PRACTICE SOLNS.

10^{-7}

7.

$4! = 24$ $7! = 5040$
$5! = 120$ $8! = 40320$
$6! = 720$

Old exam questions from Math 231

1. (i) Find the first five non-zero terms of a power series representation of

$$\int \frac{\cos x - 1}{x^2} dx.$$

(ii) Estimate $\int_0^{\frac{1}{10}} \frac{\cos x - 1}{x^2} dx$ with an error less than 10^{-9} and justify your answer.

2. Starting with a known Maclaurin series, find the Maclaurin series of

$$\frac{x^2}{x+5}.$$

Express your answer in summation notation. What is the radius of convergence of the Maclaurin series you found?

3. Suppose $g(x) = (\sin x)e^{-x^2} = \sum_{n=0}^{\infty} b_n x^n$ for all real x.

(i) (12 points) Find b_7, the seventh Maclaurin coefficient of $g(x)$. You must find a numerical value for b_7, but you *need not* simplify your answer.

(ii) (8 points) Evaluate $g^{(7)}(0)$. You *need not* simplify your answer.

4. Find the exact interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(n+1)(x-2)^n}{3^n}.$$

4. a) Use the binomial theorem to find the MacLaurin series for

$$f(x) = x(1+x^2)^{1/3}.$$

Express your answer using summation notation, but **DO NOT** expand the binomial coe cients which arise.

b) Use your answer from part (a) to evaluate the following limit. (No credit for using L'Hopital's rule.)

$$\lim_{x \to 0} \frac{f(x) - \sin(x)}{x^3}.$$

#1

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$ Note: $n=0$ term is $\boxed{1}$

$$\cos x - 1 = \sum_{\boxed{n=1}}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$ (subtracting 1 kills $n=0$ term)

$$\frac{\cos x - 1}{x^2} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-2}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n+2)!}$$

$$= -\frac{1}{2!} + \frac{x^2}{4!} - \frac{x^4}{6!} + \cdots$$

$$\int \frac{\cos x - 1}{x^2} = -\frac{x}{2!} + \frac{x^3}{3\cdot 4!} - \frac{x^5}{5\cdot 6!} + \cdots \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)(2n+2)!}$$

$$-\frac{x}{2!} + \frac{x^3}{3\cdot 4!} - \frac{x^5}{5\cdot 6!} + \frac{x^7}{7\cdot 8!} - \frac{x^9}{9\cdot 10!} + \cdots$$

by Alt series ~~...~~ estimate error is less than first omitted term. Taking $x = \frac{1}{10}$ $x^5 = 10^{-5}$ $5 \times 6! = 3600 = 3.6 \times 10^3$

$$\frac{x^5}{5\cdot 6!} \approx \frac{1}{3.6} \times 10^{-8} > 10^{-9}$$ (Not quite good enough)

Taking first three terms

$$\frac{x^7}{7\cdot 8!} \sim \frac{10^{-7}}{2.8\times 10^5} \ll 10^{-9}$$

$$\left| \int_0^{.1} \frac{\cos x - 1}{x^2} - \frac{.1}{2!} + \frac{(.1)^3}{3\cdot 24} - \frac{(.1)^5}{5\cdot 720} \right| < 10^{-9}$$

#2 $\dfrac{x^2}{x+5} = \dfrac{1}{5}\,\dfrac{x^2}{1+\frac{x}{5}}$

$$\frac{1}{1+x} = \sum (-1)^n x^n$$

$$\frac{1}{1+\frac{x}{5}} = \sum \frac{(-1)^n}{5^n} x^n$$

$$\frac{x^2}{1+\frac{x}{5}} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+2}}{5^n}$$

$$\frac{x^2}{5+x} = \sum \frac{(-1)^n x^{n+2}}{5^{n+1}}$$

#3 $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dots$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots$$

$$\sin x \, e^{-x^2} = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right)\left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots\right)$$

$-\dfrac{1}{3!}$

$-\dfrac{1}{3!}\cdot\dfrac{1}{2!}$

$-\dfrac{1}{5!}$

$-\dfrac{1}{7!}$

coefficient of x^7 is $-\dfrac{1}{3!} - \dfrac{1}{3!}\cdot\dfrac{1}{2!} - \dfrac{1}{5!} - \dfrac{1}{7!}$

<u>But</u> coeff of x^7 is also $\dfrac{g^{(7)}(0)}{7!}$

$$\therefore \quad g^{(7)}(0) = -7!\left(\frac{1}{3!} + \frac{1}{2\cdot 3!} + \frac{1}{5!} + \frac{1}{7!}\right)$$

#4 $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-2|}{3}\lim_{n\to\infty}\left|\frac{n+2}{n+1}\right| = \frac{|x-2|}{3}$

converges $\left|\frac{x-2}{3}\right| < 1$

Abs. Convergent $|x-2| < 3$ $\quad x \in (-1, 5)$

boundary pts $x=-1$, $x=5$

$x = -1$ $\quad \sum \frac{(n+1)(-3)^n}{3^n} = \sum (-1)^n (n+1)$

Diverges! (Terms don't $\to 0$)

$x = 5$ $\quad \sum (n+1)$

Diverges!

Interval converg. $(-1, 5)$

#5 $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$

$(1+x^2)^k = \sum \binom{k}{n} x^{2n}$

$(1+x^2)^{\frac{1}{3}} = \sum \binom{1/3}{n} x^{2n}$

$x(1+x^2)^{\frac{1}{3}} = \sum_{n=0}^{\infty} \binom{1/3}{n} x^{2n+1} = x + \frac{1}{3}x^3 + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!} x^5$

$\sin x = x - \frac{x^3}{3!}$

$\frac{f(x)-\sin x}{x^3} = \frac{\left(\frac{1}{3}+\frac{1}{3!}\right)x^3 + (\ )x^5}{x^3} = \frac{\frac{1}{3}+\frac{1}{3!}+(\ )x^2}{1}$

$\lim_{x\to 0} \frac{f(x)-\sin x}{x^3} = \frac{1}{3}+\frac{1}{6} = \boxed{\frac{1}{2}}$

#6) a) $f(x) = e^{3x}$ center: $x=0$

$$T_2(x) = \sum_{n=0}^{2} \frac{(3x)^n}{n!} = 1 + 3x + \frac{9x^2}{2}$$

b) $$f(x) - T_2(x) = R_2(x) = \frac{f^{(3)}(z)}{3!} x^3 = \frac{9e^{3z}}{2} x^3$$

where $0 \le z \le x \le 1/2$

so $$|R_2(x)| \le \frac{9e^{3/2}}{2} \left(\frac{1}{2}\right)^3 = \frac{9e^{3/2}}{16}$$