Group: ____________ ____________ ____________ Name: solutions

Math 231 A. Fall, 2015. Worksheet 13. 10/27/15

Determine if the following series converge absolutely, converge conditionally, or diverge. Give complete justification, and state which test or tests you are using.

1. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}\ln n}{\sqrt{n}}$.

$b_n = \dfrac{\ln(n)}{\sqrt{n}}$ is positive and decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so this series is convergent by the alternating series test.

But: $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n)}{\sqrt{n}} > \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is divergent.

so this series is <u>conditionally convergent.</u>

2. $\displaystyle\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 5\cdot 8\cdots(3n-1)}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{1\cdot 3\cdot 5\cdots(2n-1)(2(n+1)-1)}{\underbrace{2\cdot 5\cdot 8\cdots(3n-1)}_{\text{cancel.}}(3(n+1)-1)}\cdot\frac{2\cdot 5\cdot 8\cdots(3n-1)}{\underbrace{1\cdot 3\cdot 5\cdots(2n-1)}_{\text{cancel.}}}\right) = \lim_{n\to\infty}\frac{2n+1}{3n+2} = \frac{2}{3}$$

$\frac{2}{3} < 1$ so this series is <u>absolutely convergent.</u>

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$.

$\dfrac{(n+1)!}{n!} = \dfrac{1\cdot 2\cdot 3\cdot\ldots\cdot n\cdot(n+1)}{1\cdot 2\cdot 3\cdot\ldots\cdot n} = n+1$, and $\dfrac{e^{n^2}}{e^{(n+1)^2}} = \dfrac{e^{n^2}}{e^{n^2+2n+1}} = \dfrac{e^{n^2}}{e^{n^2}\cdot e^{2n+1}} = \dfrac{1}{e^{2n+1}}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{(n+1)!}{e^{(n+1)^2}}\cdot\frac{e^{n^2}}{(n)!}\right) = \lim_{n\to\infty}\left(\frac{n+1}{e^{2n+1}}\right) \overset{LR}{=} \lim_{n\to\infty}\left(\frac{1}{2e^{2n+1}}\right) = 0$$

$0 < 1$ so this series is <u>absolutely convergent</u>.

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4. You are given that $\sum c_n(-3)^n$ converges, and that $\sum c_n 5^n$ diverges.

a) What are the possible values of the radius of convergence of the power series $\sum c_n x^n$?

$3 \leq R \leq 5$

What can you say about the convergence/divergence of the following series?

b) $\sum c_n(-6)^n$ — divergent

c) $\sum c_n 2^n$ — convergent

d) $\sum c_n 4^n$ — cannot determine from given information

e) $\sum c_n(-5)^n$ — cannot determine from given info.

Use the ratio test to determine the radius of convergence. Then determine the interval of convergence.

5. $\sum_{n=1}^{\infty} \frac{n^2 x^{2n}}{(2n)!}$. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(n+1)^2 x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{n^2 \cdot x^{2n}} \right|$

$= \lim_{n\to\infty} \left(\frac{(n+1)^2 \cdot x^2}{n^2(2n+2)(2n+1)} \right) = x^2 \lim_{n\to\infty} \left(\frac{n^2 + 2n + 1}{4n^4 + 6n^3 + 2n^2} \right) = 0$, regardless of what x is.

so the radius of convergence is infinite and the interval of convergence is $(-\infty, \infty)$

6. $\sum_{n=1}^{\infty} (-1)^n \frac{(x-3)^n}{n \cdot 5^n}$. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-3)^{n+1}}{(n+1)\cdot 5^{n+1}} \cdot \frac{n \cdot 5^n}{(-1)^n (x-3)^n} \right|$

When is $\frac{|x-3|}{5} < 1$?

$= \frac{|x-3|}{5} \lim_{n\to\infty} \left(\frac{n}{n+1}\right) = \frac{|x-3|}{5}$. when $|x-3| < 5$ AKA "the distance from x to 3 is 5".

when $x = 8$, $\sum_{n=1}^{\infty} \frac{(-1)^n (8-3)^n}{n\, 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent.

when $x = -2$, $\sum_{n=1}^{\infty} \frac{(-1)^n (-2-3)^n}{n\, 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

So the radius of convergence is 5 and the interval of convergence is: $(-2, 8]$.

7. $\sum_{n=1}^{\infty} (-1)^n \frac{(x-3)^{2n}}{n \cdot 5^n}$.

converges if:

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-3)^{2n+2}}{(n+1)(5^{n+1})} \cdot \frac{n \cdot 5^n}{(-1)^n (x-3)^{2n}} \right| = \frac{|x-3|^2}{5} \lim_{n\to\infty} \left|\frac{n}{n+1}\right| = \frac{|x-3|^2}{5} < 1$,

so $|x-3| < \sqrt{5}$.

when $x = 3+\sqrt{5}$ the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n 5^n}{n\, 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, converges.

when $x = 3-\sqrt{5}$ the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n 5^n}{n\, 5^n}$ converges.

So the radius of convergence is $\sqrt{5}$ and the interval of convergence is: $[3-\sqrt{5}, 3+\sqrt{5}]$

8. $\sum_{n=1}^{\infty} \frac{x^n}{e^{n^2}}$ (the root test is also a good option for this one).

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{e^{n^2+2n+1}} \cdot \frac{e^{n^2}}{x^n} \right| = |x| \lim_{n\to\infty} \left(\frac{1}{e^{2n+1}}\right) = 0$, so (like in #5) $R = \infty$ and $I = (-\infty, \infty)$.

Similarly:

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{|x|}{(e^{n^2})^{1/n}} = |x| \lim_{n\to\infty} \frac{1}{e^n} = 0$. Same conclusion.