Thursday, February 2 $**$ *Functions of several variables; Limits.*

1. For each of the following functions $f\colon \mathbb{R}^2 \to \mathbb{R}$, draw a sketch of the graph together with pictures of some level sets.

   (a) $f(x,y) = xy$

   (b) $f(\mathbf{x}) = |\mathbf{x}|$. Please note here that $\mathbf{x}$ is a vector. In coordinates, this function is $f(x,y) = \sqrt{x^2+y^2}$.

   For (a), the result is one of the many quadric surfaces. What is the name for this type? Is the graph in (b) also a quadric surface?

2. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

   $$f(x,y) = \frac{2x^3y}{x^6+y^2} \quad \text{for } (x,y) \neq \mathbf{0}$$

   In this problem, you'll consider $\lim_{(x,y)\to\mathbf{0}} f(x,y)$.

   (a) Look at the values of f on the x- and y-axes. What do these values show the limit $\lim_{(x,y)\to\mathbf{0}} f(x,y)$ must be **if it exists**?

   (b) Show that along each line in $\mathbb{R}^2$ through the origin, the limit of f exists and is 0.

   (c) Despite this, show that the limit $\lim_{(x,y)\to\mathbf{0}} f(x,y)$ does not exist by finding a curve over which f takes on the constant value 1.

3. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

   $$f(x,y) = \frac{xy^2}{\sqrt{x^2+y^2}} \quad \text{for } (x,y) \neq \mathbf{0}$$

   In this problem, you'll show $\lim_{\mathbf{h}\to\mathbf{0}} f(\mathbf{h}) = 0$.

   (a) For $\epsilon = 1/2$, find some $\delta > 0$ so that when $0 < |\mathbf{h}| < \delta$ we have $|f(\mathbf{h})| < \epsilon$. Hint: As with the example in class, the key is to relate $|x|$ and $|y|$ with $|\mathbf{h}|$.

   (b) Repeat with $\epsilon = 1/10$.

   (c) Now show that $\lim_{\mathbf{h}\to\mathbf{0}} f(\mathbf{h}) = 0$. That is, given an arbitrary $\epsilon > 0$, find a $\delta > 0$ so that that when $0 < |\mathbf{h}| < \delta$ we have $|f(\mathbf{h})| < \epsilon$.

   (d) As the book mentions, the limit laws that appear at the start of section 2.3 also apply for limits of more than one variable, e.g., the sum of limits is the limit of sums. Explain why these laws aren't enough to compute this particular limit.