Tuesday, February 6 $*$ **Solutions** $*$ *Partial derivatives and differentiability.*

1. Consider $f(x,y) = xy^2e^{x^2}$. Compute the following:

$$\frac{\partial f}{\partial x} \qquad \frac{\partial f}{\partial y} \qquad \frac{\partial^2 f}{\partial x \partial y} \qquad \frac{\partial^2 f}{\partial y \partial x}$$

What is the relationship between your answers for the last two? This is an instance of Clairaut's Theorem and holds for most functions.

Solution.

$$\frac{\partial f}{\partial x} = 2x^2y^2e^{x^2} + y^2e^{x^2}; \qquad \frac{\partial f}{\partial y} = 2xye^{x^2}$$

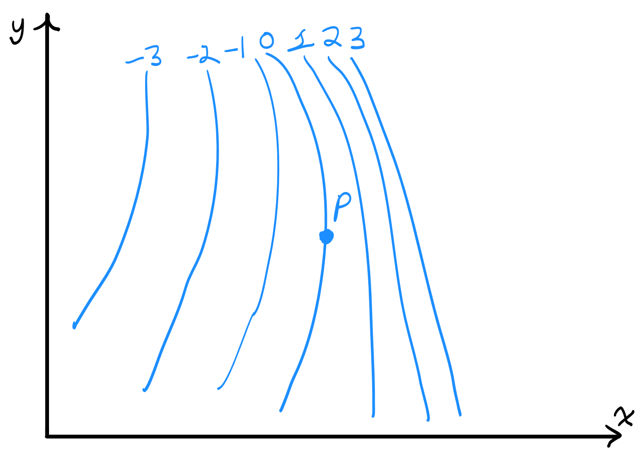
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(2xye^{x^2}) = 4x^2ye^{x^2} + 2ye^{x^2}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2x^2y^2e^{x^2} + y^2e^{x^2}) = 4x^2ye^{x^2} + 2ye^{x^2}$$

So we see that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

2. Shown are some level curves for the function $f \colon \mathbb{R}^2 \to \mathbb{R}$. Determine whether the following partial derivatives are positive, negative, or zero at the point P:

$$f_x \qquad f_y \qquad f_{xx} \qquad f_{yy} \qquad f_{xy} = f_{yx}$$



Solution.

(a) If we fix y and allow x to vary, the level curves indicate that the value of f increases as we move through P in the positive x-direction, so f_x is positive at P.

(b) If we fix x and allow y to vary, the level curves indicate that the value of f is neither increasing nor decreasing as we move through P in the positive y-direction, so f_y is zero at P.

(c) $f_{xx} = \frac{\partial}{\partial x}(f_x)$, so if we fix y and allow x to vary, f_{xx} is the rate of change of f_x as x increases. Note that at points to the right of P the level curves are closer together (in the x-direction) than at points to the left of P, demonstrating that f increases more quickly with respect to x to the right of P. So as we move through P in the positive x-direction the (positive) value of f_x increases, hence f_{xx} is positive.

(d) $f_{yy} = \frac{\partial}{\partial y}(f_y)$, so if we fix x and allow y to vary, f_{yy} is the rate of change of f_y as y increases. The level curves are closer together (in the y-direction) at points above P than at those below P, demonstrating that f increases more quickly with respect to y above P. So as we move through P in the positive y-direction the (positive) value of f_y increases, hence f_{yy} is positive.

(e) $f_{xy} = \frac{\partial}{\partial y}(f_x)$, so if we fix x and allow y to vary, f_{xy} is the rate of change of f_x as y increases. The level curves are closer together (in the x-direction) at points above P than at those below P, demonstrating that f increases more quickly with respect to x for y-values above P. So as we move through P in the positive y-direction, the (positive) value of f_x increases, hence f_{xy} is positive.

3. The wind-chill index $W = f(T, v)$ is the perceived temperature when the actual temperature is T and the wind speed is v. Here is a table of values for W.

Wind speed (km/h); Actual temperature (°C)

T \ v	20	30	40	50	60	70
−10	−18	−20	−21	−22	−23	−23
−15	−24	−26	−27	−29	−30	−30
−20	−30	−33	−34	−35	−36	−37
−25	−37	−39	−41	−42	−43	−44

(a) Use the table to estimate $\frac{\partial f}{\partial T}$ and $\frac{\partial f}{\partial v}$ at $(T, v) = (-20, 40)$.

Solution.

$$f_T(-20,40) = \lim_{h\to 0} \frac{f(-20+h,40) - f(-20,40)}{h}$$

which can be approximated by considering $h = 5$ and $h = -5$ as follows:

$$f_T(-20,40) \approx \frac{f(-15,40) - f(-20,40)}{5} = \frac{-27-(-34)}{5} = \frac{7}{5},$$
$$f_T(-20,40) \approx \frac{f(-25,40) - f(-20,40)}{-5} = \frac{-41-(-34)}{-5} = \frac{7}{5}.$$

Averaging these values, we estimate $f_T(-20,40)$ to be $7/5$.

Similarly,

$$f_v(-20,40) = \lim_{h\to 0} \frac{f(-20,40+h) - f(-20,40)}{h}$$

which can be approximated by considering $h = 10$ and $h = -10$:

$$f_v(-20,40) \approx \frac{f(-20,50) - f(-20,40)}{10} = \frac{-35-(-34)}{10} = -\frac{1}{10},$$
$$f_v(-20,40) \approx \frac{f(-20,30) - f(-20,40)}{-10} = \frac{-33-(-34)}{-10} = -\frac{1}{10}.$$

Averaging these values, we estimate $f_v(-20,40)$ to be $-1/10$.

(b) Use your answer in (a) to write down the linear approximation to f at $(-20,40)$.

Solution. The linear approximation of f at $(-20,40)$ is

$$\begin{aligned} f(T,v) &\approx f(-20,40) + f_T(-20,40)(T-(-20)) + f_v(-20,40)(v-40) \\ &= -34 + \frac{7}{5}(T+20) - \frac{1}{10}(v-40). \end{aligned}$$

(c) Use your answer in (b) to approximate $f(-22,45)$.

Solution.

$$f(-22,45) \approx -34 + \frac{7}{5}(-22+20) - \frac{1}{10}(45-40) = -37.3.$$

4. Consider $f(x,y) = \sqrt{1-x^2-y^2}$.

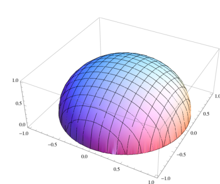
(a) What is the domain of f? That is, for which (x,y) does the function make sense?

Solution.

$$D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

(b) Describe geometrically the surface which is the graph of f.

Solution. It is the upper half of a unit sphere.



goes from $\mathbf{0}$ to the point on the graph where we just found the tangent plane. What is the angle between $\mathbf{v}$ and a normal vector to the tangent plane?

5. Consider $f(x,y) = \sqrt[3]{x^3+y^3}$.

(a) Compute $f_x(0,0)$. Note: this partial derivative exists.

Solution.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{(h^3+0)^{1/3}-0}{h} = \lim_{h\to 0} \frac{h}{h} = 1.$$