Math 241: Exam 2, March 13, 2018

Name: NetID:

Circle your discussion section:

- ADA: 8am, Nam
- ADB: 9am, Block Gorman
- ADC: 10am, Block Gorman
- ADD: 11am, Shin
- ADE: Noon, Shin
- ADF: 1pm, Mousley
- ADG: 2pm, Okano
- ADH: 3pm, Wojtalewicz
- ADI: 4pm, Wojtalewicz
- ADK: 9am, Christenson
- ADL: 10am, Field
- ADM: 2pm, Gao
- ADN: 3pm, Gao
- ADO: noon, Bavisetty
- ADP: 1pm, Bavisetty
- AD1: 11am, Weigandt
- AD2: 1pm, Rennie
- BD@: 1pm, Zhang, N.
- BDA: 8am, Huynh
- BDB: 9am, Huynh
- BDC: 10am, Park
- BDD: 11am, Han
- BDE: Noon, Park
- BDF: 1pm, Han
- BDG: 2pm, Drake
- BDH: 3pm, Zhang, Y.
- BDI: 4pm, Zhang, Y.
- BDJ: 9am, Field
- BDK: 10am, Christenson
- BDL: noon, Huang
- BDM: 2pm, Mousley
- BDN: 3pm, Okano
- BDO: 4pm, Drake
- BDR: 11am, Huang

Write and bubble in your UIN:

0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9	9

Instructions: You have **75 minutes** to complete this exam. There are **70 points** available and not all problems are weighted equally. Calculators, books, notes, and suchlike aids are **not** permitted. It is not necessary to show work for multiple-choice questions. **For all other questions, show work that justifies your answer** as in those problems **credit will not be given** for correct answers without proper justification. Work written outside of the space provided for a problem will **not** be graded. The last page of the exam contains a **table of trigonometric identities**.

Do not open exam until instructed.

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	9	7	7	9	7	8	7	8	8	70
Score:										

1. (a) **(3 points)** Find the tangent plane to the surface

$$z = 3(x-1)^2 + 2(y-3)^2 + 2$$

at the point $(2,4,7)$.

Solution: $z - 7 = 6(x-2) + 4(y-4)$.

An equation of this plane is (circle one):

(A) $z - 7 = 4(x-2) + 6(y-4)$; (B) $z - 7 = 6(x-2) + 4(y-4)$;

(C) $z - 7 = 4(x-1) + 6(y-3)$; (D) $z - 7 = 6(x-1) + 4(y-3)$;

(b) **(3 points)** Use the linear approximation to $f(x,y) = 3xe^{\sin(xy)}$ at the point $(2,0)$ to find the approximate value of $f(2.1, 0.1)$.

Solution: $L(x,y) = 6 + 3(x-2) + 12(y-0)$, so $f(2.1, .1) \approx L(2.1, .1) = 7.5$

(circle one): (A) 7.6; (B) 7.2; (C) 7.4; (D) 7.5;

(c) **(3 points)** Find $\frac{\partial y}{\partial z}$ at the point $(1,1,1)$ on the level surface $3xy^3 + zy - 3xz - 1 = 0$.

Solution: $\frac{\partial y}{\partial z} = \frac{3x-y}{9xy^2+z}$, at $(1,1,1)$ this is $\frac{1}{5}$.

(circle one): (A) $\frac{3}{10}$; (B) $\frac{1}{5}$; (C) $\frac{-1}{5}$; (D) 5 ;

2. (a) **(4 points)** Use the Chain Rule to find $\frac{\partial w}{\partial x}$ at the point $(x, y, t) = (2, 1, \pi)$, where

$$w = r^2 + \theta^2, \; r = y + x\cos t, \; \theta = x + y\sin t$$

Solution: $\frac{\partial w}{\partial x} = 2r\cos t + 2\theta$ and at the point $(x, y, t) = (2, 1, \pi)$ we have $\frac{\partial w}{\partial x} = 6$.

(circle one): (A) 2; (B) 0; (C) 6; (D) -2;

(b) **(3 points)** The tangent plane to the ellipsoid, $2x^2 + 4y^2 + 3z^2 = 6$, is parallel to the plane, $4x - 4y + 6z = 9$, at which of the following points?

Solution: The normal vector for the tangent plane to this ellipsoid at (x, y, z) is $\langle 4x, 8y, 6z \rangle$, which is parallel to the normal vector of the plane $4x - 4y + 6z = 9$ at $(1, \frac{-1}{2}, 1)$.

(circle one): (A) $(-1, \frac{1}{2}, 1)$; (B) $(1, \frac{-1}{2}, 1)$; (C) $(1, \frac{1}{2}, 1)$; (D) $(1, \frac{1}{2}, -1)$;

3. (a) **(3 points)** Find the directional derivative of $f(x,y,z) = 3xy - z^2$ at the point $(1,-2,2)$ in the direction of the vector from the point $(1,-2,2)$ to the origin.

Solution: The vector from that point to the origin in $\vec{v} = \langle -1,2,-2\rangle$, the unit vector in that direction is $\vec{u} = \langle \frac{-1}{3}, \frac{2}{3}, \frac{-2}{3}\rangle$. $D_{\vec{u}} f(1,-2,2) = \vec{u} \cdot \nabla f(1,-2,2) = \frac{20}{3}$.

(circle one): (A) $\frac{4}{3}$; (B) $\frac{20}{3}$; (C) $\frac{-20}{3}$; (D) $\frac{-4}{3}$;

(b) **(4 points)** Find the unit vector that *minimizes* the directional derivative $D_{\vec{u}} f(x,y,z)$ at the point $(1,3,1)$ where

$$f(x,y,z) = xyz + e^{3-xy} + z^2.$$

Solution: This minimum occurs in the direction of $-\nabla f(1,3,1) = \langle 0,0,-5\rangle$. The unit vector in this direction is $\langle 0,0,-1\rangle$.

(circle one):

(A) $\langle \frac{-6}{\sqrt{65}}, \frac{-2}{\sqrt{65}}, \frac{-5}{\sqrt{65}}\rangle$; (B) $\langle 0,0,-1\rangle$;

(C) $\langle 0,0,1\rangle$; (D) $\langle \frac{6}{\sqrt{65}}, \frac{2}{\sqrt{65}}, \frac{5}{\sqrt{65}}\rangle$;

4. Let $f(x,y) = x^3 + 2y^3 - 3x^2 - 3y^2 - 9x$.

(a) **(3 points)** How many critical points does $f(x,y)$ have?

Solution: Solving the system $f_x(x,y) = 3x^2 - 6x - 9 = 0$ and $f_y(x,y) = 6y^2 - 6y = 0$, gives critical points of $(3,0)$, $(3,1)$, $(-1,0)$, and $(-1,1)$. So $f(x,y)$ has 4 critical points.

(circle one): (A) 2; (B) 3; (C) 4; (D) 6;

(b) **(2 points)** At how many of these critical points does $f(x,y)$ have a saddle point?

Solution: $f_{xx} = 6x - 6$, $f_{yy} = 12y - 6$, $f_{xy} = f_{yx} = 0$, and $D = (6x-6)(12y-6)$. Using the 2^{nd} Derivative test gives that $f(x,y)$ has two saddle points at $(3,0)$ and $(-1,1)$.

(circle one): (A) 0; (B) 1; (C) 2; (D) 3;

(c) **(2 points)** At how many of these critical points does $f(x,y)$ have a local minimum?

Solution: $f_{xx} = 6x - 6$, $f_{yy} = 12y - 6$, $f_{xy} = f_{yx} = 0$, and $D = (6x-6)(12y-6)$. Using the 2^{nd} Derivative test gives that $f(x,y)$ has one local minimum at $(3,1)$.

(circle one): (A) 0; (B) 1; (C) 2; (D) 3;

(d) **(2 points)** At how many of these critical points does $f(x,y)$ have a local maximum?

Solution: $f_{xx} = 6x - 6$, $f_{yy} = 12y - 6$, $f_{xy} = f_{yx} = 0$, and $D = (6x-6)(12y-6)$. Using the 2^{nd} Derivative test gives that $f(x,y)$ has one local maximum at $(-1,0)$.

(circle one): (A) 0; (B) 1; (C) 2; (D) 3;

5. Let $f(x, y) = x^2 + y^2$ and $D = \{(x, y) \mid x^2 + y^2 + 2x - 3 \le 0\}$.

(a) **(4 points)** What is the absolute maximum value of $f(x, y)$ on the *boundary* of D?

Solution: Using Lagrange Multipliers with $g(x, y) = x^2 + y^2 + 2x - 3 = 0$, we solve the system:

$$\begin{aligned} 2x &= \lambda 2(x+1) \\ 2y &= \lambda 2y \\ x^2 + y^2 + 2x - 3 &= 0 \end{aligned}$$

The solutions to this system, (λ, x, y), are $(\frac{1}{2}, 1, 0)$ and $(\frac{3}{2}, -3, 0)$. Since $f(1, 0) = 1$ and $f(-3, 0) = 9$. The answer is 9.

(circle one): (A) 25; (B) 8; (C) 9; (D) 16;

(b) **(3 points)** What is the absolute minimum value of $f(x, y)$ on D?

Solution: Using calculations from part (a) and noting the only critical point of $f(x, y)$ is $(0, 0)$ along with the Extreme Value Theorem gives the answer of 0.

(circle one): (A) 0; (B) 1; (C) 9; (D) 16;

6. **(8 points)** Find the point on the ellipsoid $x^2 + 2y^2 + 3z^2 = 6$ where the function $f(x, y, z) = 2x + 4y + 6z$ is minimized.

Solution: Letting $g(x, y, z) = x^2 + 2y^2 + 3z^2$, we use Lagrange multipliers to find the absolute maximum and absolute minimum of $f(x, y, z)$ subject to $g(x, y, z) = 6$. The Lagrange system to solve is:

$$\begin{aligned} 2 &= \lambda 2x \\ 4 &= \lambda 4y \\ 6 &= \lambda 6z \\ x^2 + 2y^2 + 3z^2 &= 6 \end{aligned}$$

The first equation gives us that $\lambda \neq 0$, so $x = y = z = \frac{1}{\lambda}$, plugging these into the fourth equation gives $\lambda = 1$ or $\lambda = -1$, which gives solution points of $(1, 1, 1)$ and $(-1, -1, -1)$, respectively. $f(1, 1, 1) = 12$ and $f(-1, -1, -1) = -12$. So the answer is: $(-1, -1, -1)$.

The point is: (, ,)

7. (a) **(4 points)** Let C be the curve of intersection of $x^2 + y^2 = 1$ and $z = -x^2 + y$. Find the tangent line $\vec{l}(t)$ to C at the point $(1, 0, -1)$.

Solution: C can be parametrized by $\vec{r}(t) = \langle \cos t, \sin t, -\cos^2 t + \sin t \rangle$. The point $(1, 0, -1)$ corresponds to $\vec{r}(t)$ at $t = 0$. $\vec{r}\,'(t) = \langle -\sin t, \cos t, 2\cos t \sin t + \cos t \rangle$. The tangent line of interest is then: $\vec{l}(t) = \langle 1, t, -1 + t \rangle$.

(circle one):

(A) $\vec{l}(t) = \langle 1 + t, t, -1 - t \rangle$; (B) $\vec{l}(t) = \langle 1 + 2t, t, -1 - 2t \rangle$;

(C) $\vec{l}(t) = \langle 1, t, -1 \rangle$; (D) $\vec{l}(t) = \langle 1, t, -1 + t \rangle$;

(b) **(3 points)** Find the length of the curve of $\vec{r}(t) = \langle 2e^t, e^t \sin t, e^t \cos t \rangle$, for $0 \le t \le \ln 2$.

Solution: This length is given by $\int_0^{\ln 2} e^t \sqrt{6}\, dt = \sqrt{6}$

(circle one): (A) $\sqrt{5}$; (B) $2\sqrt{6}$; (C) $\sqrt{6}$; (D) $2\sqrt{5}$;

8. (a) **(4 points)** Find $\int_C x^2 y\, ds$, where C is given by $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$ for $0 \le t \le \pi$.

Solution: $\int_C x^2 y\, ds = \int_0^\pi \cos^2(t)\sin(t)\sqrt{2}\, dt = \frac{2\sqrt{2}}{3}$

(circle one): (A) $\frac{2\sqrt{2}}{3}$; (B) $\frac{\sqrt{2}}{3}$; (C) 0; (D) $\frac{-\sqrt{2}}{3}$;

(b) **(4 points)** Find $\int_C z^2 dx + x^2 dy + y^2 dz$, where C is the line segment from $(0,0,0)$ to $(2,3,1)$.

Solution: This line segment can be parametrized by $\vec{r}(t) = \langle 2t, 3t, t\rangle$ for $0 \le t \le 1$, giving $\int_C z^2 dx + x^2 dy + y^2 dz = \int_0^1 2t^2 + 12t^2 + 9t^2 dt = \frac{23}{3}$.

(circle one): (A) $\frac{23}{3}$; (B) 7; (C) 8; (D) $\frac{22}{3}$;

9. **(8 points)** Evaluate the $\int_C \vec{F}\cdot \mathrm{d}\vec{r}$, where $\vec{F}(x,y) = \langle x-y, x+y\rangle$ and C is the path given by the ellipse $16x^2 + y^2 = 16$ transversed once and oriented clockwise.

Solution:

We can parameterized the ellipse by:

$$\vec{r}(t) = \langle \cos(-t), 4\sin(-t)\rangle = \langle \cos t, -4\sin t\rangle, \quad t \in [0, 2\pi]$$

(Note the choice of $-t$ to have a clockwise path.) This path has derivative:

$$\vec{r}\,'(t) = \langle -\sin t, -4\cos t\rangle.$$

Therefore, using the definition of the integral, we find:

$$\begin{aligned}
\int_C \vec{F}\cdot \mathrm{d}\vec{r} &= \int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,\mathrm{d}t \\
&= \int_0^{2\pi} \langle \cos t + 4\sin t, \cos t - 4\sin t\rangle \cdot \langle -\sin t, -4\cos t\rangle\,\mathrm{d}t \\
&= \int_0^{2\pi} 15\cos t\sin t - 4(\sin^2 t + \cos^2 t)\,\mathrm{d}t \\
&= \int_0^{2\pi} 15\cos t\sin t - 4\,\mathrm{d}t = -8\pi.
\end{aligned}$$

$\int_C \vec{F}\cdot \mathrm{d}\vec{r} =$

Trigonometric Identities

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$
$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
$$\tan(\theta+\phi) = \frac{\tan(\theta)+\tan(\phi)}{1-\tan\theta\tan\phi}$$
$$\sin(\theta-\phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$$
$$\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$$
$$\tan(\theta-\phi) = \frac{\tan(\theta)-\tan(\phi)}{1+\tan\theta\tan\phi}$$
$$\sin(\theta)\sin(\phi) = \frac{\cos(\theta-\phi)-\cos(\theta+\phi)}{2}$$
$$\cos(\theta)\cos(\phi) = \frac{\cos(\theta-\phi)+\cos(\theta+\phi)}{2}$$
$$\sin(\theta)\cos(\phi) = \frac{\sin(\phi+\theta)-\sin(\phi-\theta)}{2}$$
$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$
$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$
$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$
$$\tan\theta = \frac{\sin(2\theta)}{1+\cos(2\theta)}$$