Math 231 - Midterm 3 Review

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Sections: ADJ/ADK

Alternating Series

Alternating Series Test (AST)

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ with $b_n > 0$ satisfies

(i) $b_{n+1} \leq b_n$ (decreasing)

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is CONVERGENT.

With the same hypothesis as above, by the **Alternating Series Estimation Theorem** we have

$$\boxed{|R_n| \leq \underline{\underline{b_{n+1}}}}$$ (NOT b_n !!)

Absolute Convergence and the Ratio and Root Tests

- Define the following notions:

a) Absolutely convergent: $\sum_{n=0}^{\infty} a_n$ is absolutely convergent if $\sum_{n=0}^{\infty} |a_n|$ is convergent.

Examples: (a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is <u>not</u> absolutely convergent.

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutely convergent.

b) Conditionally convergent: $\sum_{n=0}^{\infty} a_n$ is conditionally convergent if $\sum_{n=0}^{\infty} a_n$ converges <u>but</u> $\sum_{n=0}^{\infty} |a_n|$ diverges.

Example: $\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n}$ is conditionally convergent.

* can you think of more examples?

The Ratio Test

$$\text{If } \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \begin{cases} < 1 & \text{Absolutely convergent} \\ > 1 & \text{Divergent} \\ = 1 & \text{Inconclusive} \end{cases}$$

Don't forget to take the absolute value!

The Root Test

$$\text{If } \lim_{n\to\infty} \sqrt[n]{|a_n|} \begin{cases} < 1 & \text{Absolutely convergent} \\ > 1 & \text{Divergent} \\ = 1 & \text{Inconclusive} \end{cases}$$

STEPS:

1. Use Ratio or Root Test.
2. If that is inconclusive and the series is alternating, then do BOTH of the next:
   (a) Use AST (Alternating Series Test)
   (b) Check whether $\sum |a_n|$ is divergent or convergent.

If the AST shows that the series in convergent, BUT $\sum |a_n|$ is divergent then $\sum a_n$ is **conditionally convergent**. However, if $\sum |a_n|$ is convergent then $\sum a_n$ is **absolutely convergent**.

(no need to use AST in this case)

Determine whether the following series converge absolutely, converge conditionally or diverge.

1. $\sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\pi)}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n}$ Harmonic Series!

Therefore, the series $\boxed{\text{DIVERGES.}}$

! It's NOT an alternating series! We may be tempted to take $b_n = \frac{\cos(n\pi)}{n}$ and use AST. But we cannot because b_n is not always possitive.

↳ CONCLUSION: Remember to check the condition $b_n > 0$.

2. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+7}$ Ratio Test will be inconclusive.

(a) AST : $b_n = \frac{n}{n^2+7} > 0$, $b_{n+1} = \frac{n+1}{(n+1)^2+7} < \frac{n}{n^2+7} = b_n$ and $\lim_{n\to\infty} \frac{n}{n^2+7} = 0$.

↳ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+7}$ is convergent

(b) $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n}{n^2+7} \sim \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$\Rightarrow$ The series is $\boxed{\text{CONDITIONALLY CONVERGENT}}$

3. $\sum_{n=2}^{\infty} \frac{(-1)^{n+1} (2n)!}{5^n \, n! \, n!}$ RATIO TEST:

$$\lim_{n\to\infty} \left| \frac{[2(n+1)]!}{5^{n+1}(n+1)!\,(n+1)!} \cdot \frac{5^n \cdot n! \cdot n!}{(2n)!} \right| = \lim_{n\to\infty} \left| \frac{(2n+2)! \cdot \cancel{5^n} \cdot \cancel{n!} \cdot \cancel{n!}}{\cancel{5^n} \cdot 5\,(n+1)\cancel{n!}\,(n+1)\cancel{n!}\,(2n)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(2n+2)(2n+1)\cancel{(2n)!}}{5(n+1)(n+1)\cancel{(2n)!}} \right| = \frac{4}{5} < 1 \Rightarrow \boxed{\text{ABSOLUTELY CONVERGENT}}$$

4. $\sum_{n=2}^{\infty} (-1)^n \left(\frac{n+1}{3n+5}\right)^n$ ROOT TEST :

$$\lim_{n\to\infty} \sqrt[n]{\left| (-1)^n \left(\frac{n+1}{3n+5}\right)^n \right|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n+1}{3n+5}\right)^n} = \lim_{n\to\infty} \frac{n+1}{3n+5} = \frac{1}{3} < 1$$

$\Rightarrow \boxed{\text{ABSOLUTELY CONVERGENT}}$

Power Series

1. Find the radius of convergence for the following power series.

$$\sum_{n=1}^{\infty} \frac{n!x^n}{5\cdot 11\cdot 17\cdot \ldots \cdot (6n-1)}$$

RATIO TEST:

$$\lim_{n\to\infty}\left|\frac{(n+1)!\; x^{n+1}}{\cancel{5\cdot 11\cdot 17\cdots(6n-1)}\underbrace{(6n+5)}_{(6(n+1)-1)}}\cdot\frac{\cancel{5\cdot 11\cdot 17\cdots(6n-1)}}{n!\; x^n}\right| =$$

$$\lim_{n\to\infty}\left|\frac{(n+1)\cancel{n!}\,\cancel{x^n}\cdot x}{(6n+5)\cancel{n!}\,\cancel{x^n}}\right| = \left|\frac{n+1}{6n+5}\cdot x\right| = \frac{|x|}{6} < 1$$

Condition for Ratio Test to be abs. convergent.

$$\frac{|x|}{6} < 1 \quad \text{when} \quad |x| < 6 \implies \boxed{R = 6}$$

2. Find the interval of convergence for the following power series.

$$\sum_{n=1}^{\infty} \frac{2(x-3)^n}{3^n\cdot n}$$

RATIO TEST: (to find R)

$$\lim_{n\to\infty}\left|\frac{2(x-3)^{n+1}}{3^{n+1}\cdot(n+1)}\cdot\frac{3^n\cdot n}{2(x-3)^n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{2}\cdot\cancel{(x-3)^n}\cdot(x-3)\cdot\cancel{3^n}\cdot n}{\cancel{3^n}\cdot 3(n+1)\cancel{2}\cdot\cancel{(x-3)^n}}\right| =$$

$$= \lim_{n\to\infty}\left|\frac{(x-3)\cdot n}{3(n+1)}\right| = \frac{|x-3|}{3} < 1 \implies |x-3| < 3 \implies \boxed{R = 3}$$

→ Interval of convergence: $I = [(3-3,\ 3+3])$ We need to check end points!!

$\quad\quad\quad ?\ \ \overset{\shortparallel}{0}\quad \overset{\shortparallel}{6}\ \ ?$

• $\underline{x=0}$: $\displaystyle\sum_{n=1}^{\infty}\frac{2\cdot(0-3)^n}{3^n\cdot n} = \sum_{n=1}^{\infty}\frac{2\cdot(-3)^n}{3^n\cdot n} = \sum_{n=1}^{\infty}\frac{2\cdot(-1)^n\,\cancel{3^n}}{\cancel{3^n}\cdot n} = \sum_{n=1}^{\infty}\frac{2\cdot(-1)^n}{n}$ CONVERG. (by AST)

• $\underline{x=6}$: $\displaystyle\sum_{n=1}^{\infty}\frac{2(6-3)^n}{3^n\cdot n} = \sum_{n=1}^{\infty}\frac{2\cdot\cancel{3^n}}{\cancel{3^n}\cdot n} = \sum_{n=1}^{\infty}\frac{2}{n} = 2\cdot\sum_{n=1}^{\infty}\frac{1}{n}$ DIVERGES (Harmonic Series)

Therefore, $\boxed{I = [0, 6)}$ OR $0 \le x < 6$

Representation of Functions as Power Series

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{, for } |x| < 1}$$

REMARKS:

1. If you integrate a series to find the power series for a particular function, remember to calculate +C.
2. The radius of convergence (R) doesn't change when we derivate or integrate a series. However, the interval of convergence (I) may change (i.e., you will still need to check the endpoints).

Examples: Find the power series representation of the function and determine the interval of convergence.

1. $f(x) = \frac{x}{9+x^2} = x \cdot \frac{1}{9+x^2} = \frac{x}{9} \cdot \frac{1}{1+\frac{x^2}{9}} = \frac{x}{9} \cdot \frac{1}{1-\left(-\frac{x^2}{9}\right)} = \frac{x}{9} \cdot \sum_{n=0}^{\infty} \left(\frac{-x^2}{9}\right)^n =$

DON'T stop there!

$$= \sum_{n=0}^{\infty} \frac{x}{9} \cdot \frac{(-1)^n x^{2n}}{9^n} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{9^{n+1}}, \quad \text{for } \left|\frac{-x^2}{9}\right| < 1 \Rightarrow |x| < 3$$

$$\boxed{I = (-3, 3)}$$

2. $f(x) = \left(\frac{x}{1+4x}\right)^2 = \frac{x^2}{(1+4x)^2} = x^2 \cdot \frac{1}{(1+4x)^2}$

$$\bullet \int \frac{1}{(1+4x)^2}\,dx = \int \frac{1}{4u^2}\,du = -\frac{1}{4}u^{-1} + C = \frac{-1}{4(1+4x)} + C \Rightarrow \frac{1}{(1+4x)^2} = \frac{d}{dx}\left[\frac{-1}{4(1+4x)}\right]$$

$u = 1+4x$

$du = 4\,dx$

$$\Rightarrow \frac{1}{(1+4x)^2} = \frac{-1}{4}\frac{d}{dx}\left(\frac{1}{1+4x}\right) = \frac{-1}{4}\frac{d}{dx}\left(\sum_{n=0}^{\infty}(-4x)^n\right) = \frac{-1}{4}\frac{d}{dx}\left(\sum_{n=0}^{\infty}(-1)^n 4^n x^n\right) =$$

$$= \frac{-1}{4}\sum_{n=0}^{\infty}(-1)^n 4^n n x^{n-1} = \sum_{n=0}^{\infty}(-1)^{n+1} 4^{n-1} n x^{n-1}, \quad \text{for } |4x| < 1$$

$\hookrightarrow |x| < \frac{1}{4}$

$$\hookrightarrow \frac{x^2}{(1+4x)^2} = x^2 \cdot \sum_{n=0}^{\infty}(-1)^{n+1}4^{n-1} n x^{n-1} = \sum_{n=0}^{\infty}(-1)^{n+1}4^{n-1} n x^{n+1}$$

$$\boxed{I = \left(-\tfrac{1}{4}, \tfrac{1}{4}\right)}$$

Since we took the derivative we need to check endpoints even if R = 1/4. (CHECK THEM YOURSELF!)

3. $f(x) = \ln\left(\frac{1+x}{1-x}\right) = \ln|1+x| - \ln|1-x| = \int \frac{1}{1+x}\,dx - \int \frac{-1}{1-x}\,dx =$

$$= \int \sum_{n=0}^{\infty} (-x)^n\,dx + \int \sum_{n=0}^{\infty} x^n\,dx = \sum_{n=0}^{\infty} \int (-1)^n x^n\,dx + \sum_{n=0}^{\infty} \int x^n\,dx =$$

$$= \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}} + \overset{0}{\overset{\shortparallel}{C}} + \underbrace{\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}} + \overset{0}{\overset{\shortparallel}{C}} =$$

$$= \left(x - \cancel{\frac{x^2}{2}} + \frac{x^3}{3} - \cancel{\frac{x^4}{4}} + \cdots\right) + \left(x + \cancel{\frac{x^2}{2}} + \frac{x^3}{3} + \cancel{\frac{x^4}{4}} + \cdots\right)$$

$$= 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right] = \underbrace{\sum_{n=0}^{\infty} \frac{2 \cdot x^{2n+1}}{2n+1}} \quad \text{for } |x| < \underbrace{1 = R}$$

- Check endpoints:
  - $x = -1$: $\sum_{n=0}^{\infty} 2 \cdot \frac{\overset{= -1 \text{ always!}}{(-1)^{2n+1}}}{2n+1} = -\sum_{n=0}^{\infty} \frac{2}{2n+1}$ DIVERGENT (p-series $p=1$)
  - $x = 1 \Rightarrow \sum_{n=0}^{\infty} \frac{2}{2n+1} \Rightarrow$ DIVERGENT (p-series, $p=1$)

$\Rightarrow \boxed{I = (-1, 1)}$

4. $f(x) = \arctan(2x)$

$$\arctan(2x) = \int \frac{2}{1+(2x)^2}\,dx = \int \frac{2}{1+4x^2}\,dx = \int 2 \cdot \sum_{n=0}^{\infty} (-4x^2)^n\,dx =$$

$$= 2 \cdot \int \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n}\,dx = 2 \cdot \sum_{n=0}^{\infty} (-1)^n \overset{2^{2n}}{4^n} \frac{x^{2n+1}}{2n+1} =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1}, \quad \text{for } |-4x^2| < 1$$

$$|x^2| < \frac{1}{4}$$

$$|x| < \frac{1}{2} \Rightarrow \underbrace{R = \frac{1}{2}}$$

- Check endpoints:
  - $x = \frac{-1}{2}$: $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} \left(\frac{-1}{2}\right)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{2^{2n+1}} (-1)^{2n+1}}{\cancel{2^{2n+1}} \cdot (2n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^{3n+1}}{2n+1} \Rightarrow$ CONV. by AST
  - $x = \frac{1}{2}$: $\sum_{n=0}^{\infty} \frac{(-1)^n \cancel{2^{2n+1}} \cdot \cancel{\left(\frac{1}{2}\right)^{2n+1}}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \Rightarrow$ CONV. by AST.

Therefore, $\boxed{I = \left[\frac{-1}{2}, \frac{1}{2}\right]}$

Taylor and Maclaurin Series

n^{th} derivative!

Taylor series of the function f at a:
↑center

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad , |x-a| < R$$

Nth-degree Taylor polynomial of f at a:

$$T_N(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N$$

Then $f(x) = T_N(x) + R_N(x)$ and

$$|R_N(x)| \leq \frac{|f^{N+1}(z)|}{(N+1)!} \cdot |x-a|^{N+1}$$

for some value z between a and x.

- How is the series called when the center is $a = 0$? Maclaurin Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad R = \infty$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad R = \infty$$

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \qquad R = 1$$

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

← There are n terms on the top!

Practice problems

1. Find the Maclaurin series of $f(x) = 9(1-x)^{-2}$ using the definition of a Maclaurin series.

$$f(x) = 9(1-x)^{-2} \longrightarrow f(0) = 9 = 9 \cdot 1!$$

$$f'(x) = 9(-2)(-1)(1-x)^{-3} \longrightarrow f'(0) = 9 \cdot 2 = 9 \cdot 2!$$

$$f''(x) = 9(-2)(-1)(-3)(-1)(1-x)^{-4} \longrightarrow f''(0) = 9 \cdot 3 \cdot 2 = 9 \cdot 3!$$

$$f^{(3)}(x) = 9(-2)(-1)(-3)(-1)(-4)(-1)(1-x)^{-5} \longrightarrow f^{(3)}(0) = 9 \cdot 4 \cdot 3 \cdot 2 = 9 \cdot 4!$$

$$\vdots$$

$$f^{(n)}(x) = 9 \cdot (n+1)!(1-x)^{-n-2} \longrightarrow f^{(n)}(0) = 9 \cdot (n+1)!$$

Therefore:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{9(n+1)!}{n!} x^n = \boxed{\sum_{n=0}^{\infty} 9 \cdot (n+1) x^n}$$

2. Find the value of $f^{(8)}(0)$ given that $f(x) = \dfrac{1-\cos(2x^2)}{2}$.

- $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \longrightarrow \cos(2x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x^2)^{2n}}{(2n)!} \Rightarrow$

$$\Rightarrow \cos(2x^2) = 1 - \frac{4x^2}{2!} + \frac{2^4 x^8}{4!} - \cdots$$

- $1 - \cos(2x^2) = 1 - \left(1 - \frac{4x^2}{2!} + \frac{2^4 x^8}{4!} - \cdots\right)$

$$= \frac{4x^2}{2!} - \frac{2^4 x^8}{4!} + \cdots$$

- $f(x) = \frac{1-\cos(2x^2)}{2} = \frac{4x^2}{2 \cdot 2!} - \frac{2^4 x^8}{2 \cdot 4!} + \cdots$, where $\frac{2^4 x^8}{2 \cdot 4!}$ gives $= \frac{f^{(8)}(0)}{8!}$

By definition of Maclaurin series, $f^{8}(0)$ shows up in the coefficient of x^8: $\frac{f^{(8)}(0)}{8!} = \frac{2^4}{2 \cdot 4!} \Rightarrow \boxed{f^{(8)}(0) = \frac{2^3 \cdot 8!}{4!}}$

3. Find the value of the following limit:

$$\lim_{x\to\infty} x^2(e^{\frac{-1}{x^2}} - 1)$$

$\bullet\ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-1/x^2} = \sum_{n=0}^{\infty} \frac{(-1/x^2)^n}{n!} = 1 - \frac{1}{x^2} + \frac{1}{x^4 \cdot 2!} - \frac{1}{x^6 \cdot 3!} + \cdots$

$\bullet\ e^{-1/x^2} - 1 = -\frac{1}{x^2} + \frac{1}{2\cdot x^4} - \frac{1}{6\cdot x^6} + \cdots$

$\bullet\ x^2(e^{-1/x^2} - 1) = -1 + \frac{1}{2x^2} - \frac{1}{6x^4} + \cdots$

$\bullet\ \lim_{x\to\infty} x^2(e^{-1/x^2} - 1) = \lim_{x\to\infty}\left[-1 + \cancel{\frac{1}{2x^2}}^{\,0} - \cancel{\frac{1}{6x^4}}^{\,0} + \cancel{\cdots}^{\,0}\right] = \boxed{-1}$

4. Find the value of $\sum_{n=0}^{\infty} \frac{4^n}{5^n n!}$.

Notice that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Then $\sum_{n=0}^{\infty} \frac{4^n}{5^n \cdot n!} = \sum_{n=0}^{\infty} \frac{(4/5)^n}{n!} = \boxed{e^{4/5}}$

5. Approximate $f(x) = \sqrt{x}$ by a Taylor polynomial of degree 2 at a=9. Then use Taylor's Inequality to estimate the accuracy of the approximation when x lies in $[9, 10]$. You don't need to simplify your answer.

$f(x) = \sqrt{x} \longrightarrow f(9) = 3$

$f'(x) = \frac{1}{2} x^{-1/2} \longrightarrow f'(9) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$

$f''(x) = \frac{1}{2} \cdot \left(-\frac{1}{2}\right) x^{-3/2} \longrightarrow f''(9) = \frac{-1}{4} \cdot \frac{1}{3^3} = \frac{-1}{108}$

$\boxed{T_2(x)} = f(9) + f'(9)(x-9) + \frac{f''(9)}{2!}(x-9)^2 =$

$\boxed{= 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2}$

$\boxed{|R_2(x)| \leq} \frac{|f^{(3)}(z)|}{3!} \underbrace{|x-9|^3}_{\text{max at } x=10} \leq \frac{3/8 \cdot 9^{-5/2}}{3!} \cdot (10-9)^3 = \frac{\frac{3}{8} \cdot \frac{1}{3^5}}{3!} = \boxed{\frac{1}{8 \cdot 3^4 \cdot 3!}}$

$f^{(3)}(z) = \frac{3}{8} z^{-5/2} \rightsquigarrow$ max at $\underbrace{z = 9}$.

6. Give a series representation for $\int_0^1 x \arctan x \, dx$. Then write down enough terms to approximate $\int_0^1 x \arctan x \, dx$ to within $\frac{1}{100}$.

- $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \Rightarrow x \cdot \arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2n+1}$

- $\int_0^1 x\arctan(x)\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+1)(2n+3)} \Bigg|_0^1 = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+3)}}$

It's an alternating series, so we can use the alternating series estimation theorem: $|R_n| \leq b_{n+1} < \frac{1}{100}$

$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+3)} = \underbrace{\frac{1}{3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \frac{1}{7 \cdot 9} + \frac{1}{9 \cdot 11}}_{\text{we only need these terms}} - \underbrace{\frac{1}{11 \cdot 13}}_{\leq \frac{1}{100}} \rightarrow b_5 \leq \frac{1}{100}$

$\int_0^1 x \arctan x \, dx \approx S_4 = \sum_{n=0}^{4} \frac{(-1)^n}{(2n+1)(2n+3)} = \frac{1}{3} - \frac{1}{15} + \frac{1}{35} - \frac{1}{63} + \frac{1}{99}$

5 terms! (but n=4)