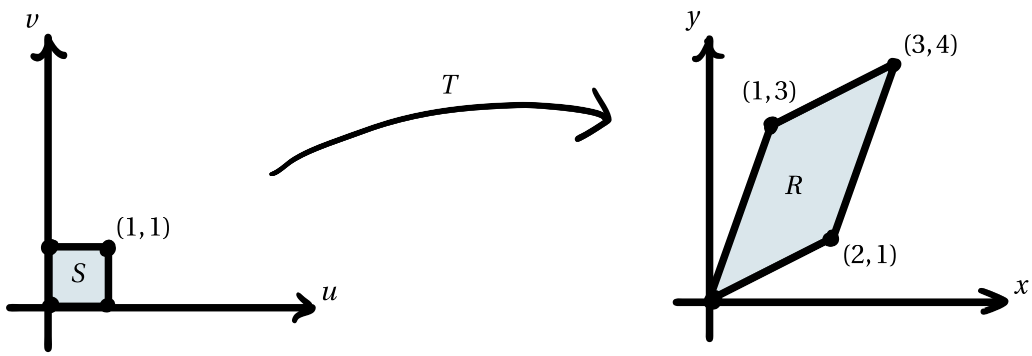
1. Consider the region R in $\mathbb{R}^2$ shown below at right. In this problem, you will do a change of coordinates to evaluate:

$$\iint_R x - 2y\, dA$$



(a) Find a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ which takes the unit square S to R.
Write you answer both as a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and as $T(u, v) = (au+bv,\ cu+dv)$, and check your answer with the instructor.

SOLUTION:

$T(u, v) = (2u + v, u + 3v)$. In matrix form,

$$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

(b) Compute $\iint_R x - 2y\, dA$ by relating it to an integral over S and evaluating that. Check your answer with the instructor.

SOLUTION:

The Jacobian of T is

$$det\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} = 6 - 1 = 5$$

So

$$\iint_R x - 2y\, dA = \iint_S [(2u + v) - 2(u + 3v)]5\, dA$$

$$= \int_0^1 \int_0^1 -25v\, du\, dv = \left[\frac{-25v^2}{2}\right]_0^1 = -25/2$$

2. Another simple type of transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a translation, which has the general form $T(u,v) = (u+a, v+b)$ for a fixed a and b.

   (a) If T is a translation, what is its Jacobian matrix? How does it distort area?

   SOLUTION:

   If $T(u,v) = (u+a, v+b)$ where a and b are constants, then the Jacobian is

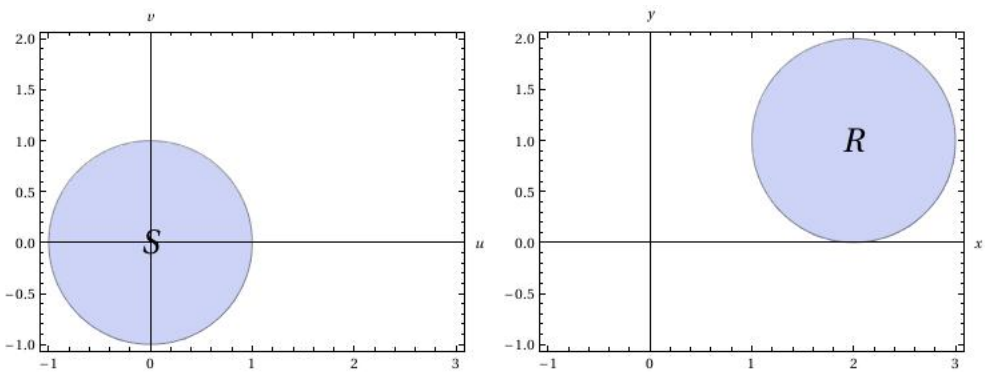
$$det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1.$$

   So T does not distort areas.

   (b) Consider the region $S = \{u^2 + v^2 \leq 1\}$ in $\mathbb{R}^2$ with coordinates (u,v), and the region $R = \{(x-2)^2 + (y-1)^2 \leq 1\}$ in $\mathbb{R}^2$ with coordinates (x,y).

   Make separate sketches of S and R.

   SOLUTION:



   (c) Find a translation T where $T(S) = R$.

   SOLUTION:

   $T(u,v) = (u+2, v+1)$

   (d) Use T to reduce

$$\iint_R x\,dA$$

   to an integral over S, and then evaluate that new integral using polar coordinates.

   SOLUTION:

   The Jacobian of T is just 1, as noted in part (a). So we have

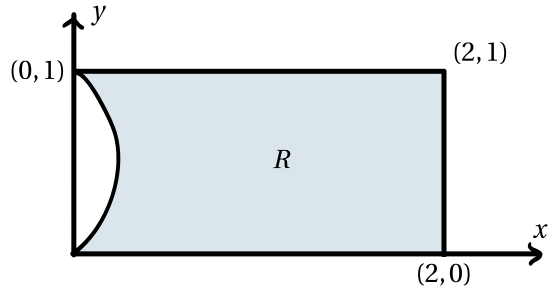
$$\iint_R x\,dA = \iint_S (u+2)\,dA$$

   Converting the second integral above to polar we have

$$\iint_S (u+2)\,dA = \int_0^{2\pi}\int_0^1 (r\cos\theta + 2)r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^3\cos\theta}{3}\right]_0^1 d\theta + 2\pi\left[r^2\right]_0^1$$

$$= 1/3 \int_0^{2\pi} \cos\theta \, d\theta + 2\pi = 1/3 \, [\sin\theta]_0^{2\pi} + 2\pi = 2\pi$$

3. Consider the region R shown below. Here the curved left side is given by $x = y - y^2$. In this problem, you will find a transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ which takes the unit square $S = [0,1] \times [0,1]$ to R.



(a) As a warm up, find a transformation that takes S to the rectangle $[0,2] \times [0,1]$ which contains R.

SOLUTION:

$$L(u,v) = (2u, v)$$

(b) Returning to the problem of finding T taking S to R, come up with formulas for $T(u,0)$, $T(u,1)$, $T(0,v)$, and $T(1,v)$. Hint: For three of these, use your answer in part (a).

SOLUTION:

$$\begin{array}{ll} T(u,0) = (2u,0) & T(u,1) = (2u,1) \\ T(1,v) = (2,v) & T(0,v) = (v - v^2, v) \end{array}$$

(c) Now extend your answer in (b) to the needed transformation T. Hint: Try "filling in" between $T(0,v)$ and $T(1,v)$ with a straight line.

SOLUTION:

$$T(u,v) = (2u + v(1-v)(1-u), v)$$

(d) Compute the area of R in two ways, once using T to change coordinates and once directly.

SOLUTION:

To change coordinates we compute the Jacobian

$$J(T) = det \begin{pmatrix} 2 - v(1-v) & (1-2v)(1-u) \\ 0 & 1 \end{pmatrix} = 2 - v(1-v)$$

So we have the area of R given by

$$\iint_R dx\,dy = \int_0^1 \int_0^1 2 - v(1-v)\,du\,dv = 11/6$$

Computing directly we have the area of R given by

$$\int_0^1 2-(y-y^2)\,dy = 11/6$$

4. If you get this far, evaluate the integrals in Problems 1 and 2 directly, without doing a change of coordinates. It's a fun-filled task...

   SOLUTION:

   For the integral in problem one, use the order $dy\,dx$. We need to split the double integral into three parts. The result is

$$\iint_R x-2y\,dA = \int_0^1 \int_{x/2}^{3x} x-2y\,dy\,dx + \int_1^2 \int_{x/2}^{x/2+5/2} x-2y\,dy\,dx + \int_2^3 \int_{3x-5}^{x/2+5/2} x-2y\,dy\,dx$$

   Evaluating this is not difficult but it is tedious. We leave it to the interested student. You should get $-25/2$.

   For the integral in problem two, again use the order $dy\,dx$. We just need one double integral.

$$\iint_R x\,dA = \int_1^3 \int_{1-\sqrt{1-(x-2)^2}}^{1+\sqrt{1-(x-2)^2}} x\,dy\,dx$$
$$= \int_1^3 2x\sqrt{1-(x-2)^2}\,dx$$

   This integral can be evaluated by making the substitution $x-2=\sin u$, yielding the integral

$$\int_{-\pi/2}^{\pi/2} (2\sin u+4)\cos^2 u\,du$$

   Now split this in two pieces as

$$\int_{-\pi/2}^{\pi/2} 2\sin u\cos^2 u\,du + \int_{-\pi/2}^{\pi/2} 4\cos^2 u\,du$$

   The first is the integral of an odd function over an interval which is symmetric about the y axis so it is 0. The second can be evaluated by using the trig identity $\cos^2 u = (1+\cos 2u)/2$. This gives

$$\int_{-\pi/2}^{\pi/2} 4\cos^2 u\,du = \int_{-\pi/2}^{\pi/2} 4(1+\cos 2u)/2\,du = 2\pi.$$