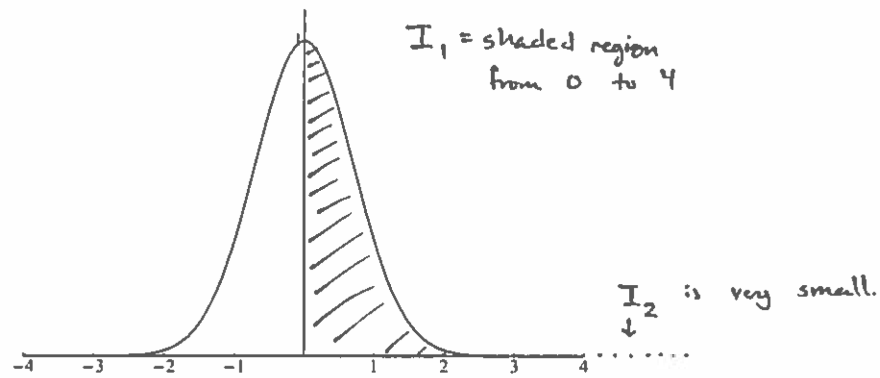
Group: ____________ ____________ ____________ Name: Solutions.

Math 231 A. Fall, 2015. Worksheet 7. 9/24/15

1. We consider the famous "Bell Curve," defined by $y = e^{-x^2}$. In particular, we will try to determine the area under the curve (from $-\infty$ to ∞). By symmetry, this area is $2\int_0^\infty e^{-x^2}\,dx$. Remember that there is no elementary way to express the antiderivative $\int e^{-x^2}\,dx$.



(a) Use a simple comparison to prove that the integral $I = \int_0^\infty e^{-x^2}\,dx$ converges.

$e^{x^2} \geq 1+x^2$ for large x, so $\frac{1}{e^{x^2}} \leq \frac{1}{1+x^2}$ for large x.

and since: $\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \arctan(b) - \arctan(0) = \frac{\pi}{2}$ converges,

so $I = \int_0^\infty e^{-x^2}\,dx$ converges too.

(b) Write $I = I_1 + I_2$, where $I_1 = \int_0^4 e^{-x^2}\,dx$ and $I_2 = \int_4^\infty e^{-x^2}\,dx$. Estimate I_1 using Simpson's rule with $n = 8$. Keep six decimal places of accuracy in your calculations.

$\Delta x = \frac{4-0}{8} = \frac{1}{2}$. so

$$\int_0^4 e^{-x^2}\,dx \approx \frac{\Delta x}{3}\left(e^0 + 4e^{-1/4} + 2e^{-1} + 4e^{-9/4} + 2e^{-4} + 4e^{-25/4} + 2e^{-9} + 4e^{-49/4} + e^{-16}\right)$$

$= .886196$

(c) Notice that $e^{-x^2} \leq e^{-4x}$ if $x \geq 4$. Use this fact to show that $I_2 \leq 0.0000001$.

$$I_2 = \int_4^\infty e^{-x^2}\,dx \leq \int_4^\infty e^{-4x}\,dx = \lim_{b\to\infty} -\frac{1}{4}e^{-4x}\Big|_{x=4}^{x=b} = \lim_{b\to\infty} \cancel{-\frac{1}{4}e^{-4b}}^{\,0} + \frac{1}{4}e^{-16} = \frac{1}{4}e^{-16} = 2.8\times10^{-8}$$

so $I_2 \leq 0.0000001$

(d) Make an educated guess as to the area under the Bell Curve. Hint: You have approximated I to high accuracy. Do you recognize the value of I? I^2? $2I^2$? etc.?

$2I = 2I_1 + 2I_2 \approx 2I_1 \approx 2(.886196) = 1.772392$

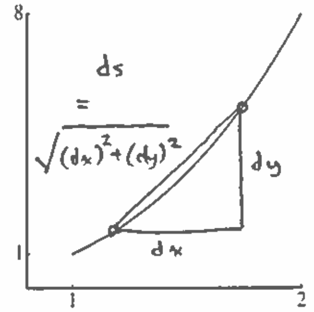
since I_2 is so small

$(2I)^2 \approx (1.772392)^2 = 3.141373$ looks close to π.

so maybe $\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}$ (in fact, this is true).

We work with the arclength differential $ds = \sqrt{(dx)^2 + (dy)^2}$ and the formula $S = \int ds$. See your lecture notes from Wednesday. This formula must be correctly interpreted in each case to produce an expression which is ready to be evaluated.

2. The curve $y = x^3$ between the points $(1, 1)$ and $(2, 8)$ is shown.



a) Indicate the meaning of the arclength differential ds on the curve.

b) Set up but do not evaluate an integral **with respect to** x for the length. All quantities involved must refer to x.

$$\text{length} = \int ds = \int \sqrt{(dx)^2 + (dy)^2} = \int \sqrt{\left(\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2\right)\cdot (dx)^2} = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

and from $x = 1$ to 2 with $y = x^3$, so $\frac{dy}{dx} = 3x^2$, we get

$$\text{length} = \int_1^2 \sqrt{1 + 9x^4}\, dx$$

c) Set up but do not evaluate an integral **with respect to** y which represents the length. All quantities involved must refer to y.

similarly, $\int ds = \int \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$, so from $y = 1$ to 8 and with $x = y^{\frac{1}{3}}$, $\frac{dx}{dy} = \frac{1}{3} y^{-\frac{2}{3}}$

$$\text{length} = \int_1^{\infty} \sqrt{\frac{1}{9} y^{-4/3} + 1}\, dy.$$

3. Find the length of the curve $y = \ln(\cos x)$, $0 \le x \le \pi/3$.

$$\frac{dy}{dx} = \frac{-\sin(x)}{\cos(x)} = -\tan(x)$$

$$\text{length} = \int_0^{\pi/3} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{\pi/3} \sqrt{1 + (-\tan(x))^2}\, dx = \int_0^{\pi/3} \sqrt{1 + \tan^2 x}\, dx$$

$$= \int_0^{\frac{\pi}{3}} \sqrt{\sec^2 x}\, dx = \int_0^{\pi/3} \sec x\, dx = \ln\left|\sec x + \tan x\right| \Bigg|_{x=0}^{x=\frac{\pi}{3}}$$

$$= \ln\left|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right| - \ln\left|\sec 0 + \tan 0\right|$$

$$= \ln\left|2 + \sqrt{3}\right| - \ln\left|1 + 0\right|$$

$$= \ln\left(2 + \sqrt{3}\right).$$