Mock Final Exam, Math 241, Spring 2018

1. (a) The plane P has normal vector $\langle -2,1,-2\rangle$ and passes through $(1,1,-1)$. Find the shortest vector $\vec{v}$ from $(2,3,5)$ to P.

   $\vec{v}$ is (circle one):

   $(A)\quad \vec{v}=\langle \frac{16}{9},\frac{-8}{9},\frac{-16}{9}\rangle;\quad (B)\quad \vec{v}=\langle \frac{-8}{3},\frac{4}{3},\frac{-8}{3}\rangle;$

   $(C)\quad \vec{v}=\langle \frac{8}{3},\frac{-4}{3},\frac{8}{3}\rangle;\quad (D)\quad \vec{v}=\langle \frac{-16}{9},\frac{8}{9},\frac{16}{9}\rangle;$

   (b) Find the area of the triangle in $\mathbb{R}^3$ which has corners at $(1,2,2)$, $(1,2,3)$, and $(1,6,6)$.

   (circle one): (A) 2; (B) $\frac{3}{2}$; (C) 1; (D) 4;

   (c) Find the volume of the parallelepiped determined by the vectors $\langle 2, 1, 3\rangle$, $\langle 1, 0, 3\rangle$, and $\langle 4, 2, 1\rangle$.

   (circle one): (A) 5; (B) 3; (C) 10; (D) 7;

2. (a) Find the limit:

$$L=\lim_{(x,y)\to(0,0)}\frac{xy}{2-\sqrt{xy+4}}$$

   (circle one): (A) $L=2$; (B) $L=4$; (C) $L=-4$; (D) DNE;

   (b) Find the limit:

$$\lim_{(x,y)\to(0,0)}\frac{7x^3y^3}{x^6+5y^6}$$

   (circle one): (A) $L=0$; (B) $L=\frac{1}{2}$; (C) $L=\frac{5}{8}$; (D) DNE;

   (c) Find the limit

$$\lim_{(x,y)\to(0,0)}\frac{2x^2y+3x^4y}{x^4+2y^3}$$

   (circle one): (A) $L=0$; (B) $L=2$; (C) $L=\frac{1}{2}$; (D) DNE;

   (d) Find the limit

$$\lim_{(x,y)\to(0,0)}\frac{5x^2y^2}{x^2+3y^2}$$

   (circle one): (A) $L=0$; (B) $L=2$; (C) $L=\frac{1}{3}$; (D) DNE;

3. Consider the surface S given by $z = f(x, y)$ where $f(x, y) = 4 + 3e^{(x^2-1)(y^2-1)}$.

(a) Find an equation of the tangent plane to the surface at the point $(1, 1, 7)$.

Equation for the tangent plane is (circle one):	
$(A)\quad z-7=3(x-1)+3(y-1);$	$(B)\quad z-7=6(x-1)+6(y-1);$
$(C)\quad z-7=0;$	$(D)\quad z-7=6(x+1)+6(y+1);$

(b) Find all five of the points on S where the tangent plane has $\langle 0, 0, 1\rangle$ as a normal vector.

The five points are:	
$(x, y, z) = (\quad,\quad,\quad)$	$(x, y, z) = (\quad,\quad,\quad)$
$(x, y, z) = (\quad,\quad,\quad)$	$(x, y, z) = (\quad,\quad,\quad)$
$(x, y, z) = (\quad,\quad,\quad)$	

4. (a) Consider the function $f(x, y) = 2x^3 - 2x^2 + 4xy + 4y^2$. Using the second derivative test, find and classify its critical points.

The critical points of $f(x, y)$ are (circle one):
$(A)\quad (0,0)$, a saddle point, and $(1, -\frac{1}{2})$, a local minimum;
$(B)\quad (0,0)$, a local minimum, and $(1, -\frac{1}{2})$, a local maximum;
$(C)\quad (0,0)$, a saddle point, and $(-1, \frac{1}{2})$, a local minimum;
$(D)\quad (0,0)$, a saddle point, and $(-1, -\frac{1}{2})$, a local maximum;

(b) In what unit direction does the maximum rate of change of $g(x, y, z) = \frac{x}{y+z}$ occur at the point $(8, 1, 3)$?

$\vec{v}$ is (circle one):	
$(A)\quad \langle \frac{1}{4}, \frac{-1}{2}, \frac{-1}{2}\rangle;$	$(B)\quad \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\rangle;$
$(C)\quad \langle \frac{1}{3}, \frac{-2}{3}, \frac{-2}{3}\rangle;$	$(D)\quad \langle \frac{-1}{3}, \frac{2}{3}, \frac{2}{3}\rangle;;$

5. Let $f(x,y,z) = 2x + 2y + 4z + 3$, and $g(x,y,z) = x^2 + y^2 + z^2 - 24$.

(a) What is the maximum value of

$$f(x,y,z)$$

subject to the constraint: $g(x,y,z) = 0$?

(circle one): (A) 24; (B) 27; (C) 11; (D) 23;

(b) What is the minimum value of

$$f(x,y,z)$$

subject to the constraint: $g(x,y,z) = 0$?

(circle one): (A) -24; (B) -27; (C) -21; (D) -14;

6. Let C be the curve of $\vec{r}(t) = \langle 3\cos t, 5\sin t, 4\cos t\rangle$ for $0 \le t \le \pi$.

(a) Evaluate $\int_C xyz\,\mathrm{d}y$.

(circle one): (A) 30; (B) 150; (C) 0; (D) 600;

(b) What is the average value of $f(x,y,z) = x^2 + y^2$ on curve C?

(circle one): (A) 85π; (B) 25; (C) 17π; (D) 17;

7. (a) Let $R = \{(x,y,z)\,|\,0 \le z \le 9 - x^2 - y^2\}$. Compute $\iiint_R 3x^2\,\mathrm{d}V$.

(circle one): (A) $\frac{81\pi}{4}$; (B) 4π; (C) $\frac{3^6\pi}{4}$; (D) $\frac{27\pi}{8}$;

(b) Let $H = \{(x,y,z)\,|\,x^2 + y^2 + z^2 \le 4,\ x \ge 0, \text{ and } y \ge 0\}$. Compute $\iiint_H 5x^2 + z\,\mathrm{d}V$.

(circle one): (A) $\frac{16\pi}{3}$; (B) $\frac{5\pi^2}{3}$; (C) $\frac{32\pi}{3}$; (D) $\frac{8\pi}{3}$;

8. (a) Let c be a constant and consider the vector field

$$\vec{F}(x,y) = \langle 2c^2(y+c) + \sin(x) - cy,\ 2cx + 2y + y^3\rangle$$

(i) For $c = 0$, $\vec{F}$ is conservative. In this case find a potential function f for $\vec{F}$.

$f(x,y) =$

(ii) Find the other value of c for which $\vec{F}$ is conservative.

(circle one): (A) $c = 0$; (B) $c = 2$; (C) $c = \dfrac{1}{3}$; (D) $c = \frac{3}{2}$;

(b) Identify whether these vector fields are conservative or not.

(i) $\vec{F}(x,y) = \langle y^2 + ye^{xy},\ 2xy + xe^{xy} \rangle$

(circle one): (A) $\vec{F}$ is conservative; (B) $\vec{F}$ is not conservative;

(ii) $\vec{G}(x,y,z) = \langle 2xyz + y + z,\ x^2 z + x + z,\ x^2 y + x + y \rangle$

(circle one): (A) $\vec{G}$ is conservative; (B) $\vec{G}$ is not conservative;

(iii) $\vec{H}(x,y,z) = \langle e^x \cos(y),\ -e^x \sin(y) + z^2 - 3z,\ 8yz \rangle$

(circle one): (A) $\vec{H}$ is conservative; (B) $\vec{H}$ is not conservative;

(iv) $\vec{J}(x,y,z) = \langle 2yz\sin(x)\cos(x),\ z - z\cos^2(x),\ y\sin^2(x) \rangle$

(circle one): (A) $\vec{J}$ is conservative; (B) $\vec{J}$ is not conservative;

(c) Consider the vector field $\vec{F}$ and the curve C parameterized by $\vec{r}(t)$ below.

$$\vec{F}(x,y,z) = \langle 2yz + 2xy^2,\ 2xz + z + 2x^2 y,\ 2xy + y \rangle,$$
$$C : \vec{r}(t) = \langle t,\ t^3 + 2,\ 2t^2 - 2t^4 \rangle, \text{ for } 0 \le t \le 1.$$

Evaluate the integral $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$.

(circle one): (A) 6; (B) 9; (C) 5; (D) 4;

9. Find the mass of the lamina that occupies the region $D = \{(x,y) \,|\, 0 \le x \le 2,\ 0 \le y \le x\}$, whose density at any point in D is $\rho(x,y) = 2y\sqrt{x^2 - y^2}$.

(circle one): (A) $\frac{4}{3}$; (B) $\frac{2}{3}$; (C) 4; (D) $\frac{8}{3}$;

10. (a) Consider the transformation $T(u,v) = (x(u,v), y(u,v)) = (v\sin u, -v\cos u)$.

i. Compute the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ of T.

(circle one): (A) u; (B) v; (C) $-u$; (D) $-v$;

ii. Let $S = \{(u,v) \,|\, 0 \le u \le \pi,\ 1 \le v \le 3\}$ and $R = T(S)$. Compute $\iint_R x^2 + y^2 \,\mathrm{d}A$.

(circle one): (A) 18π; (B) 20π; (C) 22π; (D) -18π;

(b) Let R be the parallelogram with corners at $(0,0)$, $(3,1)$, $(4,-3)$, and $(7,-2)$. Evaluate $\iint_R (3x+y), \mathrm{d}A$.

Answer =

11. (a) Find the area of the torus, with major radius 3 and minor radius 1, parameterized by

$$r(\phi,\theta) = \langle\, (3+\cos\phi)\cos\theta, (3+\cos\phi)\sin\theta\,,\, \sin\phi\rangle, \text{ for } (\phi,\theta) \in D = [0,2\pi]\times[0,2\pi]$$

Area =

(b) Let S be the cone defined by $S = \{(x,y,z) \mid z = 1-\sqrt{x^2+y^2} \ge 0\}$. Compute $\iint_S x^2\,\mathrm{d}S$.

$\iint_S x^2\,\mathrm{d}S =$

12. (a) Complete the following statement of Green's theorem:

Let C be a positively oriented, piecewise-smooth, simple, closed curve bounding a region D in $\mathbb{R}^2$, and let $\vec{F}(x,y) = \langle P(x,y),\, Q(x,y)\rangle$. If $P(x,y)$ and $Q(x,y)$ have continuous partial derivatives then:

$$\iint_{\square} \boxed{}\,\mathrm{d}A = \int_{\square} \boxed{} \cdot \mathrm{d}\vec{r}$$

(b) Suppose that D is the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$. Let C be the boundary of D, oriented counter-clockwise. Compute

$$\int_C \langle x^2, (x+1)y\rangle \cdot d\vec{r}$$

Answer =

13. (a) Complete the following statement of Stokes' theorem:

Let S be an oriented, piecewise-smooth, surface in $\mathbb{R}^3$ that is bounded by a simple, closed piecewise-smooth boundary curve C with positive orientation. If $\vec{F}(x,y,z)$ is a vector field whose components have continuous partial derivatives then:

$$\iint_{\square} \boxed{} \cdot \boxed{} = \int_{\square} \boxed{} \cdot \boxed{}$$

(b) Consider the vector field

$$\vec{F}(x,y,z) = \langle 2xy + z^2 + 1,\ 2x + x^2 + 2yz,\ y^2 + 2xz + 1\rangle$$

Compute $\int_C \vec{F}\cdot \mathrm{d}\vec{r}$, where C is the ellipse $4x^2 + y^2 = 4$ in the xy–plane which is oriented counterclockwise.

Answer =

(c) Let

$$\vec{F}(x,y,z) = \langle ze^y,\ x\cos y,\ xz\sin y\rangle,$$

and S be the hemisphere $x^2 + y^2 + z^2 = 9,\ y \geq 0$, oriented in the direction of the positive y–axis. Compute $\iint_S \operatorname{curl}(\vec{F})\cdot \mathrm{d}\vec{S}$.

Answer =

14. (a) Complete the following statement of the Divergence theorem:

Let E be a simple solid region and let S be its boundary surface with outward orientation. If $\vec{F}(x,y,z)$ is a vector field whose components have continuous partial derivatives then:

$$\iiint_{\boxed{}} \boxed{\phantom{\operatorname{div}\vec{F}\quad\quad\quad}}\ \mathrm{d}V = \iint_{\boxed{}} \boxed{\phantom{\vec{F}\ }}\cdot\boxed{\phantom{d\vec{S}}}$$

(b) Let S be the outwardly oriented boundary of the top half of the unit ball, that is,

$$S = \{(x,y,z)\,|\,x^2 + y^2 + z^2 = 1,\ \text{and } z > 0\} \cup \{(x,y,0)\,|\,x^2 + y^2 \leq 1\}$$

Compute

$$\iint_S \langle z^2 x,\ \frac{1}{3}y^3 + \cos(z),\ x^2 z + y^2\rangle \cdot \mathrm{d}\vec{S}$$

Answer =

Trigonometric Identities

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$
$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
$$\tan(\theta+\phi) = \frac{\tan(\theta)+\tan(\phi)}{1-\tan\theta\tan\phi}$$
$$\sin(\theta-\phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$$
$$\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$$
$$\tan(\theta-\phi) = \frac{\tan(\theta)-\tan(\phi)}{1+\tan\theta\tan\phi}$$
$$\sin(\theta)\sin(\phi) = \frac{\cos(\theta-\phi)-\cos(\theta+\phi)}{2}$$
$$\cos(\theta)\cos(\phi) = \frac{\cos(\theta-\phi)+\cos(\theta+\phi)}{2}$$
$$\sin(\theta)\cos(\phi) = \frac{\sin(\phi+\theta)-\sin(\phi-\theta)}{2}$$
$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$
$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$
$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$
$$\tan\theta = \frac{\sin(2\theta)}{1+\cos(2\theta)}$$