An ongoing revolution in quantum many-body theory

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‘Party is over’ or ‘A revolution is coming’?

There are two scenarios for a revolution in theoretical physics to occur.

(1) A change of the foundation or completing a badly incomplete foundation. For example, general relativity changed Newton's absolute spacetime paradigm and established a new theory of gravity based on a new spacetime geometry.

(2) A significant advance in a completely unknown territory, which is governed by new physical principles. For example, quantum mechanics is a new set of physics principles governing physics in a much shorter length scale than classical physics.

Majority's point of view.

- Most work physicists believe that no fundamental shift of paradigm is in sight for next 20 or 30 years. This opinion simply says that (1) is impossible in 20-30 years.
- C. N. Yang: "Party is over", where the party refers to the party between high energy experiments and high energy theory. If we can not explore a shorter distance physics via higher energy accelerator, there is no possibility for (2).

Question: Are we lucky enough to witness or even participate a real revolution in theoretical physics?

In order to answer this question, we need to investigate those questions that are core to the theoretical foundation of physics, and to find out whether there is a need for a revolution.

1. Is quantum mechanics complete?
2. the mathematical foundation of quantum field theories,
3. quantum gravity,
4. strong correlating systems (e.g., High Tc superconductor, FQH),
5. the theory of phase transitions.

Is quantum mechanics (QM) complete?

For Einstein, the answer is NO because he believed that 'God does not play dice with the universe'. I also think that QM is incomplete but for a different reason. QM claimed its success by reproducing the spectrum of Hydrogen Atom. However, QM is not enough to explain its spectrum!

Copy-paste from internet: "The Lamb shift splits the $2S_{\frac{1}{2}}$ and $2P_{\frac{1}{2}}$ states which are otherwise degenerate. Its origin is purely from quantum field theory. The experimental measurement of the Lamb shift stimulated theorists to develop Quantum ElectroDynamics (QED). The correction increases the energy of s states. One may think of the physical origin as the electron becoming less point-like as virtual photons are emitted and reabsorbed. "

In other words, even the Hydrogen Atom needs a QED to explain. However, the QED (or QFT in general) has infinitely many degrees of freedom and still lacks a mathematical foundation. In other words, one can not avoid the many body problem completely. The few body problem is perhaps only a classical illusion!

Moreover, different from Newton Mechanics, the founding fathers of QM did not provide a quantum calculus for quantum mechanics. As a consequence, QM lacks of geometric intuition and very often quite counter-intuitive because QM is not compatible with Newtonian spacetime intuition. For example, quantum entanglement suggests non-locality. Is the lack of geometric foundation also a sign of incompleteness?

Remark: Where does quantum entanglement come from?

1. If there is something that are qualified to be called the origin of quantum entanglement, then I claim that this "origin" must be geometric or capable of providing a geometric intuition.
2. The geometric origin of quantum entanglement or a geometric foundation of QM is essentially the same question as that of quantum geometry or quantum gravity.

Quantum gravity or quantum geometry demands us to have a radically new foundation of geometry. But where to start to find such a new foundation?

1. Is quantum gravity a many-body problem or few-body problem?
2. If it is a quantum many-body problem, this new quantum geometry (or calculus) necessarily catches some universal and key properties of all quantum many-body theories or QFT's.
3. This suggests that, until we have found the key and universal properties of all quantum many-body theories, quantum gravity is impossible.

Remark: Although the aim of string theory is quantum gravity, in reality, a large portion of string theorists' works focus on the study of all kinds of QFT's and their relations. To some extent, I believe that this is perhaps a more significant and direct value of string theory than its original motivation.

One of the most difficult problems in theoretical physics is strong correlating systems (e.g., High Tc superconductor, FQH) or strong coupling QFT's (e.g., QCD).

Although the perturbative approach towards weak coupling systems is non-trivial, in retrospect, its success is perhaps due to a discouraging fact that a weak coupling system is not that different from a system without interaction. Maybe it is a one-body problem in disguise.

But strong coupling systems are totally different. For example, the Kondo model is a strong correlating system, in which electrons are scattered by a magnetic impurity. Luckly, this model can be rigorously solved. It turns out that the collective modes split into two types: one type of modes has only charges but no spin, and the other has only spins but no charge.

This example shows that the collective modes in a strong coupling systems can be completely unpredictable or crazy. Think about high Tc superconductor and human brain. Therefore, only strong coupling systems are true many-body systems.

$\rightsquigarrow$ Strong coupling is more, and more is different!

On Aug 04, 1972, P. W. Anderson published a famous article published in "Science" with the title "More is different".

- The ability to reduce everything to simple fundamental laws does not imply the ability to start from those laws and reconstruct the universe. The key point is that even though we know the fundamental building blocks, we still have to know how they organize themselves in large quantities.
- What was not mentioned in Anderson's article is that the so-called "elementary particles" has been becoming less and less elementary. For example, even an electron is a collective mode or a one-particle state in a quantum field theory, which contains infinite number of degrees of freedom.

Again, one can not avoid the many body problem completely. The few-body problems are perhaps only an emergent illusion!

40 years later, what can we add to Anderson's point of view? YES, a lot can be added. See my online talk "Infinitely more and more different" (2020) `https://www.koushare.com/video/details/5321`

Another fundamental question in condensed matter physics:

- what is a phase and phase transition?

It is strongly intertwined with the strong coupling problem (if not the same).

1. The reason perturbative method does not work is that before the coupling constant reach a high value, it already drives the system to a phase transition, where the perturbative method fails completely.
2. Many theoretical tools in the study of phase transitions, such as conformal field theory, was originally proposed to study strong coupling hardron physics.
3. Moreover, by their transcendental nature, critical phenomena are quintessential examples of long range correlation systems illustrating the fact that infinitely more is more different. my online talk on "Infinitely more is more different" in 2020

My conclusion:

1. Finding new and systematic methods for the study of (strong coupling) quantum many body systems is (not one of) the **most fundamental** theoretical problem of our time!
2. A breakthrough in this problem will lead to a revolution in theoretical physics.

Galois' approach towards strong coupling systems

The current approach towards strong coupling systems is very similar to the computing-the-area-below-a-curve problem before the invention of calculus.

Namely, we do a case-by-case calculation by summing up the areas of rectangles with small width then taking a limit. A universal and transcendental solution to this problem is beyond imagination before the invention of calculus.

$$\text{area below the curve } \sin(x) = \int_0^{\pi} \sin(x)dx = -\cos(x)|_0^{\pi} = 2$$

Similarly, a universal and transcendental solution to strong coupling problem is certainly out of reach, but even the proposal of such a possibility already sounds quite out of your mind.

In the history of science, there are many examples that a concrete problem led to a universal and transcendental solution, which is often an abstract mathematical theory that applies to many other problems.

Another well-known example is Galois theory, which was originated from the problem of finding a radical solution to a polynomial equation: for $a_i \in \mathbb{Q}$,

$$\begin{aligned} p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0; \\ (x - s_1)(x - s_2) \cdots (x - s_n) = 0. \end{aligned}$$

In practice, such a factorization is very difficult to find even when it is possible.

Galois's deep insight is not to solve any concrete equation, but to understand the meaning of "solving" by studying certain 'solution space' and its relation to $p(x)$.

More precisely, by allowing addition, multiplication and their inverses, the so-called 'solution space' is a field extension F of $\mathbb{Q}$, i.e., a field extension of $\mathbb{Q}$ generated by the solutions $s_1, \cdots, s_n$ of the equation $p(x) = 0$.

$$F = \mathbb{Q}(s_1, \cdots, s_n).$$

Theorem

$p(x)$ is solvable by radicals if and only if the group $\mathrm{Gal}(F/\mathbb{Q})$ is solvable.

Remark: Unsolvable examples of $p(x)$ were quickly found; and once we know how $\mathrm{Gal}(F/\mathbb{Q})$ is resolved (by abelian quotients), it guides us to solve the equation $p(x) = 0$. In this sense, Galois theory provides a universal and transcendental solution to this solving-polynomial-equation-by-radicals problem.

There is an obvious similarity between solving a polynomial equation (i.e., factorizing a polynomial $p(x) = \prod_{i=1}^{n}(x - s_i)$) and solving a concrete lattice model in condensed matter physics (i.e., block-diagonalizing a large matrix $M = F(\oplus_{j=1}^{N} M_i)F^{-1}$).

(1) The difficulty of both problems lies in the fact that some fundamental ingredients are entangled. Even when the disentanglement is possible, it is hard to disentangle them in practice.

(2) The difficulty of both problems also lies in the fact that the god-given perspective, which is available when you set up the problem, is not necessarily the right perspective to approach the problem. Moreover, once we are accustomed to this god-given perspective, it becomes difficult to accept that there are other possibilities.

(3) If we are accustomed to the Hamiltonian-lattice-model perspective, it is indeed difficult to image that there is a universal and transcendental solution to all strong-coupling problems.

This 'difficulty' partially lies in a 'very natural' misunderstanding that the "diagonalizing a large matrix" is merely a technical or computational difficulty (instead of a conceptual one). Many physicists in the old generation expressed this attitude, including Feymann, even though he confessed that he has no idea how to obtain the 'love' by solving Schrödinger equations with large amount of variables.

Anderson's 'More is different' is a direct attack of this wrong attitude. 'More' is not merely a computational problem but a conceptual one.

Anderson's attack is somewhat incomplete as I explained in my online talk on "Infinitely more is more different". This incompleteness not only has a philosophical value but also has direct impacts in real life. It has triggered some misunderstandings among the new generation of working physicists who have grown up with the 'More is different' philosophy. Although these new generation physicists are passionate in their study of strong coupling systems, their insensitivity to (or reluctancy in accepting) new mathematical theories inspired by QFT's, commonly attributed to the conservative or experimental tradition of condensed matter physics, is perhaps also a consequence of an even deeper misunderstanding (or ignorance) about the difference between 'more' and 'infinitely more'. The key difference lies in the 'surprising' fact that there are many mathematical or physical (any difference?) structures lying in the world of 'infinitely more' that are impossible to see in that of 'more'. In other words, many infinite dimensional mathematical structures are not limits of finite dimensional ones. (see my online article "Impacts of mathematical physics on mathematics" in Chinese)

Similar to Galois theory, the existence of these infinite dimensional mathematical structures further confirms the fact that strong-coupling is not merely a computational problem but a conceptual one.

Moreover, the entire program of solving strong-coupling problems is to find new structures, new concepts, new principles and new languages in the world of ‘more’ or ‘infinitely more’. The consequence of this endeavor should be a new and unified mathematical theory of quantum many-body systems, and a theory of phases and phase transitions.

The world of ‘infinitely more’ is a completely uncharted territory in both physics and mathematics. Therefore, the strong-coupling or infinitely-more problem is not a pure physics problem. It demands both new physics and new mathematics.

If you decide to take a Galois' approach toward quantum many-body theory, you will ask the following natural question.

- What is the 'solution space' of a quantum many-body system?

If we consider a gapped system at zero temperature, only ground states matter.

1. Microscopic approach: A quantum phase can be defined by an equivalence class of the ground state wave function (e.g., local unitary transformations + stacking product states). **Tensor networks** are very powerful and efficient in this study.
2. Macroscopic approach: A quantum phase is also defined its the long wave length behavior. A natural and reasonable guess is that a quantum phase can be completely characterized by summarizing all '**macroscopic observables**' (in the long wave length limit). We focus on the macroscopic approach in this talk.

Galois' approach = Microscopic approach + Macroscopic approach.

Remark: If you look at the world around you through you bare eyes, you can tell the difference of the physical objects, such as leaves, tree, paper, water, very easily, even though these objects all come from similar quantum many-body systems from a microscopic perspective.

Why is it so difficult to derive their difference from their unified (perhaps boring) microscopic realizations (as lattice models with Hamiltonians), and, at the same time, so easy to detect or characterize their differences macroscopically.

1. There are a lot of deep and rich structures lying in the seemingly boring setup of lattice models with Hamiltonians but hard to see microscopically.
2. The microscopic perspective is natural and convenient because that is how you setup the problem. However, it does not mean that it is the right perspective that leads us to a solution. What it suggests is that a solution to the strong-coupling problem might comes from taking a macroscopic point of view.

A journey of simplification — from QFT's to CFT's, to TQFT's and finally to topological orders

Finding the precise mathematical characterization of the ‘solution space’ of quantum many-body systems is a highly non-trivial task. It is clear that it is more likely to succeed in a simpler quantum many-body system than in a complicated one.

It turns out that it is highly non-trivial to find a simpler, not to say the simplest, system because, for a long time, we do not understand what ‘simple’ (but ‘non-trivial’) means. Indeed, 0+1D is quantum mechanics and is not even a many-body system. So you have to start from 1+1D. To many condensed matter theorists, 1+1D Ising chain is the simplest quantum many body system, right? It is rigorously and almost trivially solved, and was believed to be completely understood as the simplest example of Landau’s symmetry-breaking theory, and it is perhaps the first example you learn in a textbook.

However, Ising chain did not lead us to the correct characterization of the ‘solution space’ because, in retrospect, we were misguided by the success of Landau’s theory. The discovery occurred in an even ‘simpler’ system.

Most of the earlier theorists who developed quantum many-body theory or QFT's were not aiming at a simplification. They are driven by the goal for a better understanding of real physical systems. However, in retrospect, we have been unconscious of but secretly guided by the hidden drive for simplification.

After the miraculous success of QED in late 40's, however, a small group of theorists set out on a journey (or an odyssey) to a theoretical foundation of QED or QFT in general. This not-so-mainstream program became very influential in the post-standard-model era, and eventually led to the recent revolution on the subject.

In this section, we briefly review this journey. This review is guided by the goal to show that each step of progress was made possible by certain 'simplification' of the system (for example, by imposing a larger symmetry).

(1) QED was established in late 40's by Feynman, Schwinger, Tomonaga and Dyson. Its success was marked by the unreasonable accuracy in the calculation of anomalous magnetic moment and the Lamb shift.

- The very abstract formalism of QED (or a QFT in general) was quickly accepted by the working physicists at that time not because it passed the re-examination from the first principle. Instead, it was persuaded by Schwinger's unreasonable accuracy in the calculation of anomalous magnetic moment, and by the urgency for new discoveries, which was largely responsible to the popularity of the shut-up-and-calculate spirit.
- The theory was shrouded in mystery by the usage of the renormalization techniques (i.e., by subtracting infinity from infinity to reach a finite number).

(2) A small group of theorists set out on a journey (or an odyssey) to a theoretical foundation of QED or QFT in general.

- Wightman introduced the so-called Wightman axioms in the early 50's. It further led to the program of constructive quantum field theory, which was somewhat successful for scalar field theories in 70's but failed for gauge theories in 80's.
- Haag-Kastler axioms for algebraic quantum field theories (AQFT) was introduced in 60's. It defines a QFT via local observable algebras (or the net of local operator algebras). AQFT was not so successful for QED (?). (It was successful in 90's in providing a mathematical foundation for the 1+1D rational CFT's based on the theory of 'conformal nets'.)

(3) S matrix theory for strong coupling systems in 60's: (taken from John Schwarz's review arXiv:0708.1917 and online talk on May 25th, 2024)

- "Chew argued that field theory was inappropriate for describing strong nuclear forces. Instead, he advocated focussing attention on physical quantities, especially the S Matrix, which describes on-mass-shell scattering amplitudes. The goal was therefore to develop a theory that would determine the S matrix."
- "Chew advocated another principle called the bootstrap. The idea was that the exchange of hadrons in crossed channels provide forces that are responsible for causing hadrons to form bound states. Thus, one has a self-consistent structure in which the entire collection of hadrons provides the forces that makes their own existence possible. It was unclear for some time how to formulate this intriguing property in a mathematically precise way."

- “The bootstrap idea had a precise formulation in the narrow resonance approximation, which was called duality. This is the statement that a scattering amplitude can be expanded in an infinite series of s-channel poles, and this gives the same result as its expansion in an infinite series of t-channel poles. ....” Chew and Frautschi:

$$A(s,t) = \sum_i \frac{\beta_i(t)}{s - M_i^2} = \sum_j \frac{\beta_j(s)}{t - M_j^2}$$

- In 1968, “Veneziano dropped a bombshell – an exact analytic formula for the amplitude that exhibited the conjectured duality”:

$$A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))},$$

where $\alpha(s) = \alpha(0) + \alpha' s$ is the so-called linear Regge trajectory.

- "Very soon after the appearance of the Veneziano amplitude, Virasoro proposed an alternative formula:

$$A(s,t)+A(s,u)+A(t,u)=\frac{\Gamma(-\frac{1}{2}\alpha(s))\Gamma(-\frac{1}{2}\alpha(t))\Gamma(-\frac{1}{2}\alpha(u))}{\Gamma(-\frac{1}{2}\alpha(t)-\frac{1}{2}\alpha(u))\Gamma(-\frac{1}{2}\alpha(s)-\frac{1}{2}\alpha(u))\Gamma(-\frac{1}{2}\alpha(s)-\frac{1}{2}\alpha(t))}$$

- "The motivation for writing down this formula was mostly phenomenological, but it turned out that formulas of this type describe tree amplitudes in a perturbatively consistent quantum theory (of strings)!" Nambu:1969, Susskind:1970
  Open strings in the first case and closed strings in the second case.

(4) Bootstrap program of 1+1D RCFT's: (see my online talks on RCFT's for more details)

- The bootstrap program on 1+1D conformal field theories in 1970's was inspired by bootstrap efforts in 60's and also motivated by the critical phenomena and the strong coupling hardron physics. Polyakov (1970), Ferrara et al. (1973b) and Polyakov (1974).

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle \quad \propto \quad \begin{cases} x_{12}^{-2\Delta_1} & \text{if } \Delta_1 = \Delta_2; \\ 0 & \text{if } \Delta_1 \neq \Delta_2. \end{cases}$$

We show in the present paper that the correlation functions at the transition point are invariant against transformation of a conformal group that includes a change of scale as a particular case. This circumstance makes it possible to calculate in explicit form any three-point correlators and greatly limit the possible form of multipoint correlators.

$$G_{III} = \text{const}\; r_{12}^{(\Delta_c - \Delta_a - \Delta_b)}\, r_{13}^{(\Delta_b - \Delta_c - \Delta_a)}\, r_{23}^{(\Delta_a - \Delta_b - \Delta_c)}. \qquad (10)$$

Formula (10) is confirmed in the flat Ising model, where an expression for the correlator <εσσ> is known, where ε is the energy density and σ is the magnetic moment [6, 7]. For a four-point function, analogous arguments yield the result

$$G_{IV} = r_{12}^{(\Delta_b + \Delta_d}\, r_{24}^{\Delta_a + \Delta_c)}\, r_{12}^{(-\Delta_a - \Delta_b}\, r_{23}^{-\Delta_b - \Delta_c)}\, r_{34}^{(-\Delta_c - \Delta_d}\, r_{41}^{-\Delta_d - \Delta_a)} \times \\ \times F\left(\frac{r_{13}r_{24}}{r_{12}r_{34}}, \frac{r_{14}r_{23}}{r_{12}r_{34}}\right), \qquad (11)$$

- 1970's is a golden age for the study of infinite dimensional Lie algebras in mathematics.
- In 1984, by combining Polyakov's bootstrap program with Feigin-Fuchs' theory on the representations of Virasoro algebra, Belavin, Polyakov and Zamolodchikov made breakthrough in the construction of 1+1D rational CFT's (i.e., minimal models).
- A flood of research works followed BPZ:1984, and culminated in Moore-Seiberg's introduction of modular tensor category (MTC) as the category of modules (E_2-modules) over a chiral algebra (i.e., a vertex operator algebra (VOA)).

The success of the bootstrap program on 1+1D RCFT is due to the fact that conformal symmetry in 1+1D is infinitely dimensional and largely determines the theory.

After Moore-Seiberg:1989, the study of RCFT was carried on mainly by mathematicians.

- There are three success program on the mathematical foundation of open-closed RCFT's (preceded by Cardy's works on boundary-bulk CFT in 80's and 90's)
    - Longo and Rehren's conformal net approach (1990-1999);
    - Fuchs-Runkel-Schweigert's state-sum construction of all "categorical correlators" of all genus from a special symmetric Frobenius algebra in a MTC (1999-2004);
    - Huang and Kong's approach based on rational VOA and its representation theory (2003-2006).
- 1+1D RCFT enriched by topological defects, which were viewed as an extension of the theory instead of a defining data. However, it was observed and highlighted that they determine the dynamical data up to E_2-Morita equivalence.
    - Frölich-Fuchs-Runkel-Schweigert developed the theory of topological defects in RCFT's (2006).
    - It was reformulated in terms of internal homs, enriched categories and center functor by Davydov-Kong-Runkel (2013).

(5) From CFT's to TQFT's:

- In 1988, Segal proposed a mathematical definition of a 1+1D CFT as a symmetric monoidal functor from the category of complex analytic bordisms to the category of (locally convex) topological vector spaces.
- It was immediately simplified to TQFT and generalized to all dimensions by Atiyah (i.e., replacing complex analytic bordisms by topological bordisms).
- It motivated Witten's construction of Chern-Simons-Witten TQFT in 1989.
- Mathematical constructions of TQFT's: Dijkgraaf-Witten theory (1990); Reshetikhin-Turaev TQFT based on modular Hopf algebras (1991); Turaev-Viro TQFT (1992) based on MTC; Turaev's construction of Reshetikhin-Turaev TQFT based on MTC (1994); Freed's works on extended TQFT in mid 90's; Baez-Dolan's cobordism hypothesis (1995); Lurie's sketch of a proof of the cobordism hypothesis (2009).

(6) From TQFT to topological orders: A topological order is gapped quantum phase without symmetry at the zero temperature. Since it is gapped, it hardly responds to any external probes, or equivalently, all correlation functions decay exponentially. It seems that it must be featureless and trivial. Therefore, it is qualified to be called the simplest phase. However, it was one of the greatest discoveries in 1980's that this seemingly 'trivial' phase is actually non-trivial. There are three sources of this great discovery.

- The experimentally discovered 2+1D fractional quantum Hall (FQH) phases.
- Chiral spin liquids emerged from the study of superconductors at high critical temperatures. It was realized that different chiral spin liquids can have the same symmetry but different numbers of heat-conducting edge modes.
- Witten's construction of 2+1D Chern-Simons TQFT in 1989.

These developments motivated Xiao-Gang Wen to introduced the notion of a 'topological order' (TO) in 1989.

In 1990s, a TQFT was viewed as the low energy effective field theory of a topological order (TO); and a TO was viewed as a physical realization of a TQFT. Mathematically, they are "in some sense equivalent (by Baez-Dolan's cobordism hypothesis and later proved by Lurie and Francis under some assumptions).

In retrospect, from a technical point of view, TO should be viewed as a 'simplification' of a TQFT. All non-trivial spacetime manifolds enter the definition of a TQFT as defining data, however, a TO is already well defined on an open disk. Therefore, roughly speaking,

$$\boxed{\text{a TO can be viewed as a TQFT on an open disk.}}$$

There are two different approaches towards quantum field theories.

- functorial quantum field theory (FQFT): A FQFT is defined by a symmetric monoidal functor from the cobordism category Bord with possible additional structures (orientation, framing, spin, G-bundle, conformal/metric structures) to the category of vector spaces (or Hilbert spaces or certain topological vector spaces).
- algebraic quantum field theory (AQFT): An AQFT is defined by the net of local observable algebras (von Neumann algebra, E_n-algebras, chiral algebra, factorization algebra, etc.).

The relation between these two approaches was not completely known. However, some partial results in special cases were known.

1. 1+1D RCFT: FQFT= VOA, AQFT= conformal net (or Beilinson-Drinfeld's chiral algebra);
2. Fully extended TQFT: FQFT=TQFT, AQFT=TO.

The partition function on a non-trivial spacetime manifold M can be obtained by integrating the local data over M. This theory of integrals is called factorization homology.

However, technically, it is indeed a highly non-trivial simplification!

- By restricting to an open disk, it allows us to ask new questions (e.g., anyon condensations and boundary-bulk relation) and discover new concepts (e.g., a morphism between TO's, center functor, gauging) that are natural (or possible?) only on an open disk. These new questions and concepts led us to many deep results that were previously unthinkable (e.g., a unified theory of gapped/gapless phases, topological holography) and completely revolutionized the field.

Remark: I grew up within the community of mathematicians that only work in the framework of FQFT's. After I encountered topological orders in 2008, I shifted my point of view from FQFT to AQFT based on the following observation. It is important to distinguish questions that are non-trivial on an open disk from those that are non-trivial only on non-trivial spacetime manifolds, and I believe that the former questions are much deeper than the latter ones. This is one of many important turning points of my life, all of which share the same feature as some kind of simplification in different contexts. This choice is exactly the opposite to that of the majority of my academic peers.

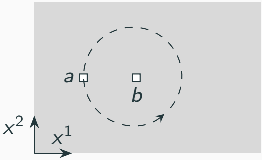
Why does simplicity lead us to a revolution?

1. In history of science, revolutions often happened when the key problem was reduced to its simplest form. For example, the problem of describing planet orbits was eventually reduced to a two-body problem of Newton's gravity! This reduction process last over 2000 years. Quantum mechanics claimed its victory only after it solved the two-body problem of Hydrogen atom. My opinion is that TO should be compared to the two-body problem in Newtonian mechanics and the Hydrogen atom in quantum mechanics.
2. Moreover, simplicity often leads to precise mathematical language and complete solutions. Vague intuitions become precise axioms or notions, upon which a new and solid theoretical framework or platform can be built. On this new and solid platform, we see further. New questions, new concepts and new tools naturally reveal themselves in front of us for free. Our imagination also fly to a new altitude.
3. When "trivial" becoming non-trivial means all previous theories are wrong or badly incomplete.

From topological orders to a unified theory of gapped/gapless quantum liquids

By Witten's Chern-Simons-Witten theory Witten:1989 and Moore-Seiberg's theory on 2D RCFT Moore-Seiberg:1989, it is already clear that Wilson lines in 2+1D Chern-Simons-Witten theory can be fused and braided, thus form a modular tensor category (MTC) (see also Fredenhagen-Rehren-Schroer:1989, Fröhlich-Gabbiani:1990). To interpret Wilson lines as anyons in Fractional Quantum Hall System need extra works. Zhang-Hansson-Kivelson:1989, Moore-Read:1991, Wen:1991, Kitaev:cond-mat/0506438.

It was established later that a 2+1D topological order can be completely characterized by a pair $(\mathcal{C}, c)$, where $\mathcal{C}$ is the modular tensor category of anyons and c is the chiral central charge. Kitaev:cond-mat/0506438 This is the first precise and complete mathematical description of a quantum phase (or the previously mentioned 'solution space' of a quantum many-body system).



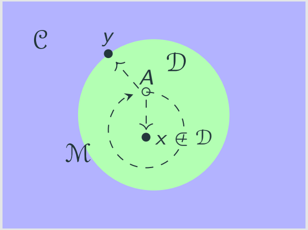
Remark: Similar to that of Galois' theory, this solution space can add $\oplus$ and multiply $\otimes$, and both operations are 'commutative'.

The mathematical theory of anyon condensation has a long history Moore-Seiberg:1988-1989, Bais-Slingerland:2002-2008, Kapustin-Saulina:1008.0654, Levin:1301.7355, Barkeshli-Jian-Qi:1305.7203, ..., Böckenhauer-Evans-Kawahigashi:math/9904109,0002154, Kirillov-Ostrik:math/0101219, Frölich-Fuchs-Runkel-Schweigert:math/0309465, K.:1307.8244, ...

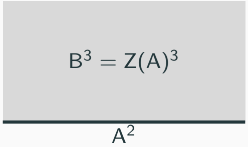
Theorem (K.:1307.8244)

Let $\mathcal{C}$ and $\mathcal{D}$ be the modular fusion categories (MFC) of anyons in two 2+1D topological orders. An anyon (or boson) condensation from $\mathcal{C}$ to $\mathcal{D}$, which produces a gapped domain wall $\mathcal{M}$, is determined by a condensable E_2-algebra A in $\mathcal{C}$.

- $\mathcal{D} = \{$*deconfined particles*$\}$ = *the category* $\mathcal{C}_A^{loc}$ *of local* A*-modules in* $\mathcal{C}$; $\mathbb{1}_{\mathcal{D}} = A \in \mathcal{C}$;
- $\mathcal{M}$ *is the category* $\mathcal{C}_A$ *of (de)-confined particles* = *the category of right* A*-modules in* $\mathcal{C}$.
- *bulk-to-wall maps:* $\mathcal{C} \xrightarrow{-\otimes A} \mathcal{M} \hookleftarrow \mathcal{D}$
- *Boundary-bulk relation:* $\mathfrak{Z}_1(\mathcal{C}_A) \simeq \mathcal{C} \boxtimes \overline{\mathcal{C}_A^{loc}}$.



By restricting 2+1D topological order B^3 to an open 2-disk, it is also natural to ask what happens on a gapped boundary A^2:



By Kitaev-K.:1104.5047, K.:1307.8244, Ogata:2308.08087, we have the following results.

1. Particles on the gapped boundary A^2 for a fusion 1-category $\mathcal{A}$. This is the first complete mathematical description of a 1+1D gapped phase (without symmetry).
2. Boundary-bulk relation: $\mathcal{B} \simeq \mathfrak{Z}_1(\mathcal{A})$, where $\mathcal{B}$ is the category of bulk anyons (i.e., a MFC) and $\mathfrak{Z}_1(\mathcal{A})$ is the Drinfeld center of $\mathcal{A}$. Such a precise relation is possible only when the mathematical description of the phases are complete (at least to some extent).

   New structures, relations and principles emerge only when the mathematical description is complete!

Theorem (Ising chain, K.-Wen-Zheng:2108.08835)

1. *the symmetric phase of 1+1D Ising chain can be mathematically characterized by* ${}^{\mathsf{TC}}\mathrm{Rep}(\mathbb{Z}_2)$*, which is a fusion category* $\mathrm{Rep}(\mathbb{Z}_2)$ *enriched a modular fusion category* TC*.*
2. *the symmetry-breaking phase of 1+1D Ising chain can be mathematically characterized by* ${}^{\mathsf{TC}}\mathrm{Vec}_{\mathbb{Z}_2}$*, which is a fusion category* $\mathrm{Vec}_{\mathbb{Z}_2}$ *enriched a modular fusion category* TC*.*

1. $\mathsf{TC} \simeq \mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2)) \simeq \mathfrak{Z}_1(\mathrm{Vec}_{\mathbb{Z}_2})$ is the modular fusion category of particles in the bulk of the 2+1D toric code model.
2. $\mathrm{Rep}(\mathbb{Z}_2)$ the fusion category of particles on the smooth boundary of the toric code.
3. $\mathrm{Vec}_{\mathbb{Z}_2}$ is the fusion category of particles on the rough boundary of the toric code.

Unique Bulk Hypothesis/Principle K.-Wen:1405.5858 A (potentially anomalous) nD topological order (or quantum liquid) A^n has a unique n+1D anomaly-free topological order as its bulk, denoted by $\mathfrak{Z}(\mathrm{A}^n) = \mathfrak{Z}(\mathrm{A})^{n+1}$ and often depicted graphically as follows.

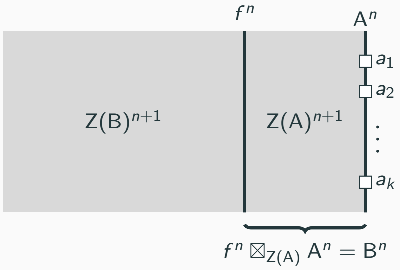
The unique bulk of A^n is also called the gravitational anomaly of A^n.

Morphism between quantum liquids

Definition: K.-Wen-Zheng:1502.01690 A morphism $f : \mathrm{A}^n \to \mathrm{B}^n$ between two topological orders (or quantum liquids) A^n and B^n (both having gapped bulks) is a gapped wall f_n between $\mathrm{Z}(\mathrm{A})$ and $\mathrm{Z}(\mathrm{B})$ such that

where $f_n \boxtimes_{\mathrm{Z}(\mathrm{A})^{n+1}} \mathrm{A}^n = \mathrm{B}^n$ denotes the fusion of f_n with A^n along $\mathrm{Z}(\mathrm{A})_{n+1}$. Such a fusion is well defined for gapped domain walls among topological orders.

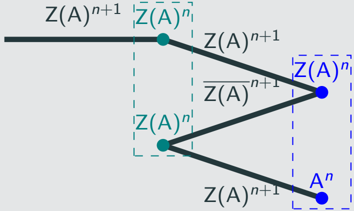
$f : A^n \to B^n$



- In mathematics, the notion of a morphism is arguably the most of important concept in mathematics because all mathematical notions can be defined by morphisms (e.g., 1, 1+1). It is necessarily one of the most important concepts in physics.
- In 2022, a part of this notion was rediscovered as the so-called sandwich construction by Freed, Moore and Teleman. Freed-Moore-Teleman:2209.07471. The Z(A) in the sandwich was later called topological symmetry or SymTFT or SymTO. Sandwich construction has been used to study almost all aspects of the theory, such as dualities, gauging, holography, etc. The reason behind its wide applications is precisely because it is a morphism.

Theorem (bulk=Z(boundary): K.-Wen-Zheng:1502.01690)

The pair $(\mathrm{Z(A)}^n, m)$, *where* $m : \mathrm{Z(A)}^n \boxtimes \mathrm{A}^n \to \mathrm{A}^n$ *is defined below, satisfies the universal property of center. That is, if* X^n *is an nD topological order equipped with a morphism* $f : \mathrm{X}_n \boxtimes \mathrm{A}^n \to \mathrm{A}^n$, *then there is a unique morphism* $f' : \mathrm{X}^n \to \mathrm{Z(A)}^n$ *such that the following diagram is commutative:*



$$\underbrace{(\mathrm{Z(A)}^n \boxtimes \mathrm{Z(A)}^n) \boxtimes_{\mathrm{Z(A)}\boxtimes\overline{\mathrm{Z(A)}}\boxtimes\mathrm{Z(A)}} (\mathrm{Z(A)}^n \boxtimes \mathrm{A}^n)} = \mathrm{A}^n$$

$$\begin{tikzcd}
 & \mathrm{Z(A)}_n \boxtimes \mathrm{A}^n \arrow[ddr, bend left, "m"] & \\
 & \mathrm{X}_n \boxtimes \mathrm{A}^n \arrow[u, "\exists!\ f'\boxtimes\mathrm{id}_\mathrm{A}"'] \arrow[dr, "f"] & \\
\mathrm{A}^n \arrow[uur, bend left, "\iota\boxtimes\mathrm{id}_\mathrm{A}"] \arrow[ur] \arrow[rr, "\mathrm{id}_\mathrm{A}"] & & \mathrm{A}^n
\end{tikzcd}$$

Mathematical characterization of an n+1D topological order

Theorem (K.-Wen-Zheng:1502.01690, Johnson-Freyd:2003.06663, K.-Zheng:2011.02859,2107.03858)

1. *The category $\mathcal{C}$ of defects in an n+1D topological order C^{n+1} is an E_1-fusion n-category* K.-Wen-Zheng:2015, Johnson-Freyd:2003.06663*; and the category $\Omega^{k-1}\mathcal{C}$ of defects of codimension k and higher is E_k-fusion* Johnson-Freyd:2003.06663, K.-Zheng:2011.02859.
2. *The gravitational anomaly of C^{n+1} is characterized by a 1-dimensional higher topological order $\mathsf{B}^{n+2} = \mathsf{Z}(\mathsf{C})^{n+2}$ such that $\Omega\mathcal{B} \simeq \mathfrak{Z}_1(\mathcal{C})$.*
3. *When C^{n+1} is anomaly-free,*
   (a) *$\mathcal{C}$ is a non-degenerate fusion n-category, i.e., $\mathfrak{Z}_1(\mathcal{C}) \simeq n\mathrm{Vec}$;*
   (b) *$\Omega\mathcal{C}$ is non-degenerate braided fusion $(n-1)$-category, i.e., $\mathfrak{Z}_2(\Omega\mathcal{C}) \simeq (n-1)\mathrm{Vec}$;*
   (c) *$\mathcal{C} = \Sigma\Omega\mathcal{C} \simeq \mathrm{RMod}_{\Omega\mathcal{C}}(n\mathrm{Vec})$.* Gaiotto-Johnson-Freyd:1905.09566

Mathematical characterization of an n+1D SPT/SET order

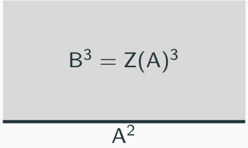
Theorem K.-Lan-Wen-Zhang-Zheng:2003.08898

For $n \geq 1$, an n+1D (spatial dimension) SET order with an internal higher symmetry given by a symmetric fusion n-category $\mathcal{R}$ and possibly with a 't Hooft anomaly (without gravitational anomaly) is characterized by a quintuple $(\mathcal{A}, \iota_\mathcal{A}; \mathcal{M}, \iota_\mathcal{M}; \phi)$, where

1. $(\mathcal{A}, \iota_\mathcal{A})$ is a unitary fusion n-category over $\mathcal{R}$ describing all topological excitations (including all condensation descendants).
2. $(\mathcal{M}, \iota_\mathcal{M})$ is a minimal modular extension of $(\mathcal{R}, \mathrm{id}_\mathcal{R})$, which determines the 't Hooft anomaly (the bulk n+2D SPT order).
3. $\phi : \mathcal{M} \to \mathfrak{Z}_1(\mathcal{A})$ is a braided equivalence such that the diagram commutes:

$$\begin{tikzcd} & \mathcal{R} \arrow[dl, hook', "\iota_\mathcal{M}"'] \arrow[dr, hook, "\iota_\mathcal{A}"] & \\ \mathcal{M} \arrow[rr, "\phi", "\simeq"'] & & \mathfrak{Z}_1(\mathcal{A}). \end{tikzcd}$$

"Bulk=Z(boundary)" applies to the case: the 2+1D bulk topological order B^3 has a gapless boundary A^2:



Again, let $\mathcal{B}$ is the category of bulk anyons (i.e., a MTC). Then we must have

$$\mathcal{B} \simeq \mathfrak{Z}_1(?)$$

where ? is something unknown but definitely associated to the gapless boundary A^2. This means that there should be a unified mathematical theory of gapped/gapless boundaries of 2+1D topological orders.

Theorem (K.-Zheng:1705.01087, 1905.04924, 1912.01760)

The gapped/gapless boundaries of a 2+1D topological order $(\mathcal{C}, c)$ can be completely characterized or classified by the triples $(V, \phi, {}^{\mathcal{B}}\mathcal{S})$, where

1. V *is the chiral/non-chiral symmetry:*
   - *for a chiral gapless boundary, V is a rational VOA of central charge c;* *Huang:math/0502533*
   - *for a non-chiral gapless boundary, V is a rational full field algebra $(c_L - c_R = c)$* *K.-Huang:math/0511328;*
   - *When $V = \mathbb{C}$, it describes a gapped boundary.* *Kitaev-K.:1104.5047*
2. $\phi : \mathrm{Mod}_V \xrightarrow{\simeq} \mathcal{B}$ *is a braided equivalence.*
3. $\mathcal{S}$ *is a fusion category equipped with $\mathcal{C} \boxtimes \overline{\mathcal{B}} \simeq^{br} \mathfrak{Z}_1(\mathcal{S})$, which determines ${}^{\mathcal{B}}\mathcal{S}$ via the so-called canonical construction, i.e. $M_{a,b} = [a, b] \in \mathcal{B}$.* *Lindner:1981,Morrison-Penneys:1701.00567*

Theorem (K.-Zheng:1704.01447, K.-Yuan-Zhang-Zheng:2104.03121)

The bulk is the center of a boundary, i.e. $\mathcal{C} \simeq \mathfrak{Z}_1({}^{\mathcal{B}}\mathcal{S})$.

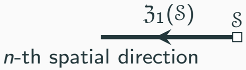
Theorem (K.-Zheng:1705.01087, 1905.04924, 1912.01760)

A gapped/gapless boundary X *of a 2+1D topological order* $(\mathcal{C}, c)$ *can be completely characterized by a triple* $\mathsf{X} = (\mathsf{X}_{\mathrm{lqs}}, \mathsf{X}_{\mathrm{top}})$*, where* $\mathsf{X}_{\mathrm{lqs}} = (V, \phi)$ *is called local quantum symmetry;* $\mathsf{X}_{\mathrm{top}} = {}^{\mathcal{B}}\mathcal{S}$ *is called topological skeleton.*

When $(\mathcal{C}, c)$ *is trivial, i.e.* $(\mathcal{C}, c) = (\mathrm{Vec}, 0)$*, we obtain a holographic duality.*

Topological Wick Rotation leads to a new type of holographic dualities

K.-Zheng:1705.01087, 1905.04924, 1912.01760, 2011.02859: (nD is the spacetime dimension.)



an n+1D topological order with a gapped boundary	an nD quantum liquid (SPT/SET/SSB/gapless) with an internal symmetry of finite type
$\mathcal{S}$ is the category of topological defects on the boundary	$\mathcal{S}$ is the category of topological defects { the superselection sectors of states }
$\mathfrak{Z}_1(\mathcal{S})$ is the category of topological defects in the bulk K.-Wen-Zheng:1502.01690,1702.00673	$\mathfrak{Z}_1(\mathcal{S})$ is the category of topological sectors of operators { spaces of non-local operators invariant under LOA}
$\mathfrak{Z}_1(\mathcal{S})$ naturally acts on $\mathcal{S}$	$\mathfrak{Z}_1(\mathcal{S})$-action on $\mathcal{S} \rightsquigarrow$ an enriched category ${}^{\mathfrak{Z}_1(\mathcal{S})}\mathcal{S}$

Theorem (Ising chain, K.-Wen-Zheng:2108.08835)

1. *the symmetric phase of 1+1D Ising chain can be mathematically characterized by* ${}^{\mathsf{TC}}\mathrm{Rep}(\mathbb{Z}_2)$, *which is a fusion category* $\mathrm{Rep}(\mathbb{Z}_2)$ *enriched a modular fusion category* TC.
2. *the symmetry-breaking phase of 1+1D Ising chain can be mathematically characterized by* ${}^{\mathsf{TC}}\mathrm{Vec}_{\mathbb{Z}_2}$, *which is a fusion category* $\mathrm{Vec}_{\mathbb{Z}_2}$ *enriched a modular fusion category* TC.

1. $\mathsf{TC} \simeq \mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2)) \simeq \mathfrak{Z}_1(\mathrm{Vec}_{\mathbb{Z}_2})$ is the modular fusion category of particles in the bulk of the 2+1D toric code model.
2. $\mathrm{Rep}(\mathbb{Z}_2)$ the fusion category of particles on the smooth boundary of the toric code.
3. $\mathrm{Vec}_{\mathbb{Z}_2}$ is the fusion category of particles on the rough boundary of the toric code.

$$\boxed{\text{2+1D toric code model} + \text{gapped boundaries}} \xrightarrow{\text{top. Wick rotation}} \boxed{\text{1+1D Ising model}}.$$

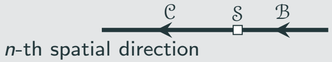
$$(\mathsf{TC}, \mathrm{Rep}(\mathbb{Z}_2)), \quad (\mathsf{TC}, \mathrm{Vec}_{\mathbb{Z}_2}) \quad \mapsto \quad {}^{\mathsf{TC}}\mathrm{Rep}(\mathbb{Z}_2), \quad {}^{\mathsf{TC}}\mathrm{Vec}_{\mathbb{Z}_2}.$$

This holographic dualities were explicitly/implicitly/partially discovered in various contexts.

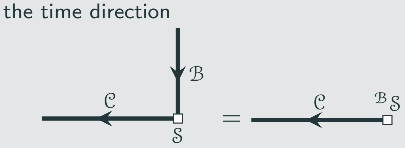
- Witten:1989, Moore-Seiberg:1988-1989, Frölich-Fuchs-Runkel-Schweigert:2006-2008, K.-Runkel:2008, K.:1307.8244, ...
- Generalized KW dualities: Freed-Teleman:1806.00008, Lootens-Delcamp-Ortiz-Verstraete:2112.09091.
- Categorical Symmetries: Ji-Wen:1912.13492, K.-Lan-Wen-Zhang-Zheng:2003.08898,2005.14178, Albert-Aasen-Xu-Ji-Alicea-Preskill:2111.12096, Chatterjee-Wen:2203.03596,2205.06244, Liu-Ji:2208.09101, Chatterjee-Ji-Wen:2212.14432
- Topological Wick Rotation: K.-Zheng:1705.01087, 1905.04924, 1912.01760, K.-Zheng:2011.02859, K-Wen-Zheng:2108.08835, Xu-Zhang:2205.09656, Lu-Yang:2208.01572
- Classical Statistical Models: Aasen-Mong-Fendley:2008.08598
- SymTFT: Gaiotto-Kulp:2008.05960, Bhardwaj-Lee-Tachikawa:2009.10099, Apruzzi-Bonetti-Etxebarria-Hosseini-Schafer-Nameki:2112.02092, Freed-Moore-Teleman:2209.07471, Apruzzi:2203.10063, Moradi-Moosavian-Tiwari:2207.10712, ...
- Strange correlators: Bal-Williamson-Vanhove-Bultinck-Haegeman-Verstraete:1801.05959

Topological Wick Rotation in higher dimensions II: K.-Zheng:1905.04924

For an n+1D topological order $\mathcal{C}$,



$\mathcal{S}$ is an "anomalous gapped boundaries" of $\mathcal{C}$



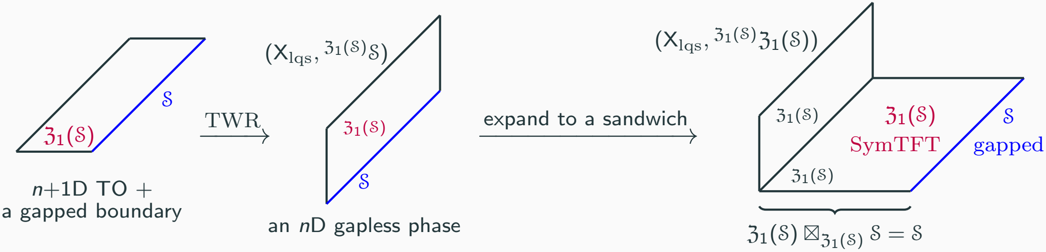
nD potentially gapless boundaries of $\mathcal{C}$

Moreover, the boundary-bulk relation holds, i.e. $\mathcal{C} \simeq \mathfrak{Z}_1({}^{\mathcal{B}}\mathcal{S})$. K.-Zheng:in preparation

This leads to a precise mathematical characterization of the topological skeleton of a quantum liquids. The topological skeleton X_{top} does not determine if X^n is gapped or gapless.

In K.-Zheng:2201.05726, we proposed a theory of "local quantum symmetry" $\mathrm{X}_{\mathrm{lqs}}$ as the mathematical theory of topological nets and defects of (symmetric) local operator algebras in nD generalizing that of conformal nets and defects in 2D.

Relation to the idea of SymTFT: Expand the gapless system to a sandwich such that the fusion category symmetry $\mathcal{S}$ lives on one side of the sandwich and leave the dynamical data on the other side. K.-Zheng:1705.01087



1. Our theory gives a more precise explanation of the ideas in "SymTFT".
2. TWR also tells you that a topological data of a gapless phase is precisely 'the half of a sandwich' or a 'quiche' named in Freed-Moore-Teleman:2209.07471.

Categories of quantum liquids

We denote the category of nD anomaly-free gapped/gapless quantum liquids by QL^n and that of the topological skeletons of nD quantum liquids by QL^n_{top}.

Theorem (K.-Zheng:2011.02859, 2201.05726)

$$\mathrm{QL}^n \simeq \mathrm{QL}^n_{top} \simeq \bullet/(n+1)\mathrm{Vec},$$

where $(n+1)\mathrm{Vec} = \Sigma^n\mathrm{Vec} = \Sigma^{n+1}\mathbb{C}$ *Gaiotto-Johnson-Freyd:1905.09566 is the category of nD topological orders (that admit gapped boundaries) and higher codimensional defects.*

(1) The category of nD topological orders and potentially gapless domain walls: $\mathcal{TO}^n \simeq (\bullet/(n+1)\mathrm{Vec})|_{\bullet\to\bullet}$.
(2) If we impose some reconstruction theorems for rational VOA's, then we obtain the category of 1+1D rational CFT's as $\mathcal{QL}^2 \simeq \bullet/3\mathrm{Vec}$.

Conclusion: We are entering a new age of studying all QFT's or quantum phases as a whole, i.e. studying the categories of QFT's or quantum phases. It provides a unified framework to study higher (invertible/non-invertible) symmetries, dualities, holography, dimension reductions, symmetry-breaking/condensation/gauging, boundary-bulk duality and anomalies, classification problems, etc. all at the same time.

The very existence of such a unified theoretical framework is completely unthinkable five years ago. New language, new concepts, new principles, new tools and new results are emerging in an unprecedented way. It is the sign of a true revolution.

Long march to a new calculus

When we encounter a new exotic phase, the first question to ask is how to characterize it. Another natural question is how to realize it by lattice models.

Indeed, in the last 30 years, the systematic study of topological orders and SPT/SET orders has focused on the following two directions

1. Find a precise and complete mathematical characterization, which is also the first step in the classification program.
2. Lattice model and tensor network realizations.

Classification Program

1. When math characterization is known, i.e., topological orders and SPT/SET/SSB orders with an finite internal symmetry:
    - Classification of modular tensor 1-category is still out of research.
    - Classification of modular 2-categories and fusion 2-categories.
    - E_k-fusion higher categories are still lack of examples.
2. Find math characterization of SPT/SET/SSB orders with non-trivial spacetime symmetries or spacetime symmetry mixed with internal symmetries.
3. Find math characterization of (symmetry enriched) gapless quantum liquids then develop a classification theory, including
    - develop a theory of higher dimensional conformal field theories,
    - develop Kong-Zheng's theory of topological nets or other alternatives.

Lattice model and tensor network realizations:

1. Lattice model realizations for non-chiral/chiral topological orders, SPT/SET/SSB orders and gapless quantum liquids. Use these lattice models to study various important questions, especially those already known in the formal theory, such as boundary/wall physics, condensations, gauging symmetry and holography.
2. Tensor network realizations of above things. I think that tensor networks are very powerful in the study of strong coupling systems. It is a framework, in which direct numerical simulations can be carried out and, at the same time, **abstract formal theory can be developed in its fully generality**. Tensor network is gradually becoming the third independent approach towards quantum many-body systems (in addition to Lagrangian QFT's and lattice models).

Galois' approach = Microscopic approach + Macroscopic approach.

Note that classification program are not enough for a new paradigm beyond Landau. A fundamental ingredient of the yet-unknown paradigm is the theory of phase transitions. Although the study of phase transitions never stopped in the past, these works are mainly case-by-case study and not as systematic and universal as the classification program. The lack of systematic study as in the classification program in last 30 years is mainly because a phase transition closes the energy gap and we lack a systematic and universal understanding of gapless many-body systems.

The situation has been changed in 2017-2019 as a unified mathematical theory of gapped and gapless quantum liquids is emerging. My prediction is that there will a significant progress in establishing a new paradigm of phase transitions in the next 10 years.

The new paradigm of phase transitions demands an entirely new calculus.

- Currently, the bootstrap program on the condensations of topological defects (or liquid-like non-topological defects) without determining the critical point is known for its general framework. Mathematically, this bootstrap program is based on higher categories, higher algebras and higher representation theory. This theory still lack of examples and is far from being developed.

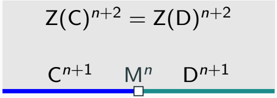
  Remark: It is not that physics needs wait for the development of higher categories, higher algebras and higher representation theory in order to make progress. The reality is that physics is pushing forward the development of the math theory in a surprisingly efficient way. In other words, physics and mathematics are developed at the same time. At least in this case, it is meaningless to distinguish physics and mathematics.

Theorem (K.-Zhang-Zheng-Zhao:2403.07813): Condensing a k-codimensional topological defect A in an n+1D (potentially anomalous) topological order C^{n+1} amounts to a k-step process.

(1) The k-codimensional defect $A \in \Omega^{k-1}\mathcal{C}$ is condensable if it is equipped with the structure of a condensable E_k-algebra, i.e. an algebra equipped with compatible multiplications in k independent directions.
We first condense A along one of the transversal directions x^k, thus obtaining a $(k-1)$-codimensional defect $\Sigma A := \mathrm{RMod}_A(\Omega^{k-1}\mathcal{C}) \in \Sigma\Omega\mathcal{C}$.

(2) It turns out that ΣA is naturally equipped with the structure of a condensable E_{k-1}-algebra, thus it can be further condensed along one of the remaining transversal direction x^{k-1}, thus obtaining a $(k-2)$-codimensional defect $\Sigma^2 A := \mathrm{RMod}_{\Sigma A}(\Omega^{k-2}\mathcal{C})$.

(3) In the k-th step, condensing the 1-codimensional defect $\Sigma^{k-1}A$ along the only transversal direction defines a phase transition to a new n+1D topological order D^{n+1}, which is Morita equivalent to C^{n+1}, and a gapped domain wall M^n.



$$\mathcal{D} \simeq \mathrm{Mod}^{E_1}_{\Sigma^{k-1}A}(\mathcal{C}) = \mathrm{BMod}_{\Sigma^{k-1}A|\Sigma^{k-1}A}(\mathcal{C})$$
$$(\mathcal{M}, m) = (\Sigma^k A := \mathrm{RMod}_{\Sigma^{k-1}A}(\mathcal{C}), \Sigma^{k-1}A).$$

- A more complete condensation theory should allow us to determine the critical point of the phase transition precisely. This question is widely open. Mathematically, it is beyond current known mathematical framework. It is possible that the language of (∞, n)-category is needed but not necessarily enough for this task.
- Largely generalize current condensation theory in order to find direction connections to other condensation phenomena, such as BEC.

Where are we in the development of the yet-unknown calculus by comparing it with the old calculus? Here is my estimation. We are only at the beginning stage of understanding 'integers' perhaps with some vague idea of the existence of $\sqrt{2}$.

The last important and missing ingredient in this Galois' approach is the following problem.

- How to combine macroscopic approach with microscopic approach?

A lot of non-trivial works are needed to fill in the gap. We do not yet know precisely how this combination can solve the strong-coupling problem. One of the possible scenario is that, once we obtain a rather complete classification of quantum phases, it might be possible to use a few important but partial characterization of a strong-coupling system, say High Tc superconductor, to limit the possible quantum phases that associated to this partial characterization to a special family, which, in the best scenarios, consists of only finitely many phases. If this is possible, it is already very close to a complete solution.

International situation and our tasks

- Our prediction of this revolution is in 2017 (K.-Wen-Zheng:1502.01690,1702.00673, K.-Zheng:1705.01087).
- In 2020 (?), two multi-million grants: Simons Collaboration on Ultra-Quantum Matter https://projects.iq.harvard.edu/ultra-qm and on Global Categorical Symmetries https://scgcs.berkeley.edu/
- In 2022, we witnessed the rediscovery of the earlier results in terms of "SymTFT/SymTO" or "sandwich construction" or "topological holography".
- In 2022-2024, we witnessed a flood of young Ph.D graduates and postdocs in this direction, a flood of international conferences on this topics and a flood of publications in this direction.
- Job positions were opened up or created for researchers in this direction, including many top schools and top institutes, such as IHÉS, RIMS (to my surprising).
- Not many activities have happened in China so far.

What do we need to do?

- Grants. Perhaps we need a grant similar to Simons Collaboration on ‘Ultra-Quantum Matter’ and ‘Global Categorical Symmetries’.
- We need serious articles, reviews and lectures on the importance of the fields.
- We need positions for new postdocs and young faculties.
- We need open and design introductory courses, lecture notes or books.
- We need workshops and conferences. I think that it is reasonable to have at least 1 annul conference and 2-3 (regional) workshops.
- Most importantly, we need cultivate a new generation of scholars, who are ready to meet the challenge of our time. What kind of person is that?

If you agree with the following claim on the nature of the strong-coupling or infinitely-more problem, i.e.,

- the strong-coupling or infinitely-more problem is not a pure physics problem, and it demands both new physics and new mathematics,

then you can see that it is impossible for those physicists, who are educated by the tradition of the previous generations, to find and develop the new theory alone. It is necessarily a joint task for both physicists and mathematicians.

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What do I mean by ‘a joint task’? Does that mean physicists raise and formulate the question then ask mathematicians to solve it? The answer to this question depends on the answers to the following two questions:

(1) are physicists capable of asking all necessary questions?

(2) are physicists capable of reformulating the question in precise mathematical language?

My answer to the (1) is ‘NO’ and to (2) is ‘YES, but such physicists are very rare’.

(2). Are physicists capable of reformulating the question in precise mathematical language?

Comments copy-pasted from a letter of recommendation I wrote years ago:

> "A not-yet-known new paradigm demands us not only to master those existing and necessary theoretical tools, but also, perhaps more importantly, to introduce new physical concepts and new mathematical language, and to develop new mathematical tools. Among all these challenges, the hardest one is to walk through a labyrinth of new phenomena, which is often shrouded by vague and sometimes misleading intuitions, by patiently reorganizing and constantly reformulating, to reach a level of clarity that is sufficient for a precise mathematical definition. The ability to meet this challenge is a true manifestation of creativity. It demands not only a deep understanding of the physics problems but also an high level of mathematical maturity. Not many theoretical physicists are able to meet this challenge. ...."

It nevertheless remains true that the finding of a 'satisfactory' model (or, if need be, several such models, which would 'connect' in as satisfactory a manner as possible...) – whether this model was 'continuous', 'discrete' or of a 'mixed' nature – would require a great conceptual imagination, and a consummate art for apprehending and updating mathematical structures of a new type. This kind of imagination or 'flair' is rare indeed, not only amongst physicists (Einstein and Schrödinger seem to be notable exceptions), but even amongst mathematicians (and there I am speaking in full knowledge of the facts).

— Alexandre GROTHENDIECK

Harvests and Seeds. Chapter 2. A Walk through my work or The Child and the Mother. 2.20 A glance over the road, p.80 (transcription by Yves Pocchiola)

(1) Are physicists capable of asking all necessary questions? My answer: NO.

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- The basic language of physics – calculus – was developed out of large scale physics, which has an intrinsic destructure feature because find structures in a small region becomes less important in a larger scale. As a consequence, the classical calculus is a destructured calculus. As we approach a microscopic world, however, we see more and more structures. Therefore, we need a structured calculus for quantum physics.
- Category theory is a structured calculus. It should not be viewed as a subfield of mathematics based on the same set-theoretical foundation as other subfields. Itself is a new foundation and a new way of thinking, and is completely missing in old physics tradition, which was largely influenced by reductionism point of view somewhat parallel to the set-theoretical philosophy. Category theory teaches us that there are new categorical truths that are impossible to state in a set-theoretical framework. Namely, there are truths living outside of the set-theoretical mathematical world.
- Physicists have walked a long way on the road of destructure (starting from their first education of math). It becomes very difficult for working physicists to accept the categorical language and, more importantly, to master the categorical way of thinking.

Therefore, what I mean by ‘a joint work’ is not limited to the scenario, in which physicists raise and formulate the question then ask mathematicians to solve it, nor to the scenario, in which mathematicians develop some new mathematical theory and ask physicists for a physical meaning.

In addition to above two scenarios, this ‘joint task’ should, at least, include another important part.

- To cultivate and nourish a new generation of men or women. Each of them is a physicist and a mathematician at the same time. They are capable of asking new questions that naturally grow out the strong-coupling or infinitely-more problem and are completely unknown in both mathematics and physics communities. Such a man or a woman used to be called a natural philosopher, a name which has been long forgotten.
- This ‘joint task’ will be led by not one but a group of natural philosophers.

To sum up, I predict that the long-awaited renewal (if it is still coming...) will come from a born mathematician well-informed about the big questions of physics rather than from a physicist. But above all, we will need a man with the kind of 'philosophical openness' necessary to take hold of the heart of the problem. This problem is by no means a technical one, but is rather a fundamental question of 'natural philosophy'.

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I am speaking in full knowledge of the fact that "a born mathematician well-informed about the big questions of physics" is not enough for this task.

Mother nature will bless those who have faith in her transcendental beauty!

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Let's make physics beautiful again. Thank you!