Math 231, Midterm 3. April 19, 2017

Full Name: ______________________________ **Section Code from table below:** | | | |

Net ID: ______________________________

For graders only	Score
Problem 11 (14 points)	
Problem 12 (18 points)	
Problem 13 (16 points)	
Total (48 points)	

TA	Section	Time and Location	TA	Section	Time and Location
Akbari, Sahand	EDD EDE	WF 11:00-11:50 -- 106B6 ENGR HALL WF 12:00-12:50 -- 1 ILL HALL	Karve, Vaibhav	AD1	WF 9:00-10:50 -- 159 ALTGELD
Caulfield, Erin	AD2	WF 1:00-2:50 -- 159 ALTGELD	Kim, Hee Yeon	EDF EDG	WF 1:00-1:50 -- 241 ALTGELD WF 2:00-2:50 -- 345 ALTGELD
Chavoshi, Amir	EDB ADC	WF 9:00-9:50 -- 441 ALTGELD WF 10:00-10:50 -- 347 ALTGELD	Kim, Heejoung	ADA ADB	WF 8:00-8:50 -- 141 ALTGELD WF 9:00-9:50 -- 145 ALTGELD
Chung, Jooyeon	BD2	WF 11:00-12:50 -- 173 ALTGELD	Linz, William	DDA DDB	WF 8:00-8:50 -- 441 ALTGELD WF 9:00-9:50 -- 141 ALTGELD
Duffy, Michael	DDE DDF	WF 12:00-12:50 -- 441 ALTGELD WF 1:00-1:50 -- 145 ALTGELD	Loving, Marissa	CDE	WF 12:00-12:50 -- 140 HENRY BLD
Ellis, Matthew	DDG DDH	WF 2:00-2:50 -- 441 ALTGELD WF 3:00-3:50 -- 441 ALTGELD	Luo, Hao	CDC CDD	WF 10:00-10:50 -- 145 ALTGELD WF 11:00-11:50 -- 143 HENRY BLD
Gramcko-Tursi, Mary Angelica	CDG CDH	WF 2:00-2:50 -- 143 ALTGELD WF 3:00-3:50 -- 145 ALTGELD	Mousley, Sarah	ADD ADE	WF 11:00-11:50 -- 447 ALTGELD ADE -- WF 12:00-12:50 -- 341 ALTGELD
Han, Xiaolong (Hans)	ADF ADG	WF 1:00-1:50 -- 141 ALTGELD WF 2:00-2:50 -- 241 ALTGELD	Ochoa de Alaiza Gracia, Itziar	BDC BDD	WF 10:00-10:50 -- 447 ALTGELD WF 11:00-11:50 -- 137 HENRY BLD
Harris, Terence	BDA BDB	WF 8:00-8:50 -- 137 HENRY BLD WF 9:00-9:50 -- 341 ALTGELD	Pratt, Kyle	DDC DDD	WF 10:00-10:50 -- 441 ALTGELD WF 11:00-11:50 -- 441 ALTGELD
Heath, Emily	BD3	WF 1:00-2:50 -- 173 ALTGELD	Tamazyan, Albert	BDE CDF	WF 12:00-12:50 -- 137 HENRY BLD WF 1:00-1:50 -- 137 HENRY BLD
Huang, Jianting (Jesse)	EDA EDC	WF 8:00-8:50 -- 143 ALTGELD WF 10:00-10:50 -- 443 ALTGELD	Wright, Benjamin	CDA CDB	WF 8:00-8:50 -- 145 ALTGELD WF 9:00-9:50 -- 443 ALTGELD

- The exam is one hour long.
- You must not communicate with other students during this exam.
- No written materials of any kind allowed.
- No phones, calculators, smart-watches or electronic devices of any kind allowed for any reason, including checking the time (you may use a simple wristwatch).
- Do not turn this page until instructed to.

(1 point) Fill in top of this page correctly. Fill in name, UIN (Student number), and Net ID on Scantron form. Fill in the following answers on the Scantron form:

Zone 1

Multiple Choice Questions. Mark answers on Scantron form.

1/1. (6 points) What is the value of $\binom{\frac{1}{3}}{3}$?

A. ★ $\frac{5}{81}$

B. $-\frac{-2}{27}$

C. $\frac{-1}{27}$

D. $-\frac{10}{81}$

E. $\frac{-35}{243}$

Solution. $\dfrac{\frac{1}{3} \cdot -\frac{2}{3} \cdot -\frac{5}{3}}{3 \cdot 2 \cdot 1} = \dfrac{5}{81}.$

1/2. (6 points) What is the value of $\binom{\frac{2}{3}}{3}$?

A. ★ $\frac{4}{81}$

B. $-\frac{8}{81}$

C. $\frac{-7}{243}$

D. $-\frac{-1}{27}$

E. $\frac{-2}{27}$

Solution. $\dfrac{\frac{2}{3} \cdot -\frac{1}{3} \cdot -\frac{4}{3}}{3 \cdot 2 \cdot 1} = \dfrac{4}{81}.$

2/1. (6 points) Find the first two non-zero terms in the Maclaurin series expansion of $\dfrac{e^{-x^2} + \cos x - 2}{x}$.

A. ★ $-\dfrac{3}{2}x + \dfrac{13}{24}x^3 + \cdots$

B. $-x + \dfrac{1}{2}x^3 + \cdots$

C. $-\dfrac{1}{2}x - \dfrac{2}{3}x^3 + \cdots$

D. $\dfrac{1}{2}x + \dfrac{13}{24}x^2 + \cdots$

E. $-\dfrac{3}{2}x + \dfrac{1}{2}x^3 + \cdots$

Solution. $\left((1 - x^2 + \dfrac{x^4}{2!} - \cdots) + (1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots) - 2\right)/x = \left(x^2\left[-\dfrac{1}{2!} - 1\right] + x^4\left[\dfrac{1}{2!} + \dfrac{1}{4!}\right] + \cdots\right)/x$
$= -\dfrac{3}{2}x + \dfrac{13}{24}x^3 + \cdots$.

2/2. (6 points) Find the first two non-zero terms in the Maclaurin series expansion of $xe^{x^2} + \sin x - 2x$.

A. ★ $\dfrac{5}{6}x^3 + \dfrac{61}{120}x^5 + \cdots$

B. $2x + 2x^2 + \cdots$

C. $2x - \dfrac{5}{6}x^2 + \cdots$

D. $-\dfrac{5}{6}x - \dfrac{1}{120}x^3 + \cdots$

E. $\dfrac{1}{2}x^3 + \dfrac{1}{24}x^5 + \cdots$

Solution. $(x + x^3 + \dfrac{x^5}{2!} + \cdots) + (x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots) - 2x = \left[1 - \dfrac{1}{6}\right]x^3 + \left[\dfrac{1}{2} + \dfrac{1}{120}\right]x^5 + \cdots = \dfrac{5}{6}x^3 + \dfrac{61}{120}x^5 + \cdots$

3/1. (6 points) Let $f(x) = x(1+4x^3)^{\frac{1}{2}}$. Find $f^{(16)}(0)$.

A. ★ $16!\binom{\frac{1}{2}}{5}4^5$

B. $5!\binom{\frac{1}{2}}{16}4^{16}$

C. $\binom{\frac{1}{2}}{5}4^5$

D. $\binom{\frac{1}{2}}{16}4^{16}$

E. $16!\binom{\frac{1}{2}}{5}$

Solution. The Maclaurin series is $f(x) = x\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(4x^3)^n = \sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}4^n x^{3n+1}$. Since the Maclaurin series is by definition $f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$, we have $\frac{f^{(16)}(0)}{16!}x^{16} = \binom{\frac{1}{2}}{5}4^5x^{16}$.

3/2. (6 points) Let $f(x) = x(1+3x^4)^{\frac{1}{5}}$. Find $f^{(17)}(0)$.

A. ★ $17!\binom{\frac{1}{5}}{4}3^4$

B. $4!\binom{\frac{1}{5}}{17}3^{17}$

C. $\binom{\frac{1}{5}}{4}3^4$

D. $\binom{\frac{1}{5}}{17}3^{17}$

E. $17!\binom{\frac{1}{5}}{4}$

Solution. The Maclaurin series is $f(x) = x\sum_{n=0}^{\infty}\binom{\frac{1}{5}}{n}(3x^4)^n = \sum_{n=0}^{\infty}\binom{\frac{1}{5}}{n}3^n x^{4n+1}$. Since the Maclaurin series is by definition $f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$, we have $\frac{f^{(17)}(0)}{17!}x^{17} = \binom{\frac{1}{5}}{4}3^4x^{17}$.

3/3. (6 points) Let $f(x) = x(1+5x^3)^{\frac{1}{7}}$. Find $f^{(13)}(0)$.

A. ★ $13!\binom{\frac{1}{7}}{4}5^4$

B. $4!\binom{\frac{1}{7}}{13}5^{13}$

C. $\binom{\frac{1}{7}}{4}5^{4}$

D. $\binom{\frac{1}{7}}{13}5^{13}$

E. $13!\binom{\frac{1}{7}}{4}$

Solution. The Maclaurin series is $f(x) = x\sum_{n=0}^{\infty}\binom{\frac{1}{7}}{n}(5x^3)^n = \sum_{n=0}^{\infty}\binom{\frac{1}{7}}{n}5^n x^{3n+1}$. Since the Maclaurin series is by definition $f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$, we have $\frac{f^{(13)}(0)}{13!}x^{13} = \binom{\frac{1}{7}}{4}5^4x^{13}$.

4/1. (6 points) Consider the series $\sum_{n=0}^{\infty} \frac{(-1)^n n}{5^n}$. Which of the following is true?

A. ★ The series converges absolutely by the ratio test.

B. The series converges by the alternating series test, but does not converge absolutely.

C. The series diverges by the test for divergence.

D. The series converges absolutely by the alternating series test.

E. The series diverges by the ratio test.

Solution. $\left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{1}{5}\right)\left(\frac{n+1}{n}\right) \to \frac{1}{5} < 1$ as $n \to \infty$, so the series converges absolutely by the Ratio Test.

4/2. (6 points) Consider the series $\sum_{n=0}^{\infty} \frac{(-1)^n n}{2^n}$. Which of the following is true?

A. ★ The series converges absolutely by the ratio test.

B. The series converges by the alternating series test, but does not converge absolutely.

C. The series diverges by the test for divergence.

D. The series converges absolutely by the alternating series test.

E. The series diverges by the ratio test.

Solution. $\left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{1}{2}\right)\left(\frac{n+1}{n}\right) \to \frac{1}{2} < 1$ as $n \to \infty$, so the series converges absolutely by the Ratio Test.

5/1. (6 points) Consider the series $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{3n^2}{7n^2+5}$. Which of the following is true?

A. ★ The series diverges by the test for divergence.

B. The series diverges by the ratio test.

C. The series converges by the alternating series test, but does not converge absolutely.

D. The series converges absolutely by the alternating series test.

E. The series converges conditionally by the ratio test.

Solution.

We have $\lim_{n\to\infty}\frac{3n^2}{7n^2+5}=\frac{3}{7}$. So $\lim_{n\to\infty}(-1)^{n+1}\frac{3n^2}{7n^2+5}$ does not exist. In particular, the limit is not zero, so the series diverges by the test for divergence.

5/2. (6 points) Consider the series $\sum_{n=0}^{\infty}(-1)^{n}\frac{n^2}{5n^2+7}$. Which of the following is true?

A. ★ The series diverges by the test for divergence.

B. The series diverges by the ratio test.

C. The series converges by the alternating series test, but does not converge absolutely.

D. The series converges absolutely by the alternating series test.

E. The series converges conditionally by the ratio test.

Solution.

We have $\lim_{n\to\infty}\frac{n^2}{5n^2+7}=\frac{1}{5}$. So $\lim_{n\to\infty}(-1)^{n}\frac{n^2}{5n^2+7}$ does not exist. In particular, the limit is not zero, so the series diverges by the test for divergence.

6/1. (6 points) Find the Maclaurin series for $f(x) = \dfrac{x^2}{3+x^2}$.

A. ★ $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} x^{2n+2}$

B. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} x^{2n}$

C. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n}} x^{2n+2}$

D. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n}} x^{2n}$

E. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^{n+1}} x^{2n+2}$

Solution.

We have $f(x) = \dfrac{x^2}{3+x^2} = x^2\left(\dfrac{1}{3}\right)\dfrac{1}{1+\frac{1}{3}x^2} = x^2\left(\dfrac{1}{3}\right)\dfrac{1}{1-(-\frac{1}{3}x^2)} = x^2\left(\dfrac{1}{3}\right)\displaystyle\sum_{n=0}^{\infty}(-1)^n\left(\frac{x^2}{3}\right)^n = x^2\left(\frac{1}{3}\right)\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n}x^{2n} =$ $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}x^{2n+2}$.

6/2. (6 points) Find the Maclaurin series for $f(x) = \dfrac{x^2}{6+x^2}$.

A. ★ $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{6^{n+1}} x^{2n+2}$

B. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{6^{n+1}} x^{2n}$

C. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{6^{n}} x^{2n+2}$

D. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{6^{n}} x^{2n}$

E. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{6^{n+1}} x^{2n+2}$

Solution.

We have $f(x) = \dfrac{x^2}{6+x^2} = x^2\left(\dfrac{1}{6}\right)\dfrac{1}{1+\frac{1}{6}x^2} = x^2\left(\dfrac{1}{6}\right)\dfrac{1}{1-(-\frac{1}{6}x^2)} = x^2\left(\dfrac{1}{6}\right)\displaystyle\sum_{n=0}^{\infty}(-1)^n\left(\frac{x^2}{6}\right)^n = x^2\left(\frac{1}{6}\right)\sum_{n=0}^{\infty}\frac{(-1)^n}{6^n}x^{2n} =$ $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{6^{n+1}}x^{2n+2}$.

Zone 2

7/1. (5 points) Consider the series $\sum_{n=1}^{\infty} \frac{\sin^3(2n)}{3^n}$. Which of the following is true?

A. ★ The series converges absolutely.

B. The series converges conditionally.

C. The series diverges.

Solution.

We have $\left|\frac{\sin^3(2n)}{3^n}\right| \leq \frac{1}{3^n}$. Since $\sum \frac{1}{3^n}$ converges (it's a geometric series with $r = \frac{1}{3}$), we find that the given series converges absolutely.

7/2. (5 points) Consider the series $\sum_{n=1}^{\infty} \frac{\cos^3(2n)}{3^n}$. Which of the following is true?

A. ★ The series converges absolutely.

B. The series converges conditionally.

C. The series diverges.

Solution.

We have $\left|\frac{\cos^3(2n)}{3^n}\right| \leq \frac{1}{3^n}$. Since $\sum \frac{1}{3^n}$ converges (it's a geometric series with $r = \frac{1}{3}$), we find that the given series converges absolutely.

8/1. (5 points) Consider the series $\sum_{n=1}^{\infty}(-1)^n \sin(n^{-2/3})$. Which of the following is true?

A. ★ The series converges conditionally.

B. The series converges absolutely.

C. The series diverges.

Solution. This is an alternating series of the form $\sum_{n=0}^{\infty}(-1)^n b_n$ with $b_n = \sin(n^{-2/3})$.

Since $b_n > 0$, $b_n \to 0$, and $\{b_n\}$ is decreasing, the series converges by the Alternating Series Test. However,

$$\lim_{n\to\infty} \frac{\sin(n^{-2/3})}{\frac{1}{n^{2/3}}} = 1,$$

and so $\sum_{n=0}^{\infty} \left|(-1)^n \sin(n^{-2/3})\right| = \sum_{n=0}^{\infty} \sin(n^{-2/3})$ diverges by the limit comparison test with the p–series with $p = 2/3 < 1$. Therefore the given series converges conditionally.

8/2. (5 points) Consider the series $\sum_{n=1}^{\infty}(-1)^n \sin(n^{-3/4})$. Which of the following is true?

A. ★ The series converges conditionally.

B. The series converges absolutely.

C. The series diverges.

Solution. This is an alternating series of the form $\sum_{n=0}^{\infty}(-1)^n b_n$ with $b_n = \sin(n^{-3/4})$.

Since $b_n > 0$, $b_n \to 0$, and $\{b_n\}$ is decreasing, the series converges by the Alternating Series Test. However,

$$\lim_{n\to\infty} \frac{\sin(n^{-3/4})}{\frac{1}{n^{3/4}}} = 1,$$

and so $\sum_{n=0}^{\infty} \left|(-1)^n \sin(n^{-3/4})\right| = \sum_{n=0}^{\infty} \sin(n^{-3/4})$ diverges by the limit comparison test with the p–series with $p = 3/4 < 1$. Therefore the given series converges conditionally.

8/3. (5 points) Consider the series $\sum_{n=1}^{\infty}(-1)^n \sin(n^{-4/5})$. Which of the following is true?

A. ★ The series converges conditionally.

B. The series converges absolutely.

C. The series diverges.

Solution. This is an alternating series of the form $\sum_{n=0}^{\infty}(-1)^n b_n$ with $b_n = \sin(n^{-4/5})$.

Since $b_n > 0$, $b_n \to 0$, and $\{b_n\}$ is decreasing, the series converges by the Alternating Series Test. However,

$$\lim_{n\to\infty} \frac{\sin(n^{-4/5})}{\frac{1}{n^{4/5}}} = 1,$$

and so $\sum_{n=0}^{\infty} \left|(-1)^n \sin(n^{-4/5})\right| = \sum_{n=0}^{\infty} \sin(n^{-4/5})$ diverges by the limit comparison test with the p–series with $p = 4/5 < 1$. Therefore the given series converges conditionally.

9/1. (5 points) Consider the series $\sum_{n=1}^{\infty} \frac{(-3)^n x^n}{(2n)!\, n^3}$. Which of the following is true?

A. ★ The series converges for all x.

B. The series converges only for $x = 0$.

C. The series has radius of convergence $R = 1$.

D. The series has radius of convergence $R = 2$.

E. The series has radius of convergence $R = n$.

Solution.

We have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{n+1}|x|^{n+1}}{(2(n+1))!\,(n+1)^3} \frac{(2n)!\,n^3}{3^n|x|^n} = \frac{3|x|}{(2n+2)(2n+1)} \frac{n^3}{(n+1)^3} \to 0,$$

for any value of x. So the series converges for all x.

9/2. (5 points) Consider the series $\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{(3n)!\, n^2}$. Which of the following is true?

A. ★ The series converges for all x.

B. The series converges only for $x = 0$.

C. The series has radius of convergence $R = 1$.

D. The series has radius of convergence $R = 3$.

E. The series has radius of convergence $R = n$.

Solution.

We have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}|x|^{n+1}}{(3(n+1))!\,(n+1)^2} \frac{(3n)!\,n^2}{2^n|x|^n} = \frac{2|x|}{(3n+3)(3n+2)(3n+1)} \frac{n^2}{(n+1)^2} \to 0,$$

for any value of x. So the series converges for all x.

Zone 3

Free response questions. Write complete solutions and show all work for full credit.

10/1. (14 points) Consider the power series $\sum_{n=1}^{\infty} \frac{(x-7)^n}{n(-6)^n}$.

a) Find the radius of convergence of the given series.

b) Find the interval of convergence of the given series.

Solution. a) Since $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-7)(n)}{6(n+1)}\right| \to \left|\frac{x-7}{6}\right|$ as $n \to \infty$, the Ratio Test tells us that the series converges when $|x-7| < 6$. The radius of convergence is 6.

b) When $x = 13$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which converges.

When $x = 1$, the series is $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges. Thus the interval of convergence is $(1, 13]$.

10/2. (14 points) Consider the power series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n(-4)^n}$.

a) Find the radius of convergence of the given series.

b) Find the interval of convergence of the given series.

Solution.

a) Since $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-2)(n)}{4(n+1)}\right| \to \left|\frac{x-2}{4}\right|$ as $n \to \infty$, the Ratio Test tells us that the series converges when $|x-2| < 4$. The radius of convergence is 4.

b) When $x = 6$, the series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which converges.

When $x = -2$, the series is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges. Thus the interval of convergence is $(-2, 6]$.

Zone 4

11/1. (18 points)

a) Find the Maclaurin series representation for $1 - e^{-x}$.

b) Use part (a) to find a representation for $\int \frac{1 - e^{-x}}{x}\, dx$ as an infinite series.

c) Use part (b) to write down a number which approximates $\int_0^1 \frac{1 - e^{-x}}{x}\, dx$ to within 1/50 and explain how you know that your approximation is accurate.

Solution.

a) $1 - e^{-x} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x^n)}{(n)!}$.

b) $\int \frac{1 - e^{-x}}{x}\, dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n \cdot n!} + C$.

c) $\int_0^1 \frac{1 - e^{-x}}{x}\, dx = 1 - \frac{1}{2 \cdot 2!} + \frac{1}{3 \cdot 3!} - \frac{1}{4 \cdot 4!} - \cdots$. This is an alternating series. Since the fourth term is less than 1/50, the alternating series test guarantees that $1 - \frac{1}{4} + \frac{1}{18} = \frac{29}{36}$ approximates the integral to within 1/50.

Zone 5

12/1. (16 points)
a) You are given the values $f(0) = 1, f'(0) = 2, f''(0) = 0, f'''(0) = 2, f^{(4)}(0) = \frac{1}{2}$. Find the degree three Maclaurin polynomial $T_3(x)$ for $f(x)$.

b) You are given that the fourth derivative of the function from part (a) is $f^{(4)}(x) = \dfrac{1}{x+2}$. Use Taylor's theorem to estimate the maximum possible error in the approximation $f(x) \approx T_3(x)$ by the degree 3 Maclaurin polynomial $T_3(x)$, for $0 \le x \le \frac{1}{2}$.

(You should give an answer in numerical form, but you do not need to simplify.)

Solution.
a) $T_3(x) = f(0) + f'(0)x + \dfrac{f''(0)}{2!}x^2 + \dfrac{f'''(0)}{3!}x^3 = 1 + 2x + \dfrac{1}{3}x^3$.
b) By Taylor's theorem, we have $R_3(x) = \dfrac{f^{(4)}(z)}{4!}x^4$ for some z between 0 and x. We have $f^{(4)}(z) = \dfrac{1}{2+z} \le \dfrac{1}{2}$.
So $|R_3(x)| \le \dfrac{\frac{1}{2}\left(\frac{1}{2}\right)^4}{4!}$.

12/2. (16 points)
a) You are given the values $f(0) = 1, f'(0) = 0, f''(0) = 2, f'''(0) = 2, f^{(4)} = \frac{1}{4}$. Find the degree three Maclaurin polynomial $T_3(x)$ for $f(x)$.

b) You are given that the fourth derivative of the function from part (a) is $f^{(4)}(x) = \dfrac{1}{x+4}$. Use Taylor's theorem to estimate the maximum possible error in the approximation $f(x) \approx T_3(x)$ by the degree 3 Maclaurin polynomial $T_3(x)$, for $0 \le x \le 1$.

(You should give an answer in numerical form, but you do not need to simplify.)

Solution.
a) $T_3(x) = f(0) + f'(0)x + \dfrac{f''(0)}{2!}x^2 + \dfrac{f'''(0)}{3!}x^3 = 1 + x^2 + \dfrac{1}{3}x^3$.
b) By Taylor's theorem, we have $R_3(x) = \dfrac{f^{(4)}(z)}{4!}x^4$ for some z between 0 and x. We have $f^{(4)}(z) = \dfrac{1}{4+z} \le \dfrac{1}{4}$.
So $|R_3(x)| \le \dfrac{1}{4}\dfrac{1}{4!}$.