Math 231 Midterm 2, March 14, 2017

Full Name: ______________________________ **Section Code from table below:** | | | |

Net ID: ______________________________

For graders only	Score
Problem 11 (18 points)	
Problem 12 (12 points)	
Problem 13 (14 points)	
Total (44 points)	

TA	Section	Time and Location	TA	Section	Time and Location
Akbari, Sahand	EDD EDE	WF 11:00-11:50 -- 106B6 ENGR HALL WF 12:00-12:50 -- 1 ILL HALL	Karve, Vaibhav	AD1	WF 9:00-10:50 -- 159 ALTGELD
Caulfield, Erin	AD2	WF 1:00-2:50 -- 159 ALTGELD	Kim, Hee Yeon	EDF EDG	WF 1:00-1:50 -- 241 ALTGELD WF 2:00-2:50 -- 345 ALTGELD
Chavoshi, Amir	EDB ADC	WF 9:00-9:50 -- 441 ALTGELD WF 10:00-10:50 -- 347 ALTGELD	Kim, Heejoung	ADA ADB	WF 8:00-8:50 -- 141 ALTGELD WF 9:00-9:50 -- 145 ALTGELD
Chung, Jooyeon	BD2	WF 11:00-12:50 -- 173 ALTGELD	Linz, William	DDA DDB	WF 8:00-8:50 -- 441 ALTGELD WF 9:00-9:50 -- 141 ALTGELD
Duffy, Michael	DDE DDF	WF 12:00-12:50 -- 441 ALTGELD WF 1:00-1:50 -- 145 ALTGELD	Loving, Marissa	CDE	WF 12:00-12:50 -- 140 HENRY BLD
Ellis, Matthew	DDG DDH	WF 2:00-2:50 -- 441 ALTGELD WF 3:00-3:50 -- 441 ALTGELD	Luo, Hao	CDC CDD	WF 10:00-10:50 -- 145 ALTGELD WF 11:00-11:50 -- 143 HENRY BLD
Gramcko-Tursi, Mary Angelica	CDG CDH	WF 2:00-2:50 -- 143 ALTGELD WF 3:00-3:50 -- 145 ALTGELD	Mousley, Sarah	ADD ADE	WF 11:00-11:50 -- 447 ALTGELD ADE -- WF 12:00-12:50 -- 341 ALTGELD
Han, Xiaolong (Hans)	ADF ADG	WF 1:00-1:50 -- 141 ALTGELD WF 2:00-2:50 -- 241 ALTGELD	Ochoa de Alaiza Gracia, Itziar	BDC BDD	WF 10:00-10:50 -- 447 ALTGELD WF 11:00-11:50 -- 137 HENRY BLD
Harris, Terence	BDA BDB	WF 8:00-8:50 -- 137 HENRY BLD WF 9:00-9:50 -- 341 ALTGELD	Pratt, Kyle	DDC DDD	WF 10:00-10:50 -- 441 ALTGELD WF 11:00-11:50 -- 441 ALTGELD
Heath, Emily	BD3	WF 1:00-2:50 -- 173 ALTGELD	Tamazyan, Albert	BDE CDF	WF 12:00-12:50 -- 137 HENRY BLD WF 1:00-1:50 -- 137 HENRY BLD
Huang, Jianting (Jesse)	EDA EDC	WF 8:00-8:50 -- 143 ALTGELD WF 10:00-10:50 -- 443 ALTGELD	Wright, Benjamin	CDA CDB	WF 8:00-8:50 – 145 ALTGELD WF 9:00-9:50 -- 443 ALTGELD

- The exam is one hour long.
- You must not communicate with other students during this exam.
- No written materials of any kind allowed.
- No phones, calculators, smart-watches or electronic devices of any kind allowed for any reason, including checking the time (you may use a simple wristwatch).
- Do not turn this page until instructed to.

(1 point) Fill in top of this page correctly. Fill in name, UIN (student number), and Net ID on Scantron form. Fill in the following answers on the Scantron form:

Zone 1

Multiple Choice Questions. Mark answers on Scantron form.

1/1. (6 points) If the infinite geometric series $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \ldots$ converges, find the sum.

A. ★ 3

B. 0

C. $\frac{2}{3}$

D. 15

E. The series diverges.

Solution. $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \ldots = 5(1 - (2/3) + (2/3)^2 - (2/3)^3 + \ldots) = 5\frac{1}{1-(-2/3)} = 5 \cdot \frac{3}{5} = 3.$

1/2. (6 points) If the infinite geometric series $3 - \frac{9}{4} + \frac{27}{16} - \frac{81}{64} + \ldots$ converges, find the sum.

A. ★ $\frac{12}{7}$

B. 0

C. $\frac{3}{4}$

D. 12

E. The series diverges.

Solution. $3 - \frac{9}{4} + \frac{27}{16} - \frac{81}{64} + \ldots = 3(1 - (3/4) + (3/4)^2 - (3/4)^3 + \ldots) = 3\frac{1}{1-(-3/4)} = 3 \cdot \frac{4}{7} = \frac{12}{7}.$

1/3. (6 points) If the infinite geometric series $4 - \frac{8}{5} + \frac{16}{25} - \frac{32}{125} + \ldots$ converges, find the sum.

A. ★ $\frac{20}{7}$

B. 0

C. $\frac{2}{5}$

D. $\frac{20}{3}$

E. The series diverges.

Solution. $4 - \frac{8}{5} + \frac{16}{25} - \frac{32}{125} + \ldots = 4(1 - (2/5) + (2/5)^2 - (2/5)^3 + \ldots) = 4\frac{1}{1-(-2/5)} = 4 \cdot \frac{5}{7} = \frac{20}{7}.$

2/1. (6 points) The series $\sum_{n=1}^{\infty} a_n$ has N^{th} partial sum $S_N = 1 - \dfrac{1}{N+1}$. Find the value of a_n for $n > 1$.

A. ★ $\frac{1}{n(n+1)}$

B. 1

C. $\frac{n}{n+1}$

D. $\frac{1}{n+1}$

E. $1 + \frac{1}{n+1}$

Solution. $a_n = S_n - S_{n-1} = (1 - \frac{1}{n+1}) - (1 - \frac{1}{(n-1)+1}) = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$.

2/2. (6 points) The series $\sum_{n=1}^{\infty} a_n$ has N^{th} partial sum $S_N = 2 - \dfrac{2}{N+1}$. Find the value of a_n for $n > 1$.

A. ★ $\frac{2}{n(n+1)}$

B. 2

C. $\frac{2n}{n+1}$

D. $\frac{2}{n+1}$

E. $2 + \frac{2}{n+1}$

Solution. $a_n = S_n - S_{n-1} = (2 - \frac{2}{n+1}) - (2 - \frac{2}{(n-1)+1}) = \frac{2}{n} - \frac{2}{n+1} = \frac{2}{n(n+1)}$.

2/3. (6 points) The series $\sum_{n=1}^{\infty} a_n$ has N^{th} partial sum $S_N = 3 - \dfrac{2}{N+1}$. Find the value of a_n for $n > 1$.

A. ★ $\frac{2}{n(n+1)}$

B. 3

C. $\frac{2n}{n+1}$

D. $\frac{2}{n+1}$

E. $3 + \frac{2}{n+1}$

Solution. $a_n = S_n - S_{n-1} = (3 - \frac{2}{n+1}) - (3 - \frac{2}{(n-1)+1}) = \frac{2}{n} - \frac{2}{n+1} = \frac{2}{n(n+1)}$.

3/1. (5 points) The series $\sum_{n=1}^{\infty} a_n$ has N^{th} partial sum $S_N = 2 + \frac{(-1)^N}{N^4}$. Find $\sum_{n=1}^{\infty} a_n$.

A. ★ 2

B. 0

C. Diverges

D. 1

E. $(-1)^n \frac{2n-1}{n(n-1)}$

Solution. $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = 2.$

3/2. (5 points) The series $\sum_{n=1}^{\infty} a_n$ has N^{th} partial sum $S_N = 3 + \frac{(-1)^N}{N^2}$. Find $\sum_{n=1}^{\infty} a_n$.

A. ★ 3

B. 0

C. Diverges

D. 1

E. $(-1)^n \frac{2n^2-2n+1}{n^2(n-1)^2}$

Solution. $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = 3.$

3/3. (5 points) The series $\sum_{n=1}^{\infty} a_n$ has N^{th} partial sum $S_N = 4 + \frac{(-1)^N}{N^3}$. Find $\sum_{n=1}^{\infty} a_n$.

A. ★ 4

B. 0

C. Diverges

D. 1

E. $(-1)^n \frac{2n^3-3n^2+3n-1}{n^3(n-1)^3}$

Solution. $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = 4.$

Zone 2

4/1. (6 points) A triangular lamina with vertices at $(0,0),(2,0),(2,1)$ has area-density 2 kg/m^2. Find the x–coordinate of the center of mass, $\overline{x}$.

A. ★ 4/3

B. 2/3

C. 6/3

D. 8/3

E. 10/3

Solution. The vertical strip at position x has height $x/2$ and width dx. So its mass is $\int_0^2 2 \cdot x/2\,dx = 2$, and

$$M_y = \int_0^2 2x \cdot x/2\,dx = 8/3, \text{ hence } \overline{x} = \frac{8/3}{2} = 4/3$$

Alternatively, the area of the triangle is $A = 1$, and $\bar{x}$ is computed by the formula

$$\bar{x} = \frac{1}{A}\int_0^2 x\frac{x}{2}\,dx = \frac{x^3}{6}\Big|_0^2 = \frac{4}{3}.$$

4/2. (6 points) A triangular lamina with vertices at $(0,0),(3,0),(3,1)$ has area-density 3 kg/m^2. Find the x–coordinate of the center of mass, $\overline{x}$.

A. ★ 2

B. 1

C. 4/3

D. 5/3

E. 7/3

Solution. The vertical strip at position x has height $x/3$ and width dx. So its mass is $\int_0^3 3 \cdot x/3\,dx = 9/2$, and

$$M_y = \int_0^3 3x \cdot x/3\,dx = 9, \text{ hence } \overline{x} = \frac{9}{9/2} = 2$$

Alternatively, the area of the triangle is $A = \frac{3}{2}$, and $\bar{x}$ is computed by the formula

$$\bar{x} = \frac{1}{A}\int_0^3 x\frac{x}{3}\,dx = \frac{2}{3}\frac{x^3}{9}\Big|_0^3 = 2.$$

4/3. (6 points) A triangular lamina with vertices at $(0,0),(4,0),(4,1)$ has area-density 4 kg/m^2. Find the x–coordinate of the center of mass, $\overline{x}$.

A. ★ 8/3

B. 2/3

C. 6/3

D. 4/3

E. 10/3

Solution. The vertical strip at position x has height $x/4$ and width dx. So its mass is $\int_0^4 4 \cdot x/4\, dx = 8$, and

$$M_y = \int_0^4 4x \cdot x/4\, dx = 64/3, \text{ hence } \overline{x} = \frac{64/3}{8} = 8/3$$

Alternatively, the area of the triangle is $A = 2$, and $\bar{x}$ is computed by the formula

$$\bar{x} = \frac{1}{A} \int_0^4 x \frac{x}{4}\, dx = \frac{1}{2} \frac{x^3}{12}\Big|_0^4 = \frac{8}{3}.$$

Zone 3

5/1. (6 points) Use the Trapezoid rule to estimate $\int_1^5 x^2 dx$, partitioning the interval of integration into $n = 4$ subintervals of equal width $\Delta x = \frac{5-1}{4} = 1$.

A. ★ 42

B. 32

C. 22

D. 60

E. 70

Solution.

$$\frac{1}{2}(1 + 2 \cdot 4 + 2 \cdot 9 + 2 \cdot 16 + 25) = 42$$

5/2. (6 points) Use the Trapezoid rule to estimate $\int_0^4 x^2 dx$, partitioning the interval of integration into $n = 4$ subintervals of equal width $\Delta x = \frac{4-0}{4} = 1$.

A. ★ 22

B. 32

C. 42

D. 60

E. 70

Solution.

$$\frac{1}{2}(0 + 2 \cdot 1 + 2 \cdot 4 + 2 \cdot 9 + 16) = 22$$

5/3. (6 points) Use the Trapezoid rule to estimate $\int_2^6 x^2 dx$, partitioning the interval of integration into $n = 4$ subintervals of equal width $\Delta x = \frac{6-2}{4} = 1$.

A. ★ 70

B. 32

C. 42

D. 60

E. 22

Solution.

$$\frac{1}{2}(4 + 2 \cdot 9 + 2 \cdot 16 + 2 \cdot 25 + 36) = 70$$

Zone 4

6/1. (5 points) If $a_n = \dfrac{3^n}{3+n!}$, then the **sequence** $\{a_n\}$

A. ★ Converges to 0.

B. Converges to 1.

C. Diverges.

Solution. We have

$$\frac{3^n}{3+n!} \le \frac{3^n}{n!} = \frac{3\cdot 3\cdot 3\cdot 3\cdot 3\cdots 3}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots n} = \frac{3\cdot 3\cdot 3}{1\cdot 2\cdot 3} \times \frac{3\cdot 3\cdot 3\cdots 3}{4\cdot 5\cdot 6\cdots n}.$$

For large n this is $< \dfrac{3}{n}$. Since $0 < a_n < \dfrac{3}{n}$, we have $\lim a_n = 0$ by the Squeeze Theorem.

6/2. (5 points) If $a_n = \dfrac{5^n}{5+n!}$, then the **sequence** $\{a_n\}$

A. ★ Converges to 0.

B. Converges to 1.

C. Diverges.

Solution. We have

$$\frac{5^n}{5+n!} \le \frac{5^n}{n!} = \frac{5\cdot 5\cdot 5\cdot 5\cdot 5\cdot 5\cdots 5}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdots n} = \frac{5\cdot 5\cdot 5\cdot 5\cdot 5}{1\cdot 2\cdot 3\cdot 4\cdot 5} \times \frac{5\cdot 5\cdot 5\cdots 5}{6\cdot 7\cdot 8\cdots n}.$$

For large n this is $< \dfrac{5}{n}$. Since $0 < a_n < \dfrac{5}{n}$, we have $\lim a_n = 0$ by the Squeeze Theorem.

6/3. (5 points) If $a_n = \dfrac{7^n}{7+n!}$, then the **sequence** $\{a_n\}$

A. ★ Converges to 0.

B. Converges to 1.

C. Diverges.

Solution. We have

$$\frac{7^n}{7+n!} \le \frac{7^n}{n!} = \frac{7\cdot 7\cdot 7\cdot 7\cdot 7\cdot 7\cdots 7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdots n} = \frac{7\cdot 7\cdot 7\cdot 7\cdot 7\cdot 7\cdot 7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} \times \frac{7\cdot 7\cdot 7\cdots 7}{8\cdot 9\cdot 10\cdots n}.$$

For large n this is $< \dfrac{7}{n}$. Since $0 < a_n < \dfrac{7}{n}$, we have $\lim a_n = 0$ by the Squeeze Theorem.

7/1. (5 points) If $a_n = 1 - \dfrac{\ln 2n}{n}$ then the sequence $\{a_n\}_{n=2}^{\infty}$

A. ★ Converges to 1

B. Converges to $1 - \ln 2$

C. Converges to 0

D. Converges to $1 - 2e$

E. Diverges

Solution. **Method 1**: By L'Hospital's rule, $\lim\limits_{x\to\infty} \dfrac{\ln 2x}{x} = \lim\limits_{x\to\infty} \dfrac{\frac{2}{2x}}{1} = 0$, and so $\lim\limits_{x\to\infty} \left(1 - \dfrac{\ln 2x}{x}\right) = \lim\limits_{x\to\infty} 1 - \lim\limits_{x\to\infty} \dfrac{\ln 2x}{x} = 1 - 0 = 1$. Therefore, we also must have $\lim\limits_{n\to\infty} 1 - \dfrac{\ln 2n}{n} = 1 - 0 = 1$.

Method 2: Since the function $g(x) = \dfrac{\ln 2x}{x}$ is decreasing for sufficiently large x (namely $g'(x) = \frac{1}{2} - \ln 2x$ is < 0 for sufficiently large x), then the function $f(x) = 1 - g(x)$ is increasing for suffciently large x (and in particular, $f(x)$ is increasing for $x \geq 2$). It follows that $a_n = f(n)$ is an increasing sequence for $n \geq 2$. Furthermore, $\{a_n\}_{n\geq 2}$ is bounded, since $|a_n| < 1$. These two facts, along with the Monotone Convergence Theorem, imply that $\{a_n\}_{n\geq 2}$ is a convergent sequence.

7/2. (5 points) If $a_n = 2 - \dfrac{\ln 3n}{n}$ then the sequence $\{a_n\}_{n=2}^{\infty}$

A. ★ Converges to 2

B. Converges to $2 - \ln 3$

C. Converges to 1

D. Converges to $2 - 3e$

E. Diverges

Solution. **Method 1**: By L'Hospital's rule, $\lim\limits_{x\to\infty} \dfrac{\ln 3x}{x} = \lim\limits_{x\to\infty} \dfrac{\frac{3}{3x}}{1} = 0$, and so $\lim\limits_{x\to\infty} \left(2 - \dfrac{\ln 3x}{x}\right) = \lim\limits_{x\to\infty} 2 - \lim\limits_{x\to\infty} \dfrac{\ln 3x}{x} = 2 - 0 = 2$. Therefore, we also must have $\lim\limits_{n\to\infty} 2 - \dfrac{\ln 3n}{n} = 2 - 0 = 2$.

Method 2: Since the function $g(x) = \dfrac{\ln 3x}{x}$ is decreasing for sufficiently large x (namely $g'(x) = \frac{1}{3} - \ln 3x$ is < 0 for sufficiently large x), then the function $f(x) = 1 - g(x)$ is increasing for suffciently large x (and in particular, $f(x)$ is increasing for $x \geq 2$). It follows that $a_n = f(n)$ is an increasing sequence for $n \geq 2$. Furthermore, $\{a_n\}_{n\geq 2}$ is bounded, since $|a_n| < 2$. These two facts, along with the Monotone Convergence Theorem, imply that $\{a_n\}_{n\geq 2}$ is a convergent sequence.

Zone 5

8/1. (6 points) For which value of C does the following series converge? $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}+\frac{C}{2n+1}\right)$.

A. ★ -1

B. 1

C. 1/2

D. 2

E. It does not converge for any value of C.

Solution. We have that $\dfrac{1}{2n-1}+\dfrac{C}{2n+1}=\dfrac{(2n+1)+C(2n-1)}{(2n-1)(2n+1)}=\dfrac{(2+2C)n+(1-C)}{4n^2-1}$.
Therefore:
If $C=-1$, the limit comparison test with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ shows that our series converges.
If $C\neq -1$, the limit comparison test with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ shows that our series diverges.

8/2. (6 points) For which value of C does the following series converge? $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{3n-2}-\frac{C}{3n+2}\right)$.

A. ★ 1

B. -1

C. 3

D. 2

E. It does not converge for any value of C.

Solution. We have that $\dfrac{1}{3n-2}-\dfrac{C}{3n+2}=\dfrac{(3n+2)-C(3n-2)}{(3n-2)(3n+2)}=\dfrac{(3-3C)n+(2+2C)}{9n^2-4}$.
Therefore:
If $C=1$, the limit comparison test with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ shows that our series converges.
If $C\neq 1$, the limit comparison test with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ shows that our series diverges.

8/3. (6 points) For which value of C does the following series converge? $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{4n+3}+\frac{C}{4n-3}\right)$.

A. ★ -1

B. 1

C. 2/3

D. 3/2

E. It does not converge for any value of C.

Solution. We have that $\frac{1}{4n+3} + \frac{C}{4n-3} = \frac{(4n-3)+C(4n+3)}{(4n+3)(4n-3)} = \frac{(4+4C)n+(-3+3C)}{16n^2-9}$.
Therefore:
If $C = -1$, the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ shows that our series converges.
If $C \neq -1$, the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$ shows that our series diverges.

Zone 6

9/1. (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan(\frac{1}{n})}{n^2+1}$

A. ★ Converges

B. Diverges

Solution. Since $0 \leq \arctan(x) < \frac{\pi}{2}$ for every $x \geq 0$, we see in particular that $0 \leq \arctan(\frac{1}{n}) < \frac{\pi}{2}$ for every n. We also note that $0 \leq n^2 \leq n^2+1$ so that $0 \leq \frac{1}{n^2+1} \leq \frac{1}{n^2}$. Combining these inequalities, we see that $0 \leq \frac{\arctan(\frac{1}{n})}{n^2+1} \leq \frac{\pi}{2} \cdot \frac{1}{n^2}$ for every n. Because the series $\frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, Comparison Test implies that the series $\sum_{n=1}^{\infty} \frac{\arctan(\frac{1}{n})}{n^2+1}$ also converges.

9/2. (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^3+5}$

A. ★ Converges

B. Diverges

Solution. Since $0 < e^x \leq e$ for every $x \leq 1$, we see in particular that $0 < e^{1/n} \leq e$ for every n. We also note that $0 \leq n^3 \leq n^3+5$ so that $0 \leq \frac{1}{n^3+5} \leq \frac{1}{n^3}$. Combining these inequalities, we see that $0 \leq \frac{e^{1/n}}{n^3+5} \leq \frac{e}{n^3}$ for every n. Because the series $e \sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, Comparison Test implies that the series $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^3+5}$ also converges.

10/1. (5 points) $\sum_{n=1}^{\infty} 1 - \frac{1}{n^3}$

A. ★ Diverges

B. Converges

Solution. Since $\lim_{n\to\infty} 1 - \frac{1}{n^3} = 1 \neq 0$, the divergence test says the series diverges.

10/2. (5 points) $\sum_{n=1}^{\infty} 1 - \frac{1}{n^2}$

A. ★ Diverges

B. Converges

Solution. Since $\lim_{n\to\infty} 1 - \frac{1}{n^2} = 1 \neq 0$, the divergence test says the series diverges.

Zone 7

Free response questions. Write complete solutions and show all work for full credit.

11/1. (18 points) Consider the curve $y = 2 - \sin(x)$ between the points $(\pi, 2)$ and $(2\pi, 2)$. Set up **but do not evaluate** integrals which represent the following quantities. **All integrals should be in terms of** x.

a) The length of the curve.

b) Surface area when the curve is rotated about the x-axis.

c) Surface area when the curve is rotated about the line $x = 1$.

Solution. a) We have $\frac{dy}{dx} = -\cos(x)$. So

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \cos^2(x)}\, dx,$$

and

$$\text{Arclength} = \int_{\pi}^{2\pi} \sqrt{1 + \cos^2(x)}\, dx.$$

b) Rotating about the x–axis, surface area is $\displaystyle\int 2\pi y\, ds$, so

$$\text{Area} = \int_{\pi}^{2\pi} 2\pi(2 - \sin(x))\sqrt{1 + \cos^2(x)}\, dx$$

c) If the curve is rotated about the curve $x = 1$, then the surface area is given by $\displaystyle\int 2\pi(x-1)\, ds$,

$$\text{Area} = \int_{\pi}^{2\pi} 2\pi(x - 1)\sqrt{1 + \cos^2(x)}\, dx$$

11/2. (18 points) Consider the curve $y = 2 + \cos(x)$ between the points $(\pi, 1)$ and $(2\pi, 3)$. Set up **but do not evaluate** integrals which represent the following quantities. **All integrals should be in terms of** x.

a) The length of the curve.

b) Surface area when the curve is rotated about the x-axis.

c) Surface area when the curve is rotated about the line $x = 2$.

Solution. a) We have $\frac{dy}{dx} = -\sin(x)$. So

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \sin^2(x)}\, dx,$$

and

$$\text{Arclength} = \int_{\pi}^{2\pi} \sqrt{1 + \sin^2(x)}\, dx.$$

b) Rotating about the x–axis, surface area is $\displaystyle\int 2\pi y\, ds$, so

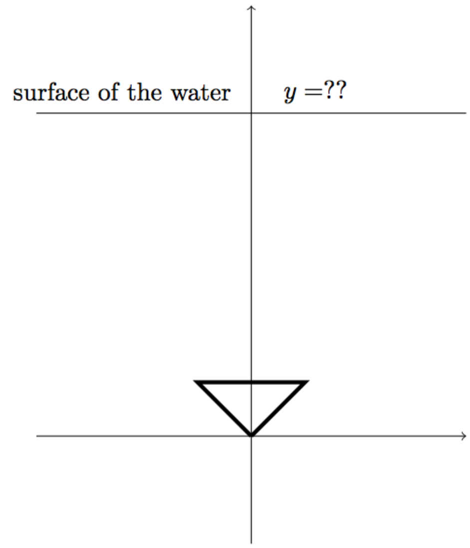
$$\text{Area} = \int_{\pi}^{2\pi} 2\pi(2 + \cos(x))\sqrt{1 + \sin^2(x)}\, dx$$

c) If the curve is rotated about the curve $x = 2$, then the surface area is given by $\int 2\pi(x-2)\,ds$,

$$\text{Area} = \int_{\pi}^{2\pi} 2\pi(x-2)\sqrt{1+\sin^2(x)}\,dx$$

Zone 8

12/1. (12 points) An aquarium has a triangular window described by $|x| \leq y \leq 1$ (in meters). The top of the window is 5 meters below the surface of the water. Recall that the acceleration of gravity is $9.8\, m/s^2$ and that water has density $1000\, kg/m^3$.



a) The surface of the water is given by $y =$ __________

b) The pressure as a function of y (in the given coordinate system)

is given by Pressure = __________ N/m^3.

c) Set up **but do not evaluate** an integral which represents the hydrostatic force on the window.

Solution.

a) The top of the window is at $y = 1$, so 5 meters above that is $y = 6$.

b) We have Pressure = (density)·(gravity)·(depth). Since the depth is given by $6 - y$ (distance to the surface of the water), the pressure is

$$\text{Pressure } = 9800(6 - y)\, N/m^3.$$

c) At height y, a strip of the window with thickness Δy has area approximately $2y\Delta y$. So, the contribution at height y to the hydrostatic force is $9800(6 - y)2y\, dy = 19600(6 - y)y\, dy$, and hence

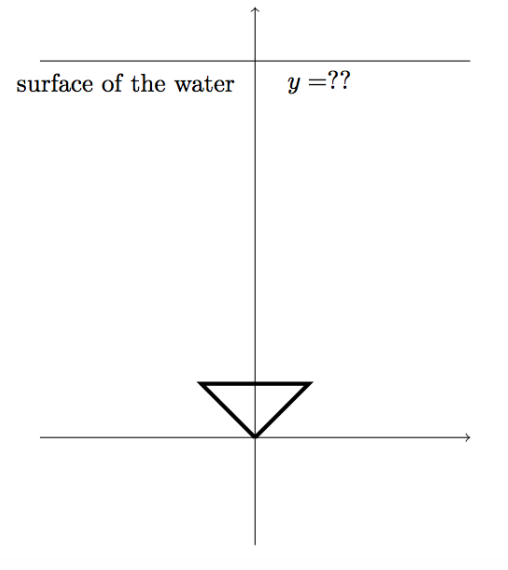
$$\text{Force} = 19600 \int_0^1 (6 - y)y\, dy.$$

12/2. (12 points) An aquarium has a triangular window described by $|x| \leq y \leq 1$ (in meters). The top of the window is 6 meters below the surface of the water. Recall that the acceleration of gravity is $9.8\, m/s^2$ and that water has density $1000\, kg/m^3$.

a) The surface of the water is given by $y =$ ____________

b) The pressure as a function of y (in the given coordinate system)

is given by Pressure = ________________ N/m^3.



c) Set up **but do not evaluate** an integral which represents the hydrostatic force on the window.

Solution.

a) The top of the window is at $y = 1$, so 6 meters above that is $y = 7$.

b) We have Pressure = (density)·(gravity)·(depth). Since the depth is given by $7 - y$ (distance to the surface of the water), the pressure is

$$\text{Pressure } = 9800(7 - y)\, N/m^3.$$

c) At height y, a strip of the window with thickness Δy has area approximately $2y\Delta y$. So, the contribution at height y to the hydrostatic force is $9800(7 - y)2y\, dy = 19600(7 - y)y\, dy$, and hence

$$\text{Force} = 19600 \int_0^1 (7 - y)y\, dy.$$

Zone 9

13/1. (14 points)

a) **Use the integral test** to show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{(n^2+5)^2}$ converges. Do not forget to check the conditions of the integral test.

b) Give an upper bound on the remainder $R_N = S - S_N$ obtained when estimating the sum S of the series by the N^{th} partial sum S_N, and find a value of N which guarantees that the remainder satisfies $R_N < \frac{1}{86}$. Provide brief but complete details.

Solution. a) Let $f(x) = \dfrac{2x}{(x^2+5)^2}$ and check the conditions of the Integral Test. This function is positive, decreasing ($f'(x) < 0$), and continuous for all x sufficiently large. The improper integral $\displaystyle\int_1^{\infty} \frac{2x}{(x^2+5)^2}\,dx$ converges. To see this, we can make the substitution $u = x^2 + 5$, $du = 2x\,dx$ to obtain

$$\int_1^{\infty} \frac{2x}{(x^2+5)^2}\,dx = \int_6^{\infty} \frac{du}{u^2}.$$

By the Integral Test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{(n^2+5)^2}$ also converges.

b) Let S_N be the Nth partial sum and let $S = S_N + R_N$. We know that

$$R_N < \int_N^{\infty} \frac{2x}{(x^2+5)^2}\,dx.$$

So choose N such that

$$\int_N^{\infty} \frac{2x}{(x^2+5)^2}\,dx = \lim_{t\to\infty} \left.\frac{-1}{x^2+5}\right|_N^t = \frac{1}{N^2+5} \leq \frac{1}{86}.$$

In other words, $N^2 + 5 \geq 86$, which is the same as $N^2 \geq 81$. In particular $N = 9$ is a valid choice.

13/2. (14 points) a) **Use the integral test** to show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{(n^2+3)^2}$ converges. Do not forget to check the conditions of the integral test.

b) Give an upper bound on the remainder $R_N = S - S_N$ obtained when estimating the sum S of the series by the N^{th} partial sum S_N, and find a value of N which guarantees that the remainder satisfies $R_N < \frac{1}{103}$. Provide brief but complete details.

Solution. a) Let $f(x) = \frac{2x}{(x^2+3)^2}$ and check the conditions of the Integral Test. This function is positive, decreasing, and continuous for all x sufficiently large. The improper integral $\int_1^\infty \frac{2x}{(x^2+3)^2}\,dx$ converges. To see this, we can make the substitution $u = x^2 + 3$, $du = 2x\,dx$ to obtain

$$\int_1^\infty \frac{2x}{(x^2+3)^2}dx = \int_4^\infty \frac{du}{u^2}.$$

By the Integral Test, the series $\sum_{n=1}^{\infty} \frac{2n}{(n^2+3)^2}$ also converges.

b) Let S_N be the Nth partial sum and let $S = S_N + R_N$. We know that

$$R_N < \int_N^\infty \frac{2x}{(x^2+3)^2}dx.$$

So choose N such that

$$\int_N^\infty \frac{2x}{(x^2+3)^2}dx = \lim_{t\to\infty} \left.\frac{-1}{x^2+3}\right|_N^t = \frac{1}{N^2+3} \leq \frac{1}{103}.$$

In other words, $N^2 + 3 \geq 103$, which is the same as $N^2 \geq 100$. In particular $N = 10$ is a valid choice.
