**Math 231 - Midterm 1 Review**

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Sections: BDJ/BDK

## Table of integrals

| | |
|---|---|
| $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ (if $n \neq -1$) | $\int \frac{1}{x} dx = \ln\lvert x\rvert + C$ |
| $\int e^x dx = e^x + C$ | $\int a^x dx = \frac{a^x}{\ln a} + C$ |
| $\int \sin x dx = -\cos x + C$ | $\int \cos x dx = \sin x + C$ |
| $\int \sec^2 x dx = \tan x + C$ | $\int \csc^2 x dx = -\cot x + C$ |
| $\int \sec x \tan x dx = \sec x + C$ | $\int \csc x \cot x dx = -\csc + C$ |
| $\int \sec x dx = \ln\lvert \sec x + \tan x\rvert + C$ | $\int \csc x dx = \ln\lvert \csc x - \cot x\rvert + C$ |
| $\int \tan x dx = \ln\lvert \sec x\rvert + C$ | $\int \cot x dx = \ln\lvert \sin x\rvert + C$ |
| $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \cdot \arctan\left(\frac{x}{a}\right) + C$ | $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin(x) + C$ |

## Integration by parts

$$\boxed{\int_a^b u dv = uv\Big|_a^b - \int_a^b v du}$$

What does LIATE stand for? It may help you choose u.

Logarithmic
Inverse trig.
Algebraic $(x, x^2, \dots)$
Trig.
Exponential

# Trigonometric Integrals

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\cos^2\theta = \frac{1}{2}[1 + \cos(2\theta)]$$

$$\sin^2\theta = \frac{1}{2}[1 - \cos(2\theta)]$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

**Case 1:** $\int \sin^m x \cos^n x dx$

(a) If n odd:

- save a copy of: $\cos x$
- use itentity: $\cos^2 x = 1 - \sin^2 x$
- u- substitution: $u = \sin x \rightarrow du = \cos x \, dx$

(b) If m odd:

- save a copy of: $\sin x$
- use itentity: $\sin^2 x = 1 - \cos^2 x$
- u- substitution: $u = \cos x \longrightarrow du = -\sin x \, dx$

(c) If both n and m are even:

- use itentities: $\cos^2\theta = \frac{1}{2}[1 + \cos(2\theta)]$ and $\sin^2\theta = \frac{1}{2}[1 - \cos(2\theta)]$
- Sometimes useful to use: $\boxed{\sin(2\theta) = 2\sin\theta\cos\theta}$

**Case 2:** $\int \tan^m x \sec^n x dx$

(a) If n even

- save a copy of: $\sec^2 x$
- use itentity: $\sec^2 x = 1 + \tan^2 x$
- u- substitution: $u = \tan x \longrightarrow du = \sec^2 x \, dx$

(b) If m odd:

- save a copy of: $\sec x \cdot \tan x$
- use itentity: $\tan^2 x = \sec^2 x - 1$
- u- substitution: $u = \sec x \longrightarrow du = \sec x \cdot \tan x \, dx$

# Trigonometric Substitutions

| Expression | Substitution | Identity | Restriction |
|---|---|---|---|
| $\sqrt{a^2-x^2}$ | $x = a\sin\theta$<br>$dx = a\cos\theta\, d\theta$ | $1-\sin^2\theta = \cos^2\theta$<br>$a^2 - a^2\sin^2\theta = a^2\cos^2\theta$ | $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ |
| $\sqrt{a^2+x^2}$ | $x = a\tan\theta$<br>$dx = a\sec^2\theta\, d\theta$ | $1+\tan^2\theta = \sec^2\theta$<br>$a^2+a^2\tan^2\theta = a^2\sec^2\theta$ | $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ |
| $\sqrt{x^2-a^2}$ | $x = a\sec\theta$<br>$dx = a\sec\theta\tan\theta\, d\theta$ | $\sec^2\theta - 1 = \tan^2\theta$<br>$a^2\sec^2\theta - a^2 = a^2\tan^2\theta$ | $0 \le \theta < \frac{\pi}{2}$ or<br>$\pi \le \theta < 3\pi/2$ |

What substitution would you use in the following examples?

1. $\sqrt{9-x^2} = \sqrt{9-(3\sin\theta)^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9(\underbrace{1-\sin^2\theta}_{\cos^2\theta})} = 3\cos\theta$ when $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

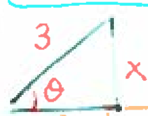
$\uparrow a = 3$

$\sin\theta = \frac{x}{3}$ ← $\boxed{x = 3\sin\theta \longrightarrow dx = 3\cos\theta\, d\theta}$

$$\int\sqrt{9-x^2}\,dx = \int 3\cos\theta \cdot \overbrace{3\cos\theta\, d\theta}^{dx} = \int 9\cos^2\theta\, d\theta = \int \frac{9}{2}\cdot[1+\cos 2\theta]\,d\theta$$

$\frac{1}{2}[1+\cos(2\theta)]$

$$= \frac{9}{2}\theta + \frac{9}{4}\underbrace{\sin(2\theta)}_{2\sin\theta\cdot\cos\theta} + C = \frac{9}{2}\arcsin\left(\frac{x}{3}\right) + \frac{9}{4}\cdot 2\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C$$




$? = \sqrt{9-x^2}$

← Don't stop here!

↑ (We know this from the triangle).

2. $\sqrt{x^2+4x} = \sqrt{(x+2)^2-4} = \sqrt{4\sec^2\theta - 4} = \sqrt{4(\underbrace{\sec^2\theta-1}_{\tan^2\theta})} = 2\tan\theta$ when $0 \le \theta \le \pi/2$ or $\pi \le \theta < 3\pi/2$

↑ complete the square

$\boxed{x+2 = 2\sec\theta \longrightarrow dx = 2\sec\theta\tan\theta\, d\theta}$

$$\int\sqrt{x^2+4x}\,dx = \int 2\tan\theta \cdot \overbrace{2\sec\theta\tan\theta\, d\theta}^{dx} = \int 4\sec\theta\tan^2\theta\, d\theta = \cdots$$

FINISH IT! Hint: • use $\tan^2\theta = \sec^2\theta - 1$
• use by parts to solve $\int \sec^3\theta$.

**Common mistakes Comments:**

1. Don't forget to replace $dx$ by the corresponding expression. It is not just $d\theta$.
2. Give the solution in term of $x$, not $\theta$. Use the triangle to do that.
3. Remember to use the expression $\boxed{\sin(2\theta) = 2\sin\theta\cos\theta}$ when necessary.
4. Complete the square if necessary!

# Integration of Rational Functions by Partial Fractions

1. **CASE I: The denominator is a product of distinct linear factors**

   *Example:* $\dfrac{x^2+2x-1}{2x^3-5x^2+2x} = \dfrac{x^2+2x-1}{x(2x-1)(x-2)} = \dfrac{A}{x} + \dfrac{B}{2x-1} + \dfrac{C}{x-2}$

2. **CASE II: The denominator is a product of linear factors, some of which are repeated**

   *Example:* $\dfrac{x^2+2x-1}{2x^4-5x^3+2x^2} = \dfrac{x^2+2x-1}{x^2(2x-1)(x-2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{2x-1} + \dfrac{D}{x-2}$

3. **CASE III: The denominator contains *irreducible* quadratic factors, none of which is repeated**

   *Example:* $\dfrac{x^2+x-5}{x^3+5x} = \dfrac{x^2+x-5}{x(x^2+5)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+5}$

4. **CASE IV: The denominator contains a repeated *irreducible* quadratic factor**

   *Example:* $\dfrac{x^2-3}{x(x^2+3)^2} = \dfrac{A}{x} + \dfrac{Bx+C}{(x^2+3)} + \dfrac{Dx+E}{(x^2+3)^2}$

# Improper Integrals

1. **Type 1: Infinite Intervals**

   (a) If $\int_a^t f(x)dx$ exists for every number $t \geq a$, then

   $$\int_a^\infty f(x)dx = \lim_{t \to \infty} \int_a^t f(x)\,dx$$

   provided this limit exists (as a finite number).

   (b) If $\int_t^b f(x)dx$ exists for every number $t \leq b$, then

   $$\int_{-\infty}^b f(x)dx = \lim_{t \to -\infty} \int_t^b f(x)\,dx$$

   provided this limit exists (as a finite number).

   The improper integrals $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

   (a) If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent, then

   $$\int_{-\infty}^\infty f(x)dx =$$

2. **Type 2: Discontinuous Integrands**

   (a) If $f$ is continuous on $[a, b)$ and is discontinuous at b, then

   $$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$

   if this limit exists (as a finite number).

   (b) If $f$ is continuous on $(a, b]$ and is discontinuous at a, then

   $$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$

   if this limit exists (as a finite number).

   The improper integral $\int_a^b f(x)dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

   (a) If $f$ has a discontinuity at $c$, where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then

   $$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

**Comparison theorem**

Suppose that $f$ and $g$ are continuous functions with $f \geq g \geq 0$ for $x \geq a$.

(a) If $f(x)$ is convergent, then $g(x)$ is convergent.

(b) If $g(x)$ is divergent, then $f(x)$ is divergent.

**Remark:** The comparison theorem only tells you whether the integral diverges or converges. If it converges, it does not tell you to which number!

**p-test**

(a) $\int_1^\infty \frac{1}{x^p}dx$ converges/diverges for $p > 1$ and converges/diverges for $p \leq 1$.

(b) $\int_0^1 \frac{1}{x^p}dx$ converges/diverges for $p \geq 1$ and converges/diverges for $p < 1$.