MATH 231. Midterm 3. November 27, 2017.

Full Name: Solutions **Section Code (from table below):** | | |

Net ID (start of your University email): ______________________

Also fill and bubble your **last name** and middle initial and **your net id (not your UIN)** on the scantron. Once the test begins, bubble in the **test version** on your scantron. **You will lose a point for each item incorrectly filled out or missing.**

Instructions

- **Do not turn this page until instructed to.**
- You must not communicate with other students during this test.
- No written materials of any kind allowed.
- No phones, calculators, smart-watches or electronic devices of any kind allowed for any reason, including checking the time (you may use a simple wristwatch).
- **When we end the exam, you must stop writing. Stay in your seat until we collect your exams.**
- Violations of academic integrity (in other words, cheating) will be taken extremely seriously.

TA	Section, Start time, Location	TA	Section, Start time, Location
Ochoa de Alazia, Itziar	ADJ, 10:00, 1 ILLINI HALL ADK, 11:00, 140 HENRY	Huynh, Chi	ADP, 8:00, 345 ALTGELD ADV, 9:00, 1 ILLINI HALL
Kim, Hee Yeon	ADC, 10:00, 147 ALTGELD ADE, 12:00, 147 ALTGELD	Mastroeni, Matthew	ADM, 1:00, 137 HENRY BLD ADR, 12:00, 143 HENRY BLD
Shin, Brian	ADL, 12:00, 136 BURRILL ADF, 1:00, 147 ALTGELD	Shinkle, Emily	ADA, 8:00, 137 HENRY BLD ADB, 9:00, 140 HENRY BLD
Valletta, Justin	ADN, 2:00, 143 HENRY BLD ADU, 3:00, 143 HENRY BLD	Wojtalewicz, Nikolas	ADG, 2:00, 145 ALTGELD ADH, 3:00, 145 ALTGELD
Merriman, Claire	ADI, 9:00, 111 DKH	Obeidin, Malik	AD1, 3:00, 173 ALTGELD
Quan, Hadrian	AD3,1:00, 159 ALTGELD	Rasekh, Nima	AD2, 9:00, 159 ALTGELD
Zhu, Heyi	ADD, 11:00, 143 HENRY BLD ADT, 1:00, 443 ALTGELD		

Question	9	10	11	12	Total
Score					
Grader					

This is version A. Bubble in test code A now on your scantron (bottom right corner).

Mark your answers for these 8 multiple choice questions on the Scantron form. Each is worth 4 points.

(1) Consider the series $\sum_{n=1}^{\infty} \frac{(2n+1)(-2)^n}{n!}$. Which of the following is true?

A. The series diverges by the ratio test.

B. The series converges by the ratio test, but does not converge absolutely.

C. The series converges absolutely by the ratio test.

D. The series converges absolutely by the alternating series test.

E. The series diverges by the alternating series test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2}{n+1} \quad \frac{2n+3}{2n+1}\right| \to 0 \text{ as } n \to \infty$$

(2) Consider the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2+3}}$. Which of the following is true?

A. The series diverges by the ratio test.

B. The series converges by the ratio test, but does not converge absolutely.

C. The series converges absolutely by the ratio test.

D. The series converges by the alternating series test but does not converge absolutely.

E. The series diverges by the alternating series test.

- $b_n = \frac{1}{\sqrt{n^2+3}} \longrightarrow 0$, $b_{n+1} < b_n$ so converges by Alternating Series Test

- $b_n > \frac{1}{2n}$ so not absolutely convergent.

(3) Consider the series $\sum_{n=0}^{\infty}(-1)^n \frac{2n+1}{6n-3}$. Which of the following is true?

A. The series diverges by the alternating series test.

B. The series converges by the ratio test, but does not converge absolutely.

C. The series converges absolutely by the ratio test.

D. The series converges absolutely by the alternating series test.

E. The series diverges by the divergence test.

$\lim_{n \to 0} (-1)^n \left(\frac{2n+1}{6n-3}\right)$ does not exist

So diverges by divergence test.

(4) The radius of convergence of the series $\sum_{n=2}^{\infty} \frac{1}{(n-1)!}\left(\frac{x}{3}\right)^n$ is

A. $R = \infty$

B. $R = 3$

C. $R = \frac{1}{3}$

D. $R = 0$

E. $R = 1$

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{1}{n}\,\frac{x}{3}\right| \longrightarrow 0$ as $n \to \infty$ for all x

(5) The coefficient of $(x-5)^3$ in the Taylor series for $f(x) = \ln(x)$ centered at $a = 5$ is

A. $\frac{1}{50}$

B. $\frac{2}{x^3}$

C. $-\frac{1}{3x^3}$

D. $\frac{1}{375}$

E. $-\frac{1}{125}$

Need $\dfrac{f'''(5)}{3!}$

$f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, $f'''(x) = \frac{2}{x^3}$

$$\frac{f'''(5)}{3!} = \frac{2}{125} \cdot \frac{1}{6} = \frac{1}{375}$$

(6) The Maclaurin series for $(1+x)^{-5/2}$ starts as

A. $1 - \dfrac{5}{2}x + \dfrac{5}{4}x^2 + \cdots$

B. $1 + \dfrac{5}{2}x - \dfrac{35}{8}x^2 + \cdots$

C. $1 - \dfrac{5}{2}x - \dfrac{15}{8}x^2 + \cdots$

D. $1 + \dfrac{5}{2}x + \dfrac{5}{4}x^2 + \cdots$

E. $1 - \dfrac{5}{2}x + \dfrac{35}{8}x^2 + \cdots$

$$(1+x)^{-5/2} = \binom{-5/2}{0} + \binom{-5/2}{1}x + \binom{-5/2}{2}x^2 + \cdots$$

$$= 1 - 5/2\, x + \frac{(-5/2)(-7/2)}{2!}x^2 + \cdots$$

$$= 1 - 5/2\, x + \frac{35}{8}x^2 + \cdots$$

(7) Which of the partial sums **in this list** has the fewest number of terms necessary to approximate $\sum_{n=1}^{\infty} \frac{(-1)^n}{10n^2}$ with error less than 1/800?

For alternating series with sum S

$|S - S_n| < b_{n+1}$

A. $\sum_{n=1}^{3} \frac{(-1)^n}{10n^2}$

B. $\sum_{n=1}^{5} \frac{(-1)^n}{10n^2}$

Need $b_{n+1} < \frac{1}{800}$

C. $\sum_{n=1}^{8} \frac{(-1)^n}{10n^2}$

$\Rightarrow \frac{1}{10(n+1)^2} < \frac{1}{800}$

D. $\sum_{n=1}^{11} \frac{(-1)^n}{10n^2}$

E. $\sum_{n=1}^{14} \frac{(-1)^n}{10n^2}$

$\Rightarrow (n+1)^2 > 80$

$\Rightarrow n \geq 8$

(8) The Maclaurin series for $f(x) = x\cos(x^2)$ is

A. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$

$\cos x = \sum (-1)^n \frac{x^{2n}}{(2n)!}$

B. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n+2}$

$\cos x^2 = \sum (-1)^n \frac{x^{4n}}{(2n)!}$

C. $\sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{4n+1}$

$x \cos x^2 = \sum (-1)^n \frac{x^{4n+1}}{(2n)!}$

D. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+1)!} x^{4n+1}$

E. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n+1}$

The next four questions are free response. Show all your work and CIRCLE your answers.

(9) (8 points) Consider the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{3^n(7n-1)}$.

(a) Find the radius of convergence.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{1}{3}\,\frac{(7n-1)}{(7n+6)}(x-1)\right| \xrightarrow{n\to\infty} \left|\frac{x-1}{3}\right|$$

$$\left|\frac{x-1}{3}\right| < 1 \Rightarrow |x-1| < 3 \quad \text{so} \quad R=3$$

(b) Find the interval of convergence.

$$|x-1|<3 \Rightarrow -2 < x < 4$$

$\underline{x=-2}$: $$\sum \frac{(-3)^n}{3^n(7n-1)} = \sum \frac{(-1)^n}{7n-1}$$

Converges by Alt. Series Test.

$\underline{x=4}$: $$\sum \frac{3^n}{3^n(7n-1)} = \sum \frac{1}{7n-1}$$

Diverges by Comparison Test.

interval of convergence is $[-2,4)$

(10) (12 points) Consider the function $f(x) = \dfrac{1}{8+x^3}$.

(a) Find the Maclaurin series for $f(x)$. Write your answer using $\sum$ notation.

$$\frac{1}{8+x^3} = \frac{1}{8}\left(\frac{1}{1+\frac{x^3}{8}}\right) = \frac{1}{8}\sum_{n=0}^{\infty}\left(-\frac{x^3}{8}\right)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{x^{3n}}{8^{n+1}}$$

(b) Find the radius of convergence of this series.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^3}{8}\right| \qquad \text{Need } \left|\frac{x^3}{8}\right| < 1 \Rightarrow |x| < 2$$

radius of conv is 2

(c) Find the Maclaurin series for $f'(x)$. Write your answer using $\sum$ notation.

$$f'(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{8^{n+1}}\,3n\,x^{3n-1}$$

(d) Write down the first two nonzero terms of the Maclaurin series for $f'(x)$.
Your coefficients should be numbers, but you do not need to simplify the numbers.

$$\frac{-3}{8^2}x^2 + \frac{6}{8^3}x^5$$

(11) (8 points) Let $f(x) = e^{-2x}$.

(a) Find the third degree Taylor polynomial, $T_3(x)$, of $f(x)$ centered about 0.

$$e^{-2x} = \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!}$$

$$T_3(x) = 1 - 2x + \frac{4x^2}{2!} - \frac{8x^3}{3!}$$

$$= 1 - 2x + 2x^2 - \tfrac{4}{3}x^3$$

(b) Use **Taylor's Remainder theorem** to estimate the maximum error of the approximation $f(x) \approx T_3(x)$ on $[-\frac{1}{10}, \frac{1}{10}]$.

Your answer should be a number, but you do not need to simplify the number.

- $R_3(x) = \dfrac{f^{(4)}(z)}{4!} x^4$ for z b/w 0 and x
- $f^{(4)}(x) = (-2)^4 e^{-2x} = 16e^{-2x}$ which is decreasing.
- For z in $[-\frac{1}{10}, \frac{1}{10}]$, $f^{(4)}(z) \leq f^{(4)}(-\frac{1}{10}) = 16e^{\frac{1}{5}}$

So $|R_3(x)| \leq \dfrac{16e^{1/5}}{4!}\left(-\frac{1}{10}\right)^4 = \frac{2}{3}e^{1/5}\, 10^{-4}$

(12) (12 points) Let $f(x) = \sin(2x)$ and let $g(x) = \cos(x)$. Recall that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$.

(a) Find the degree five Taylor polynomial for $f(x)$ centered about 0. Please put a box around your answer.

$$2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} = 2x - \frac{8}{3!}x^3 + \frac{32}{5!}x^5$$

(b) Find the degree five Taylor polynomial for $g(x)$ centered about 0. Please put a box around your answer.

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

(c) Find the degree **three** Taylor polynomial for $h(x) = f(x)g(x)$ centered about 0. Please put a box around your answer.

$$\left(2x - \frac{8}{3!}x^3 + \frac{32}{5!}x^5 - \ldots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)$$

$$= 2x + \left(-\frac{2}{2!} - \frac{8}{3!}\right)x^3 + \ldots$$

So $T_3(x)$ for $h(x)$ is $2x - \frac{7}{3}x^3$

(d) Evaluate $\lim_{x \to 0} \frac{f(x)g(x) - 2x}{x^3}$ using Taylor approximations.

(No credit for using L'Hospital's rule.)

$$\lim_{x \to 0} \frac{f(x)g(x) - 2x}{x^3} = \lim_{x \to 0} \frac{\left(2x - \frac{7}{3}x^3 + \ldots\right) - 2x}{x^3}$$

$$= \lim_{x \to 0} -\frac{7}{3} + \ldots = -\frac{7}{3}$$