University of Illinois at Urbana-Champaign
Dept. of Electrical and Computer Engineering

ECE 120: Introduction to Computing

Good Representations and
Modular Arithmetic

What About Negative Numbers?

Last time, we developed
- the **N-bit unsigned representation**
- for integers in the range **$[0, 2^N - 1]$**

Now, let's think about negative numbers.
- How should we represent them?
- Can we use a minus sign?

$$-\mathbf{11000} = -24_{10} \text{ ?}$$

There's no "–" in a bit!

One Option: The Signed-Magnitude Representation

But we can use another bit for a sign:

0 → +, and **1 → –**

Doing so gives the
N-bit signed-magnitude representation:

sign	(N-1)-bit magnitude

This representation can represent numbers in the range **$[-2^{N-1} - 1, 2^{N-1} - 1]$**.

What Happened to the Last Bit Pattern?

Signed-magnitude was used in some early computers (such as the IBM 704 in 1954).

A question for you:
- The range represented is **$[-2^{N-1} - 1, 2^{N-1} - 1]$**.
- That gives **$2^N - 1$ different numbers**.
- **What's the last pattern being used to represent?**

Signed-Magnitude Has Two Patterns for Zero

There are two bit patterns for 0!

0	00000...00000	+0
1	00000...00000	-0

This aspect made some hardware more complex than is necessary.

Modern machines do not use signed-magnitude.

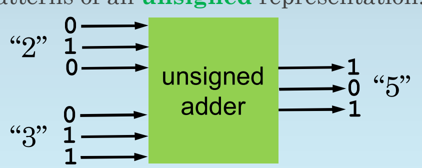
How Do We Choose Among Representations?

What makes a representation good?

- **efficient**: most bit patterns represent some item uniquely (so, not unary!)
- **easy/fast implementation of common operations**: such as arithmetic for numbers
- **shared implementation with other representations**: in this case, implementation is "free" in some sense

Representations Can be Chosen to Share Hardware

Imagine a device that performs addition on two bit patterns of an **unsigned** representation.



Can we use the same "adder" device for signed numbers? **Yes! If we choose the right representations.**

Add Unsigned Bit Patterns Using Base 2 Addition

Recall that the unsigned representation is drawn from base 2.

We use base 2 addition for unsigned patterns.

- Like base 10, we **add digit by digit**.
- Unlike base 10, the single-digit table of sums is quite small…
- What is `1 + 1 + 1`? `11`

A	B	Sum
0	0	0
0	1	1
1	0	1
1	1	10

Example: Addition of Unsigned Bit Patterns

Let's do an example with **5-bit unsigned**

```
     11
     01110  (14)
   + 00100  (4)
   -------
     10010  (18)
```

Good, we got the right answer!

Overflow Can Occur with Unsigned Addition

The unsigned representation is **fixed width**.
- If we start with **N** bits,
- we must end with **N** bits.

What is the condition under which the sum cannot be represented?
- The sum should have a 1 in the 2^N place.
- Only occurs when the most significant bits of the addends generate a carry.

We call this condition an **overflow**.

Example: Overflow of Unsigned Bit Patterns

Let's do an another example, again with **5-bit unsigned**

We have no space for that bit!

```
    111
     01110  (14)
   + 10101  (21)
   -------
     00011  (3)
```

Oops! (The carry out indicates an overflow for unsigned addition.)

Unsigned Addition is Modular Arithmetic

Modular arithmetic is related to the idea of the "remainder" of a division.

Given integers **A**, **B**, and **M**,
- **A** and **B** are said to be **equal mod M** iff*
- **A = B + kM** for some integer **k**.

Note that **k** can be negative or zero, too.

We write: **(A = B) mod M**.

* "iff" means "if and only if," an implication in both directions, and is often used for mathematical definitions

Unsigned Addition is Always Correct Mod 2^N

Let $\mathbf{SUM_N(A,B)}$ be the number represented by the sum of two **N-bit unsigned** bit patterns.

If no overflow occurs ($\mathbf{A + B < 2^N}$), we have $\mathbf{SUM_N(A,B) = A + B}$.

For sums that produce an overflow, the bit pattern of the sum is missing the $\mathbf{2^N}$ bit, so $\mathbf{SUM_N(A,B) = A + B - 2^N}$

In both cases,

$$\mathbf{(SUM_N(A,B) = A + B) \bmod 2^N}.$$

Modular Arithmetic Key to Good Integer Representations

Modular arithmetic is the key.

It allows us to define
- a **representation for signed integers**
- that **uses the same devices**
- as are **needed for unsigned arithmetic**.

The representation is called **2's complement**.

Details soon…

Modular Arithmetic on the Number Line

To understand modular arithmetic graphically, imagine breaking the number line into groups of **M** numbers, as shown above for **M=8**.

Two numbers are equal mod **M** if they occupy the same position in their respective groups.

For example, 0 is equal to an infinite number of other numbers (…, -24, -16, -8, 8, 16, 24, …).

We usually name sets of numbers that are equal mod **M** using the number in the range **[0, M-1]**.

