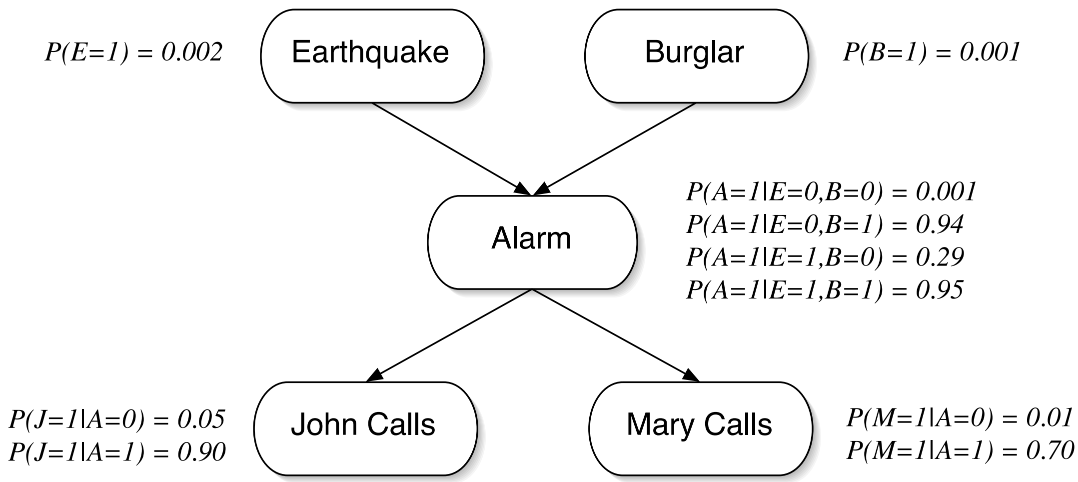
CSE 250A. Assignment 2

Out: *Tue Oct 6*
Due: *Tue Oct 13* (end of class)
Reading: Russell & Norvig, Chapters 13-14.

2.1 Probabilistic inference

Recall the alarm belief network described in class. The directed acyclic graph (DAG) and conditional probability tables (CPTs) are shown below:



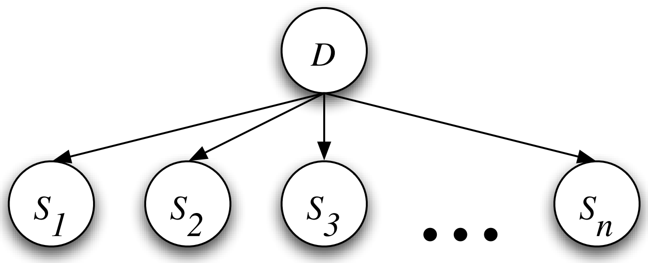
Compute numeric values for the following probabilities, exploiting relations of conditional independence as much as possible to simplify your calculations. You may re-use numerical results from lecture, but otherwise *show your work*. Be careful not to drop significant digits in your answer.

(a) $P(E\!=\!1|A\!=\!1)$ (c) $P(A\!=\!1|J\!=\!0)$ (e) $P(A\!=\!1|M\!=\!1)$
(b) $P(E\!=\!1|A\!=\!1, B\!=\!1)$ (d) $P(A\!=\!1|J\!=\!0, M\!=\!0)$ (f) $P(A\!=\!1|M\!=\!1, E\!=\!0)$

Consider your results in (b) versus (a), (d) versus (c), and (f) versus (e). Do they seem consistent with commonsense patterns of reasoning?

2.2 Probabilistic reasoning

A patient is known to have contracted a rare disease which comes in two forms, represented by the values of a binary random variable $D \in \{0, 1\}$. Symptoms of the disease are represented by the binary random variables $S_k \in \{0, 1\}$, and knowledge of the disease is summarized by the belief network:



The conditional probability tables (CPTs) for this belief network are as follows. In the absence of evidence, both forms of the disease are equally likely, with prior probabilities: $P(D=0) = P(D=1) = \frac{1}{2}$. In the first form of the disease $(D=0)$, the first symptom occurs with probability one,

$$P(S_1=1|D=0) = 1,$$

while the k^{th} symptom (with $k \geq 2$) occurs with probability

$$P(S_k=1|D=0) = \frac{f(k-1)}{f(k)},$$

where the function $f(k)$ is defined by

$$f(k) = 2^k + (-1)^k.$$

By contrast, in the second form of the disease $(D=1)$, all the symptoms are uniformly likely to be observed, with $P(S_k=1|D=1) = \frac{1}{2}$ for all k.

Suppose that on the k^{th} day of the month, a test is done to determine whether the patient is exhibiting the k^{th} symptom, and that each such test returns a positive result. Thus, on the k^{th} day, the doctor observes the patient with symptoms $\{S_1=1, S_2=1, \ldots, S_k=1\}$. Based on the cumulative evidence, the doctor makes a new diagnosis each day by computing the ratio:

$$r_k = \frac{P(D=0|S_1=1, S_2=1, \ldots, S_k=1)}{P(D=1|S_1=1, S_2=1, \ldots, S_k=1)}.$$

If this ratio is greater than 1, the doctor diagnoses the patient with the $D=0$ form of the disease; otherwise, with the $D=1$ form.

(a) Compute the ratio r_k as a function of k. How does the doctor's diagnosis depend on the day of the month? Show your work.

(b) Does the diagnosis become more or less certain as more symptoms are observed? Explain.

2.3 Sigmoid function

Let $Y \in \{0, 1\}$ denote a binary random variable that depends on k other random variables X_i as:

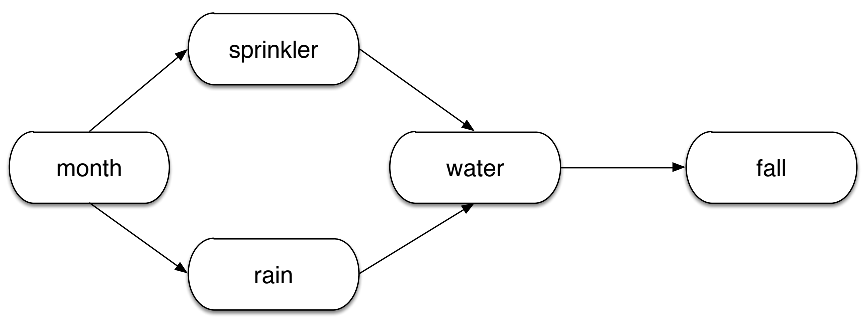
$$P(Y\!=\!1|X_1\!=\!x_1, X_2\!=\!x_2, \ldots, X_k\!=\!x_k) = \sigma\left(\sum_{i=1}^{k} w_i x_i\right) \quad \text{where} \quad \sigma(z) = \frac{1}{1+e^{-z}}.$$

The real-valued parameters w_i in this CPT are known as weights. The so-called sigmoid function $\sigma(z)$ arises in many contexts. In neural networks, it models the probability that a neuron Y fires given its input from other neurons X_i; the weights w_i describe the connections between neurons. In statistics, the sigmoid function appears in models of logistic regression. Sketch the function $\sigma(z)$, and verify the following properties:

(a) $\sigma'(z) = \sigma(z)\sigma(-z)$.
(b) $\sigma(-z) + \sigma(z) = 1$.
(c) $L(\sigma(z)) = z$, where $L(p) = \log\left(\frac{p}{1-p}\right)$ is the *log-odds* function.

2.4 Conditional independence

Consider the DAG shown below, describing the following domain. Given the `month` of the year, there is some probability of `rain`, and also some probability that the `sprinkler` is turned on. Either of these events leads to some probability that a puddle of `water` forms on the sidewalk, which in turn leads to some probability that someone has a `fall`.



List all the conditional independence relations that must hold in any probability distribution represented by this DAG. More specifically, list all tuples $\{X, Y, E\}$ such that $P(X, Y|E) = P(X|E)P(Y|E)$, where

$$\begin{aligned} X, Y &\in \{\texttt{month}, \texttt{rain}, \texttt{sprinkler}, \texttt{water}, \texttt{fall}\}, \\ E &\subseteq \{\texttt{month}, \texttt{rain}, \texttt{sprinkler}, \texttt{water}, \texttt{fall}\}, \\ X &\neq Y, \\ X, Y &\notin E. \end{aligned}$$

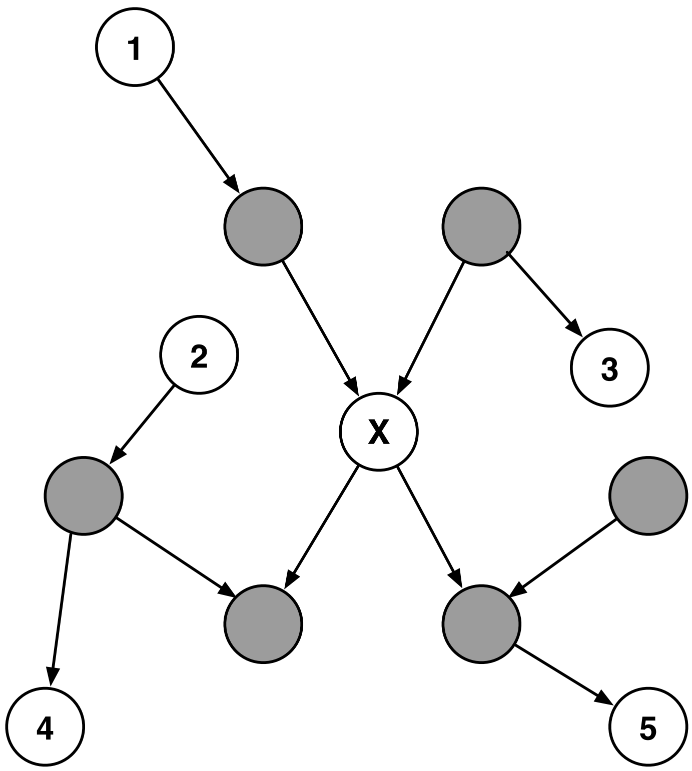
Hint: There are sixteen such tuples, not counting those that are equivalent up to exchange of X and Y. Do any of the tuples contain the case $E\!=\!\emptyset$?

2.5 Markov blanket

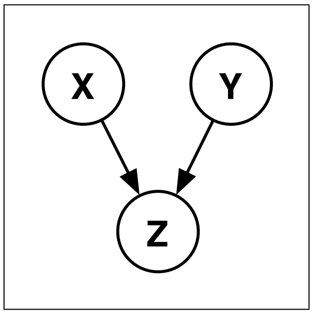
The Markov blanket B_X of a node X in a belief network includes its parents, its children, and the parents of its children (excluding the node X itself). Draw a picture of a Markov blanket; then, by appealing to the three conditions for d-separation, prove that for any node Y outside the Markov blanket of X (that is, satisfying $Y \notin B_X$ and $Y \neq X$), we have:

$$P(X, Y | B_X) = P(X | B_X) P(Y | B_X).$$

Hint: A complete solution will consider *five* different types of paths from nodes outside the Markov blanket to the node X; the picture below may be helpful.



2.6 Noisy-OR



Nodes: $X \in \{0,1\}, Y \in \{0,1\}, Z \in \{0,1\}$

Noisy-OR CPT: $P(Z=1|X,Y) = 1 - (1-p_x)^X\,(1-p_y)^Y$

Parameters: $p_x \in [0,1]$, $p_y \in [0,1]$, $p_x < p_y$

Suppose that the nodes in this network represent binary random variables and that the CPT for $P(Z|X,Y)$ is parameterized by a noisy-OR model, as shown above. Suppose also that

$$0 < P(X=1) < 1,$$
$$0 < P(Y=1) < 1,$$

while the parameters of the noisy-OR model satisfy:

$$0 < p_x < p_y < 1.$$

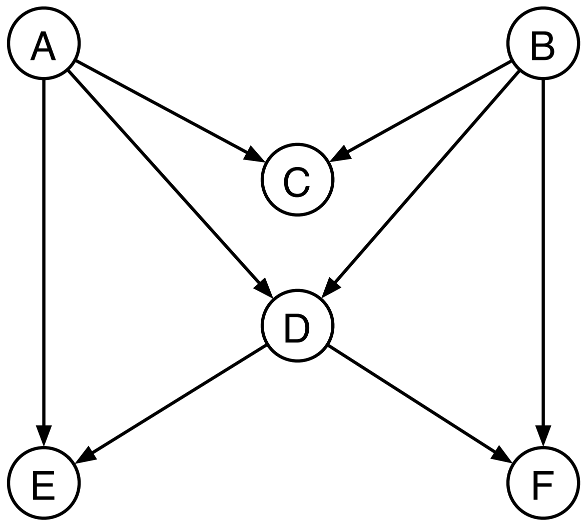
Consider the following pairs of probabilities. In each case, indicate whether the probability on the left is equal ($=$), greater than ($>$), or less than ($<$) the probability on the right. The first one has been filled in for you as an example. (You should use your intuition for these problems; you are not required to show work.)

	$P(X=1)$	$=$	$P(X=1)$
(a)	$P(Z=1\vert X=0,Y=0)$		$P(Z=1\vert X=0,Y=1)$
(b)	$P(Z=1\vert X=1,Y=0)$		$P(Z=1\vert X=0,Y=1)$
(c)	$P(Z=1\vert X=1,Y=0)$		$P(Z=1\vert X=1,Y=1)$
(d)	$P(X=1)$		$P(X=1\vert Z=1)$
(e)	$P(X=1)$		$P(X=1\vert Y=1)$
(f)	$P(X=1\vert Z=1)$		$P(X=1\vert Y=1,Z=1)$
(g)	$P(X=1)\,P(Y=1)\,P(Z=1)$		$P(X=1,Y=1,Z=1)$

Challenge (optional): for each case, prove rigorously the correctness of your answer.

2.7 More conditional independence

For the belief network shown below, indicate whether the following statements of (conditional) independence are **true (T)** or **false (F)**.



_____________ $P(E, F|D) = P(E|D)\, P(F|D)$

_____________ $P(E, F|C, D) = P(E|C, D)\, P(F|C, D)$

_____________ $P(E, F|A, B, D) = P(E|A, B, D)\, P(F|A, B, D)$

_____________ $P(D|C) = P(D)$

_____________ $P(D|A, B) = P(D|A, B, C)$

_____________ $P(A, B) = P(A)\, P(B)$

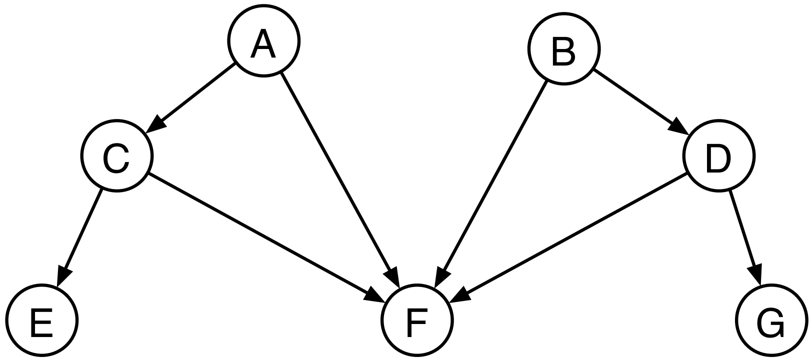
_____________ $P(A|C, D) = P(A|C, D, F)$

_____________ $P(A|B, C, D) = P(A|B, C, D, F)$

_____________ $P(B|A, C, D, F) = P(B|A, C, D, F, E)$

_____________ $P(B, F, A, E|C, D) = P(B, F|C, D)\, P(A, E|C, D)$

2.8 Even more conditional independence



Consider the following statements of (conditional) independence for the belief network shown above. Indicate the largest subset of nodes $\mathcal{S} \subset \{A, B, C, D, E, F, G\}$ for which each statement is true. Note that one possible answer is the empty set $\mathcal{S} = \emptyset$ or $\mathcal{S} = \{\}$ (whichever notation you prefer). The first has been done as an example.

$P(B)$	=	$P(B\|\mathcal{S})$	$\mathcal{S} = \{A, C, E\}$
$P(B\|D)$	=	$P(B\|\mathcal{S})$	________________
$P(B\|D, F)$	=	$P(B\|\mathcal{S})$	________________
$P(C\|D)$	=	$P(C\|\mathcal{S})$	________________
$P(C\|F, G)$	=	$P(C\|\mathcal{S})$	________________
$P(C\|A, E, F)$	=	$P(C\|\mathcal{S})$	________________
$P(E\|F)$	=	$P(E\|\mathcal{S})$	________________
$P(E\|C)$	=	$P(E\|\mathcal{S})$	________________
$P(F)$	=	$P(F\|\mathcal{S})$	________________
$P(F\|C, D)$	=	$P(F\|\mathcal{S})$	________________
$P(A, B)$	=	$P(A, B\|\mathcal{S})$	________________