- Properties of multi. normal dist

1) Normality of linear combinations of the components of $\underline{X}' = (X_1, \dots, X_p)$.

Let $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$.

$$\underline{a}'\underline{X} = a_1 X_1 + \dots + a_p X_p$$

Then $\underline{a}'\underline{X}$ is $N_1(\underbrace{\underline{a}'\underline{\mu}}_{\text{scalar}}, \underbrace{\underline{a}'\Sigma\underline{a}}_{\text{scalar}})$.

2) If $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$, then each $X_i \sim N_1(\mu_i, \sigma_{ii})$.

pf: $\underline{X} = (X_1, \dots, X_p)$. Let $\underline{a}' = (1, 0, \dots, 0)$

Then $\underline{a}'\underline{X} = X_1 \sim N_1(\underline{a}'\underline{\mu}, \underline{a}'\Sigma\underline{a})$.

Here $\underline{a}'\underline{\mu} = (1, 0, \dots, 0)\begin{bmatrix} \mu_1 \\ \vdots \\ \mu_p \end{bmatrix} = \mu_1$

$$\underline{a}'\Sigma\underline{a} = (1, \dots, 0)\begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \vdots & & & \\ \sigma_{1p} & \sigma_{2p} & \cdots & \sigma_{pp} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \sigma_{11}$$

Thus, $\underline{a}'\underline{X} = X_1 \sim N_1(\mu_1, \sigma_{11})$.

In a similar fashion, we can prove each $X_i \sim N_1(\mu_i, \sigma_{ii})$.

That is, the marginal dist of each X_i is normal.

3) If $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$ and $\underset{q \times p}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ \vdots & & & \\ a_{q1} & a_{q2} & \cdots & a_{qp} \end{bmatrix}$

where $q \le p$, then q linear combinations

$$A\underline{X} = \begin{bmatrix} a_{11}X_1 + \cdots + a_{1p}X_p \\ \vdots \\ a_{q1}X_1 + \cdots + a_{qp}X_p \end{bmatrix} \sim N_q(A\underline{\mu}, A\Sigma A')$$

4) Let $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$ and $\underline{d} = \begin{bmatrix} d_1 \\ \vdots \\ d_p \end{bmatrix}$ be a vector of constants.

Then, $\underline{X} + \underline{d} \sim N_p(\underline{\mu} + \underline{d}, \Sigma)$.

Example: Let $\underline{X} \sim N_3(\underline{\mu}, \Sigma)$ when $\underline{\mu} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$,

$\Sigma = \begin{bmatrix} 6 & 1 & -2 \\ 1 & 13 & 4 \\ -2 & 4 & 4 \end{bmatrix}$. Find the joint dist of $Y_1 = \sum X_i$

and $Y_2 = X_1 - X_2 + 2X_3$.

Sol^n: We have 2 lin. combi. Y_1 and Y_2.

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} X_1 + X_2 + X_3 \\ X_1 - X_2 + 2X_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = A\underline{X}$$

By property 3), $A\underline{X} \sim N_2(A\underline{\mu}, A\Sigma A')$

$$A\underline{\mu} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \end{bmatrix}$$

$$A\Sigma A' = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 6 & 1 & -2 \\ 1 & 13 & 4 \\ -2 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 29 & -1 \\ -1 & 9 \end{bmatrix}$$

check: $Var(Y_1) = Var(X_1 + X_2 + X_3) = \sigma_{11} + \sigma_{22} + \sigma_{33} + 2(\sigma_{12} + \sigma_{13} + \sigma_{23})$

$= 6 + 13 + 4 + 2 - 4 + 8 = 29$

$cov(Y_1, Y_2) = cov(X_1 + X_2 + X_3, X_1 - X_2 + 2X_3)$

$= 6 - 1 - 4 + 1 - 13 + 8 - 2 - 4 + 8 = -1$

5) All subsets of $\underline{X}$ are normally distributed. If we respectively partition $\underline{X}$, its mean vector μ and its covariance matrix Σ as

$$\underset{p\times 1}{\underline{X}} = \begin{bmatrix} \underline{X}_1 \quad q\times 1 \\ \underline{X}_2 \quad (p-q)\times 1 \end{bmatrix}, \quad \underline{\mu} = \begin{bmatrix} \underline{\mu}_1 \\ \underline{\mu}_2 \end{bmatrix} \quad \text{and } \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

where $\Sigma_{11} : q\times q$, $\Sigma_{22} : (p-q)\times(p-q)$,

$\Sigma_{12} : q \times (p-q)$, $\Sigma_{21} : (p-q)\times q$, $\Sigma_{12}' = \Sigma_{21}$

Then, the subset $\underline{X}_1 \sim N_q(\underline{\mu}_1, \Sigma_{11})$

$\underline{X}_2 \sim N_{(p-q)}(\underline{\mu}_2, \Sigma_{22})$.

Example: Let $\underline{X} \sim N_6(\underline{\mu}, \Sigma)$ and $\underline{X}_1 = \begin{bmatrix} X_1 \\ X_2 \\ X_4 \end{bmatrix}$

Then, $\underline{X}_1 \sim N_3(\underline{\mu}_1, \Sigma_{11})$ where

$$\underline{\mu}_1 = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_4 \end{bmatrix}, \quad \Sigma_{11} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{14} \\ \sigma_{12} & \sigma_{22} & \sigma_{24} \\ \sigma_{14} & \sigma_{24} & \sigma_{44} \end{bmatrix}$$

- Results on statistical independence and zero correlation for normal random variables

6) If $\underset{q_1 \times 1}{\underline{X_1}}$ and $\underset{q_2 \times 1}{\underline{X_2}}$ are independent, then

$Cov(\underline{X_1}, \underline{X_2}) = 0_{q_1 \times q_2}$, $Cov(\underline{X_2}, \underline{X_1}) = 0_{q_2 \times q_1}$

7) If $\begin{bmatrix} \underline{X_1} \\ \hdashline \underline{X_2} \end{bmatrix}$ is $N_{q_1+q_2}\left(\begin{bmatrix} \underline{\mu_1} \\ \hdashline \underline{\mu_2} \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \right)$

Then $\underline{X_1}$ and $\underline{X_2}$ are independent <u>iff</u>.

$\Sigma_{12} = 0_{q_1 \times q_2}$ or $\Sigma_{21} = 0_{q_2 \times q_1}$

In general, corr = 0 does not imply independence for non-normal data.

8) If $\underline{X_1} \sim N_{q_1}(\underline{\mu_1}, \Sigma_{11})$, $\underline{X_2} \sim N_{q_2}(\underline{\mu_2}, \Sigma_{22})$ and $\underline{X_1}$ and $\underline{X_2}$ are independent, then

$\begin{bmatrix} X_1 \\ \hdashline X_2 \end{bmatrix} \sim N_{q_1+q_2}\left(\begin{bmatrix} \mu_1 \\ \hdashline \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{bmatrix} \right)$

joint dist.

- A random vector $\underline{X_1}$ is independent of $\underline{X_2}$ meaning each element in $\underline{X_1}$ is ind of each element in $\underline{X_2}$.

$\underline{X_1} = \begin{bmatrix} x_1^1 \\ x_2^1 \\ \vdots \\ x_{q_1}^1 \end{bmatrix}$, $\underline{X_2} = \begin{bmatrix} x_1^2 \\ x_2^2 \\ \vdots \\ x_{q_2}^2 \end{bmatrix}$ $cov(x_i^1, x_j^2) = 0$ for all i, j.

$i = 1, 2, \ldots q_1$

$j = 1, 2, \ldots q_2$

Exercise 4.3

$\underline{X} \sim N_3(\underline{\mu}, \Sigma)$, $\underline{\mu}' = [-3, 1, 4]$, $\Sigma = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Which of the following variables are ind?

a) X_1 and X_2

For normal random variables, $\text{Cov} = 0 \Rightarrow$ ind.

$\sigma_{12} = \sigma_{21} = -2$ no.

b) $\sigma_{23} = 0$ yes

c) Cov bet (X_1, X_2) (2) and X_3 (1)

$\text{Cov} = \begin{bmatrix} \sigma_{13} \\ \sigma_{23} \end{bmatrix}_{2\times 1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ yes

d) $\text{Cov}\left[\frac{X_1 + X_2}{2}, X_3\right] = \frac{1}{2}\text{Cov}(X_1, X_3) + \frac{1}{2}\text{Cov}(X_2, X_3)$

$= 0$ yes

e) $\text{Cov}\left[X_2, X_2 - \frac{5}{2}X_1 - X_3\right]$

$= \text{Cov}(X_2, X_2) - \frac{5}{2}\text{Cov}(X_2, X_1) - \text{Cov}(X_2, X_3)$

$= 5 - \frac{5}{\cancel{2}}(-\cancel{2}) - 0 = 5 + 5 = 10$ no

Alternately:

$A = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{5}{2} & 1 & -1 \end{bmatrix}$

$\text{Cov}(X_2, X_2 - \frac{5}{2}X_1 - X_3) = A\Sigma A' = \underset{2\times 3}{\begin{bmatrix} 0 & 1 & 0 \\ -\frac{5}{2} & 1 & -1 \end{bmatrix}} \underset{3\times 3}{\begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 0 & -\frac{5}{2} \\ 1 & 1 \\ 0 & -1 \end{bmatrix}}$

$= \begin{bmatrix} 5 & 10 \\ 10 & \frac{93}{4} \end{bmatrix}$ $\text{Cov}(X_2, X_2 - \frac{5}{2}X_1 - X_3)$

Properties of Multi-variate normal dist (continued...)

9. Standardized Variables

Univariate case: $X \sim N(\mu, \sigma^2)$

$$Z = \frac{X-\mu}{\sigma} \sim N(0,1)$$

Multi case: with $|\Sigma| > 0$.

Let $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$

$$\underset{p\times 1}{\underline{Z}} = \Sigma^{-\frac{1}{2}}(\underline{X}-\underline{\mu})$$

$$\mathrm{Cov}(\underline{Z}) = \Sigma^{-\frac{1}{2}} \Sigma \Sigma^{-\frac{1}{2}} = \Sigma^{-\frac{1}{2}}\Sigma^{\frac{1}{2}}\Sigma^{\frac{1}{2}}\Sigma^{-\frac{1}{2}} = I \cdot I = I$$

$\Sigma^{\frac{1}{2}}$ is the symmetric, square root matrix of Σ defined as

$$\Sigma^{\frac{1}{2}} = \sum_{i=1}^{p} \sqrt{\lambda_i}\, \underline{e}_i \underline{e}_i' = P \Lambda^{\frac{1}{2}} P' \xleftarrow{\text{by}} \text{spectral decomposition}$$

$$\Sigma^{-\frac{1}{2}} = \sum_{i=1}^{p} \frac{1}{\sqrt{\lambda_i}} \underline{e}_i \underline{e}_i' = P \Lambda^{-\frac{1}{2}} P' \xleftarrow{\text{by}} \text{"}$$

λ_i are eigenvalues of Σ, $\lambda_i > 0$.

P is the orthogonal matrix with columns $\underline{e}_i$

Here, $\underline{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_p \end{bmatrix}$ is a vector of standardized normal random variables with $E[Z_i] = 0$, $\mathrm{Var}(Z_i) = 1$ and all covariances $= 0$. Thus, $\underline{Z} = \Sigma^{-\frac{1}{2}}(\underline{X}-\underline{\mu}) \sim N_p(\underline{0}, I_p)$.

Each $Z_i = \dfrac{X_i - \mu_i}{\sqrt{\sigma_{ii}}} \sim N(0,1)$; Z_i's are independent

$E[\underline{Z}] = E[A(\underline{X}-\underline{\mu})] = A\,E(\underline{X}-\underline{\mu}) = A.\underline{0} = \underline{0}$

$\mathrm{Cov}(\underline{Z}) = \mathrm{Cov}(A(\underline{X}-\underline{\mu})) = A\,\mathrm{Cov}(\underline{X}-\underline{\mu})A' = A\Sigma A' = \Sigma^{-\frac{1}{2}} \Sigma \Sigma^{-\frac{1}{2}} = I$

10. If $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$, then $(\underline{X}-\underline{\mu})' \Sigma^{-1} (\underline{X}-\underline{\mu}) \sim \chi^2_p$.

Proof: $(\underline{X}-\underline{\mu})' \Sigma^{-1} (\underline{X}-\underline{\mu})$

$$= (\underline{X}-\underline{\mu})' \Sigma^{-\frac{1}{2}} \cdot \Sigma^{-\frac{1}{2}} (\underline{X}-\underline{\mu})$$

$$= \left[\Sigma^{-\frac{1}{2}} (\underline{X}-\underline{\mu})\right]' \cdot \left[\Sigma^{-\frac{1}{2}} (\underline{X}-\underline{\mu})\right] \quad \left[\begin{array}{l} (AB)' = B'A' \\ (\Sigma^{-\frac{1}{2}})' = \Sigma^{-\frac{1}{2}} \end{array}\right.$$

$$= \underline{Z}' \underline{Z} = [Z_1, \cdots Z_p] \begin{bmatrix} Z_1 \\ \vdots \\ Z_p \end{bmatrix} = \sum_{i=1}^{p} Z_i^2,$$

which is the sum of squares of p independent standard normal random variables $Z_i = \frac{X_i - \mu_i}{\sqrt{\sigma_{ii}}}$

Then, $\sum_{i=1}^{p} Z_i^2 \sim \chi^2_p$.

11. Conditional Distribution:

Let $\underline{X} = \begin{bmatrix} \underline{X}_1 \\ \hdashline \underline{X}_2 \end{bmatrix}$ be distributed as $N_p(\underline{\mu}, \Sigma)$

with $\underline{\mu} = \begin{bmatrix} \underline{\mu}_1 \\ \hdashline \underline{\mu}_2 \end{bmatrix}$, $\Sigma = \left[\begin{array}{c:c} \Sigma_{11} & \Sigma_{12} \\ \hdashline \Sigma_{21} & \Sigma_{22} \end{array}\right]$, and $|\Sigma_{22}| > 0$

Then, the conditional distribution of $\underline{X}_1$, given $\underline{X}_2 = \underline{x}_2$ is normal with

$$\text{Mean} = \underline{\mu}_1 + \Sigma_{12} \Sigma_{22}^{-1} (\underline{x}_2 - \underline{\mu}_2), \text{ and}$$

$$\text{Covariance matrix} = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}$$