Multiple choice. Mark answers on scantron form. Test form A.

For each series on this page:

- Mark **A** if the series **Converges Absolutely**.
- Mark **C** if the series **Converges Conditionally**.
- Mark **D** if the series **Diverges**.

1. (4 points) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ conv. conditionally

$\displaystyle\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{\ln n}$ div. by Comp. Test $\left(\frac{1}{\ln n} \geq \frac{1}{n} \text{ for } n \geq 3 \text{ and } \sum_{n=2}^{\infty} \frac{1}{n} \text{ div} \right)$

$\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ conv. by A.S.T. $\left(b_n = \frac{1}{\ln n} \geq 0 \text{ for } n \geq 3,\ \lim_{n\to\infty} \frac{1}{\ln n} = 0,\ \{b_n\} \text{ dec.} \right)$

2. (4 points) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n \cdot n^2}{n!}$ conv. abs. by Ratio Test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-3)^{n+1}(n+1)^2}{(n+1)!} \cdot \frac{n!}{(-3)^n\, n^2} \right|$$

$$= \lim_{n\to\infty} \frac{3(n+1)^2}{(n+1)\, n^2} = \lim_{n\to\infty} \frac{3(n+1)}{n^2} = 0 < 1$$

3. (4 points) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n+2}{n^3+1}$ conv. abs.

$\displaystyle\sum_{n=1}^{\infty} \left| (-1)^n \frac{n+2}{n^3+1} \right| = \sum_{n=1}^{\infty} \underbrace{\frac{n+2}{n^3+1}}_{a_n}$ conv. by L.C.T

Let $b_n = \frac{n}{n^3} = \frac{1}{n^2}$. Then $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+2}{n^3+1} \cdot \frac{n^2}{1} = 1 > 0$

Since $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ conv. $(p = 2 > 1)$, $\displaystyle\sum_{n=1}^{\infty} a_n$ conv.

4. (5 points) Find the Maclaurin series for $x^2 \sin(3x)$.

(A) $\sum_{n=0}^{\infty} \frac{(-3)^n x^{2n+3}}{(2n+3)!}$

(B) $\sum_{n=0}^{\infty} \frac{(-3)^n (x)^{2n+3}}{(2n)!}$

(C) $\sum_{n=0}^{\infty} \frac{(-3)^{2n+1} x^{2n+1}}{(2n+1)!}$

(D) $\sum_{n=0}^{\infty} \frac{(-3)^{2n+1} x^{6n+3}}{(2n+1)!}$

(E) $\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+3}}{(2n+1)!}$

$$x^2 \sin(3x) = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{(3x)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+3}}{(2n+1)!} \quad \text{for all } x$$

5. (5 points) Find the Maclaurin series for $f(x) = \frac{x}{1+4x^2}$.

(A) $\sum_{n=0}^{\infty} (-2)^{2n} x^{2n+1}$

(B) $\sum_{n=0}^{\infty} (-1)^n (2x)^{2n+1}$

(C) $\sum_{n=0}^{\infty} (-2)^n x^{3n+3}$

(D) $\sum_{n=0}^{\infty} 4^n x^{2n+1}$

(E) $\sum_{n=0}^{\infty} (-1)^n 4^n x^{2n+1}$

$$f(x) = x \cdot \frac{1}{1-(-4x^2)}$$

$$= x \sum_{n=0}^{\infty} (-4x^2)^n \quad \text{for } |-4x^2| < 1$$

$$= \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n+1} \quad \text{for } |x| < 1/2$$

Consider the statements:

(1) The series diverges by the ratio test.
(2) The series converges by the ratio test.
(3) The ratio test is inconclusive.
(4) The series converges absolutely.
(5) The series converges conditionally.
(6) The series converges by the alternating series test.

6. (5 points) Find the correct choice for the series $\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+1}$.

(A) Statements 1 and 6 are both true.
(B) Statements 1 and 5 are both true.
(C) Statements 2 and 5 are both true.
(D) Statements 3 and 4 are both true.
(E) Statements 3 and 5 are both true.

Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n+1}{(n+1)^2+1}\cdot\frac{n^2+1}{n} = 1$$

$\to (3)$

$$\sum_{n=1}^{\infty}\left|(-1)^n\frac{n}{n^2+1}\right| = \sum_{n=1}^{\infty}\frac{n}{n^2+1} \text{ div. by L.C.T}$$

$(b_n = \frac{1}{n})$

(5) $\leftarrow$

$$\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+1} \text{ conv. by A.S.T}$$

$(b_n = \frac{n}{n^2+1})$

$\to (6)$

7. (5 points) Find the correct choice for the series $\sum_{n=0}^{\infty}\frac{(-3)^{n+1}}{(n+1)!}$.

(A) Statements 3 and 6 are both true.
(B) Statements 3 and 5 are both true.
(C) Statements 2 and 4 are both true.
(D) Statements 2 and 5 are both true.
(E) Statements 1 and 5 are both true.

Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-3)^{n+2}}{(n+2)!}\frac{(n+1)!}{(-3)^{n+1}}\right|$$

$$= \lim_{n\to\infty}\frac{3}{n+2} = 0 < 1$$

$\to$ series conv. abs.

(2) & (4)

Suppose that the power series $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = -3$ and diverges when $x = 5$. What can be said about the convergence or divergence of the following series?

8. (4 points) $\sum_{n=0}^{\infty} c_n(-6)^n$

conv. div

-3 0 5

$\Rightarrow 3 \le R \le 5$

(A) Converges.

(B) Diverges.

(C) Impossible to determine with the information given.

9. (4 points) $\sum_{n=0}^{\infty} c_n 3^n$

(A) Converges.

(B) Diverges.

(C) Impossible to determine with the information given.

It can be the end pt. of the interval of conv.

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10. (2 points) Which is the correct definition of the Taylor series of $f(x)$ centered at $\underline{\underline{a}}$?

(A) $\sum_{n=0}^{\infty} \frac{f^{(n)}\cancel{(x)}}{n!}(x-a)^n$

(B) $\sum_{\cancel{n=1}}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

(C) $\sum_{n=0}^{\infty} \frac{f^{(n)}\cancel{(0)}}{n!}(x-a)^n$

(D) $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

(E) $\sum_{n=1}^{\infty} \frac{f^{(n)}\cancel{(x)}}{n!}(x-a)^n$

11. (5 points) Let $f(x) = \arctan x$. You are given that $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ for $|x| < 1$. Use this to evaluate the derivative $f^{(23)}(0)$.

(A) $-22!$
(B) $-23!$
(C) $-1/23$
(D) $1/23$
(E) $-1/22$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$k=23$ $\quad$ $n=11$

$$\frac{f^{(23)}(0)}{23!} x^{23} = (-1)^{11} \frac{x^{23}}{23}$$

$$\Rightarrow f^{(23)}(0) = -\frac{23!}{23} = -22!$$

12. (5 points) The first nine terms of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ are used to approximate the sum of the series. According to the alternating series estimation theorem, what is the maximum error in this approximation?

(A) $1/9^3$
(B) $1/10^3$
(C) $3/9^2$
(D) $3/10^2$
(E) None of the above.

$$|R_9| \leq b_{10} = \frac{1}{10^3}$$

2. (12 points) Consider the power series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n(-2)^n}$.

a) Find the radius of convergence.

Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{(n+1)(-2)^{n+1}} \cdot \frac{n(-2)^n}{(x-3)^n}\right|$$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)\frac{|x-3|}{2} = \frac{|x-3|}{2} < 1$$

$$|x-3| < 2 = R$$

b) Find the interval of convergence.

$x=1$; $\sum_{n=1}^{\infty} \frac{(-2)^n}{n(-2)^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ div. (Harmonic series)

$x=5$; $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ conv. by A.S.T

$$\left(\begin{array}{l} b_n = \frac{1}{n} \geq 0 \\ \lim_{n\to\infty} b_n = 0 \\ \{b_n\} \text{ dec} \end{array}\right)$$

$I = (1, 5]$

3. (12 points)
a) Find the Maclaurin series for e^{3x^2} and give its radius of convergence.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x \quad (R=\infty)$$

$$e^{3x^2} = \sum_{n=0}^{\infty} \frac{(3x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n}{n!} x^{2n} \quad \text{for all } x$$

$$R = \infty$$

b) Use this to evaluate the integral $\int_0^2 e^{3x^2}\, dx$ (your answer will be an infinite series).

$$\int_0^2 e^{3x^2} dx = \int_0^2 \sum_{n=0}^{\infty} \frac{3^n}{n!} x^{2n} dx$$

$$= \sum_{n=0}^{\infty} \frac{3^n}{n!} \int_0^2 x^{2n} dx$$

$$= \sum_{n=0}^{\infty} \frac{3^n}{n!} \left[\frac{x^{2n+1}}{2n+1} \right]_0^2$$

$$= \sum_{n=0}^{\infty} \frac{3^n \, 2^{2n+1}}{n! \, (2n+1)}$$

4. (12 points)
a) Use the binomial series to find the Maclaurin series for $f(x) = (1+3x^2)^{-1/2}$ (you do not need to simplify the binomial coefficients which arise). Give the radius of convergence.

$$(1+3x^2)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} (3x^2)^n \quad \text{for } |3x^2|<1$$

$$= \sum_{n=0}^{\infty} \binom{-1/2}{n} 3^n x^{2n} \quad \text{for } |x| < 1/\sqrt{3} = R.$$

b) Write out the first three non-zero terms in this series and simplify as much as you can.

$$\binom{-1/2}{0} + \binom{-1/2}{1} 3x^2 + \binom{-1/2}{2} 3^2 x^4$$

$$= 1 + \frac{(-1/2)}{1!} 3x^2 + \frac{(-1/2)(-3/2)}{2!} 3^2 x^4$$

$$= 1 - \frac{3}{2}x^2 + \frac{27}{8}x^4$$

c) The first <u>two</u> non-zero terms are used to approximate $f(x)$ in the range $0 \le x \le \frac{1}{3}$. What is the maximum error in this approximation?

By A.S. Error Estimation,

$$|R| \le \frac{27}{8}x^4 \le \frac{27}{8}\left(\frac{1}{3}\right)^4 = \frac{1}{24}.$$