Math 231 Fall 2014. Worksheet 12. 10/23/14

Group: ____________ ____________ ____________ Name: SOLUTIONS

[redacted]

1. a) Use the alternating series test to prove that that the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ converges.

$b_n = \frac{1}{n!}$

- $b_n \geq 0$
- $\lim_{n\to\infty} \frac{1}{n!} = 0$
- since $(n+1)! > n!$, $\{b_n\}$ is dec.

$\Rightarrow$ by A.S.T, $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ converges

b) Using a calculator, find the partial sum $s_6 = \sum_{n=0}^{6} \frac{(-1)^n}{n!}$ to four decimal places. What is the maximum value of $|R_6|$?

$s_6 = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \approx 0.3681$

$|R_6| < b_7 = \frac{1}{7!} \approx 0.0002$

c) Soon we will prove that $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1/e$. Compute $1/e$ to four decimal places to check that your answer agrees with this. $\frac{1}{e} \approx 0.3679$

$|R_6| = |\frac{1}{e} - s_6| \approx |0.3679 - 0.3681| = 0.0002$

2. Consider the series

$$1 - \frac{1}{10^1} + \frac{1}{2} - \frac{1}{10^2} + \frac{1}{3} - \frac{1}{10^3} + \frac{1}{4} - \frac{1}{10^4} + \dots \qquad (\star)$$

a) Show that the series diverges.

Hint: we know that the series

$$0 + \frac{1}{10^1} + 0 + \frac{1}{10^2} + 0 + \frac{1}{10^3} + 0 + \frac{1}{10^4} + \dots \qquad (\star\star)$$

converges. If the series $(\star)$ converged as well, what would happen?

If $(\star)$ converged, then $(\star) + (\star\star) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ would converge, but we know the harmonic series diverges. This contradiction implies $(\star)$ diverges.

b) Why doesn't the Alternating Series Test apply to the series $(\star)$?

$\{b_n\} = (1, \frac{1}{10}, \frac{1}{2}, \frac{1}{10^2}, \frac{1}{3}, \dots)$ is not decreasing

Since $\frac{1}{n} > \frac{1}{10^{n-1}}$ for all n (so $b_3 = \frac{1}{2} > \frac{1}{10} = b_2$, $b_5 = \frac{1}{3} > \frac{1}{100} = b_4$, etc)

3. Around 1910, the mathematician Ramanujan discovered the formula

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}}.$$

William Gosper used this series in 1985 to compute π to 17 million digits.

(a) Verify that the series is convergent.

$$L = \lim_{n\to\infty} \frac{(4n+4)!\,(1103+26390(n+1))\,((n+1)!)^{-4}\,396^{-4n-4}}{(4n)!\,(1103+26390n)\,(n!)^{-4}\,396^{-4n}} = \lim_{n\to\infty} \frac{(4n+4)(4n+3)(4n+2)(4n+1)\,(1103+26390(n+1))}{396^4 \cdot (n+1)^4 \cdot (1103+26390n)}$$

$$= \frac{4^4}{396^4} < 1$$ so the series converges by the ratio test

(b) How many correct decimal places of π do you get if you use just the first term of the series? What if you use the first 2 terms? (Use a calculator.) ($\pi = 3.14159265358979 32\ldots$)

$$S_0 = a_0 = 1103 \Rightarrow \pi \approx \frac{9801}{2\sqrt{2}}(1103)^{-1} \approx \underbrace{3.141592}_{\text{correct}}73001\ldots$$

$$\text{using } S_1 = a_0 + a_1 : \pi \approx \frac{9801}{2\sqrt{2}}(a_0 + a_1)^{-1} \approx \underbrace{3.141592653589793}_{\text{correct}}8\ldots$$

In each problem, determine if the series converges absolutely, converges conditionally, or diverges. Show work and state explicitly which test or tests you are using (Ratio, Root, Alternating Series, etc.).

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n n^2}{n!}$.

$$L = \lim_{n\to\infty} \left| \frac{(-3)^{n+1}(n+1)^2((n+1)!)^{-1}}{(-3)^n n^2 (n!)^{-1}} \right| = \lim_{n\to\infty} \frac{3(n+1)^2}{(n+1)\,n^2} = 0 < 1$$

By the ratio test, the series converges absolutely.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n}$

$$L = \lim_{n\to\infty} \left(\frac{(-2)^{2n}}{n^n}\right)^{1/n} = \lim_{n\to\infty} \frac{4}{n} = 0 < 1$$

By the root test, the series converges absolutely

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(\ln n)^2}{n}$ $\qquad a_n = \frac{(-1)^n(\ln n)^2}{n} \qquad b_n = |a_n| = \frac{(\ln n)^2}{n}$

- Since $|a_n| > \frac{1}{n}$ for large n, $\Sigma |a_n|$ diverges by the comp. test, so Σa_n does not absolutely converge.

- A.S.T:
  - $b_n \geq 0$
  - $\lim_{n\to\infty} b_n = 0$ (Squeeze Thm: $0 < b_n < \frac{(n^{1/3})^2}{n} = n^{-1/3}$ for large n)
  - $\{b_n\}$ is decreasing for $n > e^2$ (since $\frac{d}{dx}\frac{(\ln x)^2}{x} = \frac{\ln x(2-\ln x)}{x^2} < 0$ for $x > e^2$)

so by A.S.T, the series converges conditionally.