Group: ______________ ______________ ______________ Name: Key

You will need Taylor's Theorem, which says that $f(x) = T_N(x) + R_N(x)$, where $T_N(x)$ is the degree N Taylor polynomial for f at a, and the remainder $R_N(x)$ equals

$$R_N(x) = \frac{f^{(N+1)}(z)}{(N+1)!}(x-a)^{N+1}$$

for some number z between a and x.

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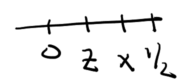
1. a) Write down the Taylor polynomial of degree 5 for e^x at 0 (should take no work).

$$T_5(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

b) Use Taylor's Theorem to find the maximum error in the approximation $e^x \approx T_5(x)$ on the interval $[0, 1/2]$.

$$f^{(N+1)}(z) = e^z$$

So $R_5(x) = \frac{e^z}{6!} \cdot x^6$ for some z between 0 and x



$$|R_5(x)| \leq \frac{e^{1/2}}{6!} \cdot (1/2)^6 < .00036$$

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2. a) Find the Taylor polynomial of degree 2 for $f(x) = x^{\frac{3}{2}}$ at 4

$f(x) = x^{3/2}$ $\quad f(4) = 8$

$f'(x) = \frac{3}{2}x^{1/2}$ $\quad f'(4) = 3$

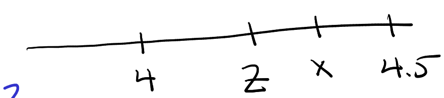
$f''(x) = \frac{3}{4}x^{-1/2}$ $\quad f''(4) = \frac{3}{8}$

$f'''(x) = -\frac{3}{8}x^{-3/2}$

$$T_2(x) = 8 + 3(x-4) + \frac{3}{16}(x-4)^2$$

b) Use Taylor's Theorem to find the maximum error in the approximation $f(x) \approx T_2(x)$ on the interval $[4, 4.5]$.

$R_2(x) = \frac{f'''(z)}{3!}(x-4)^3$ for some z between 4 and x



$$|f'''(z)| = \left|\frac{3}{8z^{3/2}}\right| \leq \frac{3}{8 \cdot 4^{3/2}} = \frac{3}{64}$$

So $|R_2(x)| \leq \frac{\left(\frac{3}{64}\right) \cdot \frac{1}{8}}{3!} < .001$

$$|x-4|^3 \leq \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

3. Show that the degree 1 Taylor polynomial for $(1+x)^K$ is $T_1(x) = 1 + Kx$ (you can just write down the first two terms of the binomial series).

So we have $(1+x)^K \approx 1 + Kx$ if x is small.

$$(1+x)^K = 1 + \binom{K}{1}x + \binom{K}{2}x^2 + \cdots\cdots$$

$$\text{So } T_1(x) = 1 + \binom{K}{1}x = 1 + Kx$$

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4. Pendulums have been used for centuries to keep time. Pendulums exhibit the property of *isochronism* when they swing through small angles. This means that the period of each swing (i.e. the amount of time which each swing takes) does not change very much as the angle of the swing changes. For small angles, the period is given by the formula

$$T \approx 2\pi\sqrt{\frac{L}{g}} \tag{1}$$

where L is the length of the pendulum and g is the gravitational constant.

However, the period of the pendulum *must* depend somehow on the size of the angle through which it swings (to convince yourself of this, imagine a pendulum which is thousands of feet long swinging through small angles, and then swinging through large angles).

Suppose that the maximum angle which the pendulum makes with vertical is $\theta_{\max}$, and set $k = \sin\left(\frac{1}{2}\theta_{\max}\right)$. Then the precise formula for the period of a pendulum is

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1-k^2\sin^2 x}}. \tag{2}$$

We will use series to reconcile equations (1) and (2).

a) Use Problem 3 to find the first two terms of the expansion of $\dfrac{1}{\sqrt{1-k^2\sin^2 x}}$. (The answer involves k and $\sin x$.)

$$(1 - k^2\sin^2 x)^{-\frac{1}{2}} \approx 1 + \frac{1}{2}k^2\sin^2 x$$

b) Use part (a) and (2) to find the first two terms of the expansion of T in terms of k (the answer involves k only). Use the fact that $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x\, dx = \frac{\pi}{4}$.

$$T \approx 4\sqrt{\frac{L}{g}}\int_0^{\frac{\pi}{2}} \left(1 + \frac{1}{2}k^2\sin^2 x\right)dx$$

$$\approx 4\sqrt{\frac{L}{g}}\left(\frac{\pi}{2} + \frac{1}{2}k^2 \cdot \frac{\pi}{4}\right)$$

c) Explain very briefly why the approximation (1) is accurate when k is small.

When k is small, $T \approx 4\sqrt{\frac{L}{g}} \cdot \frac{\pi}{2} = 2\pi\sqrt{\frac{L}{g}}$