University of Lausanne
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Assignment 2

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Course: *Empirical Methods in Finance* – Professor: *Dr. Florian Ielpo*
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Contents

1. Introduction

This report is divided into two parts. The first part aims to manipulate times series of various crypto-currencies (i.e. Bitcoin, Ethereum and Monero). We will first test for stationarity using the Dickey Fuller test, then check for co-integration among our crypto-currencies, and finally try to implement a pair-trading strategy using the results found previously. The second part will aim to test again the stationarity of four new time series (i.e. US Dollar in trade weighted terms, Gold, the level of VIX index and Bitcoin), using the Augmented Dickey Fuller test and the Kwiatkowski Phillips Schmidt Shin (KPSS) test. After transforming the non-stationary series into stationary ones, we will try to build a Vector Auto-Regressive (VAR) model, by deciding the number of lags using various information criteria model. Thereafter, we will determine the Impulse Response Function between our assets. Finally, we will estimate an unconstrained and constrained VAR(1) model, and check whether there are any statistical differences between them. All the implementations will be done using Python.

2. Part 1: Pair-Trading

2.1. Testing for Stationarity.

The first step will be for us to test for some possible stationarity in time series. We will therefore run the 1-lag regression (AR(1)) on all time series ($p_t = \log(P_t)$):

$$p_t = \alpha + \beta p_{t-1} + u_t \tag{1}$$

Having a look at the regression results will allow us to conclude on stationarity based on a Dickey-Fuller test.

Q1.1 The Dickey-Fuller test we compute here tests a null hypothesis H_0 assuming that a unit root exists in the model, that is $\beta = 1$, implying non-stationarity. On the other hand, the alternative hypothesis here is stationarity, that is $\beta < 1$.

Q1.2 To test the null hypothesis, we run the mentioned regression and look at the t-stat which can be calculated such as:

$$\tau_\mu = \frac{\hat{\beta} - 1}{std(\hat{\beta})} = \frac{\hat{\beta} - 1}{(\hat{\sigma}_u^2 / \sum_{t=1}^{T}(p_{t-1} - \bar{p})^2)^{\frac{1}{2}}} \tag{2}$$

where

$$\hat{\sigma}_u^2 = \frac{1}{T-1}\sum_{t=1}^{T}(p_t - \hat{\alpha} - \hat{\beta}p_{t-1})^2 \tag{3}$$

We know that due to the non-stationarity of the process ($\beta = 1$) the distribution of τ_μ deviates from a Gaussian. As a consequence of the degenerated distribution, we need to make our conclusions on H_0 based on some simulated (Monte Carlo) critical values. When Monte Carlo simulations are done and we have resulting critical values, we can conclude on H_0 by comparing τ_μ with those values. Whenever the value of τ_μ is smaller than the critical value, we can reject the null-hypothesis H_0 and conclude for some stationarity.

Q1.3 Setting $r_t = p_t - p_{t-1}$, if we focus on the following regression:

$$r_t = \alpha' + \beta' p_{t-1} + u_t \tag{4}$$

Then the null hypothesis would be again the unit root hypothesis of the AR(1) model, by assuming this time $\beta = 0$ implying non-stationarity. On the other hand, the alternative hypothesis would therefore assume stationarity (e.g. $\beta < 0$). To compare with Q1.1 null hypothesis, let's rewrite the first model we were looking at when subtracting the lag value on both sides of equation (1):

$$p_t - p_{t-1} = \alpha + \beta p_{t-1} - p_{t-1} + u_t = \alpha + (\beta - 1)p_{t-1} + u_t \tag{5}$$

Comparing equation (4) with equation (1), we clearly notice that testing null hypothesis assuming $\beta = 1$ in Q1.1 is equivalent to testing null hypothesis assuming $\beta' = 0$ in current question since both assume non-stationarity .

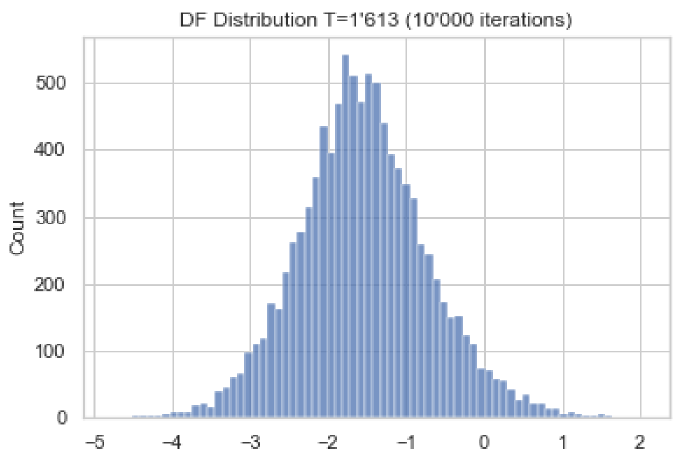
2.1.1. Critical Values.

Q1.4 Due to the unknown distribution of the t-stat under H_0 (which is neither Gaussian nor Student) we need to simulate the process and compute the t-stat for each of the many simulations to estimate the unknown distribution of the t-stat, more specifically its critical values. To conduct such Monte Carlo simulations, we have to simulate random walk processes such as:

$$p_t = p_{t-1} + \varepsilon_t \tag{6}$$

which corresponds to equation (1) under the null hypothesis H_0 and also $\alpha = 0$. The next step to test stationarity of our process will be to estimate AR(1) processes on the simulated time series, to compute the t-stat using equation (2) and estimate the distribution of the τ_μ to compare our sample t-stat with the simulated critical value to make conclusions on stationarity of our time series.

Q1.5 Having computed 10'000 different t-stats, we can observe the following distribution:



At first glance it might seem to be a normal distribution but it isn't. Indeed, the simulated t-stats show a Dickey-Fuller distribution which is different from the Gaussian one. This is the reason why we need to simulate critical values instead of doing the usual t-test procedure.

Q1.6 After having simulated the 10'000 t-stats we were able to compute 1%, 5% and 10% percentiles to get the associated critical values. Here below are the results we found:

	10%	5%	1%
Critical Value	-2.5671	-2.863282	-3.432601

Q1.7 What if we simulate the following AR(1) model this time ?

$$p_t = 0.2p_{t-1} + \varepsilon_t \tag{7}$$

Since the absolute value of the autoregressive coefficient is smaller than one, the AR(1) process given by equation (7) is covariance stationary. Hence, this time we know that when simulating a significant number of observations of this process, it will remain covariance-stationary. For this reason, we would be able to proceed with a usual t-test. Finally the t-stat distribution would be classical Student t-distribution. It would therefore be enough to look for critical values in a t-distribution table without any need for simulation.

2.1.2. Testing Non-Stationarity.

Q1.8 Having estimated equation (1) for the 3 currencies, we arrive at the following parameters:

	Bitcoin	Ethereum	Monero
$\hat{\alpha}$	0.014942	0.017359	0.018463
$\hat{\beta}$	0.998609	0.997481	0.996323

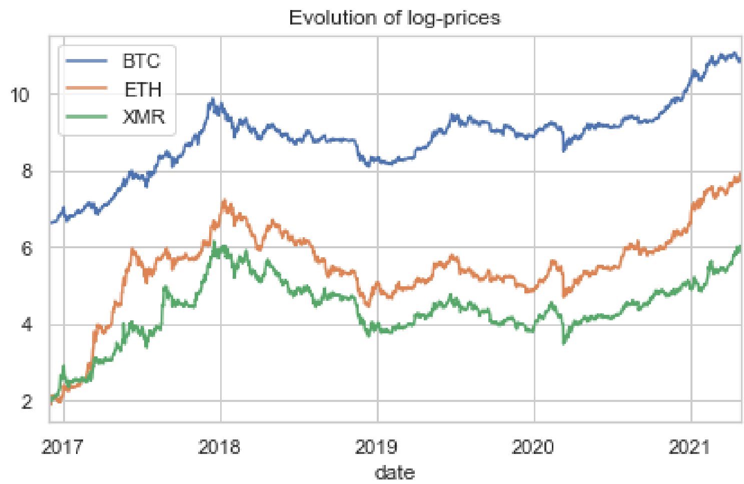
At a glance, it does indeed seem like the betas are approximately equal to one and therefore that the log-price has a unit root and is non-stationary. It is up to the t-statistics however to allow us to make such a claim.

Q1.9 In order to find these t-statistics, we compute equation (2) for each currency and find these results:

	Bitcoin	Ethereum	Monero
t-stat	-1.183806	-1.988053	-1.986522

Comparing them to the critical values given in Q1.6, we can say that the betas are not far enough from 1 to allow us to reject H_0 at any of the thresholds. In other words, we can not reject H_0 at a significance level smaller or equal than

10% since the t-stats are all smaller than the critical values for such significance level. Hence, we can assert with relative confidence that the log-price has a unit root. This means that the time series are integrated of an order higher than 0, and they are most probably I(1).
Observing the graphs of the p_t for each stock we can indeed see that the time series are not stationary since they seem to display an upward trend.



Q1.10 Oftentimes, when combining two time series which are integrated of order p, the result is also $I(p)$. However, there can exist pairs which when combined, result in an $I(q)$ process with $q < p$. Furthermore, when such a combination of non-stationary time-series leads to an $I(0)$ the time series are said to be cointegrated. In our sample, although the currencies are non-stationary on a standalone basis, it could be the case that the currencies are driven by the same factors which would lead the difference of the prices to remain equal through time which would mean that they are cointegrated. In the following part, we will attempt to find if such a relation exists between our currencies.

2.2. Cointegration.

Q2.1 In order to test for cointegration, we must first estimate the following regression for each of the 3 currency pairs BTC-ETH, BTC-XMR and ETH-XMR:

$$p_t^A = a + bp_t^B + z_t \tag{8}$$

Having done this, we are left with the following estimated parameters:

	BTC-ETH	XMR-BTC	XMR-ETH
$\hat{a}$	4.965093	-2.050610	0.723806
$\hat{b}$	0.706699	0.727231	0.666719
R^2	0.756144	0.685647	0.872525

It seems that all currency pairs are positively correlated since all $\hat{b}$s are positive. This could mean for example that there exists a causal effect between the currencies' price changes. For example, Bitcoin, which accounts for over 40% of the cryptocurrency market is sometimes used as a proxy of the market as a whole. It could be the case that investors interpret a crash in Bitcoin as a sign that all cryptos are doomed and divest which would cause the other currencies to follow the same trend. There could also be non causal effects between the prices. For example, one possible reason could be that the prices are all quoted on the dollar and thus, all fluctuate when the dollar moves. Another possible explanation could be that cryptos generally all serve the same function, as an alternative to traditional assets and therefore move in the same direction on average as a result of movements in other markets.

Something to remain weary of however is the presence of spurious correlations. Indeed, since as we have shown, the time series are non-stationary, the positive correlation might simply be a result of the fact that they all follow an upward trend. The seemingly high R^2 could support this theory.

Q2.2 After estimating the relation between pairs of currencies, we must verify if the residual z_t has a unit root or not. If it does not, the pair is cointegrated.
To check if this is the case, we must perform a Dickey-Fuller test on equation (9). This will allow us to conclude if we can reject the null-hypothesis that z_t has a unit root.
First we estimate the parameters of equation (9).

$$\Delta \hat{z}_t = \alpha + \beta \hat{z}_{t-1} + u_t \tag{9}$$

This yields the following results:

	BTC-ETH	XMR-BTC	XMR-ETH
α	-0.0001261	-0.000458183	-4.55527e-06
β	0.0025733	0.00541987	0.0120014

Thereafter, the t-stats are computed using equation (10) note that it is similar to equation (2) but this time we want to know if $\beta = 0$ since we set the regression in much the same way as for equation (4).
We then compare this t-statistic against the appropriate critical values given in the following table:

	BTC-ETH	XMR-BTC	XMR-ETH
t-stat	-1.403529	-2.177683	-3.117281
1% critical	-3.96	-3.96	-3.96
5% critical	-3.37	-3.37	-3.37
10% critical	-3.07	-3.07	-3.07

$$t\text{-stat} = \frac{\hat{\beta}}{std(\hat{\beta})} \tag{10}$$

This allows us to make the following conclusions for each currency pair:

BTC-ETH: $-1.40 > -3.07$, so we do not reject the null hypothesis that the residual has a unit root. This pair is therefore not cointegrated.

XMR-BTC: $-2.18 > -3.07$, so we do not reject the null hypothesis that the residual has a unit root. This pair is therefore not cointegrated.

XMR-ETH: $-3.07 > -3.12 > -3.37$, so we reject the null hypothesis that the residual has a unit root with $\alpha = 10\%$. There is reason to believe that the pair is cointegrated.

We also compared the t-stats against the critical values given by the adfuller function provided in the statsmodels library to make sure.

	BTC-ETH	XMR-BTC	XMR-ETH
t-stat	-1.403529	-2.177683	-3.117281
1% critical	-3.434413	-3.434413	-3.434413
5% critical	-2.863335	-2.863335	-2.863335
10% critical	-2.567725	-2.567725	-2.567725

The conclusion is identical with the exception that we reject the null hypothesis with $\alpha = 5\%$ since the critical values are a bit more lenient to reject H_0.

2.3. Pair-trading.

Q3.1 The goal of this question is to show how one can design a trading strategy based on the co-integration between two time series. Two time series are co-integrated if a linear combination of them is covariance-stationary. Indeed, if we regress one variable on the other one, one can re-write the residual of the OLS regression as:

$$z_t = p_t^A - a - bp_t^B \tag{11}$$

Thus, if A and B are co-integrated, z_t must be covariance-stationary. Recall from the last question, that the only pair of cryptocurrencies that shows co-integration is the pair XMR-ETH. Indeed, the residual of the regression of XMR of ETH is covariance-stationary, meaning that the distance between those two cryptocurrencies has no trend. Therefore, the spread between the return of both cryptocurrencies must be mean-reverting in such a case. That is, we can use this property to design a trading strategy that consists in going long in the undervalued cryptocurrency and going short into the overvalued one. Technically, we defined the spread between XMR and ETH as follows:

$$spread_t = r_{XMR,t} - r_{ETH,t} \tag{12}$$

And then we need to standardize this spread to determine which currency is overvalued and which currency is undervalued, defining the zscore as follows :

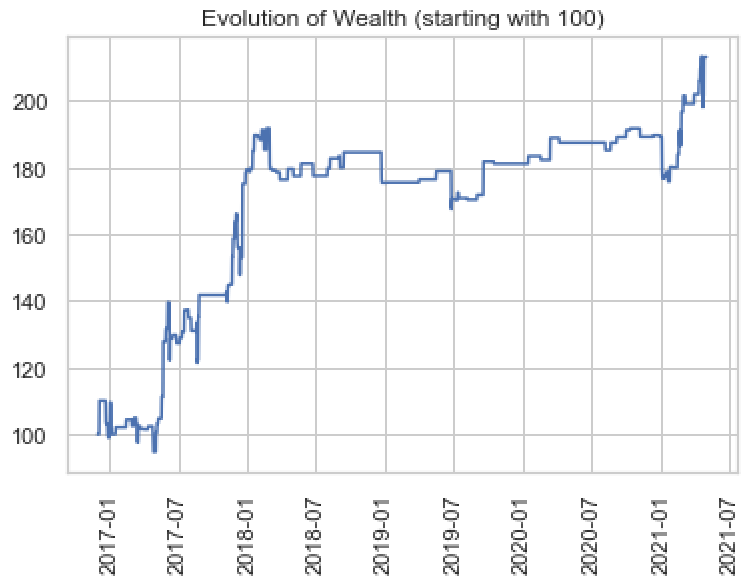
$$zscore_t = \frac{spread_t - \overline{spread}}{\sigma_{spread}} \tag{13}$$

If $zscore_t > zscore_{upper}$, this means that the return of XMR is particularly high compared to the return of ETH, hence we expect that the spread will decrease since it is stationary, meaning that $r_{XMR,t}$ is expected to fall and $r_{ETH,t}$ is expected to rise, meaning that XMR is currently overvalued and ETH is currently undervalued. Therefore, we have to go long in ETH and to go short in XMR.

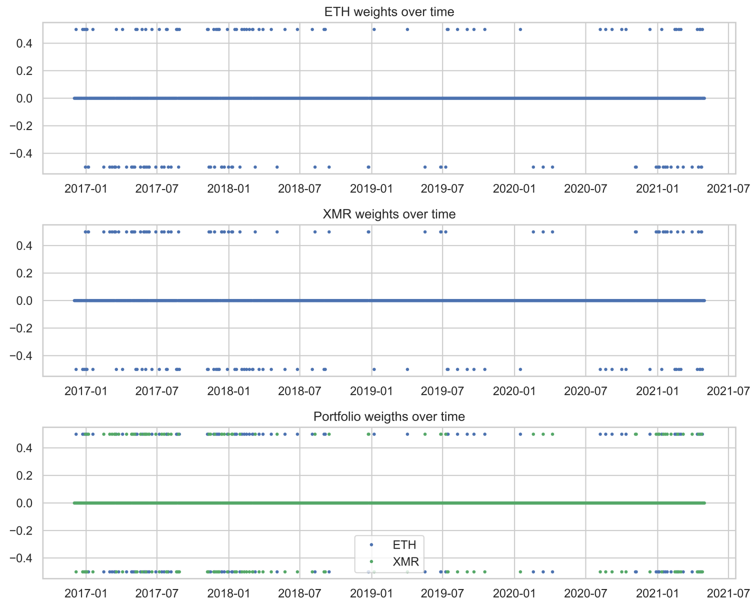
Alternatively, if $zscore_t < zscore_{lower}$, this means that the return of XMR is particularly small compared to the return of ETH, hence we expect that the spread will increase since it is stationary, meaning that $r_{XMR,t}$ is expected to rise and $r_{ETH,t}$ is expected to fall, meaning that XMR is currently undervalued and ETH is currently overvalued. Therefore, we have to go long in XMR and to go short in ETH. Then, if we have a long position in XMR and $zscore_t > -zscore_{out}$ or if we are long in ETH and $zscore_t < zscore_{out}$, we close our position since the returns are not far enough from each other to take a bet in the direction that the spread will follow.

Q3.2 No, this strategy does not exploit arbitrage opportunity because arbitrage opportunity exists only if we can earn a positive cash-flow, with no initial investment and without taking any risks.
Here, since our strategy is a long-short strategy, there's no initial investment that is needed since the strategy is self-financing, meaning that we finance the purchase of the undervalued currency with the sale of the overvalued one. However, the reason why there's no arbitrage is that this strategy is risky and doesn't provide positive cash-flow in every states of the world.
Since the strategy is a long-short strategy based on mean-reversion and provides a positive outcome, we say that it can be classified as "Statistical Arbitrage" but not as (pure) arbitrage.

Q3.3 Below we can see the cumulated performance of our strategy, with initial investment of 100 :



The following chart shows the evolution of the weights of XMR and ETH in the Pair trading strategy:



The performance of the strategy is quite impressive: starting with an initial wealth of 100 $, we end up with almost 200 $ four years later, yielding an annual return of roughly 25% over the considered time horizon. Moreover, we can summarize the following metrics of the strategy in the next table:

	Pair Trading
Annualized mean	13.32%
Annualized volatility	17.41%
Sharpe ratio	0.7646

The Sharpe ratio of this strategy is quite high even though not extraordinary. The performance of this strategy depends as usual on our benchmark. For comparison, if you look at the Sharpe ratio of the S&P 500 over the approximately same period, it has a Sharpe ratio of 0.94[1]. To assess more precisely the profitability of such a strategy, we would need to consider a longer time horizon than just 4 years. However, we can see that the strategy seems to be profitable, at least in the short-term.

[1] https://www.morningstar.com/indexes/spi/spx/risk

3. Part 2: VAR Models

Q4.1 **The price of the US Dollar in trade weighted terms**: this is an indicator used to gauge the strength of the dollar. It is a very useful indicator for international trade as an increase of the dollar means that imports are more affordable but exports become more expensive for recipients. An example when this indicator or rather its Chinese equivalent made the headlines was in 2019 when China was labeled a currency manipulator by the US. It was alleged that China was intentionally deflating its currency in order to counteract measures taken during Trump's war on trade.

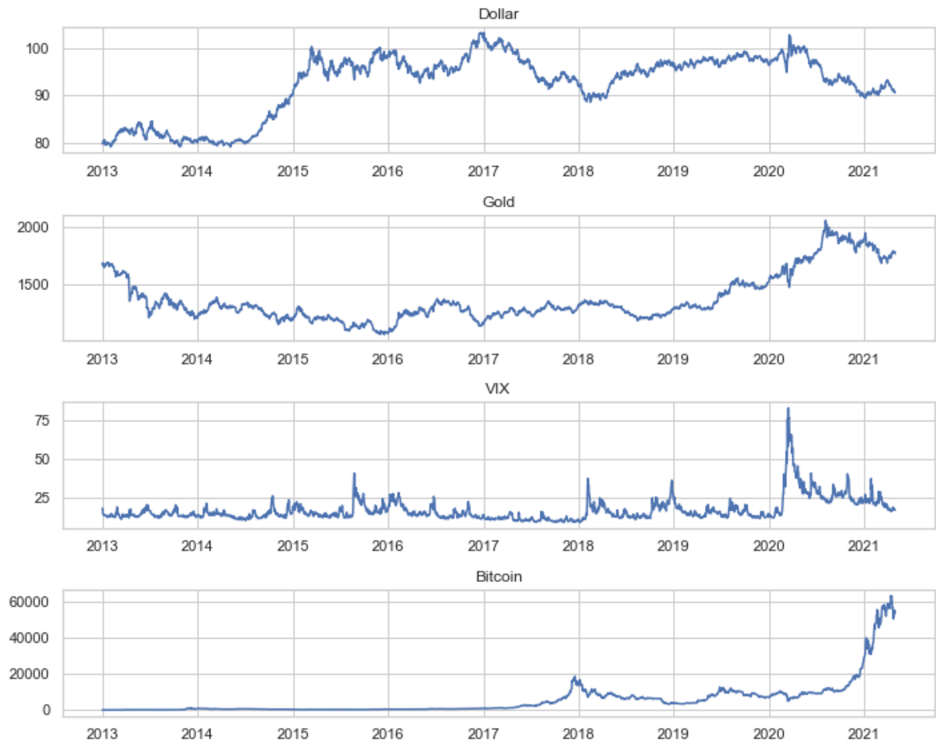
The price of Gold: Here, seemingly the dollar price of an ounce of gold, it is often used as an indicator of investor confidence. Indeed, since Gold is considered a safe-haven, investors often increase their holding of gold when the economic outlook is poor which leads to an increase in its price by law of supply and demand.

The VIX: This is the volatility index, sometimes dubbed the Fear Index, indicates the volatility anticipated by the market by extracting volatility from S&P 500 options for the 30 upcoming days. In other words, the VIX index measures the implied volatility embedded in market prices of the S&P500 index options. The Chicago Board Options Exchange (CBOE) is responsible of computing, publishing and also trading futures as well as options on the VIX.

The price of one Bitcoin expressed in US Dollar terms: The name says it all. Although very controversial, initially a fringe asset used for shady practices on the internet, Bitcoin has become a mainstream asset in recent years and has even begun to rival gold in its capacity to escape from traditional assets when markets are unstable.

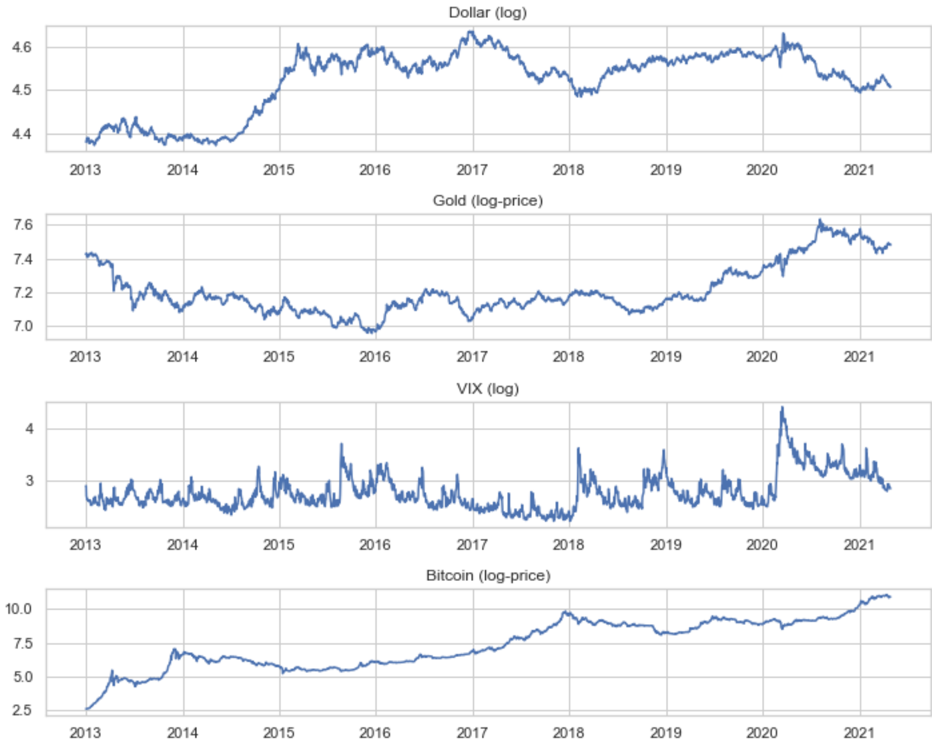
The following graph illustrates the evolution of prices from 2013 to present day.

Figure 1: Time Series Evolution



Q4.2 Just by looking at the following graphs of the log of prices, we are inclined to say that none of the time series are stationary since they all appear to have trends, although we could argue that the VIX index might show some sign of seasonality most of the time, thus concluding that it is the only stationary time series.

Figure 2: Un-transformed Time Series



To test whether the time series are stationary or not, we performed an Augmented Dickey Fuller (ADF) test. Compared to the traditional Dickey-Fuller test, this test has the advantage of taking into account potential auto-correlations in the residual of equation (12) by including the lags of Δp_{t-1} in the regression. Let's consider the following model:

$$p_t = \mu + \varphi_1 p_{t-1} + \sum_{i=1}^{k-1} \delta_i \Delta p_{t-1} + \varepsilon_t \tag{14}$$

Where the null hypothesis $H_0 : \varphi_1 = 1$, which is that the time series is not stationary. The t-stat for this null hypothesis is defined as:

$$\tau_\tau = \frac{\hat{\varphi}_1 - 1}{std(\hat{\varphi}_1)} \tag{15}$$

Therefore, we perform an Augmented Dickey Fuller test to make sure that our times series are not stationary and get the following results:

Table 1: Augmented Dickey Fuller Test using Un-Transformed Time Series

	USD	GOLD	VIX	BTC
adf-stat	-1.969288	-1.300700	-4.638760	-1.596168
p-val	0.300199	0.628852	0.000110	0.485518
critical 1%	-3.433371	-3.433363	-3.433371	-3.433387
critical 5%	-2.862875	-2.862871	-2.862875	-2.862881
critical 10%	-2.567481	-2.567479	-2.567481	-2.567484

Recall that the null hypothesis, in which series are not stationary, is rejected whenever the adf-stat is smaller than the critical value. Consequently, we cannot reject the null hypothesis for all time series except for the VIX index. Rejecting the hypothesis suggests there is reason to believe that the time series is in fact stationary. This would indicate that the log VIX is stationary since only the log VIX has a small enough t-statistic compared to the critical values to warrant us to reject the null hypothesis.

However, looking at the graph, we were not really convinced and we decided to perform a Kwiatkowski–Phillips–Schmidt–Shin (KPSS) test to double check. The KPSS test is defined as:

$$KPSS = T^{-2} \sum_{t=1}^{T} \frac{\hat{S}_t^2}{\hat{\lambda}^2} \tag{16}$$

Where $\hat{S}_t^2 = \sum_{j=1}^{t} \hat{v}_j$, with $\hat{v}_j$ being the residual of a regression of y_t on D_t, $\hat{\lambda}$ is a consistent estimate of the long run variance of v_t. The null hypothesis is that the time series is stationary. We reject the null hypothesis H_0 if the KPSS-stat is larger the than corresponding critical value.

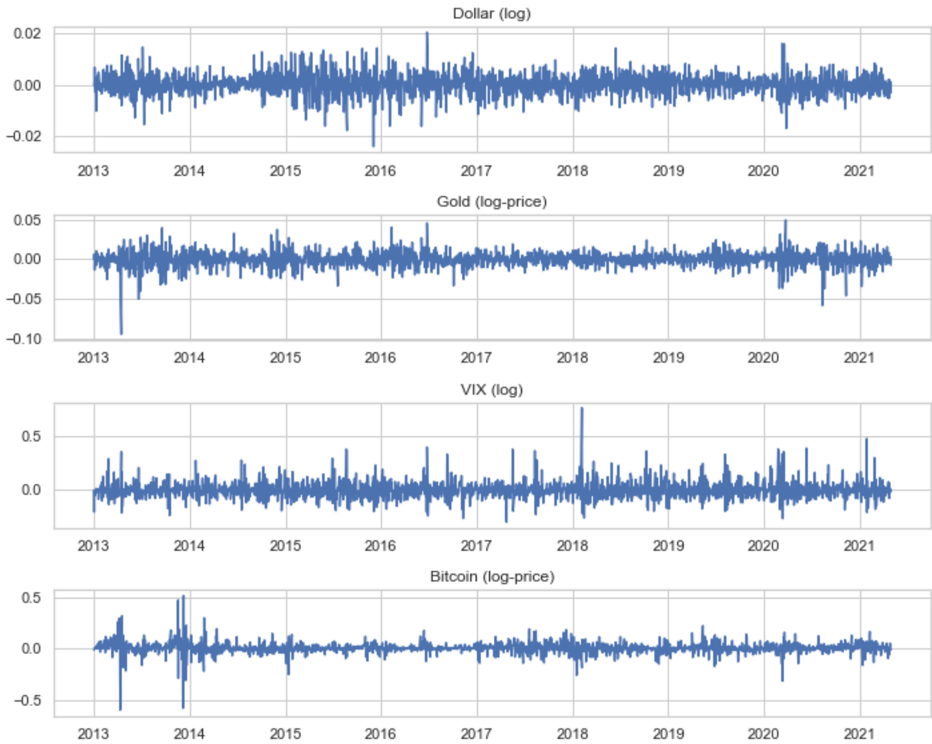
Table 2: KPSS Test using Un-Transformed Time Series

	USD	GOLD	VIX	BTC
kpss-stat	3.9357	3.385558	1.903981	7.170887
p-val	0.0100	0.010000	0.010000	0.010000
critical 1%	0.7390	0.739000	0.739000	0.739000
critical 5%	0.4630	0.463000	0.463000	0.463000
critical 10%	0.3470	0.347000	0.347000	0.347000

The results of this test are somewhat conflicting for the log VIX since we can no longer reject the null hypothesis at the 1% confidence level. To be safe, we will transform all the series.

In order to make them stationary, we take the first difference of the time series which leads to the following time series:

Figure 3: Transformed Time Series



Now they all seem to be stationary but we perform an Augmented Dickey Fuller test again to make sure which gives:

Table 3: Augmented Dickey Fuller Test using Transformed Time Series

	USD	GOLD	VIX	BTC
adf-stat	-19.880073	-46.275344	-19.215754	-9.980236e+00
p-val	0.000000	0.000000	0.000000	2.124269e-17
critical 1%	-3.433371	-3.433364	-3.433374	-3.433387e+00
critical 5%	-2.862875	-2.862872	-2.862876	-2.862881e+00
critical 10%	-2.567481	-2.567479	-2.567481	-2.567484e+00

The small p-values allow us to reject the null hypothesis of non-stationarity and claim with relative certainty that our time series are now stationary.

We also performed the KPSS test using the differentiated time series, which gives us:

Table 4: KPSS Test using Transformed Time Series

	USD	GOLD	VIX	BTC
kpss-stat	0.329736	0.467266	0.017522	0.15063
p-val	0.100000	0.049039	0.100000	0.10000
critical 1%	0.739000	0.739000	0.739000	0.73900
critical 5%	0.463000	0.463000	0.463000	0.46300
critical 10%	0.347000	0.347000	0.347000	0.34700

Once again, at a 1% level of significance, we cannot reject the hypothesis of each time series. Therefore, they are all stationary.

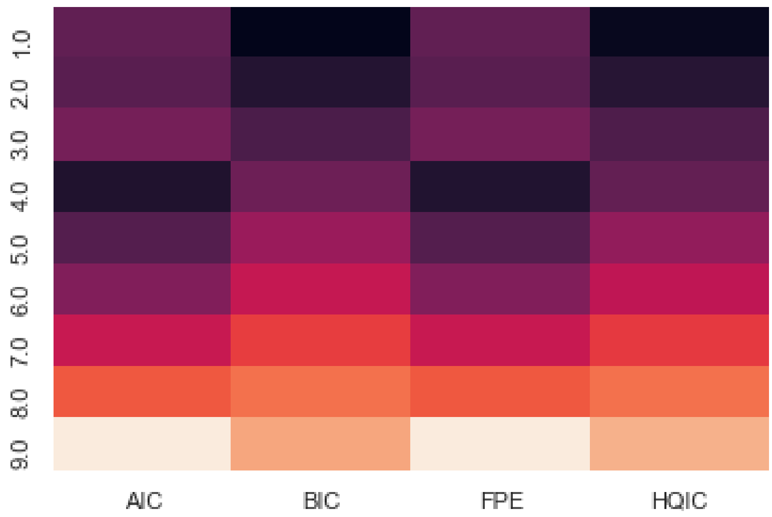
Q4.3 In order to choose the optimal number of lags, we calculated various Information Criteria which help with model selection. We tried AR processes with lags spanning from 1 to 9 and show the results in the following table:

Table 5: Model Selection Criterion

Lags	AIC	BIC	FPE	HQIC
1	-31.378953	-31.326620	2.356643e-14	-31.359817
2	-31.379571	-31.285335	2.355189e-14	-31.345113
3	-31.377540	-31.241370	2.359977e-14	-31.327747
4	-31.383963	-31.205827	2.344868e-14	-31.318824
5	-31.379930	-31.159796	2.354348e-14	-31.299431
6	-31.376748	-31.114584	2.361855e-14	-31.280877
7	-31.371851	-31.067625	2.373456e-14	-31.260595
8	-31.367059	-31.020739	2.384864e-14	-31.240407
9	-31.356703	-30.968257	2.409698e-14	-31.214642

For better visualization, we plotted this table with a heatmap.

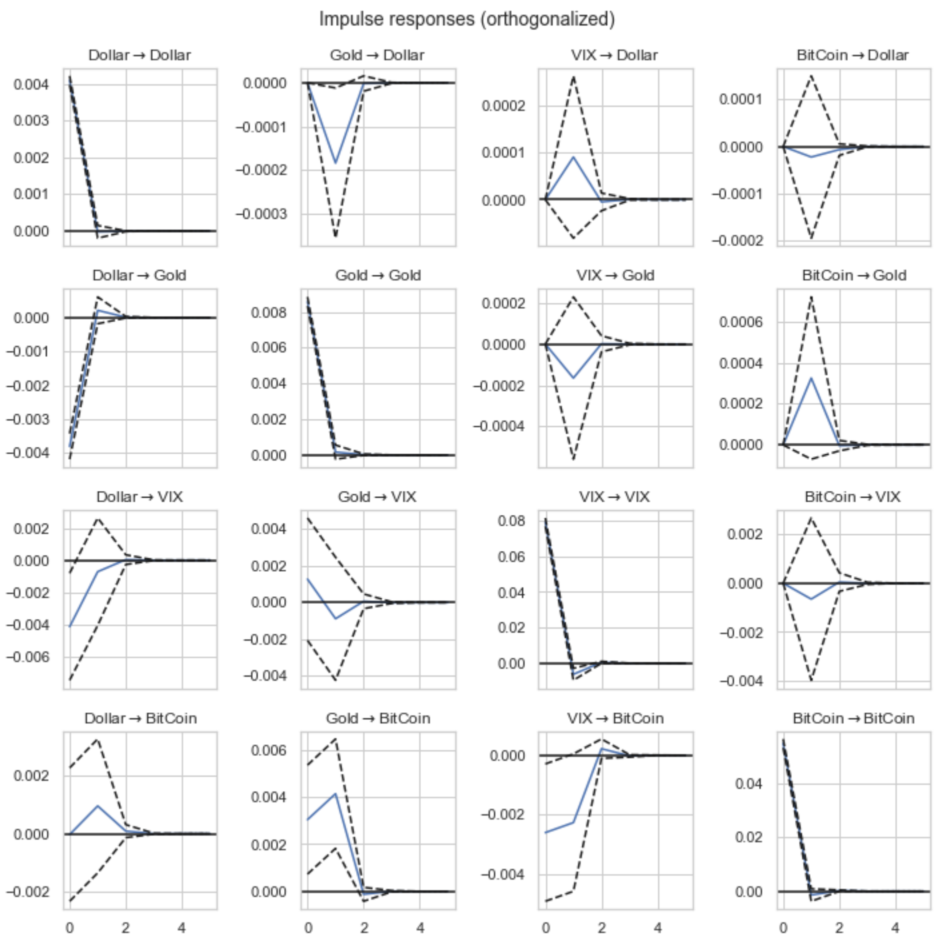
Figure 4: Model Selection Criterion (Heatmap)



Therefore, the lower the value with a given information criteria, the better the model. In this case the darkest spots represent the best performing models. Since the HQIC is supposedly the most widely used, notably because it is the one that usually leads to the most parsimonious model, we decide to follow its advice and settle for 1 lag.

Now that we selected a VAR model with one lag, we will also assess the impact of one standard deviation shock of the Dollar, Gold and the VIX on Bitcoin. To do so, we determined the Impulse Response Function (IRF):

Figure 5: IRFs of Each Time Series



The results which are of our interest are the bottom graphs.

- **Dollar - Bitcoin:** We notice that there is no contemporaneous response of Bitcoin to a change in the Dollar. Nevertheless, the crypto-currency is positively impacted on period 1, but the shock in the next period is almost fully vanished. Nevertheless, we cannot ensure with certainty that the impact of USD on BTC is positive, nor negative, as the confidence interval at 95% is fairly large, indicative that the impact can be statistically be either positive or negative.
- **Gold - Bitcoin:** This time, we notice that Bitcoin is positively and contemporaneously related to Gold, as it is impacted on period 0. This is no surprise as both Gold and BTC are quoted in USD, which might reflect the positive relationship. (Additionally, we have witnessed during periods of financial distress that both assets are being perceived as safe assets). Nevertheless, the impact is even stronger in the next period, indicating some non-simultaneous relationship. The impact is almost vanished in period 2. Although the confidence interval at 95% is fairly large, we can conclude that it is likely that the impact of Gold on BTCUSD is positive.

- **VIX - Bitcoin:** Bitcoin is negatively contemporaneously related to VIX, as the cryptocurrency is negatively impacted on period 0. This might reflect the positive correlation between the asset and the overall market, as its price plummets when the market is tanking (i.e. the VIX increases). On period 2 the shock is almost vanished. Again, although the confidence interval at 95% is fairly large, we can conclude that it is likely that the impact of Gold on BTCUSD is negative.

Q4.4 Now we will consider two VAR models: an unconstrained version and a constrained one so that Bitcoin does not influence the other assets dynamically. Let $p_{1,t}, p_{2,t}, p_{3,t}, p_{4,t}$ be the log price of US Dollar, Gold, VIX and Bitcoin respectively. Let's first define the **unconstrained** VAR(1) as being:

$$\begin{pmatrix} p_{1,t} \\ p_{2,t} \\ p_{3,t} \\ p_{4,t} \end{pmatrix} = \begin{pmatrix} \phi_{1,1} & \phi_{1,2} & \phi_{1,3} & \phi_{1,4} \\ \phi_{2,1} & \phi_{2,2} & \phi_{2,3} & \phi_{2,4} \\ \phi_{3,1} & \phi_{3,2} & \phi_{3,3} & \phi_{3,4} \\ \phi_{4,1} & \phi_{4,2} & \phi_{4,3} & \phi_{4,4} \end{pmatrix} \begin{pmatrix} p_{1,t-1} \\ p_{2,t-1} \\ p_{3,t-1} \\ p_{4,t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \\ \varepsilon_{4,t} \end{pmatrix}, \varepsilon_t \sim N(0, \sigma^2) \tag{17}$$

Then we define the **constrained** VAR(1), where Bitcoin does not influence the other assets dynamically, as being:

$$\begin{pmatrix} p_{1,t} \\ p_{2,t} \\ p_{3,t} \\ p_{4,t} \end{pmatrix} = \begin{pmatrix} \phi_{1,1} & \phi_{1,2} & \phi_{1,3} & 0 \\ \phi_{2,1} & \phi_{2,2} & \phi_{2,3} & 0 \\ \phi_{3,1} & \phi_{3,2} & \phi_{3,3} & 0 \\ \phi_{4,1} & \phi_{4,2} & \phi_{4,3} & \phi_{4,4} \end{pmatrix} \begin{pmatrix} p_{1,t-1} \\ p_{2,t-1} \\ p_{3,t-1} \\ p_{4,t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \\ \varepsilon_{4,t} \end{pmatrix}, \varepsilon_t \sim N(0, \sigma^2) \tag{18}$$

To estimate both models, we will use the Maximum Likelihood, as it is impossible to estimate the constrainted VAR model with an OLS regression since the parameters are correlated (implying biased estimations). Therefore, we define the log-likelihood function as the sum of the log-densities for a random variable X_t as being:

$$logL = \sum_{t=1}^{n} logf(X_t) \tag{19}$$

If $X_t = [x_1, x_2, ..., x_n]$ follows a multivariate Gaussian random variable with expectation Γ and a covariance Σ, then its probability density function is defined as:

$$f(X_t) = \frac{1}{(2\pi)^{k/2}|\Sigma|^{1/2}} e^{-0.5(X_t-\Gamma)^T\Sigma^{-1}(X_t-\Gamma)} \tag{20}$$

Therefore, to estimate the parameters of our model, we minimized the **negative** log-likelihood (which is equivalent to maximizing the positive log-likelihood), using a Sequential Least Squares Programming (SLSQP) optimization method.

Consequently, we obtained the following parameters estimations for the unconstrained VAR(1) model:

$$\begin{pmatrix} p_{1,t} \\ p_{2,t} \\ p_{3,t} \\ p_{4,t} \end{pmatrix} =$$
$$\begin{pmatrix} -0.0234849320 & 0.0650891208 & -0.335305441 & 0.665530056 \\ -0.0208382181 & 0.0171825492 & -0.0868157229 & 0.498160964 \\ 0.00116980840 & -0.00199791088 & -0.0788028179 & -0.0294027965 \\ -0.000362084167 & 0.00588829339 & -0.0118159345 & -0.0261247902 \end{pmatrix} \begin{pmatrix} p_{1,t-1} \\ p_{2,t-1} \\ p_{3,t-1} \\ p_{4,t-1} \end{pmatrix}$$
$$+ \begin{pmatrix} 0.0000645043017 \\ -0.00000490846590 \\ 0.0000402760135 \\ 0.00386981765 \end{pmatrix}$$

And the estimation of the constrained VAR(1) model is:

$$\begin{pmatrix} p_{1,t} \\ p_{2,t} \\ p_{3,t} \\ p_{4,t} \end{pmatrix} =$$
$$\begin{pmatrix} -0.0252121463 & 0.0655078865 & -0.335394951 & 0 \\ -0.0213856010 & 0.0167010759 & -0.0872798465 & 0 \\ 0.00112668564 & -0.00190392231 & -0.0788313753 & 0 \\ -0.000443948039 & 0.00595827609 & -0.0118197038 & -0.0201218854 \end{pmatrix} \begin{pmatrix} p_{1,t-1} \\ p_{2,t-1} \\ p_{3,t-1} \\ p_{4,t-1} \end{pmatrix}$$
$$+ \begin{pmatrix} 0.0000631484849 \\ -0.00000270785989 \\ 0.0000412284062 \\ 0.00390311647 \end{pmatrix}$$

As both models are estimated by MLE, we run a Likelihood Ratio (LR) test to compare them. Let's denote $log\mathcal{L}_U$ (resp. $log\mathcal{L}_R$) the maximum likelihood resulting from the maximization of the unrestricted log-likelihood (resp. the restricted log-likelihood). From our optimization, we obtained:

$$log\mathcal{L}_U = -7993.956 > -7993.978 = log\mathcal{L}_R \tag{21}$$

The null hypothesis of the LR test is that the "true" model is the restricted one. In other words, if H_0 is true, one expects that $log\mathcal{L}_U$ is not too different from $log\mathcal{L}_R$. Assume that J is the number of constraints set in the constrained model. Asymptotically, we have:

$$2(log\mathcal{L}_U - log\mathcal{L}_R) \xrightarrow{\text{d}} \chi^2(J) \tag{22}$$

As we obtained a p-value of 0.996 with our models, where $J = 3$, we cannot reject the null hypothesis that the constrained are un-constrained models are different.

Therefore, by not rejecting the null hypothesis, this would suggest that the true model is indeed restricted. This result would suggest that, statistically, the past price of Bitcoin does not have any impact on the current prices of the other assets we have available. This result might have a potentially positive impact in an investment portfolio. Indeed, as the past price of Bitcoin does not have any effect on the current price of the other assets, it might generate diversification benefits: hedging against adverse price movements.

4. Conclusion

This assignment consists in two parts. In the first part on pair trading, we investigate the stationarity and cointegration of 3 different crypto-currencies, Bitcoin, Monero and Ethereum.
We test whether the time series are second-order stationary using the Dickey-Fuller test. Since under the null hypothesis of non-stationarity of the time series the test-statistic follows a non-standard distribution, we had to simulate its distribution with Monte-Carlo technique.
Then, we were able to extract the critical values of the simulated distribution, and finally to test the stationarity of the different time series, that is the log of price of the aforementioned crypto-currencies. The result of the test allows us to conclude that all time series weren't second order stationary (as expected from the chart, since (log)prices tend to have positive trend over the time). Starting from this results, we were able to investigate about cointegration of these time series.
Testing for cointegration for each pairs of currencies leads us to conclude that the only cointegrated pair was the pair Monero-Ethereum. Cointegration is particularly important in finance, since it allows to design a well-known trading strategy called "Pair-trading" that uses the mean-reversion property in the difference between both time series.
From our analysis, we can see that pair-trading is quite a profitable strategy, at least over the period over which we test it.
The second part of this project deals with VAR models. We investigate the property of the log of four time series (US Dollar, Gold, VIX index and Bitcoin). Once again after concluding non-stationarity of all four time series, we were able to transform them into stationary time series by taking the first difference for each of them in order to be able to estimate a VAR model.
The VAR(1,4) model, that is the lag-1 VAR comprising the four time series allows us to investigate the relation between these four time series. In particular, using impulse response functions, we analysed the impact of a 1 standard deviation shock of the Dollar, Gold and VIX on Bitcoin.
We have seen that the shock of the Dollar has no statistically significant impact on Bitcoin, that the shock on the Gold has a statistically significant positive impact on the Bitcoin (at least one period ahead the shock) and that the shock of VIX impacts negatively the Bitcoin one period after the shock.
Finally, we test whether a constrained version of the previous VAR model, in which Bitcoin has no impact on the other time series, explains the relations between time series better compared to the previous one. We strongly cannot reject the null according to which the restricted model is the "True" one. This result suggests that Bitcoin has no statistically significant effect on the Dollar, the Gold and the VIX, meaning that the

Bitcoin is somehow uncorrelated with the other asset classes. This has important properties in terms of asset allocation, in the sense that Bitcoin could be used to increase the diversification effect in our portfolio, leading to reducing the overall portfolio's risk.