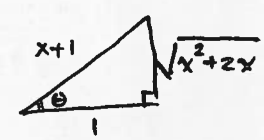
Hint: Instead of trigonometric substitution, try substituting $u = \sqrt{x^2-9}$. This trick would also work on $\int \frac{x}{\sqrt{x^2-9}}\,dx$, but would not work on $\int \frac{x^2}{\sqrt{x^2-9}}\,dx$ or $\int \frac{x^4}{\sqrt{x^2-9}}\,dx$

$$= \int \frac{x^2 \cdot x}{\sqrt{x^2-9}}\,dx = \int u^2+9\,du = \frac{1}{3}u^3 + 9u + C$$

$$= \frac{1}{3}\left(\sqrt{x^2-9}\right)^3 + 9\sqrt{x^2-9} + C$$

4. Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2+2x}}\,dx$. (Hint: Complete the square.)

$$= \int \frac{1}{\sqrt{x^2+2x+1-1}}\,dx = \int \frac{1}{\sqrt{(x+1)^2-1}}\,dx \qquad \left[\begin{array}{l} x+1 = \sec\theta \\ dx = \sec\theta\tan\theta\,d\theta \end{array}\right]$$



$\sec\theta = x+1$

$\tan\theta = \sqrt{x^2+2x}$

$0 \le \theta < \frac{\pi}{2}$

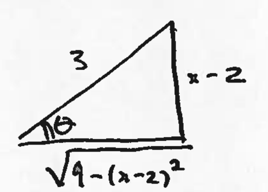
or $\pi \le \theta < \frac{3\pi}{2}$

$$= \int \frac{\sec\theta\tan\theta}{\sqrt{\sec^2\theta-1}}\,d\theta = \int \frac{\sec\theta\tan\theta}{\sqrt{\tan^2\theta}}\,d\theta = \int \sec\theta\,d\theta$$

$$= \ln\left|\sec\theta+\tan\theta\right| + C = \ln\left|x+1+\sqrt{x^2+2x}\right| + C$$

5. Evaluate $\displaystyle\int (x-2)^3\sqrt{5+4x-x^2}\,dx$.

$$= \int (x-2)^3\sqrt{9-(x-2)^2}\,dx \qquad \left[\begin{array}{l} x-2 = 3\sin\theta \\ dx = 3\cos\theta\,d\theta \end{array}\right]$$



$\cos\theta = \dfrac{\sqrt{9-(x-2)^2}}{3}$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$= \int (3\sin\theta)^3\sqrt{9-9\sin^2\theta}\;3\cos\theta\,d\theta = \int (3\sin\theta)^3\sqrt{9\cos^2\theta}\;3\cos\theta\,d\theta$$

$$= 3^5\int \sin^3(\theta)\cos^2\theta\,d\theta = 3^5\int \sin^2\theta\cos^2\theta\,\sin\theta\,d\theta$$

$$= 3^5\int (1-\cos^2\theta)\cos^2\theta\,\sin\theta\,d\theta \qquad \left[\begin{array}{l} u = \cos\theta \\ du = -\sin\theta\,d\theta \end{array}\right] = -3^5\int (1-u^2)u^2\,du$$

$$= -3^5\int u^2-u^4\,du = -3^5\left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right) + C = -3^5\left(\frac{1}{3}\cos^3(\theta) - \frac{1}{5}\cos^5(\theta)\right)$$

$$= -3^5\left(\frac{1}{3}\left(\frac{\sqrt{9-(x-2)^2}}{3}\right)^3 - \frac{1}{5}\left(\frac{\sqrt{9-(x-2)^2}}{3}\right)^5\right) + C$$