Thursday, April 19 * **Solutions** * *Parametrizations and Integrals*

1. Consider the ellipsoid with implicit equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

   (a) Parametrize this ellipsoid.

   Solution. One could use the parametrization

$$x=a\sin\phi\cos\theta,\ y=b\sin\phi\sin\theta,\ z=c\cos\phi,\quad 0\le\phi\le\pi,\ 0\le\theta\le 2\pi.$$

   (b) Set up, but do not evaluate, a double integral that computes its surface area.

   Solution. Since $\mathbf{r}(\phi,\theta)=\langle a\sin\phi\cos\theta,\ b\sin\phi\sin\theta,\ c\cos\phi\rangle$, one has

$$\mathbf{r}_\phi=\langle a\cos\phi\cos\theta,\ b\cos\phi\sin\theta,\ -c\sin\phi\rangle,\quad \mathbf{r}_\theta=\langle -a\sin\phi\sin\theta,\ b\sin\phi\cos\theta,\ 0\rangle,$$

   so

$$\mathbf{r}_\phi\times\mathbf{r}_\theta=\langle bc\sin^2\phi\cos\theta,\ ac\sin^2\phi\sin\theta,\ ab\sin\phi\cos\theta\rangle.$$

   Therefore

$$|\mathbf{r}_\phi\times\mathbf{r}_\theta|=\sqrt{b^2c^2\sin^4\phi\cos^2\theta+a^2c^2\sin^4\phi\sin^2\theta+a^2b^2\sin^2\phi\cos^2\phi},$$

   and the surface area is computed by

$$\begin{aligned}\text{Area}&=\int_0^{2\pi}\int_0^{\pi}|\mathbf{r}_\phi\times\mathbf{r}_\theta|\,d\phi d\theta\\&=\int_0^{2\pi}\int_0^{\pi}\sqrt{b^2c^2\sin^4\phi\cos^2\theta+a^2c^2\sin^4\phi\sin^2\theta+a^2b^2\sin^2\phi\cos^2\phi}\,d\phi d\theta.\end{aligned}$$

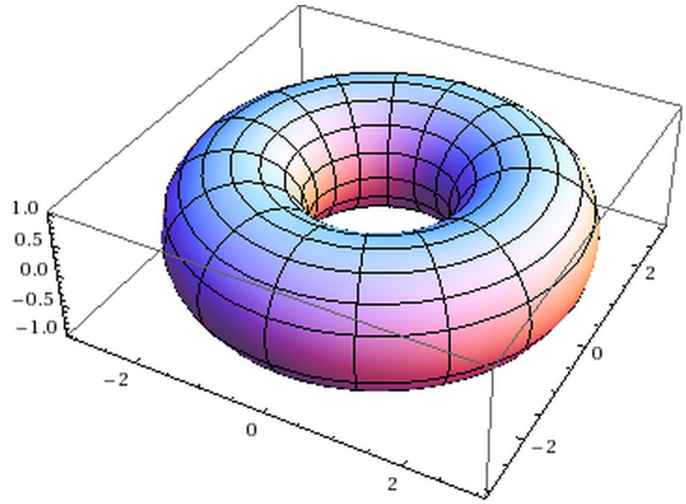
2. Let

$$\mathbf{r}(u,v)=\langle(2+\cos u)\cos v,(2+\cos u)\sin v,\sin u\rangle,$$

   where $0\le u\le 2\pi$ and $0\le v\le 2\pi$.

   (a) Sketch the surface parametrized by this function.

   Solution. The sketch of the surface is as follows.



(b) Compute its surface area.

Solution. By the parametrization, one has

$$\mathbf{r}_u = \langle -\sin u \cos v,\ -\sin u \sin v,\ \cos u\rangle,$$
$$\mathbf{r}_v = \langle -(2+\cos u)\sin v,\ (2+\cos u)\cos v,\ 0\rangle,$$

and so

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -(2+\cos u)\cos u \cos v,\ -(2+\cos u)\cos u \sin v,\ -(2+\cos u)\sin u\rangle.$$

Therefore $|\mathbf{r}_u \times \mathbf{r}_v| = 2+\cos u$, and the surface area is computed by

$$\text{Area} = \int_0^{2\pi}\int_0^{2\pi} |\mathbf{r}_u \times \mathbf{r}_v| \, du\, dv = \int_0^{2\pi}\int_0^{2\pi} (2+\cos u)\ du\, dv = 8\pi^2.$$

3. Consider the surface integral

$$\iint_\Sigma z\, dS$$

where Σ is the surface with sides S_1 given by the cylinder $x^2+y^2=1$, S_2 given by the unit disk in the xy-plane, and S_3 given by the plane $z = x+1$. Evaluate this integral as follows:

(a) Parametrize S_1 using (θ, z) coordinates.

Solution. One can parametrize S_1 by

$$x = \cos\theta,\ y = \sin\theta,\ z = z, \quad 0 \le \theta \le 2\pi,\ 0 \le z \le \cos\theta + 1.$$

(b) Evaluate the integral over the surface S_2 without parametrizing.

Solution. Since $z = 0$ on S_2, we know $\iint_{S_2} z\, dS = 0$.

(c) Parametrize S_3 in Cartesian coordinates and evaluate the resulting integral using polar coordinates.

Solution. One can parametrize S_3 in Cartesian coordinates

$$x = x,\ y = y,\ z = x+1, \quad -1 \le x \le 1,\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}.$$

Now we move to evaluate the integral $\iint_\Sigma z\,dS$. Obviously

$$\iint_\Sigma z\,dS = \iint_{S_1} z\,dS + \iint_{S_2} z\,dS + \iint_{S_3} z\,dS := I_1 + I_2 + I_3.$$

To estimate I_1, using the parametrization in (a), one has

$$\mathbf{r}(\theta, z) = \langle \cos\theta,\ \sin\theta,\ z\rangle.$$

Then

$$\mathbf{r}_\theta = \langle -\sin\theta,\ \cos\theta,\ 0\rangle, \quad \mathbf{r}_z = \langle 0,\ 0,\ 1\rangle,$$

and

$$\mathbf{r}_\theta \times \mathbf{r}_z = \langle \cos\theta,\ \sin\theta,\ 0\rangle.$$

So $|\mathbf{r}_\theta \times \mathbf{r}_z| = 1$, and

$$\begin{aligned} I_1 &= \int_0^{2\pi}\int_0^{\cos\theta+1} z\,dz d\theta = \int_0^{2\pi} \frac{(\cos\theta+1)^2}{2}\,d\theta \\ &= \int_0^{2\pi} \frac{\cos^2\theta + 2\cos\theta + 1}{2}\,d\theta = \frac{3\pi}{2}. \end{aligned}$$

In (b) we know $I_2 = 0$. To evaluate I_3, by the parametrization in (c), one has

$$\mathbf{r}(x,\ y) = \langle x,\ y,\ x+1\rangle, \quad -1 \le x \le 1,\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2},$$

and so

$$\mathbf{r}_x = \langle 1, 0, 1\rangle, \quad \mathbf{r}_y = \langle 0, 1, 0\rangle, \quad \mathbf{r}_x \times \mathbf{r}_y = \langle -1, 0, 1\rangle.$$

Thus $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{2}$, and the surface integral is

$$I_3 = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x+1)\sqrt{2}\,dydx = \iint_{x^2+y^2\le 1} (x+1)\sqrt{2}\,dydx.$$

To evaluate this integral, one can use the polar coordinates

$$x = r\cos\theta,\ y = r\sin\theta, \quad 0 \le r \le 1,\ 0 \le \theta \le 2\pi.$$

Therefore,

$$I_3 = \int_0^{2\pi}\int_0^1 (r\cos\theta + 1)\sqrt{2}\,rdrd\theta = \sqrt{2}\pi.$$

Adding up all three integrals, one gets

$$\iint_\Sigma z\,dS = I_1 + I_2 + I_3 = \frac{3\pi}{2} + \sqrt{2}\pi.$$

4. Let C be the circle in the plane with equation $x^2+y^2-2x=0$.

(a) Parametrize C as follows. For each choice of a slope t, consider the line L_t whose equation is $y=tx$. Then the intersection $L_t \cap C$ of L_t and C contains two points, one of which is $(0,0)$. Find the other point of intersection, and call its $x-$ and $y-$coordinates $x(t)$ and $y(t)$. Compute a formula for $\mathbf{r}(t) = \langle x(t), y(t)\rangle$. Check your answer with your TA.

Solution. Bring $y = tx$ into $x^2+y^2-2x=0$, then one has $x^2+t^2x^2-2x=0$, and it is easy to get $x=\frac{2}{1+t^2}$, and then $y=\frac{2t}{1+t^2}$. Thus $\mathbf{r}(t)=\langle\frac{2}{1+t^2},\frac{2t}{1+t^2}\rangle$.

(b) Suppose that $t=\dfrac{p}{q}$ is a rational number. Show that $x(p/q)$ and $y(p/q)$ are also rational numbers. Explain how, by clearing denominators in $x(p/q)-1$ and $y(p/q)$, you can find a a triple of integers U, V, and W for which $U^2+V^2=W^2$.

Solution. Plug $t=\frac{p}{q}$ into the the parametrization, one gets

$$x(p/q)=\frac{2q^2}{p^2+q^2},\quad y(p/q)=\frac{2pq}{p^2+q^2},$$

and both of them are rational numbers. Since $(x-1)^2+y^2=1$, and $x(p/q)-1=\frac{q^2-p^2}{p^2+q^2}$, then one has

$$\left(\frac{q^2-p^2}{p^2+q^2}\right)^2+\left(\frac{2pq}{p^2+q^2}\right)^2=1.$$

By setting

$$U=q^2-p^2,\quad V=2pq,\quad W=p^2+q^2,$$

one has $U^2+V^2=W^2$.

(c) Compute $\displaystyle\int_C \frac{1}{2}\langle -y,x\rangle\cdot d\mathbf{r}$ using your parametrization above.

Solution. Since $\mathbf{r}=\langle\frac{2}{1+t^2},\frac{2t}{1+t^2}\rangle$, one has $\mathbf{r}'=\langle-\frac{4t}{(1+t^2)^2},\frac{2-2t^2}{(1+t^2)^2}\rangle$. Then

$$\begin{aligned}\int_C \frac{1}{2}\langle -y,x\rangle\cdot d\mathbf{r} &= \int_{-\infty}^{\infty}\frac{1}{2}\langle-\frac{2t}{1+t^2},\frac{2}{1+t^2}\rangle\cdot\langle-\frac{4t}{(1+t^2)^2},\frac{2-2t^2}{(1+t^2)^2}\rangle\,dt\\ &= \int_{-\infty}^{\infty}\frac{2}{(1+t^2)^2}\,dt=\pi.\end{aligned}$$