University of Illinois at Urbana-Champaign
Dept. of Electrical and Computer Engineering

ECE 120: Introduction to Computing

Fixed- and Floating-Point Representations

In Binary, We Have A Binary Point

Let's talk about representations.

In decimal, we have a **decimal point**.

tenths
hundredths
3.1415...

In binary, we have a **binary point**.

2^{-1}'s place
2^{-2}'s place
11.001001...
2^{-3}'s place

Fixed-Point Representations Support Fractions

If we need fractions,
- we can use a **fixed-point representation**
- in which some number of bits
- come after the binary point.

For example, with 32 bits:

integer part (16 bits)	fractional part (16 bits)

Some signal processing and embedded processors use fixed-point representations.

What about Real Numbers?

A question for you:

Do we need anything else to support real numbers?

Note: Saying "yes" on the basis that there are uncountably many* real numbers is not a good answer. Integers are also infinite, and 2's complement is sufficient for practical use.

* An infinite number for each integer.

Isn't Fixed-Point Good Enough?

Let's do a calculation.
32-bit 2's complement has what range?
That's right: **[-2,147,483,648, 2,147,483,647]**.
You DID all know that, right?
I didn't. I usually write **[-2^{31}, $2^{31} - 1$]**.
Let's write banking software to count pennies.
2,147,483,647 pennies is **$21,474,836.47**.
Anyone here have more? If not, we're done.
If so, use **64-bit**. You don't have that much!

Anyone Here Taking Chemistry?

But maybe you want to do your Chemistry homework?

You may need **Avogadro's number**.

Anyone remember it? **6.022×10^{23} / mol**

Sure. No problem.

10^3 is around 2^{10}, so 80 bits should work.

Who can tell me Avogadro's number to 80 bits (the first 24 decimal digits will do)?

Wikipedia May Not Help as Much as You Think!

Last I checked (July 2016!), the best known experimental value was

$6.022140858 \times 10^{23}$ / mol

That's only 10 digits.

So you have some serious Chemistry research to get done for your next homework!

Good luck!

Maybe we can just be close?

What about Physics?

Some have Quantum Mechanics homework?

Your computer will need **Planck's constant**.

What is it again? **6.626×10^{-27} erg-sec***

Ok. Another 90 bits after the binary point.

170 bits total.

Don't forget to find another 90 bits (27 more decimal digits) for Avogadro.

*Use ergs, not Joules; we'll need fewer bits!

We Need More Dynamic Range, Not More Precision

Do we really need **170 bits** of precision?

Do we really need to specify the first **51 significant figures** for Avogadro's number?

Of course not!

But we do need 170 bits of range.

We need to be able to express both tiny numbers and huge numbers.

Develop a Representation Based on Scientific Notation

Let's borrow another representation from humans: scientific notation.

sign exponent

$$+\underbrace{6.022}_{\text{mantissa/significant figures (precision)}} \times 10^{23}$$

The human representation has three parts.

Modern Computers Use Standard Floating-Point

Modern digital systems implement the IEEE 754 standard for floating-point.

A single-precision floating-point number consists of 32 bits:

sign (1 bit)	exponent (8 bits)	mantissa (23 bits)

What value does a bit pattern represent?

First, let me ask you a question…

What Values Can a Leading Digit Take?

A question for you:

In the canonical form of scientific notation, what are the possible values of the leading digit?

This one

$$-4.123 \times 10^{45}$$

Any digit? 0-4? 1-7?

1-9 (not 0). Change exponent as needed.

What Values Can a Leading Digit Take?

Another question for you:

Same question, but now in binary.

1 (not 0). Change exponent as needed.

And one more:

How many bits do we need to store one possible answer?

The leading 1 is implicit in binary (0 bits)!

How to Calculate the Value of a Floating-Point Bit Pattern

The value represented by an IEEE single-precision floating-point bit pattern is…

sign (1 bit)	exponent (8 bits)	mantissa (23 bits)

$$(-1)^{\mathbf{sign}}\ 1.\underline{\mathbf{mantissa}} \times 2^{(\mathbf{exponent} - 127)}$$

Convert the exponent to decimal as if it were unsigned before subtracting 127.

Except that Exponents 0 and 255 have Special Meanings

That's almost correct. But **exponents 0 and 255 have special meanings**:

- 255 can mean **infinity** or **not-a-number** (**NaN**).
- 0 is a **denormalized** number: the leading implicit "1" is replaced with "0" (with power 2^{-126}), allowing the representation to capture numbers closer to 0.

Except for the fact that **the bit pattern of all 0s means 0**, these aspects are beyond the scope of our class.

* *

Exponent 255 is Used for Infinity and NaN

Exponent 255

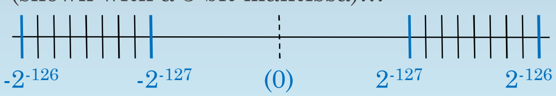
- Mantissa 0
  - Sign 0: **Positive infinity**
  - Sign 1: **Negative infinity**
- Non-zero mantissa: **NaN** (Not a Number)

These special values allow the representation to have 'correct' answers to some problems (such as 42.0 / 0.0) and to silently track the impact of missing values and incorrect computation (such as Infinity * 0).

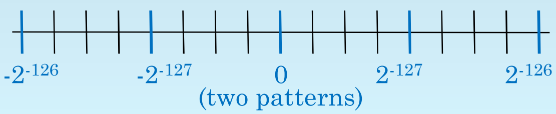
* *

Denormalization Allows Numbers Closer to 0 (and 0)

Without denormalized numbers, we have (shown with a 3-bit mantissa)…



Denormalization puts these patterns closer to 0 and gives two patterns for 0:



Converting to a Floating-Point Bit Pattern

Conversion from decimal to IEEE floating-point is not too hard:

1. Convert to binary.
2. Change to scientific notation (in binary).
3. Encode each of the three parts.

Use a Polynomial to Convert a Fraction to Binary

To convert a fraction **F** to binary, remember that **a fraction also corresponds to a polynomial**:

$$\mathbf{F = a_{-1}2^{-1} + a_{-2}2^{-2} + a_{-3}2^{-3} + a_{-4}2^{-4} + \ldots}$$

If we multiply both sides by 2

- the left side can only be ≥ 1
- if $\mathbf{a_{-1} = 1}$

We can then subtract $\mathbf{a_{-1}}$ from both sides

and repeat to get $\mathbf{a_{-2}}$, $\mathbf{a_{-3}}$, $\mathbf{a_{-4}}$, and so forth.

Example of Finding a Floating-Point Bit Pattern

For example, let's say that we want to find the bit pattern for **5.046875**.

We first write **5** in binary: **101**.

Now we need to convert the fraction

$$\mathbf{F = 0.046875}.$$

$0.046875 \times 2 = 0.09375$ (< 1, so $\mathbf{a_{-1} = 0}$)

$0.09375 - 0 = 0.09375$

Example of Finding a Floating-Point Bit Pattern

Start with 0.09375.

$0.09375 \times 2 = 0.1875$ (< 1, so $\mathbf{a_{-2} = 0}$)

$0.1875 - 0 = 0.1875$

$0.1875 \times 2 = 0.375$ (< 1, so $\mathbf{a_{-3} = 0}$)

$0.375 - 0 = 0.375$

$0.375 \times 2 = 0.75$ (< 1, so $\mathbf{a_{-4} = 0}$)

$0.75 - 0 = 0.75$

Example of Finding a Floating-Point Bit Pattern

Start with 0.75.

$0.75 \times 2 = 1.5$ (so $\mathbf{a_{-5} = 1}$)

$1.5 - 1 = 0.5$

$0.5 \times 2 = 1$ (so $\mathbf{a_{-6} = 1}$)

$1 - 1 = 0$ (done)

Putting the bits together, we find

$$\mathbf{F = 0.046875_{10} = 0.000011_2}$$

Example of Finding a Floating-Point Bit Pattern

Now we have converted to binary:

$$\mathbf{5.046875_{10} = 101.000011_2}$$

In binary scientific notation, we have

$$\mathbf{+\ 1.01000011 \times 2^2}$$

And, in single-precision floating point,
- the sign bit is **0**,
- the exponent is 2+127 = 129 = **10000001**,
- and the mantissa is **01000011…** (no leading 1, and 15 more 0s afterward).

Tricky Questions about Floating-Point

A question for you:

What is $2^{-30} + (1 - 1)$?

Quite tricky, I know. But yes, it's $\mathbf{2^{-30}}$.

Another question for you:

What is $(2^{-30} + 1) - 1$?

That's right. It's **0**.

At least it is with floating-point.

Floating-Point is Not Associative

Why?

Our first sum was $(2^{-30} + 1)$.

To hold the integer 1, the bit pattern's exponent must be 2^0.

But, the mantissa for single-precision floating point has only **23 bits**.

And thus represents powers down to 2^{-23}.

The 2^{-30} term is lost, giving $(2^{-30} + 1) = 1$.

So $2^{-30} + (1 - 1) \neq (2^{-30} + 1) - 1$.