MIII-Vorlesung **Unendliche Reihen** 4.4.24

$a_1, a_2, a_3 \ldots \in \mathbb{K}$, $\mathbb{K} \in \mathbb{R}$, $\mathbb{K} = \mathbb{C}$, $a_1 + a_2 + \ldots + a_m = \sum_{k=1}^{n} a_k$

$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \ldots$

Basles Problem: $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \ldots = \frac{\pi^2}{6}$

Sei $\underbrace{\sum_{k=1}^{\infty} a_k}_{(*)}$ eine unendliche Reihe, $\underbrace{S_n = \sum_{k=1}^{n} a_k = a_1 + \ldots + a_n}_{\text{n-te Partialsumme}}$

- Die Reihe (*) heißt konvergent (besitzt einen Grenzwert), wenn $\lim_{n\to\infty} S_n = S$ existiert

  $S = \sum_{k=1}^{\infty} a_k$ (andenfalls divergent)

- Wie finden wir heraus ob die Reihe konvergent ist?
- Wie berechnet man diesen Grenzwert

Reihen die wir berechnen können:

(1) geometrische Reihe
(2) Teleskopreihen

Sei $q \in \mathbb{R}, \mathbb{C}$ mit $|q| < 1$

$\sum_{k=0}^{n} q^k = \underbrace{q^0}_{=1} + q^1 + \ldots + q^n = \frac{1 - \overbrace{q^{n+1}}^{\to 0}}{1-q}$; geom. Summe $\xrightarrow{n\to\infty} \frac{1}{1-q}$

$\Rightarrow \sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$, Beispiele $\sum_{k=0}^{\infty} \frac{1}{7^k} = \sum_{k=0}^{\infty} \left(\frac{1}{7}\right)^k = \frac{1}{1-\frac{1}{7}} = \frac{1}{\frac{6}{7}} = \frac{7}{6}$

$|\frac{1}{7}| < 1$

$$\sum_{k=0}^{\infty} \frac{3^k + 4^{k+1}}{10^k} = \sum_{k=0}^{\infty} \left(\frac{3^k}{10^k} + \frac{4^{k+1}}{10^k}\right) = \sum_{k=0}^{\infty} \frac{3^k}{10^k} + \sum_{k=0}^{\infty} \frac{4^{k+1}}{10^k} = \sum_{k=0}^{\infty} \underbrace{\left(\frac{3}{10}\right)^k}_{q=\frac{3}{10}<1} + 4 \sum_{k=0}^{\infty} \underbrace{\left(\frac{4}{10}\right)^k}_{q=\frac{2}{5}<1} \Bigg| \frac{a^k}{b^k} = \left(\frac{a}{b}\right)^k, \; 4^{k+1} = 4 \cdot 4^k$$

$$= \frac{1}{1-\frac{3}{10}} + 4 \cdot \frac{1}{1-\frac{2}{5}} = \frac{1}{\frac{7}{10}} + 4 \cdot \frac{1}{\frac{3}{5}} = \frac{10}{7} + 4 \cdot \frac{5}{3} = \frac{10}{7} + \frac{20}{3} = \frac{30+140}{21} = \frac{170}{21}$$

Bestimme $q \in \mathbb{R}$, so dass $\sum_{k=0}^{\infty} q^k = \frac{12}{11} = \frac{1}{1-q} \quad | \cdot (1-q), \cdot 11$

$12(1-q) = 11 \Leftrightarrow 12 - 12q = 11 \quad | -12$
$\Leftrightarrow -12q = -1 \Rightarrow q = \frac{1}{12}$

a) Berechne $\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k$ b) Berechne $\sum_{k=0}^{\infty} \frac{3^k + 5^k}{6^k}$ c) Bestimme q so, dass $\sum_{k=0}^{\infty} q^k = 4$

a) da $|\frac{2}{3}| < 1 \Rightarrow \frac{1}{1-\frac{2}{3}} = 3$

b) $\sum_{k=0}^{\infty} \underbrace{\left(\frac{3}{6}\right)^k}_{\frac{1}{2}<1} + \sum_{k=0}^{\infty} \underbrace{\left(\frac{5}{6}\right)^k}_{\frac{5}{6}<1} = \frac{1}{1-\frac{1}{2}} + \frac{1}{1-\frac{5}{6}} = 8$

$\frac{3}{14}$ c) $4 = \frac{1}{1-q} \quad | \cdot (1-q)$

$4 \cdot (1-q) = 1 \Leftrightarrow 4 - 4q = 1 \quad | -4$
$-4q = -3 \quad | :(-4) \Rightarrow q = \frac{3}{4}$

Das Leibnizsche Teleskopverfahren

$$\frac{A}{k} - \frac{B}{k+1} = \frac{A(k+1)-Bk}{k(k+1)} = \frac{Ak+A-Bk}{k(k+1)} = \frac{(A-B)k+A}{k(k+1)}$$

$\Rightarrow A-B=0$

$A=1 \Rightarrow B=1$

Beispiel: $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$, $a_k = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$

n-te Partialsumme $S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right) =$

$= 1 - \frac{1}{n+1}$

$$S = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \cancelto{0}{\frac{1}{n+1}}\right) = 1$$

$\sum_{k=3}^{\infty} \frac{1}{(k+2)(k+3)(k+4)}$, $a_k = \frac{1}{(k+2)(k+3)(k+4)}$ Ansatz: $\frac{1}{(k+2)(k+3)} - \frac{1}{(k+3)(k+4)} = \frac{(k+4)-(k+2)}{(k+2)(k+3)(k+4)} =$

$= \frac{2}{(k+2)(k+3)(k+4)} = 2a_k \Rightarrow a_k = \frac{1}{2}\left(\frac{1}{(k+2)(k+3)} - \frac{1}{(k+3)(k+4)}\right)$

$S_n = \sum_{k=3}^{n} \frac{1}{(k+2)(k+3)(k+4)} = \frac{1}{2}\sum_{k=3}^{n}\left(\frac{1}{(k+2)(k+3)} - \frac{1}{(k+3)(k+4)}\right) = \frac{1}{2}\left[\left(\frac{1}{5\cdot6} - \frac{1}{6\cdot7}\right) + \left(\frac{1}{6\cdot7} - \frac{1}{7\cdot8}\right) + \left(\frac{1}{7\cdot8} - \frac{1}{8\cdot9}\right) +\right.$

$\left. \ldots + \left(\frac{1}{(n+2)(n+3)} - \frac{1}{(n+3)(n+4)}\right)\right] = \frac{1}{2}\left(\frac{1}{30} - \frac{1}{(n+3)(n+4)}\right)$

$$\Rightarrow \sum_{k=3}^{\infty} \frac{1}{(k+2)(k+3)(k+4)} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{1}{30} - \cancelto{0}{\frac{1}{(n+3)(n+4)}}\right) = \frac{1}{60}$$

Beispiel $\sum_{k=2}^{\infty} \frac{1}{(k-1)(k+1)}$ $\quad \frac{1}{(k-1)k} - \frac{1}{k(k+1)} = \frac{k+1-(k-1)}{(k-1)k(k+1)} = \frac{2}{(k-1)k(k+1)} \Rightarrow a_k = \frac{1}{2}\left(\frac{1}{(k-1)k} - \frac{1}{k(k+1)}\right)$

$\Rightarrow S_n = \sum_{k=2}^{n} \frac{1}{(k-1)k(k+1)} = \frac{1}{2}\sum_{k=2}^{\infty}\left(\frac{1}{(k-1)k} - \frac{1}{k(k+1)}\right) = \frac{1}{2}\left[\left(\frac{1}{1\cdot2} - \frac{1}{2\cdot3}\right) + \ldots + \left(\frac{1}{(n-1)n} - \frac{1}{n(n+1)}\right)\right] =$

$= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{n(n+1)}\right) \Rightarrow \sum_{k=2}^{\infty} \frac{1}{(k-1)k(k+1)} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{1}{2} - \cancelto{0}{\frac{1}{n(n+1)}}\right) = \frac{1}{4}$

Aufgabe: $\sum_{k=1}^{\infty} \frac{1}{(k+5)(k+7)}$ $\quad \frac{1}{(k+5)k} - \frac{1}{k(k+7)} = \frac{k+7-(k+5)}{(k+5)(k+7)} = \frac{2}{(k+5)(k+7)} \Rightarrow \frac{1}{2}\left(\frac{1}{(k+5)k} - \frac{1}{k(k+7)}\right)$

$\Rightarrow S_n = \sum_{k=1}^{n} \frac{1}{(k+5)k(k+7)} = \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{(k+5)k} - \frac{1}{k(k+7)}\right) = \frac{1}{2}\left[\left(\frac{1}{6\cdot1} - \cancel{\frac{1}{1\cdot8}}\right) + \cancel{\left(\frac{1}{(n+5)n} - \frac{1}{n(n+7)}\right)} =\right.$

$\left. + \left(\frac{1}{7} - \cancel{\frac{1}{9}}\right) + \left(\cancel{\frac{1}{8}} - \cancel{\frac{1}{10}}\right) + \left(\cancel{\frac{1}{9}} - \cancel{\frac{1}{11}}\right) + \ldots + \left(\cancel{\frac{1}{n+4}} - \frac{1}{n+6}\right) + \left(\cancel{\frac{1}{n+5}} - \frac{1}{n+7}\right)\right]$

$S_n = \frac{1}{2}\left(\frac{1}{6} + \frac{1}{7} - \frac{1}{n+6} - \frac{1}{n+7}\right) \xrightarrow{n\to\infty} \frac{1}{2}\left(\frac{1}{6} + \frac{1}{7}\right) = \frac{1}{2}\cdot\frac{13}{42} = \frac{13}{84}$

Die harmonische Reihe $\sum_{k=1}^{\infty} \frac{1}{k}$ ist divergent

Konvergenzkriterien Quotientenkriterium $\sum_{k=1}^{\infty} a_k$

$$\left|\frac{a_{k+1}}{a_k}\right| \xrightarrow{k\to\infty} \begin{cases} <1 \Rightarrow \text{Reihe absolut konvergent} \\ >1 \Rightarrow \text{—"— divergent} \\ =1\ ? \end{cases}$$

Beispiele: $\sum_{k=0}^{\infty} \frac{3^k}{k!}$, $a_k = \frac{3^k}{k!}$ $\quad \left|\frac{a_{k+1}}{a_k}\right| = \frac{\frac{3^{k+1}}{(k+1)!}}{\frac{3^k}{k!}} = \frac{\cancel{3^{k+1}}\, k!}{\cancel{3^k}\,(k+1)!} = \frac{3}{k+1} \xrightarrow{k\to\infty} 0 < 1 \Rightarrow$ konv.

$\sum_{k=1}^{\infty} (-1)^k \frac{2^k}{k^2}$, $a_k = (-1)^k \frac{2^k}{k^2}$ $\quad \left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(-1)^{k+1}\frac{2^{k+1}}{(k+1)^2}}{(-1)^k\frac{2^k}{k^2}}\right| = \left|\frac{(-1)\cdot 2^{k+1} k^2}{2^k (k+1)^2}\right| = \frac{2k^2}{(k+1)^2} = 2\left(\frac{k}{k+1}\right)^2 \xrightarrow{k\to\infty} 2 > 1$ div.

*abs. konv. $\sum |a_k|$ konv. $\Rightarrow$ konv.

$\not\Leftarrow$

$\sum_{k=1}^{\infty} (-1)^k \frac{k!}{k^k}$ $\quad \left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(-1)^{k+1}\frac{(k+1)!}{(k+1)^{k+1}}}{(-1)^k\frac{k!}{k^k}}\right| = \frac{(k+1)!\,k^k}{(k+1)^{k+1}\,k!} = \frac{(k+1)\,k^k}{(k+1)(k+1)^k} = \left(\frac{k}{k+1}\right)^k = \frac{1}{\left(\frac{k+1}{k}\right)^k} = \frac{1}{\left(1+\frac{1}{k}\right)^k}$

$\xrightarrow{k\to\infty} \frac{1}{e} = \frac{1}{2{,}7182\ldots} < 1$

Vorlesung MIII

Untersuche auf Konvergenz a) $\sum_{k=1}^{\infty} \frac{k!}{k^4 \cdot 4^k}$ b) $\sum_{k=1}^{\infty} \frac{2^k \cdot k^6}{k!}$

4.4.24.

a) div. geg. ∞

b) konv.

a) $\left| \frac{\frac{(k+1)!}{(k+1)^4 \cdot 4^{k+1}}}{\frac{k!}{k^4 \cdot 4^k}} \right| = \left| \frac{(k+1)! \cdot k^4 \cdot \cancel{4^k}}{(k+1)^4 \cdot 4^{k+1} \cdot \cancel{k!}} \right| = \left| \frac{(k+1)}{4} \right|$

b)

<u>Leibniz-Kriterium</u>

$\sum_{k=0}^{\infty} (-1)^k a_k$, $a_k > 0$ und $a_k \downarrow 0$ (streng monoton fallend) $\Rightarrow$ Reihe konv.

Beispiele: $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k}$ konv. $\rightarrow \ln 2$

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \rightarrow \frac{\pi}{4}$$

<u>Wurzelkriterium</u> $\sum_{k=1}^{\infty} a_k \quad \sqrt[k]{|a_k|} \xrightarrow{k \to \infty} \begin{cases} <1 & \text{abs. konv.} \\ >1 & \text{div.} \\ =1 & ? \end{cases}$

Beispiel $\sum_{k=2}^{\infty} \frac{4^k}{k^k}$ $\quad \sqrt[k]{\frac{4^k}{k^k}} = \frac{\sqrt[k]{4^k}}{\sqrt[k]{k^k}} = \frac{4}{k} \rightarrow 0$

notwendiges Kriterium $\sum_{k=1}^{\infty} a_k$ konv. $\Rightarrow a_k \rightarrow 0$

$a_k \not\rightarrow 0 \Rightarrow \sum_{k=1}^{\infty} a_k$ div. $\qquad$ z.B. $\sum_{k=1}^{\infty} (-1)^k \frac{k^{\frac{1}{2}}+1}{2k+3}$ div.

<u>Majorantenkriterium</u> Sei $\sum_{k=1}^{\infty} b_k$ konv. $b_k > 0$, $|a_k| \leq b_k \Rightarrow \sum_{k=1}^{\infty} a_k$ konv.

Bsp.: $\sum_{k=1}^{\infty} \frac{\cos(\sqrt{k})}{2^k}$ a_k $\quad |a_k| = \left| \frac{\cos(\sqrt{k})}{2^k} \right| \leq \frac{1}{2k}$, $\sum_{k=1}^{\infty} \frac{1}{2k}$ konv.

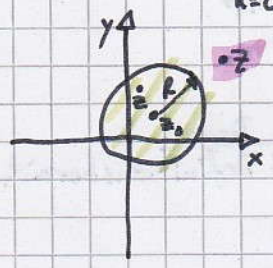
Mathe III Vorlesung

Funktionenreihen: $\mathbb{K}=\mathbb{R}$, $\mathbb{K}=\mathbb{C}$, $D\subseteq\mathbb{K}$ $f_n: D\to\mathbb{K}$ 11.04.24

$$\sum_{k=0}^{\infty} f_n(z)$$ Frage für welche $z\in D$ ist eine solche Reihe konvergent, wenn ja für welche?

Potenzreihen: $\sum_{k=0}^{\infty} a_k (z-z_0)^k$, $a_k\in\mathbb{K}$, $z_0\in\mathbb{K}$ Entwicklungspunkt der Reihe.

Keine Aussage möglich zu divergenz oder konvergenz



R Konvergenzradius
$(z_0)=\{z\in\mathbb{K} \mid |z-z_0|<R\}$
K_R Konvergenzkreis

Wie bestimmt man den Konvergenzradius? Formel von Euler $R=\lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right|$

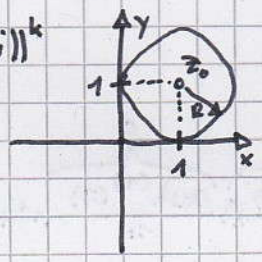
Formel von J. Hadamard $R=\limsup_{k\to\infty}\frac{1}{\sqrt[k]{|a_k|}}$

Beispiele:

1) $f(z)=\sum_{k=0}^{\infty}(-1)^k(z-(1+i))^k$

$z_0=1+i$ $R=1$

$a_k=(-1)^k$



2) $f(x)=\sum_{k=0}^{\infty}\frac{x^k}{k!}=\sum_{k=0}^{\infty}\frac{1}{k!}\cdot x^k$ $=e^x$ $\forall x\in\mathbb{R}$

$x_0=0$
$a_k=\frac{1}{k!}$ $R=\lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right|=\lim_{k\to\infty}\frac{\frac{1}{k!}}{\frac{1}{(k+1)!}}=$

$=\lim_{k\to\infty}\frac{(k+1)!}{k!}=\lim_{k\to\infty}(k+1)=\infty$

limes superior
↳ der größte Häufungspunkt der Reihe
limes inferior
↳ der kleinste Häufungspunkt

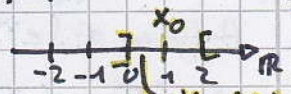
$\sin x=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}$ $R=\infty$

Weitere Beispiele:

3) $\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}(x-1)^k$, $x_0=1$, $a_k=\frac{(-1)^k}{2k+1}$, $R=\lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right|=\lim_{k\to\infty}\left|\frac{\frac{(-1)^k}{2k+1}}{\frac{(-1)^{k+1}}{2(k+1)+1}}\right|=\lim_{k\to\infty}\frac{2(k+1)-1}{2k+1}=\lim_{k\to\infty}\frac{2k+3}{2k+1}=$

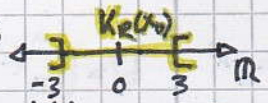
$=\lim_{k\to\infty}\frac{\cancel{k}(2+\overset{\to 0}{\frac{3}{k}})}{\cancel{k}(2+\underset{\to 0}{\frac{1}{k}})}=1$

Für $x=0{,}7$ konv.
$x=-0{,}7$ div.



$K_R(x_0)=]0;2[$

4) $f(x)=\sum_{k=0}^{\infty}\frac{x^k}{3^k}$, $x_0=0$, $a_k=\frac{1}{3^k}$, $R=\lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right|=\lim_{k\to\infty}\frac{\frac{1}{3^k}}{\frac{1}{3^{k+1}}}=\lim_{k\to\infty}\frac{3^{k+1}}{3^k}=3$



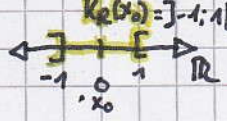
$\sum_{k=0}^{\infty}\frac{1}{3^k}(x-0)^k$

Zusammenfassung: Die Reihe ist auf dem Inneren von $K_R(x_0)$ absolut konvergent.
—— " —— Äußeren —— " —— divergent. auf den Rand.

Reihe kann in einem Randpunkt konvergent sein, aber nicht absolut!

Beispiel: $f(x)=\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k}x^k$, $x_0=0$, $a_k=(-1)^{k-1}\frac{1}{k}$, $R=\lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right|=\lim_{k\to\infty}\left|\frac{\frac{(-1)^{k-1}}{k}}{\frac{(-1)^k}{k+1}}\right|=\lim_{k\to\infty}\frac{k+1}{k}=1$

$K_R(x_0)=]-1;1[$



, $\partial K_R(x_0)=\{-1,1\}$, $f(1)=\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k}$ konvergent nach Leibnizkriterium aber nicht abs. konv.

$f(-1)=\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k}(-1)^k=\sum_{k=1}^{\infty}\underbrace{(-1)^{2k-1}}_{=-1}\frac{1}{k}=-\sum\frac{1}{k}$ div.

Aufgaben: a) $\sum_{k=1}^{\infty}\frac{1}{k^2}x^k$, b) $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{k^2}{2^k}(x-1)^k$, c) $\sum_{k=1}^{\infty}k!\,x^k$

1 a) $x_0=0$, $a_k=\frac{1}{k^2}$, $R=\lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right|=\lim_{k\to\infty}\frac{(k+1)^2}{k^2}=$

2 b) $x_0=1$, $a_k=(-1)^{k-1}\frac{k^2}{2^k}$, $R=\lim_{k\to\infty}\left|\frac{\frac{(-1)^{k-1}k^2}{2^k}}{\frac{(-1)^k(k+1)^2}{2^{k+1}}}\right|=\lim_{k\to\infty}2$

konv. nur im Entwicklungspunkt x_0 ⇒ 0 c) $x_0=0$, $a_k=k!$,

$\sum_{k=0}^{\infty} a_k (x-x_0)^k$ Bemerkung: Jede Potenzreihe kann auf ihrem Konvergenzkreis gliederweise integriert und differenziert werden. (R ändert sich dabei nicht)

Beispiel: $f(x)=\frac{1}{1-x}=\sum_{k=0}^{\infty} 1\cdot x^k$, $|x|<1$, (geom. Reihe) $\hat{=}$ Potenzreihe mit $x_0=0$, $a_k=1$, $R=1$, $K_1(0)=]-1;1[$

$$\int f(x)\,dx=\int\frac{1}{1-x}\,dx=-\ln(1-x)=\int\sum_{k=0}^{\infty}x^k\,dx=\sum_{k=0}^{\infty}\int x^k\,dx=\sum_{k=0}^{\infty}\frac{x^{k+1}}{k+1}=\sum_{k=1}^{\infty}\frac{x^k}{k},\ |x|<1$$

$$f'(x)=\frac{0\cdot(1-x)-1\cdot(-1)}{(1-x)^2}=\frac{1}{(1-x)^2}=\left(\sum_{k=0}^{\infty}x^k\right)'=\sum_{k=0}^{\infty}k\,x^{k-1}=0\cdot x^{-1}+1\cdot x^0+2x^1+3x^2+\ldots=$$

$$=\sum_{k=1}^{\infty}k\,x^{k-1}$$

Entwickeln von Funktionen in Potenzreihen mit dem Satz von Taylor:

Sei $f(x)$ eine unendlich oft differenzierbare Funktion mit Definitionsbereich $D\subseteq\mathbb{R}$, dann gilt $f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$ (Taylor-Reihe), $x_0\in D$

speziell $x_0=0 \Rightarrow f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$ (McLaurien-Reihe)

Beispiel: Sei $f(x)=e^x$, $x_0=0$, $f^{(k)}(x)=e^x\ \forall k \Rightarrow f^{(k)}(0)=1 \Rightarrow f(x)\sum_{k=0}^{\infty}\frac{1}{k!}x^k\ \forall x\in\mathbb{R}$

$\sinh x=\frac{e^x-e^{-x}}{2}$

$(\sinh x)'=\cosh x$

$\cosh 0=1$

$\cosh x=\frac{e^x+e^{-x}}{2}$

$(\cosh x)'=\sinh x$

$\sinh 0=0$

$$\sin x=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)}$$

$$\cos x=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}$$

$$\cosh x=\sum_{k=0}^{\infty}\frac{\cosh^{(k)}(0)}{k!}x^k=\overbrace{\frac{\cosh^{(0)}(0)}{0!}}^{1}x^0+\overbrace{\frac{\cosh^{(1)}(0)}{1!}}^{0}x^1+\overbrace{\frac{\cosh^{(2)}(0)}{2!}}^{1}x^2+\overbrace{\frac{\cosh^{(3)}(0)}{3!}}^{0}x^3+\ldots=$$

$$=\sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!}\ \text{(gerade Therme)}$$

$$\sinh x=\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!}$$

$$\boxed{e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}}$$

$$f(x)=e^{-x^3}=\sum_{k=0}^{\infty}\frac{(-x^3)^k}{k!}=\sum_{k=0}^{\infty}(-1)^k\frac{x^{3k}}{k!}$$

Aufgaben: a) $\int e^{-x^3}dx$ b) $(e^{-x^3})'$

$$a)=\int\left(\sum_{k=0}^{\infty}(-1)^k\frac{x^{3k}}{k!}\right)dx=\sum_{k=0}^{\infty}(-1)^k\int\frac{x^{3k}}{k!}dx=\sum_{k=0}^{\infty}(-1)^k\frac{x^{3k+1}}{(3k+1)k!}$$

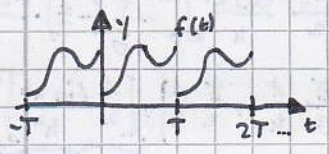
$-3x^2e^{-x^3}$

$$b)=\left(\sum_{k=0}^{\infty}(-1)^k\frac{x^{3k}}{k!}\right)'=\sum_{k=1}^{\infty}(-1)^k\,3k\frac{x^{k-1}}{\cancel{k!}\,(k-1)!}=3\sum_{k=1}^{\infty}(-1)^k\frac{x^{3k-1}}{(k-1)!}$$

Mathe III
Vorlesung

Fourier-Reihen: (Darstellung von periodischen Funktionen durch trigonometrische Reihen) 18.04.24

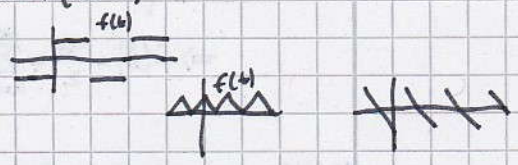
Def. $f: \mathbb{R} \longrightarrow \mathbb{K}$ ($\mathbb{K}=\mathbb{R}$, $\mathbb{K}=\mathbb{C}$) heißt periodisch mit der Periode $T>0$, wenn gilt
$\forall t \in \mathbb{R}: f(t+T) = f(t)$ (T ist stets die Minimalperiode



Ziel: $f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left\{ a_k \cos(k\omega t) + b_k \sin(k\omega t) \right\}$ Fourier-Reihen, $\omega = \frac{2\pi}{T}$ (Kreis-frequenz)

a_k, b_k Euler. Fourier-Koeffizienten (EFK)

Klausurrelevant sind nur:
- Rechtecksignale (stückweise konst.)
- Sägezahnfunktion (stückweise linear)

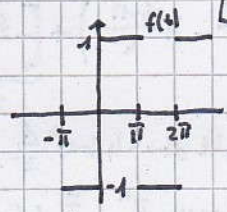


Berechnung der EFK: $a_k = \frac{2}{T} \int_0^T f(t) \cos(k\omega t)\,dt = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos(k\omega t)\,dt$

$b_k = \frac{2}{T} \int_0^T f(t) \sin(k\omega t)\,dt = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin(k\omega t)\,dt$

$\boxed{\text{part. Integration} \quad \int u' v\,dx = uv - \int u v'\,dx}$

Bsp. 1) Rechtecksignal: $f(t) = \begin{cases} +1, & 0<t<\pi \\ -1, & \pi<t<2\pi \\ 0, & t=0,\ t=\pi,\ t=2\pi \end{cases}$



$T = 2\pi$, $\omega = \frac{2\pi}{T} = 1$

Bemerkung: f(t) achsensym. $\Rightarrow b_k = 0$
f(t) punktsym. $\Rightarrow a_k = 0$

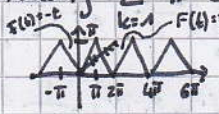
$\cos(k\pi) = (-1)^k$,
$\cos 0 = \cos(2\pi k) = 1$

$$b_k = \frac{2}{T} \int_0^T f(t) \sin(k\omega t)\,dt = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin(kt)\,dt = \frac{1}{\pi} \left(\int_0^{\pi} 1 \cdot \sin(kt)\,dt + \int_{\pi}^{2\pi} (-1) \sin(kt)\,dt \right) =$$

$$= \frac{1}{\pi} \left(\left[\frac{-\cos kt}{k} \right]_{\pi}^{2\pi} \right) = \frac{1}{\pi} \left(\frac{-\cos k\pi + 1}{k} + \frac{\cos(2\pi k) - \cos(k\pi)}{k} \right) = \frac{1}{\pi} \left(\frac{-(-1)^k + 1}{k} + \frac{1-(-1)^k}{k} \right) =$$

$$= \frac{1}{\pi} \frac{2 - 2(-1)^k}{k} = \frac{2}{\pi} \frac{1-(-1)^k}{k} \Rightarrow b_k = \frac{2}{\pi} \frac{1-(-1)^k}{k} \Rightarrow b_{2k} = 0,\ b_{2k-1} = \frac{4}{\pi} \frac{1}{2k-1} \Rightarrow$$

$$\Rightarrow f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left\{ a_k \cos(k\omega t) + b_k \sin(k\omega t) \right\} = \sum_{k=1}^{\infty} \frac{4}{\pi} \frac{1}{2k-1} \sin((2k-1)t) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2k-1)t}{2k-1}$$



$T = 2\pi$, $\omega = 1$, $b_k = 0$

2) Sägezahnsignal: $f(t) = |t|$, $-\pi \leq t \leq \pi$

Immer die trigonom. Funktion „aufwerten" bei part. Integration

$$k \geq 1,\ a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(kt)\,dt = \frac{1}{\pi} \left(\int_{-\pi}^{0} (-t) \cos(kt)\,dt + \int_0^{\pi} t \cos(kt)\,dt \right) = \frac{1}{\pi} \left(\underbrace{\left[t \cdot \frac{\sin kt}{k} \right]_{-\pi}^{0}}_{=0} - \right.$$

$$\left. - \int_{-\pi}^{0} \overset{\uparrow 2}{(-1)} \frac{\sin kt}{k}\,dt + \underbrace{\left[t \frac{-\sin kt}{k} \right]_0^{\pi}}_{=0} - \int_0^{\pi} 1 \cdot \frac{-\sin kt}{k}\,dt \right) = \frac{1}{\pi} \left(-\int_{-\pi}^{0} \frac{\sin kt}{k}\,dt + \int_0^{\pi} \frac{\sin kt}{k}\,dt \right) =$$

$$= \frac{1}{\pi} \left(-\left[\frac{-\cos kt}{k^2} \right]_{-\pi}^{0} + \left[\frac{-\cos kt}{k^2} \right]_0^{\pi} \right) = \frac{1}{\pi} \left(\left[\frac{\cos kt}{k^2} \right]_{-\pi}^{0} - \left[\frac{\cos kt}{k^2} \right]_0^{\pi} \right) = \frac{1}{\pi} \left(\frac{\cos 0 - \cos k(-\pi)}{k^2} - \frac{\cos k\pi - \cos 0}{k^2} \right)$$

$$= \frac{1}{\pi} \frac{1-(-1)^k - (-1)^k + 1}{k^2} = \frac{1}{\pi} \frac{2 - 2(-1)^k}{k^2} = \frac{2}{\pi} \frac{1-(-1)^k}{k^2} \Rightarrow a_{2k} = 0,\ a_{2k-1} = \frac{4}{\pi} \frac{1}{(2k-1)^2} \quad (k>0)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(t) \underbrace{\cos(\overset{\uparrow 2}{0} \cdot t)}_{=1}\,dt = \frac{1}{\pi} \left(\int_{-\pi}^{0} (-t)\,dt + \int_0^{\pi} t\,dt \right) = \frac{2}{\pi} \int_0^{\pi} t\,dt = \frac{2}{\pi} \left[\frac{1}{2} t^2 \right]_0^{\pi} = \frac{\cancel{2}}{\pi} \cdot \frac{1}{\cancel{2}} \pi^2 = \pi$$

$$\Rightarrow f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left\{ a_k \cos k\omega t + b_k \sin k\omega t \right\} = \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{4}{\pi} \frac{1}{(2k-1)^2} \cos(2k-1)t = \frac{\pi}{2} + \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2k-1)t}{(2k-1)^2}$$

$$f(t)=\begin{cases}\frac{\pi-t}{2}, & 0<t<2\pi\\ 0, & t>0,\ t=2\pi\end{cases}$$

punktsym., $T=2\pi$, $\omega=1$, $a_k=0$

$\left(\frac{\pi}{2}+\frac{\pi}{2}\right)\frac{1}{k}=\pi\cdot\frac{1}{k}$

$\cos(k\pi)=(-1)^k$

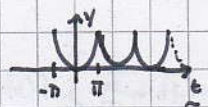
$\cos 0=\cos(2\pi k)=1$

$$b_k=\frac{2}{T}\int_0^T f(t)\sin(k\omega t)\,dt=\frac{1}{\pi}\int_0^{2\pi}\underbrace{\frac{\pi-t}{2}}_{v}\underbrace{\sin(kt)}_{u'}\,dt=\frac{1}{\pi}\left(\left[-\frac{\pi-t}{2}\,\frac{\cos kt}{k}\right]_0^{2\pi}-\int_0^{2\pi}\left(-\frac{1}{2}\right)\cdot\frac{-\cos kt}{k}\,dt\right)$$

$$=\frac{1}{\pi}\left(-\left(-\frac{\pi}{2}\right)\frac{\cos 2\pi k}{k}+\frac{\pi}{2}\,\frac{\cos 0}{k}\right)-\int_0^{2\pi}\frac{1}{2}\,\frac{\cos kt}{k}\,dt=\frac{1}{\pi}\left(\frac{\pi}{2}\,\frac{1}{k}+\frac{\pi}{2}\,\frac{1}{k}\right)-\underbrace{\left[\frac{1}{2}\,\frac{\sin kt}{k^2}\right]_0^{2\pi}}_{=0}=\frac{1}{\cancel{\pi}}\cdot\cancel{\pi}\cdot\frac{1}{k}=\frac{1}{k}$$

$$\Rightarrow f(t)=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left\{a_k\cos(k\omega t)+b_k\sin(k\omega t)\right\}=\sum_{k=1}^{\infty}\frac{\sin kt}{k}$$

Nichtklausurrelevante Bsp.aufgabe:



$f(t)=t^2$, $-\pi\le t\le\pi$, $T=2\pi$, $\omega=1$, achsensym. $\Rightarrow b_k=0$

$$a_k=\frac{1}{\pi}\int_{-\pi}^{+\pi}\underbrace{t^2}_{v}\underbrace{\cos(kt)}_{u'}\,dt=\frac{1}{\pi}\left(\left[t^2\,\frac{\sin kt}{k}\right]_{-\pi}^{\pi}-\int_{-\pi}^{\pi}\underbrace{2t}_{v'}\cdot\underbrace{\frac{\sin kt}{k}}_{u}\,dt\right)=\frac{1}{\pi}\left(\underbrace{\left[t^2\,\frac{\sin kt}{k}\right]_{-\pi}^{\pi}}_{=0}+\left[t\cdot\frac{+\cos kt}{k^2}\right]_{-\pi}^{\pi}+\int_{-\pi}^{\pi}+\frac{\cos kt}{k^2}\,dt\right)=$$

$$=\frac{1}{\pi}\left(-\pi\frac{(-1)^k}{k^2}-\pi\frac{-(-1)^k}{k^2}\right)+\underbrace{\left[-\frac{\sin kt}{k^3}\right]_{-\pi}^{\pi}}_{=0}=-4\,\frac{(-1)^k}{k^2}$$

Lösung: $a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}t^2\cos kt\,dt=-4(-1)^k\frac{1}{k^2}$, $k\ge 1$

$a_0=\frac{2}{3}\pi^2\Rightarrow f(t)=\frac{\pi^2}{3}-\sum_{k=1}^{\infty}4(-1)^k\frac{\cos(kt)}{k^2}$

<u>Parsevalsche Gleichung</u>: $\frac{|a_0|^2}{2}+\sum_{k=1}^{\infty}\left\{|a_k|^2+|b_k|^2\right\}f(t)^2\,dt$

Anwendung: $f(t)=\frac{\pi-t}{2}=\sum_{k=1}^{\infty}\frac{\sin kt}{k}$, $b_k=\frac{1}{k}$, $a_k=0$

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi-t}{2}\right)^2dt=\frac{1}{\pi}\int_0^{2\pi}\frac{\pi^2-2\pi t+t^2}{4}\,dt=\frac{1}{4\pi}\left[\pi^2t-\pi t^2+\frac{1}{3}t^3\right]_0^{2\pi}=$$

$$=\frac{1}{4\pi}\left(2\pi^3-4\pi^3+\frac{8}{3}\pi^3\right)=\frac{1}{\cancel{4\pi}_{2}}\cdot\frac{\cancel{2}^{1}}{3}\pi^{\cancel{3}^{2}}=\frac{\pi^2}{6}\qquad\boxed{\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}}$$

Mathe III
Vorlesung

25.04.24

T-periodische Funktion: $f(t)=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left\{a_k\cos(k\omega t)+b_k\sin(k\omega t)\right\}$

$a_k=\frac{2}{T}\int_0^T f(t)\cos(k\omega t)\,dt=\frac{2}{T}\int_{-T/2}^{T/2}\dots \qquad b_k=\frac{2}{T}\int_0^T f(t)\sin(k\omega t)\,dt=\frac{2}{T}\int_{-T/2}^{T/2}\dots$

punktsym. $\Rightarrow a_k=0$
Achsensym. $\Rightarrow b_k=0$

$\omega=\frac{2\pi}{T}$

$$\frac{|a_0|^2}{2}+\sum_{k=1}^{\infty}\left\{|a_k|^2+|b_k|^2\right\}=\frac{2}{T}\int_0^T|f(t)|^2dt \quad \text{Parsevalsche Gleichung}$$

Die komplexe Fourier-Reihe: $f(t)=\sum_{k=-\infty}^{\infty}c_k e^{ik\omega t}$, $e^{ik\omega t}=\cos k\omega t+i\sin k\omega t$

$c_0=\frac{a_0}{2}$, $c_k=\frac{a_k-ib_k}{2}$, $c_{-k}=\frac{a_k+ib_k}{2}$, $a_0=2c_0$, $a_k=c_k+c_{-k}$, $b_k=i(c_k-c_{-k})$

$\boxed{\text{Klirrfaktor} \quad K=\sqrt{\frac{\sum_{k=2}^{\infty}|c_k|^2}{\sum_{k=1}^{\infty}|c_k|^2}}}$ $\qquad c_k=\frac{1}{T}\int_0^T f(t)e^{-ik\omega t}dt=\frac{1}{T}\int_{-T/2}^{T/2}\dots$

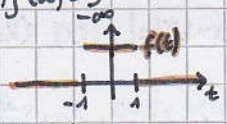
Der Übergang von einem periodischen Signal zu einem aperiodischen: $T\to\infty$

$c_k\to c(\omega)=\int_{-\infty}^{+\infty}f(t)e^{-i\omega t}dt$ (Spektralfunktion $\hat{=}$ Fourier-Transformierte von $f(t)$)

Schreibweise: $F(\omega)=\mathcal{F}\{f(t)\}(\omega)=\int_{-\infty}^{+\infty}f(t)e^{-i\omega t}dt$

$\boxed{\begin{aligned}e^{i\omega}&=\cos\omega+i\sin\omega\\ e^{-i\omega}&=\cos\omega-i\sin\omega\end{aligned}}$

Beispiel: $f(t)=\begin{cases}1, & -1\le t\le 1\\ 0, & \text{sonst}\end{cases}$



$F(\omega)=\int_{-\infty}^{+\infty}f(t)e^{-i\omega t}dt=\int_{-1}^{+1}1\cdot e^{-i\omega t}dt=$

$=\left[\frac{e^{-i\omega t}}{-i\omega}\right]_{-1}^{1}=\frac{e^{-i\omega\cdot 1}-e^{-i\omega(-1)}}{-i\omega}=\frac{e^{-i\omega}-e^{i\omega}}{-i\omega}=\frac{\cos\omega-i\sin\omega-(\cos\omega+i\sin\omega)}{-i\omega}=$

$=\frac{-2i\sin\omega}{-i\omega}=2\frac{\sin\omega}{\omega}=2\,\mathrm{sinc}\,\omega \qquad \mathrm{sinc}\,x=\mathrm{si}\,x=\begin{cases}\frac{\sin x}{x}, & x\neq 0\\ 1, & x=0\end{cases}$ Sinus Cardinalis

2. Beispiel: Der beidseitig abfallende Impuls $f(t)=e^{-|t|}$

$=\begin{cases}e^{-t}, & t\ge 0\\ e^{t}, & t\le 0\end{cases}$

$F(\omega)=\int_{-\infty}^{+\infty}f(t)e^{-i\omega t}dt=\int_0^{+\infty}e^{-t}e^{-i\omega t}dt+\int_{-\infty}^{0}e^{t}e^{-i\omega t}dt=\int_0^{\infty}e^{-(1+i\omega)t}dt+\int_{-\infty}^{0}e^{(1-i\omega)t}dt=$

$(e^{ab})'=ae^{ab}$
$\int e^{at}dt=\frac{1}{a}e^{at}$

$=\left[\frac{e^{-(1+i\omega)t}}{-(1+i\omega)}\right]_0^{t\to\infty}+\left[\frac{e^{(1-i\omega)t}}{1-i\omega}\right]_{t\to-\infty}^{0}=\lim_{t\to\infty}\frac{\cancel{e^{-(1+i\omega)t}}^{\,0}-1}{-(1+i\omega)}+\lim_{t\to-\infty}\frac{1-\cancel{e^{(1-i\omega)t}}^{\,0}}{1-i\omega}=$

$=\frac{1}{(1+i\omega)}+\frac{1}{1-i\omega}=\frac{1-\cancel{i\omega}+1+\cancel{i\omega}}{(1+i\omega)(1-i\omega)}=\frac{2}{1+\omega^2}$

$*=\cancel{e^{-t}}^{\,0}\cdot e^{-i\omega t}$

Aufgabe: $f(t)=\begin{cases}e^{-at}, & t\ge 0,\ a>0\\ 0, & \text{sonst}\end{cases}$ Bestimme $F(\omega)$, $\mathrm{Re}\,F(\omega)$, Was passiert für $a\to\infty$, $a\to 0$

$\frac{a}{b}+\frac{c}{d}=\frac{ad+cb}{bd}$

$F(\omega)=\int_{-\infty}^{+\infty}f(t)e^{-i\omega t}dt=\int_0^{+\infty}e^{-at}e^{-i\omega t}dt=\int_0^{+\infty}e^{-(a+i\omega)t}dt=\left[\frac{e^{-(a+i\omega)t}}{-(a+i\omega)}\right]_0^{t\to\infty}=\lim_{t\to\infty}\frac{\cancel{e^{-(a+i\omega)t}}^{\,0}-1}{-(a+i\omega)}=\frac{1}{a+i\omega}=$

$1^2-\cancel{i\omega}+\cancel{i\omega}-i^2\omega^2$ *

$=\frac{a-i\omega}{(a+i\omega)(a-i\omega)}=\frac{a-i\omega}{a^2+\omega^2}\Rightarrow \mathrm{Re}\,F(\omega)=\frac{a}{a^2+\omega^2},\ \mathrm{Im}\,F(\omega)=\frac{-\omega}{a^2+\omega^2}$

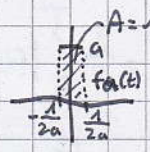
Die Umkehrformel von Fourier

$$F(\omega) = \mathcal{F}\{f(t)\}(\omega) = \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t}\, dt$$

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)\, e^{i\omega t}\, d\omega$$

Die Diracsche Deltafunktion

$$f(x) = \begin{cases} \infty, & x = 0 \\ 0, & \text{sonst} \end{cases} \qquad \int_{-\infty}^{+\infty} f(x)\, dx = 1$$



$$f_a(t) = \begin{cases} a, & -\frac{1}{2a} < t < \frac{1}{2a} \\ 0, & \text{sonst} \end{cases}$$

$$f_a \xrightarrow{a \to \infty} \delta \qquad \Rightarrow \int_{-\infty}^{+\infty} F_a(t)\, dt = 1 \quad \forall a$$

Regeln für die Deltafunktion

1) $\int_{-\infty}^{+\infty} \delta(x) f(x)\, dx = f(0)$

2) $\int_{-\infty}^{+\infty} \delta(x - x_0) f(x)\, dx = f(x_0)$

3) Poissonsche Summenformel $\delta(g(x)) = \sum_k \frac{\delta(x - x_k)}{|g'(x_k)|}$

g differenzierbar, mit einfachen Nullstellen $x_1, x_2, x_3 \ldots$

<u>Beispiel</u> $f(x) = \sin x$, Nullstellen $x_k = k\pi$, $k \in \mathbb{Z}$

$f'(x) = \cos x$, $f'(k\pi) = \cos(k\pi) = (-1)^k$

$$\Rightarrow \delta(\sin x) = \sum_{k \in \mathbb{Z}} \frac{\delta(x - k\pi)}{|(-1)^k|} = \sum_{k=-\infty}^{\infty} \delta(x - k\pi) = \sqcup\!\sqcup(x) \quad \text{(Kammfunktion)}$$

$$\int_{-\infty}^{+\infty} e^{2x}\, \underbrace{\delta(x^2 - 3x + 2)}_{g(x)}\, dx, \quad g(x) = x^2 - 3x + 2 = 0, \quad x_{1,2} = \frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2 + 2} = \frac{3}{2} \pm \frac{1}{2} \Rightarrow \begin{matrix} x_1 = 2 \\ x_2 = 1 \end{matrix}$$

$$\delta(x^2 - 3x + 2) = \sum_{k=1}^{2} \frac{\delta(x - x_k)}{|g'(x_k)|} = \frac{\delta(x-2)}{1} + \frac{\delta(x-1)}{1} = \delta(x-2) + \delta(x-1)$$

$$y'(x) = 2x - 3$$

$$= \int_{-\infty}^{+\infty} e^{2x}(\delta(x-2) + \delta(x-1))\, dx = \int_{-\infty}^{+\infty} e^{2x}\delta(x-2)\, dx + \int_{-\infty}^{+\infty} e^{2x}\delta(x-1)\, dx = e^4 + e^2$$

$$\mathcal{F}\{\delta(t)\}(\omega) = \int_{-\infty}^{+\infty} \delta(t)\, e^{-i\omega t}\, dt = 1$$

Mathe III
Vorlesung

Gewöhnliche Differentialgleichungen (DGL)

$F(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0$ gew. DGL der Ordnung n

$y^{(n)} = f(x, y(x), \dots, y^{(n-1)}(x))$ explizite DGL, sonst implizit

25.04.24

Beispiele: $e^{y \cdot y'} + \sin(y'' + x) = 0$

$y' e^{y'} + (y^3)^2 = x^2$

Da die Reibung der Federkraft entgegenwirkt.
Reibung ist proportional zur Beschleunigung, welche die erste Ableitung von v, also y' ist.

$m y'' = \sum F_x = F_{Reib} + F_{Feder} = -\beta y' - k y \Leftrightarrow m y'' = -\beta y' - k y \quad | : m$

$y'' = -\frac{\beta}{m} y' - \frac{k}{m} y \qquad \delta = \frac{\beta}{m}$ Dämpfung

$\Rightarrow y'' + \frac{\beta}{m} y' + \frac{k}{m} y = 0 \qquad \omega_0 = \sqrt{\frac{k}{m}}$ Eigenfrequenz

Gleichungen erster Ordnung

$y' = f(x, y)$

Trennbare Gleichungen (TdV) Die Gleichung heißt trennbar, wenn gilt $y' = g(x) \cdot h(y)$ # $y' = \frac{dy}{dx} \Leftrightarrow \int \frac{dy}{h(y)} = \int g(x)\,dx$

$H(y)$ Stammfunk. von $\frac{1}{h(y)}$

$G(x)$ " " $g(x)$

$\Rightarrow H(y) = G(x) + C_1 \quad | H^{-1}$

$y(x) = H^{-1}(G(x) + C_1)$ (allg. Lösung)

Die Integrationskonstante C_1 kann durch die Erfüllung eines Anfangswertproblems (AWP) bestimmt werden $\boxed{y(x_0) = y_0}$

$y' = \frac{dy}{dx}$

Beispiel: $\frac{dy}{dx} y' = xy$, $y(0) = 2$, $g(x) = x$, $h(y) = y$, $x_0 = 0$, $y_0 = 2$

TDV: $\int \frac{dy}{y} = \int x\,dx = \ln y = \frac{x^2}{2} + C_1 \quad | e^{(\cdot)} \Rightarrow y = e^{\frac{1}{2}x^2 + C_1} \qquad 2 = y(0) = e^{C_1} \Rightarrow C_1 = \ln 2$

$\Rightarrow y_2(x) = e^{\frac{1}{2}x^2 + \ln 2} = e^{\ln 2} \cdot e^{\frac{1}{2}x^2} = 2 e^{\frac{1}{2}x^2}$

4uthe III Vorlesung

Gewöhnliche DGLn → Gleichungen 1. Ordnung → Trennbare Gleichungen 02.05.24

diese haben die Form $y' = g(x) \cdot h(y)$, $y' = \frac{dy}{dx} \Leftrightarrow \frac{dy}{dx} = g(x)\, h(y) \;|\cdot dx, \div h(y)$

$\Leftrightarrow \int \frac{1}{h(y)}\, dy = \int g(x)\, dx$, sei $H(y)$ eine Stammfunktion von $\frac{1}{h(y)}$, $G(x)$ eine Stammfunktion von $g(x)$

TdV = Trennung der Variablen

$\Leftrightarrow H(y) + C_1 = G(x) + C_2 \Rightarrow H(y) = G(x) + C \quad (C = C_2 - C_1) \quad |H^{-1} \Rightarrow y(x) = H^{-1}(G(x) + C)$ allg. Lösung von ⊛

$\boxed{\text{AWP: } y_0 = y(x_0) \Rightarrow \text{Berechne } C \Rightarrow \text{Lösung des AWPs}}$

Beispiele: 1) $y' = \frac{y+1}{x}$, $y(0) = 1 \Leftrightarrow \frac{dy}{dx} = \frac{y+1}{x} \overset{TdV}{\Rightarrow} \int \frac{1}{y+1}\, dy = \int \frac{1}{x}\, dx$

$\Rightarrow \ln(y+1) = \ln x + C \Rightarrow y + 1 = e^{\ln x + C} = e^{\ln x} \cdot \underbrace{e^C}_{k} = kx \;|-1$

$\Rightarrow y(x) = kx - 1$ (allg. Lösung)

AWP: $1 = y(0) = -1$ ↯ ⇒ AWP nicht lösbar! Andres AWP: $y(1) = 4$ $\quad 4 = y(1) = k - 1 \Rightarrow k = 5 \Rightarrow \boxed{y(x) = 5x - 1}$

2) $y' = y^2(x+1)$ $\quad y(1) = 2 \overset{TdV}{\Leftrightarrow} \int \frac{1}{y^2}\, dy = \int (x+1)\, dx \Rightarrow -\frac{1}{y} = \frac{1}{2}x^2 + x + C \Rightarrow y(x) = -\frac{1}{\frac{1}{2}x^2 + x + C}$

AWP: $2 = y(1) = -\frac{1}{\frac{1}{2} + 1 + C} = -\frac{1}{\frac{3}{2} + C} \Rightarrow \frac{1}{2} = -\left(\frac{3}{2} + C\right) = -\frac{3}{2} - C \Rightarrow C = -2 \Rightarrow y(x) = -\frac{1}{\frac{1}{2}x^2 + x - 2}$

3) $y' = \frac{y^2+1}{x^2+1}$, $y(0) = 1 \overset{TdV}{\Leftrightarrow} \int \frac{1}{y^2+1}\, dy = \int \frac{1}{x^2+1}\, dx \Rightarrow \arctan y = \arctan x + C \;|\tan$

$\Rightarrow y(x) = \tan(\arctan x + C)$, AWP: $1 = y(0) = \tan C \Rightarrow \frac{\pi}{4} \Rightarrow y(x) = \tan\left(\arctan x + \frac{\pi}{4}\right)$

Löse die AWPs: 1) $y' = \frac{1}{2}xy$, $y(1) = 3$ $\quad$ 2) $y' = \frac{x^3}{y}$, $y(1) = 2$ $\quad$ 3) $y' = \frac{y^2+1}{x^2}$, $y(2) = 1$

1) $= 3e^{\frac{1}{4}x^2 - \frac{1}{4}}$

2) $= \sqrt{\frac{1}{2}x^4 + \frac{7}{2}}$

3) $= \tan\left(\frac{\pi}{4} + \frac{1}{2} - \frac{1}{x}\right)$

1) $\overset{TdV}{\Rightarrow} \int \frac{1}{y}\, dy = \int \frac{1}{2}x\, dx \Rightarrow \ln y = \frac{1}{4}x^2 + C \Rightarrow e^{\frac{1}{4}x^2 + C} = e^{\frac{1}{4}x^2} \cdot e^C = ke^{\frac{1}{4}x^2}$ allg. Lösung

AWP: $3 = y(1) = ke^{\frac{1}{4}} \Rightarrow k\, e^{-\frac{1}{4}} \Rightarrow y(x) = 3e^{-\frac{1}{4}} \Rightarrow y(x) = 3e^{-\frac{1}{4}} \cdot \underbrace{e^{\frac{1}{4}x^2}}_{k} = 3e^{\frac{1}{4}x^2 - \frac{1}{4}}$

2) $\overset{TdV}{\Rightarrow} \int y\, dy = \int x^3\, dx \Leftrightarrow \frac{1}{2}y^2 = \frac{1}{4}x^4 + C \;|\cdot 2, \Rightarrow y^2 = \frac{1}{2}x^4 + 2C \;|\sqrt{\;} \Rightarrow y(x) = \pm\sqrt{\frac{1}{2}x + 2C}$

AWP: $2 = y(1) = \sqrt{\frac{1}{2} + 2C} \;|()^2 \Rightarrow 4 = \frac{1}{2} + 2C \Rightarrow 2C = \frac{7}{2} \Rightarrow y(x) = \sqrt{\frac{1}{2}x^4 + \frac{7}{2}}$

3) $\overset{TdV}{\Rightarrow} \int \frac{1}{y^2+1}\, dy = \int \frac{1}{x^2}\, dx \Rightarrow \arctan y = -\frac{1}{x} + C \Rightarrow y(x) = \tan\left(C - \frac{1}{x}\right)$

AWP: $1 = y(2) = \tan\left(C - \frac{1}{2}\right) \Rightarrow \frac{\pi}{4} = C - \frac{1}{2} \Rightarrow C = \frac{\pi}{4} + \frac{1}{2} \Rightarrow y(x) = \tan\left(\frac{\pi}{4} + \frac{1}{2} - \frac{1}{x}\right)$

Da Lösung des AWPs positiv ist, muss das positive Vorzeichen für die Lösung benutzt werden

Änderungsrate bedeutet immer erste Ableitung

Das Gesetz des radioaktiven Zerfalls. Sei $N(t)$ die zur Zeit t vorhandene Menge einer radioaktiven Substanz.

Die Änderungsrate ist proportional zur vorhandenen Menge $N(t)$

$N'(t) \sim N(t) \rightarrow N'(t) = -\lambda N(t)$ $\quad N(0) = N_0$ (AWP) $\Rightarrow \frac{dN}{dt} = -\lambda N \overset{TdV}{\Leftrightarrow} \int \frac{1}{N}\, dN = -\int \lambda\, dt$

$\Rightarrow \ln N = -\lambda t + C \;|e \Rightarrow N(t) = e^{-\lambda t + C} = ke^{-\lambda t}$ $\quad k = e^C$

AWP: $N_0 = N(0) = k \cdot e^0 = k \Rightarrow \boxed{N(t) = N_0 e^{-\lambda t}}$ $\quad T_{1/2} = \frac{\ln 2}{\lambda}$ Halbwertszeit

Die lineare Gleichung 1. Ordnung ⊛ $y' + a(x)\, y = b(x)$ $\quad a(x), b(x)$ gegebene stetige Funktion, AWP: $y_0 = y(x_0)$ Lösungsformel: $y(x) = e^{-A(x)}\left\{y_0 + \int_{x_0}^{x} e^{A(\xi)} b(\xi)\, d\xi\right\}$, $A(x) = \int_{x_0}^{x} a(\xi)\, d\xi$

(⊛ heißt homogen, wenn $b(x) = 0$) homogene Gl. $\Leftrightarrow y' + a(x)\, y = 0 \Leftrightarrow y' = -a(x)\, y \Leftrightarrow$

$\overset{TdV}{\Leftrightarrow} \int \frac{1}{y}\, dy = -\int a(x)\, dx \Leftrightarrow \ln y = -A(x) + C \Rightarrow y(x) = k \cdot e^{-A(x)}$ $\quad k = e^C$

inhomogene Gl.: Variation der Konstanten (VdK): $k = k(x) \Rightarrow y(x) = k(x) \cdot e^{-A(x)} \Rightarrow y' = k'(x)\, e^{-A(x)} + k(x) \cdot (-a(x)) e^{-A(x)} = e^{-A(x)}(k'(x) - k(x)\, a(x))$

$\Rightarrow e^{-A(x)}(k'(x) - k(x)\, a(x)) + a(x)\, k(x)\, e^{-A(x)} = b(x) \Leftrightarrow k'(x)\, e^{-A(x)} = b(x) \Leftrightarrow k'(x) = e^{A(x)} b(x) \Rightarrow$

$\Rightarrow k(x) = \int e^{A(x)} b(x)\, dx + C \Rightarrow y(x) = e^{-A(x)}\left\{\int e^{A(x)} b(x)\, dx + C\right\} \overset{AWP}{\Rightarrow} LF$

Beispiele: 1) $y' + \frac{2}{x} y = x^2$, $y(1) = 3$, $a(x) = \frac{2}{x}$, $b(x) = x^2$, $x_0 = 1$, $y_0 = 3$, $A(x) = \int_{x_0}^{x} a(\xi)\,d\xi =$

$= \int_1^x \frac{2}{\xi}\,d\xi = [2\ln\xi]_1^x = 2\ln x$

$y(x) = e^{-A(x)}\left\{y_0 + \int_{x_0}^{x} e^{A(\xi)} b(\xi)\,d\xi\right\} = e^{-2\ln x}\left\{3 + \int_1^x e^{2\ln\xi}\cdot\xi^2\,d\xi\right\}$ NR. $e^{-2\ln x} = (e^{\ln x})^{-2} = \frac{1}{x^2}$

$= \frac{1}{x^2}\left\{3 + \int_1^x \xi^4\,d\xi\right\} = \frac{1}{x^2}\left(3 + \left[\frac{1}{5}\xi^5\right]_1^x\right) = \frac{1}{x^2}\left(3 + \frac{1}{5}x^5 - \frac{1}{5}\right) = \frac{14}{5}\frac{1}{x^2} + \frac{1}{5}x^3$

2) $y' - \frac{1}{x} y = -\frac{2}{x^2}$, $y(1) = 2$, $a(x) = -\frac{1}{x}$, $b(x) = -\frac{2}{x^2}$, $x_0 = 1$, $y_0 = 2$ $A(x) = \int_{x_0}^{x} a(\xi)\,d\xi = \int_1^x \left(-\frac{1}{\xi}\right)d\xi = -\ln x$

$\Rightarrow y(x) = e^{-A(x)}\left\{y_0 + \int_{x_0}^{x} e^{A(\xi)} b(\xi)\,d\xi\right\} = \underbrace{e^{\ln x}}_{=x}\left\{2 + \int_1^x \underbrace{e^{-\ln\xi}}_{=\frac{1}{\xi}}\left(-\frac{2}{\xi^2}\right)d\xi\right\} = x\left(2 - \int_1^x \underbrace{\frac{2}{\xi^3}}_{=2\xi^{-3}}\,d\xi\right) = x\left(2 + \left[\frac{1}{\xi^2}\right]_1^x\right)$

$= x\left(2 + \frac{1}{x^2} - 1\right) = x + \frac{1}{x}$

Aufgaben: 1) $y' + 2y = 4$, $y(0) = 1$ 2) $y' - \frac{5}{x} y = \frac{1}{x}$, $y(1) = 10$ 3) $y' + 2xy = x$, $y(0) = 4$

1) $= 2 - e^{-2x}$

2) $= \frac{51}{5}x^5 - \frac{1}{5}$

3) $= \frac{7}{2}e^{-x^2} + \frac{1}{2}$

1) $a(x) =$

2) $a(x) = -\frac{5}{x}$, $b(x) = \frac{1}{x}$, $x_0 = 1$, $y_0 = 10$, $A(x) = \int_1^x a(\xi)\,d\xi = \int_1^x \left(-\frac{5}{\xi}\right)d\xi = [-5\ln x]_1^x = -5\ln x$

$y(x) = \underbrace{e^{+5\ln x}}_{x^5}\left(10 + \int_1^x \underbrace{e^{-5\ln\xi}}_{x^{-5}}\cdot\frac{1}{\xi}\,d\xi\right) = x^5\left(10 + \int_1^x \xi^{-6}\,d\xi\right) = x^5\left(10 + \left[\frac{\xi^{-5}}{-5}\right]_1^x\right) \Rightarrow \frac{x^5}{-5} + \frac{+1}{+5} =$

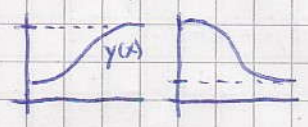
$= x^5\left(10 + \left(\frac{1}{5} - \frac{x^{-5}}{5}\right)\right) = x^5\left(\frac{51}{5} - \frac{x^{-5}}{5}\right) = \frac{51}{5}x^5 - \frac{1}{5}$

3)

Mathe III Vorlesung

Die Bernoullische DGL: $y' + g(x)\,y = h(x)\,y^{\alpha}$, $\alpha \in \mathbb{R} \setminus \{0;1\}$, $g,h:]\subseteq \mathbb{R} \to \mathbb{R}$ stetig 16.05.24

AWP: $y(x_0) = y_0$, $x_0 \in\,]$ (logistische Prozesse, beschränktes Wachstum, ...)



Substitution: $z(x) = y(x)^{1-\alpha} \Rightarrow z'(x) = (1-\alpha)\cdot y(x)^{-\alpha}\cdot y'(x)$

$\Rightarrow y' + g(x)\,y = h(x)\,y^{\alpha} \;\Big|\cdot(1-\alpha)\cdot y(x)^{-\alpha} \Rightarrow \underbrace{(1-\alpha)\,y^{-\alpha}\cdot y'}_{z'} + (1-\alpha)\,g(x)\,\underbrace{y^{1-\alpha}}_{z} =$

$= (1-\alpha)\,h(x)\,\overbrace{\cancel{y^{\alpha}\cdot y^{-\alpha}}}^{=1} \Rightarrow z' + (1-\alpha)\,g(x)\,z = (1-\alpha)\,h(x) \Rightarrow z' + a(x)\,z = b(x)$

$\xi = x$:

$\left(\text{mit } a(x) = (1-\alpha)\,g(x),\; b(x) = (1-\alpha)\,h(x)\right)$ (lin. Gl. 1. Ordnung $\Rightarrow$ Lösungsformel)

$$\text{LF: } z(x) = e^{-A(x)}\left\{z_0 + \int_{x_0}^{x} e^{A(\xi)}\,b(\xi)\,d\xi\right\},\quad A(x) = \int_{x_0}^{x} a(\xi)\,d\xi,\quad z_0 = y_0^{1-\alpha}$$

$$\Rightarrow y(x) = z(x)^{\frac{1}{1-\alpha}} \quad \square$$

Beispiele:

1) $y' + 3y - 4y^2 = 0$, $y(0) = 2$ i) Auf Bernoulli-Normalform bringen: $y' + 3y = 4y^2$

$a(x) = -3$, $A(x) = -3x$

$z_0 = y_0^{1-\alpha} = 2^{-1} = \frac{1}{2}$ ii) Feststellung der Daten: $g(x) = 3$, $h(x) = 4$, $\alpha = 2$, $x_0 = 0$, $y_0 = 2$

iii) Aufstellung der Gleichung-z: $a(x) = (1-\alpha)\,g(x) = -3$, $b(x) = (1-\alpha)\,h(x) = -4$

$z' + a(x)\,z = b(x) \Leftrightarrow z' - 3z = -4$, $z(x_0) = z_0 = \frac{1}{2} \Rightarrow$ LF: $e^{-A(x)}\left\{z_0 + \int_{x_0}^{x} e^{A(\xi)}\,b(\xi)\,d\xi\right\} = e^{3x}\left\{\frac{1}{2} + \int_0^x e^{-3\xi}(-4)\,d\xi\right\} =$

$= e^{3x}\left\{\frac{1}{2} + \left[\frac{4}{3}e^{-3\xi}\right]_0^x\right\} = e^{3x}\left\{\frac{1}{2} + \frac{4}{3}e^{-3x} - \frac{4}{3}\right\} = e^{3x}\left\{-\frac{5}{6} + \frac{4}{3}e^{-3x}\right\} = -\frac{5}{6}e^{3x} + \frac{4}{3}$

$y(x) = z(x)^{\frac{1}{1-\alpha}} = \left(-\frac{5}{6}e^{3x} + \frac{4}{3}\right)^{-1} = \frac{1}{-\frac{5}{6}e^{3x} + \frac{4}{3}} = \frac{6}{-5e^{3x} + 8}$ $\quad y(0) = \frac{6}{-5+8} = 2$

2) $y' + \frac{y}{x} - y^2 = 0$, $y(1) = \frac{1}{\pi} \Rightarrow$ NF: $y' + \frac{1}{x}y = y^2$, $g(x) = \frac{1}{x}$, $h(x) = 1$, $\alpha = 2$, $x_0 = 1$, $y_0 = \frac{1}{\pi}$,

$a(x) = (1-\alpha)\,g(x) = -\frac{1}{x}$, $b(x) = (1-\alpha)\,h(x) = -1$, $A(x) = \int_1^x \left(-\frac{1}{\xi}\right)d\xi = -\ln x$

$z_0 = y_0^{1-\alpha} = \left(\frac{1}{\pi}\right)^{-1} = \pi$ $\quad$ z-Gl.: $z' + a(x)\,z = b(x) \Rightarrow z' - \frac{1}{x}z = -1$

$\Rightarrow z(x) = e^{-A(x)}\left\{z_0 + \int_{x_0}^{x} e^{A(\xi)}\,b(\xi)\,d\xi\right\} = e^{\ln x}\left\{\pi + \int_1^x e^{-\ln\xi}(-1)\,d\xi\right\} = x\left\{\pi + \int_1^x \frac{-1}{\xi}\,d\xi\right\} =$

$= x\left(\pi + \left[-\ln(\xi)\right]_1^x\right) \Rightarrow y(x) = z(x)^{\frac{1}{1-\alpha}} = (\pi x - x\ln x)^{-1} = \frac{1}{\pi x - x\ln x}$ $\quad y(1) = \frac{1}{\pi - 1\cdot\ln 1} = \frac{1}{\pi}$

3) $y' + \frac{1}{x}y + y^3 = 0$, $y(1) = 4$, NF: $y' + \frac{1}{x}y = -y^3$, $g(x) = \frac{1}{x}$, $h(x) = -1$, $\alpha = 3$, $x_0 = 1$, $y_0 = 4$

$a(x) = (1-\alpha)\,g(x) = -\frac{2}{x}$ $\quad b(x) = (1-\alpha)\,h(x) = 2$ $\quad A(x) = -2\ln x$, $z_0 = y_0^{1-\alpha} = 4^{-2} = \frac{1}{16}$

$z' + a(x)\,z = b(x) \Leftrightarrow z' - \frac{2}{x}z = 2$

$\Rightarrow z(x) = e^{-A(x)}\left\{z_0 + \int_{x_0}^{x} e^{A(\xi)}\,b(\xi)\,d\xi\right\} = e^{2\ln x}\left\{\frac{1}{16} + \int_1^x e^{-2\ln\xi}\cdot 2\,d\xi\right\} = x^2\left\{\frac{1}{16} + \int_1^x \frac{1}{\xi^2}\cdot 2\,d\xi\right\} =$

$= x^2\left\{\frac{1}{16} + \left[-\frac{2}{\xi}\right]_1^x\right\} = x^2\left\{\frac{1}{16} - \frac{2}{x} + 2\right\} = x^2\left(\frac{33}{16} - \frac{2}{x}\right) = \frac{33}{16}x^2 - 2x$

$y(x) = z(x)^{\frac{1}{1-\alpha}} = \left(\frac{33}{16}x^2 - 2x\right)^{-\frac{1}{2}} = \frac{1}{\sqrt{\frac{33}{16}x^2 - 2x}}$, $y(1) = \frac{1}{\sqrt{\frac{33}{16} - 2}} = \frac{1}{\frac{1}{4}} = 4$

Aufgaben 1) $y' + 3y = e^x y^2$, $y(0) = 1$; 2) $y' + y + \frac{x^5}{3+x}y^5 = 0$, $y(1) = 3$; 3) $y' + 2y = xy^2$, $y(0) = 1$

1) $= y(x) = \frac{2}{e^{3x} + e^x}$

1) NF: $y' + 3y = y^2$, $g(x) = e^x$, $h(x) = 1$, $\alpha = 2$, $x_0 = 0$, $y_0 = 1$, $a(x) = (1-\alpha)\,g(x) = -\frac{1}{e^x}$, $b(x) = (1-\alpha)\,h(x) = -1$

2) $= \left(\left(\frac{x}{3}\right)^4 + \ln\frac{4}{x+3}\right)^{-\frac{1}{4}}$ $\quad A(x) = e^{-x}$ $\quad z_0 = y_0$

3) $\frac{4}{5e^{2x} + 2x - 1}$

2) NF: $y' + g(x)y = h(x)y^\alpha \Rightarrow y' + \frac{1}{x}y = -\frac{x^4}{3+x}y^5$, $g(x) = \frac{1}{x}$, $h(x) = -\frac{x^4}{3+x}$, $\alpha = 5$, $x_0 = 1$, $y_0 = 3$

$a(x) = (1-\alpha)g(x) = -\frac{4}{x}$, $b(x) = (1-\alpha)h(x) = \frac{4x^4}{3+x}$, $A(x) = \int_{x_0}^{x} a(\xi)d\xi = \int_1^x \left(-\frac{4}{\xi}\right)d\xi = -4\ln x$

$z_0 = y_0^{1-\alpha} = 3^{-4} = \frac{1}{3^4} \Leftrightarrow z' + a(x)z = b(x) \Leftrightarrow z' - \frac{4}{x}z = \frac{4x^4}{3+x} \Rightarrow z(x) = e^{-A(x)}\left\{z_0 + \int_{x_0}^{x} e^{A(\xi)} b(\xi)d\xi\right\} =$

$= e^{4\ln x}\left\{\frac{1}{3^4} + \int_1^x e^{-4\ln\xi} \cdot \frac{4\xi^4}{3+\xi}d\xi\right\} = x^4\left\{\frac{1}{3^4} + \int_1^x \frac{1}{\xi^4}\frac{4\xi^4}{3+\xi}d\xi\right\} = x^4\left\{\frac{1}{3^4} + \left[4\ln(\xi+3)\right]_1^x\right\} =$

$\ln a - \ln b = \ln\frac{a}{b}$

$= x^4\left\{\frac{1}{3^4} + 4\ln(x+3) - 4\ln 4\right\} = x^4\left\{\frac{1}{3^4} + 4\ln\left(\frac{x+3}{4}\right)\right\} = \left(\frac{x}{3}\right)^4 + 4x^4\ln\frac{x+3}{4}$

$y(x) = z(x)^{\frac{1}{1-\alpha}} = z(x)^{-\frac{1}{4}} = \left(\left(\frac{x}{3}\right)^4 + 4x^4\ln\frac{x+3}{4}\right)^{-1/4}$ $\quad y(1) = \left(\left(\frac{1}{3}\right)^4 + \underbrace{4\ln\frac{1+3}{4}}_{=0}\right)^{-\frac{1}{4}} =$

$= \left(\left(\frac{1}{3}\right)^4\right)^{-1/4} = \left(\frac{1}{3}\right)^{-1} = 3$

3) $y' + 2y = xy^2$, $y(0) = 1$, $g(x) = 2$, $h(x) = x$, $x_0 = 0$, $y_0 = 1$, $\alpha = 2$, $a(x) = (1-\alpha)g(x) = -2$, $b(x) = (1-\alpha)h(x) = -x$

$A(x) = -2x$, $z_0 = y_0^{1-\alpha} = 1^{-1} = 1$ $\quad z' + a(x)z = b(x) \Leftrightarrow z' - 2z = -x$

$z(x) = e^{-A(x)}\left\{z_0 + \int_{x_0}^{x} e^{A(\xi)} b(\xi)d\xi\right\} = e^{2x}\left\{1 + \int_0^x e^{-2\xi}(-\xi)d\xi\right\}$

$\left(\text{NR: } \int_0^x (-\xi e^{-2\xi})d\xi = \left[\frac{1}{2}\xi e^{-2\xi}\right]_0^x - \int_0^x (1)\left(+\frac{1}{2}e^{-2\xi}\right)d\xi = \frac{1}{2}xe^{-2x} - \left[-\frac{1}{4}e^{-2\xi}\right]_0^x = \frac{1}{2}xe^{-2x} + \frac{1}{4}e^{-2x} - \frac{1}{4}\right)$

$\Rightarrow e^{2x}\left\{1 + \frac{1}{2}xe^{-2x} + \frac{1}{4}e^{-2x} - \frac{1}{4}\right\} = \frac{3}{4}e^{2x} + \frac{1}{2}x + \frac{1}{4} \Rightarrow y(x) = z(x)^{\frac{1}{1-\alpha}} = \left(\frac{3}{4}e^{2x} + \frac{1}{2}x + \frac{1}{4}\right)^{-1} =$

$= \frac{4}{3e^{2x} + 2x + 1}$

Die Riccatische DGL: $(*)\ y' = \overbrace{P(x)y^2}^{\text{Quadratisch}} + \overbrace{Q(x)y}^{\text{Linear}} + R(x)$, $P, Q, R:]\subseteq\mathbb{R} \to \mathbb{R}$ stetige Funktionen

$y(x_0) = y_0$

$y' = y^2 + x^2$, $P(x) = 1$, $Q(x) = 0$, $R(x) = x^2$ (nicht durch elementare Funktionen lösbar)

Sei $v(x)$ (irgendeine) spezielle Lösung von $(*)$, dann kann man auch die allg. Lösung finden und das AWP lösen.

Subst. $u(x) = \frac{1}{y(x) - v(x)} \Leftrightarrow y(x) = v(x) + \frac{1}{u(x)} \Rightarrow u' + (2Pv + Q)u = -P$

$\Rightarrow$ lin. Gl. 1. O.d.

$\Rightarrow$ LF

Ansätze für $v(x)$:

$v(x) = c = \text{const}$

$v(x) = ax + b$

$v(x) = x^\alpha$

$v(x) = e^{ax}$

$v(x) = \sin ax$