Group: ______________ ______________ ______________ Name: solutions

Math 231 A. Fall, 2015. Worksheet 15. 11/5/15

1. a) Use the Maclaurin series for $\cos x$ to find the Maclaurin series for $f(x) = x^3\cos(x^2)$.

$$\cos(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}, \text{ so } f(x) = x^3\cos(x^2) = x^3\sum_{n=0}^{\infty}(-1)^n\frac{(x^2)^{2n}}{(2n)!}$$

$$= x^3\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+3}}{(2n)!}$$

b) Use part a) and the definition of Maclaurin series to find the value of $f^{(11)}(0)$.

The coefficient on x^{11} in the Maclaurin series for $f(x)$ is $\frac{f^{(11)}(0)}{11!}$

$4n+3 = 11$ when $n=2$, so the x^{11} term in the series is

$$(-1)^2\frac{x^{4(2)+3}}{(2\cdot 2)!} = \frac{1}{24}x^{11}. \quad \text{so} \quad \frac{f^{(11)}(0)}{11!} = \frac{1}{24}, \text{ hence } f^{(11)}(0) = \frac{11!}{24}$$

2. Find the Taylor series (centered at the given point) for each of the following functions by differentiating the given functions and finding the pattern.

(a) $f(x) = \sin x$ centered at $a = \frac{\pi}{2}$.

$f'(x) = \cos(x)$

$f''(x) = -\sin(x)$

$f^{(3)}(x) = -\cos(x)$

$f^{(4)}(x) = \sin(x)$

(repeats)

n	$f^{(n)}(x)$ at $x = \frac{\pi}{2} = a$
0	1
1	0
2	-1
3	0
4	1
5	0
6	-1
7	0
8	1
9	0
⋮	

so $\sin(x) = 1 + \frac{0}{1!}(x-\frac{\pi}{2}) + \frac{-1}{2!}(x-\frac{\pi}{2})^2 + \frac{0}{3!}(x-\frac{\pi}{2})^3$

$+ \frac{1}{4!}(x-\frac{\pi}{2})^4 + \frac{0}{5!}(x-\frac{\pi}{2})^5 \ldots$

$= 1 - \frac{1}{2!}(x-\frac{\pi}{2})^2 + \frac{1}{4!}(x-\frac{\pi}{2})^4 - \frac{1}{6!}(x-\frac{\pi}{2})^6 + \ldots$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{(x-\frac{\pi}{2})^{2n}}{(2n)!}$$

(b) $f(x) = \ln x$ centered at $a = 2$.

$f(x) = \ln(x)$

$f'(x) = \frac{1}{x}$

$f''(x) = \frac{-1}{x^2}$

$f^{(3)}(x) = \frac{2}{x^3}$

$f^{(4)}(x) = \frac{-6}{x^4}$

$f^{(5)}(x) = \frac{24}{x^5}$

$f^{(n)} = (-1)^{n-1}\frac{(n-1)!}{x^n}$ for $n \geq 1$ so $f^{(n)}(2) = (-1)^{n-1}\frac{(n-1)!}{2^n}$ for $n \geq 1$, so:

$$\ln(x) = \ln(2) + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(n-1)!}{2^n\cdot n!}(x-2)^n$$

$$= \ln(2) + \sum_{n=1}^{\infty}(-1)^n\frac{1}{2^n n}(x-2)^n$$

3. a) Write down the first three terms of the Maclaurin series for $f(x) = \dfrac{\sin(x^2) - x^2\cos x}{x^4}$.

$$\frac{1}{x^4}\left(\left(x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \ldots\right) - x^2\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\ldots\right)\right)$$

$$= \frac{1}{x^4}\left(\frac{x^4}{2!} + \left(\frac{-1}{3!} - \frac{1}{4!}\right)x^6 + \frac{x^8}{8!}\ \ldots\right) = \frac{1}{2} + \left(\frac{-1}{3!} - \frac{1}{4!}\right)x^2 + \frac{x^4}{8!} + \ldots$$

so as $x \to 0$, $\lim_{x\to 0} f(x) = \frac{1}{2} + 0 + 0 + \ldots$

$= \frac{1}{2}.$

b) Use this to evaluate $\lim_{x\to 0} f(x)$.

4. Recall the binomial series:

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n, \qquad |x| < 1,$$

where

$$\binom{k}{0} = 1, \qquad \binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!} \quad \text{for } n \geq 1.$$

a) Use this to write down the Maclaurin series for $\sqrt{1+x^2}$. **No simplification necessary–leave the binomial coefficients in your answer.**

$$(1+x^2)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^{2n}$$

b) Write out the first four terms ($n = 0$ through $n = 3$) in this series and simplify the coefficients as much as you can.

$$\binom{1/2}{0}x^0 + \binom{1/2}{1}x^2 + \binom{1/2}{2}x^4 + \binom{1/2}{3}x^6 + \dots$$

$$= 1 + \frac{1}{2}x^2 + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!}x^4 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}x^6 + \dots$$

$$= 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6$$

c) Write out the first four terms of the Maclaurin series for $\dfrac{1}{\sqrt[4]{1+x}}$ and simplify the coefficients.

$$(1+x)^{-1/4} = \sum_{n=0}^{\infty} \binom{-1/4}{n} x^n$$

$$= \binom{-1/4}{0}x^0 + \binom{-1/4}{1}x^1 + \binom{-1/4}{2}x^2 + \binom{-1/4}{3}x^3 + \dots$$

$$= 1 + \frac{-1}{4}x + \frac{(-\frac{1}{4})(-\frac{1}{4}-1)}{2!}x^2 + \frac{(-\frac{1}{4})(-\frac{1}{4}-1)(-\frac{1}{4}-2)}{3!}x^3 + \dots$$

$$= 1 - \frac{1}{4}x + \frac{5}{4^2 \cdot 2!}x^2 - \frac{5 \cdot 9}{4^3 \cdot 3!}x^3 + \dots$$

5. Use the Maclaurin series for e^x, $\sin x$, or $\cos x$ to find the sum of each series.

a) $1 - \dfrac{\pi^2}{2!} + \dfrac{\pi^4}{4!} - \dfrac{\pi^6}{6!} + \dots. = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!} = \cos(\pi) =$

b) $1 - \dfrac{\pi^2}{2^2 2!} + \dfrac{\pi^4}{2^4 4!} - \dfrac{\pi^6}{2^6 6!} + \dots = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(\frac{\pi}{2})^{2n}}{(2n)!} = \cos\left(\frac{\pi}{2}\right) =$

c) $\dfrac{1}{2!} - \dfrac{1}{3!} + \dfrac{1}{4!} - \dfrac{1}{5!} + \dots = \displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n!}$ ← this is most of the Maclaurin Series for e^x, but is missing the first 2 terms:

$$= \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\right] - \left[\underset{n=0}{1} - \underset{n=1}{1}\right]$$

$$= e^{-1} - 1 + 1 = \frac{1}{e}.$$