Math 231 Midterm 1. Feb 14, 2017

Full Name: ______________________________ **Section Code from table below:** | | | |

Net ID: ______________________________

For graders only	Score
Problem 10 (16 points)	
Problem 11 (16 points)	
Problem 12 (16 points)	
Total (48 points)	

TA	Section	Time and Location	TA	Section	Time and Location
Akbari, Sahand	EDD EDE	WF 11:00-11:50 -- 106B6 ENGR HALL WF 12:00-12:50 -- 1 ILL HALL	Karve, Vaibhav	AD1	WF 9:00-10:50 -- 159 ALTGELD
Caulfield, Erin	AD2	WF 1:00-2:50 -- 159 ALTGELD	Kim, Hee Yeon	EDF EDG	WF 1:00-1:50 -- 241 ALTGELD WF 2:00-2:50 -- 345 ALTGELD
Chavoshi, Amir	EDB ADC	WF 9:00-9:50 -- 441 ALTGELD WF 10:00-10:50 -- 347 ALTGELD	Kim, Heejoung	ADA ADB	WF 8:00-8:50 -- 141 ALTGELD WF 9:00-9:50 -- 145 ALTGELD
Chung, Jooyeon	BD2	WF 11:00-12:50 -- 173 ALTGELD	Linz, William	DDA DDB	WF 8:00-8:50 -- 441 ALTGELD WF 9:00-9:50 -- 141 ALTGELD
Duffy, Michael	DDE DDF	WF 12:00-12:50 -- 441 ALTGELD WF 1:00-1:50 -- 145 ALTGELD	Loving, Marissa	CDE	WF 12:00-12:50 -- 140 HENRY BLD
Ellis, Matthew	DDG DDH	WF 2:00-2:50 -- 441 ALTGELD WF 3:00-3:50 -- 441 ALTGELD	Luo, Hao	CDC CDD	WF 10:00-10:50 -- 145 ALTGELD WF 11:00-11:50 -- 143 HENRY BLD
Gramcko-Tursi, Mary Angelica	CDG CDH	WF 2:00-2:50 -- 143 ALTGELD WF 3:00-3:50 -- 145 ALTGELD	Mousley, Sarah	ADD ADE	WF 11:00-11:50 -- 447 ALTGELD ADE -- WF 12:00-12:50 -- 341 ALTGELD
Han, Xiaolong (Hans)	ADF ADG	WF 1:00-1:50 -- 141 ALTGELD WF 2:00-2:50 -- 241 ALTGELD	Ochoa de Alaiza Gracia, Itziar	BDC BDD	WF 10:00-10:50 -- 447 ALTGELD WF 11:00-11:50 -- 137 HENRY BLD
Harris, Terence	BDA BDB	WF 8:00-8:50 -- 137 HENRY BLD WF 9:00-9:50 -- 341 ALTGELD	Pratt, Kyle	DDC DDD	WF 10:00-10:50 -- 441 ALTGELD WF 11:00-11:50 -- 441 ALTGELD
Heath, Emily	BD3	WF 1:00-2:50 -- 173 ALTGELD	Tamazyan, Albert	BDE CDF	WF 12:00-12:50 -- 137 HENRY BLD WF 1:00-1:50 -- 137 HENRY BLD
Huang, Jianting (Jesse)	EDA EDC	WF 8:00-8:50 -- 143 ALTGELD WF 10:00-10:50 -- 443 ALTGELD	Wright, Benjamin	CDA CDB	WF 8:00-8:50 -- 145 ALTGELD WF 9:00-9:50 -- 443 ALTGELD

- The exam is one hour long.
- You must not communicate with other students during this exam.
- No written materials of any kind allowed.
- No phones, calculators, smart-watches or electronic devices of any kind allowed for any reason, including checking the time (you may use a simple wristwatch).
- Do not turn this page until instructed to.

Trig identities: $\sin^2 x + \cos^2 x = 1$, $\tan^2 x + 1 = \sec^2 x$, $\sin 2x = 2 \sin x \cos x$, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$, $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

(1 point) Fill in top of this page correctly. Fill in name, UIN (student number), and Net ID on Scantron form. Fill in the following answers on the Scantron form:

Zone 1

Multiple Choice Questions. Mark answers on Scantron form.

1/1. (7 points) Evaluate $\displaystyle\int_1^3 \frac{\ln x}{x^2}\,dx$.

A. ★ $(2-\ln 3)/3$

B. $(2+\ln 3)/3$

C. $(2+\ln 3)/9$

D. $(2-\ln 3)/9$

E. $\frac{\ln 3}{9}$

Solution. Use integration by parts with $u=\ln x$ and $dv=\frac{1}{x^2}\,dx$, so that $du=\frac{1}{x}\,dx$ and $v=-\frac{1}{x}$. Then

$$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} - \int -\frac{1}{x^2}\,dx = \frac{-\ln x}{x} - \frac{1}{x} + C,$$

and

$$\left[\frac{-\ln x}{x} - \frac{1}{x}\right]_1^3 = \frac{-\ln 3}{3} - \frac{1}{3} + 1 = \frac{2}{3} - \frac{\ln 3}{3}.$$

1/2. (7 points) Evaluate $\displaystyle\int_1^7 \frac{\ln x}{x^2}\,dx$.

A. ★ $(6-\ln 7)/7$

B. $(6+\ln 7)/7$

C. $(6+\ln 7)/49$

D. $(6-\ln 7)/49$

E. $\frac{\ln 7}{49}$

Solution. Use integration by parts with $u=\ln x$ and $dv=\frac{1}{x^2}\,dx$, so that $du=\frac{1}{x}\,dx$ and $v=-\frac{1}{x}$. Then

$$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} - \int -\frac{1}{x^2}\,dx = \frac{-\ln x}{x} - \frac{1}{x} + C,$$

and

$$\left[\frac{-\ln x}{x} - \frac{1}{x}\right]_1^7 = \frac{-\ln 7}{7} - \frac{1}{7} + 1 = \frac{6}{7} - \frac{\ln 7}{7}.$$

1/3. (7 points) Evaluate $\displaystyle\int_1^5 \frac{\ln x}{x^2}\,dx$.

A. ★ $(4-\ln 5)/5$

B. $(4+\ln 5)/5$

C. $(4 + \ln 5)/25$

D. $(4 - \ln 5)/25$

E. $\frac{\ln 5}{25}$

Solution. Use integration by parts with $u = \ln x$ and $dv = \frac{1}{x^2}\,dx$, so that $du = \frac{1}{x}\,dx$ and $v = -\frac{1}{x}$. Then

$$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} - \int -\frac{1}{x^2}\,dx = \frac{-\ln x}{x} - \frac{1}{x} + C,$$

and

$$\left[\frac{-\ln x}{x} - \frac{1}{x}\right]_1^5 = \frac{-\ln 5}{5} - \frac{1}{5} + 1 = \frac{4}{5} - \frac{\ln 5}{5}.$$

2/1. (7 points) Evaluate $\int_0^{\frac{\pi}{3}} 2\tan^3(x)\sec(x)\,dx$.

A. ★ 8/3

B. 7/3

C. 7/2

D. 14/3

E. 15/2

Solution. Let $u = \sec x$. Then $du = \sec(x)\tan(x)\,dx$ and

$$\int_0^{\frac{\pi}{3}} 2\tan^3(x)\sec(x)\,dx = 2\int_0^{\frac{\pi}{3}} \tan^2(x)\cdot\sec(x)\tan(x)\,dx$$

$$= 2\int_0^{\frac{\pi}{3}} (\sec^2(x)-1)\cdot\sec(x)\tan(x)\,dx = 2\int_1^2 (u^2-1)\,du = 2\left[\frac{u^3}{3}-u\right]_1^2 = \frac{8}{3}.$$

2/2. (7 points) Evaluate $\int_0^{\frac{\pi}{3}} 4\tan^3(x)\sec(x)\,dx$.

A. ★ 16/3

B. 14/3

C. 7

D. 4/3

E. 15

Solution. Let $u = \sec x$. Then $du = \sec(x)\tan(x)\,dx$ and

$$\int_0^{\frac{\pi}{3}} 4\tan^3(x)\sec(x)\,dx = 4\int_0^{\frac{\pi}{3}} \tan^2(x)\cdot\sec(x)\tan(x)\,dx$$

$$= 4\int_0^{\frac{\pi}{3}} (\sec^2(x)-1)\cdot\sec(x)\tan(x)\,dx = 4\int_1^2 (u^2-1)\,du = 4\left[\frac{u^3}{3}-u\right]_1^2 = \frac{16}{3}.$$

2/3. (7 points) Evaluate $\int_0^{\frac{\pi}{3}} 5\tan^3(x)\sec(x)\,dx$.

A. ★ 20/3

B. 7/3

C. 75/4

D. 40/3

E. 15/2

Solution. Let $u = \sec x$. Then $du = \sec(x)\tan(x)\,dx$ and

$$\int_0^{\frac{\pi}{3}} 5\tan^3(x)\sec(x)\,dx = 5\int_0^{\frac{\pi}{3}} \tan^2(x)\cdot\,\sec(x)\tan(x)\,dx$$

$$= 5\int_0^{\frac{\pi}{3}} (\sec^2(x)-1)\cdot\,\sec(x)\tan(x)\,dx = 5\int_1^2 (u^2-1)\,du = 5\left[\frac{u^3}{3}-u\right]_1^2 = \frac{20}{3}.$$

3/1. (7 points) Evaluate $\int_0^2 \frac{2t+9}{(t+1)(t+5)}\,dt$.

A. ★ $\frac{7}{4}\ln 3 + \frac{1}{4}\ln 7 - \frac{1}{4}\ln 5$

B. $\frac{1}{4}\ln 3 + \frac{7}{4}\ln 7 - \frac{7}{4}\ln 5$

C. $\frac{7}{4}\ln 3 + \frac{1}{4}\ln 7 + \frac{1}{4}\ln 5$

D. $\frac{1}{4}\ln 3 + \frac{7}{4}\ln 7 + \frac{7}{4}\ln 5$

E. $-\frac{1}{4}\ln 3 + \frac{7}{4}\ln 7 + \frac{1}{4}\ln 5$

Solution. Use a partial fractions decomposition of the form $\frac{2t+9}{(t+1)(t+5)} = \frac{A}{t+1} + \frac{B}{t+5}$. Clearing denominators gives $2t+9 = A(t+5) + B(t+1)$ so that plugging in $t=-1$ and $t=-5$ yield $A = \frac{7}{4}$ and $B = \frac{1}{4}$ respectively. Then integrate:

$$\int_0^2 \frac{2t+9}{(t+1)(t+5)}\,dt = \int_0^2 \frac{7}{4}\cdot\frac{1}{t+1} + \frac{1}{4}\cdot\frac{1}{t+5}\,dt = \left[\frac{7}{4}\ln|t+1| + \frac{1}{4}\ln|t+5|\right]_0^2$$
$$= \left(\frac{7}{4}\ln 3 + \frac{1}{4}\ln 7\right) - \left(0 + \frac{1}{4}\ln 5\right) = \frac{7}{4}\ln 3 + \frac{1}{4}\ln 7 - \frac{1}{4}\ln 5$$

3/2. (7 points) Evaluate $\int_2^3 \frac{5t+6}{(t-1)(t+3)}\,dt$.

A. ★ $\frac{11}{4}\ln 2 - \frac{9}{4}\ln 5 + \frac{9}{4}\ln 6$

B. $\frac{9}{4}\ln 2 - \frac{11}{4}\ln 5 + \frac{11}{4}\ln 6$

C. $\frac{11}{4}\ln 2 + \frac{9}{4}\ln 5 + \frac{9}{4}\ln 6$

D. $\frac{11}{4}\ln 2 - \frac{9}{4}\ln 5 - \frac{9}{4}\ln 6$

E. $-\frac{9}{4}\ln 2 + \frac{11}{4}\ln 5 + \frac{9}{4}\ln 6$

Solution. Use a partial fractions decomposition of the form $\frac{5t+6}{(t-1)(t+3)} = \frac{A}{t-1} + \frac{B}{t+3}$. Clearing denominators gives $5t+6 = A(t+3) + B(t-1)$ so that plugging in $t=1$ and $t=-3$ yield $A = \frac{11}{4}$ and $B = \frac{9}{4}$ respectively. Then integrate:

$$\int_2^3 \frac{5t+6}{(t-1)(t+3)}\,dt = \int_2^3 \frac{11}{4}\cdot\frac{1}{t-1} + \frac{9}{4}\cdot\frac{1}{t+3}\,dt = \left[\frac{11}{4}\ln|t-1| + \frac{9}{4}\ln|t+3|\right]_2^3$$
$$= \left(\frac{11}{4}\ln 2 + \frac{9}{4}\ln 6\right) - \left(0 + \frac{9}{4}\ln 5\right) = \frac{11}{4}\ln 2 + \frac{9}{4}\ln 6 - \frac{9}{4}\ln 5$$

3/3. (7 points) Evaluate $\displaystyle\int_0^1 \frac{3t+4}{(t+1)(t+3)}\,dt$.

A. ★ $\frac{1}{2}\ln 2 + \frac{5}{2}\ln 4 - \frac{5}{2}\ln 3$

B. $\frac{1}{2}\ln 2 - \frac{5}{2}\ln 4 - \frac{5}{2}\ln 3$

C. $\frac{5}{2}\ln 2 + \frac{1}{2}\ln 4 + \frac{5}{2}\ln 3$

D. $\frac{5}{2}\ln 2 + \frac{1}{2}\ln 4 - \frac{1}{2}\ln 3$

E. $-\frac{1}{2}\ln 2 + \frac{1}{2}\ln 4 - \frac{5}{2}\ln 3$

Solution. Use a partial fractions decomposition of the form $\dfrac{3t+4}{(t+1)(t+3)} = \dfrac{A}{t+1} + \dfrac{B}{t+3}$. Clearing denominators gives $3t+4 = A(t+3) + B(t+1)$ so that plugging in $t=-1$ and $t=-3$ yield $A = \frac{1}{2}$ and $B = \frac{5}{2}$ respectively. Then integrate:

$$\begin{aligned}\int_0^1 \frac{3t+4}{(t+1)(t+3)}\,dt &= \int_0^1 \frac{1}{2}\cdot\frac{1}{t+1} + \frac{5}{2}\cdot\frac{1}{t+3}\,dt = \left[\frac{1}{2}\ln|t+1| + \frac{5}{2}\ln|t+3|\right]_0^1 \\ &= \left(\frac{1}{2}\ln 2 + \frac{5}{2}\ln 4\right) - \left(0 + \frac{5}{2}\ln 3\right) = \frac{1}{2}\ln 2 + \frac{5}{2}\ln 4 - \frac{5}{2}\ln 3\end{aligned}$$

4/1. (7 points) Evaluate $\int_0^4 \cos(\sqrt{x})\,dx$.

A. ★ $4\sin(2) + 2\cos(2) - 2$

B. $2\sin(2) + 2\cos(2)$

C. $2\sin(2) + 4\cos(2) - 2$

D. $\cos(2) - 1$

E. $\sin(2)$

Solution. Let $u = \sqrt{x}$, so that $u^2 = x$ and $2u\,du = dx$. Then $\int_0^4 \cos(\sqrt{x})\,dx = \int_0^2 \cos(u)2u\,du$. Integrate by parts with $U = u$ and $dV = \cos(u)du$ so that $dU = du$ and $V = \sin(u)$. This gives $2(u\sin(u)|_0^2 - \int_0^2 \sin(u)du) = 2(u\sin(u) + \cos(u))|_0^2 = 4\sin(2) + 2\cos(2) - 2$.

4/2. (7 points) Evaluate $\int_0^9 \cos(\sqrt{x})\,dx$.

A. ★ $6\sin(3) + 2\cos(3) - 2$

B. $2\sin(3) + 2\cos(3)$

C. $2\sin(3) + 6\cos(3) - 2$

D. $\cos(3) - 1$

E. $\sin(3)$

Solution. Let $u = \sqrt{x}$, so that $u^2 = x$ and $2u\,du = dx$. Then $\int_0^9 \cos(\sqrt{x})\,dx = \int_0^3 \cos(u)2u\,du$. Integrate by parts with $U = u$ and $dV = \cos(u)du$ so that $dU = du$ and $V = \sin(u)$. This gives $2(u\sin(u)|_0^3 - \int_0^3 \sin(u)du) = 2(u\sin(u) + \cos(u))|_0^3 = 6\sin(3) + 2\cos(3) - 2$.

4/3. (7 points) Evaluate $\int_0^{16} \cos(\sqrt{x})\,dx$.

A. ★ $8\sin(4) + 2\cos(4) - 2$

B. $2\sin(4) + 2\cos(4)$

C. $2\sin(4) + 8\cos(4) - 2$

D. $\cos(4) - 1$

E. $\sin(4)$

Solution. Let $u = \sqrt{x}$, so that $u^2 = x$ and $2u\,du = dx$. Then $\int_0^{16} \cos(\sqrt{x})\,dx = \int_0^4 \cos(u)2u\,du$. Integrate by parts with $U = u$ and $dV = \cos(u)du$ so that $dU = du$ and $V = \sin(u)$. This gives $2(u\sin(u)|_0^4 - \int_0^4 \sin(u)du) = 2(u\sin(u) + \cos(u))|_0^4 = 8\sin(4) + 2\cos(4) - 2$.

5/1. (7 points) Using a trig substitution, the integral $\displaystyle\int \frac{x^4}{\sqrt{x^2-9}}\,dx$ can be transformed into one of the integrals below. Select the correct one.

A. ★ $\int 81\sec^5\theta\,d\theta$

B. $\int 81\sec\theta\tan^4\theta\,d\theta$

C. $\int 9\sec\theta\tan^2\theta\,d\theta$

D. $\int 81\sec\theta\tan\theta\,d\theta$

E. $\int 9\sec^5\theta\tan\theta\,d\theta$

Solution. Use the substitution $x = 3\sec\theta$, so that $dx = 3\sec\theta\tan\theta\,d\theta$. We have

$$\int \frac{x^4}{\sqrt{x^2-9}}\,dx = \int \frac{81\sec^4\theta}{3\tan\theta}\,(3\sec\theta\tan\theta\,d\theta) = \int 81\sec^5\theta\,d\theta.$$

5/2. (7 points) Using a trig substitution, the integral $\displaystyle\int \frac{x^4}{\sqrt{x^2+9}}\,dx$ can be transformed into one of the integrals below. Select the correct one.

A. ★ $\int 81\sec\theta\tan^4\theta\,d\theta$

B. $\int 81\sec^5\theta\,d\theta$

C. $\int 9\sec\theta\tan^2\theta\,d\theta$

D. $\int 81\sec\theta\tan\theta\,d\theta$

E. $\int 9\sec^5\theta\tan\theta\,d\theta$

Solution. Use the substitution $x = 3\tan\theta$, so that $dx = 3\sec^2\theta\,d\theta$. We have

$$\int \frac{x^4}{\sqrt{x^2+9}}\,dx = \int \frac{81\tan^4\theta}{3\sec\theta}\,(3\sec^2\theta\,d\theta) = \int 81\sec\theta\tan^4\theta\,d\theta.$$

5/3. (7 points) Using a trig substitution, the integral $\displaystyle\int \frac{x^2}{\sqrt{x^2+9}}\,dx$ can be transformed into one of the integrals below. Select the correct one.

A. ★ $\int 9\sec\theta\tan^2\theta\,d\theta$

B. $\int 81\sec^5\theta\,d\theta$

C. $\int 81\sec\theta\tan^4\theta\,d\theta$

D. $\int 81\sec\theta\tan\theta\,d\theta$

E. $\int 9\sec^5\theta\tan\theta\,d\theta$

Solution. Use the substitution $x = 3\tan\theta$, so that $dx = 3\sec^2\theta\,d\theta$. We have

$$\int \frac{x^2}{\sqrt{x^2+9}}\,dx = \int \frac{9\tan^2\theta}{3\sec\theta}\,(3\sec^2\theta\,d\theta) = \int 9\sec\theta\tan^2\theta\,d\theta.$$

Zone 2

Determine if each improper integral converges or diverges.

6/1. (4 points) $\displaystyle\int_1^\infty \frac{(\ln x)^4 + \sin^2(x)}{x^3}\,dx$

A. ★ Converges

B. Diverges

Solution. For large x we have $0 < (\ln x)^4 + \sin^2(x) < x + 1 < 2x$, so that $((\ln x)^4 + \sin^2(x))/x^3 < 2x/x^3 = 2/x^2$. $\displaystyle\int_1^\infty \frac{2}{x^2}\,dx = 2\int_1^\infty \frac{1}{x^2}\,dx$ converges ($p = 2$) and thus $\displaystyle\int_1^\infty \frac{(\ln x)^4 + \sin^2(x)}{x^3}\,dx$ converges (Comparison Theorem).

6/2. (4 points) $\displaystyle\int_1^\infty \frac{(\ln x)^3 + \sin^2(x)}{x^4}\,dx$

A. ★ Converges

B. Diverges

Solution. For large x we have $0 < \ln(x)^3 + \sin^2(x) < x$, so that $((\ln x)^3 + \sin^2(x))/x^4 < 2x/x^4 = 2/x^3$. $\displaystyle\int_1^\infty \frac{2}{x^3}\,dx = 2\int_1^\infty \frac{1}{x^3}\,dx$ converges ($p = 3$) and thus $\displaystyle\int_1^\infty \frac{(\ln x)^3 + \sin^2(x)}{x^4}\,dx$ converges (Comparison Theorem).

7/1. (4 points) $\displaystyle\int_0^2 \frac{x^2+1}{x^4}\,dx$

A. ★ Diverges

B. Converges

Solution.

$$\int_0^2 \frac{x^2+1}{x^4}\,dx = \lim_{t\to 0^+} \int_t^2 \frac{1}{x^2} + \frac{1}{x^4}\,dx = \lim_{t\to 0^+} \frac{-1}{x} - \frac{1}{3x^3}\bigg|_t^2 = \lim_{t\to 0^+} \frac{-1}{2} - \frac{1}{24} + \frac{1}{t} + \frac{1}{3t^3} = \infty$$

So the given integral diverges.

7/2. (4 points) $\displaystyle\int_0^5 \frac{x^3+1}{x^3}\,dx$

A. ★ Diverges

B. Converges

Solution.

$$\int_0^5 \frac{x^3+1}{x^3}\,dx = \lim_{t\to 0^+} \int_t^5 1 + \frac{1}{x^3}\,dx = \lim_{t\to 0^+} x - \frac{1}{2x^2}\bigg|_t^5 = \lim_{t\to 0^+} 5 - \frac{1}{50} - t + \frac{1}{2t^2} = \infty$$

So the given integral diverges.

8/1. (4 points) $\displaystyle\int_1^\infty \frac{1+\sin^2(x)}{3x-1}\,dx$

A. ★ Diverges

B. Converges

Solution. Since $1+\sin^2(x) \geq 1$ and $3x-1 < 3x$, we have $\dfrac{1+\sin^2(x)}{3x-1} \geq \dfrac{1}{3x-1} > \dfrac{1}{3x}$ for large x. Since $\displaystyle\int_1^\infty \frac{1}{3x}\,dx$ diverges, so does $\displaystyle\int_1^\infty \frac{1+\sin^2(x)}{3x-1}\,dx$, by the Comparison Theorem.

8/2. (4 points) $\displaystyle\int_1^\infty \frac{1+\cos^2(x)}{4x-2}\,dx$

A. ★ Diverges

B. Converges

Solution. Since $1+\cos^2(x) \geq 1$ and $4x-2 < 4x$, we have $\dfrac{1+\cos^2(x)}{4x-2} \geq \dfrac{1}{4x-2} > \dfrac{1}{4x}$ for large x. Since $\displaystyle\int_1^\infty \frac{1}{4x}\,dx$ diverges, so does $\displaystyle\int_1^\infty \frac{1+\cos^2(x)}{4x-2}\,dx$, by the Comparison Theorem.

9/1. (4 points) $\displaystyle\int_1^\infty \frac{1-\cos^4(x)}{e^x+x^4}\,dx$

A. ★ Converges

B. Diverges

Solution. Since $1-\cos^4(x) \le 1$ and $e^x + x^4 > e^x$, we have $\dfrac{1-\cos^4(x)}{e^x+x^4} \le \dfrac{1}{e^x+x^4} < \dfrac{1}{e^x}$. Since $\displaystyle\int_1^\infty \frac{1}{e^x}\,dx$ converges, so does $\displaystyle\int_1^\infty \frac{1-\cos^4(x)}{e^x+x^4}\,dx$, by the Comparison Theorem.

9/2. (4 points) $\displaystyle\int_1^\infty \frac{1-\sin^4(x)}{e^{2x}+x}\,dx$

A. ★ Converges

B. Diverges

Solution. Since $1-\sin^4(x) \le 1$ and $e^{2x} + x > e^{2x}$, we have $\dfrac{1-\sin^4(x)}{e^{2x}+x} \le \dfrac{1}{e^{2x}+x} < \dfrac{1}{e^{2x}}$. Since $\displaystyle\int_1^\infty \frac{1}{e^{2x}}\,dx$ converges, so does $\displaystyle\int_1^\infty \frac{1-\sin^4(x)}{e^{2x}+x}\,dx$, by the Comparison Theorem.

Zone 3

Free response questions. Write complete solutions and show all work for full credit.

10/1. (16 points) Evaluate the improper integral or show that it diverges.

$$\int_0^\infty \frac{10}{(2x+3)^3}\,dx$$

Solution. The denominator is never zero on the interval of integration, so the integral is improper only at ∞. By definition,

$$\int_0^\infty \frac{10\ dx}{(2x+3)^3} = \lim_{t\to\infty} \int_0^t \frac{10\ dx}{(2x+3)^3} = \lim_{t\to\infty} -\frac{5}{2(2x+3)^2}\Bigg|_0^t = \lim_{t\to\infty} -\frac{5}{2(2t+3)^2} + \frac{5}{18} = \frac{5}{18}.$$

10/2. (16 points) Evaluate the improper integral or show that it diverges.

$$\int_0^\infty \frac{12}{(3x+1)^4}\,dx$$

Solution. The denominator is never zero on the interval of integration, so the integral is improper only at ∞. By definition,

$$\int_0^\infty \frac{12\ dx}{(3x+1)^4} = \lim_{t\to\infty} \int_0^t \frac{12\ dx}{(3x+1)^4} = \lim_{t\to\infty} -\frac{4}{3(3x+1)^3}\Bigg|_0^t = \lim_{t\to\infty} -\frac{4}{3(3t+1)^3} + \frac{4}{3} = \frac{4}{3}.$$

10/3. (16 points) Evaluate the improper integral or show that it diverges.

$$\int_0^\infty \frac{10}{(2x+7)^3}\,dx$$

Solution. The denominator is never zero on the interval of integration, so the integral is improper only at ∞. By definition,

$$\int_0^\infty \frac{10\ dx}{(2x+7)^3} = \lim_{t\to\infty} \int_0^t \frac{10\ dx}{(2x+7)^3} = \lim_{t\to\infty} -\frac{5}{2(2x+7)^2}\Bigg|_0^t = \lim_{t\to\infty} -\frac{5}{2(2t+7)^2} + \frac{5}{98} = \frac{5}{98}.$$

Zone 4

11/1. (16 points) Evaluate $\displaystyle\int \frac{\sqrt{x^2-4}}{x^4}\,dx$.

Solution. Let $x = 2\sec\theta$. Then $dx = 2\sec\theta\tan\theta\;d\theta$ and

$$\int \frac{\sqrt{x^2-4}}{x^4}\,dx = \int \frac{(2\tan\theta)(2\sec\theta\tan\theta)}{(2\sec\theta)^4}\,d\theta = \int \frac{\tan^2\theta}{4\sec^3\theta}\,d\theta = \frac{1}{4}\int \sin^2\theta\cos\theta\,d\theta = \frac{1}{12}\sin^3\theta + C = \frac{1}{12}\frac{(x^2-4)^{3/2}}{x^3} + C$$

11/2. (16 points) Evaluate $\displaystyle\int \frac{\sqrt{x^2-9}}{x^4}\,dx$.

Solution. Let $x = 3\sec\theta$. Then $dx = 3\sec\theta\tan\theta\;d\theta$ and

$$\int \frac{\sqrt{x^2-9}}{x^4}\,dx = \int \frac{(3\tan\theta)(3\sec\theta\tan\theta)}{(3\sec\theta)^4}\,d\theta = \int \frac{\tan^2\theta}{9\sec^3\theta}\,d\theta = \frac{1}{9}\int \sin^2\theta\cos\theta\,d\theta = \frac{1}{27}\sin^3\theta + C = \frac{1}{27}\frac{(x^2-9)^{3/2}}{x^3} + C$$

11/3. (16 points) Evaluate $\displaystyle\int \frac{\sqrt{x^2-16}}{x^4}\,dx$.

Solution. Let $x = 4\sec\theta$. Then $dx = 4\sec\theta\tan\theta\;d\theta$ and

$$\int \frac{\sqrt{x^2-16}}{x^4}\,dx = \int \frac{(4\tan\theta)(4\sec\theta\tan\theta)}{(4\sec\theta)^4}\,d\theta = \int \frac{\tan^2\theta}{16\sec^3\theta}\,d\theta = \frac{1}{16}\int \sin^2\theta\cos\theta\,d\theta = \frac{1}{48}\sin^3\theta + C = \frac{1}{48}\frac{(x^2-16)^{3/2}}{x^3} + C$$

Zone 5

12/1. (16 points) Evaluate $\displaystyle\int \frac{e^{2x}}{e^x+1}\,dx$.

Solution. Let $u = e^x + 1$, so that $du = e^x\,dx$. Then

$$e^{2x}\,dx = e^x \cdot e^x\,dx = (u-1)\,du.$$

So the integral becomes

$$\int \frac{u-1}{u}\,du = \int (1-u^{-1})\,du = u - \ln|u| + C.$$

Converting back to x, we end up with $e^x + 1 - \ln(e^x+1) + C$.

12/2. (16 points) Evaluate $\displaystyle\int \frac{e^{2x}}{e^x+2}\,dx$.

Solution. Let $u = e^x + 2$, so that $du = e^x\,dx$. Then

$$e^{2x}\,dx = e^x \cdot e^x\,dx = (u-2)\,du.$$

So the integral becomes

$$\int \frac{u-2}{u}\,du = \int (1-2u^{-1})\,du = u - 2\ln|u| + C.$$

Converting back to x, we end up with $e^x + 2 - 2\ln(e^x+2) + C$.

12/3. (16 points) Evaluate $\displaystyle\int \frac{e^{2x}}{e^x+3}\,dx$.

Solution. Let $u = e^x + 3$, so that $du = e^x\,dx$. Then

$$e^{2x}\,dx = e^x \cdot e^x\,dx = (u-3)\,du.$$

So the integral becomes

$$\int \frac{u-3}{u}\,du = \int (1-3u^{-1})\,du = u - 3\ln|u| + C.$$

Converting back to x, we end up with $e^x + 3 - 3\ln(e^x+3) + C$.
