Group: ____________ ____________ ____________ Name: solutions.

Math 231 A. Fall, 2015. Worksheet 10. 10/8/15

1. Recall the fundamental geometric series

$$1 + r + r^2 + r^3 + \cdots = \begin{cases} \frac{1}{1-r} & \text{if } |r| < 1 \\ \text{Divergent} & \text{if } |r| \geq 1 \end{cases}$$

Write each of the following series in the form $a(1 + r + r^2 + r^3 + \ldots)$. Identify the value of r in each case. Find the sum of the series, or write "Diverges".

a) $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots = \frac{1}{4}\left(1 + \frac{1}{4} + \frac{1}{16} + \cdots\right) = \frac{1}{4}\left(1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots\right)$

$= \frac{1}{4}\left(\frac{1}{1-\frac{1}{4}}\right) = \frac{1}{3}.$ so: $r = \frac{1}{4}$, $|r| < 1$.

b) $\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \ldots + \frac{1}{768} + \cdots = \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right) = \frac{1}{3}\left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots\right)$

$r = \frac{1}{2}$, $|r| < 1$

$= \frac{1}{3}\left(\frac{1}{1-\frac{1}{2}}\right) = \frac{2}{3}.$

c) $\sum_{n=1}^{\infty} 5(-2)^{n-1} = 5\left(1 + (-2) + (-2)^2 + (-2)^3 + \ldots\right)$ $r = (-2)$, $|r| \geq 1$

so the series diverges

d) $\sum_{n=2}^{\infty} \frac{2^{2n-1}}{7^n} = \frac{1}{2}\sum_{n=2}^{\infty} \frac{2^{2n}}{7^n} = \frac{1}{2}\sum_{n=2}^{\infty} \frac{4^n}{7^n} = \frac{8}{49}\left(1 + \left(\frac{4}{7}\right) + \left(\frac{4}{7}\right)^2 + \ldots\right)$ $r = \frac{4}{7}$, $|r| < 1$

$= \frac{8}{49}\left(\frac{1}{1-\frac{4}{7}}\right) = \frac{8}{21}$

2. Show that the following series all diverge:

$\sum_{n=0}^{\infty} \frac{n^2}{n^2+1}$ $\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 \neq 0.$

series diverges by the divergence test.

$\sum_{n=0}^{\infty} e^{-\frac{n}{n^2+1}}$ $\lim_{n\to\infty} e^{\frac{-n}{n^2+1}} = e^{\lim_{n\to\infty}\frac{-n}{n^2+1}} = e^0 = 1 \neq 0$, series diverges by the divergence test.

$\sum_{n=0}^{\infty} \cos\left(\frac{n\pi}{2}\right)$ $\cos\left(\frac{n\pi}{2}\right) = \{1, 0, -1, 0, 1, \ldots\}$ is not a convergent sequence,

so the series diverges by the divergence test.

3. Given the partial sum $S_n = \frac{n}{n+1}$, find a_n and $\sum_{n=1}^{\infty} a_n$. (It's easy if you know the definitions.)

$S_N = a_1 + a_2 + \ldots + a_N = \frac{N}{N+1}$, so $a_N = (a_1 + \ldots + a_N) - (a_1 + \cdots + a_{N-1})$, so:

then: $a_N = S_N - S_{N-1} = \frac{N}{N+1} - \frac{N-1}{N-1+1} = \frac{1}{N^2+N}$, and

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2+n} = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{N}{N+1} = 1$

(definition)

To use the integral test, you should check that the function $f(x)$ in question is positive and decreasing. Remember that you only need to check whether or not $\int_1^\infty f(x)\,dx$ converges or diverges. **You do not have to evaluate the integral.**

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4. Use the integral test to determine if $\sum_{n=1}^{\infty} \frac{n^4}{e^n}$ converges or diverges.

$f(x) = \frac{x^4}{e^x}$ is positive,

$f'(x) = \frac{(4-x)x^3}{e^x} < 0$ for $x \geq 1$, so $f(x)$ is decreasing, and

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{x^4}{e^x}\,dx \leq \int_1^\infty \frac{x^4}{x^6}\,dx = \int_1^\infty \frac{1}{x^2}\,dx$$ converges, so $\int_1^\infty f(t)\,dt$ converges by comparison, so the series converges by the integral test.

5. Use the integral test to determine if $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ converges or diverges.

$f(x) = \frac{x}{x^3+1}$ is positive, and $f'(x) = \frac{1-2x^3}{(x^3+1)^2} < 0$ for $x \geq 1$, so f is decreasing,

and $$\int_1^\infty \frac{x}{x^3+1}\,dx \leq \int_1^\infty \frac{x}{x^3}\,dx = \int_1^\infty \frac{1}{x^2}\,dx$$ converges, so $\int_1^\infty f(t)\,dt$ converges by comparison, so the series converges by the integral test

6. Use the integral test to determine if $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$ converges or diverges.

$f(x) = \frac{1}{x\ln(x)}$ is positive, and decreasing for $x \geq 1$ since the denominator is increasing while the numerator is constant.

$$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{1}{x\ln x}\,dx = \int_{\ln(2)}^\infty \frac{1}{u}\,du$$ diverges, so the series diverges by the integral test.

7. Use the integral test to show that $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges for $p > 1$. What does problem (6) tell you about what happens when $p \leq 1$?

$f(x) = \frac{1}{x(\ln(x))^p}$ is positive and decreasing, (for large x)

if $p \neq 1$ then:

$$\int_2^\infty \frac{1}{x(\ln(x))^p}\,dx = \int_{\ln(2)}^\infty \frac{1}{u^p}\,du = \lim_{t\to\infty} \left. \frac{u^{-p+1}}{-p+1} \right|_{u=\ln(2)}^{t},$$ which diverges if $p<1$ and converges if $p>1$

$$= \lim_{t\to\infty} \frac{t^{1-p}}{1-p} - \frac{\ln(2)^{1-p}}{1-p}$$