University of Illinois at Urbana-Champaign
Dept. of Electrical and Computer Engineering

ECE 120: Introduction to Computing

Signed Integers and 2's Complement

Strategy: Use Common Hardware for Two Representations

Recall:

- addition of bit patterns in **N-bit unsigned** representations
- corresponds to arithmetic mod 2^N.

Using this arithmetic, we develop the **2's complement representation** for signed integers.

The same hardware can then perform arithmetic for both representations.

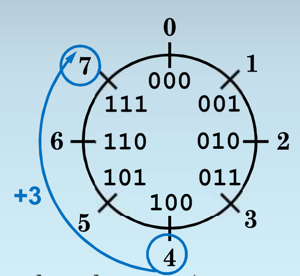
What about the name? Later.

Graphical Illustration of Modular Arithmetic

The circle illustrates **3-bit unsigned**.

Adding a number corresponds to counting clockwise.

The answer is always correct mod 8.



(For subtraction, count the other way.)

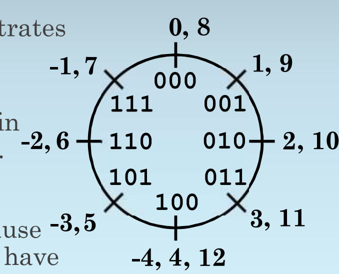
Representations Must be Unambiguous

The same circle illustrates **equality mod 8**.

For example, we can extend the numbers in a clockwise direction.

Or the other way.

Overflow occurs because a representation can have only **one value per bit pattern**.



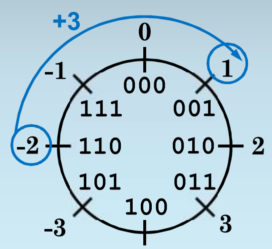
We Can Choose Any Meaning for a Bit Pattern

But what if we pick a different set of labels?

The arithmetic doesn't change.

Let's include both positive and negative numbers!

And try some addition.



That's One Way to Define 2's Complement

Draw a circle for **N** bits (**2^N** points).

Starting at 0 at the top.

Write unsigned bit patterns clockwise around the circle.

Starting again from 0,
- find bit patterns for negative numbers
- by moving counter-clockwise.

What about the name? Later.

2's Complement Can Also be Derived Algebraically

We can also define **N-bit 2's complement** algebraically.

An adder for **N-bit unsigned** gives

$$\mathbf{SUM_N(A,B) = A + B \bmod 2^N}$$

N-bit 2's complement includes positive numbers in the range **$[1, 2^{N-1} - 1]$**. These bit patterns all start with a "0" bit.

We need to find bit patterns for negative numbers.

Properties Needed for Negative Number Bit Patterns

For each number **K**, **$0 < K < 2^{N-1}$**,
- we want to find an **N**-bit pattern **P_K**, **$0 \le P_K < 2^N$**,
- such that **for any integer M,**

$$\mathbf{(-K + M = P_K + M) \bmod 2^N}$$

The bit pattern **P_K** then produces the same results as **-K** when used with unsigned arithmetic.

Also, **P_K must not be used by a number ≥ 0**.

Do Algebra to Define Negative Patterns

Starting with our property,

$$(-K + M = P_K + M) \bmod 2^N,$$

subtract M from both sides to obtain

$$(-K = P_K) \bmod 2^N.$$

Next, note that

$$(2^N = 0) \bmod 2^N.$$

Now add the last two equations to obtain

$$(2^N - K = P_K) \bmod 2^N.$$

Final Answer: -K is Represented by $2^N - K$

One easy solution to $(2^N - K = P_K) \bmod 2^N$ is $P_K = 2^N - K$.

Since $0 < K < 2^{N-1}$, this solution gives $2^{N-1} < P_K < 2^N$.

But these are all unused bit patterns—the patterns starting with "1!"

So we're done:
-K is represented by the pattern $2^N - K$.

What about the name? Are you really ready?

Negating Twice Gives an Identity Operation

Let's do a sanity check.

What is the bit pattern for **- (-K)**?

We know that **-K** is $2^N - K$.

Substituting once, we obtain $-(2^N - K)$.

Substituting again, we obtain $2^N - (2^N - K)$.

But that's just **K**, as we expect.

What name? Oh, "2's complement?"

Is There an Easy Way to Find -K?

How do we calculate $2^N - K$?

We can subtract (for example, with **N=5**)…

```
  100000  (2^N)
-  ?????  (K)
```

But that seems painful.

Instead, notice that $2^N = (2^N - 1) + 1$.

So we can calculate $(2^N - 1) - K + 1$.

2's Complement is 1's Complement Plus One!

Again for **N=5**:

```
  11111  (2^N – 1)
– ?????  (K)
 (answer)
+     1
```

The first step is trivial: replace 0 with 1, and 1 with 0. The result (**(2^N – 1) – K**) is called the **1's complement of K**.

Adding 1 more gives the **2's complement**.

Distinguish 2's Complement from Negation

Here or elsewhere, you will hear the phrase "take the 2's complement."

We will **try not to use "2's complement" in that way**.

Students get confused between the **2's complement representation** for signed integers and the operation of **negation** on a bit pattern for a number represented with 2's complement.

For clarity, we suggest that you do the same.

Example: Negating a Number in 2's Complement

Let's do an example of negation with **8-bit 2's complement**.

As you know, I like 42.

As you may remember, 42_{10} = `00101010`.

So what's -42?

First, complement the bits: `11010101`.

Then add 1: `11010110` = -42_{10}!

2's Complement Conversion Can Be Same as Unsigned

For **non-negative numbers** (bit patterns starting with 0),

conversion between decimal value and **2's complement** bit pattern

is **identical to conversion for the unsigned representation**.

Use Two Negations to Convert Negative Numbers

To convert decimal **D < 0** to **2's complement**,
- first **convert –D** (as **unsigned**),
- then **negate the resulting bit pattern**.

To convert a negative **2's complement** bit pattern (a bit pattern starting with 1) to decimal,
- first **negate the bit pattern**,
- then **convert to decimal D** (as unsigned).
- The **answer is -D**.

Alternate Method for Calculating 2's Complement Values

If we have a negative number **-K**, we can use the base 2 polynomial to calculate $2^N - K$:

$$2^N - K = a_{N-1}2^{N-1} + a_{N-2}2^{N-2} + \ldots + a_0 2^0$$

We know that $a_{N-1} = 1$ for a negative number. Substituting and subtracting 2^N gives:

$$-K = (2^{N-1} - 2^N) + a_{N-2}2^{N-2} + \ldots + a_0 2^0$$

$$-K = -a_{N-1}2^{N-1} + a_{N-2}2^{N-2} + \ldots + a_0 2^0$$

This polynomial also works when $a_{N-1} = 0$.

What about the Last Bit Pattern?

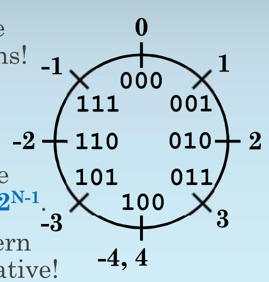
We didn't define a value for one of the bit patterns!

What should it be?

2^{N-1}? -2^{N-1}? Undefined?

In **2's complement**, the pattern always means -2^{N-1}.

Why? So that any pattern starting with a 1 is negative!



Extend Unsigned Bit Patterns by …

In some cases, we need
- to convert a bit pattern
- from a smaller representation (fewer bits)
- to a larger one (more bits)

How do we convert **N-bit unsigned** to **(N+k)-bit unsigned** (for **k** > 0)?

Hint: We already had to solve a similar problem when a number does not require **N** bits in base 2.

Add k more leading 0s (called **zero extension**).

What about 2's Complement?

How do we convert **N-bit 2's complement** to **(N+k)-bit 2's complement** (for **k** > 0)?

For **non-negative values**,

- **2's complement** is the same as **unsigned** (with an extra 0 for the sign)
- So **add k more leading 0s**.

What about negative values?

Extend 2's Complement Bit Patterns by …

In **5-bit 2's complement**,

-5_{10} has bit pattern 11011
-10_{10} has bit pattern 10110

(spaces added to help us humans)

And in **8-bit 2's complement**?

-5_{10} has bit pattern 111 11011
-10_{10} has bit pattern 111 10110

So how do we convert **N-bit 2's complement** to **(N+k)-bit 2's complement** (for **k** > 0)?

Add k copies of the sign bit (called **sign extension**).