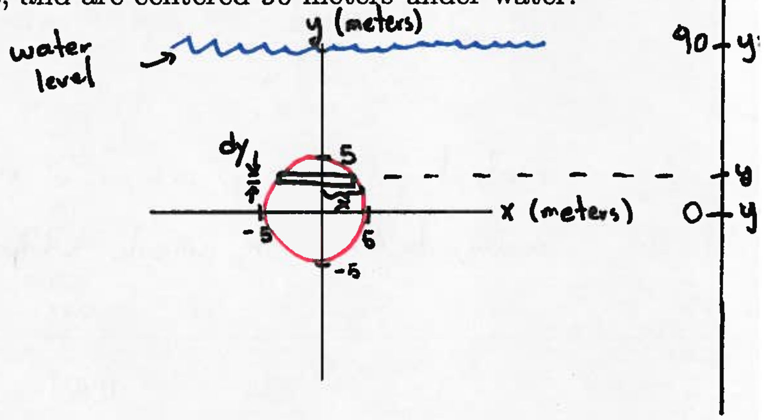
Group: ____________ ____________ ____________ Name: Solutions

Math 231 A. Fall, 2014. Worksheet 9. 10/2/14

1. The Hoover Dam near Las Vegas has "penstock gates" to control the flow of water. They are circular, approximately 5 meters in radius, and are centered 90 meters under water.

a) Make a clear diagram of this problem. Include a "ruler" to the right which clearly indicates the meaning of your coordinates.



b) Compute the hydrostatic force on on of these gates to two significant figures. Use 9.8 m/s^2 for the gravitational constant and 1000 kg/m^3 for the density of water. **Hint:** You can evaluate all integrals which arise in your head, without any hard work.

Equation of circle $x^2+y^2=25$, so $x=\sqrt{25-y^2}$

Look at strip at y: area = length x width $= (2x)\,dy = 2\sqrt{25-y^2}\,dy$

pressure $= \rho\, g\, (90-y)$

density, grav. constant, depth

force $= \rho g (90-y)\, 2\sqrt{25-y^2}\,dy$

So, total force on gate:

$$\text{Force} = \int_{-5}^{5} \rho g (90-y)\, 2\sqrt{25-y^2}\,dy$$

$$= 180\int_{-5}^{5} \rho g \sqrt{25-y^2}\,dy - \underbrace{2\int_{-5}^{5} \rho g y\sqrt{25-y^2}\,dy}_{\text{an odd function, so integral}=0}$$

$$= 180\rho g \underbrace{\int_{-5}^{5}\sqrt{25-y^2}\,dy}_{\text{area of a half-disk of radius 5}}$$

$$= 180\rho g\,\frac{25\pi}{2}\,m^2 = 180\left(1{,}000\ \tfrac{kg}{m^3}\right)\left(9.8\ \tfrac{m}{s^2}\right)\left(\tfrac{25\pi}{2}\ m^2\right) \approx \boxed{6.9\times 10^7\ N}$$

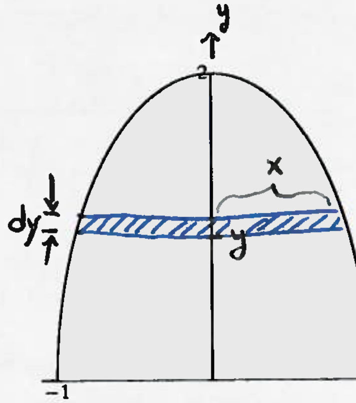
c) The mass of a loaded 747 airplane is approximately 400, 000 kg. Find the weight of a 747 in Newtons. How many 747s would it take to provide the force you computed in part (b)?

Weight = mass $\cdot$ g $= (4\times10^5\,kg)(9.8\,m/s^2)$

$\approx 3.9\times 10^6$ N

So, # of planes $= \dfrac{6.9\times10^7\ N}{3.9\times10^6\ N} \approx 17.7 \Rightarrow$ $\boxed{\text{Need 18 747 planes}}$

2. A lamina with area density λ kg/m^2 occupies the top half of the ellipse $4x^2 + y^2 = 4$ as shown. You may use the fact that the area of the lamina is π m^2. Find the moments M_x and M_y about the x and y axes, respectively. Then find the coordinates $(\overline{x}, \overline{y})$ of the centroid. You may use any available symmetries.



eqn of ellipse $x^2 + \left(\frac{y}{2}\right)^2 = 1$

First, by symmetry, $\bar{x} = 0$, so moment about y-axis, $\boxed{M_y = 0.}$

Second, to find M_x, consider strip at y:

$$\text{area} = (2x)\,dy = 2\sqrt{1-\left(\tfrac{y}{2}\right)^2}\,dy = \sqrt{4-y^2}\,dy$$

$$\text{mass} = \text{density} \times \text{area} = \lambda\sqrt{4-y^2}\,dy$$

$$dM_x = y \cdot \text{mass} = \lambda y\sqrt{4-y^2}\,dy$$

$$\text{So, } M_x = \int dM_x = \int_0^2 \lambda y\sqrt{4-y^2}\,dy \underset{\uparrow}{=} \frac{\lambda}{2}\int_0^4 u^{1/2}\,du = \boxed{\frac{8\lambda}{3} = M_x}$$

$u = 4-y^2$ substitution

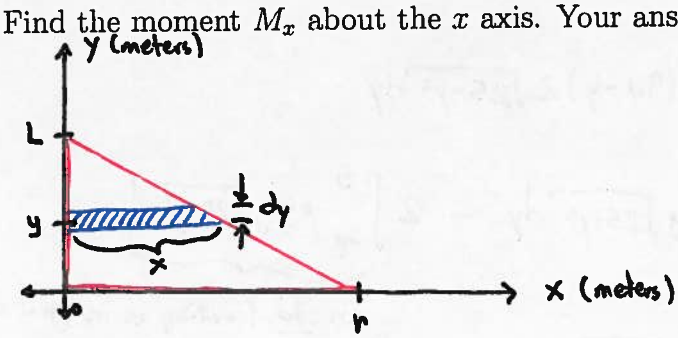
To find $\bar{y}$ need total mass, $\lambda \cdot \text{area} = \lambda\pi$. So $\bar{y} = \frac{M_x}{\text{mass}} = \frac{8\lambda/3}{\pi\lambda} = \frac{8}{3\pi}$

Then centroid $= \boxed{(\bar{x}, \bar{y}) = \left(0, \frac{8}{3\pi}\right)}$

3. A lamina has the shape of a right triangle of height L and base r (meters). The base lies along the x-axis. It has density ρ kg/m^2.

a) Make a careful diagram of the problem (like the one on the last page).

b) Find the moment M_x about the x axis. Your answer will involve ρ, L, and r.



eqn of line $y = L - \frac{L}{r}x$

To find M_x consider strip at y:

$$\text{area} = x\,dy = r\left(1 - \frac{y}{L}\right)dy$$

$$\text{mass} = \rho \cdot \text{area} = \rho r\left(1-\frac{y}{L}\right)dy$$

$$dM_x = y \cdot \text{mass} = \rho r y\left(1-\frac{y}{L}\right)dy$$

$$\text{So, } M_x = \int dM_x = \int_0^L \rho r y\left(1-\frac{y}{L}\right)dy = \rho r\int_0^L y\left(1-\frac{y}{L}\right)dy = \rho r\left(\frac{y^2}{2} - \frac{y^3}{3L}\right)\Bigg|_0^L$$

$$\boxed{= \frac{\rho r L^2}{6}}$$