Group: solutions ______ ______ ______ Name: ______

Math 231 A. Fall, 2015. Worksheet 6. 9/15/15

1. Evaluate each of the improper integrals or show that it diverges.

a) $\int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{b\to\infty} \int_1^b \frac{dx}{\sqrt{x}} = \lim_{b\to\infty} \left[2\sqrt{x} \Big|_1^b \right] = \lim_{b\to\infty} (2\sqrt{b} - 2)$ diverges.

b) $\int_1^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_1^b \frac{dx}{1+x^2} = \lim_{b\to\infty} \left[\arctan(x) \Big|_1^b \right]$

$= \lim_{b\to\infty} (\arctan(b) - \arctan(1))$

$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$ convergent.

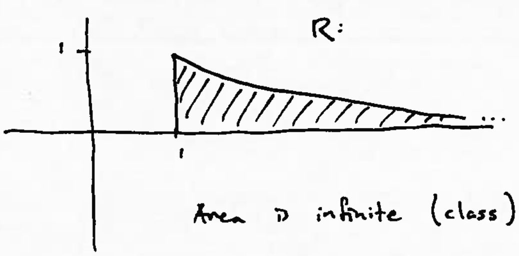
c) $\int_e^\infty \frac{1}{x \ln x}\,dx$ $\left[\begin{array}{ll} \text{sub:} & \\ u = \ln(x) & x=e \Rightarrow u=1, \\ du = \frac{1}{x}dx, & \text{upper bound still infinity} \end{array} \right] = \int_1^\infty \frac{1}{u}\,du$

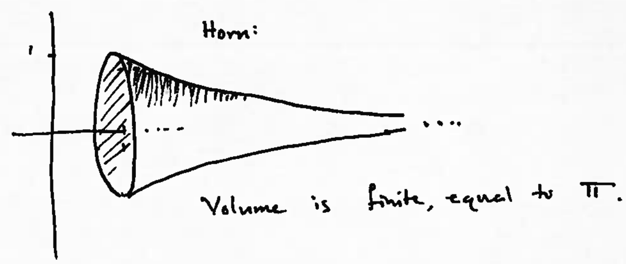
$= \lim_{b\to\infty} \int_1^b \frac{1}{u}\,du = \lim_{b\to\infty} \left[\ln(u) \Big|_1^b \right] = \lim_{b\to\infty} (\ln(b) - \cancel{\ln(1)}^{\,0})$ diverges.

You can also get this to:

$\lim_{b\to\infty} (\ln(\ln(b)) - \cancel{\ln(\ln(e))}^{\,0})$ which also diverges..

2. As we saw in class, the region $R = \{(x,y) : x \geq 1, 0 \leq y \leq 1/x\}$ has infinite area. The *Horn of Gabriel* is formed by rotating this region about the x-axis. Make a careful sketch of R and of the Horn. Then find the volume of the Horn of Gabriel.





Volume $= \int_1^\infty \pi \left(\frac{1}{x}\right)^2 dx = \lim_{b\to\infty} \int_1^b \frac{\pi}{x^2}\,dx = \lim_{b\to\infty} \left(-\frac{\pi}{x} \Big|_1^b \right) = \lim_{b\to\infty} \left(\frac{-\pi}{b} + \frac{\pi}{1} \right) = \pi.$

3. Which of the following statements shows a correct use of the Comparison Theorem?

a) Since $\int_2^\infty \frac{dx}{\sqrt{x}}$ diverges, and $\frac{1}{\sqrt{x}} < \frac{1}{\sqrt{x-1}}$ for all $x > 2$, $\int_2^\infty \frac{dx}{\sqrt{x-1}}$ must diverge.

true / true / follows from comparison Theorem

b) Since $\int_2^\infty \frac{dx}{\sqrt{x}}$ diverges, and $\frac{1}{\sqrt{x}} > \frac{1}{\sqrt{x+1}}$ for all $x > 2$, $\int_2^\infty \frac{dx}{\sqrt{x+1}}$ must diverge.

true / true. / cannot be concluded from Comparison Theo

(a) is correct, while (b) is not.

"larger than something divergent: still divergent.
smaller than something convergent: still convergent."

4. Use a comparison to a known integral to determine if these improper integrals converge or diverge.

a) $\int_0^\infty \frac{1}{x^4+27}\,dx$. Is the integral improper at 0 or at ∞?

$x^4+27 > x^4+1 > x^2+1$ for large enough x, so $\frac{1}{x^4+27} < \frac{1}{x^2+1}$.

$\int_0^\infty \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} (\arctan(b) - \arctan(0)) = \frac{\pi}{2}$ Converges,

so $\int_0^\infty \frac{1}{x^4+27}\,dx$ converges, by comparison

b) $\int_0^\infty e^{-x}\sin^2(x)\,dx$ Hint: How big can $\sin^2 x$ be? $\sin^2(x) \le 1$ so $e^{-x}\sin^2(x) \le e^{-x}$.

$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty} (-e^{-b} + e^{0}) = \lim_{b\to\infty} (1 - e^{-b}) = 1$, convergent.

so $\int_0^\infty e^{-x}\sin^2(x)\,dx$ converges by comparison.

c) $\int_3^\infty \frac{x}{x^3+e^x+\cos^2 x}\,dx$

parts
$\left[\begin{matrix} u=x & v=-e^{-x} \\ du=dx & dv=e^{-x}dx \end{matrix}\right]$

$\frac{x}{x^3+e^x+\cos^2 x} \le \frac{x}{e^x}$, and $\int \frac{x}{e^x}\,dx = -xe^{-x} - e^{-x}$,

so $\int_3^\infty \frac{x}{e^x}\,dx = \lim_{b\to\infty} \left(\left(\cancel{-be^{-b}}^{\,0} - \cancel{e^{-b}}^{\,0}\right) - \left(-3e^{-3} - e^{-3}\right)\right)$ converges

so $\int_3^\infty \frac{x}{x^3+e^x+\cos^2 x}\,dx$ converges, by comparison.