- Convergence of sequences of r.v.s
- Limit Theorems

Sequences of R.V.s, Examples

Let $X_1, X_2, ..., X_n, ...$ be a sequence of r.v. on the same Probability Space.

Examples:

- We toss a coin, infinitely many times, and let X_k be 0, it the k-th toss resulted in Heads, and $X_k = 1$ otherwise.
- Let X_k be the Closing price for day k calculated from today for the AMZN Stock.
- Let X_k be the height (in cm) of the k-th person I will meet tomorrow.
- Let X_k be the number of downloads for the Supper-Pupper inc. mobile app for the day k.
- Let X_k be the blood pressure for the patient k for some clinic.

I know, almost all examples are examples of *finite* sequences, but for theoretical (and practical) reasons we can assume they are infinite.

Sequences of R.V.s, Motivation

In the rest of the course we will work a lot with sequences of r.v.s. We will construct Statistics for the unknown parameters of Distribution, something to estimate that parameters.

And one of the important questions will be: is our Statistic good enough to estimate the parameter? The point is that since the parameter value is unknown, we need to have some theoretical guarantees that our estimators are working well.

So we will use different notions of r.v. sequence convergence to assess the quality of our estimator, Statistics.

Also, we will use a lot the CLT, say, to construct Confidence Intervals and design Tests for Hypotheses, so we need to know the CLT, and CLT is about the limit of a r.v. sequence.

Convergence of a Sequence of r.v.

There are different notions of a convergence for a r.v. sequence.

This is because, a sequence of r.v., besides being just a sequence of functions[1], also encloses randomness behind, and we need to deal with that randomness.

Say, what it means for r.v.s X and Y that X is close to Y?

Aha, that's the problem - it is not so easy to define the closedness $\ddot\smile$

[1] And we have different notions for the convergence of functional sequences like pointwise, uniform, a.e., L^p, ... convergences

Convergence a.s.

Assume X_n is a sequence of r.v. and X is a r.v. over the same Probability Space.

Definition: We will say that $X_n \to X$ **almost sure**, and we will write $X_n \to X$ a.s. or $X_n \xrightarrow{a.s.} X$, if

$$\mathbb{P}\Big(\omega \in \Omega : \lim_{n \to +\infty} X_n(\omega) = X(\omega)\Big) = 1,$$

or, for short,

$$\mathbb{P}\Big(X_n \to X\Big) = 1$$

Equivalently, we can write

$$X_n \xrightarrow{a.s.} X \qquad \text{iff} \qquad \mathbb{P}\Big(X_n \not\to X\Big) = 0.$$

Convergence in Probability

Definition: We will say that $X_n \to X$ **in Probability**, and we will write $X_n \xrightarrow{\mathbb{P}} X$, if

$$\text{for any } \varepsilon > 0,\ \mathbb{P}\Big(|X_n - X| \geq \varepsilon\Big) \to 0, \qquad \text{when} \quad n \to \infty.$$

Equivalently, we can write

$$X_n \xrightarrow{\mathbb{P}} X \qquad \text{iff} \qquad \mathbb{P}\Big(|X_n - X| < \varepsilon\Big) \to 1 \text{ for any } \varepsilon > 0.$$

Convergence in the Mean Square Sence

Definition: We will say that $X_n \to X$ **in the Quadratic Mean Sense or in L^2 (or in the Mean Square Sense)**, and we will write $X_n \xrightarrow{L^2} X$ or $X_n \xrightarrow{qm} X$, if

$$MSE(X_n, X) = \mathbb{E}\Big((X_n - X)^2\Big) \to 0, \qquad \text{when} \quad n \to \infty.$$

Here $MSE(X_n, X)$ is the *Mean Square Error* (of the approximation of X by X_n).

Convergence in Distributions

Now we assume that X_n and X are arbitrary r.v.'s, not necessarily defined on the same probability space, and $F_{X_n}(x)$ and $F_X(x)$ are their CDF's, respectively.

Definition: We will say that $X_n \to X$ **in Distribution (or in Law)**, and we will write $X_n \xrightarrow{D} X$, if

$$F_{X_n}(x) \to F_X(x) \quad \text{as} \quad n \to \infty \text{ at any point of continuity } x \text{ of } F_X(x).$$

Remark: This is equivalent to saying that for (almost) any subsets $A \subset \mathbb{R}$

$$\mathbb{P}(X_n \in A) \to \mathbb{P}(X \in A).$$

Remark on the notation: Usually, in the case of the Convergence in Distribution, we write the Distribution as the limit, e.g., we write

$$X_n \xrightarrow{D} \mathcal{N}(0,1)$$

instead of writing $X_n \xrightarrow{D} X$, $X \in \mathcal{N}(0,1)$.

Cauchy Principle for a.e, $\mathbb{P}$ and L^2 Convergence

Now, for checking the convergence of a sequence of r.v. X_n, we can use the following Theorem (Cauchy Principle):

Theorem:

- If $X_n - X_m \to 0$ a.e. when $m, n \to +\infty$, then there exists a r.v. X such that $X_n \to X$ a.e.;
- If for any $\varepsilon > 0$, $\mathbb{P}(|X_n - X_m| \geq \varepsilon) \to 0$ when $m, n \to +\infty$, then there exists a r.v. X such that $X_n \xrightarrow{\mathbb{P}} X$;
- If $\mathbb{E}\Big((X_n - X_m)^2\Big) \to 0$ when $m, n \to +\infty$, then there exists a r.v. X such that $X_n \xrightarrow{L^2} X$.

Example

Example: We have a sequence of infinitely many (independent) tosses of a fair coin, and let X_n be the result of the n-th trial ($Head = 1,\ Tail = 0$). So the Distribution of X_n is

$$X_n \sim Bernoulli(0.5).$$

- Is X_n convergent in the sense of Distributions ?
- Is X_n convergent in the Probability sense ?
- Is X_n convergent in the MS sense ?
- Is X_n convergent in the a.s. sense?