Math 231 - Midterm 2 Review

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Sections: ADJ/ADK

Approximate Integration

$\Delta x = \frac{b-a}{n}$

$\bar{x}_i = \frac{1}{2}(x_{i-1}+x_i)$ = midpoint of $[x_{i-1}, x_i]$

$x_i = a + i\Delta x$

Midpoint Rule

$$\int_a^b f(x)dx = \Delta x[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)]$$

Trapezoidal Rule

$$\int_a^b f(x)dx = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

Simpsons Rule

$$\int_a^b f(x)dx = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(x_n)]$$

Arc Length

- Write the formula **in terms of x**: $y = f(x), a \le x \le b$

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

- Write the formula **in terms of y**: $x = g(y), c \le y \le d$

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$

Area of a Surface of Revolution

General formula:

$y = f(x)$, $a \leq x \leq b$

$\Updownarrow$ $\Updownarrow$

$x = g(y)$ $\underbrace{f(a)}_{c} \leq y \leq \underbrace{f(b)}_{d}$

$$\boxed{S = \int 2\pi R\, ds}$$

	integral in terms of x	integral in term of y
rotate about x-axis (R= $y = f(x)$)	$\int_a^b 2\pi f(x)\sqrt{1+\left[\frac{dy}{dx}\right]^2}\,dx$	$\int_c^d 2\pi\, y\sqrt{1+\left[\frac{dx}{dy}\right]^2}\,dy$
rotate about y-axis (R= $x = g(y)$)	$\int_a^b 2\pi x\sqrt{1+\left[\frac{dy}{dx}\right]^2}\,dx$	$\int_c^d 2\pi\, g(y)\sqrt{1+\left[\frac{dx}{dy}\right]^2}\,dy$

Example:
Set up an integral for the area of the surface obtained by rotation the curve $y = \tan(x)$, $0 \leq x \leq \pi/3$

$x = \arctan(y)$ $0 \leq y \leq \sqrt{3}$

- about the x-axis in terms of x:

$$\int_0^{\pi/3} 2\pi \tan(x)\sqrt{1+[\sec^2 x]^2}\,dx$$

- about the x-axis in terms of y:

$$\int_0^{\sqrt{3}} 2\pi\, y\sqrt{1+\left[\frac{1}{1+y^2}\right]^2}\,dy$$

- about the y-axis in terms of x:

$$\int_0^{\pi/3} 2\pi x\sqrt{1+[\sec^2 x]^2}\,dx$$

- about the y-axis in terms of y:

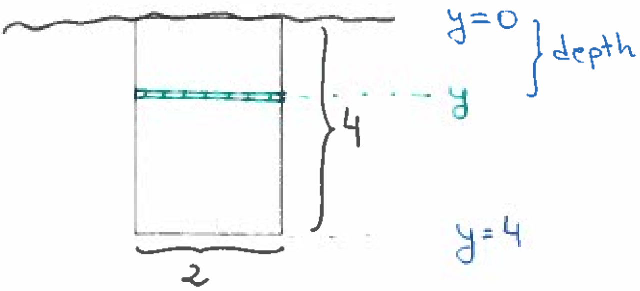
$$\int_0^{\sqrt{3}} 2\pi \arctan(y)\sqrt{1+\left[\frac{1}{1+y^2}\right]^2}\,dy$$

Hydrostatic force

General formula:

$$\boxed{F = \int_a^b \rho g (\text{depth})(\text{width})\, dy}$$

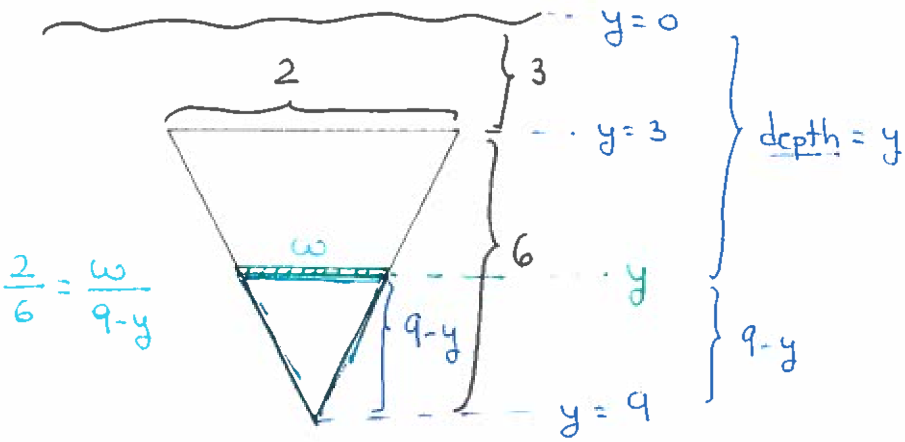
1. Rectangle:



$$F = \int_0^4 \rho g\, y \cdot 2\, dy$$

depth = y
width = 2

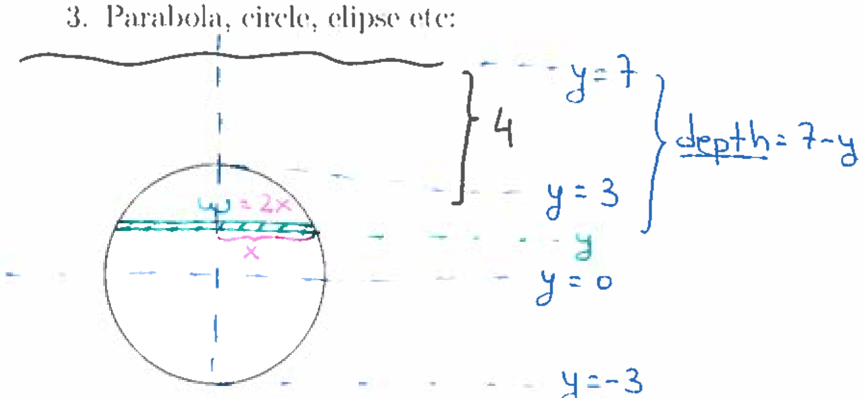
2. Triangle:



$$\frac{2}{6} = \frac{\omega}{9-y}$$

$$F = \int_3^9 \rho g\, y\, \overbrace{\tfrac{2}{6}(9-y)}^{\omega}\, dy$$

3. Parabola, circle, elipse etc:



$$F = \int_{-3}^{3} \rho g\, (7-y)\, \overbrace{2\cdot\sqrt{9-y^2}}^{\omega}\, dy$$

$$x^2 + y^2 = 9 \rightsquigarrow x = \sqrt{9-y^2} \longrightarrow \omega = 2x = 2\cdot\sqrt{9-y^2}$$

Moments and Centers of Mass

$$M_y = \rho \int_a^b x \cdot f(x) \quad dx$$

$$M_x = \rho \int_a^b \frac{1}{2}\left[f(x)\right]^2 \quad dx$$ OR $M_x = \rho \int_c^d y \cdot (\text{width})\, dy$

(in terms of y)

$$\bar{x} = \frac{M_y}{\rho \cdot A}$$

$$\bar{y} = \frac{M_x}{\rho \cdot A}$$

What if the region lies between two curves $y = f(x)$ and $y = g(x)$ where $f(x) \geq g(x)$?

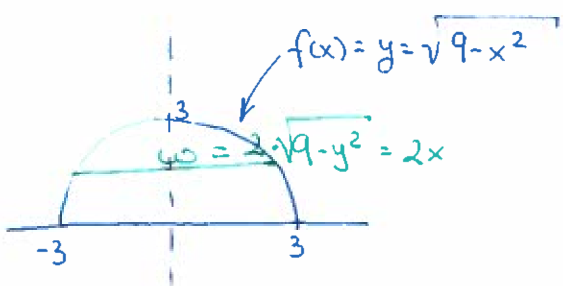
$$M_y = \rho \int_a^b x\left[f(x) - g(x)\right] dx$$

$$M_x = \rho \int_a^b \frac{1}{2}\left[(f(x))^2 - (g(x))^2\right] dx$$

Example : A lamina of density ρ kg/m^2has the shape of the half circle defined by

$$x^2 + y^2 = 9, \qquad y \geq 0.$$

Set up but **do not evaluate** an integral to compute the moment M_x about the x-axis.



Method 1:

$$M_x = \rho \int_a^b \frac{1}{2}\left[f(x)\right]^2 dx = \rho \int_{-3}^{3} \frac{1}{2}\left(\sqrt{9-x^2}\right)^2 dx$$

$$= \rho \int_{-3}^{3} \frac{1}{2}(9-x^2)\, dx$$

Method 2: $M_x = \rho \int_c^d y\,(\text{width})\, dy$

$$M_x = \rho \int_0^3 y\left(2\sqrt{9-y^2}\right) dy$$

Sequences

If $\lim_{n\to\infty} a_n$ exists (as a finite number), we say the the sequence $\{a_n\}$ CONVERGES . Otherwise we say that the sequence is DIVERGENT .

Examples: Are the following sequences convergent or divergent?

a. $a_n = \dfrac{\sqrt{9n^2 - 2n}}{2n + 3}$

$$\lim_{n\to\infty} \frac{\sqrt{9n^2-2n}}{2n+3} = \lim_{n\to\infty} \frac{\sqrt{9n^2-2n}\cdot 1/n}{(2n+3)\cdot 1/n} = \lim_{n\to\infty} \frac{\sqrt{\dfrac{9n^2-2n}{n^2}}}{2+\frac{3}{n}} = \lim_{n\to\infty} \frac{\sqrt{9-\overbrace{\frac{2}{n}}^{\to 0}}}{2+\underbrace{\frac{3}{n}}_{0}} = \frac{\sqrt{9}}{2} = \frac{3}{2}$$

The sequence converges to $\frac{3}{2}$.

b. $a_n = \ln(n+6) - \ln(n)$ $= \ln\left|\dfrac{n+6}{n}\right|$

$$\lim_{n\to\infty} \ln\left|\frac{n+6}{n}\right| = \ln(1) = 0$$

The sequence converges to 0.

c. $a_n = \dfrac{\cos^2 n}{4^n}$

$$0 \leq \cos^2 n \leq 1$$

$$\underbrace{\frac{0}{4^n}}_{0} \leq \frac{\cos^2 n}{4^n} \leq \underbrace{\frac{1}{4^n}}_{0}$$

By the Squeeze theorem,

$\lim_{n\to\infty} \dfrac{\cos^2 n}{4^n} = 0$. So the sequence converges to 0.

Series

Given a series $\sum_{n=1}^{\infty} a_n$, let s_n denote its nth partian sum:

$$s_n = a_1 + a_2 + \dots + a_n$$

If the sequence $\{s_n\}$ is convergent, then the series $\sum_{n=1}^{\infty} a_n$ is called and we write

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$$

$\neq \lim_{n\to\infty} a_n$

- How can we find a_n if s_n is given?

$$a_n = s_n - s_{n-1}$$

- Examples of series we know well:

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots$$

is $\boxed{\text{convergent}}$/divergent if $\boxed{|r| < 1}$ and its sum is: $\frac{a}{1-r}$

If $|r| \geq 1$, the geometric series is **divergent**.

p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{CONVERGES IF } p > 1 \\ \text{DIVERGES IF } p \leq 1 \end{cases}$$

- Tests we can use to find convergence or divergence:

Test for Divergence Given $\sum_{n=1}^{\infty} a_n$,

If $\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ DIVERGES

The Integral Test

- What are the hypothesis for f? $f(x)$ continuous, positive, decreasing
- What is the conclusion?

$\int_1^{\infty} f(x)$ and $\sum_{n=1}^{\infty} f(n)$ do the same (if one conv. the other does too).

The Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

- If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.
- If $\sum b_n$ is divergent and $b_n \leq a_n$ for all n, then $\sum a_n$ is also divergent.

The Limit Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = C > 0$ $\neq 0$, then either both series converge or both diverge.

Remarks:

- $\boxed{\text{Careful}}$ when using the divergence test: If $\lim_{n\to\infty} a_n = 0$, we **cannot** conclude anything from the divergence test.

  Example: Look at $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Both limits are 0, but the first one diverges and the second one converges.

- If using the Integral test make sure you check the hypothesis.

Example: Use the integral test to show whether $\sum_{n=1}^{\infty} e^{-n}$ converges: $f(x) = e^{-x} = \frac{1}{e^x}$ → continuous ✓, positive ✓, decreasing ✓

$\int_1^{\infty} \frac{1}{e^x}\,dx \le \int_1^{\infty} \frac{1}{\cancel{x}^{10}}\,dx$ converges $\Rightarrow \int_1^{\infty} \frac{1}{e^x}\,dx$ conv. by comparison.

Therefore, by the integral test, $\sum_{n=1}^{\infty} e^{-n}$ converges too.

Examples: Are the following series convergent or divergent?

a. $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{2n^2+3} \le \sum_{n=1}^{\infty} \frac{1}{2n^2+3} \le \sum_{n=1}^{\infty} \frac{1}{2n^2} \le \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series $(p=2)$

By comparison test, $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{2n^2+3}$ converges.

b. $\sum_{n=1}^{\infty} \arctan(n)$

Divergence test:

$\lim_{n\to\infty} \arctan(n) = \frac{\pi}{2} \ne 0 \Rightarrow \sum_{n=1}^{\infty} \arctan(n)$ diverges.

c. $\sum_{n=1}^{\infty} \frac{2^{3n+1}}{3^n} = \sum_{n=1}^{\infty} \frac{2 \cdot 2^{3n}}{3^n} = \sum_{n=1}^{\infty} \frac{2 \cdot (2^3)^n}{3^n} = \sum_{\boxed{n=1}}^{\infty} 2 \cdot \left(\frac{8}{3}\right)^n = \sum_{n=1}^{\infty} \underbrace{2 \cdot \frac{8}{3}}_{a} \underbrace{\left(\frac{8}{3}\right)}_{r}^{n-1}$

The series is geometric with $r = \frac{8}{3} > 1$, therefore the series diverges.

$\boxed{\text{OR}}$

$\lim_{n\to\infty} \frac{2^{3n+1}}{3^n} = \lim_{n\to\infty} 2\left(\frac{8}{3}\right)^n = \infty \Rightarrow$ The series diverges by the Divergence Test.

d. $\sum_{n=1}^{\infty} \sin\left(\frac{4}{n}\right)$ → compare to $\sum_{n=1}^{\infty} \frac{4}{n}$

Limit comparison: $\lim_{n\to\infty} \frac{\sin(4/n)}{4/n} \overset{L'H}{=} \lim_{n\to\infty} \frac{\cos(4/n)\cdot\cancel{\left(\frac{-4}{n^2}\right)}}{\cancel{\left(\frac{-4}{n^2}\right)}} = 1 > 0 \Rightarrow$ Since $\sum_{n=1}^{\infty} \frac{4}{n}$ diverges, $\sum_{n=1}^{\infty} \sin\left(\frac{4}{n}\right)$ diverges

e. $\sum_{n=1}^{\infty} \frac{n^3+5n}{e^n} \le \sum_{n=1}^{\infty} \frac{n^3+5n}{n^{10}} \le \sum_{n=1}^{\infty} \frac{n^3+5n^3}{n^{10}} \le \sum_{n=1}^{\infty} \frac{6n^3}{n^{10}} = \sum_{n=1}^{\infty} \frac{6}{n^7}$

$\boxed{e^n \gg n^a}$

converges by p-series $(p=7)$

By comparison test, $\sum_{n=1}^{\infty} \frac{n^3+5n}{e^n}$ converges

Remember: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \ldots$ and we define

$$S_n = a_1 + \cdots + a_n \qquad \text{and} \qquad R_n = a_{n+1} + a_{n+2} + \ldots$$

When we approximate $\sum_{n=1}^{\infty} a_n$ by S_n we make and "error" R_n and we want to know how big this error is.

Reminder Estimate for the Integral Test

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx$$

$$S_n + \int_{n+1}^{\infty} f(x)\,dx \le S = \sum_{n=1}^{\infty} a_n \le S_n + \int_{n}^{\infty} f(x)\,dx$$

- How many terms of the series $\sum_{n=1}^{\infty} \frac{5}{n^3}$ would we need to add to estimate the sum to within 0.1?

We want $R_n \le \int_n^{\infty} \frac{5}{x^3}\,dx \le 0.1$

$$\int_n^{\infty} \frac{5}{x^3}\,dx = \lim_{t\to\infty} \int_n^{t} \frac{5}{x^3}\,dx = \lim_{t\to\infty} \left. \frac{-5}{2x^2} \right|_n^t = \lim_{t\to\infty} \left[\underbrace{\frac{-5}{2t^2}}_{\to 0} - \frac{-5}{2n^2} \right] = \frac{5}{2n^2} \le 0.1$$

$$\frac{5}{2n^2} \le 0.1 \iff \frac{2n^2}{5} \ge 10 \iff n^2 \ge \frac{50}{2} = 25 \Rightarrow \boxed{n \ge 5}$$

We need to add at least 5 terms.

- Approximate $\sum_{n=1}^{\infty} \frac{5}{n^3}$ within 0.1.

Since we know that $n \ge 5$, we want to approximate the series by $S_5 = a_1 + a_2 + a_3 + a_4 + a_5$:

$$\boxed{S_5 = \frac{5}{1} + \frac{5}{2^3} + \frac{5}{3^3} + \frac{5}{4^3} + \frac{5}{5^3}}$$