- Convergence of sequences of r.v.s
- Limit Theorems

Example: Assume X_n is a Discrete r.v. with the following PMF, defined on the same Probability Space:

X_n	$3 + \frac{1}{n^2}$	n
$\mathbb{P}(X_n = x)$	$1 - \dfrac{1}{n}$	$\dfrac{1}{n}$.

Which of the followings are true (use only the definitions):

- $X_n \xrightarrow{\mathbb{P}} 3$;
- $X_n \xrightarrow{qm} 3$;
- $X_n \xrightarrow{D} 3$?

Example: Assume

$$X_n \sim Unif\left[0, \frac{1}{n}\right]$$

and X_n are defined on the same Probability Space. Which of the followings are true (use only the definitions):

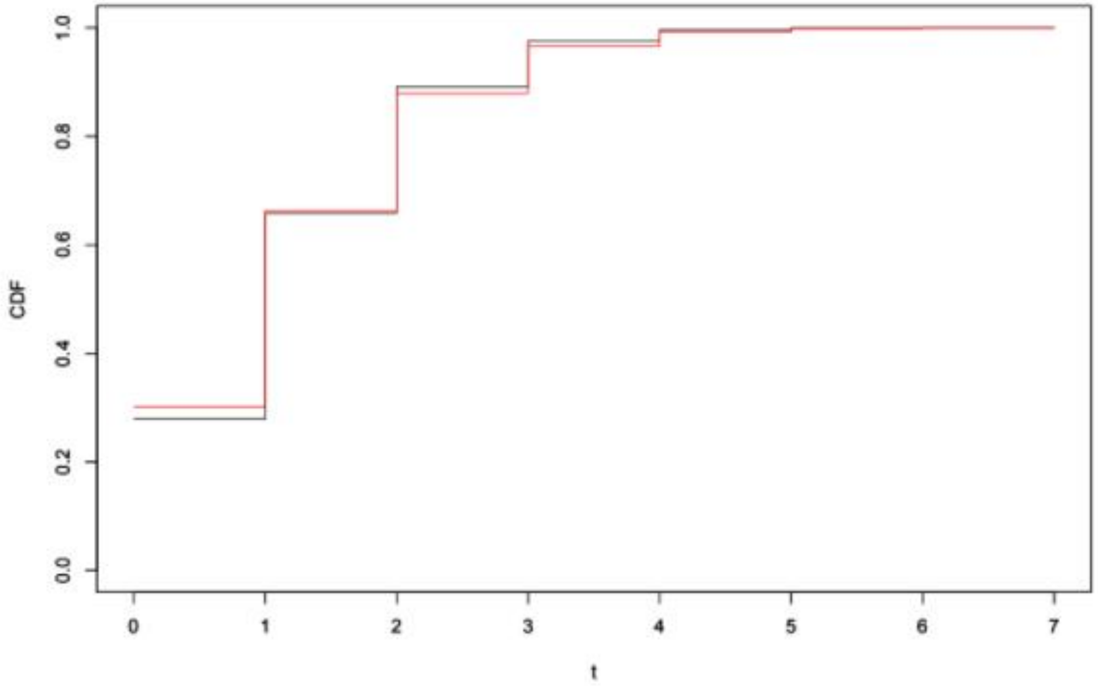
- $X_n \xrightarrow{\mathbb{P}} 0$;
- $X_n \xrightarrow{qm} 0$;
- $X_n \xrightarrow{D} 0$?

Example

Example: Show that if $X_n \sim Binom\left(n, \frac{\lambda}{n}\right)$, then $X_n \xrightarrow{D} Pois(\lambda)$.

Note: Note that when using $X_n \xrightarrow{D} Pois(\lambda)$ we mean $X_n \xrightarrow{D} X$, where $X \sim Pois(\lambda)$.

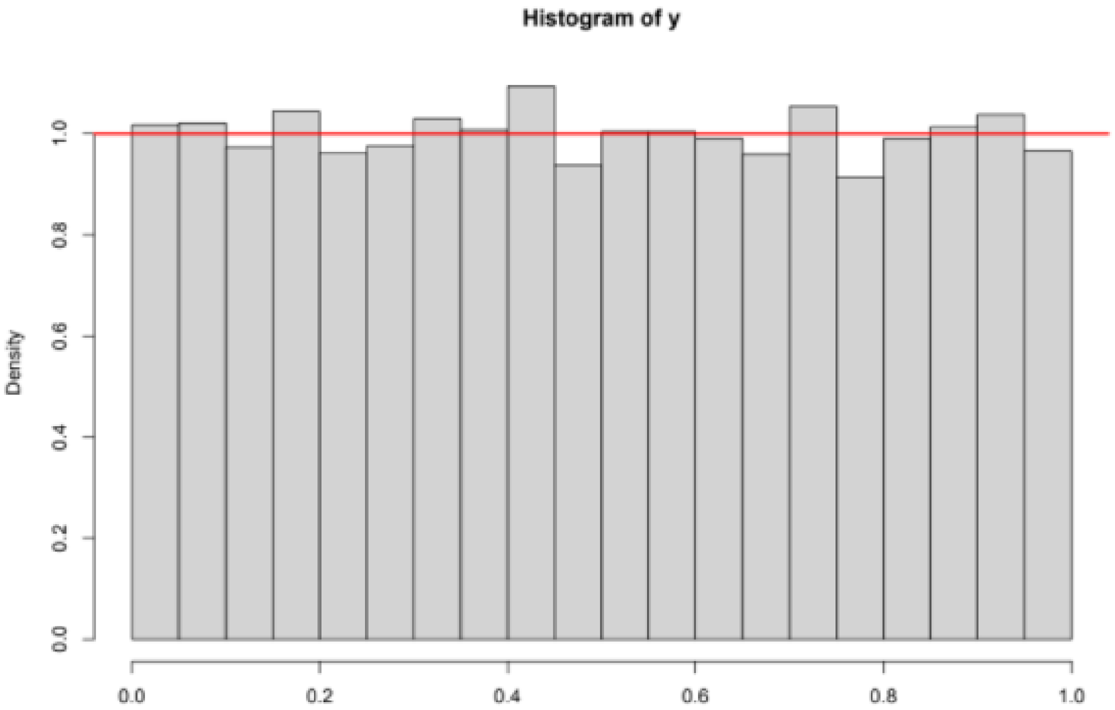
```
lambda <- 1.2; n <- 10; t <- seq(0,7, 0.1)
plot(t,pbinom(t, size = n, prob = lambda/n), type = "s", ylim = c(0,1), ylab = "CDF")
par(new = T)
plot(t, ppois(t, lambda = lambda), type = "s", col = "red", ylim = c(0,1), ylab = "CDF")
```



Example

Example: Let $X_n \sim Unif[0, n]$. Find the limit in Distributions of $Y_n = \frac{X_n}{n}$. Let us visually show that $Y_n \xrightarrow{D} Y$, where $Y \sim Unif[0, 1]$.

```
n <- 10000 ## We use Y_n
m <- 10000 ## No. of generated numbers
y <- runif(m, min = 0, max = n)/n
hist(y, freq = F)
abline(h = 1, col = "red", lwd = 2)
```



Some Properties

Theorem: Assume $X_n \xrightarrow{a.s.(P,qm,D)} X$ and $Y_n \xrightarrow{a.s.(P,qm,D)} Y$. Then

- $X_n + Y_n \xrightarrow{a.s.(P,qm)} X + Y$;
- $X_n \cdot Y_n \xrightarrow{a.s.(P)} X \cdot Y$; and $\alpha \cdot X_n \xrightarrow{qm} \alpha \cdot X$ for any constant α;
- If $g \in C(\mathbb{R})$, then $g(X_n) \xrightarrow{a.s.(P,D)} g(X)$

Note: In the general case, if $X_n \xrightarrow{D} X$ and $Y_n \xrightarrow{D} Y$, then not necessarily $X_n + Y_n \xrightarrow{D} X + Y$.

But we have:

Theorem (Slutsky) Assume $X_n \xrightarrow{D} X$ and $Y_n \xrightarrow{P} c$, where $c \in \mathbb{R}$ is a constant. Then

- $X_n + Y_n \xrightarrow{D} X + c$;
- $X_n \cdot Y_n \xrightarrow{D} c \cdot X$.

Relationship between Convergence Types

Theorem: (Convergence Relationship Diagram)

$$
\begin{tikzcd}
X_n \xrightarrow{a.s.} X \arrow[dr] & & \\
& X_n \xrightarrow{\mathbb{P}} X \arrow[r] & X_n \xrightarrow{D} X \\
X_n \xrightarrow{qm} X \arrow[ur] & &
\end{tikzcd}
$$

Note: Inverse implications are not always correct. But, say, the following holds: If $X_n \xrightarrow{D} X$ and $X \equiv \mathit{constant}$, then $X_n \xrightarrow{P} X$ (X_n and X are defined on the same Probability space).

Note

Note: Mostly, in our course, we will deal with the following type of sequences of r.v.s:

$$\overline{X}_n = \frac{X_1 + X_2 + ... + X_n}{n},$$

and to calculate the limit of this sequence $\overline{X}_1, \overline{X}_2, ..., \overline{X}_n, ...$, we will use our famous Limit Theorems: LLN and CLT.

Sequence of IID r.v.

Assume X_n is a sequence of **Independent, Identically Distributed (IID)** r.v.s. This means that:

- All X_n-s have the same Distribution. In particular, all numerical partial characteristics of X_n coincide. In particular,

$$\mathbb{E}(X_1) = \mathbb{E}(X_2) = \dots = \mathbb{E}(X_n) = \dots,$$

$$Var(X_1) = Var(X_2) = \dots = Var(X_n) = \dots.$$

We will use this many-many-many-many-... times.

- X_n-s are independent. Say, in particular,

$$Var(X_1+X_2+\dots+X_n) = Var(X_1)+Var(X_2)+\dots+Var(X_n) = n\cdot Var(X_1).$$

Idea of the LLN and CLT

Assume we have a sequence X_n of IID rvs. We want to study the behavior of either the sum

$$S_n = X_1 + X_2 + \dots + X_n$$

or the average

$$\overline{X}_n = \frac{X_1 + X_2 + \dots + X_n}{n}.$$

Note: Not an easy task to find the Distribution of S_n or $\overline{X}_n$. Even for $n = 2$. We need Convolutions!

What we know about S_n and $\overline{X}_n$

Some important known facts about S_n and $\overline{X}_n$ in the general case:

$$\mathbb{E}(S_n) = n \cdot \mathbb{E}(X_1), \qquad \mathbb{E}(\overline{X}_n) = \mathbb{E}(X_1);$$

so *the mean of the means is the mean* ⌣̈, and

$$Var(S_n) = n \cdot Var(X_1), \qquad Var(\overline{X}_n) = \frac{Var(X_1)}{n}.$$

The last property is the mathematical proof of the effectivness of "7 angam chapir, mek angam ktrir" ⌣̈

The interpretation of $\mathbb{E}(\overline{X}_n) = \mathbb{E}(X_1)$ and $Var(\overline{X}_n) = \frac{Var(X_1)}{n}$:the values of $\overline{X}_n$ are centered at $\mathbb{E}(X_1)$ and are becoming more and more concentrated around that number as n increases.