Math 241: Exam 3, April 17, 2018

Name: NetID:

Circle your discussion section:

- ADA: 8am, Nam
- ADB: 9am, Block Gorman
- ADC: 10am, Block Gorman
- ADD: 11am, Shin
- ADE: Noon, Shin
- ADF: 1pm, Mousley
- ADG: 2pm, Okano
- ADH: 3pm, Wojtalewicz
- ADI: 4pm, Wojtalewicz
- ADK: 9am, Christenson
- ADL: 10am, Field
- ADM: 2pm, Gao
- ADN: 3pm, Gao
- ADO: noon, Bavisetty
- ADP: 1pm, Bavisetty
- AD1: 11am, Weigandt
- AD2: 1pm, Rennie
- BD@: 1pm, Zhang, N.
- BDA: 8am, Huynh
- BDB: 9am, Huynh
- BDC: 10am, Park
- BDD: 11am, Han
- BDE: Noon, Park
- BDF: 1pm, Han
- BDG: 2pm, Drake
- BDH: 3pm, Zhang, Y.
- BDI: 4pm, Zhang, Y.
- BDJ: 9am, Field
- BDK: 10am, Christenson
- BDL: noon, Huang
- BDM: 2pm, Mousley
- BDN: 3pm, Okano
- BDO: 4pm, Drake
- BDR: 11am, Huang

Write and bubble in your UIN:

0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9	9

Instructions: You have **75 minutes** to complete this exam. There are **70 points** available and not all problems are weighted equally. Calculators, books, notes, and suchlike aids are **not** permitted. It is not necessary to show work for multiple-choice questions. **For all other questions, show work that justifies your answer** as in those problems **credit will not be given** for correct answers without proper justification. Work written outside of the space provided for a problem will **not** be graded. The last page of the exam contains a **table of trigonometric identities**.

Do not open exam until instructed.

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	7	6	8	8	8	8	10	6	9	70
Score:										

1. (a) **(3 points)** Let $\vec{G} = \nabla g$, where $g(x, y) = y^2 \cos(xy) + x$, and C be the curve parametrized by $\vec{r}(t) = \langle -3\cos t, 2\sin t\rangle$, for $0 \le t \le \frac{\pi}{2}$. Evaluate the integral

$$\int_C \vec{G} \cdot \mathrm{d}\vec{r}.$$

Solution:
By the Fundamental Theorem of Line Integrals

$$\int_C \vec{G} \cdot \mathrm{d}\vec{r} = \int_C \nabla g \cdot \mathrm{d}\vec{r} = g(\vec{r}(\frac{\pi}{2})) - g(\vec{r}(0)) = 7$$

(circle one): (A) -4; (B) 4; (C) 3; (D) 7;

(b) **(4 points)** Consider the vector field $\vec{F}$ and the curve C parameterized by $\vec{r}(t)$ below.

$$\vec{F}(x, y, z) = \langle yz + 2xz^2 + y, xz + x, xy + 2x^2 z\rangle,$$
$$C : \vec{r}(t) = \langle t + 1, t^3 + 2, 2t^2 - 2t^4\rangle, \text{ for } 0 \le t \le 1.$$

Evaluate the integral $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$.

Solution:
Solving for $\nabla f = \vec{F}$, gives $f(x, y, z) = xyz + x^2 z^2 + xy$.
Since $\vec{F}(x, y, z)$ is a conservative vector field, with $\nabla f = \vec{F}$,
$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_C \nabla f \cdot \mathrm{d}\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0))$ (by the Fundamental Theorem of Line Integrals) $= f(2,3,0) - f(1,2,0) = 4$.

(circle one): (A) 4; (B) 3; (C) 1; (D) 5;

2. (a) **(3 points)** Determine whether or not $\vec{F}$ is a conservative vector field. If it is, find a function f such that $\vec{F} = \nabla f$. If the vector field is not conservative, enter "DNE" or "does not exist".

$$\vec{F}(x,y) = \langle -y^2 e^{-x} + 4x, 2ye^{-x} \rangle$$

Solution:
Writing $\vec{F}$ as $\vec{F} = \langle P, Q \rangle$, since $P_y = -2ye^{-x} = Q_x$, $\vec{F}$ is conservative. Solving $\vec{F} = \nabla f$, gives $f(x,y) = y^2 e^{-x} + 2x^2$.

$f(x,y) =$

(b) **(3 points)** Determine whether or not $\vec{G}$ is a conservative vector field. If it is, find a function g such that $\vec{G} = \nabla g$. If the vector field is not conservative, enter "DNE" or "does not exist".

$$\vec{G}(x,y) = \langle e^{xy} \cos y, e^{xy} \sin y \rangle$$

Solution:
Writing $\vec{G}$ as $\vec{G} = \langle P, Q \rangle$, since $P_y = xe^{xy} \cos y - e^{xy} \sin y \neq ye^{xy} \sin y = Q_x$,
$\vec{G}$ is **not** conservative. ($g(x,y) =$ "DNE".)

$g(x,y) =$

3. (a) **(4 points)** Calculate the volume of the solid occupying the region under the plane $-2x - 2y + z = 1$ and above the rectangle $R = \{(x, y, 0) \mid 0 \le x \le 2, 0 \le y \le 1\}$.

Solution:

The solid lies under the plane $z = 2x + 2y + 1$ and above the rectangle, so its volume is:

$$\iint_R 2x + 2y + 1 \, \mathrm{d}A = \int_0^1 \int_0^2 2x + 2y + 1 \, \mathrm{d}x \mathrm{d}y = 8$$

(circle one): (A) 12; (B) 8; (C) 10; (D) 11;

(b) **(4 points)** Let R be the region in the 1^{st} quadrant of the xy-plane between $y = 1$ and $y = x^{1/5}$. Evaluate

$$\iint_R 6\sqrt{y^6 + 1} \, \mathrm{d}A.$$

Solution:

$$\iint_R 6\sqrt{y^6 + 1} \, \mathrm{d}A = \int_0^1 \int_0^{y^5} 6\sqrt{y^6 + 1} \mathrm{d}x \mathrm{d}y = \int_0^1 6y^5 \sqrt{y^6 + 1} \, \mathrm{d}y = \frac{2(2^{\frac{3}{2}} - 1)}{3}$$

(circle one): (A) $(2\sqrt{2} - 1)$; (B) $\frac{(2\sqrt{2}-1)}{6}$; (C) $\frac{2(2\sqrt{2}-1)}{15}$; (D) $\frac{2(2\sqrt{2}-1)}{3}$;

4. (a) **(4 points)** Evaluate

$$\int_{-2}^{0}\int_{-\sqrt{4-x^2}}^{0} 2\sin(x^2+y^2)\,\mathrm{d}y\mathrm{d}x.$$

Hint: Sketch the region of integration.

Solution:

Using polar coordinates:

$$\int_{-2}^{0}\int_{-\sqrt{4-x^2}}^{0} 2\sin(x^2+y^2)\,\mathrm{d}y\mathrm{d}x = \int_{\pi}^{\frac{3\pi}{2}}\int_{0}^{2} 2r\sin(r^2)\,\mathrm{d}r\mathrm{d}\theta \;=\; \frac{\pi(1-\cos(4))}{2}$$

(circle one): (A) $\frac{\pi}{2}(1-\cos(4))$; (B) $\frac{-\pi}{2}\cos(4)$; (C) $\frac{\pi\cos(4)}{2}$; (D) $\frac{3\pi\cos(4)}{2}$;

(b) **(4 points)** Find the mass of the lamina that occupies the region $D = \{(x, y) \mid y \le 0,\, 1 \le x^2 + y^2 \le 9\}$, whose density at any point in D is $\rho(x, y) = x^2$.

Solution:

Using polar coordinates:

$$\text{mass} = \int_{\pi}^{2\pi}\int_{1}^{3} r^3\cos^2\theta\,\mathrm{d}r\mathrm{d}\theta \;=\; \frac{80}{4}\int_{\pi}^{2\pi}\cos^2\theta\,\mathrm{d}\theta \;=\; 10\pi$$

(circle one): (A) $\frac{13\pi}{3}$; (B) 20π; (C) 10π; (D) $\frac{26\pi}{3}$;

5. (a) **(4 points)** Let $E = \{(x, y, z) \mid x \geq 0, 0 \leq y \leq 3, 0 \leq z \leq 4 - x^2\}$. Evaluate the triple integral

$$\iiint_E 2y \, \mathrm{d}V.$$

Solution:

$$\iiint_E 2y \, \mathrm{d}V = \int_0^2 \int_0^{4-x^2} \int_0^3 2y \, \mathrm{d}y \mathrm{d}z \mathrm{d}x =$$

$$\int_0^2 \int_0^{4-x^2} 9 \, \mathrm{d}z \mathrm{d}x = 9 \int_0^2 (4 - x^2) \, \mathrm{d}x = 48$$

(circle one): (A) 32; (B) 48; (C) $\frac{32}{3}$; (D) 96;

(b) **(4 points)** Set up (but do **NOT** evaluate) an integral to find the mass of the solid T with density function $\rho(x, y, z) = 4xy + 2z$, where T is the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 4$.

Note: This integral must be set up in the specified order of integration.

Solution:

$$\text{Mass of } T = \int_0^4 \int_0^{4-x} \int_0^{4-x-y} 4xy + 2z \, \mathrm{d}z \mathrm{d}y \mathrm{d}x$$

$$\int_{\square}^{\square} \int_{\square}^{\square} \int_{\square}^{\square} \square \, \mathrm{d}z \mathrm{d}y \mathrm{d}x$$

6. (a) **(4 points)** Let E be the solid in the first octant that lies beneath the paraboloid $z = 16 - x^2 - y^2$. Evaluate the triple integral $\iiint_E (x^2 y + y^3)\,\mathrm{d}V$.

Solution:

$$\iiint_E (x^2 y + y^3)\,\mathrm{d}V = \int_0^{\frac{\pi}{2}} \int_0^4 \int_0^{16-r^2} r^4 \sin\theta\,\mathrm{d}z\mathrm{d}r\mathrm{d}\theta =$$

$$\int_0^{\frac{\pi}{2}} \int_0^4 (16 - r^2) r^4 \sin\theta\,\mathrm{d}r\mathrm{d}\theta = (\frac{4^7}{5} - \frac{4^7}{7}) \int_0^{\frac{\pi}{2}} \sin\theta\,\mathrm{d}\theta = \frac{4^7}{5} - \frac{4^7}{7}.$$

(circle one): (A) $\frac{1}{5} - \frac{1}{7}$; (B) $\frac{4^7}{5} - \frac{4^7}{7}$; (C) $\frac{4^6}{4} - \frac{4^6}{6}$; (D) $\frac{4^7}{7} - \frac{4^7}{5}$;

(b) **(4 points)** Let $H = \{(x, y, z) \,|\, x^2 + y^2 + z^2 \leq 4,\ y \geq 0, \text{ and } z \geq 0\}$. Evaluate the triple integral

$$\iiint_H \frac{z}{2}\,\mathrm{d}V.$$

Solution:

$$\iint_H \frac{z}{2}\,\mathrm{d}V = \int_0^{\pi} \int_0^{\frac{\pi}{2}} \int_0^2 \frac{1}{2} \rho \cos\phi \rho^2 \sin\phi\,\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta = 2 \int_0^{\pi} \int_0^{\frac{\pi}{2}} \cos\phi \sin\phi\,\mathrm{d}\phi\mathrm{d}\theta = \pi$$

(circle one): (A) 0; (B) 2π; (C) π; (D) 4π;

7. Let E be the solid which lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 2z$.

Note: The integrals in parts (a) and (b) must be set up in the specified order of integration.

(a) **(4 points)** Set up (but do **NOT** evaluate) the integral $\iiint_E 1 \, \mathrm{d}V$ using cylindrical coordinates.

Solution:

$$\iiint_E 1 \, \mathrm{d}V = \int_0^{2\pi} \int_0^1 \int_r^{1+\sqrt{1-r^2}} r \, \mathrm{d}z \mathrm{d}r \mathrm{d}\theta$$

$$\int_{\boxed{}}^{\boxed{}} \int_{\boxed{}}^{\boxed{}} \int_{\boxed{}}^{\boxed{}} \boxed{} \, \mathrm{d}z \mathrm{d}r \mathrm{d}\theta$$

(b) **(4 points)** Set up (but do **NOT** evaluate) the integral $\iiint_E 1 \, \mathrm{d}V$ using spherical coordinates.

Solution:

$$\iiint_E 1 \, \mathrm{d}V = \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \int_0^{2\cos\phi} \rho^2 \sin(\phi) \, \mathrm{d}\rho \mathrm{d}\theta \mathrm{d}\phi$$

$$\int_{\boxed{}}^{\boxed{}} \int_{\boxed{}}^{\boxed{}} \int_{\boxed{}}^{\boxed{}} \boxed{} \, \mathrm{d}\rho \mathrm{d}\theta \mathrm{d}\phi$$

(c) **(2 points)** Now, compute $\iiint_E 1 \, \mathrm{d}V$.

Solution:

Since E is a cone with a hemisphere on top, its volume is

$$\frac{\pi}{3}(\text{radius})^2(\text{height}) + \frac{1}{2}\left(\frac{4}{3}\pi(\text{radius})^3\right) = \pi.$$

$$\boxed{\iiint_E 1 \, \mathrm{d}V = }$$

8. (a) **(2 points)** Find the Jacobian of the transformation given by

$$x = u^2 + 2uv, \quad y = uv^2.$$

Solution: $\frac{\partial(x,y)}{\partial(u,v)} = 4u^2v + 2uv^2$.

$\frac{\partial(x,y)}{\partial(u,v)}$ is (circle one):
$(A)\quad 4u^2v + 2uv^2;\qquad (B)\quad 4u^2v;$
$(C)\quad 4u^2v + 6uv^2;\qquad (D)\quad 4uv^2 + 4u^2v;$

(b) **(4 points)** Let R be the region bounded by the ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$. Use the transformation $x = 5u, \ y = 2v$ to evaluate

$$\iint_R 100e^{4x^2+25y^2}\,\mathrm{d}A.$$

Solution: $\frac{\partial(x,y)}{\partial(u,v)} = 10$, and the transformation maps the region $S = \{(u,v) \mid u^2 + v^2 \leq 1\}$ to R. Using change of variables, we have $\iint_R 100e^{4x^2+25y^2}\,\mathrm{d}A = \iint_S 100e^{100(u^2+v^2)}10\,\mathrm{d}A = 10\int_0^{2\pi}\int_0^1 100re^{100r^2}\,\mathrm{d}r\mathrm{d}\theta = 10\pi(e^{100}-1)$.

(circle one):
$(A)\quad 20\pi(e^{100}-1);\qquad (B)\quad 10\pi(e^{100}-1);$
$(C)\quad 10\pi e^{100};\qquad (D)\quad \pi(e^{100}-1);$

9. Let $S = \{(u,v) : 0 \le u \le 1, 0 \le v \le 1\}$ be the unit square and let $R \subset \mathbb{R}^2$ be the parallelogram with vertices $(0,0)$, $(2,-2)$, $(3,3)$ and $(5,1)$.

(a) **(4 points)** Find a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(S) = R$.

Solution: If we set $\vec{a} = \langle 3,3\rangle - \langle 0,0\rangle = \langle 3,3\rangle$ and $\vec{b} = \langle 2,-2\rangle - \langle 0,0\rangle = \langle 2,-2\rangle$, then the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by:

$$T(u,v) = u\vec{a} + v\vec{b} = (3u, 3u) + (2v, -2v) = (3u+2v, 3u-2v).$$

$$\boxed{T(u,v) = (\qquad\qquad , \qquad\qquad)}$$

(b) **(5 points)** Evaluate the integral:

$$\iint_R xy \, \mathrm{d}A.$$

Hint: It may be easier to use the change of variables $T(u,v) = (x,y)$ that you found in part (a).

Solution: Using the hint, we make the change of variables provided by T:

$$x = 3u + 2v, \quad y = 3u - 2v.$$

Its Jacobian is given by:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 3 & -2 \end{vmatrix} = -12$$

so the change of variable formula gives:

$$\iint_R xy \, \mathrm{d}A = \iint_S (3u+2v)(3u-2v)\ 12\ \mathrm{d}u\mathrm{d}v =$$

$$12\int_0^1 \left(\int_0^1 (9u^2 - 4v^2)\ \mathrm{d}u\right)\mathrm{d}v = 12\int_0^1 \left(3 - 4v^2\right)\mathrm{d}v = 12\left(3 - \frac{4}{3}\right) = 20.$$

$$\boxed{\iint_R xy \, \mathrm{d}A = \qquad\qquad}$$

Trigonometric Identities

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$
$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
$$\tan(\theta+\phi) = \frac{\tan(\theta)+\tan(\phi)}{1-\tan\theta\tan\phi}$$
$$\sin(\theta-\phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$$
$$\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$$
$$\tan(\theta-\phi) = \frac{\tan(\theta)-\tan(\phi)}{1+\tan\theta\tan\phi}$$
$$\sin(\theta)\sin(\phi) = \frac{\cos(\theta-\phi)-\cos(\theta+\phi)}{2}$$
$$\cos(\theta)\cos(\phi) = \frac{\cos(\theta-\phi)+\cos(\theta+\phi)}{2}$$
$$\sin(\theta)\cos(\phi) = \frac{\sin(\phi+\theta)-\sin(\phi-\theta)}{2}$$
$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$
$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$
$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$
$$\tan\theta = \frac{\sin(2\theta)}{1+\cos(2\theta)}$$