Report for Computer Project of Applied Stochastic Analysis—Problem 7

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1 Problem Setup

Consider the following combined Dirichlet-Possion problem:

$$b\nabla u + \frac{1}{2}\Delta u = f(x, y),\ (x, y) \in B_1(0)$$
$$u = \frac{1}{2} \text{ on } (x, y) \in \mathcal{S}^1$$

, where $b = (x, y)$, $f(x, y) = x^2 + y^2 + 1$. We would like to solve this PDE numerically via the simulation of SDEs.

2 Detailed Solution

Let L denote a semi-elliptic partial differential operator on $C^2(\mathbb{R}^n)$ of the form

$$L = \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}$$

. Then we may rewrite the original PDE as:

$$Lu = -g \text{ in } B_1(0)$$
$$u = \phi \text{ on } \mathcal{S}^1$$

where $b = (x, y)^T$, $[a_{ij}] = \frac{1}{2}I_2$, $g = -(x^2 + y^2 + 1)$, $\phi = \frac{1}{2}$. Now consider an Ito diffusion$\{(X_t, Y_t)\}$ whose generator A coincides with L on $C_0^2(\mathbb{R}^2)$. To achieve this we simply choose $b = (x, y)$ and $\sigma = I_2$. According to **Theorem 9.3.3** in [1], the solution of the concerned PDE can be represented as:

$$u(x, y) = \frac{1}{2} - E^{(x,y)}[\int_0^{\tau_D} (X_s^2 + Y_s^2 + 1)ds]$$

,where τ_D is the exit time. Since we have the fact that $E^x(\tau_D) < \infty$ (as shown in prof.Li's lecture notes) and $g(x,y) = -(x^2 + y^2 + 1)$ is bounded in $B_1(0)$, it's then trivial to verify that $E^{(x,y)}[\int_0^{\tau_D} (X_s^2 + Y_s^2 + 1)ds] < \infty$.

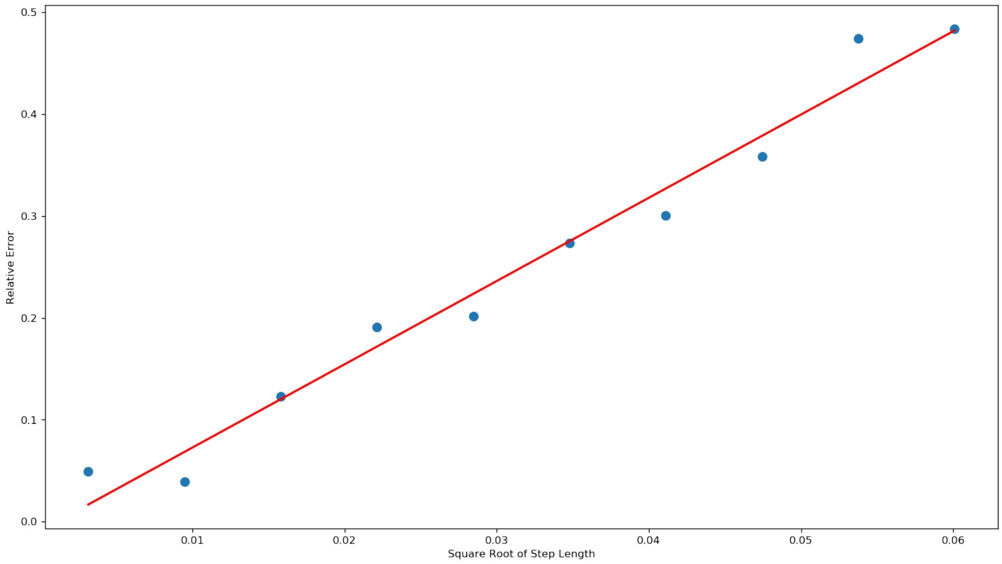
With the discussion above, to evaluate the solution of the PDE at some point, say, (x_0, y_0), we just need to compute $E^{(x,y)}[\int_0^{\tau_D} (X_s^2 + Y_s^2 + 1)ds]$. Here we use the standard Euler-Maruyama scheme to simulate the Ito diffusion $\{X_t, Y_t\}$, and then approximate the integral with the simulated track. And the operation described above would be performed multiple times as to estimate the expectation. Detailed algorithm is shown as follows:

Algorithm 1: ESTIMATION OF $u(x_0, y_0)$

```
Input: (x0, y0), δt as the step length
Output: û(x0, y0)
sum ← 0
i ← 0
while i < N do
    X_0 ← x_0
    Y_0 ← y_0
    while True do
        X_{t+1} ← X_t + X_t * δt + N(0, √δt)
        Y_{t+1} ← Y_t + Y_t * δt + N(0, √δt)
        if X_{t+1}^2 + Y_{t+1}^2 > 1 then
            break
        end
    end
    Int ← Σ_t (X_t^2 + Y_t^2 + 1)δt
    Sum ← Sum + Int
    i ← i + 1
end
return Sum/N
```

3 Numerical Results

The algorithm is implemented in C++, using a random number generator based on hardware entropy. The numerical results are quite satisfactory. Compared to the analytical solution $u(x,y) = (x^2 + y^)/2$, the relative error is less than 1% when we set δt to be 10^{-6} and N to be 10000. And clearly the error comes from simulating the Ito diffusion by Euler-Maruyama scheme, estimating the integral by finite sum and approximating the expectation by empirical mean. The numerical convergence order in step length is approximately 1/2, namely, $error \sim \sqrt{\delta t}$, which can be seen from the following picture, where the red line is obtained by performing linear regression to the relative error and square root of step length.



References

[1] Bernt Øksendal. Stochastic differential equations: An introduction with applications. Springer, 2000.