

An invitation to topological orders and category theory

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Abstract

Although it has been a well-known fact, for more than two decades, that category theory is needed for the study of topological orders, it is still a non-trivial challenge for students and working physicists to master the abstract language of category theory. In this work, for those who have no background in category theory, we explain in great details how the structure of a (braided) fusion category naturally emerges from lattice models and physical intuitions. Moreover, we show that nearly all mathematical notions and constructions in fusion categories and its representation theory, such as (monoidal) functors, Drinfeld center, module categories, Morita equivalence, condensation completion and fusion 2-categories, naturally emerge from lattice models and physical intuitions. In this process, we also introduce some basic notions and important results of topological orders.

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1 Introduction

In the last two decades, we have witnessed unprecedented interactions between topics in topological phases of matter (such as topological orders) and those in pure mathematics. It is not surprising to see the applications of geometry, topology and representation theory in physics because it is a long and old tradition. What is surprising is that category theory, one of the most abstract subjects in mathematics, has deep applications in the study of topological phases. These applications are not accidental but rooted in the very heart of quantum many-body theory or quantum field theory (QFT). As we show in this work, categorical structures are the fundamental ingredients of every quantum many-body system or QFT. Category theory is becoming an indispensable language and tool, or a new calculus, in the study of all quantum many-body systems and QFT's.

Although there are many excellent textbooks on category theory (see for example [Mac78, Awo10, Rie17]) and a modern textbook on tensor categories [EGNO15], unfortunately, none of them are written for physicists. Many years have passed since the first application of category theory in the study of topological orders. It remains a daunting task for working physicists and students to master or even to appreciate the language of category theory by first reading these books. In fact, there are still many mathematicians regarding category theory as a boring and useless subject or finding it difficult to adapt themselves to the abstract way of thinking provided by category theory. Taking into account the fact that all mathematicians were educated in the abstract language of modern mathematics from their first days of professional training, we can then imagine how difficult it can be for working physicists, who accustomed themselves to a concrete way of thinking (or computing) based on physical intuitions, to find category theory inspiring or useful. Underneath this discouraging abstractness lies an ironic fact that physical intuitions or realities are completely categorical.

Therefore, in order to help students and working physicists to enter this fast developing field and to catch up with the latest developments, it is highly desirable to have some introductory articles or books focusing on the application of category theory in physics. The goal of this work is to partially fill the gap. For those who have no background in category theory, we explain in great details how categorical structures naturally emerge from simple lattice models and physical intuitions. In other words, instead of first introducing categorical notions then giving their physical application, we reverse the order by first analyzing structures hidden in a lattice model then summarizing them as categorical notions. As we proceed, almost all ingredients in the mathematical theory of monoidal categories, such as (braided) fusion categories, (monoidal) functors, natural transformations, module categories, module functors, Drinfeld centers, Morita equivalence, condensation completion and fusion 2-categories, etc., naturally emerge. Although the mathematical theory that we can cover in this work seems quite rich already, it is still just the tip of the iceberg. At the end of this work (see Section 5), we provide a brief outline of a few advanced topics.

Although the field of topological orders is more than 30 years old, our understanding of subject is still quite limited and unsatisfying. One of the manifestation of this fact is the constant evolution of the basic notions and the occasional radical changes of the point of view. This phenomenon causes a lot of difficulties and confusion in reading the literature. In this work, we try our best to distinguish notions with subtle differences and clarify misleading statements that are common in the literature. In a few places, we adopt some modern points of view available only recently [KZ20, KZ22b]. At the same time, we try to be open to future evolution, an attitude which is reflected in many of our remarks. For this reason, we believe that this work can be useful to experts in the fields. It might also be useful to mathematicians who want to get some ideas on how category theory is used in physics.

The popularity of topological orders in recent years is not the scientific reason why we are interested in this subject. In addition to the obvious and practical reason of understanding (or finding more) exotic new materials that exhibit topological features, we are interested more in the reasons lying in the theoretical foundation of quantum many-body physics or QFT's. In Section 1.1 and 1.2, we explain the theoretical motivations for and the theoretical significance of the study of topological orders, some of which are becoming apparent only recently. In Section 1.3, we briefly explain the significance and impacts of category theory in both mathematics and physics. In Section 1.4, we briefly explain the content and the layout of this paper, and provide some suggestions on how to read this paper.

Throughout this work, we use nd to represent the spatial dimension and nD to represent the space-time dimension.

1.1 Original motivations

We start from introducing some terminologies used in this work. A quantum matter is a state of matter at zero temperature. A quantum phase is a universal class of quantum matters. There are two types of quantum phases: gapped and gapless. In this work, we are interested in a special family of gapped quantum phases called *gapped quantum liquids* [ZW15] (see Section 2.1). A gapped quantum liquid without symmetry is called a topological order [Wen89, Wen90]. Gapped quantum liquids with symmetries include gapped spontaneous symmetry breaking (SSB) orders, symmetry protected trivial (SPT) orders [GW09, CLW11, CGLW13] and symmetry enriched topological (SET) orders [CGW10]. In this work, we focus on 2d and 1d topological orders and only briefly comment on other quantum phases in Remarks and in Section 5.

At the zero temperature, due to the finite energy gap, a gapped quantum phase hardly responds to any external probes. It seems that it must be featureless and trivial. It was one of the greatest discoveries in 1980's that this seemingly 'trivial' phase is actually non-trivial. There are three sources of this great discovery. The first one is in the study of the experimentally discovered 2d fractional quantum Hall (FQH) phases [TSG82]. It was realized that different FQH phases have the same symmetry but different in other features, such as non-trivial ground state degeneracies (GSD), fractionalized excitations and chiral gapless boundaries (see [Sto92] and references therein). The second source is chiral spin liquids emerged from the study of superconductors at high critical temperatures [KL87]. It was realized that different chiral spin liquids can have the same symmetry but different numbers of heat-conducting edge modes [WWZ89]. The third source is Witten's construction of 2+1D Chern-Simons topological quantum field theories (TQFT) [Wit89]. In these TQFT's, the spaces of states are those of ground states on different closed 2d manifolds, and are different for different TQFT's; and they are non-trivial also in that there are non-trivial Wilson line defects leading to infinitely many new topological invariants [RT91]. These TQFT's should be viewed as the low energy effective quantum field theories of exotic gapped quantum phases. Therefore, these new exotic gapped phases are highly non-trivial, and are beyond Landau's paradigm of spontaneous symmetry breaking theory [Wen89, Wen90]. They provide a gateway to an entirely new world. These three sources motivated Xiao-Gang Wen to introduced the notion of a 'topological order' in 1989 [Wen89, Wen90] (see recent reviews [Wen17, Wen19] and references therein). Another commonly stated motivation is that it provides the physical foundation of the so-called fault-tolerant quantum computing [Kit03, FLW02, Fre01]. These motivations have already attracted many condensed matter physicists, mathematical physicists, computer scientists and mathematicians working jointly in this field.

Interestingly, as the study of topological orders or gapped quantum liquids deepens, we have encountered more surprises. The significances and the impacts of this study have gone far beyond condensed matter physics and have already reached out to many other fields, such as particle physics, quantum gravity, quantum computing and even pure mathematics. In the next subsection, we discuss some of the significances and the impacts of this study.

1.2 Significances and impacts

The first significance of the study of gapped quantum liquids is that in order to go beyond Landau's paradigm, this study demands us to return to perhaps the most fundamental question in condensed matter physics:

What is a phase or phase transition?

Landau's theory was developed from the study of concrete phase transitions. The tools and the language developed for this study automatically provide a way to distinguish different phases, more precisely, a partial characterization of a phase. This approach does not provide a prior reason for the completeness of this characterization. Landau's theory is not developed from the first principle, by which we mean first defining the notion of a phase precisely, then finding a mathematical characterization of a phase transition. The discovery of new exotic phases beyond Landau's paradigm provides us with a unique chance and motivation to study the notion of a quantum phase from the first principle. Indeed, it has already motivated many attempts to define the notion of a gapped quantum liquid precisely from both the microscopic perspective [CGW10, ZW15] and the macroscopic perspective [Kit06, KW14, KWZ15, JF22, KLVZZ20b, KZ22a, KZ22b]. Microscopically, a gapped quantum liquid is an equivalence class of gapped lattice models. The physical description of the equivalence relation

between two models is essentially known as a path connecting two models in the space of ‘models’ without closing the gap nor changing the ground state degeneracy anywhere on the path. By “moving along the path” we mean deforming the theory by enlarging the Hilbert space and adding local perturbations allowed by symmetries [ZW15]. The real challenge lies in how to formulate this equivalence relation mathematically precise and prove its compatibility with the macroscopic description of gapped quantum liquids. In this work, we focus our attention on the macroscopic description. Note that a macroscopic description of a topological order is possible simply because the notion of a quantum phase is itself a macroscopic notion. Therefore, it is reasonable that the precise characterization of a quantum phase can be obtained by summarizing the complete¹ set of ‘observables’ in the thermodynamic limit and in the long wave length limit. This point of view is both fundamental and natural, and has already played an important role in the study of topological orders. Interestingly, it also provided a new way of characterizing the symmetry-breaking phases in old Landau’s paradigm [KLWZZ20a, KZ22a, KWZ22], and led to a grand unification of topological orders, SPT/SET orders, gapped symmetry-breaking phases and CFT-type gapless phases [KZ20, KZ21, KZ22a, KWZ22, XZ24].

In retrospect, to be able to study topological orders from the first principle is because topological orders are the simplest quantum phases. Indeed, a topological order is a gapped quantum many-body system at zero temperature without any symmetry. It is significantly simpler than SPT/SET orders or gapless quantum liquids. Even the spontaneous symmetry breaking (SSB) orders within the Landau’s paradigm are more complicated than topological orders because the symmetry in a SSB order, however broken, still plays a non-trivial role in the theory. A direct justification of this fact is that the categorical description of an SSB order [KZ22a, KWZ22, XZ24] is much more complicated than that of a topological order. It is perhaps not just ‘more complicated’. It is nearly impossible, even in retrospect, before the advent of topological orders, because the key mathematical structure in this categorical description was new [MP17, KZ18b, KYZZ24] and was emergent from the study of the gapless boundaries of 2+1D topological orders [KZ18c]. This simplicity is the key to make precise definitions, complete characterizations and other precise results possible. We believe that the role played by topological orders in the study of quantum many-body physics is similar to that of the two-body problem in Newtonian mechanics and that of the Hydrogen atom in quantum mechanics. Moreover, this simplicity also makes it possible, for the first time, to study all topological orders or quantum liquids as a whole, i.e., the categories of topological orders (or quantum liquids) [KWZ15, KZ18a, KLWZZ20b, KZ22a, KZ24, KZ22b]. This is a completely new and uncharted territory. Many new questions from this global perspective, focusing on the relations² between quantum liquids and those between the categories of quantum liquids in different dimensions [KWZ15, KZ18a, KYZ21, JF22, KZ22a, KZ24, LYW23, KZZZ24], have been raised and are currently under investigation. As many new results are gradually emerging, it becomes clear that we are only at the beginning stage of a great revolution of quantum many-body theory.

The second significance of the study of gapped quantum liquids is that it leads to a new approach towards the study of gapless quantum phases. Now we follow the discussion in [KZ21, Section 7]. This new approach is both surprising and ironically obvious in retrospect. It is in everyone’s mind that gapless quantum phases are significantly richer and harder than gapped quantum phases. This richness lead people to a natural impression that the intuitions and methods developed in the study of the gapped cases might not be useful, if not misleading, to that of the gapless cases. From this perspective, it was somewhat surprising to realize that the mathematical characterization of a gapped quantum liquid is an indispensable ingredient of that of a gapless quantum phase, and the theory of gapped liquids might provide a systematic approach to the study of gapless phases. On the other hand, it is also completely obvious. A generic gapless phase can be obtained by stacking a gapless phase X with a gapped quantum liquid A as illustrated in the first picture in (1.2.1). Therefore, the mathematical characterization of a gapped quantum liquid, such as the higher category of topological defects, is also an indispensable ingredient of that of a gapless quantum phase. This categorical structure might be changed if we introduce coupling (or the glue C) between two layers as illustrated in the second picture

¹We want to emphasize that the mathematical description of the complete set of ‘observables’ is fundamentally different from that of a proper subset because certain structures and results, such as the boundary-bulk relation of topological excitations (see Section 4.2.4 and 5.6) and the mathematical theory of anyon condensation (see Section 5.4), are possible only when the description is complete (at least to certain extent), and are impossible to see within a partial description. In other words, new structures emerge in the limit of the completion.

²These relations include the usual dualities and those that are far more sophisticated than dualities.

in (1.2.1), but its categorical nature remains intact.

$$A \boxtimes X := \left\{ \begin{array}{c} \xrightarrow{\text{gapped}} A \\ \xrightarrow{\text{gapless}} X \end{array} \right. \quad B \boxtimes_C Y := \left\{ \begin{array}{c} \xrightarrow{\text{gapped}} B \\ \xrightarrow{\text{gapless}} Y \end{array} \right. \quad (1.2.1)$$

Instead of using stacking, we can also describe this categorical structure intrinsically. In a gapless quantum phase, there are, in general, gapped excitations, which are topological (or superselection) sectors in the Hilbert space. These topological sectors should form a higher categorical structure similar to those in a gapped quantum liquid. We call this categorical structure the ‘topological skeleton’ of the gapless quantum phase [KZ21, Section 7]. The first examples of topological skeletons of gapless quantum phases are explicitly constructed for the chiral/non-chiral gapless edges of 2d topological orders [KZ18c, KZ20, KZ21]. Although the topological skeleton is not enough to determine a gapless quantum phase, it has already provided sufficient amount of information to study certain types of questions that are irrelevant to the missing data, such as the boundary-bulk relation, Morita equivalence of two boundaries, computing dimensional reduction processes and condensations [KZ21]. For example, a condensation from the gapped phase B in (1.2.1) to a new gapped phase B’ automatically defines a phase transition from the gapless phase $B \boxtimes_C Y$ to $B' \boxtimes_C Y$. One can play the same game for C, including introducing a condensation that is confined to a 1-codimensional submanifold in C. In addition to these obvious applications, more sophisticated applications can be found in the study of topological phase transitions [CJKYZ20, CW23, LY23]. Moreover, since all the missing data must be compatible with the topological skeleton, this compatibility provides non-trivial clues to the construction of the missing data. All of these suggest that the theory of gapped quantum liquids provides a brand new and systematic approach towards the study of gapless quantum phases [KZ21, Section 7][KZ22a, KWZ22].³ Indeed, among many other things, it leads to a new mathematical theory of all gapped/gapless quantum liquids (with/without symmetries) [KZ22a, KZ24, KZ22b].

The third significance is its impact on the study of problems in the field of high energy physics. Indeed, it has stimulated an unexpected and unprecedented interaction between high energy theory and condensed matter theory. The new results and methods developed in study of gapped quantum liquids, such as many-body entanglement, string-net condensations [LW05] and emergent gauge fields and gapless modes on the boundaries, turn out to be useful in the study of many other fundamental problems in high energy physics, such as emergent chiral fermions in standard model [Wen03, Wen13, YX15, AC16, CS17], beyond standard model [Wan21] (and references therein), string theory [KPMT20, TY23], emergent gravitational modes [Xu06, GW12, Gu17], applications to loop quantum gravity [DG17, Zuo17], many of the recent developments of dualities among 2+1D CFT’s (see for example [SSWW16, SSWX19] and references therein), and holography through the study of entanglement entropy (see for example [RT06, Swi12, Qi18] and references therein). Moreover, the bulk-boundary correspondence [LH08, QKL12] and boundary-bulk relation of topological excitations [KK12, KWZ17] in the study of gapped phases echo the open-closed duality in string theory, and the holographic duality implied by topological Wick rotation [KZ20, KZ22a] echoes AdS/CFT correspondence [tH93, Mal99] (see a recent unification of these two holographical phenomena [BHJL24]). It seems that we are only at a beginning stage to grasp some universal features of quantum many-body systems. In retrospect, perhaps the reason for its influence and connection to quantum gravity is the fact that the what-is-a-phase question cannot be completely separated from the question: *What are the space and time?*, if they are not the same question (are they?). After all, they are all quantum many-body problems.

Above discussion focuses on its impacts on real physical problems in high energy physics (e.g., standard models) instead of its obvious “impact” on the study of formal QFTs. On the one hand, from the perspective of its mathematical or physical contents, this obvious “impact” can hardly be called an impact because quantum phases and QFTs are essentially the same thing. The only difference is

³A few months after the appearance of this paper on arXiv, the authors were exposed to a parallel but partially independent development from high energy physics community on generalized (higher form and non-invertible) symmetries [GKSW15] and the actions of TQFT’s on general QFT’s [FMT24] or SymTFT [GK21, BLT20, ABGEHSN23]. In particular, $B \boxtimes_C Y$ in (1.2.1) is recently called a *sandwich construction* and is viewed as a (B, C)-action on the gapless phase $B \boxtimes_C Y$ [FMT24]. The pair (B, C) was previously viewed in [KWZ15, KWZ17, KZ22a] as the defining data of a morphism from Y to $B \boxtimes_C Y$. In literature, C is called a “background category” [KZ18c, KZ20], a “categorical symmetry” [JW20, KLWZZ20a], “topological symmetry” [FMT24], “SymTFT” [ABGEHSN23] and “SymTO” [CJW22]; and the pair (B, C) is called the “topological skeleton” (of a quantum liquid) [KZ21, KZ22b] or a “quiche” [FMT24].

that a QFT studied by high energy physicists often involves a Lagrangian and path integrals on non-trivial spacetime manifolds, while a quantum phase can be defined by either a lattice model or the local observables of an effective field theory restricted to an open disk⁴. However, this difference is superficial in that the partition function on a non-trivial spacetime manifold M can be recovered by integrating local observables over M (see [Lur17, AFT17, AKZ17])⁵. Therefore, results on quantum phases should be viewed directly as results on QFT's. On the other hand, from a social or cultural perspective, this impact indeed happened, and some translations are certainly needed, especially for QFT's defined by Lagrangians and for computations involving non-trivial spacetime manifolds⁶. Since 2017, we have witnessed a movement of translating results on topological defects in topological/SPT/SET/SSB orders into results on generalized symmetries [GKSW15] in QFT's⁷, and, at the same time, a movement of applying generalized symmetries to the study of topological phases. In particular, a topological order can be understood as the consequence of breaking a generalized symmetry (see for example [McG23]). As a consequence, two fields are merging into one. In particular, the condensed matter theorists' goal of establishing a new paradigm of phase transitions beyond Landau's symmetry-breaking theory is now actively pursued by many high energy theorists.

The fourth significance is its impact on pure mathematics. It was known at the dawn of modern science that mathematics is efficient in solving problems in physics [Wig60]. There is no exception in our case. The mathematical notions of a 1+1D conformal field theory (CFT) [Seg88] and a TQFT [Ati88] were introduced earlier than that of a topological order [Wen89]. So it is not surprising to see that many mathematical results about TQFT's, CFT's and related subjects, such as quantum groups, vertex operator algebras and tensor categories, have played important roles in the development of the theory of topological orders. In recent years, more advanced mathematical topics, such as higher algebras [Lur17] and factorization homology [AF20], also joined the party [AKZ17, JF22]. A natural question is: "Can physics return the favor?". Although it has already been a few decades since physics returned the favor through string theory (see Remark 1.2.1), it is still not obvious if the favor can be returned through the study of topological orders. But it did happen. The physical problems and intuitions of topological orders have already directly produced or inspired many new mathematical results in topology (see for example [FH20, FH21, FT21]) and in category theory (see for examples [DMNO13, BNRW15, KZ18a, LKW16a, AKZ17, KZ18b, KTZ20b, GJF19a, VR19, KZ21, KYZ21, JF22, KLWZZ20b, KZ22a, JF20, JFY21b, JFR23, KZ24]) and even new mathematical theories (see for example [DR18, GJF19a, JF22, KYZZ24, KZ22b]). *A lot more new references need to be added.*

A direct reason behind this return of favor is that the study of topological orders has raised many important new questions and provided surprisingly powerful physical intuitions that are not so obvious from the mathematical perspective. For example, questions related to topological phase transitions between topological orders are completely new to mathematicians who are experts in the mathematical theory of TQFT's. These new questions demand and have already inspired many new mathematical theories. As the new approach towards the study of gapless quantum phases just starts, it is only reasonable to expect more impacts on pure mathematics in the coming future.

Remark 1.2.1. Why can physics help to discover new mathematical structures and solve mathematical problems? We recall a partial answer to this philosophical question appeared in [Kon17]. An infinite dimensional mathematical object, viewed from generators and relations, is very complicated and lacks of global intuition. However, if it describes a quantum many-body system, say a solid material, our physical intuition, including the ordinary visual effect of a solid material in the sight of human eyes, is

⁴More precisely, a QFT defined by path integrals is essentially a functorial quantum field theories (FQFT) defined by a functor from a cobordism category to a linear category [Ati88]; while a quantum phase is essentially an algebraic quantum field theory (AQFT) defined by the net of local observables algebras [HK64]. More discussions on the difference between a quantum phase and a path-integral QFT, reformulated as a H-type and a L-type QFT's, respectively, can be found in [MW12, KW14, FH21].

⁵In order for this statement to be rigorously true, one can consider those quantum phases that are called quantum liquids [KZ22a], which can be viewed as a QFT that is fully dualizable in a proper sense [Lur17]. The mathematical foundation of this integral is the mathematical theory of factorization homology [Lur17, AFT17, AFT16, AFR18, AF20] (see also [KZZZ24]).

⁶The relation between FQFT and AQFT and the theory of factorization homology are not so well known in physics community.

⁷Sometimes the 'translation' is trivial. Sometimes it is non-trivial. Let us not forget that this 'translation' is only a return of favor because the early developments in topological orders [Wen90, Kit03, LW05] were preceded and inspired by those in TQFT's [Wit89, DW90, RT91, TV92, Tur20]. In particular, as one of the main results we would like to explain in this review, the mathematical characterization of the anyons in a 2+1D topological order by a unitary modular tensor category [Kit06, Appendix E] is a direct translation (or a summarization) of the results in 2+1D TQFT's and 1+1D RCFT's [Wit89, MS89a, FR589, FG90, Reh90, RT91, Tur20] and their connections to anyons in real systems [MR91, Wen91].

the consequence of a highly non-trivial RG computation, i.e. integrating out all microscopic degrees of freedom. Similarly, all phenomenological theories, Hamiltonian or Lagrangian descriptions and even physical measurement carried out in physics labs can all be regarded as powerful computations of systems with infinite number of degrees of freedom. Although we are still far from understanding the true nature of these transcendental computations, it is not hard to imagine that they, as gifts from mother nature, can be surprisingly powerful in solving mathematical problems. Moreover, quantum field theories have revealed an entirely new world of infinite dimensional mathematics. A lot of new mathematical theories developed in mathematics were inspired by quantum field theories. \diamond

1.3 Category theory as a new language

The mathematical theory of symmetries (in the the classical sense) is that of groups. The new exotic phases of matter challenge us to find radically new mathematical language to characterize gapped quantum liquids. The enormous efforts have been made to meet this challenge (see for example [Kit06, LW05, KK12, BBCW19, KWZ15, LKW16a, LKW18, LW19, KTZ20b, JF22, KIWZZ20b, KZ22a, JF20, JFY21b, KZ22b] and references therein). The result of these efforts is a successful one. Up to invertible gapped quantum liquids, the mathematical structure that characterizes a gapped quantum liquid was found to be a (higher) category equipped with some additional structures and satisfying some properties [KIWZZ20b, KZ22a]. This (higher) category summarizes all the topological defects (and their properties) living in the gapped quantum liquid.

This success is, however, somewhat bittersweet to many working physicists. The mathematical theory of higher categories, or category theory, is far more exotic than the group theory, and is notorious for its abstractness even in mathematics. As a consequence, even after 30 years of development, it is still a big challenge for a working physicist to master the abstract language of category theory. Although it is obvious to some experts in this field that category theory is indispensable (also powerful) in the study of topological orders, ironically, if a physical paper on topological orders uses a lot of category theory, it is often considered by many physicists as mathematics or perhaps non-physics, and is not accepted by physics journals. The main goal of this paper is to bridge the gap between physical/geometrical intuitions and mathematical notions. More precisely, for physicists with no background in category theory, we explain in great details how the mathematical notions of a unitary modular tensor category (UMTC), a fusion category, a monoidal functor, Drinfeld center, a module category, Morita equivalence, condensation completion and fusion 2-categories, etc. naturally emerge from physical/geometrical intuitions or explicit lattice models calculations. We also clarify some subtle issues, misleading statements and confusions that are abundant in literature along the way. Although all results in this work were known, many parts of our explanation are hard to find in literature. Some of them should be interesting and useful even to experts.

From a physical perspective, we have already argued in Section 1.2 that the theory of gapped quantum liquids is not just a corner in physics. It leads us to a new approach towards the study of the gapless quantum phases, and links to many fundamental questions in physics, including particle physics and quantum gravity. Similarly, from a mathematical perspective, category theory is also far more than just a generalization of group theory. It is as basic as set theory, and provides a new foundation of mathematics. It is capable of supplying many different kinds of new ‘calculus’ and ‘linear algebras’. It has already become the basic languages or powerful tools in a variety of fields, such as logic, functional analysis, algebraic number theory, algebraic geometry, algebraic topology and representation theory. Moreover, it unifies ‘continuous’ with ‘discrete’, ‘algebraic’ with ‘analytic or geometric’ and ‘finite’ with ‘infinite’. Perhaps, the most important fact is that there are a lot of new mathematical truths that can only be expressed in the categorical language. In other words, these mathematical truths are beyond set theory. There has been a new wave in mathematics of replacing set theory by category theory as the new foundation of mathematics since the rise of the new paradigm of algebraic geometry developed in 1960’s by Alexander Grothendieck and his schools. Therefore, the real question is: can category theory becomes the language of a new calculus for quantum many-body physics? We believe that the answer to this question is YES.

The applications of category theory in physics started no later than 1980’s and were largely influenced by the bootstrap program of 1+1D rational conformal field theories (RCFT). This program was initiated in 1970’s and went through a great breakthrough by Belavin, Polyakov and Zamolodchikov in

1984 [BPZ84] and an intensive development afterwards. We briefly summarize some of these applications in a few relatively independent lines of developments.

- (1) **1+1D RCFT's:** One of the crowning achievements in the study of 1+1D RCFT's in 80's is Moore and Seiberg's discovery of the notion of a modular tensor category⁸ (MTC) as the conjectured category of modules over the so-called 'chiral algebras' [MS89a, MS90]⁹. Moore-Seiberg's axioms of a MTC are slightly overcomplete and was later simplified in [Reh90, Tur20] (see also [BK01] and Remark 3.5.7). The notion of a MTC played a fundamental role in later developments of RCFT's, especially in Fuchs, Runkel and Schweigert's state-sum construction of all correlators in a RCFT based on a simple special symmetric Frobenius algebra in a MTC [FS03, FRS02, FRS04, FRS05, FFRS06] (preceded by an important earlier work [FFFS02]), and in Huang and Kong's mathematical foundation of open-closed RCFT's based on VOA and its representation theory [HK04, HK07, Kon07, Kon08b, Kon08a], and in the study of open-closed duality [FFRS08, KR08, KR09, Dav10], and in that of topological defects in RCFT's [FFRS04, FFRS07, DKR15, KZ18c, KZ20, KZ21, KYZ21]. Tensor categorical methods have also been applied to 1+1D irrational (open-closed) CFT's (see for example [Hua17, FS19] and references therein).
- (2) **$n+1$ D TQFT's:** The mathematical definition of a CFT was introduced by Segal [Seg88] and Kontsevich. It was based on the idea of formalizing the factorization property of path integrals as a symmetric monoidal functor defined on a complex cobordism category. This immediately led to Atiyah's definition of an $n+1$ D TQFT's [Seg88, Ati88] as a symmetric monoidal functor from topological cobordism category to the category of vector spaces. After Witten's construction of Chern-Simons-Witten TQFT's [Wit89]¹⁰, a construction of 2+1D TQFT's based on modular Hopf algebras was obtained by Reshetikhin and Turaev in [RT91]. It was later generalized by Turaev to a construction based on MTC's (i.e., the so-called Reshetikhin-Turaev TQFT's) in [Tur20]. In 1992, a state-sum construction of 2+1D TQFT's based on MTC's was obtained by Turaev and Viro in [TV92], and it was later generalized to a state-sum construction based on spherical fusion categories by Barrett and Westbury [BW96, BW99]. Extended TQFT's were also systematically studied in 1990's by Lawrence [Law93] and by Freed [Fre94, Fre99, Fre09] and by Baez and Dolan [BD95], where Baez-Dolan's Cobordism Hypothesis was proposed. Lurie provided a sketched proof of this hypothesis in [Lur08], and Ayala and Francis provided a rigorous proof under some assumptions [AF17]. Category theory was systematically used in the program of constructing TQFT's with defects by Carqueville, Runkel, Schaumann and Mulevicius in a series of works [CR16, CRS18, CRS19, NC20, CMRSS21, Mul22, MR23, CMRSS24], preceded by some earlier works on QFTs with defects [FFRS07, FFRS10, DKR11].
- (3) **Topological phases:** Category theory entered the study of topological phases as the category of topological excitations (or anyons) in a 2d topological order from very beginning [MS89c, Wit89, FRS89, FG90, Reh90, MR91, Wen91]. However, it remains largely unnoticed by condensed matter theorists until Kitaev's famous review in [Kit06, Appendix E]. Through the constructions of Kitaev's quantum double models [Kit03], Levin-Wen models [LW05] and their gapped boundaries and domain walls [KK12], the anyon condensation theory [Kon14a] and the categorical description of 2+1D SET orders [BBCW19], nearly all modern developments of tensor categories (see a book [EGNO15]) have entered the field of topological orders in their full strength. After that, the applications of the tensor category theory to the study of gapped quantum liquids (including topological orders) has entered a golden age (see for example [KZ18a, LKW16b, LKW16a, LKW17, BGHN⁺17, AKZ17, CCW17, KZ18b, BJLP19, CJKYZ20, KZ20, KZ21, HBJP23, CGHP23, GHKP⁺24]). Recently, the new wave of applying higher category theory to the study of gapped/gapless quantum liquids is picking up momentum [KW14, KWZ15, BHW17, DR18, KTZ20b, BD19, GJF19a, BD20, JF22, KLWZZ20b, BD21, KTZ20a, JFY21a, KZ22a, BD22, JFY21b, JFR23, KZ24, XLLC21, KZ22b, ZLZH⁺23, LYW23, KZZZ24]. *A lot of more references are emerging in 2024-2025.*

⁸The name was suggested by Igor Frenkel and first appeared in [MS90]. We recommend Moore's recent online talk on the history of the discovery of MTC in the study of rational CFT's [Moo24].

⁹Moore-Seiberg's discovery was based on a bootstrap analysis of the structure of a rational CFT, which was later fully established as a rigorous mathematical theorem by Huang [Hua08b, Hua08a]. In this mathematical theory, the physical notion of a chiral algebra was replaced by a more precise mathematical notion of a vertex operator algebra (VOA). Similarly, that of a chiral vertex operator (introduced in [MS89a]) was replaced by that of an intertwining operator [FHL93].

¹⁰Witten's construction is based on Lagrangian or the action principle in physics.

Remark 1.3.1. In another line of development, category theory also played an important role in the study of homological mirror symmetry [Kon94]. In particular, both the bounded derived categories of coherent sheaves and (derived) Fukaya categories appear as the defining data of the homological mirror symmetry. They are both certain limits of the categories of D-branes in certain topologically twisted $N = 2$ super-conformal field theories (see [ABCD⁺09] for a physical review and references therein). Although this line of development is not directly related to this work, it partially influenced Lurie’s theory of higher algebras and higher representations [Lur17], which further influenced the study of topological phases in many different ways (for example, in the condensation theory of topological defects [KZZZ24]). It is not clear how many details we should give and how many references we should include here. \diamond

Remark 1.3.2. It is beneficial to reflect upon the reason why category theory is so useful in the study of quantum many-body physics. In general, a physical system with infinite degrees of freedom is very complicated if you measure it via the inefficient device of the 1-dimensional ground field \mathbb{C} . It is often expressed in terms of certain complicated infinite dimensional \mathbb{C} -linear algebraic structures, such as vertex operator algebras and algebras over operads or PROPs. However, if we change our measuring device to more efficient one: an infinite dimensional algebras over operad A , then above algebraic structures turn themselves into familiar (even ‘finite dimensional’) notions of ‘ A -linear’ algebras in certain highly non-trivial monoidal higher categories (e.g. E_k -monoidal higher categories). This fact is deeply related to or perhaps not different from the fact that all gapped defects in a quantum many-body system or a quantum field theory automatically form a higher category. \diamond

The applications of category theory have also reached out to computer science, linguistics, philosophy and economics, etc. Just as the calculus is once the universal language of all sciences, this is not hard to imagine that category theory, which can be viewed as a structuralized calculus, will eventually become a universal language of all sciences. Our humble hope is that this paper might help to speed up this process.

Remark 1.3.3. We would like to give a remark that might help some readers to overcome the psychological barrier in the study of category theory. We have heard from some physicists and students that they have spent a lot of time studying category theory, but only found it very formal, dry, uncomputable and useless. This feeling and reaction is very common and natural. It reflects a rather deep truth. In fact, category theory is as basic as set theory. Both of them can serve as the foundations of mathematics. Physicists’ reaction to set theory is likely to be the same: formal, dry, uncomputable and useless. But physicists really do not need set theory because all they need are calculus and linear algebra. Similarly, it is impossible to see the power of category theory until you see ‘categorical calculus’ and ‘categorical linear algebras’. Physicists can study calculus directly without going through set theory because all they need from set theory is automatically covered by the theory of functions in calculus. But for the ‘categorical calculus or linear algebra’, as far as we can tell, it is impossible to bypass the basic notions of a category, a functor, a natural transformation, Yoneda Lemma, adjoint functors, etc. This work provides a friendly way to get a glimpse of the ‘categorical linear algebra’ through concrete physics models. We hope that this work can help physicists to overcome the language barrier. \diamond

1.4 Layout and how to read

We explain the layout of this paper: in Section 2, we briefly review some basic notions related to topological orders; in Section 3, we explain in details how the structure of a unitary modular tensor category emerges from physical and geometric intuitions of topological excitations; in Section 4, we explain the emergence of many mathematical notions associated to the 1d gapped boundaries of 2d topological orders and the boundary-bulk relation, such as unitary fusion categories and Drinfeld center, etc; in Section 5, we give a brief guideline of some advanced topics.

In the main text of this paper, we try to be self-contained, slowly paced and not technical. We hide many technical parts in Remarks. One of the purposes of this paper to let readers to get a glimpse of the incredible richness of category theory and its application in the study of topological orders. Therefore, we choose to comment on some advanced physical or mathematical topics only in remarks, where intrigued readers can find relevant references for further reading. That is also to say that if you are a beginner and do not understand those technical and advanced Remarks, you can simply ignore them.

Throughout this work, we use “Theorem^{ph}” to highlight a physical result, and use “Theorem” to represent a mathematically rigorous result.

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1.5 Plan for the next version

1. Incorporate the new developments from non-invertible symmetries and topological symmetries. Not only one can use them in the understanding of topological defects, but also discuss its significants in Introduction.

Somewhere in the introduction, one should add comments on how to see so many categorical structures via a simple model of the toric code model.

2. Section 2 might need some rewriting.
3. Write a subsection on the comparison of the various points of view of topological defects, including the generalized symmetry point of view and entanglement bootstrap point of view. This will affect the Remarks associated to the definition of the fusion product of two particles.
4. Add some discussion of anyon condensation theory.
5. Add a subsection on the category of topological orders. This can help to explain the meaning of the direct sum of two topological orders (or two domain walls).
- 6.

2 Basic notions for topological orders

In this section, we briefly review a few basic notions associated to topological orders and introduce some notations along the line.

2.1 Gapped quantum phases

A quantum phase is a universality class of quantum many-body systems defined in the thermodynamic limit and at zero temperature. Since a *quantum phase* can be realized by lattice models, it can be defined microscopically as an equivalence class of lattice models. In this subsection, we briefly sketch this microscopic perspective of gapped quantum phases without giving details.

By a *lattice model* we mean a collection of the following data:

- There are some discrete sites in the space, and the number of sites is called the size of the lattice model.
- For each site i we associate to it a Hilbert space \mathcal{H}_i (called a *local Hilbert space*), which represents the local degrees of freedom.
- The *total Hilbert space* is defined by $\mathcal{H} := \bigotimes_i \mathcal{H}_i$.
- The *Hamiltonian* H acting on \mathcal{H} is the sum of local interaction terms. A term in H is called local if it acts as identities on all sites except some spatially local sites.

For convenience, we also use the total Hilbert space \mathcal{H} to denote a lattice model. The thermodynamic limit is defined for a quantum (many-body) system¹¹, which is a sequence $\{\mathcal{H}^{(k)}\}$ of lattice models and the size of $\mathcal{H}^{(k)}$ tends to infinity as $k \rightarrow \infty$. In order for the limit to be well-defined, additional data and conditions are needed. The literature on this subject is gigantic. We avoid to go into detail here but recommend a review [GJ87] and references therein. In principle, a *quantum phase* could be defined microscopically as an equivalence class of quantum systems. However, it is hard to define the equivalence relation in this generality.

In this work, we are only interested in a special kind of quantum systems, which satisfy the following conditions (see Figure 1):

- there exists a fixed $\Delta > 0$ independent of k such that
 1. there is no eigenvalue of $H^{(k)}$ in an energy window of size Δ as $k \rightarrow \infty$;
 2. the number of eigenstates below the energy window is finite and independent of k as $k \rightarrow \infty$;
 3. the energy splitting of those eigenstates below the energy window tends to zero as $k \rightarrow \infty$.

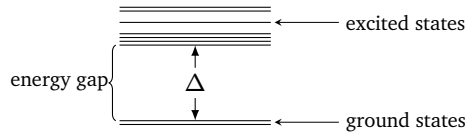


Figure 1: the energy spectrum of a typical gapped system

Such a quantum system is called a *gapped liquid-like*¹² *quantum system*. The subspace spanned by those eigenstates below the energy window is called the *ground state subspace*, and its dimension is called the *ground state degeneracy* (GSD). The quantum phase determined by a gapped liquid-like quantum system is called a *gapped quantum liquid* (or a *gapped liquid* for short), and is defined by an equivalence class of gapped liquid-like quantum systems [ZW15]. In principle, the equivalence relation could be defined by properly enlarging the Hilbert space and followed by the (symmetry allowed) perturbations of the

¹¹When a lattice model \mathcal{H} is translation invariant, it induces a canonical sequence $\{\mathcal{H}^{(k)}\}$ of lattice models. In this case, ‘a lattice model’ and ‘a quantum system’ have the same meaning.

¹²Gapped *non-liquid-like* quantum systems are known (see for example [Cha05, Haa11]).

quantum system without closing the gap and without changing GSD. A gapped liquid without symmetry is called a *topological order*. We prefer not to go into the details of the microscopic definition of a topological order or a gapped liquid (see Remark 2.1.1). Instead, we discuss some macroscopic properties of topological orders that are intuitively natural thus should be viewed as natural requirements for any reasonable microscopic definition of a topological order.

Remark 2.1.1. To define the equivalence relation precisely is a very subtle problem. Note that we have not yet made the notion of a quantum system very precise. Since only the ground state subspace is physically relevant at zero temperature, one way is to define a gapped liquid directly by an equivalence class of the ground state wave functions. The proposals of such an equivalence relation can be found in [CGW10, ZW15, SM16]. \diamond

Example 2.1.2. Suppose \mathcal{V} is a finite dimensional Hilbert space and P is a Hermitian operator on \mathcal{V} with a unique ground state $|\psi\rangle$. For each $n \in \mathbb{N}$, there is an nd lattice model defined as follows:

- Each local Hilbert space is $\mathcal{H}_i := \mathcal{V}$.
- The Hamiltonian is $H := \sum_i P_i$, where $P_i = \cdots \otimes 1 \otimes P \otimes 1 \otimes \cdots$ is the operator acting on \mathcal{H}_i as P and acting on other local Hilbert spaces as identity.

This defines a gapped liquid-like quantum system with a unique the ground state given by the *product state* $|\psi\rangle \otimes |\psi\rangle \otimes \cdots$. If we do not impose any symmetry, this quantum system realizes the *trivial topological order*, denoted by 1_n . \heartsuit

Definition 2.1.3. The stacking of two nd topological orders A_n and B_n is defined by stacking one on the top of the other without introducing any coupling between two layers. This stacking produces a new topological order denoted by $A_n \boxtimes B_n$ as illustrated in the following picture:

$$A_n \boxtimes B_n := \left\{ \begin{array}{c} \xrightarrow{\hspace{1cm}} A_n \\ \xrightarrow{\hspace{1cm}} B_n \end{array} \right.$$

■

We denote the set of all nd topological orders by TO_n . By Example 2.1.2, there is at least one topological order in each dimension: the *trivial topological order*, i.e. $1_n \in \text{TO}_n$. The stacking operation defines a binary multiplication on TO_n , which is

1. commutative, i.e. $A_n \boxtimes B_n = B_n \boxtimes A_n$;
2. associative $(A_n \boxtimes B_n) \boxtimes C_n = A_n \boxtimes (B_n \boxtimes C_n)$;
3. unital, i.e. $1_n \boxtimes A_n = A_n$. This follows from Zeng and Wen's definition of a topological order [ZW15], which requires that an nd topological order A_n is invariant if we enlarge the Hilbert space but project the enlarged part to the product state.

This result can be summarized as follows.

Theorem^{ph} 2.1.4. The set TO_n of all nd topological orders, together with the commutative multiplication \boxtimes and the unit 1_n , defines a commutative monoid.

Remark 2.1.5. The notion of a topological order is closely related to that of a topological quantum field theory (TQFT) but with a subtle difference (see Remark 3.5.15). It was generally believed that the low energy effective theory of a topological order is a TQFT [Wit89]. Indeed, the low energy effective theories of chiral spin liquids [WWZ89] and fraction quantum Hall liquids [ZHK89] are Chern-Simons TQFT's; those of Kitaev's quantum double models [Kit03] are 3+1D Dijkgraaf-Witten TQFT's; those of Levin-Wen models [LW05] are Turaev-Viro-Barrett-Westbury TQFT's [TV92, BW99]. \diamond

Remark 2.1.6. Gapped liquids with symmetries include gapped spontaneous symmetry breaking (SSB) orders, symmetry enriched topological (SET) orders [CGW10] and symmetry protected trivial (SPT) orders [GW09, CLW11, CGLW13] (see also Section 5.5). \diamond

2.2 \mathbb{R}^n -observables

At zero temperature, regardless gapped or gapless, only physically relevant data are those survived in the low energy limit, or equivalently, in the long wave length limit. In other words, the notion of a quantum phase is actually a macroscopic notion defined in the long wave length limit. Therefore,

in principle, a quantum phase should be characterized by all possible ‘observables’ in the long wave length limit.

When the quantum phase is gapless, typical examples of such ‘observables’ are the correlation functions as those in quantum field theories (QFT), which should be viewed as the low energy effective field theories of quantum many-body systems. When the quantum phase is gapped, all the correlation functions decay exponentially. They are not observables in the long wave length limit. In this case, there are other ‘observables’ which are discussed in Section 2.4.

One more thing can be said about the notion of a phase. It is often stated in literature to consider a phase defined on a closed manifold. There is nothing wrong about this statement. But it is wrong to regard the notion of a phase as a global notion defined on a closed manifold.

- The only reason we can regard the same FQH state on topological surfaces with arbitrary genus as the same FQH phase is because they are the same phase on each open 2-disk covering the surfaces (see Figure 2). This fact becomes even more obvious by considering a 2-sphere with a 2d quantum phase A defined on the northern hemisphere, a different phase B defined on the southern hemisphere and a 1d domain wall on the equator. Therefore, the notion of an nd phase is actually a ‘local’ notion in the sense that it is defined on an open n -disk¹³.
- But this term ‘local’ is misleading because each n -disk must be in the thermodynamic limit thus ‘non-local’ with respect to certain length scale. Therefore, the notion of a (quantum) phase is a notion defined on an open disk of infinite size, or on \mathbb{R}^n for simplicity. To distinguish those ‘observables’ defined on \mathbb{R}^n (in the long wave length limit) from those global observables defined on a closed manifold, we refer to the former as \mathbb{R}^n -observables. In other words, a quantum phase should be characterized by all \mathbb{R}^n -observables in the long wave length limit.

In Section 5.2, we explain briefly on how to obtain global observables (or invariants) defined on a closed manifold Σ by integrating \mathbb{R}^n -observables on Σ via the so-called factorization homology [AKZ17].

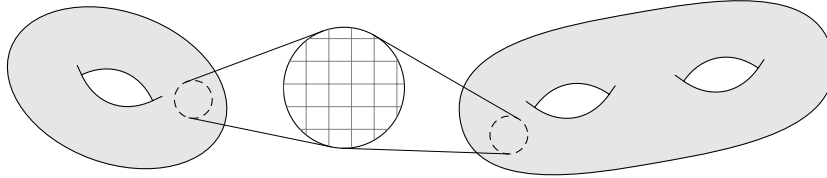


Figure 2: The notion of a phase is local in the sense that it is defined on an open disk.

2.3 Anomaly-free/anomalous topological orders

The notion of an anomaly-free/anomalous topological order was first introduced in [KW14].

Definition 2.3.1. An nd topological order is called *anomaly-free* if it can be realized by an nd lattice model with only local interactions. Otherwise it is called *anomalous*. ■

An nd anomalous topological order A_n , if exists, must be realized as a defect in a higher (but still finite) dimensional lattice model. By a process of dimensional reduction illustrated in Figure 3, one can always realize the topological order A_n as a boundary of an $n+1$ d lattice model, which realizes an $n+1$ d anomaly-free topological order C_{n+1} (see also [KW14, Section VI.B]).

¹³In Remark 3.5.14, we explain that the notion of a double braiding only makes sense on an open 2-disk instead of a 2-sphere.

Theorem^{ph} 2.3.2. An nd anomalous topological order can always be realized by a boundary of a 1-dimension-higher lattice model, or equivalently, a boundary of a 1-dimension-higher anomaly-free topological order.

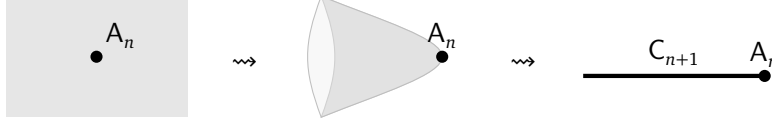


Figure 3: a dimensional reduction process

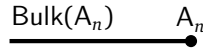
Although the realizations of A_n as a higher codimensional defect are almost never unique, it was shown in [KW14] that if A_n is a topological order, after the dimensional reduction, the anomaly-free phase C_{n+1} is unique (see [KW14, Lemma 1] for a physical proof).

Unique bulk principle: An nd topological order A_n can always be realized as a boundary of an anomaly-free $n+1d$ topological order C_{n+1} . Moreover, such C_{n+1} is unique and is called the *bulk* of A_n or the *gravitational anomaly* of A_n [KW14] and denoted by $\text{Bulk}(A_n)$.

Remark 2.3.3. This principle was later proposed as a defining property for much larger family of (potentially gapless) quantum phases, called quantum liquids [KZ22a, Section 5] (see Remark 2.3.3). When the quantum liquid A_n is gapped (i.e. a gapped quantum liquid), the proof given in [KW14, Lemma 1] does apply to this case (if its bulk C_{n+1} have the same symmetry as that of A_n). When A_n is gapless, the proof given in [KW14, Lemma 1] does not apply. Nevertheless, we propose the uniqueness of the bulk as a defining property of all quantum liquids.

There is no precise definition of a gapless quantum liquid. We use the term in the sense of [KZ22a, Section 1]. Roughly speaking, a gapless quantum liquid is a gapless quantum phase that is ‘soft’ enough so that it does not rigidly depend on the local spacetime geometry. Gapless quantum liquids in 1+1D include all rational CFT’s. For 1+1D bulk-boundary rational CFT’s, the uniqueness of the bulk was known [FFRS08, KR09]. \diamond

We depict the geometric relation between the boundary phase A_n and its bulk as follows:



where the $n+1d$ bulk phase $\text{Bulk}(A_n)$ is depicted as a spatial open interval to emphasize that a quantum phase is defined on the spatial manifold \mathbb{R}^n . The time axis is not shown. It is clear that $\text{Bulk}(1_n) = 1_{n+1}$. Moreover, we have the following result.

Theorem^{ph} 2.3.4. An nd topological order A_n is anomaly-free if and only if $\text{Bulk}(A_n) = 1_{n+1}$.

2.4 Topological defects and topological skeletons

For a topological order, all its correlation functions decay exponentially thus do not define observables in the long wave length limit. In this case, there are other \mathbb{R}^n -observables given by topological defects.

Microscopically, a *defect* in a lattice model can be realized as a local modification of the lattice, local Hilbert spaces, or the Hamiltonian. We give explicit examples in Section 3 (see also [KW14, Section IV]). A defect may break the uniformity of the phase, and determines an observable in the long wave length limit after coarse-graining. In a topological order, the equivalence class of such an observable with the equivalence relation defined by all admitted perturbations is called a *topological defect* (see Figure 4).

A defect itself can be viewed as a lower-dimensional lattice model embedded in a higher-dimensional lattice model. We only consider gapped defects in this work. In the long wave length limit, a topological defect in a non-trivial topological order can be viewed as a lower-dimensional anomalous topological order.

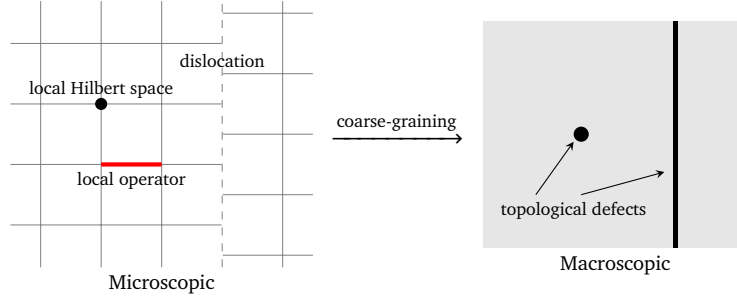


Figure 4: A defect in a lattice model can be given by introducing dislocations, enlarging local Hilbert spaces, or adding local operators to the Hamiltonian. Topological defects are defects in the long wave length limit.

In an nd topological order C_n , two adjacent topological defects of the same codimension can be fused into a single topological defect. Two examples are illustrated in Figure 5. The collection of all topological defects of codimension one or higher in C_n , together with the fusions among them in various codimensions, form a complicated structure, which is called the *topological skeleton* of C_n . The main goal of this work is to explain that the topological skeleton of a topological order is naturally a (higher) category.

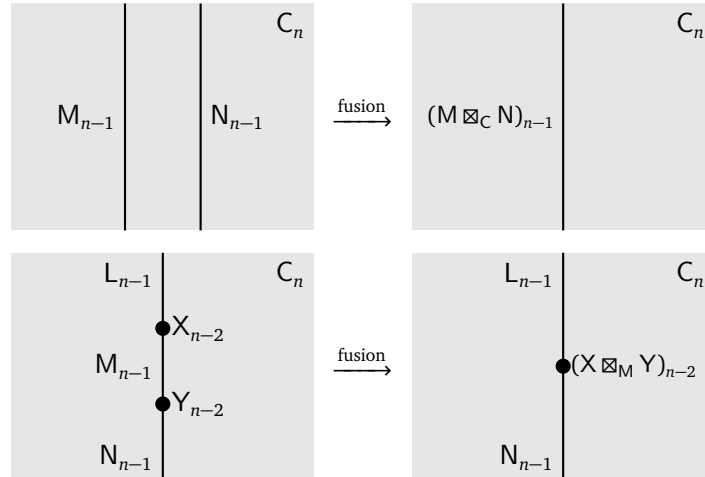


Figure 5: The topological defects in various dimensions can be fused together.

Remark 2.4.1. In literature (see for example [KW14, KWZ15, JF22]), the notion of a topological skeleton is often identified with that of a topological order (up to invertible topological orders). This point of view can sometimes cause confusion (see Remark 4.3.9). In this work, we carefully distinguish them even in their notations. \diamond

3 Topological orders in 2d

In 1982, the fractional quantum Hall effect (FQHE) was experimentally discovered by Tsui, Stormer and Gossard [TSG82]. Soon after, it was realized that there are two important features of the FQHE:

1. The quasi-particles in a fractional quantum Hall states can carry fractional charges [Lau83] and have fractional statistics [ASW84, Hal84] or even non-abelian statistics [Wu84, Wen91, RM92]. These quasi-particles are also called anyons [Wil82a, Wil82b] due to their anyonic statistics. It is worth pointing out that, without the guidance from FQHE, the existence of ‘anyonic’ particles in 2+1D were actually first predicted theoretically in 1977 by Leinaas and Myrheim in [LM77], and independently by Goldin, Menikoff and Sharp in [GMS80, GMS81] (see [Gol23] for the early history of predicting anyons¹⁴).
2. Tao and Wu [TW84] first proved that a fractional quantum Hall state defined on a nontrivial surface has nontrivial ground state degeneracy (GSD). The source of this nontrivial GSD, however, was unclear at first and caused some confusion, which was often misguided by symmetry-breaking tradition (see for example [And83, NTW85, Tho85]). It was finally clarified by Wen and Niu [Wen90, WN90] that the GSD is robust against arbitrary weak perturbations. Hence the nontrivial GSD is not related to any symmetry and beyond Landau’s paradigm. Moreover, it was known that the nontrivial GSD is directly related to the fractional statistics of quasi-particles [WN90].

In 1990, Xiao-Gang Wen proposed that the emergence of anyons and the robust GSD indicate a new kind of order that is beyond Landau’s paradigm [Wen90]. Partially inspired by Witten’s Chern-Simons TQFT’s [Wit89], Wen named this new kind of order as a “topological order” because its low energy effective theory is a TQFT (see Remark 2.1.5).

The anyons are special cases of topological defects introduced in Section 2.4 because they are particle-like. The main goal of this section is to explain in detail that particle-like topological defects in a 2d topological order form a mathematical structure called a unitary modular tensor category.

3.1 Toric code model

In this subsection we briefly review the 2d toric code model, which was introduced by Kitaev [Kit03], and explicitly compute the GSD.

3.1.1 Definition of the 2d toric code model

The 2d toric code model on a square lattice is defined as follows. There is a spin-1/2 on each edge (or link) of the lattice. In other words, the local degree of freedom \mathcal{H}_i on each edge i is a two-dimensional Hilbert space \mathbb{C}^2 . The total Hilbert space is $\mathcal{H}_{\text{tot}} := \bigotimes_i \mathcal{H}_i = \bigotimes_i \mathbb{C}^2$.

Let us recall the Pauli matrices acting on \mathbb{C}^2 :

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For each vertex v and plaquette p we define a vertex operator $A_v := \prod_i \sigma_x^i$ and a plaquette $B_p := \prod_j \sigma_z^j$ acting on adjacent edges. Here $\sigma_x^i = \cdots \otimes 1 \otimes \sigma_x \otimes 1 \otimes \cdots$ is the operator that acts on \mathcal{H}_i as σ_x and acts on other local Hilbert spaces as identities. For example, the operators in Figure 6 are

$$A_v = \sigma_x^1 \sigma_x^2 \sigma_x^3 \sigma_x^4, \quad B_p = \sigma_z^3 \sigma_z^4 \sigma_z^5 \sigma_z^6.$$

The Hamiltonian of the toric code model is defined to be

$$H := \sum_v (1 - A_v) + \sum_p (1 - B_p), \quad (3.1.1)$$

where the summation takes over all vertices v and all plaquettes p .

¹⁴The braid statistics of fields in 1+1D was first introduced by Frölich in 1976 in [Frö76] (see [FGM90] for the early history of braid statistics in quantum field theories).

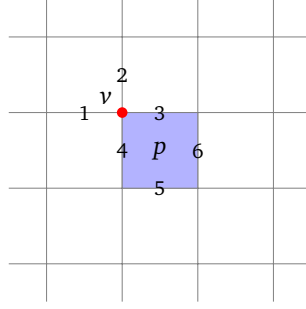


Figure 6: the toric code model

Remark 3.1.1. The toric code model can be defined on an arbitrary lattice, and its behavior in the long wave length limit is independent of the choice of the lattice. This is due to the fact that the toric code model is at a fixed point of the renormalization flow. But for simplicity, we only consider square lattice here. \diamond

A simple observation is that all A_v and B_p operators mutually commute. Indeed, any two plaquette operators commute because they consist of σ_z operators. Similarly, any two vertex operators commute. So we only need to consider a plaquette operator B_p and a vertex operator A_v . If v and p are not adjacent, they have no common edges and thus commute. If they are adjacent (for example see Figure 6), there are exactly two common edges labeled by 3 and 4. Since σ_x^3 anti-commutes with σ_z^3 and σ_x^4 anti-commutes with σ_z^4 , we conclude that A_v commutes with B_p .

It follows that the total Hilbert space can be decomposed as the direct sum of common eigenspaces of all A_v and B_p operators, whose eigenvalues are ± 1 . In particular, the ground state subspace has energy 0 and is the common eigenspace of all A_v and B_p operators with eigenvalues $+1$. Moreover, the system is gapped, because the first excited state has energy at least 2, no matter how large the system size is.

3.1.2 The ground state degeneracy and ground state wave functions

Let us compute the GSD of the toric code model defined on a genus g closed surface. Suppose the lattice has V vertices, E edges and F plaquettes. By Euler's formula, we have

$$V - E + F = 2 - 2g.$$

The dimension of the total Hilbert space is 2^E , and the ground state subspace is determined by the constraints that all $A_v = 1$ and $B_p = 1$. Note that these constraints are not independent. We have

$$\prod_v A_v = \prod_p B_p = 1$$

because the model is defined on a closed surface. So there are only $(V + F - 2)$ independent constraints, and each one halves the Hilbert space. Hence the GSD is $2^{E-(V+F-2)} = 2^{2g}$, which is a topological invariant.

In the rest of this section we explicitly compute the ground state wave functions on a torus ($g = 1$).

Choose eigenstates $|\pm\rangle$ of σ_x with eigenvalue ± 1 . We can decorate the lattice by red strings and construct a wave function from each string configuration (see Figure 7): if there is a red string on an edge, we put a state $|-\rangle$ on this edge; if there is no red string on an edge, we put a $|+\rangle$ on it. Thus each string configuration corresponds to a wave function is $|+\rangle \otimes |+\rangle \otimes |-\rangle \otimes \dots$, where $|-\rangle$ only appears on red edges.

The condition that all A_v operators acting on such a wave function as $+1$ means there is no open string. Indeed, if v is an end point of an open string, then A_v acts on the wave function as -1 . It follows that a ground state wave function is the superposition of closed string wave functions. Moreover, the condition that all B_p operators acting on the ground state wave function as $+1$ means that, if a closed string configuration can be obtained from another one by the action of several B_p operators,

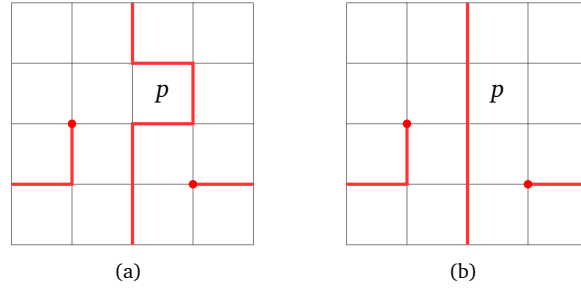


Figure 7: A string configuration corresponds to a wave function. We impose periodic boundary conditions in two directions so that the lattice model is defined on a torus. There is a long closed string and a short open string in these two figures. The action of the B_p operator locally deforms the string from (a) to (b).

their superposition coefficients must be equal. Clearly, the action of B_p operators is locally creating, annihilating or deforming closed strings (see Figure 7). If there is a non-contractible closed string, such a configuration can not be obtained from the vacuum configuration by B_p operators. Hence, by considering different non-contractible loops on a torus, we get four different (orthogonal) ground state wave functions:

$$\begin{aligned} |\psi_{00}\rangle &= \left| \begin{array}{cc} \square & \square \\ \square & \square \end{array} \right\rangle + \left| \begin{array}{cc} \square & \square \\ \square & \square \end{array} \right\rangle + \left| \begin{array}{cc} \square & \square \\ \square & \square \end{array} \right\rangle + \cdots, & |\psi_{01}\rangle &= \left| \begin{array}{cc} \square & \square \\ \square & \square \end{array} \right\rangle + \left| \begin{array}{cc} \square & \square \\ \square & \square \end{array} \right\rangle + \left| \begin{array}{cc} \square & \square \\ \square & \square \end{array} \right\rangle + \cdots, \\ |\psi_{10}\rangle &= \left| \begin{array}{cc} \square & \square \\ \square & \square \end{array} \right\rangle + \left| \begin{array}{cc} \square & \square \\ \square & \square \end{array} \right\rangle + \left| \begin{array}{cc} \square & \square \\ \square & \square \end{array} \right\rangle + \cdots, & |\psi_{11}\rangle &= \left| \begin{array}{cc} \square & \square \\ \square & \square \end{array} \right\rangle + \left| \begin{array}{cc} \square & \square \\ \square & \square \end{array} \right\rangle + \left| \begin{array}{cc} \square & \square \\ \square & \square \end{array} \right\rangle + \cdots. \end{aligned}$$

The closed strings in them belong to different \mathbb{Z}_2 -homology classes of a torus. Recall that $H_1(\text{torus}; \mathbb{Z}_2) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Remark 3.1.2. The ground state wave functions of the toric code model defined on an genus g surface Σ_g are similar. For each first homology class of Σ_g (with coefficient \mathbb{Z}_2) we can similarly construct a closed string wave function and hence there are $|H_1(\Sigma_g; \mathbb{Z}_2)| = |(\mathbb{Z}_2)^{\oplus 2g}| = 2^{2g}$ orthogonal ground state wave functions. \diamond

This example also illustrates the following general principle [LW05]:

The ground states are obtained from closed string condensation; the excitations are the end points of open strings.

3.2 The category of particle-like topological defects

3.2.1 Local operators and topological excitations

Recall Section 2.4 that a topological defect is an observable in the long wave length limit. How do we obtain a topological defect from microscopic data?

The way to read macroscopic information from microscopic data is to do coarse-graining, which is a process of integrating and averaging out the microscopic degrees of freedom. For the study of topological defects, the effect of coarse-graining can be stated as the screening by local operators. A local operator is an operator defined in a bounded region. Local operators in a bounded region act on the local Hilbert space in the bounded region. Any macroscopic observables are those survived after the screening by the cloud of local operators (see Figure 8).

The simplest way to obtain a defect is to add a local term in the Hamiltonian. Suppose δH_ξ is an operator acting around a site ξ . Then the ground state of the new Hamiltonian $(H + \delta H_\xi)$ is different from that of the original Hamiltonian H in general. With respect to the original Hamiltonian H , the new ground state looks like an excitation around the site ξ , but coincides with the original ground state far from the site ξ . Intuitively, the local term δH_ξ traps an excitation located at the site ξ . Conversely,

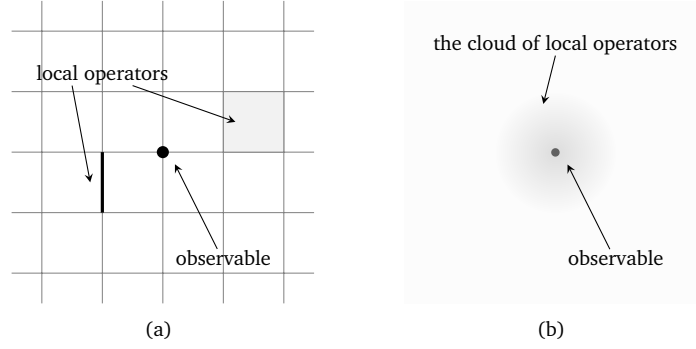


Figure 8: Observables are screened by the cloud of local operators. (a): an observable in the lattice model; (b): what we see in the long wave length limit.

any excitation located at a site ξ can be trapped by some operators δH_ξ . Therefore, such a defect is characterized by an excitation of the Hamiltonian H .

What is the topological defect corresponding to a trap δH_ξ ? Suppose $|\psi\rangle$ is an excitation trapped by δH_ξ (i.e., $|\psi\rangle$ is a ground state of $(H + \delta H_\xi)$) and A is a local operator, then $A|\psi\rangle$ should also be trapped by δH_ξ after coarse-graining. Thus all trapped states in the long wave length limit form a subspace which is invariant under the action of local operators. Such a subspace is called a *topological excitation*. The topological defect corresponding to δH_ξ is characterized by this subspace [KW14].

Definition 3.2.1. A *topological excitation* (or a particle-like topological defect) is a subspace of the total Hilbert space that is invariant under the action of local operators. ■

Given a state $|\psi\rangle$, the minimal topological excitation containing $|\psi\rangle$ is

$$\{A|\psi\rangle \mid A \text{ is a local operator}\},$$

called the topological excitation *generated* by $|\psi\rangle$. In particular, the topological excitation generated by the ground state subspace is called the *trivial topological excitation*, denoted by $\mathbb{1}$.

Remark 3.2.2. A *nontrivial* topological excitation is usually defined to be an excitation which cannot be created from or annihilated to the ground state by local operators. This definition is equivalent to Definition 3.2.1. Indeed, if a state $|\psi\rangle$ can be created from the ground state by local operators, the topological excitation generated by $|\psi\rangle$ should be the same as the trivial topological excitation.

As shown in Remark 3.2.8, a nontrivial topological excitation can be created from or annihilated to the ground state by a non-local operator. Adding a local trap δH_ξ to the original Hamiltonian H is equivalent to introducing a non-local operator on the world line of ξ . ◇

Remark 3.2.3. The topological excitation given by a local trap δH_ξ located at a site ξ is particle-like. Particle-like topological excitations in 2d are also called *anyons* due to their statistics (see Remark 3.4.68). We can also consider a local trap acting around a higher-dimensional submanifold in the space, and such a local trap gives a higher-dimensional topological excitation [KW14]. ◇

Remark 3.2.4. A particle-like topological defect is a special 0d topological defect, a name which is reserved for more general type of 0d defects, such as 0d domain walls between two 1d domain walls. ◇

Remark 3.2.5. In Definition 3.2.1, we use the hand-waving term “the action of local operators” without defining it precisely. There are two different approaches to treat ‘local operators’ precisely. In the first approach, the ‘local operators’ were explicitly constructed in the quantum double models [Kit03] and the Levin-Wen models [LW05, KK12, Kon13] and were shown to form a Hopf-like algebra. Then a topological excitation can be defined to be a module over this algebra (see Section 3.3.7). In the second approach, the ‘local operator algebras’ are replaced by the nets of ‘local operator algebras’ [Haa96]. Then a topological excitation can be defined by a (superselection) sector of this net [Naa11, KZ22b, Wal23]. A lot of new references are emerging in this direction, including the works of Ogata, Jones, Penneys, etc. ◇

This section needs a significant expansion to include a brief review of many different approaches towards the meaning of a particle-like topological defect or topological excitations or anyons.

3.2.2 Particle-like topological defects of the toric code model

Let us find the particle-like topological defects of the toric code model.

The Hamiltonian of the toric code model is the sum of mutually commuting operators, so an excitation, as an energy eigenstate, is determined by its eigenvalue of all A_v and B_p operators. For example, given a vertex v_0 , there is a state $|\psi_{v_0}\rangle$ satisfying the following property:

$$\begin{cases} A_{v_0}|\psi_{v_0}\rangle = -|\psi_{v_0}\rangle, \\ A_v|\psi_{v_0}\rangle = |\psi_{v_0}\rangle \text{ for all vertices } v \neq v_0, \\ B_p|\psi_{v_0}\rangle = |\psi_{v_0}\rangle \text{ for all plaquettes } p. \end{cases}$$

In other words, $|\psi_{v_0}\rangle$ is an excitation located at v_0 . Then we try to apply a local operator σ_z^1 on $|\psi_{v_0}\rangle$ (see Figure 9). Since σ_z^1 commutes with all plaquette operators and vertex operators except A_{v_0} and A_{v_1} , we still have

$$\begin{aligned} B_p \cdot \sigma_z^1 |\psi_{v_0}\rangle &= \sigma_z^1 |\psi_{v_0}\rangle \\ A_v \cdot \sigma_z^1 |\psi_{v_0}\rangle &= \sigma_z^1 |\psi_{v_0}\rangle \end{aligned}$$

for all plaquettes p and vertices $v \neq v_0, v_1$. But σ_z^1 anti-commutes with A_{v_0} and A_{v_1} , so we have

$$\begin{aligned} A_{v_0} \cdot \sigma_z^1 |\psi_{v_0}\rangle &= -\sigma_z^1 A_{v_0} |\psi_{v_0}\rangle = \sigma_z^1 |\psi_{v_0}\rangle, \\ A_{v_1} \cdot \sigma_z^1 |\psi_{v_0}\rangle &= -\sigma_z^1 A_{v_1} |\psi_{v_0}\rangle = -\sigma_z^1 |\psi_{v_0}\rangle. \end{aligned}$$

Therefore, $\sigma_z^1 |\psi_{v_0}\rangle$ is an excitation located at v_1 .

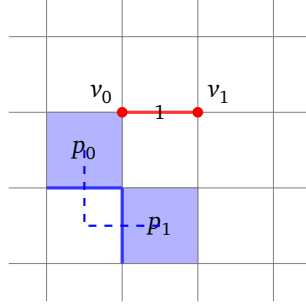


Figure 9: topological excitations of toric code

As illustrated by the above discussion, local operators can not annihilate such an excitation $|\psi_{v_0}\rangle$. So $|\psi_{v_0}\rangle$ generates a nontrivial topological excitation, denoted by e . All $|\psi_{v_0}\rangle$'s generate the same topological excitation e , because they can be connected by a string of σ_z operators.

Remark 3.2.6. The state $|\psi_{v_0}\rangle$ is the ground state of the Hamiltonian

$$H' = H + 2A_{v_0} = \sum_{v \neq v_0} (1 - A_v) + \sum_p (1 - B_p) + (1 + A_{v_0}).$$

Thus the corresponding defect is given by the local trap $\delta H_{v_0} = 2A_{v_0}$. In the long wave length limit, all these defects give the same topological defect e . \diamond

Similarly, for each plaquette p_0 there is a state $|\psi_{p_0}\rangle$ satisfying the following property:

$$\begin{cases} B_{p_0}|\psi_{p_0}\rangle = -|\psi_{p_0}\rangle, \\ B_p|\psi_{p_0}\rangle = |\psi_{p_0}\rangle \text{ for all plaquettes } p \neq p_0, \\ A_v|\psi_{p_0}\rangle = |\psi_{p_0}\rangle \text{ for all vertices } v. \end{cases}$$

Then all states $|\psi_{p_0}\rangle$ generate the same topological excitation, denoted by m . Moreover, for every vertex v_0 and plaquette p_0 there is a state $|\psi_{v_0,p_0}\rangle$ satisfying the following property:

$$\begin{cases} A_{v_0}|\psi_{v_0,p_0}\rangle = -|\psi_{v_0,p_0}\rangle, \\ A_v|\psi_{v_0,p_0}\rangle = |\psi_{v_0,p_0}\rangle \text{ for all vertices } v \neq v_0, \\ B_{p_0}|\psi_{v_0,p_0}\rangle = -|\psi_{v_0,p_0}\rangle, \\ B_p|\psi_{v_0,p_0}\rangle = |\psi_{v_0,p_0}\rangle \text{ for all plaquettes } p \neq p_0. \end{cases}$$

Then all states $|\psi_{v_0,p_0}\rangle$ generate the same topological excitation, denoted by f . Intuitively, if an e particle and an m particle are adjacent, they can be viewed as a single topological excitation f . Finally, there is always the trivial topological excitation $\mathbb{1}$, i.e., the topological excitation generated by the ground state. Consequently, we have found 4 different topological excitations: $\mathbb{1}, e, m, f$.

Remark 3.2.7. There are more topological excitations than $\mathbb{1}, e, m, f$. Recall that a topological excitation is a subspace of the total Hilbert space, thus we can talk about the direct sum of topological excitations. For example, $\mathbb{1} \oplus m$ can be generated by the ground state subspace of the following Hamiltonian:

$$H' = H + B_{p_0} = \sum_v (1 - A_v) + \sum_{p \neq p_0} (1 - B_p) + 1.$$

We provide more examples in Section 3.4.1. \diamond

Remark 3.2.8. The nontrivial topological excitations e, m, f can be created from or annihilated to the ground state by non-local operators. As depicted in Figure 10, an e particle can be created or annihilated by an infinitely long string operator $\cdots \sigma_z^4 \sigma_z^3 \sigma_z^2 \sigma_z^1$, and an m particle can be created or annihilated by $\cdots \sigma_x^7 \sigma_x^6 \sigma_x^5$. \diamond

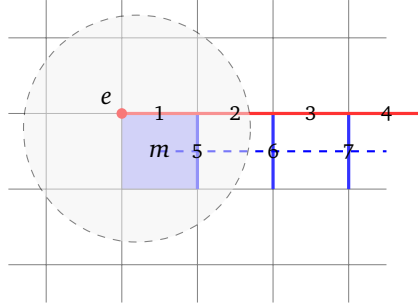


Figure 10: Topological excitations can be created or annihilated by non-local operators.

Remark 3.2.9. We do not really need an infinitely long string operators to create or annihilate topological excitations, because the notion of locality depends on the length scale. For example, if we treat operators in the shaded region in Figure 10 as local, then $\sigma_z^3 \sigma_z^2 \sigma_z^1$ is already non-local and can create or annihilate an e particle.

This example suggests that the notion of a local operator, as well as a topological excitation, depend on the length scale. We can choose an arbitrary region R , provided that it is particle-like when we look at it from far away, and define local operators and topological excitations in the region R . So a particle-like topological excitation does not necessarily located at a point. Its ‘shape’ is the same as the chosen region R . \diamond

3.2.3 Instantons

Topological excitations are subspaces of the total Hilbert space. A natural question is whether operators on these spaces are observables in the long wave length limit.

Suppose x, y are two topological excitations located at a site ξ and $f : x \rightarrow y$ is an operator between these two Hilbert spaces. The operator f should be viewed as a 0D (spacetime dimension) defect between the world lines of x and y . Thus it gives a 0D topological defect in the long wave length limit.

As depicted in Figure 11 (a), a local operator A can act on the world line before or after f . In the long wave length limit, all these actions are averaged out, thus what we can see is not necessarily f , but an screened operator which commutes with all local operators. Hence, a 0D topological defect in the spacetime is an operator which commutes with all local operators. These operators are called *instantons* because they are localized on time axis. Figure 11 (b) is an intuitive picture of an instanton. We denote the space of all instantons from x to y by $\text{Hom}(x, y)$.

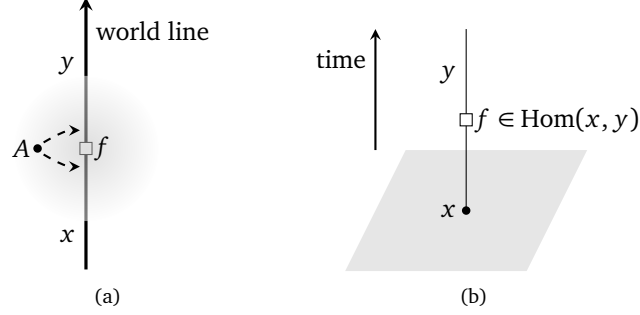


Figure 11: (a): the screening on time axis; (b) an instanton can freely pass through the cloud of local operators.

Instantons can be fused together. Intuitively, if two instantons $f \in \text{Hom}(x, y)$ and $g \in \text{Hom}(y, z)$ are very close to each other, they can be viewed as a single instanton from x to z , denoted by $g \circ f$ (see Figure 12). The fusion of instantons defines a map:

$$\begin{aligned} \circ: \text{Hom}(y, z) \times \text{Hom}(x, y) &\rightarrow \text{Hom}(x, z) \\ (g, f) &\mapsto g \circ f. \end{aligned}$$

In the language of operators, the fusion of instantons is just the composition (or product) of operators.

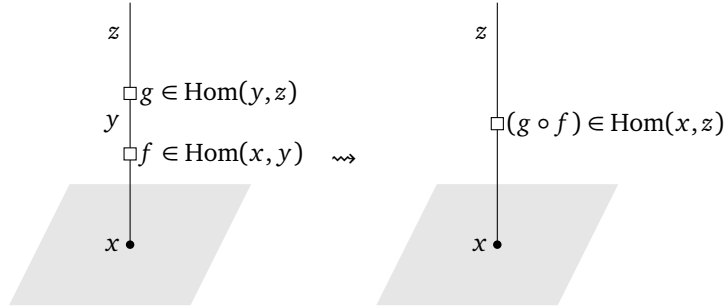


Figure 12: the fusion of instantons

We can also fuse more instantons. Suppose there are three instantons $f_i: x_{i-1} \rightarrow x_i$ for $i = 1, 2, 3$. By fusing them three together we get an instanton denoted by $f_3 \circ f_2 \circ f_1: x_0 \rightarrow x_3$. On the other hand, fusing them two by two is also possible: we can first fuse f_1 and f_2 to get $f_2 \circ f_1$, then fuse it with f_3 to get $f_3 \circ (f_2 \circ f_1)$; or fuse f_2 and f_3 first then fuse f_1 with the result to get $(f_3 \circ f_2) \circ f_1$. It is physically obvious that these different ways of fusion should give the same result:

$$f_3 \circ f_2 \circ f_1 = f_3 \circ (f_2 \circ f_1) = (f_3 \circ f_2) \circ f_1.$$

In other words, the fusion of instantons is associative. By considering instantons as operators on Hilbert spaces, the associativity is also obvious.

Remark 3.2.10. The associativity implies that the fusion of three instantons can be defined via the fusion of two instantons. Similarly, there are many different ways to fuse n instantons two by two, and all the results are the same by the associativity. One can easily prove this statement by induction on n . Hence, it is enough to consider the fusion of two instantons and the associativity condition. \diamond

Moreover, for any topological defect x , clearly ‘nothing happening’ is also an instanton. As an operator on the Hilbert space, it is just the identity operator. So we denote it by id_x . It is trivial in the sense that any instanton $f \in \text{Hom}(x, y)$ does not change by fusing with a trivial instanton (see Figure 13):

$$\text{id}_y \circ f = f = f \circ \text{id}_x.$$

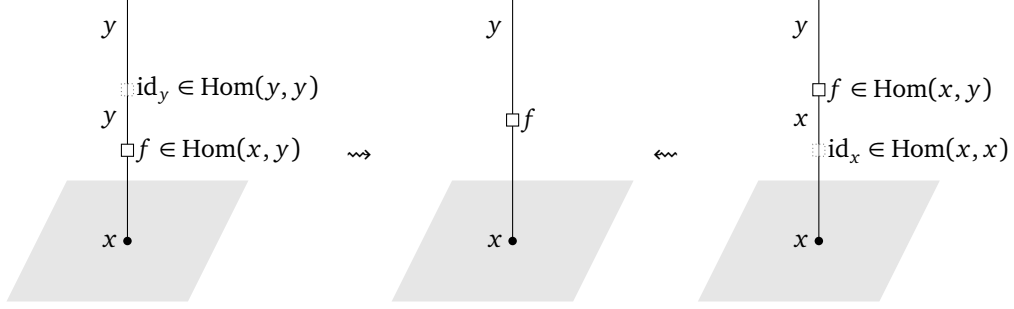


Figure 13: The fusion of an instanton f with a trivial instanton is still f .

3.2.4 The structure of a category

We summarize the data and properties developed in previous subsections to a formal structure, called a category.

Definition 3.2.11. A category \mathcal{C} consists of the following data:

- a set $\text{ob}(\mathcal{C}) = \{x, y, \dots\}$, whose elements are called *objects* of \mathcal{C} ;
- a set $\text{Hom}_{\mathcal{C}}(x, y)$ for every $x, y \in \text{ob}(\mathcal{C})$, whose elements are called *morphisms* from x to y ;
- a map $\circ: \text{Hom}_{\mathcal{C}}(y, z) \times \text{Hom}_{\mathcal{C}}(x, y) \rightarrow \text{Hom}_{\mathcal{C}}(x, z): (g, f) \mapsto g \circ f$ for any $x, y, z \in \text{ob}(\mathcal{C})$, called the *composition* of morphisms;
- a distinguished morphism $\text{id}_x \in \text{Hom}_{\mathcal{C}}(x, x)$ for each $x \in \text{ob}(\mathcal{C})$, called the *identity* morphism;

and these data satisfy the following conditions:

1. (**associativity**) $(h \circ g) \circ f = h \circ (g \circ f)$ for any $f \in \text{Hom}_{\mathcal{C}}(x, y)$, $g \in \text{Hom}_{\mathcal{C}}(y, z)$, $h \in \text{Hom}_{\mathcal{C}}(z, w)$ and $x, y, z \in \text{ob}(\mathcal{C})$;
2. (**unitality**) $\text{id}_y \circ f = f = f \circ \text{id}_x$ for any $f \in \text{Hom}_{\mathcal{C}}(x, y)$ and $x, y \in \text{ob}(\mathcal{C})$. ■

Then the discussion in previous subsections can be summarized to the following physical theorem.

Theorem^{ph} 3.2.12. Given a 2d topological order C , its particle-like (i.e., 0+1D) topological defects and (0D) instantons form a category \mathcal{C} :

- The set $\text{ob}(\mathcal{C})$ of objects is the set of particle-like topological defects (topological excitations).
- The set $\text{Hom}_{\mathcal{C}}(x, y)$ of morphisms is the space of instantons from x to y .
- The composition of morphisms is given by the fusion of instantons.
- The identity morphisms are trivial instantons.

Indeed, the above discussions are independent of the dimension of the topological order C . So this result holds for all nd topological orders where $n \geq 1$. When C is a 1d topological order (either anomaly-free or anomalous), the category \mathcal{C} is the topological skeleton of C . When the dimension of C is higher than one, there are also higher-dimensional topological defects, so the category \mathcal{C} is only a part of the topological skeleton. For 0d topological orders it is not reasonable to talk about particle-like topological defects, but such categorical structures still exist (see Section 4.4.1).

Example 3.2.13. All particle-like topological defects and instantons of the toric code model form a category, denoted by \mathcal{TC} . By definition, $1, e, m, f$ are objects in \mathcal{TC} , but there are also other objects in \mathcal{TC} . The detail of the category \mathcal{TC} is discussed in Section 3.4.1. \heartsuit

Notation 3.2.14. If x is an object of a category \mathcal{C} , we write $x \in \mathcal{C}$ for simplicity instead of $x \in \text{ob}(\mathcal{C})$.

Notation 3.2.15. In this work, we use the same letter in different fonts to denote a topological order and its category of particle-like topological defects. Topological orders are labeled by $\text{\texttt{\textsf{font}}}$: C, D, E, \dots , and categories are labeled by $\text{\texttt{\textsf{font}}}$: $\mathcal{C}, \mathcal{D}, \mathcal{E}, \dots$. However, 0d topological orders (for example, particle-like topological defects in another topological order) are also labeled by lower case letters: a, b, c, x, y, z, \dots .

3.3 Basic category theory

3.3.1 Examples of categories

Example 3.3.1. The category of sets, denoted by Set , is defined by the following data:

- An object in Set is a set.
- Given two sets X and Y , the set $\text{Hom}_{\text{Set}}(X, Y)$ is the set of maps from X to Y .
- The composition of morphisms is the usual composition of maps.
- The identity morphism id_X on a set X is the usual identity map. \heartsuit

Example 3.3.2. The category of groups, denoted by Grp , is defined by the following data:

- An object in Grp is a group.
- Given two groups G and H , the set $\text{Hom}_{\text{Grp}}(G, H)$ is the set of group homomorphisms from G to H .
- The composition of morphisms is the usual composition of maps.
- The identity morphism id_G on a group G is the usual identity map. \heartsuit

Example 3.3.3. The category of finite-dimensional vector spaces over \mathbb{C} (the field of complex numbers), denoted by $\text{Vec}_{\mathbb{C}}$ or simply Vec , is defined by the following data:

- An object in Vec is a finite-dimensional vector space over \mathbb{C} .
- Given two finite-dimensional vector spaces V and W , the set $\text{Hom}_{\text{Vec}}(V, W)$ is the set of \mathbb{C} -linear maps from V to W .
- The composition of morphisms is the usual composition of maps.
- The identity morphism id_V on a finite-dimensional vector space V is the usual identity map.

Similarly, all (not necessarily finite-dimensional) vector spaces over \mathbb{C} and linear maps also form a category. \heartsuit

Example 3.3.4. Let G be a group. The category of finite-dimensional G -representations (over \mathbb{C}), denoted by $\text{Rep}(G)$, is defined by the following data:

- An object in $\text{Rep}(G)$ is a finite-dimensional G -representation (over \mathbb{C}), i.e., a finite-dimensional vector space V equipped with a group homomorphism $G \rightarrow \text{GL}(V)$. Here $\text{GL}(V)$ denotes the group of linear automorphisms of V (i.e., invertible linear maps from V to itself).
- Given two finite-dimensional G -representations (V, ρ) and (W, σ) , a morphism $f : (V, \rho) \rightarrow (W, \sigma)$ is a \mathbb{C} -linear map $f : V \rightarrow W$ satisfying $f \circ \rho(g) = \sigma(g) \circ f$ for all $g \in G$.

- The composition of morphisms is the usual composition of maps.
- The identity morphism $\text{id}_{(V,\rho)}$ on a finite-dimensional G -representation V is the usual identity map. \heartsuit

Example 3.3.5. Let G be a group. The category of locally finite-dimensional G -graded vector spaces (over \mathbb{C}), denoted by Vec_G , is defined by the following data:

- An object in Vec_G is a locally finite-dimensional G -graded vector space (over \mathbb{C}), i.e., a collection of finite-dimensional vector spaces $\{V_g \in \text{Vec}\}_{g \in G}$. The direct sum $V := \bigoplus_{g \in G} V_g$ is called the total space. By abuse of notation, we also use V to denote this G -graded vector space.
- Given two locally finite-dimensional G -graded vector spaces $V = \bigoplus_{g \in G} V_g$ and $W = \bigoplus_{g \in G} W_g$, a morphism from V to W is a collection of \mathbb{C} -linear maps $\{f_g : V_g \rightarrow W_g\}_{g \in G}$. Note that $f := \bigoplus_{g \in G} f_g : V \rightarrow W$ is a linear map between two total spaces.
- The composition of morphisms is given by the usual composition of maps in each degree $g \in G$.
- The identity morphism id_V on a locally finite-dimensional G -graded vector space $V = \bigoplus_{g \in G} V_g$ is given by the usual identity map id_{V_g} on each degree $g \in G$. \heartsuit

Exercise 3.3.6. Let G be a group. For every $g \in G$, we define a G -graded vector space $\mathbb{C}_{(g)}$ by the total space \mathbb{C} equipped with the G -grading

$$(\mathbb{C}_{(g)})_h := \begin{cases} \mathbb{C}, & h = g, \\ 0, & h \neq g. \end{cases}$$

Find $\text{Hom}_{\text{Vec}_G}(\mathbb{C}_{(g)}, \mathbb{C}_{(h)})$ for $g, h \in G$.

Example 3.3.7. Let G be a group. There is a category BG defined by the following data:

- There is only one object in BG , i.e., $\text{ob}(BG) := \{*\}$.
- $\text{Hom}_{BG}(*, *) := G$.
- The composition is defined by the multiplication of G :

$$\text{Hom}_{BG}(*, *) \times \text{Hom}_{BG}(*, *) = G \times G \xrightarrow{\text{multiplication}} G = \text{Hom}_{BG}(*, *).$$

- The identity id_* is the unit of G (usually denoted by $e \in G$).

The category BG is called the *delooping* of G . \heartsuit

3.3.2 Construct new categories from old ones

There are some method to construct new categories from old ones.

Definition 3.3.8. Let \mathcal{C} be a category. Its *opposite category*, denoted by \mathcal{C}^{op} , is defined by the following data:

- $\text{ob}(\mathcal{C}^{\text{op}}) := \text{ob}(\mathcal{C})$.
- $\text{Hom}_{\mathcal{C}^{\text{op}}}(x, y) := \text{Hom}_{\mathcal{C}}(y, x)$ for $x, y \in \mathcal{C}^{\text{op}}$.
- The composition of morphisms in \mathcal{C}^{op} is induced by that of \mathcal{C} :

$$\begin{aligned} \text{Hom}_{\mathcal{C}^{\text{op}}}(y, z) \times \text{Hom}_{\mathcal{C}^{\text{op}}}(x, y) &= \text{Hom}_{\mathcal{C}}(z, y) \times \text{Hom}_{\mathcal{C}}(y, x) \\ &\simeq \text{Hom}_{\mathcal{C}}(y, x) \times \text{Hom}_{\mathcal{C}}(z, y) \xrightarrow{\circ \text{ of } \mathcal{C}} \text{Hom}_{\mathcal{C}}(z, x) = \text{Hom}_{\mathcal{C}^{\text{op}}}(x, z). \end{aligned}$$

- The identity morphisms in \mathcal{C}^{op} are the same as those in \mathcal{C} . \blacksquare

Intuitively, the opposite category \mathcal{C}^{op} is obtained by reversing the morphisms in \mathcal{C} .

Definition 3.3.9. Let \mathcal{C}, \mathcal{D} be categories. Their *Cartesian product*, denoted by $\mathcal{C} \times \mathcal{D}$, is defined by the following data:

- $\text{ob}(\mathcal{C} \times \mathcal{D}) := \text{ob}(\mathcal{C}) \times \text{ob}(\mathcal{D})$.
- $\text{Hom}_{\mathcal{C} \times \mathcal{D}}((x, y), (x', y')) := \text{Hom}_{\mathcal{C}}(x, x') \times \text{Hom}_{\mathcal{D}}(y, y')$ for $x, x' \in \mathcal{C}$ and $y, y' \in \mathcal{D}$.
- The composition of morphisms in $\mathcal{C} \times \mathcal{D}$ is induced by those of \mathcal{C} and \mathcal{D} :

$$\begin{aligned} & \text{Hom}_{\mathcal{C} \times \mathcal{D}}((x', y'), (x'', y'')) \times \text{Hom}_{\mathcal{C} \times \mathcal{D}}((x, y), (x', y')) \\ &= \text{Hom}_{\mathcal{C}}(x', x'') \times \text{Hom}_{\mathcal{D}}(y', y'') \times \text{Hom}_{\mathcal{C}}(x, x') \times \text{Hom}_{\mathcal{D}}(y, y') \\ &\simeq \text{Hom}_{\mathcal{C}}(x', x'') \times \text{Hom}_{\mathcal{C}}(x, x') \times \text{Hom}_{\mathcal{D}}(y', y'') \times \text{Hom}_{\mathcal{D}}(y, y') \\ &\xrightarrow{(\circ \text{ of } \mathcal{C}) \times (\circ \text{ of } \mathcal{D})} \text{Hom}_{\mathcal{C}}(x, x'') \times \text{Hom}_{\mathcal{D}}(y, y'') = \text{Hom}_{\mathcal{C} \times \mathcal{D}}((x, y), (x'', y'')). \end{aligned}$$

- $\text{id}_{(x, y)} := (\text{id}_x, \text{id}_y)$ for $x \in \mathcal{C}, y \in \mathcal{D}$. ■

Definition 3.3.10. Let \mathcal{C} be a category. Suppose \mathcal{D} is a collection of the following data:

- a subset $\text{ob}(\mathcal{D}) \subseteq \text{ob}(\mathcal{C})$;
- a subset $\text{Hom}_{\mathcal{D}}(x, y) \subseteq \text{Hom}_{\mathcal{C}}(x, y)$ for every $x, y \in \text{ob}(\mathcal{D}) \subseteq \text{ob}(\mathcal{C})$;

such that \mathcal{D} is a category with the composition and identity morphisms induced from \mathcal{C} (i.e., the hom sets in \mathcal{D} is closed under composition and $\text{id}_x \in \text{Hom}_{\mathcal{D}}(x, x)$ for each $x \in \mathcal{D}$). Then \mathcal{D} is called a *subcategory* of \mathcal{C} . We say the subcategory $\mathcal{D} \subseteq \mathcal{C}$ is *full* if $\text{Hom}_{\mathcal{D}}(x, y) = \text{Hom}_{\mathcal{C}}(x, y)$ for all $x, y \in \mathcal{D}$. ■

3.3.3 Morphisms in a category

The equations of morphisms in a category can be represented by *commutative diagrams*. Here we do not give the definition of a commutative diagram, but intuitively explain this notion by some examples. If there are three morphisms $f: x \rightarrow y$, $g: y \rightarrow z$ and $h: x \rightarrow z$, then the equation $g \circ f = h$ is equivalent to say that the following diagram is commutative:

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ & \searrow h & \downarrow g \\ & & z \end{array}$$

Also, the following commutative diagram

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ h \downarrow & & \downarrow g \\ z & \xrightarrow{k} & w \end{array}$$

is equivalent to say that $g \circ f = k \circ h$.

Definition 3.3.11. Let \mathcal{C} be a category and $x, y \in \mathcal{C}$. A morphism $f: x \rightarrow y$ is called an *isomorphism* if there exists a morphism $g: y \rightarrow x$ such that $g \circ f = \text{id}_x$ and $f \circ g = \text{id}_y$. The morphism g , if exists, is called the *inverse* of f and denoted by f^{-1} . Two objects x, y are said to be *isomorphic* if there exists an isomorphism between them and we denote $x \simeq y$. ■

Exercise 3.3.12. Prove that the inverse of an isomorphism is unique.

Example 3.3.13. In the category Set of sets, an isomorphism is a bijective map. ♡

Definition 3.3.14. Let \mathcal{C} be a category and $x \in \mathcal{C}$. A morphism $e: x \rightarrow x$ is called an *idempotent* if it satisfies $e = e^2 := e \circ e$. ■

Example 3.3.15. Let \mathcal{C} be a category and $x, y \in \mathcal{C}$. Suppose there are morphisms $r: x \rightarrow y$ and $s: y \rightarrow x$ such that $r \circ s = \text{id}_y$. Then $s \circ r: x \rightarrow x$ is an idempotent because $(s \circ r) \circ (s \circ r) = s \circ (r \circ s) \circ r = s \circ r$. ♥

Definition 3.3.16. Let \mathcal{C} be a category and $x \in \mathcal{C}$. We say that an idempotent $e: x \rightarrow x$ *splits* if there exists an object $y \in \mathcal{C}$ equipped with two morphisms $r: x \rightarrow y, s: y \rightarrow x$ such that $r \circ s = \text{id}_y$ and $s \circ r = e$. The triple (y, r, s) (or simply the object y) is called an *image* of the idempotent e . A category \mathcal{C} is *idempotent complete* if every idempotent in \mathcal{C} splits. ■

Remark 3.3.17. An image of an idempotent, if exists, is unique up to a unique isomorphism. More precisely, suppose (y, r, s) and (y', r', s') are both images of an idempotent $e: x \rightarrow x$. Then there exists a unique isomorphism $f: y \rightarrow y'$ such that the following diagram commutes:

$$\begin{array}{ccc} x & \xrightarrow{r} & y \\ & \searrow r' & \downarrow f \\ & & y' \\ & & \xrightarrow{s'} x \end{array} \quad \begin{array}{c} s \\ \nearrow \\ \end{array}$$

i.e., $f \circ r = r'$ and $s' \circ f = s$. Indeed, the morphism f is equal to $r' \circ s$. Thus we can talk about *the* image of an idempotent. ◇

Example 3.3.18. All categories in Section 3.3.1 are idempotent complete. ♥

3.3.4 \mathbb{C} -linear categories and direct sums

The categories Vec , $\text{Rep}(G)$ and Vec_G have more structures than an ordinary category. Recall for every vector spaces V and W , the hom space $\text{Hom}_{\text{Vec}}(V, W)$ is naturally a vector space:

- The addition is defined by $(f + g)(v) := f(v) + g(v)$ for $f, g \in \text{Hom}_{\text{Vec}}(V, W)$.
- The scalar product is defined by $(\lambda \cdot f)(v) := \lambda \cdot f(v)$ for $\lambda \in \mathbb{C}$ and $f \in \text{Hom}_{\text{Vec}}(V, W)$.

Moreover, the composition map $\text{Hom}_{\text{Vec}}(V, W) \times \text{Hom}_{\text{Vec}}(U, V) \rightarrow \text{Hom}_{\text{Vec}}(U, W)$ is \mathbb{C} -bilinear.

Definition 3.3.19. A \mathbb{C} -linear category is a category in which each hom set is equipped with a structure of a finite-dimensional vector space over \mathbb{C} , such that the composition of morphisms is \mathbb{C} -bilinear. ■

Example 3.3.20. The category Vec is naturally a \mathbb{C} -linear category. For any group G , both $\text{Rep}(G)$ and Vec_G are \mathbb{C} -linear categories. ♥

Exercise 3.3.21. Let \mathcal{C} be a \mathbb{C} -linear category and $x, y \in \mathcal{C}$. Since the hom space $\text{Hom}_{\mathcal{C}}(x, y)$ is a vector space, there is a zero vector in it, denoted by $0: x \rightarrow y$. This distinguished morphism is called the *zero morphism*.

- (1) For any object $z \in \mathcal{C}$ and morphisms $f: y \rightarrow z, g: z \rightarrow x$, prove that $f \circ 0 = 0 = 0 \circ g$. Note that these 0's are different morphisms in different hom spaces.
- (2) Prove that the following four conditions are equivalent for an object $a \in \mathcal{C}$:
 - (a) $\text{Hom}_{\mathcal{C}}(a, b) = 0$ for all $b \in \mathcal{C}$.
 - (b) $\text{Hom}_{\mathcal{C}}(b, a) = 0$ for all $b \in \mathcal{C}$.
 - (c) $\text{Hom}_{\mathcal{C}}(a, a) = 0$.
 - (d) $\text{id}_a = 0$.

An object satisfying one (hence all) of the above conditions is called a *zero object*.

- (3) Prove that any two zero objects are isomorphic via a unique isomorphism. Thus we can talk about *the* zero object, which is denoted by 0 .

(4) Find the zero object in Vec , $\text{Rep}(G)$ and Vec_G where G is a group.

Given a topological order \mathcal{C} , it is expected that the category \mathcal{C} of particle-like topological defects is \mathbb{C} -linear, because the instantons (morphisms) are microscopically realized by linear operators. By Remark 3.2.7, it is expected we can talk about direct sums in \mathcal{C} .

What is the direct sum in a \mathbb{C} -linear category? Recall that the direct sum of two vector spaces V and W is

$$V \oplus W := \{(v, w) \mid v \in V, w \in W\}$$

(which is just the Cartesian product $V \times W$ as a set) equipped with component-wise addition and scalar product. There are some obvious embedding and projection maps

$$\iota_V : V \rightarrow V \oplus W, \pi_V : V \oplus W \rightarrow V, \iota_W : W \rightarrow V \oplus W, \pi_W : V \oplus W \rightarrow W,$$

and these linear maps satisfy the following equations:

$$\pi_V \circ \iota_V = \text{id}_V, \pi_W \circ \iota_W = \text{id}_W, \pi_V \circ \iota_W = 0, \pi_W \circ \iota_V = 0, \iota_V \circ \pi_V + \iota_W \circ \pi_W = \text{id}_{V \oplus W}.$$

These maps and equations fully characterize the direct sum. Indeed, if there is a vector space X equipped with such linear maps satisfying the above equations, one can prove that X is canonically isomorphic to $V \oplus W$.

Definition 3.3.22. Let \mathcal{C} be a \mathbb{C} -linear category and $x_1, \dots, x_n \in \mathcal{C}$. A direct sum of x_1, \dots, x_n is an object $x \in \mathcal{C}$ equipped with morphisms $\iota_i : x_i \rightarrow x$ and $\pi_i : x \rightarrow x_i$ for every $1 \leq i \leq n$, such that the following $(n^2 + 1)$ equations hold:

$$\pi_i \circ \iota_j = \delta_{ij} \cdot \text{id}_{x_j} \text{ (for all } 1 \leq i, j \leq n), \quad \sum_{j=1}^n \iota_j \circ \pi_j = \text{id}_x.$$

■

Exercise 3.3.23. Let \mathcal{C} be a \mathbb{C} -linear category such that the zero object exists. Prove that $x \oplus 0 \simeq x \simeq 0 \oplus x$ for every $x \in \mathcal{C}$, i.e., x is the direct sum of x and 0 .

Exercise 3.3.24. Let \mathcal{C} be a \mathbb{C} -linear category and $x_1, \dots, x_n \in \mathcal{C}$. Suppose $(x, \{\iota_i\}, \{\pi_i\})$ is a direct sum of x_1, \dots, x_n .

- (1) Prove that x satisfies the following universal property of a *product*: for any object $y \in \mathcal{C}$ and morphisms $\{f_i : y \rightarrow x_i\}_{i=1}^n$, there exists a unique morphism $f : y \rightarrow x$ such that $\pi_i \circ f = f_i$ for every $1 \leq i \leq n$. Hint: $\pi_i \circ f = f_i$ implies that $\iota_i \circ \pi_i \circ f = \iota_i \circ f_i$.
- (2) Prove that x satisfies the following universal property of a *coproduct*: for any object $z \in \mathcal{C}$ and morphisms $\{g_i : x_i \rightarrow z\}_{i=1}^n$, there exists a unique morphism $g : x \rightarrow z$ such that $g \circ \iota_i = g_i$ for every $1 \leq i \leq n$.
- (3) Use the above universal properties to conclude that a morphism $f : x_1 \oplus \dots \oplus x_n \rightarrow y_1 \oplus \dots \oplus y_m$ can be equivalently represented as an m -by- n matrix of morphisms $(f_{ji} : x_i \rightarrow y_j)$. Find the precise relation between the morphism f and the matrix (f_{ji}) .

Remark 3.3.25. A direct sum of some objects does not necessarily exist. But if it exists, it is unique up to a unique isomorphism. More precisely, suppose $(x, \{\iota_i\}, \{\pi_i\})$ and $(x', \{\iota'_i\}, \{\pi'_i\})$ are both direct sums of x_1, \dots, x_n . Then there exists a unique isomorphism $f : x \rightarrow x'$ such that the following diagrams commute for all $1 \leq i \leq n$:

$$\begin{array}{ccccc} & x_i & & x & \xrightarrow{\quad f \quad} & x' \\ & \swarrow \iota_i & & \searrow \iota'_i & & \nearrow \pi'_i \\ & & & x & \xrightarrow{\quad \pi_i \quad} & x_i \\ & & & \swarrow f & & \searrow \pi_i \\ x & \xrightarrow{\quad f \quad} & x' & & & \end{array}$$

i.e., $f \circ \iota_i = \iota'_i$ and $\pi'_i \circ f = \pi_i$. Thus we can talk about *the* direct sum of objects.

◇

Example 3.3.26. The direct sum of n objects is defined for every non-negative integer $n \in \mathbb{N}$. When $n = 0$, the direct sum of no object is an object x equipped with no morphism, but satisfies a nontrivial equation (note that the sum of no morphism is 0)

$$0 = \text{id}_x.$$

By Exercise 3.3.21, such an object x , i.e., the direct sum of no object, is the zero object. \heartsuit

Example 3.3.27. Let \mathcal{C} be a \mathbb{C} -linear category and $e: x \rightarrow x$ be an idempotent. Note that $e' := (\text{id}_x - e)$ is also an idempotent and $e' \circ e = 0 = e \circ e'$. Suppose (y, r, s) and (y', r', s') are the images of e and e' , respectively. Then we have

$$r' \circ s = r' \circ s' \circ r' \circ s \circ r \circ s = r' \circ e' \circ e \circ s = 0.$$

Similarly, $r \circ s' = 0$. Hence (x, s, s', r, r') is the direct sum of y and y' . \heartsuit

Definition 3.3.28. Let \mathcal{C}, \mathcal{D} be \mathbb{C} -linear categories. Their *direct sum*, denoted by $\mathcal{C} \oplus \mathcal{D}$, is the \mathbb{C} -linear category defined by the following data:

- The underlying category of $\mathcal{C} \oplus \mathcal{D}$ is the Cartesian product $\mathcal{C} \times \mathcal{D}$.
- The \mathbb{C} -linear structure on a hom space $\text{Hom}_{\mathcal{C} \oplus \mathcal{D}}((x, y), (x', y'))$ is given by the direct sum of vector spaces $\text{Hom}_{\mathcal{C}}(x, x') \oplus \text{Hom}_{\mathcal{D}}(y, y')$. \blacksquare

Exercise 3.3.29. Let \mathcal{C}, \mathcal{D} be \mathbb{C} -linear categories. Suppose $x \in \mathcal{C}$ is the direct sum of $x_1, \dots, x_n \in \mathcal{C}$ and $y \in \mathcal{D}$ is the direct sum of $y_1, \dots, y_n \in \mathcal{D}$. Prove that $(x, y) \in \mathcal{C} \oplus \mathcal{D}$ is the direct sum of $(x_1, y_1), \dots, (x_n, y_n)$.

3.3.5 Functors and natural transformations

A structure-preserving map between categories is called a functor.

Definition 3.3.30. Let \mathcal{C}, \mathcal{D} be categories. A *functor* F from \mathcal{C} to \mathcal{D} , denoted by $F: \mathcal{C} \rightarrow \mathcal{D}$, consists of the following data:

- a map $F: \text{ob}(\mathcal{C}) \rightarrow \text{ob}(\mathcal{D})$;
- a map $F_{x,y}: \text{Hom}_{\mathcal{C}}(x, y) \rightarrow \text{Hom}_{\mathcal{D}}(F(x), F(y))$ for each pair of objects $x, y \in \mathcal{C}$ (for simplicity, we also denote $F_{x,y}(f)$ by $F(f)$ for a morphism $f: x \rightarrow y$);

and these data satisfy the following conditions:

1. $F(g) \circ F(f) = F(g \circ f)$ for any $f \in \text{Hom}_{\mathcal{C}}(x, y), g \in \text{Hom}_{\mathcal{C}}(y, z)$;
2. $F(\text{id}_x) = \text{id}_{F(x)}$. \blacksquare

Example 3.3.31. Let \mathcal{C} be a category. The *identity functor* on \mathcal{C} is the functor $\text{id}_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{C}$ defined by $\text{id}_{\mathcal{C}}(f: x \rightarrow y) := (f: x \rightarrow y)$. \heartsuit

Example 3.3.32. Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories. The *composition* of two functors $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{E}$ is the functor $G \circ F: \mathcal{C} \rightarrow \mathcal{E}$ defined by

$$(G \circ F)(f: x \rightarrow y) := (G(F(f)): G(F(x)) \rightarrow G(F(y))).$$

\heartsuit

Exercise 3.3.33. Prove that all categories and functors also form a category, denoted by Cat .

Exercise 3.3.34. Prove that every functor preserves isomorphisms. More explicitly, let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor and $f: x \rightarrow y$ be an isomorphism in \mathcal{C} . Then $F(f): F(x) \rightarrow F(y)$ is an isomorphism in \mathcal{D} .

Example 3.3.35. Suppose \mathcal{C} is a category and \mathcal{D} is a subcategory of \mathcal{C} . Then the inclusion maps of object set and hom sets define a functor $\mathcal{D} \rightarrow \mathcal{C}$. \heartsuit

Example 3.3.36. Let G be a group. A functor $F: BG \rightarrow \text{Vec}$ from the delooping of G (see Example 3.3.7) to Vec is equivalent to a finite-dimensional G -representation. \heartsuit

Example 3.3.37. Let G, H be groups. Given a group homomorphism $f: G \rightarrow H$, there is a functor $Bf: BG \rightarrow BH$ defined by $Bf(*) := *$ and $Bf(g) := f(g)$ for $g \in G = \text{Hom}_{BG}(*, *)$. Conversely, every functor $BG \rightarrow BH$ is equal to Bf for some group homomorphism $f: G \rightarrow H$. It is not hard to see that the delooping construction defines a functor $B: \text{Grp} \rightarrow \text{Cat}$. \heartsuit

Definition 3.3.38. Let \mathcal{C}, \mathcal{D} be \mathbb{C} -linear categories. A \mathbb{C} -linear functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ such that $F_{x,y}: \text{Hom}_{\mathcal{C}}(x, y) \rightarrow \text{Hom}_{\mathcal{D}}(F(x), F(y))$ is a \mathbb{C} -linear map for every $x, y \in \mathcal{C}$. \blacksquare

Example 3.3.39. Let G be a finite group. There is a \mathbb{C} -linear functor $\text{Rep}(G) \rightarrow \text{Vec}$ defined by $(V, \rho) \mapsto V$. It is called the forgetful functor because it ‘forgets’ the G -action. Similarly, there is a \mathbb{C} -linear forgetful functor $\text{Vec}_G \rightarrow \text{Vec}$ defined by $\{V_g\}_{g \in G} \mapsto V = \bigoplus_{g \in G} V_g$. \heartsuit

A structure-preserving map between functors is called a natural transformation.

Definition 3.3.40. Let \mathcal{C}, \mathcal{D} be categories and $F, G: \mathcal{C} \rightarrow \mathcal{D}$ be functors from \mathcal{C} to \mathcal{D} . A *natural transformation* $\alpha: F \Rightarrow G$ from F to G is a family of morphisms $\{\alpha_x: F(x) \rightarrow G(x)\}_{x \in \mathcal{C}}$ in \mathcal{D} , such that the following diagram commutes for any morphism $f: x \rightarrow y$ in \mathcal{C} :

$$\begin{array}{ccc} F(x) & \xrightarrow{\alpha_x} & G(x) \\ \downarrow F(f) & & \downarrow G(f) \\ F(y) & \xrightarrow{\alpha_y} & G(y) \end{array}$$

A natural transformation α is called a *natural isomorphism* if every morphism α_x is an isomorphism. \blacksquare

Example 3.3.41. Let \mathcal{C}, \mathcal{D} be categories. The *identity natural transformation* on a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is the natural transformation $\text{id}_F: F \Rightarrow F$ defined by $(\text{id}_F)_x := \text{id}_{F(x)}$ for all $x \in \mathcal{C}$. \heartsuit

Example 3.3.42. Let \mathcal{C}, \mathcal{D} be categories and $F, G, H: \mathcal{C} \rightarrow \mathcal{D}$ be functors. The *composition*, or *vertical composition*, of two natural transformations $\alpha: F \Rightarrow G$ and $\beta: G \Rightarrow H$ is the natural transformation $\beta \cdot \alpha: F \Rightarrow H$ defined by $(\beta \cdot \alpha)_x := \beta_x \circ \alpha_x$ for all $x \in \mathcal{C}$. \heartsuit

Exercise 3.3.43. Let \mathcal{C}, \mathcal{D} be \mathbb{C} -linear categories. Prove that all \mathbb{C} -linear functors from \mathcal{C} to \mathcal{D} and natural transformations between them form a \mathbb{C} -linear category, denoted by $\text{Fun}(\mathcal{C}, \mathcal{D})$.

Example 3.3.44. Let G be a finite group and $F: \text{Rep}(G) \rightarrow \text{Vec}$ be the forgetful functor defined in Example 3.3.39. For any $g \in G$ there is a natural isomorphism $\alpha^g: F \Rightarrow F$ defined by

$$\alpha_{(V, \rho)}^g := \rho(g): V \rightarrow V, \quad (V, \rho) \in \text{Rep}(G).$$

Moreover, we have $\alpha^g \cdot \alpha^h = \alpha^{gh}$ for $g, h \in G$. \heartsuit

Exercise 3.3.45. Let \mathcal{C}, \mathcal{D} be categories and $F, G: \mathcal{C} \rightarrow \mathcal{D}$ be functors. Prove that a natural transformation $\alpha: F \Rightarrow G$ is a natural isomorphism if and only if there exists a natural transformation $\beta: G \Rightarrow F$ such that $\beta \cdot \alpha = \text{id}_F$ and $\alpha \cdot \beta = \text{id}_G$.

Exercise 3.3.46. Let \mathcal{C}, \mathcal{D} be \mathbb{C} -linear categories. Suppose $x_1, \dots, x_n \in \mathcal{C}$ and $(x, \{\iota_i\}, \{\pi_i\})$ is a direct sum of x_1, \dots, x_n .

- (1) Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a \mathbb{C} -linear functor. Prove that $(F(x), \{F(\iota_i)\}, \{F(\pi_i)\})$ is a direct sum of $F(x_1), \dots, F(x_n)$. In other words, F preserves direct sums.
- (2) Let $F, G: \mathcal{C} \rightarrow \mathcal{D}$ be \mathbb{C} -linear functors and $\alpha: F \Rightarrow G$ be a natural transformations. Prove that α_x and $\{\alpha_{x_i}\}_{i=1}^n$ can determine each other. Hint: use Exercise 3.3.24 (3).

3.3.6 Equivalence of categories

Definition 3.3.47. Let \mathcal{C}, \mathcal{D} be categories. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is called an *equivalence* if there exists a functor $G: \mathcal{D} \rightarrow \mathcal{C}$ such that $G \circ F$ is naturally isomorphic to $\text{id}_{\mathcal{C}}$ and $F \circ G$ is naturally isomorphic to $\text{id}_{\mathcal{D}}$. In this case G is called the *quasi-inverse* of F . We say \mathcal{C} and \mathcal{D} are *equivalent* if there exists an equivalence between them. ■

Equivalent categories can be viewed as having the same structure of a category. There is a useful theorem to prove whether a functor is an equivalence.

Definition 3.3.48. Let \mathcal{C}, \mathcal{D} be categories and $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor. We say F is *faithful* if every map $F_{x,y}: \text{Hom}_{\mathcal{C}}(x, y) \rightarrow \text{Hom}_{\mathcal{D}}(F(x), F(y))$ is injective, and *full* if every map $F_{x,y}$ is surjective. A *fully faithful* functor is a functor that is both full and faithful. ■

Example 3.3.49. Let \mathcal{C} be a category and \mathcal{D} be a subcategory of \mathcal{C} . Then the inclusion functor $\mathcal{D} \rightarrow \mathcal{C}$ is always faithful. It is full if and only if \mathcal{D} is a full subcategory. ♥

Example 3.3.50. Let G be a finite group. The forgetful functors $\text{Rep}(G) \rightarrow \text{Vec}$ and $\text{Vec}_G \rightarrow \text{Vec}$ (see Example 3.3.39) are faithful. They are full if and only if G is the trivial group. ♥

Example 3.3.51. The delooping functor $B: \text{Grp} \rightarrow \text{Cat}$ defined in Example 3.3.37 is fully faithful. ♥

Theorem 3.3.52. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between two categories \mathcal{C}, \mathcal{D} is an equivalence if and only if it satisfies the following conditions:

1. **(fully faithful)** For any $x, y \in \mathcal{C}$, the map $F_{x,y}: \text{Hom}_{\mathcal{C}}(x, y) \rightarrow \text{Hom}_{\mathcal{D}}(F(x), F(y))$ is a bijection (i.e., F is fully faithful).
2. **(essentially surjective)** For any $z \in \mathcal{D}$, there exists an object $x \in \mathcal{C}$ such that $F(x) \simeq z$.

We do not give a proof here, but briefly explain why a functor F satisfying these two conditions is an equivalence. Intuitively, an equivalence should be an identification of two categories, i.e., an identification on both morphisms and (isomorphism classes of) objects. The first condition that F is fully faithful means that F is an identification on morphisms. The second condition that F is essentially surjective means that F induces a surjective map from the set of isomorphism classes of objects of \mathcal{C} to that of \mathcal{D} . This map is also injective by the first condition. Indeed, suppose $x, y \in \mathcal{C}$ and $g: F(x) \rightarrow F(y)$ is an isomorphism. Since F is fully faithful, there exists a unique morphism $f: x \rightarrow y$ such that $F(f) = g$. Consider the inverse of g , it is not hard to see that f is also an isomorphism. Thus x is isomorphic to y .

3.3.7 Appendix: local operator algebras

Definition 3.3.53. An *algebra* (over \mathbb{C}) is a vector space A equipped with a \mathbb{C} -bilinear map (called the *multiplication*)

$$\begin{aligned} A \times A &\rightarrow A \\ (a, b) &\mapsto a \cdot b \end{aligned}$$

and a distinguished vector $1 \in A$ (called the *identity*) such that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ and $1 \cdot a = a = a \cdot 1$ for every $a, b, c \in A$. ■

Example 3.3.54. The field \mathbb{C} is an algebra with the usual multiplication of complex numbers. ♥

Example 3.3.55. Let G be a group. The *group algebra* $\mathbb{C}[G]$ of G is defined as follows. Its underlying vector space is freely generated by G , or equivalently, spanned by a family of symbols $\{v_g\}_{g \in G}$. The multiplication is defined by $v_g \cdot v_h := v_{gh}$ for $g, h \in G$. The identity is v_e where $e \in G$ is the unit. ♥

Example 3.3.56. Let V be a finite-dimensional vector space. Then $\text{End}(V) := \text{Hom}_{\text{Vec}}(V, V)$ is an algebra with the multiplication given by the composition of linear maps and the identity given by the identity map. More generally, for any \mathbb{C} -linear category \mathcal{C} and an object $x \in \mathcal{C}$, the space $\text{Hom}_{\mathcal{C}}(x, x)$ is an algebra. ♥

Exercise 3.3.57. Prove that the notion of a finite-dimensional algebra is equivalent to a \mathbb{C} -linear category with only one object. What is the precise meaning of ‘equivalent’ here?

Definition 3.3.58. Let A be an algebra. A *left module* over A or a *left A -module* is a vector space M equipped with a bilinear map (called the left A -action)

$$\begin{aligned} A \times M &\rightarrow M \\ (a, m) &\mapsto a \cdot m \end{aligned}$$

such that $(a \cdot b) \cdot m = a \cdot (b \cdot m)$ and $1 \cdot m = m$ for every $a, b \in A$ and $m \in M$. ■

Example 3.3.59. An algebra A is a left module over itself. ♡

Example 3.3.60. Let G be a group. Suppose M is a left module over the group algebra $\mathbb{C}[G]$. The left $\mathbb{C}[G]$ action is determined by $v_g \cdot m =: \rho(g)(m)$ for all $g \in G$ and $m \in M$. Then we have

$$\rho(gh)(m) = \rho(g)(\rho(h)(m)) = (\rho(g) \circ \rho(h))(m), \quad \rho(e)(m) = m$$

for all $g, h \in G$ and $m \in M$. In other words, $\rho : G \rightarrow \text{GL}(M)$ is a group homomorphism. Thus the notion of a left $\mathbb{C}[G]$ -module is equivalent to a G -representation. ♡

Example 3.3.61. Let V be a finite-dimensional vector space. Then V is a left module over $\text{End}(V)$. If W is also a finite-dimensional vector space, then $\text{Hom}_{\text{Vec}}(W, V)$ is a left module over $\text{End}(V)$ with the left $\text{End}(V)$ -action given by the composition of linear maps. More generally, for any \mathbb{C} -linear category \mathcal{C} and objects $x, y \in \mathcal{C}$, the space $\text{Hom}_{\mathcal{C}}(y, x)$ is a left $\text{Hom}_{\mathcal{C}}(x, x)$ -module. ♡

Remark 3.3.62. Let A be a finite-dimensional algebra. A *homomorphism* or an *A -module map* between two left A -modules M, N is a linear map $f : M \rightarrow N$ satisfying $f(a \cdot m) = a \cdot f(m)$ for all $a \in A$ and $m \in M$. All finite-dimensional left A -modules and homomorphisms between them form a \mathbb{C} -linear category, denoted by $\text{LMod}_A(\text{Vec})$. ◇

Now we consider a lattice model that realizes a 2d topological order \mathcal{C} . The local operators generates an algebra denoted by \mathcal{A}_{loc} . By Definition 3.2.1, the local operator algebra acts on a topological excitation invariantly. In other words, a topological excitation is a left module over the local operator algebra \mathcal{A}_{loc} . Thus Definition 3.2.1 can be equivalently reformulated as follows.

Definition 3.3.63. A topological excitation is a module over the local operator algebra. ■

Remark 3.3.64. An operator between topological excitations commuting with all local operators (i.e., an instanton) is a homomorphism of modules. Thus the category \mathcal{C} of particle-like topological excitations of \mathcal{C} is equivalent to the category $\text{LMod}_{\mathcal{A}_{\text{loc}}}(\text{Vec})$. ◇

In the following we directly give the local operator algebra [Kit03] of the toric code model and show its modules coincide with topological excitations $\mathbb{1}, e, m, f$. However, we do not know a general principle to derive the local operator algebra from the Hamiltonian.

By a *site* in the toric code model we mean a pair (v, p) of adjacent vertex and plaquette (see Figure 14). This is a minimal choice of a local region in the toric code model (see Remark 3.2.9). For a given site $\xi = (v, p)$, the local operator algebra \mathcal{A}_{ξ} is generated by two operators A_v and B_p , subject to the following relations:

$$A_v^2 = B_p^2 = 1, \quad A_v B_p = B_p A_v.$$

Clearly A_v generates a subalgebra which is isomorphic to $\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$ and so does B_p . Since A_v commutes with B_p , the algebra \mathcal{A}_{ξ} is isomorphic to tensor product of these two subalgebras, that is $\mathbb{C}^2 \otimes \mathbb{C}^2 \simeq \mathbb{C}^4$. It is a semisimple algebra with 4 different simple modules (irreducible representations) up to isomorphism, which correspond to its 4 primitive central idempotents:

$$P_{\pm\pm} := \frac{1 \pm A_v}{2} \frac{1 \pm B_p}{2}.$$

These idempotents are projectors to the common eigenspaces of A_v and B_p and two \pm signs are eigenvalues of A_v and B_p respectively. Then the module corresponding to P_{-+} is an e particle at v , the one corresponding to P_{+-} is an m particle at p , the one corresponding to P_{--} is an f particle at ξ , and the remaining one corresponding to P_{++} is the trivial topological excitation $\mathbb{1}$.

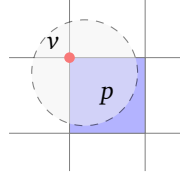


Figure 14: a site $\xi = (v, p)$

Remark 3.3.65. A general module over A_ξ may not be simple and isomorphic to a direct sum of simple modules. These non-simple modules are also topological excitations. So $\mathbb{1}, e, m, f$ are only simple topological excitations. We discuss the direct sum of topological excitations in Section 3.4.1. \diamond

In Figure 9, we say that the operator σ_z^1 is local, but why does it not appear in the local operator algebra \mathcal{A}_ξ ? This is because a minimal local region (i.e., a site ξ) does not contain the edge 1. If we choose a larger local region R , for example that in Figure 10, the local operator algebra \mathcal{A}_R should be generated by all $A_v, B_p, \sigma_z, \sigma_x$ operators contained in R . For different regions, the local operator algebras may not be isomorphic, but one can prove that their modules are the same: there are always 4 different simple modules denoted by $\mathbb{1}, e, m, f$ labeled by the eigenvalues of the product of all A_v operators and the product of all B_p operators. We do not show details here.

Remark 3.3.66. A \mathbb{C} -linear category is called a *finite category* if it is equivalent to $\text{LMod}_A(\text{Vec})$ for some finite-dimensional \mathbb{C} -algebra A . If $\text{LMod}_A(\text{Vec})$ is equivalent to $\text{LMod}_B(\text{Vec})$ for two finite-dimensional \mathbb{C} -algebras A and B , we say that A and B are *Morita equivalent*.

So the above discussion can be reformulated as follows. The local operator algebras of the toric code model depend on the length scale, but the local operator algebras in different length scales are Morita equivalent (i.e., the categories of modules over different local operator algebras are equivalent). Hence the category \mathcal{C} of particle-like topological defects is a finite category and is an observable in the long wave length limit. \diamond

Remark 3.3.67. In a general lattice model, the local operator algebras in different length scales may not be Morita equivalent. The toric code model is at a fixed point of the renormalization flow, and this is why we see the same topological excitations in all length scales. For an arbitrary lattice model, its local operator algebra and the modules should also flow under the renormalization flow. \diamond

3.4 More structures and properties

We have known that particle-like topological defects in a topological order \mathcal{C} form a category \mathcal{C} . When \mathcal{C} is a 2d topological order, \mathcal{C} is not only a category, but also equipped with more structures and has some nice properties.

In this subsection, we use \mathcal{C} to denote a 2d topological order. The category of particle-like topological defects of \mathcal{C} is denoted by \mathcal{C} .

3.4.1 Semisimplicity

As we have mentioned in Remark 3.2.7, it is reasonable to talk about the direct sum of topological excitations. First we consider an example in the toric code model. Given a plaquette p_0 , we add a local trap B_{p_0} to the Hamiltonian of the toric code model. So the new Hamiltonian is

$$H + B_{p_0} = \sum_v (1 - A_v) + \sum_{p \neq p_0} (1 - B_p) + 1.$$

Intuitively, this defect can be viewed as removing the plaquette p_0 in the lattice model. A simple calculation shows that the new ground state subspace is two-fold degenerate, and two ground states can be labeled by the eigenvalues $B_{p_0} = \pm 1$. The state with $B_{p_0} = +1$ is just the original ground state which generates the topological excitation $\mathbb{1}$, and the state with $B_{p_0} = -1$ generates an m particle. In other words, the topological excitation given by the local trap B_{p_0} , as a subspace of the total Hilbert space, is the direct sum $\mathbb{1} \oplus m$ of $\mathbb{1}$ and m .

Exercise 3.4.1. Consider the nd lattice model ($n \geq 1$) with the local Hilbert space \mathbb{C}^2 and the Hamiltonian

$$H = \sum_i (1 - \sigma_z^i).$$

The space dimension n is not important in this exercise.

- (1) Prove that the ground state of H is a product state. Thus H realizes an nd trivial topological order (see Example 2.1.2). The trivial topological excitation (i.e., the topological excitation generated by the ground state), is denoted by $\mathbb{1}$.
- (2) Fix a site j and add a local trap $2\sigma_z^j$ to the Hamiltonian H . Find the ground state of $H' = H + 2\sigma_z^j$. Show that the (particle-like) topological excitation realized by the local trap $2\sigma_z^j$ is $\mathbb{1}$. Hint: σ_z^j is a local operator.
- (3) Fix a site j and add a local trap σ_z^j to the Hamiltonian H . Find the ground state subspace of $H' = H + \sigma_z^j$. Show that the (particle-like) topological excitation realized by the local trap σ_z^j is $\mathbb{1} \oplus \mathbb{1}$.
- (4) Fix two sites j, j' . Show that the local trap $(\sigma_z^j + \sigma_z^{j'} \pm \sigma_z^j \sigma_z^{j'})$ realizes the topological excitation $\mathbb{1} \oplus \mathbb{1}$.
- (5) For each positive integer k , find a local trap that realizes the topological excitation $\mathbb{1}^{\oplus k}$ (this notation means the direct sum of k copies of $\mathbb{1}$). Hint: consider the term $-\prod_{a=1}^k (1 \pm \sigma_z^{j_a})$.

Exercise 3.4.2. Consider local traps in the toric code model.

- (1) Fix a vertex v_0 and a plaquette p_0 . Show that the local trap A_{v_0} realizes the topological excitation $\mathbb{1} \oplus e$, and the local trap $(A_{v_0} + B_{p_0})$ realizes the topological excitation $\mathbb{1} \oplus e \oplus m \oplus f$, and the local trap $(A_{v_0} + B_{p_0} + A_{v_0} B_{p_0})$ realizes the topological excitation $e \oplus m$. What is the topological excitation realized by $(A_{v_0} + B_{p_0} - A_{v_0} B_{p_0})$?
- (2) For any non-negative integers k_0, k_1, k_2, k_3 , find a local trap that realizes $\mathbb{1}^{\oplus k_0} \oplus e^{\oplus k_1} \oplus m^{\oplus k_2} \oplus f^{\oplus k_3}$. Hint: It can be achieved in two steps. (1) find a local trap for $A := (\mathbb{1} \oplus e)^{\otimes m} \otimes (\mathbb{1} \oplus m)^{\otimes n}$; (2) Add projectors $P := -\prod_{i=1}^m (1 \pm A_{v_i}) \prod_{j=1}^n (1 \pm B_{p_j})$ to the trap to select the direct summands of A .

Remark 3.4.3. The first purpose of above exercises is to justify that generic objects in a modular tensor category, i.e. the direct sums of simple ones (sometimes called composite anyons), are not just mathematical convenience but physical realities. Composite anyons play crucial roles in the theory of anyon condensations [Kon14a]. Our second purpose is to draw attention from those quantum computing oriented readers to the possibility that these composite anyons might provide more computational power than using only the simple anyons. We also want to point out that the local traps that realize the same topological excitation are never unique (see for example [KK12] for different realizations of $e \oplus m$). It is an important question to find experimentally convenient local traps. \diamond

Remark 3.4.4. One can see that a local trap at a given site i can also be viewed as defining a ‘0d boundary condition’ of the 2d model at the site i . Therefore, the category \mathcal{C} of particle-like topological defects can also be viewed as that of ‘0d boundary conditions’ at a given site i , which is denoted by \mathcal{C}_i . This point of view is sometimes very useful. We can specify a given ‘0d boundary condition’ x at the site i by a pair (\mathcal{C}_i, x) . \diamond

The direct sum usually implies instability. In the above example, we can perturb the local trap B_{p_0} by replacing it with $(1 + \varepsilon)B_{p_0}$. When $\varepsilon = 0$, the trapped topological excitation is $\mathbb{1} \oplus m$ as discussed above. When $\varepsilon < 0$, the new ground state is the same as the original ground state. When $\varepsilon > 0$, the new ground state is given by $B_{p_0} = -1$ and thus gives the topological excitation m . Therefore, the topological defect $\mathbb{1} \oplus m$ is unstable, and by applying a perturbation it collapses to $\mathbb{1}$ or m . Such an unstable topological defect is also called a *composite topological defect* (or a *composite anyon*), while a stable topological defect is called a *simple topological defect* (or a *simple anyon*). Each composite topological defect is a direct sum of simple topological defects. We say an nd topological order (where $n \geq 1$) is *stable* if its ground state degeneracy on an open disk is 1, or equivalently, the trivial topological defect $\mathbb{1}$ is stable.

We summarize the above discussions to the following definition.

Definition 3.4.5. Let \mathcal{C} be a \mathbb{C} -linear category. An object $x \in \mathcal{C}$ is called *simple* if $\text{Hom}_{\mathcal{C}}(x, x) \simeq \mathbb{C}$. Two objects $x, y \in \mathcal{C}$ are called *disjoint* if $\text{Hom}_{\mathcal{C}}(x, y) = 0 = \text{Hom}_{\mathcal{C}}(y, x)$. We say \mathcal{C} is *semisimple* if it satisfies the following conditions:

1. The direct sum of finitely many objects in \mathcal{C} exists.
2. There exists a collection of mutually disjoint simple objects $\{x_i\}_{i \in I}$ such that every object in \mathcal{C} is a finite direct sum of objects in $\{x_i\}_{i \in I}$.

If the index set I is a finite set, then we say \mathcal{C} is a *finite semisimple category*. ■

Remark 3.4.6. There are many different definitions of simple objects and semisimple categories (for example, see [Bai17, Pen21] and references therein). ◇

The category \mathcal{C} of particle-like topological defects of a 2d topological order \mathcal{C} (where $n \geq 1$) is a finite semisimple category. The simple objects in \mathcal{C} are simple topological defects in \mathcal{C} . The topological order \mathcal{C} is stable if and only if the trivial topological defect $\mathbb{1} \in \mathcal{C}$ is simple.

Remark 3.4.7. Our analysis of particle-like topological defects works for any nd topological orders. In other words, the category of particle-like topological defects of an nd topological order is also a finite semisimple category. ◇

Remark 3.4.8. Let \mathcal{C} be a \mathbb{C} -linear category. A nonzero object $x \in \mathcal{C}$ is called *indecomposable* if any decomposition $x = x_1 \oplus x_2$ is trivial, i.e., either $x_1 = 0$ or $x_2 = 0$. A simple object is indecomposable, but the converse is not true in general. When \mathcal{C} is a semisimple category, an object $x \in \mathcal{C}$ is indecomposable if and only if it is simple. ◇

Remark 3.4.9. Let \mathcal{C} be a semisimple category. Suppose $\{x_i\}_{i \in I}$ is a collection of mutually disjoint simple objects such that every object in \mathcal{C} is a direct sum of objects in $\{x_i\}_{i \in I}$. By Exercise 3.3.24 (3), every morphism in \mathcal{C} can be written as a block-diagonal matrix with coefficients in \mathbb{C} . More precisely, we have

$$\text{Hom}_{\mathcal{C}}\left(\bigoplus_{i \in I} x_i^{\oplus n_i}, \bigoplus_{j \in I} x_j^{\oplus m_j}\right) \simeq \bigoplus_{i \in I} M_{m_i \times n_i}(\mathbb{C}),$$

where $M_{m \times n}(\mathbb{C})$ denotes the space of m -by- n matrices with coefficients in \mathbb{C} . Then we immediately see that every simple object in \mathcal{C} is isomorphic to exactly one object in $\{x_i\}_{i \in I}$. In other words, each isomorphism classes of simple objects in \mathcal{C} contains exactly one object in $\{x_i\}_{i \in I}$. Thus \mathcal{C} is finite semisimple if and only if there are finitely many simple objects in \mathcal{C} (up to isomorphism). We denote the set of isomorphism classes of simple objects in \mathcal{C} by $\text{Irr}(\mathcal{C})$. ◇

Example 3.4.10. Let \mathcal{C}, \mathcal{D} be semisimple categories. Then their direct sum $\mathcal{C} \oplus \mathcal{D}$ is also semisimple. The simple objects in $\mathcal{C} \oplus \mathcal{D}$ are of the form $(x, 0)$ for $x \in \mathcal{C}$ simple or $(0, y)$ for $y \in \mathcal{D}$ simple. Moreover, $\mathcal{C} \oplus \mathcal{D}$ is finite semisimple if and only if both \mathcal{C} and \mathcal{D} are finite semisimple. ◇

Remark 3.4.11. By Theorem 3.3.52 and Remark 3.4.9, one can prove that a finite semisimple category \mathcal{C} is equivalent to the direct sum $\text{Vec}^{\oplus n}$ of n copies of Vec , where n is the number of isomorphism classes of simple objects of \mathcal{C} . It follows that every finite semisimple category is idempotent complete. In practice, we can also define a finite semisimple category to be a category that is equivalent to the direct sum of several copies of Vec . ◇

Example 3.4.12. The category \mathcal{TC} of particle-like topological defects of the toric code model is finite semisimple and has 4 simple objects: $\mathbb{1}, e, m, f$. The other objects are direct sums of these simple objects. In other words, \mathcal{TC} is equivalent to $\text{Vec}^{\oplus 4}$. ◇

Example 3.4.13. Let G be a finite group. By Maschke's theorem [Mas98, Mas99], $\text{Rep}(G)$ is a finite semisimple category. A simple object in $\text{Rep}(G)$ is also called an *irreducible G -representation*. The number of elements in $\text{Irr}(\text{Rep}(G))$ is equal to the number of conjugacy classes of G . However, there is no canonical bijection between isomorphism classes of irreducible G -representations and conjugacy classes of G . ◇

Exercise 3.4.14. Let G be a finite group. Prove that Vec_G is a finite semisimple category. Hint: use Exercise 3.3.6.

3.4.2 Monoidal structure

Roughly speaking, if two particle-like topological defects x and y are close to each other, one can view them as a single topological defect denoted by $x \otimes y$ (see Figure 15).¹⁵

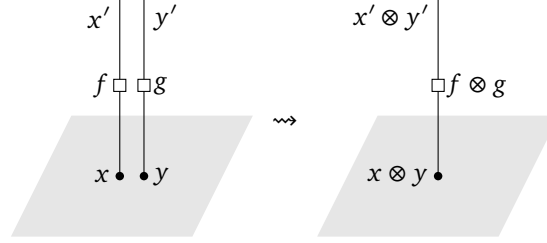


Figure 15: the fusion of particle-like topological defects

Example 3.4.15. In the toric code model, two e particles can be annihilated to the ground state by a string of σ_z operators which connects them (see Figure 9). Thus we have $e \otimes e = \mathbb{1}$. Similarly $m \otimes m = f \otimes f = \mathbb{1}$. By the definition of f we have $e \otimes m = f = m \otimes e$. Thus the fusion rule of simple topological excitations in the toric code model is the same as the multiplication of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$.¹⁶

Moreover, the instantons attached on the world lines of topological defects are also fused together. Suppose $f: x \rightarrow x'$ and $g: y \rightarrow y'$ are two instantons. Then they fuse into an instanton from $x \otimes y$ to $x' \otimes y'$, denoted by $f \otimes g$. This defines a map

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(x, x') \times \text{Hom}_{\mathcal{C}}(y, y') &\rightarrow \text{Hom}_{\mathcal{C}}(x \otimes y, x' \otimes y') \\ (f, g) &\mapsto f \otimes g. \end{aligned}$$

It follows that the fusion of particle-like topological defects defines a functor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$.

Exercise 3.4.16. Use physical intuitions to check the functoriality of \otimes .

If there are three topological defects $x, y, z \in \mathcal{C}$, there are different ways to fuse them together. We can first fuse x and y to get $x \otimes y$, then fuse it with z to get $(x \otimes y) \otimes z$; we can also fuse y and z first then fuse the result with x to get $x \otimes (y \otimes z)$. It is physically obvious that these two results should be the same. However, we do not require that $(x \otimes y) \otimes z$ and $x \otimes (y \otimes z)$ are equal on the nose, but require a distinguished isomorphism between them. We denote this isomorphism by $\alpha_{x,y,z}: (x \otimes y) \otimes z \rightarrow x \otimes (y \otimes z)$, called the associator. As an instanton, the associator $\alpha_{x,y,z}$ is depicted in Figure 16.

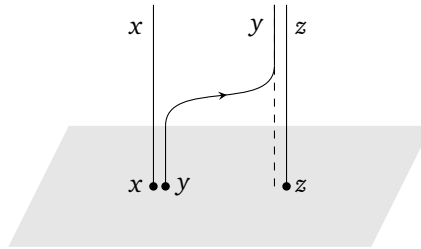


Figure 16: the associator $\alpha_{x,y,z}: (x \otimes y) \otimes z \rightarrow x \otimes (y \otimes z)$

The family $\alpha = \{\alpha_{x,y,z}\}_{x,y,z \in \mathcal{C}}$ is natural in all three variables because instantons can be freely moved along the world lines (see Figure 17). More precisely, for any morphisms $f: x \rightarrow x'$, $g: y \rightarrow y'$ and $h: z \rightarrow z'$ in \mathcal{C} , we have

$$\alpha_{x',y',z'} \circ ((f \otimes g) \otimes h) = (f \otimes (g \otimes h)) \circ \alpha_{x,y,z}.$$

¹⁵This statement is a radical simplification of the real story. Since it has already been accepted by most working physicists in the field, we also adopt it and move on from here. However, the real story is much richer and physically more natural. We choose to discuss this technical but important issue in Remark 3.4.19.

Hence, the associator is a natural isomorphism $\alpha: \otimes \circ (\otimes \times \text{id}_{\mathcal{C}}) \Rightarrow \otimes \circ (\text{id}_{\mathcal{C}} \times \otimes)$, where these two functors are both from $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$ to \mathcal{C} .

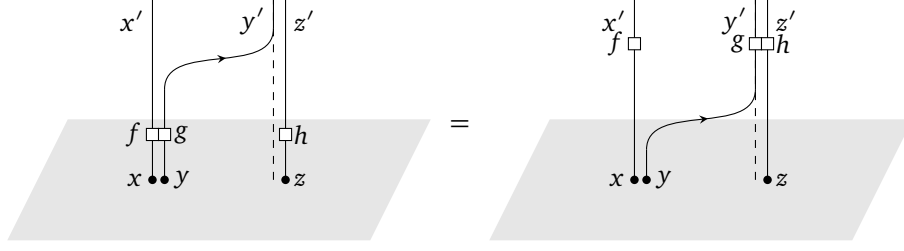


Figure 17: the naturality of the associator α

If there are four topological defects x, y, z, w , there are five different ways to fuse them two by two. The associators provide many isomorphisms between them. For example, Figure 18 depicts two different isomorphisms (instantons) from $((x \otimes y) \otimes z) \otimes w$ to $x \otimes (y \otimes (z \otimes w))$. These two instantons should be equal because their world lines are homotopy:

$$(\text{id}_x \otimes \alpha_{y,z,w}) \circ \alpha_{x,y \otimes z,w} \circ (\alpha_{x,y,z} \otimes \text{id}_w) = \alpha_{x,y,z \otimes w} \circ \alpha_{x \otimes y,z,w}.$$

This is also called the pentagon equation because it can be interpreted as a pentagon-like commutative diagram (3.4.1).

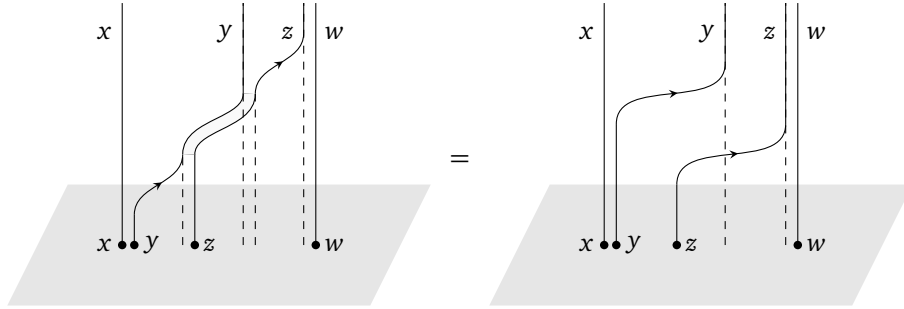


Figure 18: the pentagon equation

Remark 3.4.17. The associators and pentagon equation implies that all different ways to fuse $n \geq 4$ topological defects are isomorphic, and there is a unique isomorphism between two different fusion functors. This is called the coherence theorem [Mac63]. \diamond

Moreover, fusing a topological defect x with the trivial topological defect $\mathbb{1}$ should not change it. So there are two isomorphisms $\lambda_x: \mathbb{1} \otimes x \rightarrow x$ and $\rho_x: x \otimes \mathbb{1} \rightarrow x$, called the left and right unitor. Also they are natural in x and satisfy some coherence conditions.

We summarize the above data and properties to the following formal structure [Bén63, Mac63].

Definition 3.4.18. A *monoidal category* consists of the following data:

- a category \mathcal{C} ;
- a functor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, where $\otimes(x, y)$ is also denoted by $x \otimes y$;
- a distinguished object $\mathbb{1} \in \mathcal{C}$, called the *tensor unit*;
- a natural isomorphism $\alpha_{x,y,z}: (x \otimes y) \otimes z \rightarrow x \otimes (y \otimes z)$, called the *associator*;

- a natural isomorphism $\lambda_x: \mathbb{1} \otimes x \rightarrow x$, called the *left unitor*;
- a natural isomorphism $\rho_x: x \otimes \mathbb{1} \rightarrow x$, called the *right unitor*;

and these data satisfy the following conditions:

1. **(pentagon equation)** For any $x, y, z, w \in \mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc}
 & ((x \otimes y) \otimes z) \otimes w & \\
 \alpha_{x,y,z} \otimes \text{id}_w \swarrow & & \searrow \alpha_{x \otimes y, z, w} \\
 (x \otimes (y \otimes z)) \otimes w & & (x \otimes y) \otimes (z \otimes w) \\
 \alpha_{x, y \otimes z, w} \downarrow & & \downarrow \alpha_{x, y, z \otimes w} \\
 x \otimes ((y \otimes z) \otimes w) & \xrightarrow{\text{id}_x \otimes \alpha_{y, z, w}} & x \otimes (y \otimes (z \otimes w))
 \end{array} \tag{3.4.1}$$

2. **(triangle equation)** For any $x, y \in \mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc}
 (x \otimes \mathbb{1}) \otimes y & \xrightarrow{\alpha_{x, \mathbb{1}, y}} & x \otimes (\mathbb{1} \otimes y) \\
 \rho_x \otimes \text{id}_y \searrow & & \swarrow \text{id}_x \otimes \lambda_y \\
 & x \otimes y &
 \end{array} \tag{3.4.2}$$

A \mathbb{C} -linear monoidal category \mathcal{C} is both a \mathbb{C} -linear category and a monoidal category such that the tensor product functor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is \mathbb{C} -bilinear, i.e., both $x \otimes -: \mathcal{C} \rightarrow \mathcal{C}$ and $- \otimes x: \mathcal{C} \rightarrow \mathcal{C}$ are \mathbb{C} -linear functors for each $x \in \mathcal{C}$. \blacksquare

As a conclusion, the category \mathcal{C} of particle-like topological defects of a 2d topological order \mathcal{C} is a \mathbb{C} -linear monoidal category.

Remark 3.4.19. We have derived the data and the axioms of a monoidal category from the physical intuitions behind the fusion product \otimes , which is a radical simplification of the real story. We briefly outline the real and much richer story in this remark. In a lattice model realization of a 2d topological order, suppose that two particle-like topological defects x and y are realized in the model at two different sites ξ, η , respectively. This physical configuration already defines a fusion product $x \otimes_{(\xi, \eta)} y$, which can be viewed a subspace of the total Hilbert space. Then the lattice model realization of x and y at different sites ξ', η' is a subspace of a potentially different total Hilbert space (depending on how one realize the defects, see Remark ??). ?? In other words, there are infinitely many fusion products parameterized by the elements in the configuration space $\{(\xi, \eta) \in \mathbb{R}^2 \mid \xi \neq \eta\}$. An adiabatic move of these two defects from (ξ, η) to (ξ', η') along a path γ in the configuration space defines an isomorphism between two associated subspaces of the potentially different total Hilbert spaces, ?? i.e., a linear and invertible map $T_{x,y}^\gamma: x \otimes_{(\xi, \eta)} y \Rightarrow x \otimes_{(\xi', \eta')} y$. Using Remark 3.4.4, $\otimes_{(\xi, \eta)}$ defines a tensor product functor $\otimes_{(\xi, \eta)}: \mathcal{C}_\xi \times \mathcal{C}_\eta \rightarrow \mathcal{C}$, then $T^\gamma := \{T_{x,y}^\gamma\}_{x,y \in \mathcal{C}}: \otimes_{(\xi, \eta)} \Rightarrow \otimes_{(\xi', \eta')}$ defines a natural isomorphism. The most important physical properties of a topological order (almost a defining property) is stated as follows:

$$\text{if two paths } \gamma_1 \text{ and } \gamma_2 \text{ are homotopy equivalent in the configuration space, then } T^{\gamma_1} = T^{\gamma_2}. \quad (*)$$

This property holds for adiabatic moves or systems with an infinitely large gap¹⁶, and allows us to radically reduce the number of the tensor products $\otimes_{(\xi, \eta)}$ that need to be considered. In 1d space, all fusion products can be reduced to a single one $\otimes := \otimes_{(0,1)}$. A complete and rigorous proof of the monoidal structure on \mathcal{C} can be derived from here. It is equivalent to the proof of the well-known

¹⁶In reality, the gap of a physical system is finite and sometimes small. Then an adiabatic move is practically impossible. In this case, there might be physically detectable difference between T^{γ_1} and T^{γ_2} due to the finiteness of the gap. In this case, it can be practically useful to replace the monoidal category by a monoidal A_∞ -category (or $(\infty, 1)$ -category [Lur09]) in order to encode the information of higher homotopies. For gapless phases, it is not clear how to define an adiabatic move. However, in 2D rational CFT's, the natural isomorphisms T^{γ_1} can be naturally defined and the condition $(*)$ still make sense [HL94, Hua95, Hua08a].

mathematical theorem that an E_1 -algebra in the $(2,1)$ -category of categories is a monoidal category [SW03, Lur17, Fre17]. In 2d space, these fusion functors can be reduced to two compatible fusion functors, or equivalently, a single fusion functor \otimes equipped with a braiding structure (see Remark 3.4.66). A physical work on adiabatic moves and braided monoidal structures can be found in [KL20]. \diamond

Remark 3.4.20. In other words, the monoidal structure on the category \mathcal{C} of particle-like topological defects of a 2d topological order \mathcal{C} depends on many artificial choices. For example, the monoidal structure can be obtained by microscopic realizations of adiabatic moves [KL20], and different realizations give rise to different (but equivalent) monoidal structures. Hence a topological order does not determine a single monoidal category \mathcal{C} , but an equivalence class of monoidal categories. Physicists usually say that the monoidal structure (or simply the associator) is not ‘gauge invariant’. \diamond

Remark 3.4.21. For a monoidal category \mathcal{C} , the triangle equation (3.4.2) implies that another two triangle equations hold. More explicitly, the following two diagrams commute for any $x, y \in \mathcal{C}$:

$$\begin{array}{ccc} (1 \otimes x) \otimes y & \xrightarrow{\alpha_{1,x,y}} & 1 \otimes (x \otimes y) \\ \lambda_x \otimes \text{id}_y \searrow & & \swarrow \lambda_{x \otimes y} \\ & x \otimes y & \end{array} \quad \begin{array}{ccc} (x \otimes y) \otimes 1 & \xrightarrow{\alpha_{x,y,1}} & x \otimes (y \otimes 1) \\ \rho_{x \otimes y} \searrow & & \swarrow \text{id}_x \otimes \rho_y \\ & x \otimes y & \end{array}$$

These equations further implies that λ_1 and ρ_1 are the same morphism $1 \otimes 1 \rightarrow 1$. \diamond

Example 3.4.22. The tensor product $V \otimes_{\mathbb{C}} W$ of two (not necessarily finite-dimensional) vector spaces V, W is defined by the following universal property: $V \otimes_{\mathbb{C}} W$ is equipped with a \mathbb{C} -bilinear map $\otimes_{\mathbb{C}} : V \times W \rightarrow V \otimes_{\mathbb{C}} W$, and for any vector space X and \mathbb{C} -bilinear map $f : V \times W \rightarrow X$, there exists a unique \mathbb{C} -linear map $\underline{f} : V \otimes_{\mathbb{C}} W \rightarrow X$ such that $\underline{f} \circ \otimes_{\mathbb{C}} = f$, i.e., the following diagram commutes:

$$\begin{array}{ccc} V \times W & \xrightarrow{\otimes_{\mathbb{C}}} & V \otimes_{\mathbb{C}} W \\ & \searrow f & \downarrow \underline{f} \\ & & X \end{array}$$

This definition may be too abstract. In practice, it is enough to know the following facts:

1. For every $v \in V$ and $w \in W$, there is a vector $v \otimes_{\mathbb{C}} w \in V \otimes_{\mathbb{C}} W$. In general, a vector in $V \otimes_{\mathbb{C}} W$ is the sum of vectors of this form.
2. Some obvious equations hold:

$$\begin{aligned} (v + v') \otimes_{\mathbb{C}} w &= v \otimes_{\mathbb{C}} w + v' \otimes_{\mathbb{C}} w, \\ v \otimes_{\mathbb{C}} (w + w') &= v \otimes_{\mathbb{C}} w + v \otimes_{\mathbb{C}} w', \\ (\lambda \cdot v) \otimes_{\mathbb{C}} w &= \lambda \cdot (v \otimes_{\mathbb{C}} w) = v \otimes_{\mathbb{C}} (\lambda \cdot w). \end{aligned}$$

3. Suppose $\{v_i \mid 1 \leq i \leq n\}$ is a basis of V and $\{w_j \mid 1 \leq j \leq m\}$ is a basis of W . Then $\{v_i \otimes_{\mathbb{C}} w_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ is a basis of $V \otimes_{\mathbb{C}} W$. In particular, we have $\dim(V \otimes_{\mathbb{C}} W) = \dim(V) \dim(W)$.

The tensor product of vector spaces induces a \mathbb{C} -linear monoidal structure on Vec . Both the tensor product functor $\otimes_{\mathbb{C}} : \text{Vec} \times \text{Vec} \rightarrow \text{Vec}$, the associator and left/right unitor are induced by the universal property. For example, the associator $\alpha_{U,V,W} : (U \otimes_{\mathbb{C}} V) \otimes_{\mathbb{C}} W \rightarrow U \otimes_{\mathbb{C}} (V \otimes_{\mathbb{C}} W)$ is defined by $(u \otimes_{\mathbb{C}} v) \otimes_{\mathbb{C}} w \mapsto u \otimes_{\mathbb{C}} (v \otimes_{\mathbb{C}} w)$. The tensor unit is \mathbb{C} . \heartsuit

Example 3.4.23. Let G be a finite group. For $(V, \rho), (W, \sigma) \in \text{Rep}(G)$, their tensor product is the G -representation defined by the vector space $V \otimes_{\mathbb{C}} W$ equipped with the G -action

$$g \mapsto \rho(g) \otimes_{\mathbb{C}} \sigma(g), \quad g \in G.$$

This tensor product induces a \mathbb{C} -linear monoidal structure on $\text{Rep}(G)$. The tensor unit is the trivial G -representation, i.e., the vector space \mathbb{C} equipped with the G -action $g \mapsto \text{id}_{\mathbb{C}}$. \heartsuit

Example 3.4.24. Let G be a finite group. For $V, W \in \text{Vec}_G$, their tensor product is the G -graded vector space defined by the total space $V \otimes_{\mathbb{C}} W$ equipped with the G -grading

$$(V \otimes_{\mathbb{C}} W)_g := \bigoplus_{h \in G} V_h \otimes_{\mathbb{C}} W_{h^{-1}g}, \quad g \in G.$$

This tensor product is well-defined because the tensor product of vector spaces preserves direct sums (see Exercise 3.3.46). In particular, we have $\mathbb{C}_{(g)} \otimes_{\mathbb{C}} \mathbb{C}_{(h)} \simeq \mathbb{C}_{(gh)}$.

There are also different \mathbb{C} -linear monoidal structures on Vec_G with the same tensor product functor. Suppose the associator $\alpha_{g,h,k} : (\mathbb{C}_{(g)} \otimes_{\mathbb{C}} \mathbb{C}_{(h)}) \otimes_{\mathbb{C}} \mathbb{C}_{(k)} \rightarrow \mathbb{C}_{(g)} \otimes_{\mathbb{C}} (\mathbb{C}_{(h)} \otimes_{\mathbb{C}} \mathbb{C}_{(k)})$ is defined by

$$(1 \otimes_{\mathbb{C}} 1) \otimes_{\mathbb{C}} 1 \mapsto \omega(g, h, k) \cdot 1 \otimes_{\mathbb{C}} (1 \otimes_{\mathbb{C}} 1)$$

for some nonzero complex number $\omega(g, h, k) \in \mathbb{C}^\times$ (by Exercise 3.3.6, we only need to consider these morphisms). Then the pentagon equation (3.4.1) translates to

$$\omega(h, k, l)\omega(g, hk, l)\omega(g, h, k) = \omega(gh, k, l)\omega(g, h, kl), \quad \forall g, h, k, l \in G.$$

We say that $\omega : G \times G \times G \rightarrow \mathbb{C}^\times$ is a *3-cocycle* valued in the abelian group \mathbb{C}^\times . The space of 3-cocycles valued in \mathbb{C}^\times is denoted by $Z^3(G; \mathbb{C}^\times)$. Similarly, the triangle equation determines the left/right unitor up to a nonzero complex number $\varepsilon \in \mathbb{C}^\times$:

$$\lambda_g = \varepsilon \cdot \omega(e, e, g)^{-1}, \quad \rho_g = \varepsilon \cdot \omega(g, e, e).$$

Different choices of ε give equivalence monoidal categories (see Example 4.2.5 for details). So usually we take $\varepsilon = 1$ for simplicity. By the pentagon equation we have $\omega(e, e, e) = 1$, which shows that $\lambda_e = \rho_e$ (recall Remark 3.4.21).

As a conclusion, for each 3-cocycle $\omega \in Z^3(G; \mathbb{C}^\times)$ there is a monoidal category, denoted by Vec_G^ω , whose underlying category is Vec_G and associator is determined by ω . \heartsuit

Example 3.4.25. Let G be a group. Suppose there is a monoidal structure on the delooping category BG (see Example 3.3.7). The tensor product functor $\otimes : BG \times BG \rightarrow BG$ is determined by its action on morphisms $\otimes = \otimes_{(*, *), (*, *)} : G \times G \rightarrow G$. The functoriality implies that the map $\otimes : G \times G \rightarrow G$ satisfies $e \otimes e = e$ and the following equation for $g, h, k, l \in G$:

$$(g \otimes h) \cdot (k \otimes l) = (g \cdot k) \otimes (h \cdot l).$$

The naturality of the left and right unitor means that the following equations hold for every $g \in G$:

$$\lambda_* \cdot (e \otimes g) = g \cdot \lambda_*, \quad \rho_* \cdot (g \otimes e) = g \cdot \rho_*.$$

Since $\lambda_* = \rho_*$ (see Remark 3.4.21), we have

$$\begin{aligned} g \otimes l &= (g \cdot e) \otimes (e \cdot l) = (g \otimes e) \cdot (e \otimes l) = \lambda_*^{-1} \cdot g \cdot l \cdot \lambda_*, \\ k \otimes h &= (e \cdot k) \otimes (h \cdot e) = (e \otimes h) \cdot (k \otimes e) = \lambda_*^{-1} \cdot h \cdot k \cdot \lambda_*. \end{aligned}$$

It follows that both the map \otimes and the multiplication \cdot of G are commutative. The commutativity of the multiplication \cdot also implies that the map \cdot is the same as \otimes . In particular, we have $e \otimes g = g = g \otimes e$ for every $g \in G$. By the pentagon equation, the associator $\alpha_{*, *, *}$ must be identity. The unitors $\lambda_* = \rho_*$ can be arbitrary because the triangle equation gives no constraint. Hence we conclude that BG is a monoidal category if and only if G is an abelian group. \heartsuit

Remark 3.4.26. Let \mathcal{C} be a \mathbb{C} -linear monoidal category. Recall that the map

$$\otimes : \text{Hom}_{\mathcal{C}}(x, x') \times \text{Hom}_{\mathcal{C}}(y, y') \rightarrow \text{Hom}_{\mathcal{C}}(x \otimes y, x' \otimes y')$$

is \mathbb{C} -bilinear. By the universal property of the tensor product of vector spaces, it induces a \mathbb{C} -linear map

$$\text{Hom}_{\mathcal{C}}(x, x') \otimes_{\mathbb{C}} \text{Hom}_{\mathcal{C}}(y, y') \rightarrow \text{Hom}_{\mathcal{C}}(x \otimes y, x' \otimes y').$$

In general, this linear map is not an isomorphism and we say there is a *spatial fusion anomaly*. Physically, the presence of a spatial fusion anomaly means that the spatial fusion $\text{Hom}_{\mathcal{C}}(x, x') \otimes_{\mathbb{C}} \text{Hom}_{\mathcal{C}}(y, y')$ of two hom spaces is not at a fixed point of the renormalization flow, and it finally flows to the fixed point $\text{Hom}_{\mathcal{C}}(x \otimes y, x' \otimes y')$.

Microscopically, the presence of spatial fusion anomalies is due to the existence of ‘non-local’ instantons in $\text{Hom}_{\mathcal{C}}(x \otimes y, x' \otimes y')$. For example, if $f: x \rightarrow y'$ and $g: y \rightarrow x'$ are two instantons, then the instanton depicted in Figure 19 is an instanton in $\text{Hom}_{\mathcal{C}}(x \otimes y, x' \otimes y')$. However, it does not come from the spatial fusion $\text{Hom}_{\mathcal{C}}(x, x') \otimes_{\mathbb{C}} \text{Hom}_{\mathcal{C}}(y, y')$. When two sites are separated far enough, this instanton becomes non-local. \diamond

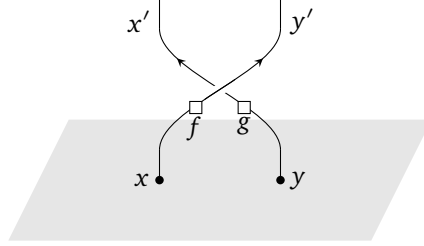


Figure 19: a ‘non-local’ instanton

3.4.3 Unitarity

Recall that an instanton can be realized as a linear map commuting with all local operators. Thus we can talk about the Hermitian conjugate of an instanton.

Remark 3.4.27. We briefly review the Hermitian conjugate of a linear map between Hilbert spaces. Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces and $f: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a linear map. The Hermitian conjugate of f is a linear map $f^\dagger: \mathcal{H}_2 \rightarrow \mathcal{H}_1$ defined by the following equation:

$$\langle f(v), w \rangle_2 = \langle v, f^\dagger(w) \rangle_1, \quad \forall v \in \mathcal{H}_1, w \in \mathcal{H}_2,$$

where $\langle -, - \rangle_i$ is the inner product of \mathcal{H}_i for $i = 1, 2$. We list some important facts about the Hermitian conjugate:

1. The map $\dagger: \text{Hom}_{\text{vec}}(\mathcal{H}_1, \mathcal{H}_2) \rightarrow \text{Hom}_{\text{vec}}(\mathcal{H}_2, \mathcal{H}_1)$ defined by $f \mapsto f^\dagger$ is anti-linear, i.e., $(f + g)^\dagger = f^\dagger + g^\dagger$ and $(\lambda \cdot f)^\dagger = \bar{\lambda} \cdot f^\dagger$ for any $f, g \in \text{Hom}_{\text{vec}}(\mathcal{H}_1, \mathcal{H}_2)$ and $\lambda \in \mathbb{C}$.
2. The map \dagger is involutive, i.e., $(f^\dagger)^\dagger = f$.
3. The identity map $\text{id}_{\mathcal{H}}$ on a Hilbert space \mathcal{H} is invariant under the Hermitian conjugate, i.e., $\text{id}_{\mathcal{H}}^\dagger = \text{id}_{\mathcal{H}}$.
4. Suppose $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ are Hilbert space and $f: \mathcal{H}_1 \rightarrow \mathcal{H}_2, g: \mathcal{H}_2 \rightarrow \mathcal{H}_3$ are linear maps. Then $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$.
5. Suppose $f: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a linear map between two Hilbert spaces and satisfies $f^\dagger \circ f = 0$. Then $f = 0$. \diamond

Since the Hermitian conjugate of a local operator is still a local operator, the Hermitian conjugate of an instanton is still an instanton. Thus taking the Hermitian conjugate of an instanton defines a map

$$\dagger: \text{Hom}_{\mathcal{C}}(x, y) \rightarrow \text{Hom}_{\mathcal{C}}(y, x)$$

for every $x, y \in \mathcal{C}$. These maps form an additional structure on the category \mathcal{C} , called a unitary structure. Intuitively, taking the Hermitian conjugate of an instanton is like a time-reversal operation (see Figure 20).

The unitary structure on \mathcal{C} should satisfy the usual properties of the Hermitian conjugate. We summarize these data and properties to the following definition.

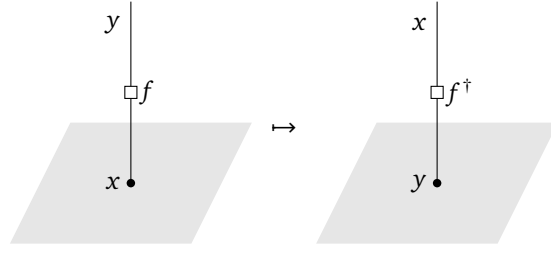


Figure 20: the dagger structure

Definition 3.4.28. Let \mathcal{C} be a \mathbb{C} -linear category. A *dagger structure* on \mathcal{C} is an involutive anti-linear functor $\dagger: \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$ which acts on objects as identity. Equivalently, a dagger structure on \mathcal{C} is a collection of anti-linear maps $\dagger: \text{Hom}_{\mathcal{C}}(x, y) \rightarrow \text{Hom}_{\mathcal{C}}(y, x)$ for all $x, y \in \mathcal{C}$, satisfying the following conditions:

1. $(f^\dagger)^\dagger = f$ for every morphism f in \mathcal{C} .
2. $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$ for all $f \in \text{Hom}_{\mathcal{C}}(x, y)$, $g \in \text{Hom}_{\mathcal{C}}(y, z)$ and $x, y, z \in \mathcal{C}$.
3. $\text{id}_x^\dagger = \text{id}_x$ for all $x \in \mathcal{C}$.

Moreover, a dagger structure is called a *unitary structure* if it satisfies the following condition:

4. A morphism f in \mathcal{C} satisfies $f^\dagger \circ f = 0$ if and only if $f = 0$.

A *unitary category* is a \mathbb{C} -linear category equipped with a unitary structure. ■

Remark 3.4.29. A unitary category is automatically semisimple. ◇

Remark 3.4.30. The notion of a unitary category is equivalent to the notion of a C^* -category [Müg00, Proposition 2.1]. ◇

For any topological order \mathcal{C} , the above discussion implies that the category \mathcal{C} of particle-like topological defects is a unitary category. When \mathcal{C} is a 2d topological order, the monoidal structure and the unitary structure on \mathcal{C} should be compatible. For example, let us consider the Hermitian conjugate of the associator. The intuition of the time-reversal operation provides a picture of $\alpha_{x,y,z}^\dagger$, as depicted in Figure 21.

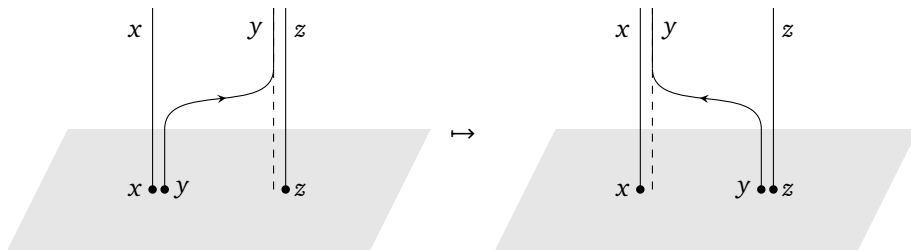


Figure 21: the Hermitian conjugate of the associator: $\alpha_{x,y,z} \mapsto \alpha_{x,y,z}^\dagger$

Then the physical intuition implies that the following equations hold because their world lines are homotopy (see Figure 22):

$$\alpha_{x,y,z}^\dagger \circ \alpha_{x,y,z} = \text{id}_{(x \otimes y) \otimes z}, \quad \alpha_{x,y,z} \circ \alpha_{x,y,z}^\dagger = \text{id}_{x \otimes (y \otimes z)}.$$

In other words, $\alpha_{x,y,z}^\dagger$ is the inverse of $\alpha_{x,y,z}$.

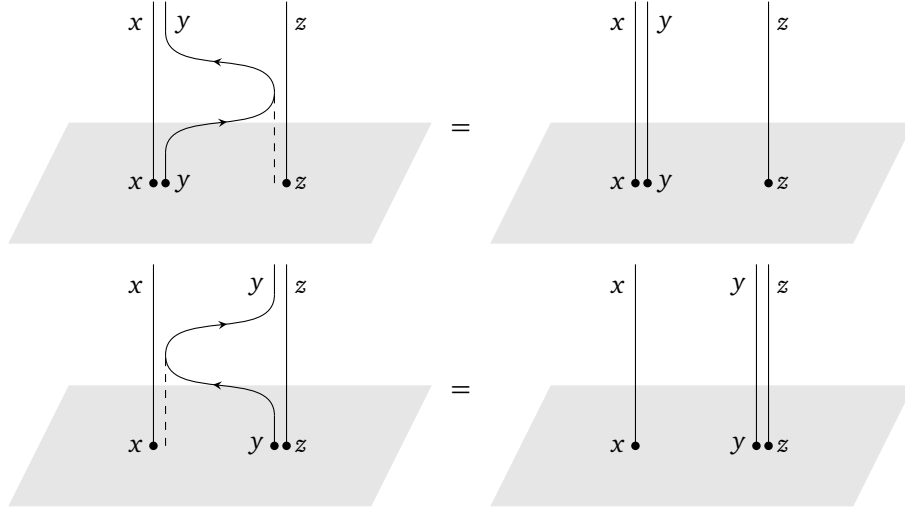


Figure 22: $\alpha_{x,y,z}^\dagger$ is the inverse of $\alpha_{x,y,z}$

Definition 3.4.31. A morphism f in a unitary category is called *unitary* if it is an isomorphism and satisfies $f^{-1} = f^\dagger$. ■

Definition 3.4.32. A dagger structure on a \mathbb{C} -linear monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, 1, \alpha, \lambda, \rho)$ is *compatible* with the monoidal structure if

1. $f^\dagger \otimes g^\dagger = (f \otimes g)^\dagger$ for all morphisms f, g in \mathcal{C} .
2. The morphisms $\alpha_{x,y,z}, \lambda_x, \rho_x$ are unitary for all $x, y, z \in \mathcal{C}$.

A *unitary* monoidal category is a \mathbb{C} -linear monoidal category equipped with a compatible unitary structure. ■

Remark 3.4.33. The condition that a dagger structure is compatible with the monoidal structure is equivalent to that the functor $\dagger: \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$ is a monoidal functor. The notion of a monoidal functor is defined in Definition 4.2.3. ◇

As a conclusion, the category \mathcal{C} of particle-like topological defects of a 2d topological order \mathcal{C} is a unitary monoidal category.

Example 3.4.34. Let Hilb be the category of finite-dimensional Hilbert spaces and linear maps. As explained in Remark 3.4.27, Hilb is a unitary category with the unitary structure defined by the usual Hermitian conjugate. Moreover, Hilb is a unitary monoidal category with the tensor product of two Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ defined by the underlying vector space $\mathcal{H}_1 \otimes_{\mathbb{C}} \mathcal{H}_2$ and the inner product

$$\langle v_1 \otimes_{\mathbb{C}} v_2, w_1 \otimes_{\mathbb{C}} w_2 \rangle := \langle v_1, w_1 \rangle_1 \cdot \langle v_2, w_2 \rangle_2.$$

The category Hilb can be viewed as a unitary version of Vec . Indeed, these two categories are equivalent. ♡

Exercise 3.4.35. Let $F: \text{Hilb} \rightarrow \text{Vec}$ be the functor that ‘forgets’ the inner products of Hilbert spaces. Use Theorem 3.3.52 to prove that F is an equivalence.

Example 3.4.36. Let G be a finite group. A *unitary G -representation* is a Hilbert space \mathcal{H} equipped with a group homomorphism $\rho: G \rightarrow \text{U}(\mathcal{H})$, where $\text{U}(\mathcal{H})$ is the group of unitary operators on \mathcal{H} . The category of finite-dimensional unitary G -representations is denoted by $\text{Rep}^{\text{u}}(G)$. It is a unitary monoidal category with the unitary structure defined by the usual Hermitian conjugate. Also, the forgetful functor $\text{Rep}^{\text{u}}(G) \rightarrow \text{Rep}(G)$ is an equivalence. Thus $\text{Rep}^{\text{u}}(G)$ is a unitary version of $\text{Rep}(G)$. ♡

Example 3.4.37. Let G be a finite group. The unitary version of Vec_G is the category Hilb_G of finite-dimensional G -graded Hilbert spaces. Moreover, by considering functions $G \times G \times G \rightarrow \text{U}(1)$ we can similarly define the group $Z^3(G; \text{U}(1))$ of 3-cocycles valued in $\text{U}(1)$. Then for every $\omega \in Z^3(G; \text{U}(1))$, there is a unitary monoidal category Hilb_G^ω whose underlying category is Hilb_G and the associator is determined by ω . \heartsuit

3.4.4 Rigidity

Recall that a nontrivial topological excitation can not be created alone by local operators from the ground state. Usually a non-local operator creates a pair of nontrivial topological excitations. Then one of them can be viewed as the ‘anti-particle’ of another one.

The simplest way to create a nontrivial topological defect x is to ‘bend’ the world line, or the path of an adiabatic move, of x (see Figure 23 (a)). Microscopically, this instanton is realized by a string operator of x acting on the ground state, which creates a topological excitation x and another one denoted by x^* , called the dual (or anti-particle) of x . Similarly there is an annihilation instanton $d_x: x^* \otimes x \rightarrow \mathbb{1}$, as depicted in Figure 23 (b).

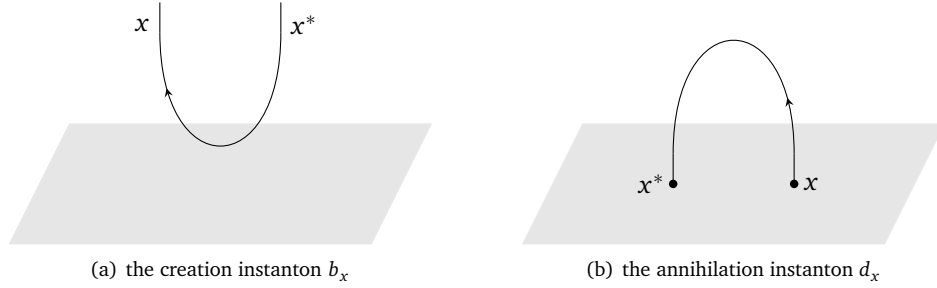


Figure 23: the creation and annihilation instantons associated to x

Example 3.4.38. Microscopically, both the adiabatic move, the creation and the annihilation instantons are realized by the same string operator. The difference is how the operator act on different states. In the toric code model, for example, an e string operator $\prod_i \sigma_z^i$ acts on an e particle as adiabatically moving it to another site. However, the same e string operator acting on the ground state creates an e particle on each end of the string. This gives the creation instanton associated to e . The annihilation instanton can be obtained similarly by acting an e string operator on two e particles. It follows that $e^* = e$ in the toric code model. Similarly $m^* = m$ and $f^* = f$. The dual of the trivial topological excitation $\mathbb{1}^* = \mathbb{1}$ is itself in any topological order \mathcal{C} . \heartsuit

The physical intuition as depicted in Figure 24 implies that the creation and annihilation instantons should satisfy the following two zig-zag equations:

$$\begin{aligned} & \left(x \xrightarrow{\lambda_x^{-1}} \mathbb{1} \otimes x \xrightarrow{b_x \otimes \text{id}_x} (x \otimes x^*) \otimes x \xrightarrow{\alpha_{x, x^*, x}} x \otimes (x^* \otimes x) \xrightarrow{\text{id}_x \otimes d_x} x \otimes \mathbb{1} \xrightarrow{\rho_x} x \right) = \text{id}_x, \\ & \left(x^* \xrightarrow{\rho_{x^*}^{-1}} x^* \otimes \mathbb{1} \xrightarrow{\text{id}_{x^*} \otimes b_x} x^* \otimes (x \otimes x^*) \xrightarrow{\alpha_{x^*, x, x^*}^{-1}} (x^* \otimes x) \otimes x^* \xrightarrow{d_x \otimes \text{id}_{x^*}} \mathbb{1} \otimes x^* \xrightarrow{\lambda_{x^*}} x^* \right) = \text{id}_{x^*}. \end{aligned}$$

We summarize the properties of the dual of a topological defect in the following definition.

Definition 3.4.39. Let \mathcal{C} be a monoidal category and $x \in \mathcal{C}$. A *left dual* of x is an object $x^L \in \mathcal{C}$ equipped with two morphisms $b_x: \mathbb{1} \rightarrow x \otimes x^L$ and $d_x: x^L \otimes x \rightarrow \mathbb{1}$ satisfying the following two zig-zag equations:

$$(\text{id}_x \otimes d_x) \circ (b_x \otimes \text{id}_x) = \text{id}_x, \quad (d_x \otimes \text{id}_{x^L}) \circ (\text{id}_{x^L} \otimes b_x) = \text{id}_{x^L}.$$

Here we ignore associators and unitors for simplicity. Similarly, a *right dual* of x is an object $x^R \in \mathcal{C}$ equipped with two morphisms $b'_x: \mathbb{1} \rightarrow x^R \otimes x$ and $d'_x: x \otimes x^R \rightarrow \mathbb{1}$ satisfying the following two zig-zag equations:

$$(d'_x \otimes \text{id}_x) \circ (\text{id}_x \otimes b'_x) = \text{id}_x, \quad (\text{id}_{x^R} \otimes d'_x) \circ (b'_x \otimes \text{id}_{x^R}) = \text{id}_{x^R}.$$

An object in \mathcal{C} is called *dualizable* if it admits both left and right duals. If every object in \mathcal{C} is dualizable, we say that \mathcal{C} is a *rigid monoidal category*. \blacksquare

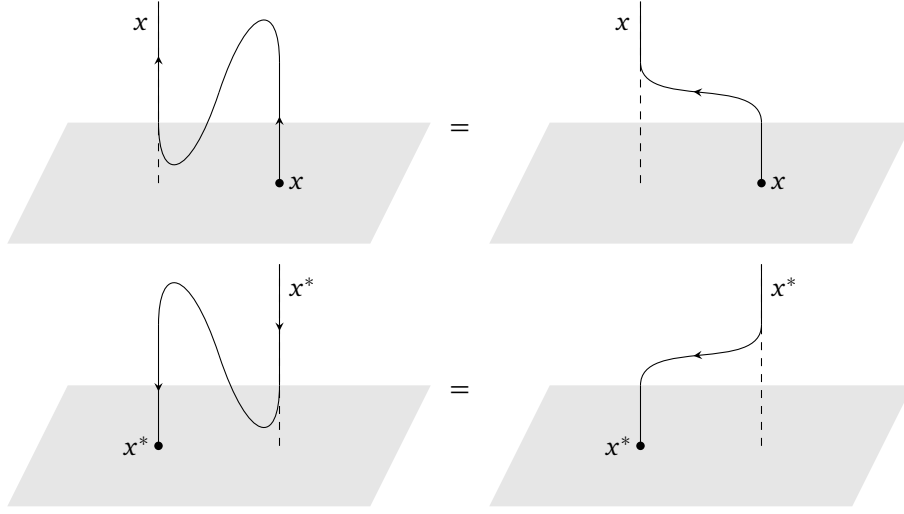


Figure 24: the zig-zag equations

Remark 3.4.40. A left/right dual of an object, if exists, is unique up to a unique isomorphism. Thus we can talk about *the* left/right dual of an object. It follows that there exists canonical isomorphisms $(x \otimes y)^L \simeq y^L \otimes x^L$ and $(x \otimes y)^R \simeq y^R \otimes x^R$ for every $x, y \in \mathcal{C}$. \diamond

Example 3.4.41. Let \mathcal{C} be a monoidal category. The tensor unit $\mathbb{1}$ is both the left and right dual of itself. \heartsuit

Remark 3.4.42. In a rigid monoidal category \mathcal{C} , the left dual $f^L: y^L \rightarrow x^L$ of a morphism $f: x \rightarrow y$ is defined by

$$f^L := (d_y \otimes \text{id}_{x^L}) \circ (\text{id}_{y^L} \otimes f \otimes \text{id}_{x^L}) \circ (\text{id}_{x^L} \otimes b_x),$$

as depicted in Figure 25. Similarly we can define the right dual of f . The zig-zag equations imply that taking left duals defines a functor $\delta^L: \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$. Note that this functor depends on many choices: we need to explicitly choose a left dual (y^L, b_y, d_y) for every object $y \in \mathcal{C}$. So indeed there are many different left dual functors, but these functors are canonically isomorphic. Similarly, we can define the right dual of a morphism and the right dual functor $\delta^R: \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$. \diamond

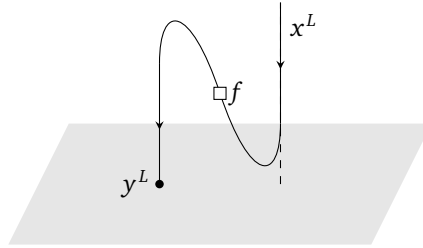


Figure 25: The left dual of a morphism $f: x \rightarrow y$ is a morphism $f^L: y^L \rightarrow x^L$.

Exercise 3.4.43. Let \mathcal{C} be a \mathbb{C} -linear rigid monoidal category. Prove that taking left duals defines a \mathbb{C} -linear functor $\delta^L: \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$. In particular, taking left duals preserves direct sums, i.e., there is a canonical isomorphism $(x_1 \oplus \dots \oplus x_n)^L \simeq x_1^L \oplus \dots \oplus x_n^L$ for $x_1, \dots, x_n \in \mathcal{C}$.

Remark 3.4.44. The instantons can be freely moved along the world line. However, the instantons b_x and d_x are not natural transformations, but satisfy the following property: for any morphism $f: x \rightarrow y$, we have

$$(f \otimes \text{id}_{x^L}) \circ b_x = (\text{id}_y \otimes f^L) \circ b_y, \quad d_y \circ (\text{id}_{y^L} \otimes f) = d_x \circ (f^L \otimes \text{id}_x).$$

Also, there are two similar equations for right duals. \diamond

Example 3.4.45. Let $V \in \text{Vec}$. Its left dual is given by the usual dual space $V^* := \text{Hom}_{\text{Vec}}(V, \mathbb{C})$. The annihilation map $d_V: V^* \otimes_{\mathbb{C}} V \rightarrow \mathbb{C}$ is defined by $\phi \otimes_{\mathbb{C}} v \mapsto \phi(v)$. Note that under the natural identification $V^* \otimes_{\mathbb{C}} V \simeq \text{Hom}_{\text{Vec}}(V, V)$ the annihilation map d_V is taking the trace. The creation map $b_V: \mathbb{C} \rightarrow V \otimes_{\mathbb{C}} V^*$ is defined by

$$1 \mapsto \sum_{i=1}^n v_i \otimes_{\mathbb{C}} v^i,$$

where $\{v_i\}_{i=1}^n$ is a basis of V and $\{v^i\}_{i=1}^n$ is the dual basis in V^* defined by $v^i(v_j) = \delta_j^i$. It is straightforward to check that the creation map is independent of the choice of the basis.

Similarly, one can verify that V^* is also the right dual of V . Therefore, Vec is a rigid monoidal category. It is worth noting that in the monoidal category of all vector spaces, an object is dualizable if and only if it is finite-dimensional. \heartsuit

Exercise 3.4.46. Let $V, W \in \text{Vec}$. For every linear map $f: V \rightarrow W$, we can define a linear map $f^*: W^* \rightarrow V^*$ by $f^*(\phi) := \phi \circ f$. Prove that both the left and right dual of $f: V \rightarrow W$ (see Remark 3.4.42) is $f^*: W^* \rightarrow V^*$.

Example 3.4.47. Let G be a finite group. Both the left and right dual of a finite-dimensional G -representation $(V, \rho) \in \text{Rep}(G)$ is defined by the dual vector space V^* equipped with the G -action

$$g \mapsto \rho(g^{-1})^*, \quad g \in G.$$

Here $\rho(g^{-1})^*$ denotes the dual linear map of $\rho(g^{-1})$ (see Exercise 3.4.46), not the complex conjugate. Therefore, $\text{Rep}(G)$ is a rigid monoidal category. \heartsuit

Example 3.4.48. Let G be a finite group and $\omega \in Z^3(G; \mathbb{C}^\times)$. Both the left and right dual of a locally finite-dimensional G -graded vector space $V \in \text{Vec}_G^\omega$ is defined by the total space V^* equipped with the G -grading

$$(V^*)_g := (V_{g^{-1}})^*, \quad g \in G.$$

This dual space is well-defined because taking duals in Vec preserves direct sums (see Exercise 3.4.43). Equivalently, both the left and right dual of $\mathbb{C}_{(g)}$ is $\mathbb{C}_{(g^{-1})}$ for every $g \in G$. Therefore, Vec_G^ω is a rigid monoidal category. \heartsuit

Exercise 3.4.49. Let G be a finite group and $\omega \in Z^3(G; \mathbb{C}^\times)$. Find the creation and annihilation morphisms of $\mathbb{C}_{(g)}$ for every $g \in G$. The answer is unique up to a nonzero complex number.

In the above discussion we only consider the left dual x^* of a topological defect x . What about the right dual? One may wish to rotate or braid the instantons in Figure 23 to get the creation and annihilation instantons for a right dual. However, these operations do not give the correct answer, and they can not be defined in 1d space.

Another solution which works in any dimension is to bend the world line in another direction. But then it is not clear whether the left dual and right dual of x coincide. If not, the theory would be complicated: a topological defect could have different anti-particles, and each one could have different anti-anti-particles. Mathematically, we need a so-called pivotal structure or spherical structure to solve this problem. For unitary theories, the answer is simpler.

For every topological defect x , the unitary structure provides another two instantons $d_x^\dagger: \mathbb{1} \rightarrow x^* \otimes x$ and $b_x^\dagger: x \otimes x^* \rightarrow \mathbb{1}$, which are time-reversal of d_x and b_x (see Figure 26). So they are nothing but the world line of x bending in another direction. One may immediately realize that $(x^*, d_x^\dagger, b_x^\dagger)$ is a right dual of x , because it satisfies the time-reversal of the definition of a left dual. Mathematically, we have

$$\begin{aligned} (b_x^\dagger \otimes \text{id}_x) \circ (\text{id}_x \otimes d_x^\dagger) &= [(\text{id}_x \otimes d_x) \circ (b_x \otimes \text{id}_x)]^\dagger = \text{id}_x^\dagger = \text{id}_x, \\ (\text{id}_{x^*} \otimes b_x^\dagger) \circ (d_x^\dagger \otimes \text{id}_{x^*}) &= [(d_x \otimes \text{id}_{x^*}) \circ (\text{id}_{x^*} \otimes b_x)]^\dagger = \text{id}_{x^*}^\dagger = \text{id}_{x^*}. \end{aligned}$$

Therefore, x^* is both the left and right dual of x . For simplicity, we say that x^* is the dual of x .



Figure 26: the time-reversal of the creation and annihilation instantons

Remark 3.4.50. Moreover, by properly choosing the left dual (x, b_x, d_x) for every $x \in \mathcal{C}$ (see Section 3.4.5), the left and right dual of every morphism in \mathcal{C} are also the same, denoted by f^* (see Figure 27):

$$f^* := (d_y \otimes \text{id}_{x^*}) \circ (\text{id}_{y^*} \otimes f \otimes \text{id}_{x^*}) \circ (\text{id}_{x^*} \otimes b_x) = (\text{id}_{x^*} \otimes b_y^\dagger) \circ (\text{id}_{x^*} \otimes f \otimes \text{id}_{y^*}) \circ (d_x^\dagger \otimes \text{id}_{x^*}).$$

Thus the left and right dual functors are equal, denoted by $\delta^*: \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$. Moreover, in this case taking the dual commutes with taking the dagger, i.e., $(f^*)^\dagger = (f^\dagger)^*$ for every morphism f . \diamond

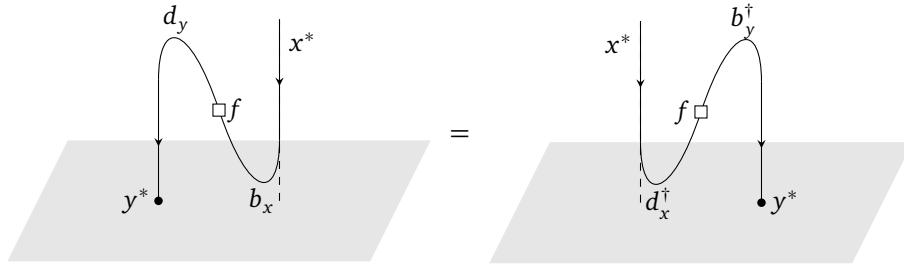


Figure 27: The left and right dual of a morphism $f : x \rightarrow y$ are equal by properly choosing the dual.

3.4.5 Fusion categories and dimensions

We summarize the structures developed in previous subsections to the following definition.

Definition 3.4.51. A *multi-fusion category* is a \mathbb{C} -linear rigid monoidal category that is also finite semisimple. A *fusion category* is a multi-fusion category such that the tensor unit $\mathbb{1}$ is a simple object. A *unitary (multi-)fusion category* is a (multi-)fusion category equipped with a compatible unitary structure. \blacksquare

Hence, the category \mathcal{C} of particle-like topological defects of a 2d topological order \mathcal{C} is a unitary multi-fusion category. If \mathcal{C} is stable, \mathcal{C} is a unitary fusion category.

Exercise 3.4.52. Prove that Vec is a fusion category.

Exercise 3.4.53. Let G be a finite group. Prove that $\text{Rep}(G)$ is a fusion category.

Exercise 3.4.54. Let G be a finite group. Prove that Vec_G^ω is a fusion category for every $\omega \in Z^3(G; \mathbb{C}^\times)$.

Remark 3.4.55. Let \mathcal{C} be a multi-fusion category. Since \mathcal{C} is semisimple, the tensor product of two simple objects $x, y \in \mathcal{C}$ is the direct sum of simple objects. Thus we have

$$x \otimes y \simeq \bigoplus_{z \in \text{Irr}(\mathcal{C})} N_{xy}^z \cdot z$$

for some non-negative integers $N_{xy}^z \in \mathbb{N}$, where $n \cdot z$ denotes the direct sum of n copies of z . These numbers $\{N_{xy}^z\}_{x,y,z \in \text{Irr}(\mathcal{C})}$ are called the *fusion rules* of \mathcal{C} .

Equivalently, the isomorphism classes of simple objects of \mathcal{C} generate a ring, where the multiplication is given by the tensor product of \mathcal{C} . Then its structure constants are the fusion rules of \mathcal{C} . This ring is called the *fusion ring* or *Grothendieck ring* of \mathcal{C} , denoted by $\text{Gr}(\mathcal{C})$.

The Grothendieck ring does not determine the multi-fusion category. As an example, the fusion ring of Vec_G^ω for every $\omega \in Z^3(G; \mathbb{C}^\times)$ is isomorphic to the group ring $\mathbb{Z}[G]$. \diamond

In the following we consider the case that \mathcal{C} is stable, i.e., \mathcal{C} is a unitary fusion category. With the help of four creation and annihilation instantons, we can close the world line of a particle-like topological defect $x \in \mathcal{C}$ to a loop. As depicted in Figure 28, for every simple object $x \in \mathcal{C}$ there are two ways to close the world line of x . The results are morphisms $b_x^\dagger \circ b_x, d_x \circ d_x^\dagger: \mathbb{1} \rightarrow \mathbb{1}$. Since $\text{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1}) \simeq \mathbb{C}$, both $b_x^\dagger \circ b_x$ and $d_x \circ d_x^\dagger$ can be viewed as positive real numbers. We define the *quantum dimension* of x by

$$\dim(x) := \sqrt{(b_x^\dagger \circ b_x) \cdot (d_x \circ d_x^\dagger)} > 0.$$

The *quantum dimension* of \mathcal{C} is defined to be

$$\dim(\mathcal{C}) := \sum_{x \in \text{Irr}(\mathcal{C})} \dim(x)^2.$$

Note that the morphisms b_x and d_x can be properly rescaled such that

$$b_x^\dagger \circ b_x = d_x \circ d_x^\dagger = \dim(x).$$

We say that a choice of duals in \mathcal{C} are *normalized* if the above equation holds for every object in \mathcal{C} . With a normalized choice of duals, the left and right dual functors are equal (see Remark 3.4.50) [Yam04].



Figure 28: the quantum dimension of x with a normalized choice of duals

Exercise 3.4.56. Prove that the quantum dimension of a finite-dimensional Hilbert space $\mathcal{H} \in \text{Hilb}$ is equal to the usual dimension $\dim(\mathcal{H})$.

Remark 3.4.57. Let \mathcal{C} be a unitary fusion category. For every $x, y \in \mathcal{C}$ we have:

$$\dim(x \oplus y) = \dim(x) + \dim(y), \quad \dim(x \otimes y) = \dim(x) \cdot \dim(y).$$

In other words, \dim defines a ring homomorphism $\text{Gr}(\mathcal{C}) \rightarrow \mathbb{R}$. In particular, we have

$$\dim(x) \cdot \dim(y) = \sum_{z \in \text{Irr}(\mathcal{C})} N_{xy}^z \cdot \dim(z)$$

for every simple objects $x, y \in \mathcal{C}$. \diamond

More generally, given an instanton $f \in \text{Hom}_{\mathcal{C}}(x, x)$, there are two ways to close the world line of x (see Figure 29). With a normalized choice of duals, these two instantons are equal and called the *trace* of f :

$$\text{tr}(f) := b_x^\dagger \circ (f \otimes \text{id}_{x^*}) \circ b_x = d_x \circ (\text{id}_{x^*} \otimes f) \circ d_x^\dagger \in \text{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1}) \simeq \mathbb{C}.$$

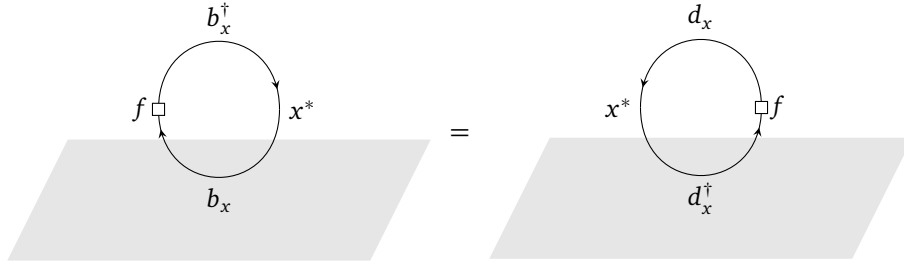


Figure 29: the trace of an instanton $f \in \text{Hom}_{\mathcal{C}}(x, x)$ with a normalized choice of duals

Remark 3.4.58. In a rigid monoidal category without unitary structure, given an object x , possibly there is no (canonical) isomorphism between x^L and x^R , or equivalently, no isomorphism between x and x^{LL} . A *pivotal structure* on a rigid monoidal category \mathcal{C} is a natural isomorphism $a_x : x \rightarrow x^{LL}$ satisfying $a_x \otimes a_y = a_{x \otimes y}$ (this condition means that a is a monoidal natural isomorphism; see Definition 4.2.10).

In a pivotal fusion category, we can similarly close the world line of an object x . However, in this case there are two different ways to close the world line and they may not be equal, and these two results are called left and right quantum dimensions of x , respectively:

$$\dim^L(x) := d_{x^L} \circ (a_x \otimes \text{id}_{x^L}) \circ b_x, \quad \dim^R(x) := d_x \circ (\text{id}_{x^L} \otimes a_x^{-1}) \circ b_{x^L}.$$

A pivotal structure on a fusion category is called *spherical* if the left and right quantum dimensions are equal [BW99]. A unitary fusion category admits a unique pivotal structure defined by (with a normalized choice of duals)

$$a_x := (b_x^\dagger \otimes \text{id}_{x^{LL}}) \circ (\text{id}_x \otimes b_{x^L}) = (\text{id}_{x^{LL}} \otimes d_x) \circ (d_{x^L}^\dagger \otimes \text{id}_x).$$

This pivotal structure is automatically spherical and compatible with the unitary structure [Yam04, Müg03a] (this fact is also mentioned in a footnote in [Kit06]). \diamond

Remark 3.4.59. The quantum dimension of a fusion category is independent of the choice of pivotal or spherical structures, but the quantum dimensions of objects depend on the choice of pivotal or spherical structures. \diamond

Example 3.4.60. For every vector space V , there is a canonical \mathbb{C} -linear map $a_V : V \rightarrow V^{**}$ defined by $v \mapsto \text{ev}_v$, where $\text{ev}_v : V^* \rightarrow \mathbb{C}$ is defined by $\text{ev}_v(\phi) := \phi(v)$. Moreover, a_V is an isomorphism if and only if V is finite-dimensional. Hence each finite-dimensional vector space is canonically isomorphic to its double dual space.

The family $\{a_V\}_{V \in \text{Vec}}$ defines a spherical structure on Vec . With this spherical structure, the quantum dimension of $V \in \text{Vec}$ is equal to the usual dimension of V . Also, such canonical isomorphisms induce spherical structures on $\text{Rep}(G)$ and Vec_G for every finite group G , and the quantum dimensions are equal to the usual dimensions. \heartsuit

In a fusion category \mathcal{C} , there is another version of dimension. Here is the physical intuition. Suppose \mathcal{C} is the fusion category of particle-like topological defects of a stable topological order. If we put n copies of a topological defect $x \in \mathcal{C}$ on a disk, the GSD grows exponentially on n :

$$\text{GSD} \sim \text{constant} \cdot d^n.$$

Mathematically, this number d is equal to the largest non-negative eigenvalue of the matrix N_x defined by $(N_x)_{zy} := N_{xy}^z$, and the existence is provided by the Frobenius-Perron theorem [Per07, Fro12]. Therefore, it is called the *Frobenius-Perron dimension* of x and denoted by $\text{FPdim}(x)$ [ENO05]. The Frobenius-Perron dimension of a fusion category \mathcal{C} is defined by

$$\text{FPdim}(\mathcal{C}) := \sum_{x \in \text{Irr}(\mathcal{C})} \text{FPdim}(x)^2.$$

Remark 3.4.61. Let \mathcal{C} be a fusion category. The Frobenius-Perron dimension also defines a ring homomorphism $\text{FPdim}: \text{Gr}(\mathcal{C}) \rightarrow \mathbb{R}$. Moreover, it is the unique ring homomorphism $\text{Gr}(\mathcal{C}) \rightarrow \mathbb{C}$ that takes non-negative values (indeed, positive values) on simple objects. So the Frobenius-Perron dimensions can be directly obtained from the fusion rules. For example, the Frobenius-Perron dimensions in $\text{Rep}(G)$ or Vec_G^ω for a finite group G and $\omega \in Z^3(G; \mathbb{C}^\times)$ have to be equal to the usual dimensions. \diamond

Remark 3.4.62. A fusion category \mathcal{C} is called *pseudo-unitary* if $\dim(\mathcal{C}) = \text{FPdim}(\mathcal{C})$. A unitary fusion category is pseudo-unitary because the quantum dimensions of simple objects are positive and thus coincide with the Frobenius-Perron dimensions (see Remark 3.4.61). Any pseudo-unitary fusion category admits a unique spherical structure such that the quantum dimensions of objects are equal to the Frobenius-Perron dimensions [ENO05]. It is believed that for any pseudo-unitary fusion category there is an equivalent unitary fusion category. So in the following, a *unitary fusion category* can be understood as a pseudo-unitary fusion category equipped with the unique spherical structure such that the quantum dimensions of objects are equal to the Frobenius-Perron dimensions. \diamond

3.4.6 Braiding structure

In a 2d topological order, a topological defect x can be adiabatically moved around another y (see Figure 30). This process should not change the fusion of x and y , and thus induces an isomorphism from $x \otimes y$ to itself.

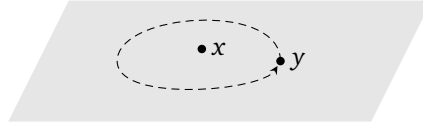


Figure 30: a double braiding or a full braiding

In particular, as depicted in Figure 31 (a), x and y can be adiabatically exchanged in the counter-clockwise direction. This process gives an isomorphism (instanton) $c_{x,y}: x \otimes y \rightarrow y \otimes x$, called the braiding of x and y . Also, x and y can be adiabatically exchanged in the clockwise direction. This process gives another isomorphism, called the anti-braiding of x and y . Clearly the anti-braiding is the inverse of the braiding.



Figure 31: the braiding and anti-braiding of x and y

The family $\{c_{x,y}\}_{x,y \in \mathcal{C}}$ is natural in two variables because instantons can be freely moved along the world lines. More precisely, for any morphisms $f: x \rightarrow x'$ and $g: y \rightarrow y'$, the physical intuition implies the following equation (see Figure 32):

$$c_{x',y'} \circ (f \otimes g) = (g \otimes f) \circ c_{x,y}.$$

Therefore, the braiding is a natural isomorphism $c: \otimes \rightarrow \otimes \circ \tau$, where $\tau: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ is the permutation of two components defined by $\tau(x, y) := (y, x)$. Intuitively, the existence of the braiding means that the tensor product is ‘commutative’, although there are different commutativities given by the braiding or anti-braiding.



Figure 32: the naturality of the braiding

Example 3.4.63. Let us compute the double braiding of e and m in the toric code model. As depicted in Figure 33, an e particle can be moved around an m particle by an e string operator $\prod_j \sigma_z^j$. This string operator is equal to the product of all B_p operators on the plaquettes which are encircled by the path of e , because all σ_z on inner links appear twice in the product. Since there is only one m particle, the action of this string operator is equal to -1 . Consequently, the double braiding $c_{m,e} \circ c_{e,m} : e \otimes m \rightarrow e \otimes m$ is equal to $-\text{id}_{e \otimes m}$. \heartsuit

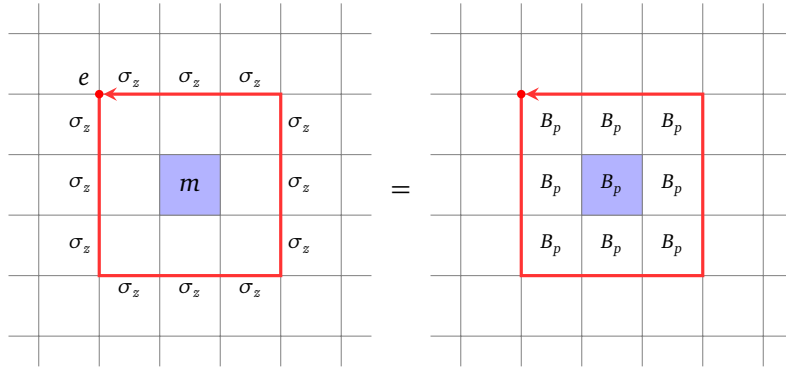


Figure 33: the double braiding of e and m in the toric code model

Now suppose there are three topological defects $x, y, z \in \mathcal{C}$. There are two ways to braid x with y and z : we can either braid x with y and z one by one, or braid x with the fusion $y \otimes z$. These two instantons should be equal because their world lines are homotopy (see Figure 34). Thus we get the following two equations, called the hexagon equations:

$$\begin{aligned} \alpha_{y,z,x} \circ c_{x,y \otimes z} \circ \alpha_{x,y,z} &= (\text{id}_y \otimes c_{x,z}) \circ \alpha_{y,x,z} \circ (c_{x,y} \otimes \text{id}_z), \\ \alpha_{z,x,y}^{-1} \circ c_{x \otimes y,z} \circ \alpha_{x,y,z}^{-1} &= (c_{x,z} \otimes \text{id}_y) \circ \alpha_{x,z,y}^{-1} \circ (\text{id}_x \otimes c_{y,z}). \end{aligned}$$

We summarize the properties of the braiding to the following formal structure.

Definition 3.4.64. A *braided monoidal category* consists of the following data:

- a monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$;
- a natural isomorphism $c_{x,y} : x \otimes y \rightarrow y \otimes x$;

and these data satisfy the following conditions:

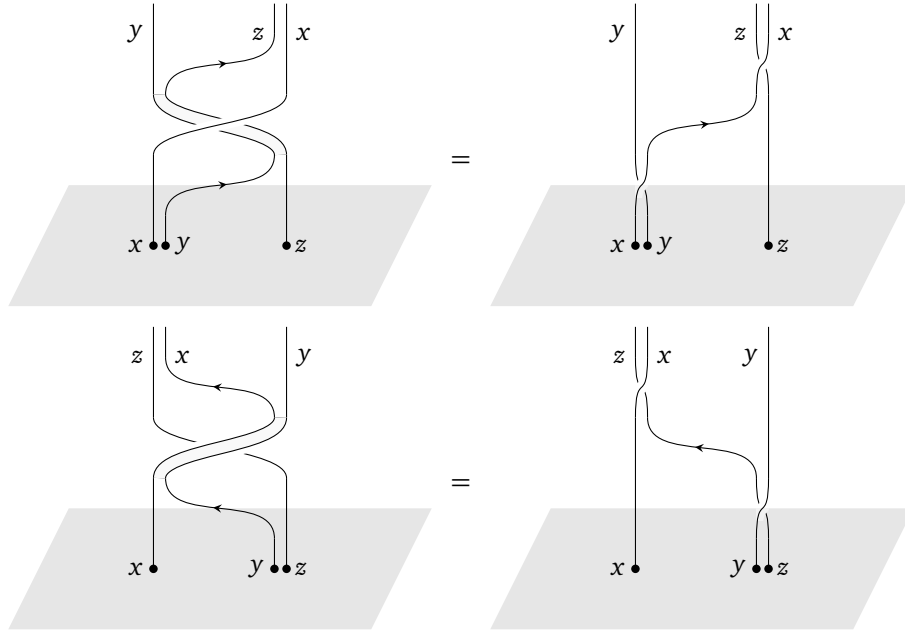


Figure 34: the hexagon equations

1. **(hexagon equation 1)** For any $x, y, z \in \mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccccc}
 & & x \otimes (y \otimes z) & \xrightarrow{c_{x,y \otimes z}} & (y \otimes z) \otimes x \\
 & \nearrow \alpha_{x,y \otimes z} & & & \searrow \alpha_{y \otimes z, x} \\
 (x \otimes y) \otimes z & & & & y \otimes (z \otimes x) \\
 & \searrow c_{x,y} \otimes \text{id}_z & & & \nearrow \text{id}_y \otimes c_{x,z} \\
 & & (y \otimes x) \otimes z & \xrightarrow{\alpha_{y,x \otimes z}} & y \otimes (x \otimes z)
 \end{array}$$

2. **(hexagon equation 2)** For any $x, y, z \in \mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccccc}
 & & (x \otimes y) \otimes z & \xrightarrow{c_{x \otimes y, z}} & z \otimes (x \otimes y) \\
 & \nearrow \alpha_{x,y \otimes z}^{-1} & & & \searrow \alpha_{z, x \otimes y}^{-1} \\
 x \otimes (y \otimes z) & & & & (z \otimes x) \otimes y \\
 & \searrow \text{id}_x \otimes c_{y,z} & & & \nearrow c_{x,z} \otimes \text{id}_y \\
 & & x \otimes (z \otimes y) & \xrightarrow{\alpha_{x,z \otimes y}^{-1}} & (x \otimes z) \otimes y
 \end{array}$$

Moreover, we say \mathcal{C} is a *symmetric monoidal category* if $c_{y,x} \circ c_{x,y} = \text{id}_{x \otimes y}$ for all $x, y \in \mathcal{C}$.

If \mathcal{C} is a \mathbb{C} -linear braided monoidal category equipped with a dagger structure, we say that the dagger structure is *compatible* with the braiding if $c_{x,y}$ is unitary for every $x, y \in \mathcal{C}$. A *unitary braided monoidal category* is a \mathbb{C} -linear braided monoidal category equipped with a compatible unitary structure. ■

Remark 3.4.65. The condition that a dagger structure is compatible with the braiding structure is equivalent to that the functor $\dagger: \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$ is a braided monoidal functor. The notion of a braided monoidal functor is defined in Definition 4.2.9. ◇

As a conclusion, the category \mathcal{C} of particle-like topological defects of a stable 2d topological order \mathcal{C} is a unitary braided fusion category.

Remark 3.4.66. As we have mentioned in Remark 3.4.19, the above discussion of braidings is a radical simplification of the real story. Using adiabatic moves and the property $(*)$ in Remark 3.4.19, we can reduce all different but isomorphic fusion functors $\otimes_{(\xi,\eta)}$ to a single one $\otimes := \otimes_{(0,1)}$. The braiding structure or the double braidings naturally appear because moving one anyon x around another y defines a homotopy non-trivial path in the configuration space. The complete and rigorous proof of the natural emergence of a braided monoidal structure on \mathcal{C} is equivalent to the proof of the well-known mathematical theorem that an E_2 -algebra in the $(2,1)$ -category of categories is a braided monoidal category [SW03, Lur17, Fre17] (see also [Hua08a] in the complex analytic setting for 2D CFT's). A physical work on adiabatic moves and braided monoidal structures can be found in [KL20]. \diamond

Remark 3.4.67. In other words, the braiding structure on the category \mathcal{C} of particle-like topological defects of a 2d topological order \mathcal{C} depends on many artificial choices (see Remark 3.4.20). Hence a topological order does not determine a single braided fusion category \mathcal{C} , but an equivalence class of braided fusion categories. Physicists usually say that the braiding is not ‘gauge invariant’. Only the double braiding $c_{y,x} \circ c_{x,y}$ is gauge invariant and this is because that the initial and final states are exactly the same $x \otimes y$.

In the toric code model, we have known that the double braiding of e and m is -1 (see Example 3.4.63), but there are (at least) two different braiding structures on \mathcal{TC} : one satisfies $c_{e,m} = +1, c_{m,e} = -1$, and the other one satisfies $c_{e,m} = -1, c_{m,e} = +1$.

To determine the braiding structure explicitly, we need to ‘fix the gauge’. One way to fix the gauge is to choose a boundary and use the half-braiding to compute the braiding (see Remark 4.1.2 and Definition 4.2.13). \diamond

Remark 3.4.68. Physically, the braiding is related to the statistics of particle-like topological defects. It is well-known that there are two types of particles: bosons and fermions. They are defined by the phase factor ± 1 of exchanging two particles. Such a phase factor must be ± 1 because exchanging twice returns to the original state. However, this is the story in the space with dimension higher than two. In 2d, the double braiding $c_{y,x} \circ c_{x,y}$ of x and y is not necessarily equal to the identity because the loop of x around y can not contract to a trivial loop (see Figure 30). So in 2d the phase factor of exchanging two particles can be *any* complex number (or even a matrix), which gives a representation of the braid group [Wu84]. This is why a particle-like topological defect in a 2d topological order is called an *anyon* [Wil82a]. *A lot of references need to be added.* \diamond

Remark 3.4.69. Suppose \mathcal{C} is a braided monoidal category. For any $x, y, z \in \mathcal{C}$, we have the following Yang-Baxter equation:

$$(c_{y,z} \otimes \text{id}_x) \circ (\text{id}_y \otimes c_{x,z}) \circ (c_{x,y} \otimes \text{id}_z) = (\text{id}_z \otimes c_{x,y}) \circ (c_{x,z} \otimes \text{id}_y) \circ (\text{id}_x \otimes c_{y,z}).$$

Here we ignore associators. This equation is a direct corollary of the hexagon equations and the naturality of the braiding (for example, see [EGNO15, Proposition 8.1.10]). The physical intuition of the Yang-Baxter equation is depicted in Figure 35. \diamond

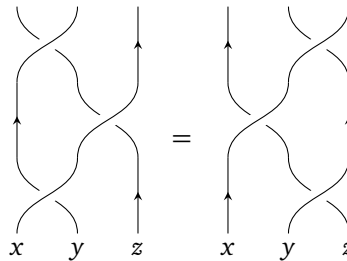


Figure 35: The Yang-Baxter equation

Example 3.4.70. There is a natural braiding on Vec defined by

$$\begin{aligned} c_{V,W} : V \otimes W &\rightarrow W \otimes V \\ v \otimes w &\mapsto w \otimes v. \end{aligned}$$

Moreover, this braiding is symmetric. ♡

Example 3.4.71. Let G be a finite group G . There is a natural braiding on $\text{Rep}(G)$ defined by

$$\begin{aligned} c_{V,W} : V \otimes W &\rightarrow W \otimes V \\ v \otimes w &\mapsto w \otimes v. \end{aligned} \tag{3.4.3}$$

Then $\text{Rep}(G)$ equipped with this braiding structure is a symmetric fusion category.

In general, there are other braiding structures on $\text{Rep}(G)$. Suppose $z \in G$ is a central element (i.e., z commutes with all elements in G) and satisfies $z^2 = e$. For any $(V, \rho) \in \text{Rep}(G)$ we define

$$V_0 := \{v \in V \mid \rho(z)(v) = v\}, \quad V_1 := \{v \in V \mid \rho(z)(v) = -v\}.$$

We denote $|v| = i$ if $v \in V_i$ for $i = 0, 1$. Then both V_0 and V_1 are G -representations and $V = V_0 \oplus V_1$. In other words, any G -representation can be decomposed with respect to the eigenvalues of z , and thus admits a canonical \mathbb{Z}_2 -grading. Then the following morphisms define a braiding structure on $\text{Rep}(G)$:

$$\begin{aligned} c_{V,W} : V \otimes W &\rightarrow W \otimes V \\ v \otimes w &\mapsto (-1)^{|v||w|} \cdot w \otimes v, \end{aligned} \tag{3.4.4}$$

The fusion category $\text{Rep}(G)$ equipped with this braiding structure is denoted by $\text{Rep}(G, z)$. It is clear that $\text{Rep}(G, z)$ is a symmetric fusion category. When $z = e$, the braiding (3.4.4) is the obvious one (3.4.3), and we denote $\text{Rep}(G, e)$ by $\text{Rep}(G)$ for simplicity. Also, $\text{Rep}(\mathbb{Z}_2, z)$ where $z = 1 \in \mathbb{Z}_2$ is the nontrivial element is usually denoted by sVec , because a \mathbb{Z}_2 -graded vector space is also called a *super vector space*. Physically, the symmetric fusion category sVec describes the fermion parity symmetry.

There is a mathematical theorem [Del07, Del02] stated that every symmetric fusion category is equivalent to $\text{Rep}(G, z)$ for some finite group G and central element $z \in G$ satisfying $z^2 = e$. ♡

Example 3.4.72. Let G be a finite group and $\omega \in Z^3(G; \mathbb{C}^\times)$. When G is non-abelian, there is no braiding structure on Vec_G^ω . When G is abelian, the braiding structures on Vec_G^ω can be equivalently described by quadratic forms on G (see Section 3.6.5). ♡

Definition 3.4.73. Let \mathcal{C} be a unitary braided fusion category. For two simple objects $x, y \in \mathcal{C}$, define (see Figure 36)

$$S_{xy} := \text{tr}(c_{y, x^*} \circ c_{x^*, y}) = \text{tr}(c_{y^*, x} \circ c_{x, y^*}).$$

The matrix $S := (S_{xy})_{x, y \in \text{Irr}(\mathcal{C})}$ is called the S matrix of \mathcal{C} . ■

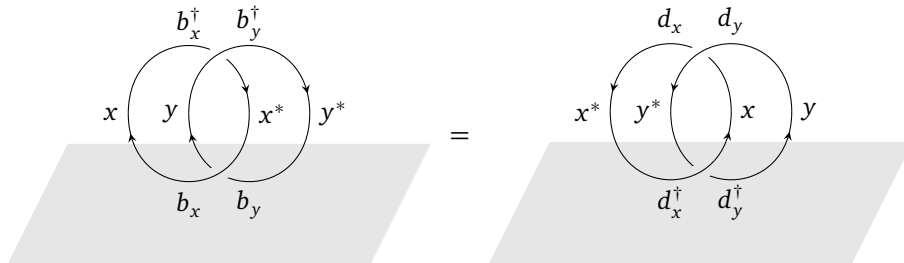


Figure 36: the S matrix S_{xy}

The S matrix encodes the information of mutual-statistics.

Example 3.4.74. In the basis $(\mathbb{1}, e, m, f)$, the S matrix of the toric code model is

$$S = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

♡

Remark 3.4.75. The S matrix can also be defined for a braided fusion category equipped with a spherical structure. ◇

3.4.7 Ribbon structure

Let \mathcal{C} be a unitary braided fusion category and $x \in \mathcal{C}$. Consider the following two morphisms as depicted in Figure 37 (with a normalized choice of duals):

$$x \xrightarrow{\text{id}_x \otimes b_x} x \otimes x \otimes x^* \xrightarrow{c_{x,x} \otimes \text{id}_{x^*}} x \otimes x \otimes x^* \xrightarrow{\text{id}_x \otimes b_x^\dagger} x, \quad (3.4.5)$$

$$x \xrightarrow{d_x^\dagger \otimes \text{id}_x} x^* \otimes x \otimes x \xrightarrow{\text{id}_{x^*} \otimes c_{x,x}} x^* \otimes x \otimes x \xrightarrow{d_x \otimes \text{id}_x} x. \quad (3.4.6)$$

By taking traces one can easily show that these two morphisms are equal. This morphism is denoted by $\theta_x : x \rightarrow x$ and called the twist (or topological spin) of x . The family $\{\theta_x\}_{x \in \mathcal{C}}$ is natural.



Figure 37: the topological spin θ_x of x

Exercise 3.4.76. Use the physical intuition in Figure 37 to prove the following statements:

- (1) Each θ_x is an isomorphism.
- (2) $\theta_x^* = \theta_{x^*}$.
- (3) $\theta_{x \otimes y} = (\theta_x \otimes \theta_y) \circ c_{y,x} \circ c_{x,y}$.

Remark 3.4.77. One may find that two world lines in Figure 37 are homotopy to the trivial paths and conclude that θ_x is just id_x . This is not true. The world line should be viewed as a ribbon (indeed, a line equipped with a framing), not just a line. As depicted in Figure 38, these two process are both equal to twisting the ribbon counterclockwisely. Intuitively, this twisting process is rotating a topological defect by 360 degrees, and that is why we call θ_x the topological spin of x . ◇

For each simple object $x \in \mathcal{C}$, since $\text{Hom}_{\mathcal{C}}(x, x) \simeq \mathbb{C}$ we know that $\theta_x = T_x \cdot \text{id}_x$ for some scalar $T_x \in \mathbb{C}$. The diagonal matrix $T := (T_x \delta_{xy})_{x,y \in \text{Irr}(\mathcal{C})}$ is called the T matrix of the unitary braided fusion category \mathcal{C} . It encodes the information of self-statistics.

Example 3.4.78. Let us compute the topological spin of f in the toric code model. Consider an f ribbon which consists of an e string and an m string (see Figure 39). The f ribbon operator is the product of the e string operator and m string operator. Intuitively, the action of this ribbon operator is rotating the f particle by 360 degrees.

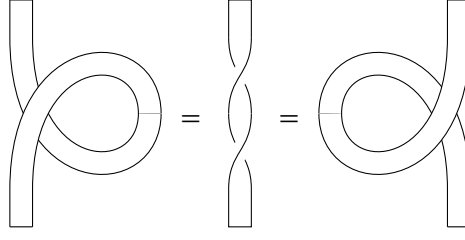


Figure 38: the twist of a ribbon

Note that a closed e string (or m string) operator is equal to the product of B_p (or A_v) operators encircled by the closed loop. Therefore, this ribbon operator is equal to the product of two B_p operators and six A_v operators. Only the upper left A_v operator acts as -1 because there is an f particle, and all other operators act as $+1$. Hence the topological spin of f is -1 .

Similarly we can compute the T matrix of the toric code model:

$$T = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

♡

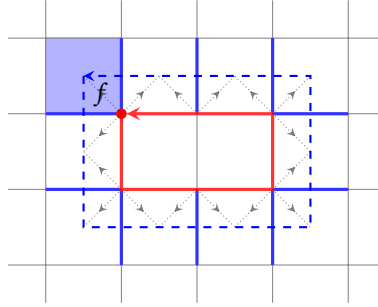


Figure 39: the topological spin of f in the toric code model

Remark 3.4.79. Let \mathcal{C} be a braided fusion category. A *ribbon structure* on \mathcal{C} is a natural isomorphism $\theta_x : x \rightarrow x$ such that $\theta_{x \otimes y} = (\theta_x \otimes \theta_y) \circ c_{y,x} \circ c_{x,y}$ and $\theta_x^* = \theta_{x^*}$. There is a one-to-one correspondence between the ribbon structures and spherical structures on a braided fusion category [Yet92]. By Remark 3.4.58, a unitary braided fusion category is equipped with a canonical ribbon structure [LR97, Müg00] (i.e., the one defined by (3.4.5) or (3.4.6)), and this ribbon structure is compatible with the unitary structure, in the sense that θ_x is unitary. ◇

Definition 3.4.80 ([Bru00]). A *pre-modular tensor category* (or simply a *pre-modular category*) is a braided fusion category equipped with a ribbon structure, or equivalently, a braided fusion category equipped with a spherical structure. ■

In particular, a unitary braided fusion category is automatically a pre-modular category.

Example 3.4.81. Let G be a finite group. Since $\text{Rep}(G)$ is symmetric, $\theta_{(V,\rho)} := \text{id}_V$ defines a ribbon structure on $\text{Rep}(G)$. The corresponding spherical structure on $\text{Rep}(G)$ is the one defined in Example 3.4.60.

More generally, suppose $z \in G$ is a central element satisfying $z^2 = e$. Then the family $\theta_{(V,\rho)} := \rho(z)$ also defines a ribbon structure on $\text{Rep}(G, z)$. The corresponding spherical structure on $\text{Rep}(G, z)$, which is the same as $\text{Rep}(G)$ as a fusion category, is also the one defined in Example 3.4.60. ♡

3.5 Unitary modular tensor categories

3.5.1 Nondegeneracy

We have known that the category \mathcal{C} of particle-like topological defects of a stable 2d topological order C is a unitary braided fusion category. If we require that C is anomaly-free (see Section 2.3), the unitary braided fusion category \mathcal{C} should satisfy a nondegeneracy condition. The following principle first appeared in [Lev13, KW14] (see [KZZZ24, Section 2.1.2] for a recent review).

Principle 3.5.1 (Remote-detectable principle). A topological order is anomaly-free if and only if all topological defects of codimension 2 and higher are able to detect themselves via braidings.

Note that the braiding can only be defined between topological defects of codimension 2 or higher. We focus on 2d topological orders. For a stable 2d topological order C , the braiding can be used to remotely detect topological defects. As depicted in Figure 30, if we want to detect whether there is a nontrivial topological defect $x \in \mathcal{C}$ in a region, we can move a topological defect y around this region and do a double braiding $c_{y,x} \circ c_{x,y}$. If there is at least one $y \in \mathcal{C}$ such that the double braiding is nontrivial, the topological defect x is detectable by this double braiding.

The trivial topological defect $x = \mathbb{1}$ is not detectable in this sense because for all $y \in \mathcal{C}$ the double braiding $c_{y,\mathbb{1}} \circ c_{\mathbb{1},y} = \text{id}_y$ is trivial. However, if all nontrivial simple topological defect $x \neq \mathbb{1}$ are detectable, then $\mathbb{1}$ can also be detected in the following sense: $\mathbb{1}$ is the only simple topological defect which has trivial double braiding with all $y \in \mathcal{C}$.

Hence, the condition that all topological defects are able to detect themselves via braidings can be reformulated as follows: if a simple topological defect $x \in \mathcal{C}$ has trivial mutual statistics with all $y \in \mathcal{C}$, i.e., $c_{y,x} \circ c_{x,y} = \text{id}_{x \otimes y}$ for all $y \in \mathcal{C}$, then $x = \mathbb{1}$ is trivial.

Above discussion naturally motivates the following mathematical definitions.

Definition 3.5.2. Let \mathcal{C} be a braided fusion category. The *Müger center* or *symmetric center* $\mathfrak{Z}_2(\mathcal{C})$ of \mathcal{C} is the full subcategory consisting of the objects which have trivial double braiding with all objects in \mathcal{C} , i.e.,

$$\mathfrak{Z}_2(\mathcal{C}) := \{x \in \mathcal{C} \mid c_{y,x} \circ c_{x,y} = \text{id}_{x \otimes y}, \forall y \in \mathcal{C}\}. \quad \blacksquare$$

It is not hard to see that the Müger center $\mathfrak{Z}_2(\mathcal{C})$ is a symmetric fusion category.

Definition 3.5.3. A braided fusion category \mathcal{C} is called *nondegenerate* if its Müger center $\mathfrak{Z}_2(\mathcal{C})$ contains only one simple object $\mathbb{1}$, or equivalently, $\mathfrak{Z}_2(\mathcal{C}) \simeq \text{Vec}$. ■

Hence the remote-detectable principle 3.5.1 implies the following physical theorem.

Theorem^{ph} 3.5.4. A stable 2d topological order C is anomaly-free if and only if the unitary braided fusion category \mathcal{C} of particle-like topological defects of C is nondegenerate.

The following proposition provides an alternative but equivalent definition of the non-degeneracy condition in Definition 3.5.3. The proof can be found in [Reh90] (see also [Bru00, Müg00, BB01, Müg03b, Müg03c] and a review [Müg12]). This alternative definition is widely used in the literature.

Proposition 3.5.5 ([Reh90]). A premodular category is nondegenerate if and only if its S -matrix is non-degenerate.

Definition 3.5.6. A *modular tensor category* (or a *modular category*) is a nondegenerate pre-modular category. A *unitary modular tensor category* (UMTC) is a nondegenerate unitary braided fusion category. ■

Remark 3.5.7. The notion of a modular tensor category was first introduced by Moore and Seiberg in [MS88, MS89a] as the category of modules over a chiral algebra in a rational 1+1D CFT. The name was suggested by Igor Frenkel and first appeared in [MS90]. Although Moore and Seiberg defined this notion via the polynomial equations of the F -matrices, the R -matrices, the S -matrix and the T -matrix, which were called Moore-Seiberg data in [BK01], they showed in [MS89a, Appendix C] that this data defines a finite semisimple rigid (braided) monoidal category. Moore-Seiberg's equations

are complete¹⁷ but slightly over-determined¹⁸, and were later simplified¹⁹ by Rehren in [Reh90] (see also [BK01, Theorem 3.1.16, 3.1.17]). The modern definition of modular tensor category in purely categorical language was first introduced by Turaev in [Tur20]. We recommend Müger’s review of modular tensor categories [Müg12] for further reading. \diamond

Example 3.5.8. Let us consider a symmetric fusion category equipped with a ribbons structure. Its S matrix satisfies $S_{x,y} = \dim(x)\dim(y)$ and has rank 1. By contrast, the S matrix of a modular tensor category has the maximal rank. Thus symmetric fusion categories and modular tensor categories are two extreme cases of pre-modular categories. \heartsuit

Remark 3.5.9. The complete classification of unitary modular tensor categories with the number of isomorphism classes of simple objects less than 5 can be found in [RSW09]. \diamond

Remark 3.5.10. In the following, a *unitary modular tensor category* can be understood as a nondegenerate pseudo-unitary braided fusion category equipped with the unique spherical structure such that the quantum dimensions of objects are equal to the Frobenius-Perron dimensions (see Remark 3.4.62). \diamond

Finally, we get the following physical theorem which is one of the most important results in this review. It provides a precise and complete summarization of particle-like topological defects in an anomaly-free stable 2d topological order [Wit89, MS89a, FRS89, FG90, Reh90, MR91, Wen91] (see also [Kit06, Appendix E] for a review).

Theorem^{ph} 3.5.11. The particle-like topological defects of an anomaly-free stable 2d topological order \mathcal{C} form a unitary modular tensor category \mathcal{C} .

Remark 3.5.12. If we view anyons in a 2+1D topological order as the Wilson lines in a 2+1D TQFT, then Theorem^{ph} 3.5.11 already follows from the results in [Wit89, MS89a, FRS89, Reh90, FG90]. Its connections to realistic physical systems were studied in [ZHK89, MR91, Wen91]. Most condensed matter physicists learned this fact from Kitaev’s review [Kit06, Appendix E]. \diamond

Remark 3.5.13. Given an anomaly-free stable 2d topological order \mathcal{C} , the unitary modular tensor category \mathcal{C} of particle-like topological defects can be viewed as a categorical description of \mathcal{C} . If \mathcal{C} is not stable, this is not an appropriate description. \diamond

Remark 3.5.14. This unitary braided fusion category summarizes all observables living on a disk (instead of any closed surfaces). For example, a double braiding of topological defects is not well-defined on a sphere or other closed surfaces. Figure 40 illustrate the fact that a double braiding on a sphere is trivial because a loop on a sphere is contractible. \diamond

Remark 3.5.15. The anomaly-free condition for a 2d topological order X is different from that of the TQFT associated to X . More precisely, a UMTC is anomaly-free from the perspective of a 2d topological order; but is anomalous from the perspective of an Atiyah’s TQFT [Ati88] due to the so-called framing anomaly [Tur20]. This subtle difference is due to the fact that the spacetime manifolds considered in a TQFT is much richer than those considered in a topological order. In particular, a topological order can be viewed as a H-type TQFT [MW12, KW14, FH21]. More precisely, the spacetime 3-manifolds in a 2+1D topological order is restricted to the type $\Sigma^2 \times S^1$ (or $\Sigma^2 \times \mathbb{R}^1$), where Σ^2 is an arbitrary spatial 2-manifold and S^1 (or \mathbb{R}^1) is the temporal 1-manifold. \diamond

Remark 3.5.16. There is also a 3+1D generalization of the toric code model [HZW05]. For intrigued readers, it is beneficial to read [KTZ20a] for the ‘modular 2-category’ of topological defects of codimension 2 and higher in the 3+1D toric code model. \diamond

¹⁷Some subsets of data and equations (or axioms) had already appeared in many earlier works. For example, the notion of a braided monoidal category was first introduced by Joyal and Street in 1986 in the preprint of the published version [JS93]. The R -matrices and hexagon identities also appeared or used in some earlier physical literature [Reh88, Fro88, Vaf88].

¹⁸More precisely, Moore-Seiberg’s definition consists of three identities of the S -matrix and the T -matrix, all obtained from different realizations of the same genus-1 surfaces by sewing genus-0 surfaces [MS89a, MS90]. They can be reduced to an explicit expression of the invertible $S(p)$ -matrices, which already appeared in [MS90, Eq. (4.3)] as the unique solution (up to a scalar) of the third Moore-Seiberg’s identity (obtained from genus-1 surfaces with two holes). The axioms can be further reduced to the non-degeneracy of the S -matrix (defined in Definition 3.4.73), where the S -matrix is just the $S(p)$ -matrix in [MS90, Eq. (4.3)] when $p = 1$.

¹⁹Different from Moore-Seiberg’s approach via the representation theory of chiral algebras, Rehren’s results were mainly inspired by the study of superselection sectors with braid group statistics in algebraic quantum field theories [Reh88, FRS89].

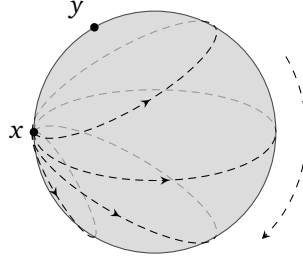


Figure 40: The braiding is not well-defined on a sphere.

3.5.2 Basic properties of modular tensor categories

In the following we list some important properties of a (not necessarily unitary) modular tensor category \mathcal{C} (see [Tur20, BK01, Müg12, EGNO15]):

1. The S matrix satisfies $S_{xy} = S_{yx} = \overline{S_{xy^*}} = \overline{S_{yx^*}}$ and $S_{x\mathbb{1}} = S_{\mathbb{1}x} = d_x := \dim(x)$.
2. For each simple object $x \in \mathcal{C}$, the assignment $h_x: y \mapsto S_{xy}/S_{x\mathbb{1}}$ defines a ring homomorphism $\text{Gr}(\mathcal{C}) \rightarrow \mathbb{C}$, i.e.,

$$S_{xy}S_{xz} = d_x \sum_{w \in \text{Irr}(\mathcal{C})} N_{yz}^w S_{xw}.$$

Note that $h_{\mathbb{1}}(y) = \dim(y)$ is the quantum dimension.

3. We have $S^2 = \dim(\mathcal{C}) \cdot C$, where C is the charge-conjugate matrix defined by $C_{xy} := \delta_{x^*y}$. Thus, $(S^{-1})_{xy} = \dim(\mathcal{C})^{-1} \cdot S_{xy^*}$.
4. The S matrix determines the fusion rule:

$$\sum_{x \in \text{Irr}(\mathcal{C})} \frac{S_{xy}S_{xz}S_{xw^*}}{S_{x\mathbb{1}}} = \dim(\mathcal{C}) \cdot N_{yz}^w.$$

This equation is called the *Verlinde formula* [Ver88, MS89a, MS90, Hua08b].²⁰

5. The *Gauss sums* of \mathcal{C} are defined by

$$\tau^{\pm}(\mathcal{C}) := \sum_{x \in \text{Irr}(\mathcal{C})} T_x^{\pm 1} d_x^2.$$

Then $\tau^+(\mathcal{C})\tau^-(\mathcal{C}) = \dim(\mathcal{C})$. The *multiplicative central charge* of \mathcal{C} is defined by

$$\xi(\mathcal{C}) := \frac{\tau^+(\mathcal{C})}{\sqrt{\dim(\mathcal{C})}} = \frac{\sqrt{\dim(\mathcal{C})}}{\tau^-(\mathcal{C})}.$$

Usually we also write $\xi(\mathcal{C}) = \exp(2\pi i c(\mathcal{C})/8)$, where $c(\mathcal{C})$ is called the *additive central charge* of \mathcal{C} . It can be proved that $\xi(\mathcal{C})$ is a root of unity [AM88, Vaf88], so $c(\mathcal{C}) \in \mathbb{Q}/8\mathbb{Z}$.

6. We have $(ST)^3 = \tau^+(\mathcal{C}) \cdot S^2$. Recall that the group $\text{SL}(2, \mathbb{Z})$, which consists of 2-by-2 integral matrices of determinant 1, is generated by

$$\mathfrak{s} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathfrak{t} := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

subject to the relations $(\mathfrak{st})^3 = \mathfrak{s}^2$ and $\mathfrak{s}^4 = 1$. Hence, the assignment

$$\mathfrak{s} \mapsto \frac{S}{\sqrt{\dim(\mathcal{C})}}, \mathfrak{t} \mapsto T \tag{3.5.1}$$

defines a projective $\text{SL}(2, \mathbb{Z})$ -representation. The obstruction of this projective representation being a linear representation is the multiplicative central charge $\xi(\mathcal{C})$.

²⁰An interesting generalization of the Verlinde formula in the presence of topological defects was found in [SH19].

7. The matrix $(\sqrt{\dim(\mathcal{C})})^{-1} \cdot S$ is unitary [ENO05]. Also each T_x is a root of unity [AM88, Vaf88], thus the T -matrix is also unitary. So for any (not necessarily unitary) modular tensor category \mathcal{C} the above projective $\mathrm{SL}(2, \mathbb{Z})$ -representation is unitary.

Remark 3.5.17. If we put an anomaly-free 2d topological order \mathcal{C} on a torus, the automorphisms of the torus induce transformations of the ground state subspace. This defines a projective representation of the mapping class group of the torus, which is isomorphic to the modular group $\mathrm{SL}(2, \mathbb{Z})$. This representation is isomorphic to the representation (3.5.1) associated to \mathcal{C} and was used to characterize 2d topological orders [Wen90]. These modular representations can also be generalized to higher dimensions [WL14, JMR14, WW15, BD22]. \diamond

Remark 3.5.18. In many lower rank examples, the S and T matrices determine the modular tensor category completely. It provides an efficient way to determine or compute the modular tensor category in many physical applications [Wen17]. In general, however, the structures of modular tensor category are not determined by the S and T matrices [MS21]. Recently, it was shown that the S and T matrices determine the braiding structures of a modular tensor category but not the fusion structures [NRW24]. \diamond

3.5.3 Construct new modular tensor categories from old ones

There are some method to construct new modular tensor categories from old ones.

Definition 3.5.19 ([Del07]). Let \mathcal{C}, \mathcal{D} be two finite semisimple categories. Their *Deligne tensor product*, denoted by $\mathcal{C} \boxtimes \mathcal{D}$, is a finite semisimple category satisfying the following universal property: $\mathcal{C} \boxtimes \mathcal{D}$ is equipped with a \mathbb{C} -bilinear functor $\boxtimes: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{C} \boxtimes \mathcal{D}$, and for any finite semisimple category \mathcal{X} and \mathbb{C} -bilinear functor $F: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{X}$, there exists a unique \mathbb{C} -linear functor $\underline{F}: \mathcal{C} \boxtimes \mathcal{D} \rightarrow \mathcal{X}$ (up to isomorphism) such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{D} & \xrightarrow{\boxtimes} & \mathcal{C} \boxtimes \mathcal{D} \\ & \searrow F & \downarrow \underline{F} \\ & & \mathcal{X} \end{array}$$

or more precisely, the functor $- \circ \boxtimes$ from $\mathrm{Fun}(\mathcal{C} \boxtimes \mathcal{D}, \mathcal{X})$ to the category of \mathbb{C} -bilinear functors from $\mathcal{C} \times \mathcal{D}$ to \mathcal{X} . \blacksquare

This definition may be too abstract. In practice, it is enough to know the following facts:

1. For every $x \in \mathcal{C}$ and $y \in \mathcal{D}$, there is an object $x \boxtimes y \in \mathcal{C} \boxtimes \mathcal{D}$. In general, an object in $\mathcal{C} \boxtimes \mathcal{D}$ is the direct sum of objects of this form.
2. $\mathrm{Hom}_{\mathcal{C} \boxtimes \mathcal{D}}(x \boxtimes y, x' \boxtimes y') \simeq \mathrm{Hom}_{\mathcal{C}}(x, x') \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathcal{D}}(y, y')$ for $x, x' \in \mathcal{C}$ and $y, y' \in \mathcal{D}$.
3. An object $x \boxtimes y \in \mathcal{C} \boxtimes \mathcal{D}$ is simple if and only if $x \in \mathcal{C}$ and $y \in \mathcal{D}$ are both simple. Thus $\mathrm{Irr}(\mathcal{C} \boxtimes \mathcal{D}) \simeq \mathrm{Irr}(\mathcal{C}) \times \mathrm{Irr}(\mathcal{D})$. In particular, $\mathrm{Vec} \boxtimes \mathcal{C} \simeq \mathcal{C} \boxtimes \mathrm{Vec}$.

When \mathcal{C} and \mathcal{D} are both pre-modular categories, their Deligne tensor product $\mathcal{C} \boxtimes \mathcal{D}$ is also a pre-modular category:

- The tensor product of $\mathcal{C} \boxtimes \mathcal{D}$ is defined by

$$(x \boxtimes y) \otimes (x' \boxtimes y') := (x \otimes x') \boxtimes (y \otimes y'), \quad x, x' \in \mathcal{C}, y, y' \in \mathcal{D}.$$

The tensor unit of $\mathcal{C} \boxtimes \mathcal{D}$ is $\mathbb{1} \boxtimes \mathbb{1}$, which is a simple object. The associator and left/right unitor are induced by those of \mathcal{C} and \mathcal{D} .

- The left/right dual of an object $x \boxtimes y \in \mathcal{C} \boxtimes \mathcal{D}$ is $x^{L/R} \boxtimes y^{L/R}$. Thus $\mathcal{C} \boxtimes \mathcal{D}$ is a fusion category. Also we have $\mathrm{FPdim}(x \boxtimes y) = \mathrm{FPdim}(x) \cdot \mathrm{FPdim}(y)$ and $\mathrm{FPdim}(\mathcal{C} \boxtimes \mathcal{D}) = \mathrm{FPdim}(\mathcal{C}) \cdot \mathrm{FPdim}(\mathcal{D})$.
- The spherical structure on $\mathcal{C} \boxtimes \mathcal{D}$ is defined by $a_{x \boxtimes y}^{\mathcal{C} \boxtimes \mathcal{D}} := a_x^{\mathcal{C}} \boxtimes a_y^{\mathcal{D}}$. It follows that $\dim(x \boxtimes y) = \dim(x) \cdot \dim(y)$ and $\dim(\mathcal{C} \boxtimes \mathcal{D}) = \dim(\mathcal{C}) \cdot \dim(\mathcal{D})$.

- The braiding structure on $\mathcal{C} \boxtimes \mathcal{D}$ is defined by

$$c_{x \boxtimes y, x' \boxtimes y'}^{\mathcal{C} \boxtimes \mathcal{D}} = ((x \boxtimes y) \otimes (x' \boxtimes y')) := (x \otimes x') \boxtimes (y \otimes y') \xrightarrow{c_{x, x'}^{\mathcal{C}} \boxtimes c_{y, y'}^{\mathcal{D}}} (x' \otimes x) \boxtimes (y' \otimes y) = (x' \boxtimes y') \otimes (x \boxtimes y).$$

So the S matrix of $\mathcal{C} \boxtimes \mathcal{D}$ is

$$S_{x \boxtimes y, x' \boxtimes y'}^{\mathcal{C} \boxtimes \mathcal{D}} = S_{x, x'}^{\mathcal{C}} \cdot S_{y, y'}^{\mathcal{D}}. \quad (3.5.2)$$

- The ribbon structure on $\mathcal{C} \boxtimes \mathcal{D}$ is defined by $\theta_{x \boxtimes y}^{\mathcal{C} \boxtimes \mathcal{D}} := \theta_x^{\mathcal{C}} \boxtimes \theta_y^{\mathcal{D}}$. So the T matrix of $\mathcal{C} \boxtimes \mathcal{D}$ is

$$T_{x \boxtimes y}^{\mathcal{C} \boxtimes \mathcal{D}} = T_x^{\mathcal{C}} \cdot T_y^{\mathcal{D}}.$$

It is not hard to see that $\mathfrak{Z}_2(\mathcal{C} \boxtimes \mathcal{D}) \simeq \mathfrak{Z}_2(\mathcal{C}) \boxtimes \mathfrak{Z}_2(\mathcal{D})$. Thus $\mathcal{C} \boxtimes \mathcal{D}$ is a modular tensor category if \mathcal{C} and \mathcal{D} are modular. Equivalently, the S matrix of $\mathcal{C} \boxtimes \mathcal{D}$ (3.5.2) is nondegenerate if the S matrices of \mathcal{C} and \mathcal{D} are both nondegenerate.

Exercise 3.5.20. Suppose \mathcal{C} and \mathcal{D} are modular tensor categories. Prove that the Gauss sums satisfy $\tau^\pm(\mathcal{C} \boxtimes \mathcal{D}) = \tau^\pm(\mathcal{C}) \cdot \tau^\pm(\mathcal{D})$, the multiplicative central charge satisfies $\xi(\mathcal{C} \boxtimes \mathcal{D}) = \xi(\mathcal{C}) \cdot \xi(\mathcal{D})$, and the additive central charge satisfies $c(\mathcal{C} \boxtimes \mathcal{D}) = c(\mathcal{C}) \cdot c(\mathcal{D})$.

Physically, the Deligne tensor product corresponds to the stacking of 2d topological orders (see Example 2.1.3). As depicted in Figure 41, suppose x and y are particle-like topological defects in anomaly-free stable 2d topological orders \mathcal{C} and \mathcal{D} , respectively. Then their stacking is a particle-like topological defect, denoted by $x \boxtimes y$, in the stacking topological order $\mathcal{C} \boxtimes \mathcal{D}$. It is believed that the modular tensor category of particle-like topological defects of $\mathcal{C} \boxtimes \mathcal{D}$ is the Deligne tensor product $\mathcal{C} \boxtimes \mathcal{D}$.



Figure 41: stacking of 2d topological orders

Definition 3.5.21. Let \mathcal{C} be a braided monoidal category. Define $\overline{\mathcal{C}}$ to be the same underlying monoidal category of \mathcal{C} equipped with the braiding structure

$$c_{x, y}^{\overline{\mathcal{C}}} := (c_{y, x}^{\mathcal{C}})^{-1}, \quad x, y \in \mathcal{C}. \quad \blacksquare$$

Remark 3.5.22. By definition a braided monoidal category \mathcal{C} is symmetric if and only if $\mathcal{C} = \overline{\mathcal{C}}$. However, it is possible that $\mathcal{C} \simeq \overline{\mathcal{C}}$ as braided monoidal categories but \mathcal{C} is not symmetric. The modular tensor category \mathcal{TC} is such an example. \diamond

When \mathcal{C} is a (pre-)modular tensor category, $\overline{\mathcal{C}}$ is also a (pre-)modular tensor category:

- By definition $\overline{\mathcal{C}}$ is a braided fusion category and $\mathfrak{Z}_2(\overline{\mathcal{C}}) = \mathfrak{Z}_2(\mathcal{C})$.
- The spherical structure on $\overline{\mathcal{C}}$ is the same as \mathcal{C} . In particular, the quantum dimension of objects in $\overline{\mathcal{C}}$ are the same as \mathcal{C} . The S matrix of $\overline{\mathcal{C}}$ is

$$S_{xy}^{\overline{\mathcal{C}}} = S_{xy^*}^{\mathcal{C}}.$$

- Equivalently, the ribbon structure on $\overline{\mathcal{C}}$ is defined by $\theta_x^{\overline{\mathcal{C}}} := (\theta_x^{\mathcal{C}})^{-1}$. The T matrix of $\overline{\mathcal{C}}$ is

$$T_x^{\overline{\mathcal{C}}} = (T_x^{\mathcal{C}})^{-1}.$$

Exercise 3.5.23. Suppose \mathcal{C} is a modular tensor category. Prove that the Gauss sums satisfy $\tau^\pm(\bar{\mathcal{C}}) = \tau^\mp(\mathcal{C})$, the multiplicative central charge satisfies $\xi(\bar{\mathcal{C}}) = \xi(\mathcal{C})^{-1}$, and the additive central charge satisfies $c(\bar{\mathcal{C}}) = -c(\mathcal{C})$.

Physically, $\bar{\mathcal{C}}$ corresponds to the time-reversal of a 2d topological order. Here time-reversal means reversing the orientation of the spacetime. In 0+1D, there is only one way to reverse the orientation of the spacetime, i.e., time-reversal. If \mathcal{C} is the category of particle-like topological defects of a topological order C , then the opposite category \mathcal{C}^{op} (see Definition 3.3.8) is the category of particle-like topological defects of the time-reversal \bar{C} of C .

In 1+1D, there are two ways to reverse the orientation of the spacetime. If we choose to reverse the orientation of the 1d space, the fusion of topological defects is also reversed. The following definition is the precise meaning of reversing the fusion.

Definition 3.5.24. Let \mathcal{C} be a monoidal category. Its *reversed category* \mathcal{C}^{rev} is the same underlying category as \mathcal{C} equipped with the tensor product \otimes^{rev} defined by

$$x \otimes^{\text{rev}} y := y \otimes x, \quad x, y \in \mathcal{C}.$$

Therefore, if \mathcal{C} is the fusion category of particle-like topological defects of a 1d topological order C , then the reversed category \mathcal{C}^{rev} is the fusion category of particle-like topological defects of the time-reversal \bar{C} of C . When \mathcal{C} is a fusion category, \mathcal{C}^{op} is also a fusion category and $\mathcal{C}^{\text{op}} \simeq \mathcal{C}^{\text{rev}}$ as fusion categories.

In 2+1D, there are two ways to reverse the orientation of the 2d space, as depicted in Figure 42. The first way reverses the fusion of topological defects. Another way does not reverse the fusion of topological defects, but reverses the braiding to the anti-braiding. Therefore, if \mathcal{C} is the braided fusion category of particle-like topological defects of a stable 2d topological order C , then $\bar{\mathcal{C}}$ is the braided fusion category of particle-like topological defects in the time-reversal \bar{C} of C . When \mathcal{C} is a modular tensor category, both \mathcal{C}^{op} and \mathcal{C}^{rev} are also modular tensor categories and $\mathcal{C}^{\text{op}} \simeq \mathcal{C}^{\text{rev}} \simeq \bar{\mathcal{C}}$ as modular tensor categories.

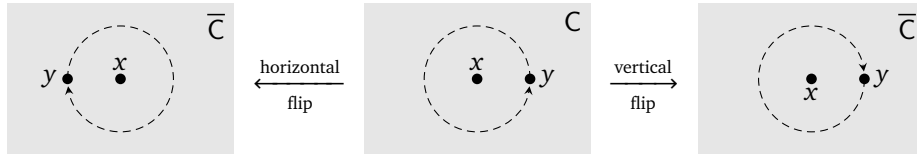


Figure 42: two ways to reverse the orientation of the 2d space

3.6 Examples of (unitary) modular tensor categories

In this subsection we give some examples of (unitary) modular tensor categories.

3.6.1 The categories Vec and Hilb

The simplest modular tensor category is the category Vec of finite-dimensional vector spaces. Its structure of a modular tensor category has been defined in Section 3.4. We list the structure in the following:

- The objects of Vec are finite-dimensional vector spaces (over \mathbb{C}), and the morphisms are linear maps. It is a finite semisimple category, and the only simple object (up to isomorphism) is \mathbb{C} .
- The tensor product of Vec is the usual tensor product $\otimes_{\mathbb{C}}$ of vector spaces. The associator and left/right unitor are induced by the universal property of $\otimes_{\mathbb{C}}$.
- Both the left and right dual of a finite-dimensional vector space V are given by the usual dual space $V^* := \text{Hom}(V, \mathbb{C})$.

- The spherical structure is

$$\begin{aligned} a_V : V &\rightarrow V^{**} \\ v &\mapsto \text{ev}_v \end{aligned}$$

where $\text{ev}_v : V^* \rightarrow \mathbb{C}$ is defined by $\text{ev}_v(\phi) := \phi(v)$.

- Both the quantum dimension and Frobenius-Perron dimension are equal to the usual dimension of vector spaces.
- The braiding structure is given by

$$\begin{aligned} c_{V,W} : V \otimes W &\rightarrow W \otimes V \\ v \otimes w &\mapsto w \otimes v. \end{aligned}$$

Note that Vec is also a symmetric fusion category.

- The ribbon structure is given by $\theta_V = \text{id}_V$.

The unitary version of Vec is the Hilb, which describes the particle-like topological defects of the trivial 2d topological order.

3.6.2 The category \mathcal{TC}

In Section 3.4 we have computed the unitary modular tensor category \mathcal{TC} of particle-like topological defects of the toric code model. We list the structure of \mathcal{TC} in the following:

- There are only four simple objects $\mathbb{1}, e, m, f$.
- The fusion rules are given by $e \otimes m = f = m \otimes e$ and $e \otimes e = m \otimes m = f \otimes f = \mathbb{1}$. This implies that each simple objects is self-dual and has Frobenius-Perron dimension 1. The associator and left/right unitor are identities.
- The S matrix is

$$S = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

- The T matrix is

$$T = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

As explained in Remark 3.4.20 and 3.4.67, the lattice model only determines \mathcal{TC} up to equivalence.

3.6.3 Quantum double category of a finite group

For any finite group G there is a (unitary) modular tensor category \mathcal{D}_G , called the quantum double category of G :

- An object in \mathcal{D}_G is a vector space V equipped with a G -action $\rho : G \rightarrow \text{GL}(V)$ and a G -grading $V = \bigoplus_{g \in G} V_g$, such that $\rho(g)$ maps V_h to $V_{ghg^{-1}}$ for every $g, h \in G$. A morphism in \mathcal{D}_G is a \mathbb{C} -linear map that is both a morphism in $\text{Rep}(G)$ and a morphism in Vec_G .
- Given an element $g \in G$, we denote the conjugacy class of g by $[g]$ and the centralizer of g by $Z(g)$. If π is an irreducible $Z(g)$ -representation, the induced representation $\text{Ind}_{Z(g)}^G(\pi) =: X_{(g,\pi)}$ is a G -representation and admits a canonical $G/Z(g) \simeq [g]$ -grading. This $[g]$ -grading on $X_{(g,\pi)}$ can be viewed as a G -grading with h -components being 0 for all $h \notin [g]$. Then one can verify that $X_{(g,\pi)}$ is a simple object in \mathcal{D}_G . Conversely, every simple object in \mathcal{D}_G is isomorphic to $X_{(g,\pi)}$ for some $g \in G$ and $\pi \in \text{Irr}(\text{Rep}(Z(g)))$. The isomorphism classes of simple objects of \mathcal{D}_G are labeled by the pairs $([g], [\rho])$ where $[g]$ is a conjugacy class in G and $[\rho]$ is an isomorphism class of irreducible representation of $Z(g)$. Note that $Z(g)$ only depends on the conjugacy class of g .

- The tensor product of two objects in \mathcal{D}_G is the usual tensor product as a G -representation, equipped with the usual tensor product G -grading. In particular, the tensor unit of \mathcal{D}_G is $\mathbb{C}_{(e)}$ equipped with the trivial G -action.
- The dual of an object in \mathcal{D}_G is the usual dual representation equipped with the usual dual G -grading. In particular, the dual of $X_{(g,\pi)}$ is $X_{(g^{-1},\pi^*)}$, where π^* is the dual representation of π . Note that $Z(g) = Z(g^{-1})$.
- The spherical structure on \mathcal{D}_G is given by that on Vec . Thus the Frobenius-Perron dimension and quantum dimension are both equal to the usual dimension of a vector space. In particular, both two dimension of $X_{(g,\pi)}$ are equal to $||g|| \cdot \dim(\pi)$. The Frobenius-Perron dimension and quantum dimension of \mathcal{D}_G are both equal to $|G|^2$.
- The braiding $c_{(V,\rho),(W,\sigma)}: V \otimes_{\mathbb{C}} W \rightarrow W \otimes_{\mathbb{C}} V$ is defined by $v \otimes_{\mathbb{C}} w \mapsto \sigma(g)(y) \otimes_{\mathbb{C}} x$ for $v \in V_g$ and $w \in W$. The S matrix is given by

$$S_{([g],[\pi]),([g'],[\pi'])} = \frac{|G|}{|Z(g)||Z(g')|} \sum_{\{h \in G | hg'h^{-1} \in Z(g)\}} \text{tr}_{\pi}(h(g')^{-1}h^{-1}) \text{tr}_{\pi'}(h^{-1}g^{-1}h).$$

- The ribbon structure on \mathcal{D}_G is defined by $\theta_{(V,\rho)}(v) := \rho(g)(v)$ for $v \in V_g$. The T matrix is given by

$$T_{([g],[\pi])} = \frac{\text{tr}_{\pi}(g)}{\text{tr}_{\pi}(e)}.$$

- The multiplicative central charge of \mathcal{D}_G is 1 and the additive central charge is 0 (mod 8).

Remark 3.6.1. Let G be a finite group. There is an exactly solvable 2d lattice model associated to G , called Kitaev's quantum double model [Kit03]. The unitary modular tensor category of particle-like topological defects of Kitaev's quantum double model is equivalent to \mathcal{D}_G . When $G = \mathbb{Z}_2$, it is the toric code model. In particular, \mathcal{TC} is equivalent to $\mathcal{D}_{\mathbb{Z}_2}$ as a modular tensor category. \diamond

Remark 3.6.2. Let G be a finite group. There is a quasi-triangular Hopf algebra $D(G)$ associated to G , called the quantum double algebra of G . The category of finite-dimensional representations of $D(G)$ is equivalent to \mathcal{D}_G as modular tensor categories (see for example [Kas95, EGNO15]). Moreover, $D(G)$ is the minimal local operator algebra of Kitaev's quantum double model (see Section 3.3.7). \diamond

Remark 3.6.3. The low energy effective theory of the quantum double model associated to a finite group G is the 2+1D Dijkgraaf-Witten theory [DW90] of G with the trivial 3-cocycle. \diamond

Remark 3.6.4. Kitaev's quantum double lattice model associated to a group can be generalized to a lattice model associated to a (weak) Hopf algebra [BMCA13, BK12, Cha14]. \diamond

3.6.4 Ising type categories

There is an important family of (unitary) modular tensor categories called Ising type (unitary) modular tensor categories.

An Ising type fusion category is a fusion category with three simple objects $\mathbb{1}, \sigma, \psi$ and fusion rules

$$\psi \otimes \psi = \mathbb{1}, \psi \otimes \sigma = \sigma = \sigma \otimes \psi, \sigma \otimes \sigma = \mathbb{1} \oplus \psi.$$

An Ising type modular tensor category is a modular tensor category whose underlying fusion category is of Ising type. We list the classification of Ising type modular tensor categories as follows [DGNO10]:

- The fusion rules imply that each simple object in an Ising type fusion category is self-dual, and $\text{FPdim}(\mathbb{1}) = \text{FPdim}(\psi) = 1$, $\text{FPdim}(\sigma) = \sqrt{2}$.

- There are 2 Ising type fusion categories (up to equivalence) labeled by a sign \pm . Their nonzero associators are given by

$$\begin{aligned}\alpha_{\psi,\sigma,\psi} &: (\psi \otimes \sigma) \otimes \psi = \sigma \xrightarrow{-1} \sigma = \psi \otimes (\sigma \otimes \psi), \\ \alpha_{\sigma,\psi,\sigma} &: (\sigma \otimes \psi) \otimes \sigma = \mathbb{1} \oplus \psi \xrightarrow{1 \oplus -1} \mathbb{1} \oplus \psi = \sigma \otimes (\psi \otimes \sigma), \\ \alpha_{\sigma,\sigma,\sigma} &: (\sigma \otimes \sigma) \otimes \sigma = (\mathbb{1} \otimes \sigma) \oplus (\psi \otimes \sigma) \xrightarrow{A_{\pm}} (\sigma \otimes \mathbb{1}) \oplus (\sigma \otimes \psi) = \sigma \otimes (\sigma \otimes \sigma),\end{aligned}$$

where

$$A_{\pm} = \frac{\pm 1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

is the only difference between these two monoidal structures. The Ising type fusion categories are also the \mathbb{Z}_2 Tambara-Yamagami categories [TY98].

- Each Ising type fusion category has 4 different braiding structures (up to equivalence), so there are 8 different Ising type braided fusion categories (up to equivalence). These braided fusion categories are labeled by a complex number ζ satisfying $\zeta^8 = -1$, and the underlying Ising type fusion categories are determined by the sign \pm in the following equation:

$$\zeta^2 + \zeta^{-2} = \pm \sqrt{2}.$$

Explicitly, the braiding structures of Ising type braided fusion categories are given by

$$\begin{aligned}c_{\psi,\psi} &: \psi \otimes \psi = \mathbb{1} \xrightarrow{-1} \mathbb{1} = \psi \otimes \psi, \\ c_{\sigma,\sigma} &: \sigma \otimes \sigma = \mathbb{1} \oplus \psi \xrightarrow{\zeta \oplus \zeta^{-3}} \mathbb{1} \oplus \psi = \sigma \otimes \sigma, \\ c_{\psi,\sigma} &: \psi \otimes \sigma = \sigma \xrightarrow{\zeta^4} \sigma = \sigma \otimes \psi, \\ c_{\sigma,\psi} &: \sigma \otimes \psi = \sigma \xrightarrow{\zeta^4} \sigma = \psi \otimes \sigma.\end{aligned}$$

The other braidings are identities. Since $c_{\psi,\sigma} \circ c_{\sigma,\psi} = \zeta^8 = -1$, all Ising type braided fusion categories have the trivial Müger center and thus are nondegenerate.

- For each Ising type braided fusion category, there are 2 different spherical structures, labeled by a number $\epsilon = \pm 1$. Their difference is the quantum dimension

$$\dim(\mathbb{1}) = \dim(\psi) = 1, \dim(\sigma) = \epsilon(\zeta^2 + \zeta^{-2}).$$

So there are 16 different Ising type modular tensor categories (up to equivalence) labeled by a pair (ζ, ϵ) , and only 8 of them with $\dim(\sigma) = +\sqrt{2}$ are unitary. The S matrix of an Ising type modular category is

$$S = \begin{pmatrix} 1 & 1 & \epsilon(\zeta^2 + \zeta^{-2}) \\ 1 & 1 & -\epsilon(\zeta^2 + \zeta^{-2}) \\ \epsilon(\zeta^2 + \zeta^{-2}) & -\epsilon(\zeta^2 + \zeta^{-2}) & 0 \end{pmatrix}.$$

If we only consider Ising type unitary modular tensor categories, the S matrix is

$$S = \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}.$$

- The T matrix of an Ising type modular category is

$$T = \begin{pmatrix} 1 & & \\ & -1 & \\ & & \epsilon \zeta^{-1} \end{pmatrix}.$$

- The multiplicative central charge of an Ising type modular category is equal to $T_\sigma = \epsilon\zeta^{-1}$. All 8 Ising type unitary modular tensor categories have different additive central charges, so we can label them by their additive central charge $c = 1/2, \dots, 15/2 \pmod{8}$.

Remark 3.6.5. The well-known Ising vertex operator algebra, which appears in the minimal model $\mathcal{M}(4,3)$, has central charge $1/2$. Its module category is the Ising type unitary modular tensor category with additive central charge $1/2 \pmod{8}$, i.e., the one corresponding to $\zeta = \exp(-2\pi i/16)$ and $\epsilon = +1$. \diamond

Remark 3.6.6. In a 2+1D topological superconductor (which can be viewed as an SPT order with only the fermion parity symmetry), the particle-like topological defects form a symmetric fusion category equivalent to sVec (see Example 3.4.71). These particle-like topological defects, together with vortices form a unitary modular tensor category \mathcal{M} which contains sVec . The well-known Kitaev's 16-fold way states that \mathcal{M} depends on the Chern number ν modulo 16 [Kit06]. When ν is odd, \mathcal{M} is the Ising type unitary modular tensor category with additive central charge $\nu/2 \pmod{8}$; when ν is even, \mathcal{M} is a pointed unitary modular tensor category (see Section 3.6.5). All 16 different unitary modular tensor categories are minimal modular extensions of sVec [Kit06, DGNO10]. \diamond

3.6.5 Pointed modular tensor categories and metric groups

Theorem 3.6.7. Let \mathcal{C} be a fusion category and $x \in \mathcal{C}$. Then the following statements are equivalent:

- The creation morphism $b_x: \mathbb{1} \rightarrow x \otimes x^L$ and annihilation morphism $d_x: x^L \otimes x \rightarrow \mathbb{1}$ are both isomorphisms.
- There exists an object $y \in \mathcal{C}$ such that $x \otimes y \simeq \mathbb{1} \simeq y \otimes x$.
- $\text{FPdim}(x) = 1$.

Proof. Clearly (a) implies (b). Suppose there exists an object $y \in \mathcal{C}$ such that $x \otimes y \simeq \mathbb{1} \simeq y \otimes x$. Then the matrix N_x of left multiplication by x is a permutation matrix, whose largest non-negative eigenvalue must be 1. Thus (b) implies (c). To prove that (c) implies (a), note that $\text{FPdim}(x^L) = \text{FPdim}(x)$ (because the matrix N_{x^L} is the transpose of N_x) and thus $\text{FPdim}(x \otimes x^L) = \text{FPdim}(x)\text{FPdim}(x^L) = 1$. Then the nonzero morphisms b_x and d_x have to be isomorphisms. \square

Definition 3.6.8. Let \mathcal{C} be a fusion category. An object in \mathcal{C} is called *invertible* if it satisfies one (and hence all) of the conditions in Theorem 3.6.7. We say \mathcal{C} is *pointed* if every simple object in \mathcal{C} is invertible. \blacksquare

Given a pointed fusion category \mathcal{C} , its isomorphism classes of simple object form a finite group $\text{Irr}(\mathcal{C}) =: G$ under the tensor product. Thus $\mathcal{C} \simeq \text{Vec}_G$ as categories and the tensor product functor of \mathcal{C} is the same as Vec_G . Moreover, the pentagon equation implies that the associator is determined by a 3-cocycle $\omega \in Z^3(G; \mathbb{C}^\times)$ (see Example 3.4.24). Thus $\mathcal{C} \simeq \text{Vec}_G^\omega$ as fusion categories.

Now suppose \mathcal{C} is a pointed braided fusion category. As a fusion category we can assume that $\mathcal{C} = \text{Vec}_G^\omega$ for some finite group G and $\omega \in Z^3(G; \mathbb{C}^\times)$. The existence of braiding implies that G is an abelian group. Similar to the associator, the braiding structure is determined by a function $c: G \times G \rightarrow \mathbb{C}^\times$ that satisfies the following two hexagon equations for any $g, h, k \in G$:

$$\begin{aligned} \omega(g, h, k)c(g, hk)\omega(h, k, g) &= c(g, h)\omega(h, g, k)c(g, k), \\ \omega(g, h, k)^{-1}c(gh, k)\omega(k, g, h)^{-1} &= c(h, k)\omega(g, k, h)^{-1}c(g, k). \end{aligned}$$

Such a pair of functions (ω, c) is called an *abelian 3-cocycle*. This braided fusion category is also denoted by $\text{Vec}_G^{(\omega, c)}$.

In the following we give another description of pointed braided fusion categories.

Definition 3.6.9. Let G be an abelian group. A *quadratic form* on G is a map $q: G \rightarrow \mathbb{C}^\times$ such that $q(g) = q(g^{-1})$ and the symmetric function $b: G \times G \rightarrow \mathbb{C}^\times$ defined by

$$b(g, h) := \frac{q(gh)}{q(g)q(h)}$$

is a *bicharacter*, i.e., $b(g_1 g_2, h) = b(g_1, h) b(g_2, h)$ for all $g_1, g_2, h \in G$. The symmetric bicharacter b is called the *associated bicharacter* of q . We say q is *nondegenerate* if b is, i.e., $b(g, h) = 1$ for all $h \in G$ if and only if $g = e$ is the unit.

A *pre-metric group* is a finite abelian group G equipped with a quadratic form q . If q is nondegenerate, then (G, q) is called a *metric group*. ■

Consider a pointed braided fusion category \mathcal{C} . By the above discussion, we can assume that $\mathcal{C} = \text{Vec}_G^{(\omega, c)}$. Define

$$q(g) := c(g, g), \quad g \in G.$$

It follows from the pentagon and hexagon equations that q is a quadratic form and the associated bicharacter is

$$b(g, h) := c(g, h) c(h, g), \quad g, h \in G.$$

Then (G, q) is called the *associated pre-metric group* of \mathcal{C} .

Conversely, two pointed braided fusion categories are equivalent if and only if their associated pre-metric groups are isomorphic [JS93, Qui99, DGNO10]. In other words, for any pre-metric group (G, q) there exists a unique (up to equivalence) pointed braided fusion category, denoted by $\mathcal{C}(G, q)$, such that its associated pre-metric group is isomorphic to (G, q) .

Exercise 3.6.10. Let (G, q) be a pre-metric group. Prove that $\mathcal{C}(G, q)$ is symmetric if and only if $q: G \rightarrow \mathbb{C}^\times$ is a group homomorphism satisfying $q(g) = \pm 1$ for all $g \in G$. Furthermore, in this case we have $\mathcal{C}(G, q) \simeq \text{Rep}(\hat{G}, q)$ as symmetric fusion categories, where $\hat{G} := \text{Hom}(G, \mathbb{C}^\times)$ is the dual group of G .

There is a unique spherical structure on $\mathcal{C}(G, q)$ such that the quantum dimension of every simple object is 1 (see Remark 3.4.62). Thus $\mathcal{C}(G, q)$ is a pre-modular category. Its S matrix is given by

$$S_{g, h} = b(g, h^{-1}) = b(g, h^{-1}),$$

and the T matrix is given by

$$T_g = q(g).$$

Hence $\mathcal{C}(G, q)$ is a modular tensor category if and only if q is nondegenerate, i.e., (G, q) is a metric group.

Remark 3.6.11. An anomaly-free 2d topological order \mathcal{C} is called an *abelian topological order* if the modular tensor category \mathcal{C} of particle-like topological defects is pointed. This name is because the mutual-statistics, i.e., the braiding $c_{x, y}$ for simple topological defects x, y , are complex numbers (abelian phase), not matrices (non-abelian phase). ◇

Remark 3.6.12. The toric code model realizes an abelian topological order. Mathematically, \mathcal{TC} is a pointed modular tensor category and the corresponding metric group is $(\mathbb{Z}_2 \times \mathbb{Z}_2, q)$ where $q(a, b) := (-1)^{ab}$, $a, b \in \{0, 1\} = \mathbb{Z}_2$. More generally, the quantum double category \mathcal{D}_A of a finite abelian group A is a pointed modular tensor category and the corresponding metric group is $(A \times \hat{A}, q)$, where $\hat{A} := \text{Hom}(A, \mathbb{C}^\times)$ is the dual group of A and the quadratic form q is defined by $q(a, \chi) := \chi(a)$. ◇

Remark 3.6.13. Physically, every anomaly-free 2d abelian topological order can be realized by a 2+1D U(1) Chern-Simons theory [BW90a, BW90b, BW91, Rea90]:

$$S = \frac{1}{4\pi} \int K_{IJ} a^I \wedge da^J, \quad (3.6.1)$$

where a^I for $I = 1, \dots, n$ are U(1) gauge fields (1-forms), and the n -by- n matrix $K = (K_{IJ})$ satisfies the following conditions:

1. All components of K are integers, and the diagonal components of K are even.
2. K is symmetric.
3. The determinant of K is non-zero.

Mathematically, K can be equivalently replaced by a nondegenerate even integral lattice L . There is a metric group (G, q) associated to such a matrix K . First, the matrix K can be viewed as a homomorphism $K: \mathbb{Z}^n \rightarrow \mathbb{Z}^n$, and we define $G := \text{coker}(K) = \mathbb{Z}^n / \text{im}(K)$, called the *discriminant group* of K . In other words, G is the group of equivalence classes of integral column vectors where the equivalence relation is generated by $v \sim v + Kl$. In the language of lattices, G is the quotient L^\vee / L , where $L^\vee := \text{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ is the dual of L . The quadratic form q is defined by

$$q([v]) := \exp(\pi i \cdot v^T K^{-1} v), \quad v \in \mathbb{Z}^n.$$

The associated bicharacter is

$$b([v], [w]) = \exp(2\pi i \cdot v^T K^{-1} w), \quad v, w \in \mathbb{Z}^n.$$

It is not hard to verify that q is nondegenerate. The category of particle-like topological defects of the topological order realized by (3.6.1) is equivalent to the pointed modular tensor category $\mathcal{C}(G, q)$ [WZ92, BW90b].

Some properties of $\mathcal{C}(G, q)$ can be obtained directly from the matrix K . The number of isomorphism classes of simple objects is $|G| = |\det K|$. The additive central charge of $\mathcal{C}(G, q)$ is equal to the number of positive eigenvalues of K minus the number of negative eigenvalues modulo 8 [Mil74] (see also [MH73, Appendix 4] for a proof).

Different K matrices are said to be equivalent if they realize physically equivalent theories. The equivalence relation is generated by:

- (a) $K \sim PKP^{-1}$ where $P \in \text{GL}(n, \mathbb{Z})$. This relation means the basis change of gauge fields is irrelevant.
- (b) $K \sim K \oplus \sigma_x$. The matrix σ_x realizes the trivial 2d topological order. So this relation means the trivial 2d topological order is the unit under stacking.

Every metric group can be realized as the discriminant group of a K matrix [Wal63]. As an example, the modular tensor category \mathcal{TC} can be constructed from

$$K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

In general, different K matrices may have isomorphic discriminant groups. ◇

3.6.6 Categories associated to the Lie algebra \mathfrak{sl}_2

Let us recall the basic representation theory of the Lie algebra $\mathfrak{sl}_2 := \mathfrak{sl}(2, \mathbb{C})$. The category $\text{Rep}(\mathfrak{sl}_2)$ of finite-dimensional representations of \mathfrak{sl}_2 is almost a fusion category, except that there are infinite simple objects up to isomorphism. Indeed, for any $n \in \mathbb{N}$, there exists a unique (up to isomorphism) irreducible representation of dimension $(n+1)$, denoted by V_n . Usually V_n is called the spin- $(n/2)$ representation. The tensor product of two irreducible representations is decomposed as the direct sum of irreducible representations by the Clebsch-Gordan rule:

$$V_n \otimes V_m = \bigoplus_{i=0}^{\min\{n,m\}} V_{n+m-2i} = V_{|n-m|} \oplus V_{|n-m|+2} \oplus \cdots \oplus V_{n+m}.$$

We may expect to ‘truncate’ $\text{Rep}(\mathfrak{sl}_2)$ to get a fusion category. One way is to ‘deform’ the enveloping algebra of \mathfrak{sl}_2 and consider certain representations of the deformed algebra.

Let $k \in \mathbb{N}$ be a non-negative integer (called the level). For every integer $a \in \mathbb{Z}$ such that the greatest common divisor of a and $(k+2)$ is 1, let $q = e^{a\pi i/(k+2)}$ (i.e., q^2 is a primitive $(k+2)$ -th root of unity) and t be a square root of q . Then there is an associated pre-modular category denoted by $\mathcal{C}(\mathfrak{sl}_2, q, t)$ [RT91]. We list some data in the following:

- There are $(k+1)$ simple objects (up to isomorphism) denoted by V_0, \dots, V_k .
- The fusion rule is given by the truncated Clebsch-Gordan rule:

$$V_n \otimes V_m = \bigoplus_{i=\max\{n+m-k, 0\}}^{\min\{n,m\}} V_{n+m-2i} = \begin{cases} V_{|n-m|} \oplus V_{|n-m|+2} \oplus \cdots \oplus V_{n+m}, & n+m \leq k; \\ V_{|n-m|} \oplus V_{|n-m|+2} \oplus \cdots \oplus V_{2k-(n+m)}, & n+m > k. \end{cases}$$

In other words, the fusion rule is the same as the usual Clebsch-Gordan rule if $n + m \leq k$; when $n + m > k$, the undefined terms (V_i with $i > k$) are deleted and also their mirror images with respect to the point $(k + 1)$. The tensor unit is V_0 and each V_n is self-dual.

- Define

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \dots + q^{-n+1}.$$

The Frobenius-Perron dimension of V_n is $\text{FPdim}(V_n) = [n + 1]_{q_1}$, where $q_1 := e^{\pi i/(k+2)}$. The Frobenius-Perron dimension of the fusion category $\mathcal{C}(\mathfrak{sl}_2, q, t)$ is

$$\text{FPdim}(\mathcal{C}(\mathfrak{sl}_2, q, t)) = \frac{k + 2}{2 \sin^2(\frac{\pi}{k+2})}.$$

The quantum dimension of V_n is $\dim(V_n) = [n + 1]_q$, and the quantum dimension of $\mathcal{C}(\mathfrak{sl}_2, q)$ is

$$\dim(\mathcal{C}(\mathfrak{sl}_2, q, t)) = \frac{k + 2}{2 \sin^2(\frac{a\pi}{k+2})}.$$

- The S matrix is

$$S_{nm} := S_{V_n, V_m} = [(n + 1)(m + 1)]_q = \frac{\sin(\frac{a(n+1)(m+1)\pi}{k+2})}{\sin(\frac{a\pi}{k+2})}.$$

- The T matrix is

$$T_n := T_{V_n} = t^{n(n+2)}.$$

The underlying fusion category of $\mathcal{C}(\mathfrak{sl}_2, q, t)$ only depends on q but not t . The pre-modular category $\mathcal{C}(\mathfrak{sl}_2, q, t)$ is modular if and only if a is odd [Bru00]. In particular, when k is even, a must be odd and hence $\mathcal{C}(\mathfrak{sl}_2, q, t)$ is modular.

Remark 3.6.14. Let us prove that $\mathcal{C}(\mathfrak{sl}_2, q, t)$ is modular if and only if a is odd by direct computations. We have

$$\begin{aligned} \sin\left(\frac{a\pi}{k+2}\right)^2 \sum_{l=0}^k S_{nl} S_{ml} &= \sum_{l=1}^{k+1} \sin\left(\frac{a(n+1)l\pi}{k+2}\right) \sin\left(\frac{a(m+1)l\pi}{k+2}\right) \\ &= -\frac{1}{2} \sum_{l=0}^{k+1} \left(\cos\left(\frac{a(n+m+2)l\pi}{k+2}\right) - \cos\left(\frac{a(n-m)l\pi}{k+2}\right) \right). \end{aligned}$$

Note that for $\theta = ar\pi/(k+2)$ where $r \in \mathbb{Z}$, we have

$$\begin{aligned} \sum_{l=0}^{k+1} \cos(l\theta) &= \text{Re}\left(\sum_{l=0}^{k+1} e^{il\theta}\right) \\ &= \begin{cases} k+2, & e^{i\theta} = 1, \\ \text{Re}\left(\frac{1-e^{i(k+2)\theta}}{1-e^{i\theta}}\right), & e^{i\theta} \neq 1, \end{cases} \\ &= \begin{cases} k+1, & e^{i\theta} = 1, \\ 0, & e^{i\theta} \neq 1, ar \text{ even}, \\ \text{Re}\left(\frac{2}{1-e^{i\theta}}\right) = 1, & e^{i\theta} \neq 1, ar \text{ odd}. \end{cases} \end{aligned}$$

Thus when a is odd we have

$$\sum_{l=0}^k S_{nl} S_{ml} = \sin\left(\frac{a\pi}{k+2}\right)^{-2} \cdot \frac{k+2}{2} \delta_{n,m} = \dim(\mathcal{C}(\mathfrak{sl}_2, q, t)) \delta_{n,m}.$$

This orthogonality relation implies the nondegeneracy of the S matrix. When a is even we have $S_{n,m} = -S_{n,k-m}$, thus the S matrix is degenerate. \diamond

Remark 3.6.15. Note that $V_k \otimes V_k = V_0$ and the only invertible objects in $\mathcal{C}(\mathfrak{sl}_2, q)$ are V_0 and V_k (up to isomorphism). The fusion subcategory generated by V_0 and V_k is denoted by $\mathcal{C}(\mathfrak{sl}_2, q, t)^\times$. We have

$$\dim(V_k) = [k+1]_q = (-1)^{a+1},$$

and

$$T_k = t^{k(k+2)}.$$

Hence, when k is even, the pre-modular category $\mathcal{C}(\mathfrak{sl}_2, q, t)^\times$ only depends on q but not t . It is symmetric and equipped with the canonical ribbon structure (see Example 3.4.81). When k is odd, the pre-modular category $\mathcal{C}(\mathfrak{sl}_2, q, t)^\times$ is modular if and only if a is odd. \diamond

Let us consider the fusion subcategory of $\mathcal{C}(\mathfrak{sl}_2, q, t)$ generated by integer-spin representations $\{V_n \mid n \text{ even}\}$. We denote this fusion subcategory by $\mathcal{C}(\mathfrak{sl}_2, q, t)_+$. It only depends on q but not t . When k is odd, $\mathcal{C}(\mathfrak{sl}_2, q, t)_+$ is modular. When k is even, the pre-modular category $\mathcal{C}(\mathfrak{sl}_2, q, t)_+$ is not modular because $S_{n,m} = S_{n,k-m}$ for even n, m .

Remark 3.6.16. Let us prove that $\mathcal{C}(\mathfrak{sl}_2, q, t)_+$ is modular when k is odd by direct computations. For n, m even, we have

$$\begin{aligned} \sin\left(\frac{a\pi}{k+2}\right)^2 \sum_{l=0}^{(k-1)/2} S_{n,2l} S_{m,2l} &= \sum_{l=0}^{(k-1)/2} \sin\left(\frac{a(n+1)(2l+1)\pi}{k+2}\right) \sin\left(\frac{a(m+1)(2l+1)\pi}{k+2}\right) \\ &= -\frac{1}{2} \sum_{l=0}^{(k-1)/2} \left(\cos\left(\frac{a(n+m+2)(2l+1)\pi}{k+2}\right) - \cos\left(\frac{a(n-m)(2l+1)\pi}{k+2}\right) \right). \end{aligned}$$

Note that for $\theta = ar\pi/(k+2)$ where $r \in 2\mathbb{Z}$, we have

$$\begin{aligned} \sum_{l=0}^{(k-1)/2} \cos((2l+1)\theta) &= \operatorname{Re}\left(\sum_{l=0}^{(k-1)/2} e^{i(2l+1)\theta}\right) \\ &= \begin{cases} \operatorname{Re}(e^{i\theta} \cdot (k+1)/2), & e^{i2\theta} = 1, \\ \operatorname{Re}\left(\frac{e^{i\theta} - e^{i(k+2)\theta}}{1 - e^{i2\theta}}\right), & e^{i2\theta} \neq 1, \end{cases} \\ &= \begin{cases} (k+1)/2, & r/(k+2) \in \mathbb{Z}, \\ \operatorname{Re}\left(\frac{-1}{1+e^{i\theta}}\right) = -1/2, & r/(k+2) \notin \mathbb{Z}. \end{cases} \end{aligned}$$

Thus we have

$$\sum_{l=0}^{(k-1)/2} S_{n,2l} S_{m,2l} = \sin\left(\frac{a\pi}{k+2}\right)^{-2} \cdot \frac{k+2}{4} \delta_{n,m} = \frac{1}{2} \dim(\mathcal{C}(\mathfrak{sl}_2, q, t)) \delta_{n,m}.$$

Also we know that $2 \dim(\mathcal{C}(\mathfrak{sl}_2, q, t))_+ = \dim(\mathcal{C}(\mathfrak{sl}_2, q, t))$. \diamond

Remark 3.6.17. When k is odd, we also have $\mathcal{C}(\mathfrak{sl}_2, q, t) \simeq \mathcal{C}(\mathfrak{sl}_2, q, t)_+ \boxtimes \mathcal{C}(\mathfrak{sl}_2, q, t)^\times$. \diamond

Example 3.6.18. Let us consider the case $k = 3$. In this case the fusion subcategory $\mathcal{C}(\mathfrak{sl}_2, q, t)_+$ has two simple objects (up to isomorphism) $\mathbb{1} := V_0, X := V_2$. The fusion rule is given by $X \otimes X = \mathbb{1} \oplus X$. It follows that the Frobenius-Perron dimension of X is $\operatorname{FPdim}(X) = (1 + \sqrt{5})/2$.

When $q = e^{\pi i/5}$, this fusion subcategory is unitary and the quantum dimension of X is $1 + 2 \cos(\frac{2\pi}{5}) = (1 + \sqrt{5})/2$. When $q = e^{2\pi i/5}$, this fusion subcategory is not unitary and the quantum dimension of X is $1 + 2 \cos(\frac{4\pi}{5}) = (1 - \sqrt{5})/2$. Both of them are modular categories. \heartsuit

We are mainly interested in the case that $q = e^{\pi i/(k+2)}$ and $t = e^{\pi i/2(k+2)}$. In this case $\mathcal{C}(\mathfrak{sl}_2, q, t)$ is a unitary modular tensor category and also denoted by $\mathcal{C}(\mathfrak{sl}_2, k)$. It is also equivalent to the category of integrable modules of level k over the affine Lie algebra $\widehat{\mathfrak{sl}_2}$ [BK01]. Its additive central charge is $3k/(k+2) \pmod{8}$. Some examples are listed below:

- For $k = 0$, $\mathcal{C}(\mathfrak{sl}_2, 0)$ is the trivial unitary modular tensor category Hilb .

- For $k = 1$, $\mathcal{C}(\mathfrak{sl}_2, 1)$ is a pointed modular category, and the associated metric group is (\mathbb{Z}_2, ϕ) where $\phi(n) = i^n$ for $n \in \mathbb{Z}_2$.
- For $k = 2$, $\mathcal{C}(\mathfrak{sl}_2, 2)$ is an Ising type unitary modular tensor category with $\mathbb{1} := V_0, \sigma := V_1, \psi := V_2$. The additive central charge is $3/2 \pmod{8}$. This Ising type unitary modular tensor category corresponds to $\zeta = \exp(10\pi i/16)$ and $\epsilon = -1$.

Remark 3.6.19. The above constructions can also be generalized to any finite-dimensional simple Lie algebras. See the appendix of [TW97] for references of constructions. \diamond

Remark 3.6.20. The $SU(2)$ Chern-Simons theory

$$S = \frac{k}{4\pi} \int \text{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

realizes a 2d topological order (2+1D topological quantum field theory [Wit89]) and the corresponding modular tensor category is $\mathcal{C}(\mathfrak{sl}_2, k)$ [RT91]. \diamond

Remark 3.6.21. The modular tensor category $\mathcal{C}(\mathfrak{sl}_2, k)$ can also be obtained from the vertex operator algebra (VOA) associated to the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ at level k [Hua08a, Hua08b]. More generally, it was proved by Huang that the category of modules over a VOA, which satisfies certain properties (i.e. a rational VOA), is a modular tensor category [Hua08a, Hua08b]. \diamond

3.7 Chiral central charges

Besides the topological defects, there is another observable of an anomaly-free stable 2d topological order, called the chiral central charge.

An anomaly-free stable 2d topological order C can have gapless 1d boundaries. The gapless modes on a boundary can carry energy and leads to thermal transport. The energy current I along the boundary is proportional to the square of the temperature T (which is assumed to be much smaller than the energy gap) [KF97, CHZ02]:

$$I = \frac{\pi}{12} c T^2.$$

This is called the thermal Hall effect. The coefficient c is called the *chiral central charge*. Intuitively, it counts the number of chiral modes minus the number of anti-chiral modes. Surprisingly, it only depends on the bulk topological order C .

There is another interpretation of the chiral central charge. Usually the low energy effective theory of a gapless 1d boundary of C is a conformal field theory (CFT). A CFT gives a representation of the Virasoro algebra and thus determines a chiral central charge c :

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}.$$

All gapless 1d boundaries of C have the same chiral central charge [KZ20, KZ21]. Therefore, the chiral central charge is an observable of the bulk topological order C and we denote it by $c(C)$. Moreover, the additive central charge $c(\mathcal{C})$ of the modular tensor category \mathcal{C} of particle-like topological defects of C is equal to $c(C)$ modulo 8 [FG90, Reh90, KZ20, KZ21].

It is believed that an anomaly-free 2d topological order C is determined by the pair $(\mathcal{C}, c(C))$, where \mathcal{C} is the unitary modular tensor category of particle-like topological defects of C and $c(C)$ is the chiral central charge of C .

Example 3.7.1. Consider an abelian topological order realized by a $U(1)$ Chern-Simons theory (3.6.1). Its chiral central charge is equal to the number of positive eigenvalues of the K matrix minus the number of negative eigenvalues. \heartsuit

Remark 3.7.2. An nd anomaly-free topological order C_n is called *invertible* if it is invertible under the stacking operation [KW14], i.e., there exists an nd topological D_n such that the stacking $C_n \boxtimes D_n = \mathbb{1}_n$ is the trivial nd topological order. The invertible topological orders (or invertible topological field theories), as well as symmetry protected topological (SPT) orders, can be classified by using methods in algebraic topology (for example, see [FT14, Fre14, FH21, Kap14c, Kap14a, KTTW15, Kit13, Kit15, GJF19b]). \diamond

Remark 3.7.3. In 2d, it is believed that a topological order \mathcal{C} is invertible if and only if there is no nontrivial particle-like topological defect, i.e., $\mathcal{C} \simeq \text{Vec}$. There is a nontrivial invertible 2d topological order called the E_8 topological order [Kit06, Kit11]. It has a gapless boundary described by the so-called E_8 CFT, whose chiral central charge is 8.

If two topological orders \mathcal{C}, \mathcal{D} have the same unitary modular tensor category of particle-like topological defects $\mathcal{C} = \mathcal{D}$, then it is believed that they are only differed by several copies of the E_8 topological order. More precisely, the stacking of \mathcal{C} and n copies of the E_8 topological order is \mathcal{D} , where $n = (c(\mathcal{D}) - c(\mathcal{C}))/8 \in \mathbb{Z}$. \diamond

Remark 3.7.4. The E_8 topological order can be realized by a 2+1D $U(1)$ Chern-Simons theory (3.6.1) with the K matrix given by the Cartan matrix of the E_8 Lie algebra (or equivalently, the E_8 root lattice). Conversely, an invertible 2d topological order is an abelian topological order, thus can be realized by a $U(1)$ Chern-Simons theory with $|\det(K)| = 1$. Such K matrices (or equivalently, unimodular even lattices), up to equivalence, is the direct sum of several copies of the E_8 Cartan matrix (or equivalently, the E_8 root lattice) [Ser73]. \diamond

4 Topological orders in 1d

4.1 Boundary theories of the toric code model

There are two different gapped boundaries of the toric code model, which was proposed by Bravyi and Kitaev [BK98]. As depicted in Figure 43, the top one is called the smooth boundary, while the bottom one is called the rough boundary. The Hilbert space of the model is the same as the original toric code model, i.e., there is a spin-1/2 on each edge. But there is no spin on the dashed edges on the rough boundary. The A_v and B_p operators in the bulk are the same as in the original toric code model, and the A_v and B_p operators on the boundaries are defined to be, for example, $A_v = \sigma_x^1 \sigma_x^2 \sigma_x^3$ and $B_p = \sigma_z^5 \sigma_z^6 \sigma_z^7$ in Figure 43.

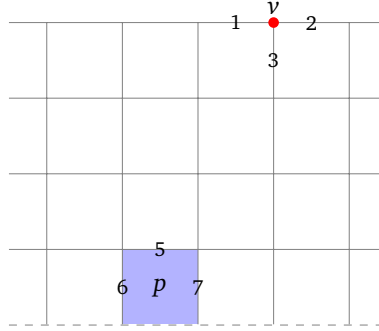


Figure 43: the smooth and rough boundary of the toric code model

In the presence of two types of boundaries, all A_v and B_p operators are still mutually commutative. Thus we can easily find the topological excitations as in the original toric code model. If there is an m particle near the smooth boundary, it can be annihilated to the ground state by a σ_x operator, which is a local operator on the smooth boundary. But if there is an e particle near the smooth boundary, it cannot be annihilated by local operators on the boundary (see Figure 44). It follows that there are two simple topological excitations on the smooth boundary: the trivial one $\mathbb{1}$ and the nontrivial one, which is denoted by E .

Similarly, on the rough boundary one can use a σ_z to annihilate an e particle; an m particle stays at the boundary and is denoted by M (see Figure 44).

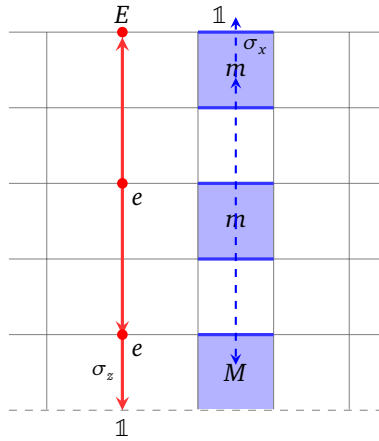


Figure 44: topological excitations on two boundaries

Remark 4.1.1. The smooth boundary is also called the m -condensed boundary while the rough boundary is called the e -condensed boundary, because they can be obtained by condensing m particles or e particles respectively. Mathematically, these condensation processes are described by Lagrangian al-

gebras $\mathbb{1} \oplus m$ and $\mathbb{1} \oplus e$ in the modular tensor category \mathcal{TC} , respectively. More generally, all gapped boundaries of a 2d topological order can be obtained by anyon condensation [Kon14a]. \diamond

To compare the topological defects in the bulk and on the boundaries, we can move bulk topological defects to boundaries by using string operators. Such operations define two maps: the bulk-to-smooth-boundary map is

$$\mathbb{1}, m \mapsto \mathbb{1}, \quad e, f \mapsto E, \quad (4.1.1)$$

and the bulk-to-rough-boundary map is

$$\mathbb{1}, e \mapsto \mathbb{1}, \quad m, f \mapsto M. \quad (4.1.2)$$

It seems that moving bulk topological defects to boundaries loses information. For example, both $\mathbb{1}$ and e become trivial after moving to the rough boundary. Is it possible to distinguish e from $\mathbb{1}$ by using the data on the rough boundary?

Consider the following configuration as depicted in Figure 45. There is an M particle on the rough boundary and an e particle on the right hand side of M , which is created by a local operator σ_z near the rough boundary. Then we use an e string operator to move it from the right hand side of M to the left. Finally we annihilate it by a σ_z operator near the rough boundary.

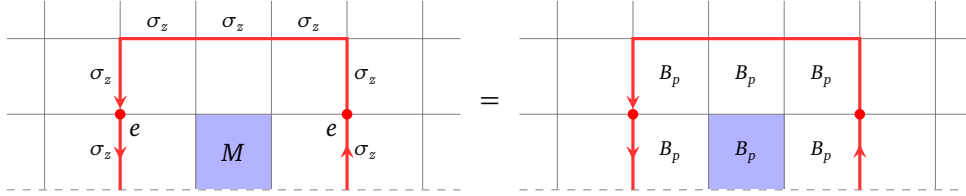


Figure 45: the half-braiding of e and M

This process can be viewed as happening on the rough boundary, thus defines an instanton from $M = M \otimes \mathbb{1}$ to $M = \mathbb{1} \otimes M$. Let us compute this instanton explicitly. As an operator, this process is an e string operator, which is precisely equal to the product of all B_p operators encircled by the path of e . So it acts on the M particle as -1 . In other words, this instanton is

$$M \otimes \mathbb{1} = M \xrightarrow{-1} M = \mathbb{1} \otimes M.$$

We say the half-braiding of e with M is -1 . Similarly, we can consider the case that there is no M particle in the region enclosed by the path of e . This gives the half-braiding of e with $\mathbb{1}$, which should be $+1$.

We list these properties of e particles in the following:

$$e = (\mathbb{1}, \mathbb{1} \otimes \mathbb{1} \xrightarrow{+1} \mathbb{1} \otimes \mathbb{1}, M \otimes \mathbb{1} \xrightarrow{-1} \mathbb{1} \otimes M).$$

- (a) The first component $\mathbb{1}$ means an e particle becomes $\mathbb{1}$ after moving to the rough boundary.
- (b) The second component means the half-braiding of e with $\mathbb{1}$ is $+1$.
- (c) The third component means the half-braiding of e with M is -1 .

Here the equal sign means we label e by these data. Note that this triple only involves the data on the rough boundary.

Similarly, all particle-like topological defects in the bulk can be labeled in this way [HZZK⁺22]:

$$\begin{cases} \mathbb{1} = (\mathbb{1}, \mathbb{1} \otimes \mathbb{1} \xrightarrow{+1} \mathbb{1} \otimes \mathbb{1}, M \otimes \mathbb{1} \xrightarrow{+1} \mathbb{1} \otimes M), \\ e = (\mathbb{1}, \mathbb{1} \otimes \mathbb{1} \xrightarrow{+1} \mathbb{1} \otimes \mathbb{1}, M \otimes \mathbb{1} \xrightarrow{-1} \mathbb{1} \otimes M), \\ m = (M, \mathbb{1} \otimes M \xrightarrow{+1} M \otimes \mathbb{1}, M \otimes M \xrightarrow{+1} M \otimes M), \\ f = (M, \mathbb{1} \otimes M \xrightarrow{+1} M \otimes \mathbb{1}, M \otimes M \xrightarrow{-1} M \otimes M). \end{cases} \quad (4.1.3)$$

Although different bulk topological defects may become the same after moving to the boundary, we can still distinguish them by their half-braidings. In this sense, we recover the bulk topological order from the rough boundary!

We can also use the smooth boundary to recover the bulk topological order:

$$\begin{cases} \mathbb{1} = (\mathbb{1}, \mathbb{1} \otimes \mathbb{1} \xrightarrow{+1} \mathbb{1} \otimes \mathbb{1}, E \otimes \mathbb{1} \xrightarrow{+1} \mathbb{1} \otimes E), \\ m = (\mathbb{1}, \mathbb{1} \otimes \mathbb{1} \xrightarrow{+1} \mathbb{1} \otimes \mathbb{1}, E \otimes \mathbb{1} \xrightarrow{-1} \mathbb{1} \otimes E), \\ e = (E, \mathbb{1} \otimes E \xrightarrow{+1} E \otimes \mathbb{1}, E \otimes E \xrightarrow{+1} E \otimes E), \\ f = (E, \mathbb{1} \otimes E \xrightarrow{+1} E \otimes \mathbb{1}, E \otimes E \xrightarrow{-1} E \otimes E). \end{cases} \quad (4.1.4)$$

These examples suggest that an anomaly-free stable 2d topological order can be recovered from a gapped boundary.

Remark 4.1.2. With respect to the rough boundary, we have $c_{e,m} = +1$ and $c_{m,e} = -1$ because the half-braiding of e with M is -1 and the half-braiding of m with $\mathbb{1}$ is $+1$. Similarly, with respect to the smooth boundary, we have $c_{e,m} = -1$ and $c_{m,e} = +1$. \diamond

Remark 4.1.3. A topological phase transition between two boundaries of the toric code model was studied in [CJKYZ20]. \diamond

4.2 Gapped boundaries of 2d topological orders

Most structures and properties developed in Section 3.4 are still valid in 1d, except the braiding and ribbon structures. So we conclude that the topological defects of a stable 1d topological order A form a (unitary) fusion category \mathcal{A} . By definition, \mathcal{A} is the topological skeleton of A .

Remark 4.2.1. In 1d we are also interested in unstable topological orders. Without the stability, the topological skeleton of a 1d topological order is a (unitary) multi-fusion category, whose tensor unit $\mathbb{1}$ is potentially non-simple: $\mathbb{1} = \bigoplus_i \mathbb{1}_i$. Physically it means that the ground state subspace is degenerate, and each simple direct summand $\mathbb{1}_i$ is generated by a ground state. Unstable topological orders and multi-fusion categories usually appear in dimensional reduction processes, which are discussed later. \diamond

4.2.1 Bulk-to-boundary map as a monoidal functor

Consider an anomaly-free stable 2d topological order C and a stable gapped 1d boundary A , whose particle-like topological defects form a (unitary) modular tensor category \mathcal{C} and a (unitary) fusion category \mathcal{A} , respectively. Let $x \in \mathcal{C}$ be a bulk topological defect. If we move it to the boundary, it becomes a boundary topological defect, denoted by $F(x) \in \mathcal{A}$ (see Figure 46). This defines a map F from the set of bulk topological defects (i.e., objects of \mathcal{C}) to the set of boundary topological defects (i.e., objects of \mathcal{A}).

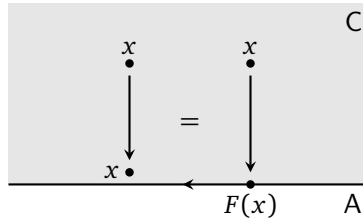


Figure 46: the bulk-to-boundary map F

Remark 4.2.2. The boundary of C is not a geometric boundary (the bottom straight line in Figure 46), but the neighborhood of this line. The ‘width’ of the boundary depends on the length scale and is not important at a fixed point of the renormalization flow. When we say “moving x to the boundary”, what we mean is “moving x very close to the boundary”, and how x is close to the boundary depends on the length scale. So the boundary topological defect $F(x)$ is indeed a topological defect x near the boundary. \diamond

We do not only move topological defects, but also the world lines and instantons attached to them. Hence F is indeed a functor $F: \mathcal{C} \rightarrow \mathcal{A}$. Moreover, F should preserve the fusion of topological defects, i.e., two different ways of moving two topological defects to the boundary should give the same result. In other words, $F(x) \otimes F(y)$ should be the same as $F(x \otimes y)$. We do not require that they are precisely equal, but expect an isomorphism $F_{x,y}^2: F(x) \otimes F(y) \rightarrow F(x \otimes y)$, which is a part of the structure of the bulk-to-boundary-map. Moreover, $F_{x,y}^2$ should satisfy some obvious coherence conditions. These data and properties of the bulk-to-boundary map can be summarized to the following definition.

Definition 4.2.3. Let \mathcal{C}, \mathcal{D} be monoidal categories. A *monoidal functor* $F: \mathcal{C} \rightarrow \mathcal{D}$ consists of the following data:

- a functor $F: \mathcal{C} \rightarrow \mathcal{D}$;
- a natural isomorphism $F_{x,y}^2: F(x) \otimes F(y) \rightarrow F(x \otimes y)$;
- an isomorphism $F^0: \mathbb{1}_{\mathcal{D}} \rightarrow F(\mathbb{1}_{\mathcal{C}})$;

and these data satisfy the following conditions:

1. For any $x, y, z \in \mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc}
 (F(x) \otimes F(y)) \otimes F(z) & \xrightarrow{\alpha_{F(x), F(y), F(z)}^{\mathcal{D}}} & F(x) \otimes (F(y) \otimes F(z)) \\
 \downarrow F_{x,y}^2 \otimes \text{id}_{F(z)} & & \downarrow \text{id}_{F(x)} \otimes F_{y,z}^2 \\
 F(x \otimes y) \otimes F(z) & & F(x) \otimes F(y \otimes z) \\
 \downarrow F_{x \otimes y, z}^2 & & \downarrow F_{x, y \otimes z}^2 \\
 F((x \otimes y) \otimes z) & \xrightarrow{F(\alpha_{x,y,z}^{\mathcal{C}})} & F(x \otimes (y \otimes z))
 \end{array} \tag{4.2.1}$$

2. For any $x \in \mathcal{C}$, the following two diagrams commute:

$$\begin{array}{ccc}
 F(\mathbb{1}_{\mathcal{C}}) \otimes F(x) & \xrightarrow{F_{\mathbb{1},x}^2} & F(\mathbb{1}_{\mathcal{C}} \otimes x) \\
 \uparrow F^0 \otimes \text{id}_{F(x)} & & \downarrow F(\lambda_x^{\mathcal{C}}) \\
 \mathbb{1}_{\mathcal{D}} \otimes F(x) & \xrightarrow{\lambda_{F(x)}^{\mathcal{D}}} & F(x)
 \end{array}
 \quad
 \begin{array}{ccc}
 F(x) \otimes F(\mathbb{1}_{\mathcal{C}}) & \xrightarrow{F_{x,\mathbb{1}}^2} & F(x \otimes \mathbb{1}_{\mathcal{C}}) \\
 \uparrow \text{id}_{F(x)} \otimes F^0 & & \downarrow F(\rho_x^{\mathcal{C}}) \\
 F(x) \otimes \mathbb{1}_{\mathcal{D}} & \xrightarrow{\rho_{F(x)}^{\mathcal{D}}} & F(x)
 \end{array} \tag{4.2.2}$$

A monoidal functor that is also an equivalence is called a *monoidal equivalence*. ■

Hence, the bulk-to-boundary map F is a monoidal functor.

Example 4.2.4. Let G be a finite group and H be a subgroup of G . Each G -representation is naturally an H -representation by ‘forgetting’ the action of the group elements that are not in H . This defines a forgetful functor

$$\begin{aligned}
 F: \text{Rep}(G) &\rightarrow \text{Rep}(H) \\
 (V, \rho) &\mapsto (V, \rho|_H).
 \end{aligned}$$

Then F is a monoidal functor with F^2 and F^0 are both identities. Note that when H is the trivial group this recovers the forgetful functor $\text{Rep}(G) \rightarrow \text{Vec}$ defined in Example 3.3.39. ♥

Example 4.2.5. Let G be a finite group and $\omega_1, \omega_2 \in Z^3(G; \mathbb{C}^\times)$. Suppose $F: \text{Vec}_G^{\omega_1} \rightarrow \text{Vec}_G^{\omega_2}$ is a monoidal functor whose underlying functor is identity. The natural isomorphism F^2 is defined by

$$\begin{aligned}
 F_{g,h}^2: \mathbb{C}_{(g)} \otimes_{\mathbb{C}} \mathbb{C}_{(h)} &\rightarrow \mathbb{C}_{(g)} \otimes_{\mathbb{C}} \mathbb{C}_{(h)} \\
 1 \otimes_{\mathbb{C}} 1 &\mapsto \beta(g, h) \cdot 1 \otimes_{\mathbb{C}} 1
 \end{aligned}$$

for some nonzero complex number $\beta(g, h) \in \mathbb{C}^\times$. Then the commutative diagram (4.2.1) translates to

$$\omega_1(g, h, k)\beta(gh, k)\beta(g, h) = \beta(g, hk)\beta(h, k)\omega_2(g, h, k), \quad \forall g, h, k \in G. \quad (4.2.3)$$

Define

$$(d\beta)(g, h, k) := \frac{\beta(h, k)\beta(g, hk)}{\beta(gh, k)\beta(g, h)}.$$

Then (4.2.3) can be rewritten as $\omega_1 = \omega_2 \cdot d\beta$. The operator d is usually called the *differential*. It is straightforward to check that $d\beta \in Z^3(G; \mathbb{C}^\times)$ is a 3-cocycle. A 3-cocycle obtained by the differential is called a *3-coboundary*. The space of 3-coboundaries (valued in \mathbb{C}^\times) is denoted by $B^3(G; \mathbb{C}^\times)$. The quotient group $H^3(G; \mathbb{C}^\times) := Z^3(G; \mathbb{C}^\times)/B^3(G; \mathbb{C}^\times)$ is called the *third group cohomology group* of G (with coefficients in \mathbb{C}^\times).

Similarly, the other two commutative diagrams in (4.2.2) imply that the morphism F^0 is equal to $(\varepsilon_2/\varepsilon_1)\beta(e, e)^{-1} \cdot \text{id}_{\mathbb{C}_{[e]}}$. Hence we conclude that such a monoidal equivalence F exists if and only if ω_1 and ω_2 are contained in the same cohomology class. In other words, the monoidal structures on the category Vec_G with the tensor product induced by the multiplication of G are classified by $H^3(G; \mathbb{C}^\times)$ up to equivalence. \heartsuit

Remark 4.2.6. For each integer $n > 1$ we have $H^3(\mathbb{Z}_n; \mathbb{C}^\times) \simeq \mathbb{Z}_n$. For $n = 2$, the function defined as follows is a 3-cocycle:

$$\omega(a, b, c) = \begin{cases} -1, & a = b = c = 1, \\ +1, & \text{otherwise.} \end{cases}$$

This 3-cocycle $\omega \in Z^3(\mathbb{Z}_2; \mathbb{C}^\times)$ is not a 3-coboundary. In other words, it represents the nontrivial cohomology class in $H^3(\mathbb{Z}_2; \mathbb{C}^\times) \simeq \mathbb{Z}_2$. \diamond

Remark 4.2.7. For every finite group G , the inclusion $U(1) \hookrightarrow \mathbb{C}^\times$ induces a group isomorphism $H^3(G; U(1)) \rightarrow H^3(G; \mathbb{C}^\times)$. \diamond

Exercise 4.2.8. Let G be an abelian group. Recall that the delooping category BG is monoidal and the left and right unitors $\lambda_* = \rho_*$ can be arbitrary (see Example 3.4.25). Show that different choices of the unitors give equivalent monoidal categories.

Definition 4.2.9. Let \mathcal{C}, \mathcal{D} be braided monoidal categories. A *braided monoidal functor* (or simply a *braided functor*) $F: \mathcal{C} \rightarrow \mathcal{D}$ is a monoidal functor $F: \mathcal{C} \rightarrow \mathcal{D}$ such that the following diagram commutes for any $x, y \in \mathcal{C}$:

$$\begin{array}{ccc} F(x) \otimes F(y) & \xrightarrow{F_{x,y}^2} & F(x \otimes y) \\ c_{F(x), F(y)}^{\mathcal{D}} \downarrow & & \downarrow F(c_{x,y}^{\mathcal{C}}) \\ F(y) \otimes F(x) & \xrightarrow{F_{y,x}^2} & F(y \otimes x) \end{array}$$

A braided monoidal functor that is also an equivalence is called a *braided monoidal equivalence*. \blacksquare

Definition 4.2.10. Let \mathcal{C}, \mathcal{D} be monoidal categories and $F, G: \mathcal{C} \rightarrow \mathcal{D}$ be monoidal functors. A *monoidal natural transformation* $\alpha: F \Rightarrow G$ is a natural transformation $\alpha: F \Rightarrow G$ satisfying the following conditions:

1. For any $x, y \in \mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} F(x) \otimes F(y) & \xrightarrow{F_{x,y}^2} & F(x \otimes y) \\ \alpha_x \otimes \alpha_y \downarrow & & \downarrow \alpha_{x \otimes y} \\ G(x) \otimes G(y) & \xrightarrow{G_{x,y}^2} & G(x \otimes y) \end{array}$$

2. The following diagram commutes:

$$\begin{array}{ccc} \mathbb{1}_{\mathcal{D}} & \xrightarrow{F^0} & F(\mathbb{1}_{\mathcal{C}}) \\ & \searrow G^0 & \downarrow \alpha_1 \\ & & G(\mathbb{1}_{\mathcal{C}}) \end{array}$$

A monoidal natural transformation that is also a natural isomorphism is called a *monoidal natural isomorphism*. ■

Example 4.2.11. Let G be a finite group and $F : \text{Rep}(G) \rightarrow \text{Vec}$ be the forgetful functor. Let us find the group of monoidal isomorphisms from F to itself (also called a *monoidal natural automorphism* on F).

In Example 3.3.44 we have constructed a natural isomorphism $\alpha^g : F \Rightarrow F$ for each $g \in G$ defined by

$$\alpha_{(V,\rho)}^g := \rho(g) : V \rightarrow V, \quad (V, \rho) \in \text{Rep}(G).$$

It is easy to see that every α^g is a monoidal natural isomorphism. Conversely, every monoidal natural automorphism on F has to be of the form α^g for some $g \in G$. The proof is not very hard but we do not give it here. In other words, the assignment $g \mapsto \alpha^g$ defines a group isomorphism from G to the group $\text{Aut}_{\otimes}(F)$ of monoidal natural automorphisms on F . This is a special case of the Tannaka-Krein duality [Tan39, Kre49]. ♡

4.2.2 Half-braidings and central functors

Now suppose there is a bulk topological defect $x \in \mathcal{C}$ and a boundary topological defect $a \in \mathcal{A}$. Recall that the boundary topological defect $F(x)$ is just x near the boundary, so we can freely move $F(x)$ along the boundary, regardless of the topological defects ‘on’ the boundary. In other words, moving x to the left or right side of a and then fusing them together should give the same result (see Figure 47). Thus we get an isomorphism (instanton) $\sigma_{a,x} : a \otimes F(x) \rightarrow F(x) \otimes a$. The family $\{\sigma_{a,x} : a \otimes F(x) \rightarrow F(x) \otimes a\}$ should be natural in both $a \in \mathcal{A}$ and $x \in \mathcal{C}$ because instantons can be freely moved along the worlds lines.

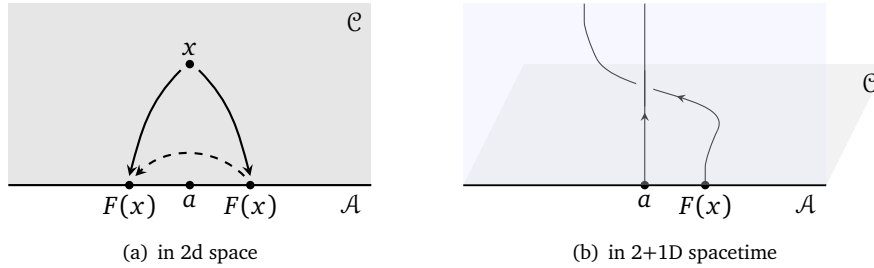


Figure 47: the half-braiding of $F(x)$ with a

The following figures show that the family $\{\sigma_{a,x}\}$ should satisfy some properties. Figure 48 shows that if there are two boundary topological defects and one bulk topological defects, two different ways of half-braidings should be the same.

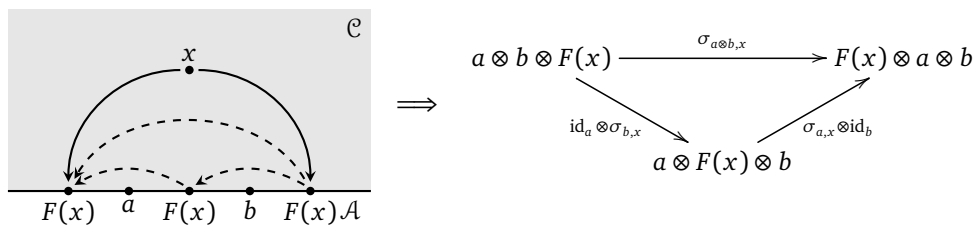


Figure 48: the first property of the half-braiding

Figure 49 shows that if there are two bulk topological defects and one boundary topological defects, two different ways of half-braidings should be the same.

$$\begin{array}{ccc}
 a \otimes F(x) \otimes F(y) & \xrightarrow{\text{id}_a \otimes F^2_{x,y}} & a \otimes F(x \otimes y) \\
 \downarrow \sigma_{a,x} \otimes \text{id}_{F(y)} & & \downarrow \sigma_{a,x \otimes y} \\
 F(x) \otimes a \otimes F(y) & & \\
 \downarrow \text{id}_{F(x)} \otimes \sigma_{a,y} & & \\
 F(x) \otimes F(y) \otimes a & \xrightarrow{F^2_{x,y} \otimes \text{id}_a} & F(x \otimes y) \otimes a
 \end{array}$$

Figure 49: the second property of the half-braiding

Figure 50 shows that by moving bulk topological defects to the boundary, the braiding of two bulk topological defects becomes the half-braiding of two boundary topological defects.

$$\begin{array}{ccc}
 F(x) \otimes F(y) & \xrightarrow{F^2_{x,y}} & F(x \otimes y) \\
 \downarrow \sigma_{F(x),y} & & \downarrow F(c_{x,y}) \\
 F(y) \otimes F(x) & \xrightarrow{F^2_{y,x}} & F(y \otimes x)
 \end{array}$$

Figure 50: the third property of the half-braiding

In the above three commutative diagrams, we ignore the monoidal structures (associators) of \mathcal{C} and \mathcal{A} . The half-braidings and their properties are summarized to the following definition [Bez04].

Definition 4.2.12. Let \mathcal{C} be a braided monoidal category and \mathcal{A} be a monoidal category. A *central structure* of a monoidal functor $F: \mathcal{C} \rightarrow \mathcal{A}$ is a natural isomorphism $\sigma_{a,x}: a \otimes F(x) \rightarrow F(x) \otimes a$, where $x \in \mathcal{C}$ and $a \in \mathcal{A}$, such that the following conditions hold:

1. For any $x \in \mathcal{C}$ and $a, b \in \mathcal{A}$, the following diagram commutes:

$$\begin{array}{ccc}
 a \otimes b \otimes F(x) & \xrightarrow{\sigma_{a \otimes b, x}} & F(x) \otimes a \otimes b \\
 \searrow \text{id}_a \otimes \sigma_{x,b} & & \nearrow \sigma_{a,x} \otimes \text{id}_b \\
 & a \otimes F(x) \otimes b &
 \end{array} \tag{4.2.4}$$

2. For any $x, y \in \mathcal{C}$ and $a \in \mathcal{A}$, the following diagram commutes:

$$\begin{array}{ccc}
 a \otimes F(x) \otimes F(y) & \xrightarrow{\text{id}_a \otimes F^2_{x,y}} & a \otimes F(x \otimes y) \\
 \downarrow \sigma_{a,x} \otimes \text{id}_{F(y)} & & \downarrow \sigma_{a,x \otimes y} \\
 F(x) \otimes a \otimes F(y) & & \\
 \downarrow \text{id}_{F(x)} \otimes \sigma_{a,y} & & \\
 F(x) \otimes F(y) \otimes a & \xrightarrow{F^2_{x,y} \otimes \text{id}_a} & F(x \otimes y) \otimes a
 \end{array} \tag{4.2.5}$$

3. For any $x, y \in \mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc}
 F(x) \otimes F(y) & \xrightarrow{F^2_{x,y}} & F(x \otimes y) \\
 \downarrow \sigma_{F(x),y} & & \downarrow F(c_{x,y}) \\
 F(y) \otimes F(x) & \xrightarrow{F^2_{y,x}} & F(y \otimes x)
 \end{array} \tag{4.2.6}$$

A *central functor* is a monoidal functor equipped with a central structure. ■

Hence the bulk-to-boundary map $F : \mathcal{C} \rightarrow \mathcal{A}$ is a central functor.

4.2.3 Drinfeld center

We have known that a bulk topological defect $x \in \mathcal{C}$ determines the following data:

- a boundary topological defect $F(x) \in \mathcal{A}$;
- a natural isomorphism $\sigma_{-,x} := \{\sigma_{a,x} : a \otimes F(x) \rightarrow F(x) \otimes a\}_{a \in \mathcal{A}}$, called the half-braidings of $F(x)$, satisfying the property given by Figure 48, i.e., the following diagram commutes for all $a, b \in \mathcal{A}$:

$$\begin{array}{ccc} a \otimes b \otimes F(x) & \xrightarrow{\sigma_{x,a \otimes b}} & F(x) \otimes a \otimes b \\ & \searrow \text{id}_a \otimes \sigma_{x,b} \quad \nearrow \sigma_{x,a} \otimes \text{id}_b & \\ & a \otimes F(x) \otimes b & \end{array}$$

Motivated by this construction and the example in the toric code model (4.1.3) (4.1.4), we propose the following definition [Maj91, JS91].

Definition 4.2.13. Let \mathcal{A} be a monoidal category. The *Drinfeld center* or *monoidal center* of \mathcal{A} is the braided monoidal category $\mathfrak{Z}_1(\mathcal{A})$ defined as follows:

- The objects are all pairs $(a, \gamma_{-,a})$, where $a \in \mathcal{A}$ and $\gamma_{-,a} : - \otimes a \Rightarrow a \otimes - : \mathcal{A} \rightarrow \mathcal{A}$ is a natural isomorphism (called a half-braiding) such that the following diagram commutes for all $b, c \in \mathcal{A}$,

$$\begin{array}{ccccc} & & b \otimes (a \otimes c) & \xrightarrow{\alpha_{b,a,c}^{-1}} & (b \otimes a) \otimes c \\ & \nearrow \text{id}_b \otimes \gamma_{c,a} & & & \searrow \gamma_{b,a} \otimes \text{id}_c \\ b \otimes (c \otimes a) & & & & (a \otimes b) \otimes c \\ & \searrow \alpha_{b,c,a}^{-1} & & & \nearrow \alpha_{a,b,c}^{-1} \\ & & (b \otimes c) \otimes a & \xrightarrow{\gamma_{b \otimes c, a}} & a \otimes (b \otimes c) \end{array}$$

where α is the associator of \mathcal{A} .

- The morphisms from $(a, \gamma_{-,a})$ to $(b, \gamma_{-,b})$ are morphisms $f \in \text{Hom}_{\mathcal{A}}(a, b)$ such that the following diagram commutes for all $c \in \mathcal{A}$.

$$\begin{array}{ccc} c \otimes a & \xrightarrow{\gamma_{c,a}} & a \otimes c \\ \text{id}_c \otimes f \downarrow & & \downarrow f \otimes \text{id}_c \\ c \otimes b & \xrightarrow{\gamma_{c,b}} & b \otimes c \end{array}$$

- The tensor product of two objects $(a, \gamma_{-,a})$ and $(b, \gamma_{-,b})$ is $(a \otimes b, \gamma_{-,a \otimes b})$, where $a \otimes b$ is the tensor product in \mathcal{A} and the half-braiding $\gamma_{c,a \otimes b} : c \otimes a \otimes b \rightarrow a \otimes b \otimes c$ is the composition

$$c \otimes a \otimes b \xrightarrow{\gamma_{c,a} \otimes \text{id}_b} a \otimes c \otimes b \xrightarrow{\text{id}_a \otimes \gamma_{c,b}} a \otimes b \otimes c.$$

Here we omit the associators of \mathcal{A} .

- The tensor unit is $(\mathbb{1}, \gamma_{-, \mathbb{1}})$, where $\mathbb{1} \in \mathcal{A}$ is the tensor unit of \mathcal{A} and the half-braiding $\gamma_{a, \mathbb{1}} : a \otimes \mathbb{1} \rightarrow \mathbb{1} \otimes a$ is

$$a \otimes \mathbb{1} \xrightarrow{r_a} a \xrightarrow{l_a^{-1}} \mathbb{1} \otimes a,$$

or simply $\gamma_{a, \mathbb{1}} = \text{id}_a$ if we ignore the unitors.

- The associator and left/right unitor of $\mathfrak{Z}_1(\mathcal{A})$ are induced by those of \mathcal{A} .
- The braiding of $(a, \gamma_{-,a})$ and $(b, \gamma_{-,b})$ is given by $\gamma_{a,b} : a \otimes b \rightarrow b \otimes a$. Note that it is independent of the half-braiding of a . ■

Remark 4.2.14. The Drinfeld center is a kind of E_1 -center [Lur17]. ◇

Remark 4.2.15. Given an object $(a, \gamma_{-,a}) \in \mathfrak{Z}_1(\mathcal{A})$, the following diagram commutes:

$$\begin{array}{ccc} \mathbb{1} \otimes a & \xrightarrow{\gamma_{\mathbb{1},a}} & a \otimes \mathbb{1} \\ \lambda_a \searrow & & \swarrow \rho_a \\ & a & \end{array}$$

where λ, ρ are the left/right unitor of \mathcal{A} . In other words, $\gamma_{\mathbb{1},a} = \text{id}_a$ if we ignore the unitors. ◇

Remark 4.2.16. Note that the definition of a half-braiding is similar to the hexagon equations in the definition of a braiding structure. It follows that for any braided monoidal category \mathcal{C} and $x \in \mathcal{C}$, the braiding structure induces a half-braiding $c_{-,x} : - \otimes x \rightarrow x \otimes -$ on x . Moreover, there is a canonical braided functor $\mathcal{C} \rightarrow \mathfrak{Z}_1(\mathcal{C})$ defined by $x \mapsto (x, c_{-,x})$. Similarly, using the anti-braiding there is a canonical braided functor $\bar{\mathcal{C}} \rightarrow \mathfrak{Z}_1(\mathcal{C})$ defined by $x \mapsto (x, c_{x,-}^{-1})$. When \mathcal{C} is symmetric, these two functors are equal. ◇

Example 4.2.17. The Drinfeld center $\mathfrak{Z}_1(\text{Vec})$ is equivalent to Vec as braided monoidal categories. Indeed, suppose $V \in \text{Vec}$ and $\gamma_{-,V} : - \otimes V \rightarrow V \otimes -$ is a half-braiding. Then we have the following commutative diagram for every vector space W and linear map $f : \mathbb{C} \rightarrow W$:

$$\begin{array}{ccccc} & & \mathbb{C} \otimes V & \xrightarrow{f \otimes 1} & W \otimes V \\ & \nearrow \simeq & \downarrow \gamma_{\mathbb{C},V} & & \downarrow \gamma_{W,V} \\ V & & V \otimes \mathbb{C} & \xrightarrow{1 \otimes f} & V \otimes W \\ & \searrow \simeq & & & \end{array}$$

Therefore, we have $\gamma_{W,V}(w \otimes v) = v \otimes w$ for every $W \in \text{Vec}$ and $w \in W$. Hence the canonical inclusion $\text{Vec} \rightarrow \mathfrak{Z}_1(\text{Vec})$ (see Remark 4.2.16) is an equivalence of braided monoidal categories. ◇

Example 4.2.18. Recall there are two simple topological defects $\mathbb{1}$ and E on the smooth boundary of the toric code model. Clearly the fusion rule is given by $E \otimes E = \mathbb{1}$. Thus the topological skeleton of the smooth boundary is equivalent to $\text{Rep}(\mathbb{Z}_2)$ as fusion categories, where $\mathbb{1}$ corresponds to the trivial representation of \mathbb{Z}_2 and E corresponds to the nontrivial irreducible representation of \mathbb{Z}_2 . Then (4.1.4) means \mathcal{TC} is equivalent to $\mathfrak{Z}_1(\text{Rep}(\mathbb{Z}_2))$ as a braided fusion category.

Similarly, the topological skeleton of the rough boundary is equivalent to $\text{Vec}_{\mathbb{Z}_2}$ as fusion categories, where $\mathbb{1}$ corresponds to \mathbb{C} with the trivial grading and M corresponds to \mathbb{C} with the nontrivial grading, and (4.1.3) is precisely a braided equivalence $\mathcal{TC} \simeq \mathfrak{Z}_1(\text{Vec}_{\mathbb{Z}_2})$. ◇

Remark 4.2.19. For any monoidal category \mathcal{A} , there is an obvious functor

$$\begin{aligned} I : \mathfrak{Z}_1(\mathcal{A}) &\rightarrow \mathcal{A} \\ (a, \gamma_{-,a}) &\mapsto a, \end{aligned}$$

called the forgetful functor because it ‘forgets’ the half-braiding. Clearly it is a faithful monoidal functor. Moreover, I is a central functor with the central structure defined by

$$\sigma_{a,(b,\gamma_{-,b})} := \gamma_{a,b}.$$

This central structure satisfies (4.2.4) by the definition of $\gamma_{-,b}$; (4.2.5) and (4.2.6) follow from the definition of tensor product and braiding of $\mathfrak{Z}_1(\mathcal{A})$, respectively.

Furthermore, suppose \mathcal{C} is a braided monoidal category, \mathcal{A} is a monoidal category and $F : \mathcal{C} \rightarrow \mathcal{A}$ is a monoidal functor. Then a central structure on F is equivalent to a braided monoidal functor $F' : \mathcal{C} \rightarrow$

$\mathfrak{Z}_1(\mathcal{A})$ such that $I \circ F' = F$. Indeed, given a braided monoidal functor $F': \mathcal{C} \rightarrow \mathfrak{Z}_1(\mathcal{A})$ such that $I \circ F' = F$, the central structure of I induces that of F , i.e.,

$$\sigma_{a,x} := \gamma_{a,F'(x)}.$$

Conversely, given a central structure $\sigma_{a,x}: a \otimes F(x) \rightarrow F(x) \otimes a$, it is not hard to see that

$$\begin{aligned} F': \mathcal{C} &\rightarrow \mathfrak{Z}_1(\mathcal{A}) \\ x &\mapsto (F(x), \sigma_{-,x}) \end{aligned}$$

is a braided monoidal functor and $I \circ F' = F$. Hence a central structure of a monoidal functor $F: \mathcal{C} \rightarrow \mathcal{A}$ can be equivalently defined as a braided monoidal functor $F': \mathcal{C} \rightarrow \mathfrak{Z}_1(\mathcal{A})$ such that $I \circ F' = F$. For example, (4.1.4) and (4.1.3) are central structures of the bulk-to-smooth-boundary map (4.1.1) and the bulk-to-rough-boundary map (4.1.2), respectively. \diamond

Remark 4.2.20. It is possible that different central functors have the same underlying monoidal functor. \diamond

4.2.4 Boundary-bulk relation

We have also shown model-independently in Section 4.2.2 and 4.2.3 that the bulk modular tensor category necessarily factors through the Drinfeld center of the boundary fusion category. It turns out that a stronger result holds for all 2d topological orders and their gapped boundaries.

Theorem^{ph} 4.2.21 (Boundary-bulk relation in 2d, 1st version). Suppose \mathcal{C} is an anomaly-free stable 2d topological order with a stable gapped 1d boundary \mathcal{A} , and their particle-like topological defects form a (unitary) modular tensor category \mathcal{C} and a (unitary) fusion category \mathcal{A} , respectively.

- (a) The bulk-to-boundary map $F: \mathcal{C} \rightarrow \mathcal{A}$ is a central functor.
- (b) The central structure $F': \mathcal{C} \rightarrow \mathfrak{Z}_1(\mathcal{A})$ is a braided equivalence.

Remark 4.2.22. In the 2d toric code model, the braided equivalences (4.1.4) and (4.1.3) imply that the bulk modular tensor category \mathcal{TC} is precisely the Drinfeld center of the boundary fusion categories (see Example 4.2.18). \diamond

Remark 4.2.23. The boundary-bulk relation in 2d was first discovered and physically proved in the Levin-Wen model [KK12] (see also [Kon13, LW14, HWW17, CCW17] for further developments). Model independently, it was shown that the bulk modular tensor category factors through the Drinfeld center of a boundary fusion category [FSV13]. Since a gapped boundary of a 2d topological order is necessarily the result of a 2d anyon condensation to the trivial phase, we obtain a complete model-independent proof of the boundary-bulk relation for 2d topological orders and their gapped boundaries based on the anyon condensation theory [Kon14a] (see also Theorem^{ph} 5.4.1). More general boundary-bulk relation for potentially gapless higher dimensional quantum liquid phases and their boundaries was proposed and proved in [KWZ15, KWZ17, KZ22a] (see also Theorem^{ph} 5.6.1). \diamond

Remark 4.2.24. The smooth and rough boundaries of the toric code model are the same as (anomalous) 1d topological orders, but they are different as 1d boundaries. Mathematically, there is a monoidal equivalence (by identifying E and M) of $\text{Rep}(\mathbb{Z}_2)$ and $\text{Vec}_{\mathbb{Z}_2}$, but this equivalence does not preserve the bulk-to-boundary maps.

In Kitaev's quantum double model associated to a finite group G , there are also (at least) two types of boundaries [BSW11, CCW17], whose topological skeletons are $\text{Rep}(G)$ and Vec_G respectively. If G is non-abelian, $\text{Rep}(G)$ and Vec_G are not equivalent as categories. \diamond

Remark 4.2.25. For any unitary (or spherical) fusion category \mathcal{A} , there is a 2d fixed point lattice model associated to \mathcal{A} , called the Levin-Wen model or the string-net model [LW05]. The (unitary) modular tensor category particle-like topological defects of the Levin-Wen model is equivalent to $\mathfrak{Z}_1(\mathcal{A})$. When $\mathcal{A} = \text{Vec}_G$ or $\text{Rep}(G)$ where G is a finite group, the associated Levin-Wen model realizes the same topological order as the quantum double model associated to G . Mathematically, the Drinfeld center of $\text{Rep}(G)$ and Vec_G are equivalent as modular tensor categories (see Example 4.2.32). \diamond

Remark 4.2.26. Our analyses do not depend on the stability of the boundary. So the boundary-bulk relation still holds when A is unstable, i.e., \mathcal{A} is a multi-fusion category. We say a multi-fusion category \mathcal{A} is *indecomposable* if it can not be written as a direct sum $\mathcal{A} \simeq \mathcal{A}_1 \oplus \mathcal{A}_2$ for nonzero multi-fusion categories $\mathcal{A}_1, \mathcal{A}_2$. The Drinfeld center $\mathfrak{Z}_1(\mathcal{A})$ of a multi-fusion category \mathcal{A} is a fusion category (indeed, a nondegenerate braided fusion category) if and only if \mathcal{A} is indecomposable. A fusion category is an indecomposable multi-fusion category because the tensor unit $\mathbb{1}$ is an indecomposable object. Moreover, we have $\mathfrak{Z}_1(\mathcal{A}_1 \oplus \mathcal{A}_2) \simeq \mathfrak{Z}_1(\mathcal{A}_1) \oplus \mathfrak{Z}_1(\mathcal{A}_2)$. \diamond

Remark 4.2.27. As a corollary of the boundary-bulk relation in 2d (Theorem^{ph} 4.2.21), the topological skeleton of an anomaly-free 1d topological order A is a (unitary) multi-fusion category \mathcal{A} satisfying $\mathfrak{Z}_1(\mathcal{A}) \simeq \text{Vec}$ as braided fusion categories. \diamond

Remark 4.2.28. Suppose C is an anomaly-free stable 2d topological order that admits a gapped boundary. Then the chiral central charge of C is 0. \diamond

4.2.5 Properties of the Drinfeld center

Since the particle-like topological defects of an anomaly-free stable 2d topological order form a (unitary) modular tensor category, the boundary-bulk relation (Theorem^{ph} 4.2.21) immediately implies that the Drinfeld center of a (unitary) fusion category should be a (unitary) modular tensor category. This consequence can be made into a mathematically rigorous theorem.

Theorem 4.2.29 ([Müg03b]). Let \mathcal{A} be a spherical fusion category. Then its Drinfeld center $\mathfrak{Z}_1(\mathcal{A})$ is a modular tensor category.

Remark 4.2.30. There are several variants of the above theorem. If \mathcal{A} is only a fusion category, $\mathfrak{Z}_1(\mathcal{A})$ is a nondegenerate braided fusion category [DGNO10]. If \mathcal{A} is a unitary fusion category, $\mathfrak{Z}_1(\mathcal{A})$ is a unitary modular tensor category [Müg03b, Gal14]. \diamond

Remark 4.2.31. We do not go through the details of the above theorem, but list some data and properties of the Drinfeld center $\mathfrak{Z}_1(\mathcal{A})$ of a spherical fusion category \mathcal{A} :

1. The left dual of $(x, \gamma_{-,x}) \in \mathfrak{Z}_1(\mathcal{A})$ is $(x^L, \tilde{\gamma}_{-,x^L})$, where the half-braiding $\tilde{\gamma}_{y,x^L} : y \otimes x^L \rightarrow x^L \otimes y$ is the left dual of $\gamma_{y^R,x} : y^R \otimes x \rightarrow x \otimes y^R$.
2. The pivotal structure $a_{(x,\gamma_{-,x})} : (x, \gamma_{-,x}) \rightarrow (x, \gamma_{-,x})^{LL}$ is given by the pivotal structure a_x of \mathcal{A} . It is spherical and $\dim(x, \gamma_{-,x}) = \dim(x)$. Therefore, the forgetful functor $I : \mathfrak{Z}_1(\mathcal{A}) \rightarrow \mathcal{A}$ preserves the quantum dimension of objects.
3. The Gauss sums $\tau^\pm(\mathfrak{Z}_1(\mathcal{C}))$ are both equal to $\dim(\mathcal{C})$ [Müg03b]. Thus $\dim(\mathfrak{Z}_1(\mathcal{C})) = \dim(\mathcal{C})^2$ [Müg03b, ENO05] and the additive central charge of $\mathfrak{Z}_1(\mathcal{C})$ is $c = 0 \pmod{8}$.
4. The Frobenius-Perron dimension of $\mathfrak{Z}_1(\mathcal{A})$ is $\text{FPdim}(\mathfrak{Z}_1(\mathcal{A})) = \text{FPdim}(\mathcal{A})^2$ [EO04]. \diamond

Example 4.2.32. Let G be a finite group. The Drinfeld center of both $\text{Rep}(G)$ and Vec_G are equivalent to the quantum double category \mathcal{D}_G (see Section 3.6.3) as modular tensor categories. \heartsuit

Exercise 4.2.33. Prove that $\mathfrak{Z}_1(\text{Vec}_G)$ is equivalent to \mathcal{D}_G as categories. Hint: use Exercise 3.3.46 (2) to determine a half-braiding in $\mathfrak{Z}_1(\text{Vec}_G)$.

Example 4.2.34. Let G be a finite group and $\omega \in Z^3(G; \mathbb{C}^\times)$. The Drinfeld center $\mathfrak{Z}_1(\text{Vec}_G^\omega)$ is not equivalent to \mathcal{D}_G in general (see [DS17] for an equivalent description of $\mathfrak{Z}_1(\text{Vec}_G^\omega)$ that is similar to \mathcal{D}_G).

Suppose $(V, \rho) \in \text{Rep}(G)$ is a finite-dimensional G -representation. The vector space V can be viewed as a G -graded vector space with the trivial G -grading so that $V \in \text{Vec}_G^\omega$, i.e.,

$$V_g = \begin{cases} V, & g = e, \\ 0, & g \neq e. \end{cases}$$

For every $g \in G$, define a morphism in Vec_G^ω by

$$\begin{aligned} \gamma_{g,V} : \mathbb{C}_{(g)} \otimes_{\mathbb{C}} V &\rightarrow V \otimes_{\mathbb{C}} \mathbb{C}_{(g)} \\ 1 \otimes_{\mathbb{C}} v &\mapsto \rho(g)(v) \otimes_{\mathbb{C}} \omega(g, e, g). \end{aligned}$$

Then $\{\gamma_{g,V}\}_{g \in G}$ defines a half-braiding on $V \in \text{Vec}_G^\omega$. In other words, $(V, \{\gamma_{g,V}\}_{g \in G}) \in \mathfrak{Z}_1(\text{Vec}_G^\omega)$.

It is straightforward to verify that the above construction defines a fully faithful \mathbb{C} -linear braided monoidal functor $\text{Rep}(G) \rightarrow \mathfrak{Z}_1(\text{Vec}_G^\omega)$. We say that $\mathfrak{Z}_1(\text{Vec}_G^\omega)$ is a *minimal modular extension* [Müg00] of $\text{Rep}(G)$. Moreover, all minimal modular extensions of $\text{Rep}(G)$ are of this form [DGNO10]. \heartsuit

Remark 4.2.35. If \mathcal{A} is a fusion category, the forgetful functor $I : \mathfrak{Z}_1(\mathcal{A}) \rightarrow \mathcal{A}$ is *surjective*, in the sense that any simple object $a \in \mathcal{A}$ appears as a direct summand of $I(x)$ for some object $x \in \mathfrak{Z}_1(\mathcal{A})$. However, this is not true if \mathcal{A} is a multi-fusion category. \diamond

Remark 4.2.36. A simple topological defect in the bulk may not be simple after moving to the boundary. Mathematically, if \mathcal{A} is a fusion category, the forgetful functor $I : \mathfrak{Z}_1(\mathcal{A}) \rightarrow \mathcal{A}$ may carry a simple object to a non-simple object. \diamond

4.3 Gapped 1d domain walls

A gapped boundary of an anomaly-free stable 2d topological order \mathcal{C} can be viewed as a gapped domain wall between \mathcal{C} and the trivial 2d topological order 1_2 . In this subsection, we study general gapped 1d domain walls between two anomaly-free stable 2d topological orders by the boundary-bulk relation in 2d (Theorem^{ph} 4.2.21).

4.3.1 General theory of 1d domain walls

The study of general 1d domain walls can be reduced to the study of boundaries by the folding trick (see Figure 51).

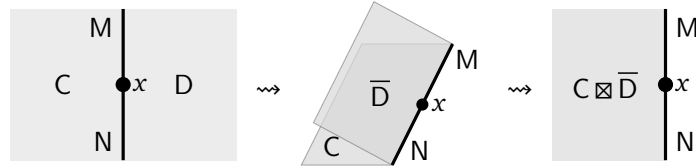


Figure 51: the folding trick

Suppose M is a gapped 1d domain wall between two anomaly-free stable 2d topological orders \mathcal{C} and \mathcal{D} .

- (a) The topological skeleton \mathcal{M} of the 1d topological order M is a (unitary) multi-fusion category. The particle-like topological defects in \mathcal{C} and \mathcal{D} form unitary modular tensor categories \mathcal{C} and \mathcal{D} , respectively.
- (b) The left-bulk-to-domain-wall map $F : \mathcal{C} \rightarrow \mathcal{M}$ is a central functor. Similarly the right-bulk-to-domain-wall map is a central functor $G : \mathcal{D} \rightarrow \mathcal{M}$. Note that the orientation is reversed.
- (c) The folding trick shows that M is a boundary of $\mathcal{C} \boxtimes \overline{\mathcal{D}}$, and the bulk-to-boundary map is given by $F \boxtimes G : \mathcal{C} \boxtimes \overline{\mathcal{D}} \rightarrow \mathcal{M}$. By the boundary-bulk relation in 2d, $F \boxtimes G$ is a central functor whose central structure is a braided equivalence $\mathcal{C} \boxtimes \overline{\mathcal{D}} \rightarrow \mathfrak{Z}_1(\mathcal{M})$.

These discussions directly give rise to the following definition [Lur17, AFT16, KZ18a, HPT16].

Definition 4.3.1. Let \mathcal{C}, \mathcal{D} be braided fusion categories.

- (1) A *(multi-)fusion left \mathcal{C} -module* is a (multi-)fusion category \mathcal{M} equipped with a braided functor $F: \mathcal{C} \rightarrow \mathfrak{Z}_1(\mathcal{M})$. It is called *closed* if F is an equivalence.
 - (2) A *(multi-)fusion right \mathcal{D} -module* is a (multi-)fusion category \mathcal{M} equipped with a braided functor $G: \overline{\mathcal{D}} \rightarrow \mathfrak{Z}_1(\mathcal{M})$. It is called *closed* if G is an equivalence.
 - (3) A *(multi-)fusion $(\mathcal{C}, \mathcal{D})$ -bimodule* is a (multi-)fusion category \mathcal{M} equipped with braided functors $F: \mathcal{C} \rightarrow \mathfrak{Z}_1(\mathcal{M})$ and $G: \overline{\mathcal{D}} \rightarrow \mathfrak{Z}_1(\mathcal{M})$. It is called *closed* if $F \boxtimes G: \mathcal{C} \boxtimes \overline{\mathcal{D}} \rightarrow \mathfrak{Z}_1(\mathcal{M})$ is an equivalence.
-

Using the new language of multi-fusion modules, we conclude that \mathcal{M} is a closed multi-fusion $(\mathcal{C}, \mathcal{D})$ -bimodule. In particular, the boundary-bulk relation in 2d (Theorem^{ph} 4.2.21) simply says that \mathcal{A} is a closed multi-fusion left \mathcal{C} -module.

Example 4.3.2. Given a stable 2d topological order C , by restricting it to a line we get the trivial 1d domain wall in C , denoted by C_1 (see Figure 52). Clearly the particle-like topological defects on the trivial domain wall are the same as those in C . In other words, the topological skeleton of the trivial domain wall is the fusion category \mathcal{C} , which is obtained by forgetting the braiding of the braided fusion category \mathcal{C} of particle-like topological defects in C . Both the left-bulk-to-domain-wall-map and right-bulk-to-domain-wall-map are the identity functor $\text{id}_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{C}$. Their central structures, i.e., the half-braiding of bulk topological defects with domain wall topological defects, are given by the braiding of \mathcal{C} , because there is essentially no domain wall.

Mathematically, given a braided fusion category \mathcal{C} , two canonical braided functors defined in Remark 4.2.16 give rise to a single braided functor $\mathcal{C} \boxtimes \overline{\mathcal{C}} \rightarrow \mathfrak{Z}_1(\mathcal{C})$, which equips \mathcal{C} with a canonical fusion $(\mathcal{C}, \mathcal{C})$ -bimodule structure. Moreover, the braided fusion category \mathcal{C} is nondegenerate (i.e., $\mathfrak{Z}_2(\mathcal{C}) = \text{Vec}$ is trivial) if and only if this canonical functor $\mathcal{C} \boxtimes \overline{\mathcal{C}} \rightarrow \mathfrak{Z}_1(\mathcal{C})$ is an equivalence [Müg03b, DGNO10], i.e., this fusion $(\mathcal{C}, \mathcal{C})$ -bimodule \mathcal{C} is closed. ♥

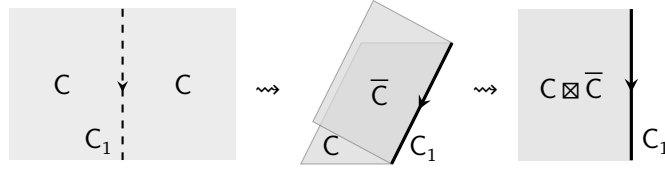


Figure 52: By restricting C to a line we get the trivial domain wall C_1 in C . The folding trick shows that $\mathcal{C} \boxtimes \overline{\mathcal{C}}$ is the bulk of C if C is anomaly-free.

Example 4.3.3. Let us give a more general construction. Given a braided fusion category \mathcal{C} and a braided autoequivalence $\phi: \mathcal{C} \rightarrow \mathcal{C}$, there is a fusion $(\mathcal{C}, \mathcal{C})$ -bimodule structure on \mathcal{C} defined by

$$\mathcal{C} \boxtimes \overline{\mathcal{C}} \xrightarrow{\text{id}_{\mathcal{C}} \boxtimes \phi} \mathcal{C} \boxtimes \overline{\mathcal{C}} \rightarrow \mathfrak{Z}_1(\mathcal{C}),$$

where the second functor is the canonical one defined in Example 4.3.2. So the left-bulk-to-boundary-map is $\text{id}_{\mathcal{C}}$ and the right-bulk-to-boundary-map is ϕ . We denote this fusion $(\mathcal{C}, \mathcal{C})$ -bimodule by \mathcal{C}_{ϕ} . Similarly, \mathcal{C}_{ϕ} is closed if and only if \mathcal{C} is nondegenerate. ♥

As depicted in Figure 53, if two domain walls M and N are very close to each other, they can be viewed as a single domain wall between C and E , denoted by $M \boxtimes_D N$. This is an example of dimensional reduction. Clearly, if D is the trivial 2d topological order, such a fusion process coincides with the stacking operation. The topological skeleton of $M \boxtimes_D N$ is denoted by $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$, called the relative tensor product of \mathcal{M} and \mathcal{N} over \mathcal{D} .

Remark 4.3.4. We only consider the case that D is stable. Surprisingly, the relative tensor product $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$ only depends on the multi-fusion categories \mathcal{M}, \mathcal{N} , the braided fusion category \mathcal{D} and the

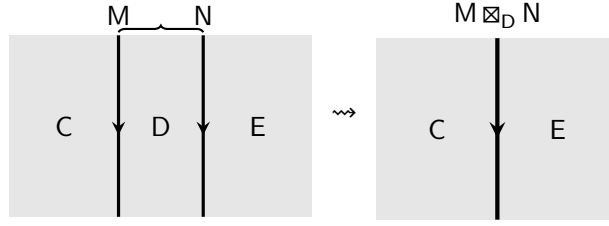


Figure 53: the fusion of two 1d domain walls

multi-fusion module structures of \mathcal{M}, \mathcal{N} over \mathcal{D} . Similar to the Deligne tensor product (see Definition 3.5.19), the relative tensor product $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$ is defined by a universal property (for example, see [Tam01, Müg03a, JL09, ENO10, Gre10, GS12, DNO12, DN13, GS15, FSV13, DSPS19, KZ18a]). In particular, $\mathcal{M} \boxtimes_{\text{vec}} \mathcal{N}$ agrees with the Deligne tensor product $\mathcal{M} \boxtimes \mathcal{N}$. \diamond

In particular, when $C = D = E$, the fusion of 1d domain walls defines a multiplication on the set of gapped 1d domain walls in the 2d topological order C (see Figure 5).

Remark 4.3.5. We say a 1d domain wall M between two 2d topological orders C and D is *invertible* if there exists another domain wall N between D and C such that their fusions are trivial domain walls:

$$M \boxtimes_D N = C_1, \quad N \boxtimes_C M = D_1.$$

Also N is called a inverse of M . Indeed, the inverse N , if exists, must be the time-reversal \overline{M} by a general argument.

The particle-like topological defects can be moved across the invertible domain wall. Suppose M is an invertible domain wall between two anomaly-free stable 2d topological orders C and D . As depicted in Figure 54, if we move a topological defect $x \in \mathcal{D}$ across M to C , this process produces a ‘bubble’ in C . Two horizontal domain walls can be annihilated because $M \boxtimes_D \overline{M} = C_1$. We denote bubble containing the topological defect x by $\phi(x)$. Then $x \mapsto \phi(x)$ defines a braided equivalence $\phi: \mathcal{D} \rightarrow \mathcal{C}$. Similarly, moving topological defects from C to D defines a braided equivalence $\psi: \mathcal{C} \rightarrow \mathcal{D}$, which is clearly a quasi-inverse of ϕ . This argument also show that the topological skeleton of an invertible domain wall M in a 2d topological order C must have the form \mathcal{C}_ϕ for some braided autoequivalence ϕ of \mathcal{C} (see Example 4.3.3).

Conversely, let \mathcal{C} be a nondegenerate braided fusion category. Suppose ϕ, ψ are two braided autoequivalences of \mathcal{C} , then we have an equivalence

$$\mathcal{C}_\phi \boxtimes_{\mathcal{C}} \mathcal{C}_\psi \simeq \mathcal{C}_{\phi \circ \psi}$$

as fusion $(\mathcal{C}, \mathcal{C})$ -bimodules. Hence, the assignment $\phi \mapsto \mathcal{C}_\phi$ is a group homomorphism from the group of isomorphism classes of braided autoequivalences of \mathcal{C} to the group of equivalence classes of invertible fusion $(\mathcal{C}, \mathcal{C})$ -bimodules. Indeed, this is a group isomorphism and both two groups are isomorphic to the Picard group of \mathcal{C} [ENO10, KZ18a]. \diamond

4.3.2 Gapped 1d domain walls in the toric code model

There are 6 different simple (i.e., stable) gapped 1d domain walls in the toric code model [LWW15] (see Remark 4.3.6). Four of them as depicted in Figure 55 are non-invertible with respect to the fusion of 1d domain walls.

Both four non-invertible domain walls are obtained by fusing two boundaries of the toric code model. We denote them by ‘ss’, ‘sr’, ‘rs’ and ‘rr’, where ‘s’ stands for the smooth boundary and ‘r’ stands for the rough boundary. For the ss domain wall, it is clear that there are 4 simple topological excitations on this domain wall: the trivial one (denoted by $\mathbb{1} \boxtimes \mathbb{1}$), an E particle on the left boundary (denoted by $E \boxtimes \mathbb{1}$), an E particle on the right boundary (denoted by $\mathbb{1} \boxtimes E$), and two E particles on both sides

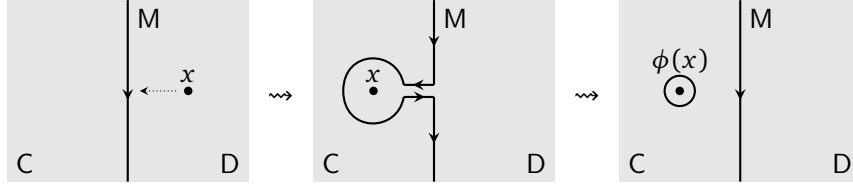


Figure 54: A topological defect $x \in \mathcal{D}$ can be moved across an invertible domain M and becomes a topological defect $\phi(x) \in \mathcal{C}$. This process defines a braided equivalence $\phi : \mathcal{D} \rightarrow \mathcal{C}$.

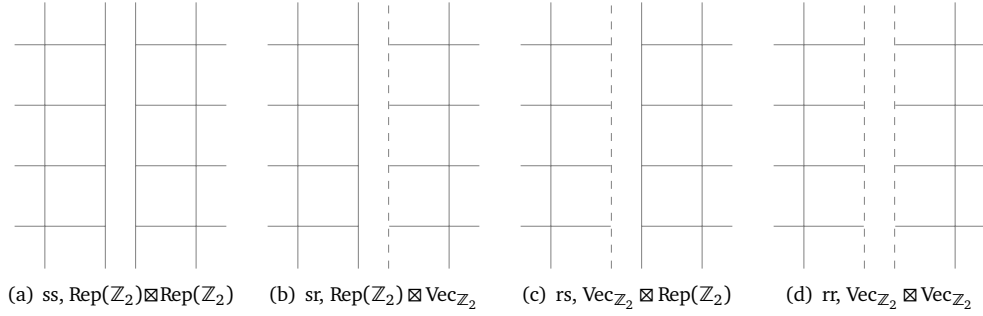


Figure 55: four non-invertible gapped 1d domain walls in the toric code model and their topological skeletons

(denoted by $E \boxtimes E$). Thus the topological skeleton of this domain wall is $\text{Rep}(\mathbb{Z}_2) \boxtimes \text{Rep}(\mathbb{Z}_2)$. The left-bulk-to-domain-wall map is

$$\mathbb{1}, m \mapsto \mathbb{1} \boxtimes \mathbb{1}, \quad e, f \mapsto E \boxtimes \mathbb{1},$$

and the right-bulk-to-domain wall map is

$$\mathbb{1}, m \mapsto \mathbb{1} \boxtimes \mathbb{1}, \quad e, f \mapsto \mathbb{1} \boxtimes E.$$

Similarly one can find the topological skeletons of the other three domain walls. It is easy to show that these four domain walls are different by checking their bulk-to-domain-wall maps.

The other two simple 1d domain walls in the toric code model are invertible. One is the trivial domain wall (see Example 4.3.2), another one is the dislocation as depicted in Figure 56. There is still a spin-1/2 on each edge, except those on the dashed line. The B_p operators on the domain wall are defined to be, for example, $B_p = \sigma_z^1 \sigma_z^2 \sigma_z^3 \sigma_x^4$ and $B_q = \sigma_z^5 \sigma_z^6 \sigma_z^7 \sigma_x^8$. The A_v and B_p operators in the bulk are the same as before.

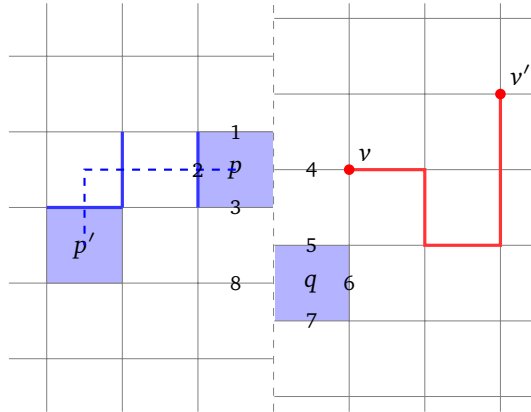


Figure 56: the e - m -exchange domain wall in toric code

There are 4 simple topological excitations on the domain wall: the trivial one $\mathbb{1}$, only one $B_p = -1$ on the left of the domain wall (denoted by X), only one $B_q = -1$ on the right of the domain wall (denoted by Y), and two $B_p = -1$ on both sides (denoted by Z). It is clear that the fusion rules are given by

$$X \otimes X = Y \otimes Y = Z \otimes Z = \mathbb{1}, \quad X \otimes Y = Y \otimes X = Z.$$

As depicted in Figure 56, by using an m string operator, an m particle in the left bulk can be moved to the domain wall and becomes an X particle. If we further apply a σ_z^4 operator, an X particle at p becomes an e particle at v , which be moved into the right bulk. Therefore, the left-bulk-to-domain-wall map is

$$\mathbb{1} \mapsto \mathbb{1}, \quad m \mapsto X, \quad e \mapsto Y, \quad f \mapsto Z,$$

and the right-bulk-to-domain-wall map is

$$\mathbb{1} \mapsto \mathbb{1}, \quad e \mapsto X, \quad m \mapsto Y, \quad f \mapsto Z.$$

This domain wall is also called the e - m -exchange domain wall, because if we move an e particle across this domain wall, it becomes an m particle. The exchange of e and m defines a braided autoequivalence $\phi : \mathcal{TC} \rightarrow \mathcal{TC}$, and the topological skeleton of this domain wall is precisely \mathcal{TC}_ϕ (see Example 4.3.3).

Remark 4.3.6. We have constructed six gapped domain walls in the 2d toric code model explicitly. We can not yet claim that we have found all gapped domain walls. How do we know if they are complete? Such a completeness question can only be answered in a model-independent way. Indeed, a positive answer to this question was provided by the bootstrap theory of 1d condensation developed in [Kon14a] and the boundary-bulk relation [KWZ15]. \diamond

Remark 4.3.7. The gapped domain wall ‘rr’ can be obtained by condensing e particles [KW14], or more precisely, condensing the algebra $\mathbb{1} \oplus e$. Similarly, the domain wall ‘ss’ can be obtained by condensing m particles or the algebra $\mathbb{1} \oplus m$, and the e - m -exchange domain wall can be obtained by condensing f particles or the algebra $\mathbb{1} \oplus f$. The other two domain walls ‘rs’ and ‘sr’ can also be obtained by condensing certain algebras whose underlying objects are $\mathbb{1} \oplus e \oplus m \oplus f$, and two different algebra structures on $\mathbb{1} \oplus e \oplus m \oplus f$ correspond to $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2; U(1)) \simeq \mathbb{Z}_2$. \diamond

4.3.3 Fusion rules of gapped 1d domain walls in the toric code model

Let us compute the fusion rules of gapped 1d domain walls in the toric code model.

First, let us consider the fusion rules of e - m -exchange domain wall with other domain walls. The fusion of two e - m -exchange domain walls is the trivial domain wall (see Figure 57 (a)). Indeed, if we move an e particle across the domain wall, it changes twice and remains to be e . Similarly, any particle-like topological defects in the bulk can be moved across the domain wall and does not change. Mathematically, we have $\mathcal{TC}_\phi \boxtimes_{\mathcal{TC}} \mathcal{TC}_\phi \simeq \mathcal{TC}$.

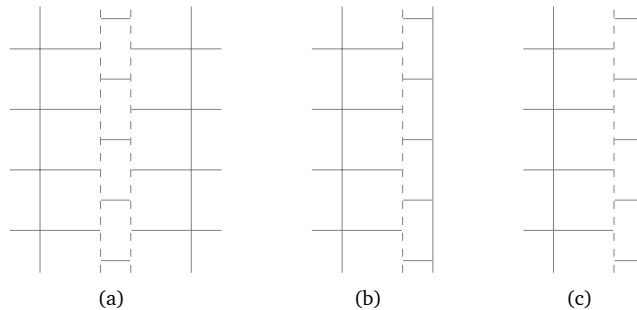


Figure 57: the fusion of the e - m -exchange domain wall and other domain walls

The fusion rules of the e - m -exchange domain wall with four non-invertible domain walls reduce to the cases in Figure 57 (b) and (c). The fusion of the e - m -exchange domain wall with the smooth

boundary is the rough boundary, because an e particle becomes trivial after moving to the boundary, and an m particle becomes an E particle on the boundary. Similarly, the fusion of the e - m -exchange domain wall with the rough boundary is the smooth boundary. Mathematically, we have

$$\mathcal{TC}_\phi \boxtimes_{\mathcal{TC}} \text{Rep}(\mathbb{Z}_2) \simeq \text{Vec}_{\mathbb{Z}_2}, \quad \mathcal{TC}_\phi \boxtimes_{\mathcal{TC}} \text{Vec}_{\mathbb{Z}_2} \simeq \text{Rep}(\mathbb{Z}_2).$$

The fusion rules of four non-invertible domain walls in the toric code model reduce to the fusion of two boundaries.

Figure 58 (a) depicts a narrow strip of the toric code model with a smooth boundary and a rough boundary. This quasi-1d system can be viewed as an anomaly-free 1d topological order. There is no nontrivial topological excitations in this topological order. For example, an e particle in the bulk can be annihilated on the rough boundary, and an m particle can be annihilated on the smooth boundary. Similarly, an E particle on the smooth boundary can be moved into the bulk and then annihilated on the rough boundary by a string operators. Such a string operator is a local operator because the strip is so narrow (with respect to the length scale). Hence the fusion of a smooth boundary with a rough boundary is the trivial 1d topological order:

$$\text{Rep}(\mathbb{Z}_2) \boxtimes_{\mathcal{TC}} \text{Vec}_{\mathbb{Z}_2} \simeq \text{Vec}.$$

Since $\mathfrak{Z}_1(\text{Vec}) \simeq \text{Vec}$, the result is consistent with the boundary-bulk relation in 2d.

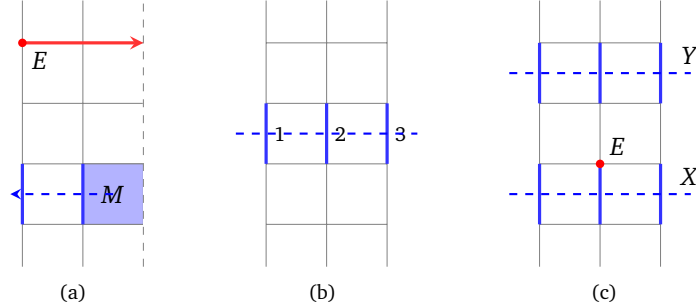


Figure 58: (a): a narrow strip with a smooth and a rough boundary; (b): a narrow strip with two smooth boundaries; (c): the degeneracy of E .

Figure 58 (b) depicts another narrow strip of the toric code model with two smooth boundaries. In this case, an m particle in the bulk can be annihilated on the boundary, but an e particles (or E particle) remains there. So we find two topological excitations $\mathbb{1}$ and E . Clearly we have $E \otimes E = \mathbb{1}$, as in the smooth boundary of the toric code model. It seems that the topological skeleton of this 1d topological order is $\text{Rep}(\mathbb{Z}_2)$. However, the Drinfeld center $\mathfrak{Z}_1(\text{Rep}(\mathbb{Z}_2))$ is not equivalent to Vec . So the result contradicts with the boundary-bulk relation in 2d. What is wrong?

The answer is that the fusion of two smooth boundaries is an unstable 1d topological order. Mathematically, $\text{Rep}(\mathbb{Z}_2) \boxtimes_{\mathcal{TC}} \text{Rep}(\mathbb{Z}_2)$ is a multi-fusion category. Let us compute the GSD of this quasi-1d lattice model on a circle [WW15]. Suppose there is an $L \times N$ square lattice and we impose a periodic boundary condition vertically. So the number of vertices is $V = (N + 1)L$, the number of edges is $E = (2N + 1)L$, and the number of plaquettes is $F = NL$. Moreover, we have $\prod_v A_v = 1$. Thus the GSD is $2^{E-(V-1)-F} = 2$.

We can also see this two-fold degeneracy in the following way [KWZ15]. Consider an m string operator $X := \sigma_x^1 \sigma_x^2 \sigma_x^3$ (the dashed line in Figure 58 (b)), which is a local operator because the strip is so narrow (with respect to the length scale). Clearly X commutes with all A_v and B_p operators. It follows that X acts on the ground state subspace invariantly, and two ground states correspond to two eigenspaces with $X = \pm 1$.

Remark 4.3.8. The existence of the operator X implies that the ground state subspace is degenerate on an open interval (without boundary), not only on a circle. More generally, the GSD of an anomaly-free 1d topological order on a circle and on an open interval should be equal. \diamond

The ground state subspace can be decomposed as the direct sum of two eigenspaces of X . Therefore, the trivial topological excitation $\mathbb{1}$ can be decomposed as the direct sum of simple ones $\mathbb{1} = \mathbb{1}_+ \oplus \mathbb{1}_-$, where X acts on $\mathbb{1}_\pm$ as ± 1 .

The fusion rules are given by $\mathbb{1}_\pm \otimes \mathbb{1}_\pm = \mathbb{1}_\pm$ and $\mathbb{1}_\pm \otimes \mathbb{1}_\mp = 0$. Indeed, if the states on both the upper and lower half strip has eigenvalue $X = 1$, then so is the state on the whole strip; if the state on the lower half strip has $X = 1$ but the state on the upper half strip has $X = -1$, they can not be combined to a state on the whole strip unless there are some defects. This is because the string operator X can be moved by applying A_v operators. If the system is at the ground state thus all $A_v = 1$, then moving X should not change the eigenvalue. The fusion rule $\mathbb{1}_+ \otimes \mathbb{1}_- = 0$ means this configuration is physically forbidden.

Similarly, the topological excitation E can also be decomposed as the direct sum of simple ones. Suppose there is an E particle located at a vertex v . The subspace of states with $A_v = -1$ also has dimension 2 and can be decomposed by the eigenvalue of X (see Figure 58 (c)). But it can also be decomposed by the eigenvalues of Y . It can be verified that if a state in this subspace has eigenvalue $X = 1$, it must have eigenvalue $Y = -1$ because there is only one $A_v = -1$. If we view the state on the lower half strip as a topological excitation $\mathbb{1}_+$ and the state on the upper half strip as $\mathbb{1}_-$, this fact means $\mathbb{1}_+ \otimes E = E \otimes \mathbb{1}_- = \mathbb{1}_+ \otimes E \otimes \mathbb{1}_-$. We denote $E_{+-} := \mathbb{1}_+ \otimes E \otimes \mathbb{1}_-$.

Also we define $E_{-+} := \mathbb{1}_- \otimes E \otimes \mathbb{1}_+$, then clearly $E = E_{+-} \oplus E_{-+}$. It follows that the fusion rules are

$$\begin{aligned} E_{+-} \otimes E_{+-} &= \mathbb{1}_+, & E_{-+} \otimes E_{-+} &= \mathbb{1}_-, & E_{+-} \otimes E_{-+} &= 0, & E_{-+} \otimes E_{+-} &= 0, \\ \mathbb{1}_+ \otimes E_{+-} &= E_{+-} = E_{-+} \otimes \mathbb{1}_-, & \mathbb{1}_- \otimes E_{-+} &= E_{-+} = E_{+-} \otimes \mathbb{1}_+, \\ \mathbb{1}_+ \otimes E_{-+} &= 0 = E_{-+} \otimes \mathbb{1}_-, & \mathbb{1}_- \otimes E_{+-} &= 0 = E_{+-} \otimes \mathbb{1}_+. \end{aligned}$$

These complicated fusion rules can be written in a more compact way. Define a multi-fusion category $M_2(\text{Vec})$ as follows:

- As a category, $M_2(\text{Vec})$ is the direct sum of 4 copies of Vec . Its objects are written as

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$$

where each $V_{ij} \in \text{Vec}$.

- The tensor product of $M_2(\text{Vec})$ is given by the matrix product and the usual tensor product. More precisely, we have

$$\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \otimes \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} = \begin{pmatrix} V_{11} \otimes W_{11} \oplus V_{12} \otimes W_{21} & V_{11} \otimes W_{12} \oplus V_{12} \otimes W_{22} \\ V_{21} \otimes W_{11} \oplus V_{22} \otimes W_{21} & V_{21} \otimes W_{12} \oplus V_{22} \otimes W_{22} \end{pmatrix}.$$

- The tensor unit of $M_2(\text{Vec})$ is

$$\begin{pmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{pmatrix} = \begin{pmatrix} \mathbb{C} & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{C} \end{pmatrix},$$

which is not a simple object.

Then we have an equivalence $\text{Rep}(\mathbb{Z}_2) \boxtimes_{\mathbb{Z}\mathbb{C}} \text{Rep}(\mathbb{Z}_2) \simeq M_2(\text{Vec})$ of multi-fusion categories:

$$\mathbb{1}_+ \mapsto \begin{pmatrix} \mathbb{C} & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{+-} \mapsto \begin{pmatrix} 0 & \mathbb{C} \\ 0 & 0 \end{pmatrix}, \quad E_{-+} \mapsto \begin{pmatrix} 0 & 0 \\ \mathbb{C} & 0 \end{pmatrix}, \quad \mathbb{1}_- \mapsto \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{C} \end{pmatrix}.$$

Moreover, by directly computing the half-braidings one can prove that $\mathfrak{Z}_1(M_2(\text{Vec})) \simeq \text{Vec}$, which gives the correct boundary-bulk relation in 2d (see Exercise 4.3.12).

In other words, the fusion of two smooth boundaries is the direct sum of two copies of the trivial 1d topological order.

Remark 4.3.9. A topological order is not the same as its topological skeleton. This distinction becomes manifest when we consider the category of topological orders [KWZ15, KZ22a]. For example, the topological skeleton of the trivial 1d topological order 1_1 is Vec , but the topological skeleton of $1_1 \oplus 1_1$ is $M_2(\text{Vec})$. This is similar to the fact that $\text{End}(\mathbb{C} \oplus \mathbb{C}) \simeq M_2(\mathbb{C})$. \diamond

Remark 4.3.10. The construction of $M_2(\text{Vec})$ can be directly generalized to $M_n(\mathcal{C})$ for any fusion category \mathcal{C} and $n \geq 1$. Similarly we have $\mathfrak{Z}_1(M_n(\mathcal{C})) \simeq \mathfrak{Z}_1(\mathcal{C})$ as braided fusion categories. As an example, the relative tensor product of $\text{Rep}(\mathbb{Z}_2) \boxtimes \text{Vec}_{\mathbb{Z}_2}$ and $\text{Vec}_{\mathbb{Z}_2} \boxtimes \text{Vec}_{\mathbb{Z}_2}$ is

$$\text{Rep}(\mathbb{Z}_2) \boxtimes \text{Vec}_{\mathbb{Z}_2} \boxtimes_{\mathcal{T}\mathcal{C}} \text{Vec}_{\mathbb{Z}_2} \boxtimes \text{Vec}_{\mathbb{Z}_2} \simeq \text{Rep}(\mathbb{Z}_2) \boxtimes M_2(\text{Vec}) \boxtimes \text{Vec}_{\mathbb{Z}_2} \simeq M_2(\text{Rep}(\mathbb{Z}_2) \boxtimes \text{Vec}_{\mathbb{Z}_2}).$$

So the fusion of 1d domain walls ‘sr’ and ‘rr’ is not a simple gapped 1d domain wall, but the superposition of two copies of ‘sr’. The complete fusion rules of simple 1d domain walls in the toric code model are listed in Table 1. \diamond

Remark 4.3.11. The multi-fusion category $M_2(\text{Vec})$ can also be realized by the 1+1D quantum Ising model (without transversal field):

$$H = - \sum_i \sigma_z^i \sigma_z^{i+1}.$$

There are two ground states with all $\sigma_z^i = +1$ or $\sigma_z^i = -1$. Two 0d domain walls between them are also clear. Fix a site j , consider the state with $\sigma_z^i = +1$ for $i < j$ and $\sigma_z^i = -1$ for $i \geq j$, then there is a 0d domain wall at the site j , which can not be annihilated by local operators. Similarly, the state with $\sigma_z^i = -1$ for $i < j$ and $\sigma_z^i = +1$ for $i \geq j$ gives another 0d domain wall. \diamond

Exercise 4.3.12. For all $n \geq 1$, prove that $\mathfrak{Z}_1(M_n(\text{Vec})) \simeq \text{Vec}$ as braided fusion categories. Hint: use Exercise 3.3.46 (2) to determine a half-braiding in $\mathfrak{Z}_1(M_n(\text{Vec}))$. Use the same method to prove that $\mathfrak{Z}_1(M_n(\mathcal{C})) \simeq \mathfrak{Z}_1(\mathcal{C})$ for any fusion category \mathcal{C} .

Remark 4.3.13. The readers might wonder why an unstable 1d topological order, often regarded as something ‘unnatural’ or completely ignored in the most classifications, deserves a beautiful mathematical description of its topological skeleton. The reason is that the strip depicted in Figure 58 is stable when it is sufficiently wide. It is important to include them when we consider the fusion of topological orders, especially in the study of the category of topological orders [KWZ15, KZ22a, KZ22b]. \diamond

4.3.4 Witt equivalence

Definition 4.3.14. We say that two anomaly-free stable 2d topological orders \mathcal{C} and \mathcal{D} are *Witt equivalent* if there exists a gapped 1d domain wall between them. \blacksquare

Definition 4.3.15 ([DMNO13]). We say that two nondegenerate braided fusion categories \mathcal{C} and \mathcal{D} are *Witt equivalent* if there exists a closed fusion $(\mathcal{C}, \mathcal{D})$ -bimodule \mathcal{M} . \blacksquare

Let us verify that the Witt equivalence of 2d topological orders is indeed an equivalence relation:

- (a) An anomaly-free stable 2d topological order \mathcal{C} is Witt equivalent to itself, because there is always the trivial domain wall C_1 in \mathcal{C} (see Remark 4.3.2).
- (b) If there is a gapped 1d domain wall M between \mathcal{C} and \mathcal{D} , then we immediately get a domain wall \overline{M} between \mathcal{D} and \mathcal{C} by reversing the orientation.
- (c) Suppose M is a gapped 1d domain wall between \mathcal{C} and \mathcal{D} , and N is a gapped 1d domain wall between \mathcal{D} and \mathcal{E} . Their fusion $M \boxtimes_{\mathcal{D}} N$ is a gapped 1d domain wall between \mathcal{C} and \mathcal{E} (see Figure 53).

Similarly, the Witt equivalence of nondegenerate braided fusion categories is also an equivalence relation.

The stacking operation should preserve the Witt equivalence relation. More precisely, if M is a domain wall between \mathcal{C} and \mathcal{D} , and M' is a domain wall between \mathcal{C}' and \mathcal{D}' , then $M \boxtimes M'$ is a domain wall between $\mathcal{C} \boxtimes \mathcal{C}'$ and $\mathcal{D} \boxtimes \mathcal{D}'$ (see Figure 59). So the stacking operation defines a multiplication on the set of Witt equivalence classes of anomaly-free stable 2d topological orders.

- The multiplication is defined by $[C] \cdot [D] := [C \boxtimes D]$.
- There is an identity element given by the trivial 2d topological order $[1_2]$. Indeed $[1_2] \cdot [C] = [1_2 \boxtimes C] = [C] = [C] \cdot [1_2]$.

- The inverse of $[C]$ is given by the time-reversal $[\bar{C}]$. Indeed, $[C] \cdot [\bar{C}] = [C \boxtimes \bar{C}] = [1_2]$, where the gapped domain wall between $C \boxtimes \bar{C}$ and 1_2 is given by C_1 (see Figure 52).
- Moreover, this multiplication is commutative because $C \boxtimes D$ and $D \boxtimes C$ are the same topological order.

As a consequence, the Witt equivalence classes of 2d anomaly-free stable topological orders form an abelian group, where the multiplication is defined by the stacking. This abelian group is called the *Witt group* of anomaly-free stable 2d topological orders. Similarly we can define the *Witt group* of nondegenerate braided fusion categories [DMNO13].



Figure 59: the stacking of 2d topological orders and 1d domain walls between them

All topological orders in a given Witt equivalence class must have the same chiral central charge. Thus taking the chiral central charge defines a group homomorphism from the Witt group of anomaly-free stable 2d topological orders to \mathbb{Q} . Similarly, taking the additive central charge defines a group homomorphism from the Witt group of pseudo-unitary modular tensor categories (which is a subgroup of the Witt group of nondegenerate braided fusion categories) to $\mathbb{Q}/8\mathbb{Z}$.

4.4 Gapped 0d domain walls

In this subsection, we give a categorical description of 0d domain walls between gapped 1d domain walls and the boundary-bulk relation for 1d topological orders and their boundaries.

4.4.1 The module category of 0d topological orders

Recall that 0+1D topological defects and instantons in an nd topological order form a category for $n \geq 1$ (see Theorem^{ph} 3.2.12). For a 0d topological order x , we cannot talk about 0+1D topological defects of x , but 0+1D topological defects (or equivalently, 0d topological orders) living on the world line of x . Then Theorem^{ph} 3.2.12 still holds for a 0d topological order x in the following sense: all 0d topological orders living on the world line of x , together with instantons between them, form a category \mathcal{X} . By definition $x \in \mathcal{X}$. We always assume that \mathcal{X} is finite semisimple.

Remark 4.4.1. Here we only consider gapped 0d topological defects. However, it is subtle to say that a 0d system is gapped. In practice, a stable 0d system is said to be gapped if its GSD is 1. In other words, a ‘gapless’ 0d system has nontrivial GSD which is robust against perturbations. Usually this nontrivial GSD is protected by some symmetries. \diamond

Example 4.4.2. An anomaly-free 0d topological order is simply a quantum mechanic system because there is no thermodynamic limit. At zero temperature, the only observable is the ground state subspace. In other words, each 0d topological order is characterized by its ground state subspace, which is a finite-dimensional vector space (indeed, a Hilbert space). Thus the category of anomaly-free 0d topological order is equivalent to Vec . \heartsuit

Example 4.4.3. For an anomalous 0d topological order x , it is reasonable to talk about the thermodynamic limit if we choose an embedding of x into an ambient higher-dimensional topological order. In this case the objects in the category \mathcal{X} are 0d topological orders which are able to live in the same ambient topological order. For example, let C be an anomaly-free stable 2d topological order and \mathcal{C} be the modular tensor category of 0+1D topological defects of C . Then each $y \in \mathcal{C}$ is an anomalous 0d topological order embedding in C . Clearly the category of 0d topological orders living on the world line of y , i.e., the category of 0d topological orders embedding in C , is \mathcal{C} . \heartsuit

Now let us study the boundary-bulk relation of a 1d topological order and its 0d boundaries. Consider an anomaly-free 1d topological order \mathcal{P} and a 0d boundary x (see Figure 60). The topological skeleton of \mathcal{P} is a multi-fusion category \mathcal{P} and the category \mathcal{X} of 0d boundaries of \mathcal{P} (i.e., the category of 0d topological orders living on the boundary of \mathcal{P}) is finite semisimple.

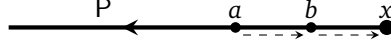


Figure 60: a 1d topological order \mathcal{P} and a 0d boundary x

As in the 2d case, a bulk topological defect $a \in \mathcal{P}$ can be moved to the boundary. This process may change the boundary topological order from x to another one, which is denoted by $a \odot x$. We do not only move topological defects, but also the world lines and instantons attached to them. Hence \odot is indeed a functor:

$$\begin{aligned} \odot: \mathcal{P} \times \mathcal{X} &\rightarrow \mathcal{X} \\ (a, x) &\mapsto a \odot x. \end{aligned}$$

The physical intuition implies that two different ways of fusing two bulk topological defects a, b with the boundary topological order x should be the same (see Figure 60). So we get an isomorphism (instanton) $s_{a,b,x}: (a \otimes b) \odot x \rightarrow a \odot (b \odot x)$. Finally, moving the trivial bulk topological defect $\mathbb{1} \in \mathcal{P}$ to the boundary should not change the boundary topological order x . Thus we get an isomorphism (instanton) $l_x: \mathbb{1} \odot x \rightarrow x$ for each $x \in \mathcal{X}$.

The above discussion leads to the following definition [Bén65].

Definition 4.4.4. Let \mathcal{P} be a monoidal category. A *left \mathcal{P} -module* or a *left module over \mathcal{P}* consists of the following data:

- a category \mathcal{X} ;
- a functor $\odot: \mathcal{P} \times \mathcal{X} \rightarrow \mathcal{X}$, where $\odot(a, x)$ is also denoted by $a \odot x$;
- a natural isomorphism $s_{a,b,x}: (a \otimes b) \odot x \rightarrow a \odot (b \odot x)$, called the *associator*;
- a natural isomorphism $l_x: \mathbb{1} \odot x \rightarrow x$, called the *left unitor*;

and these data satisfy the following conditions:

1. **(pentagon equation)** For any $a, b, c \in \mathcal{P}$ and $x \in \mathcal{X}$, the following diagram commutes:

$$\begin{array}{ccc} & ((a \otimes b) \otimes c) \odot x & \\ \alpha_{a,b,c} \odot \text{id}_x \swarrow & & \searrow \alpha_{a \otimes b, c, x} \\ (a \otimes (b \otimes c)) \odot x & & (a \otimes b) \odot (c \odot x) \\ \downarrow s_{a,b \otimes c, x} & & \downarrow s_{a, b, c \odot x} \\ a \odot ((b \otimes c) \odot x) & \xrightarrow{\text{id}_a \odot s_{b, c, x}} & a \odot (b \odot (c \odot x)) \end{array}$$

2. **(triangle equation)** For any $a \in \mathcal{P}$ and $x \in \mathcal{X}$, the following diagram commutes:

$$\begin{array}{ccc} (a \otimes \mathbb{1}) \odot x & \xrightarrow{s_{a, \mathbb{1}, x}} & a \otimes (\mathbb{1} \odot x) \\ \downarrow \rho_a \odot \text{id}_y & & \downarrow \text{id}_a \odot l_y \\ & a \odot x & \end{array}$$

When \mathcal{P} is a multi-fusion category, a left \mathcal{P} -module is called *finite semisimple* if it satisfies the following two conditions:

1. \mathcal{X} is a finite semisimple category.

2. The left action functor $\odot: \mathcal{P} \times \mathcal{X} \rightarrow \mathcal{X}$ is \mathbb{C} -bilinear (i.e., \mathbb{C} -linear in each variable). \blacksquare

Remark 4.4.5. Similarly we can define a *right module* over a monoidal category \mathcal{Q} , which is essentially the same as a left module over \mathcal{Q}^{rev} . Moreover, for two monoidal categories \mathcal{P}, \mathcal{Q} , a $(\mathcal{P}, \mathcal{Q})$ -bimodule is a category equipped with both a left \mathcal{P} -module structure and a right \mathcal{Q} -module structure, as well as the compatibility data/properties of these two structures. The details can be found in, for example, [EGNO15, Definition 7.1.7]. \diamond

Example 4.4.6. Let \mathcal{P} be a monoidal category. Then \mathcal{P} itself is a left \mathcal{P} -module with the left action defined by $a \odot b := a \otimes b$ for $a, b \in \mathcal{P}$. Similarly, \mathcal{P} is a $(\mathcal{P}, \mathcal{P})$ -bimodule. \heartsuit

Example 4.4.7. Let \mathcal{A}, \mathcal{B} be monoidal categories and $F: \mathcal{A} \rightarrow \mathcal{B}$ be a monoidal functor. Then \mathcal{B} is a $(\mathcal{A}, \mathcal{A})$ -bimodule with

$$a \odot b \odot a' := F(a) \otimes b \otimes F(a'), \quad a, a' \in \mathcal{A}, b \in \mathcal{B}.$$

More generally, every left \mathcal{B} -module \mathcal{M} can be viewed as a left \mathcal{A} -module with the left \mathcal{A} -action defined by $a \odot m := F(a) \odot m$ for $a \in \mathcal{A}$ and $m \in \mathcal{M}$. \heartsuit

Example 4.4.8. Let \mathcal{C} be a braided fusion category and \mathcal{M} be a multi-fusion left \mathcal{C} -module defined by a braided functor $F: \mathcal{C} \rightarrow \mathfrak{Z}_1(\mathcal{M})$. Then \mathcal{M} is a left \mathcal{C} -module with the left action defined by $a \odot x := F(a) \otimes x$ for $a \in \mathcal{C}$ and $x \in \mathcal{M}$. Also, \mathcal{M} is a right \mathcal{C} -module with the right action defined by $x \odot a := x \otimes F(a)$ for $a \in \mathcal{C}$ and $x \in \mathcal{M}$. This can be viewed as a special case of Example 4.4.7. \heartsuit

Example 4.4.9. Let \mathcal{P} be a rigid monoidal category and \mathcal{X} be a left \mathcal{P} -module. Then \mathcal{X}^{op} is naturally a right \mathcal{P} -module with

$$x \odot a := a^L \odot x, \quad a \in \mathcal{P}, x \in \mathcal{X}.$$

There is also another right \mathcal{P} -module structure defined by

$$x \odot a := a^R \odot x, \quad a \in \mathcal{P}, x \in \mathcal{X}.$$

If \mathcal{P} is equipped with a pivotal structure, these two right \mathcal{P} -module structures on \mathcal{X} are equivalent. \heartsuit

Remark 4.4.10. Let \mathcal{P} be a multi-fusion category. Suppose \mathcal{M}, \mathcal{N} are finite semisimple left \mathcal{P} -modules. Then their direct sum $\mathcal{M} \oplus \mathcal{N}$ (see Definition 3.3.28) is naturally a finite semisimple left \mathcal{P} -module. A finite semisimple left \mathcal{P} -module is called *indecomposable* if it can not be written as a direct sum of two nonzero finite semisimple left \mathcal{P} -modules. Given a multi-fusion category \mathcal{P} , there are only finitely many indecomposable finite semisimple left \mathcal{P} -modules (up to equivalence) [EGNO15, Corollary 9.1.6]. \diamond

Remark 4.4.11. Let \mathcal{P} be a monoidal category and \mathcal{M}, \mathcal{N} be left \mathcal{P} -modules. A *left \mathcal{P} -module functor* from \mathcal{M} to \mathcal{N} is a functor $F: \mathcal{M} \rightarrow \mathcal{N}$ equipped with a natural isomorphism

$$F_{x,m}^2: x \odot F(m) \rightarrow F(x \odot m), \quad x \in \mathcal{P}, m \in \mathcal{M},$$

that satisfies certain coherence conditions. Given two left \mathcal{P} -module functors $F, G: \mathcal{M} \rightarrow \mathcal{N}$, a *left \mathcal{P} -module natural transformation* from F to G is a natural transformation $\alpha: F \Rightarrow G$ satisfying certain coherence conditions. Similarly one can define right module functors and right module natural transformations, as well as bimodule functors and bimodule natural transformations. \diamond

In the language of module categories, we conclude that the category \mathcal{X} of 0d boundaries of a bulk 1d topological order \mathcal{P} is a finite semisimple left module over the topological skeleton \mathcal{P} (which is a multi-fusion category).

Note that the requirements that \mathcal{P} is anomaly-free and x is a boundary are not necessary. For example, given a potentially anomalous 1d topological order \mathcal{P} and a gapped domain wall x between \mathcal{P} and other 1d topological orders, \mathcal{X} is still a left module over \mathcal{P} .

Example 4.4.12. Consider an anomaly-free 2d topological order \mathcal{C} with two 1d boundaries \mathcal{P} and \mathcal{Q} , which are connected by a 0d domain wall x (see Figure 61). Mathematically, the category \mathcal{X} of 0d topological orders living on the world line of x , i.e., the category of 0d domain walls between \mathcal{P} and \mathcal{Q} , is a $(\mathcal{P}, \mathcal{Q})$ -bimodule, and both \mathcal{P} and \mathcal{Q} are closed multi-fusion left \mathcal{C} -modules.

When $\mathcal{P} = \mathcal{Q}$, the category \mathcal{X} is equal to the category \mathcal{P} . Both the left and right \mathcal{P} -module action on \mathcal{X} are given by the tensor product of \mathcal{P} (see Example 4.4.6). \heartsuit

Example 4.4.13. Let C be an anomaly-free 2d topological order. As depicted in Figure 61, a 0+1D topological defect $y \in \mathcal{C}$ is a 0d domain wall between the trivial 1d domain wall C_1 (see Example 4.3.2) and itself. \heartsuit

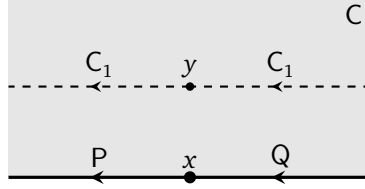


Figure 61: a typical configuration of 0d topological orders

Example 4.4.14. Let G be a finite group. A *projective G -representation* is a vector space V equipped with a group homomorphism $\bar{\rho} : G \rightarrow \text{PGL}(V)$, where $\text{PGL}(V) := \text{GL}(V)/\{\lambda \cdot \text{id}_V \mid \lambda \in \mathbb{C}^\times\}$ is the group of projective linear transformations on V . It is also the automorphism group of the projective space $\mathbb{P}(V)$.

Given a projective G -representation $(V, \bar{\rho})$, since the canonical projection map $\pi : \text{GL}(V) \rightarrow \text{PGL}(V)$ is surjective, one can always find a lifting of $\bar{\rho}$, i.e., a map (not necessarily a group homomorphism) $\rho : G \rightarrow \text{GL}(V)$ such that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \text{GL}(V) \\ & \searrow \bar{\rho} & \downarrow \pi \\ & & \text{PGL}(V) \end{array}$$

In general, one can not find a group homomorphism ρ such that the above diagram commutes. However, the equation $\bar{\rho}(g)\bar{\rho}(h) = \bar{\rho}(gh)$ implies that the map ρ satisfies the equation

$$\rho(g)\rho(h) = \beta(g, h) \cdot \rho(gh)$$

for some nonzero complex number $\beta(g, h) \in \mathbb{C}^\times$. In other words, ρ is ‘almost’ a group homomorphism up to some scalar $\beta(g, h)$. The associativity $(\rho(g)\rho(h))\rho(k) = \rho(g)(\rho(h)\rho(k))$ implies that the function $\beta : G \times G \rightarrow \mathbb{C}^\times$ satisfies the equation

$$\beta(g, h)\beta(gh, k) = \beta(h, k)\beta(g, hk),$$

or equivalently, $d\beta = 1$ (see Example 4.2.5). Such a function β is called a *2-cocycle* (valued in \mathbb{C}^\times). The space of 2-cocycles is denoted by $Z^2(G; \mathbb{C}^\times)$. We also call the pair (V, ρ) a *β -twisted G -representation*.

This 2-cocycle β depends on the choice of the lifting ρ . If we choose a different lifting ρ' of $\bar{\rho}$, the corresponding 2-cocycle β' may be different from β . Assume that $\rho'(g) = \phi(g) \cdot \rho(g)$ for some nonzero complex number $\phi(g) \in \mathbb{C}^\times$, then

$$\rho'(g)\rho'(h) = \phi(g)\phi(h) \cdot \rho(g)\rho(h) = \phi(g)\phi(h)\beta(g, h) \cdot \rho(gh).$$

It follows that

$$\beta'(g, h)\phi(gh) = \phi(g)\phi(h)\beta(g, h). \quad (4.4.1)$$

Define

$$(d\phi)(g, h) := \frac{\phi(h)\phi(g)}{\phi(gh)}, \quad g, h \in G.$$

Then (4.4.1) can be rewritten as $\beta' = \beta \cdot d\phi$. The operator d is usually called the *differential*. It is straightforward to check that $d\phi \in Z^2(G; \mathbb{C}^\times)$ is a 2-cocycle. A 2-cocycle obtained by the differential is called a *2-coboundary*. The space of 2-coboundaries (valued in \mathbb{C}^\times) is denoted by $B^2(G; \mathbb{C}^\times)$. The quotient group $H^2(G; \mathbb{C}^\times) := Z^2(G; \mathbb{C}^\times)/B^2(G; \mathbb{C}^\times)$ is called the *second group cohomology group* of G (with coefficients in \mathbb{C}^\times). Then the above discussion implies that there is a well-defined cohomology class $[\beta] \in H^2(G; \mathbb{C}^\times)$ associated to each projective representation $(V, \bar{\rho})$.

Now we fix a 2-cocycle $\beta \in Z^2(G; \mathbb{C}^\times)$. Given two β -twisted G -representation (V, ρ) and (W, σ) , a *homomorphism* between them is a linear map $f: V \rightarrow W$ satisfying $f \circ \rho(g) = \sigma(g) \circ f$ for all $g \in G$. All finite-dimensional β -twisted G -representations and homomorphisms between them form a \mathbb{C} -linear category, denoted by $\text{Rep}(G, \beta)$. Indeed, $\text{Rep}(G, \beta)$ is finite semisimple.

Suppose $(V, \rho) \in \text{Rep}(G)$ and $(W, \sigma) \in \text{Rep}(G, \beta)$. It is easy to see that the tensor product space $V \otimes_{\mathbb{C}} W$ equipped with the G -action defined by

$$g \mapsto \rho(g) \otimes_{\mathbb{C}} \sigma(g), \quad g \in G,$$

is also a finite-dimensional β -twisted G -representation, denoted by $(V, \rho) \odot (W, \sigma)$. This defines a left $\text{Rep}(G)$ -module structure on $\text{Rep}(G, \beta)$ with the left action $\odot: \text{Rep}(G) \times \text{Rep}(G, \beta) \rightarrow \text{Rep}(G, \beta)$. \heartsuit

Remark 4.4.15. For each integer $n > 1$ the cohomology group $H^2(\mathbb{Z}_n; \mathbb{C}^\times)$ is trivial. Also we have $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2; \mathbb{C}^\times) \simeq \mathbb{Z}_2$, and the nontrivial cohomology class can be induced from the following projective $\mathbb{Z}_2 \times \mathbb{Z}_2$ -representation on \mathbb{C}^2 :

$$(0, 0) \mapsto \text{id}_{\mathbb{C}^2}, \quad (1, 0) \mapsto \sigma_x, \quad (0, 1) \mapsto \sigma_y, \quad (1, 1) \mapsto \sigma_z.$$

Moreover, let β be the 2-cocycle induced by the above projective representation. Then $\text{Rep}(\mathbb{Z}_2 \times \mathbb{Z}_2, \beta) \simeq \text{Vec}$, and the only simple object (up to isomorphism) is the above one. \diamond

Remark 4.4.16. For every finite group G , the inclusion $U(1) \hookrightarrow \mathbb{C}^\times$ induces a group isomorphism $H^2(G; U(1)) \rightarrow H^2(G; \mathbb{C}^\times)$. \diamond

Remark 4.4.17. More generally, suppose H is a subgroup of G . Since the forgetful functor $F: \text{Rep}(G) \rightarrow \text{Rep}(H)$ is a monoidal functor (see Example 4.2.4), every left $\text{Rep}(H)$ -module is naturally a left $\text{Rep}(G)$ -module (see Example 4.4.7). In particular, $\text{Rep}(H, \beta)$ is a left $\text{Rep}(G)$ -module for each $\beta \in Z^2(H; \mathbb{C}^\times)$.

Indeed, every indecomposable finite semisimple left $\text{Rep}(G)$ -module is equivalent to $\text{Rep}(H, \beta)$ for some subgroup H of G and 2-cocycle $\beta \in Z^2(H; \mathbb{C}^\times)$ [Ost03a]. \diamond

Remark 4.4.18. Let G be a finite group and $\beta \in Z^2(G; \mathbb{C}^\times)$. The β -twisted group algebra $\mathbb{C}[G, \beta]$ is defined as follows. Its underlying vector space is freely generated by G , or equivalently, spanned by a family of symbols $\{v_g\}_{g \in G}$. The multiplication is defined by $v_g \cdot v_h := \beta(g, h) \cdot v_{gh}$ for $g, h \in G$. Then $\text{Rep}(G, \beta)$ is equivalent to the category $\text{LMod}_{\mathbb{C}[G, \beta]}(\text{Vec})$ of finite-dimensional modules over $\mathbb{C}[G, \beta]$. \diamond

4.4.2 The boundary-bulk relation in 1d

Let \mathcal{P} be an anomaly-free 1d topological order and \mathcal{P} be its topological skeleton. The category of 0d boundaries of \mathcal{P} is denoted by \mathcal{X} . Moving a bulk topological excitation $a \in \mathcal{P}$ to the boundary changes the boundary topological order and defines a functor from \mathcal{X} to itself. Mathematically, by fixing the first variable of the module action $\odot: \mathcal{P} \times \mathcal{X} \rightarrow \mathcal{X}$ we get a \mathbb{C} -linear functor $a \odot -: \mathcal{X} \rightarrow \mathcal{X}$ for each $a \in \mathcal{P}$.

We use $\mathfrak{Z}_0(\mathcal{X})$ to denote the category $\text{Fun}(\mathcal{X}, \mathcal{X})$ of \mathbb{C} -linear functors from a finite semisimple category \mathcal{X} to itself (see Exercise 3.3.43). It is a monoidal category where the tensor product is defined by the composition of functors. Then we see that the left \mathcal{P} -module structure on \mathcal{X} induces a monoidal functor

$$\begin{aligned} \mathcal{P} &\rightarrow \text{Fun}(\mathcal{X}, \mathcal{X}) \\ a &\mapsto a \odot -. \end{aligned}$$

Moreover, \mathcal{X} is a left module over $\mathfrak{Z}_0(\mathcal{X})$ with the left action defined by

$$\begin{aligned} \mathfrak{Z}_0(\mathcal{X}) \times \mathcal{X} &\rightarrow \mathcal{X} \\ (F, x) &\mapsto F(x). \end{aligned}$$

Example 4.4.19. Let \mathcal{X} be a finite semisimple category with n non-isomorphic simple objects, i.e., $\mathcal{X} \simeq \text{Vec}^{\oplus n}$. Then $\mathfrak{Z}_0(\mathcal{X}) \simeq M_n(\text{Vec})$ as monoidal categories. Hence $\mathfrak{Z}_0(\mathcal{X})$ is indeed a multi-fusion category. \heartsuit

Given a multi-fusion category \mathcal{P} and a finite semisimple category \mathcal{X} , a left \mathcal{P} -module structure on \mathcal{X} is indeed equivalent to a \mathbb{C} -linear monoidal functor $\mathcal{P} \rightarrow \text{Fun}(\mathcal{X}, \mathcal{X})$. In other words, any left action of a monoidal category on \mathcal{X} factors through $\text{Fun}(\mathcal{X}, \mathcal{X})$, in the sense that there exists a unique (up to isomorphism) \mathbb{C} -linear monoidal functor $\varphi: \mathcal{P} \rightarrow \mathfrak{Z}_0(\mathcal{X})$ such that the following diagram commutes up to isomorphism:

$$\begin{array}{ccc} & \mathfrak{Z}_0(\mathcal{X}) \times \mathcal{X} & \\ \varphi \times \text{id}_{\mathcal{X}} \nearrow & & \searrow \\ \mathcal{P} \times \mathcal{X} & \xrightarrow{\quad \odot \quad} & \mathcal{X} \end{array}$$

Remark 4.4.20. The functor category $\mathfrak{Z}_0(\mathcal{X})$ is a kind of E_0 -center [Lur17]. \diamond

Remark 4.4.21. It is also convenient to write a 0d boundary $x \in \mathcal{X}$ as a pair (\mathcal{X}, x) to emphasize the category \mathcal{X} of 0d boundaries. Mathematically, the pair (\mathcal{X}, x) is a kind of E_0 -algebra. \diamond

Similar to the 2d case, there is a boundary-bulk relation for 1d topological orders and their 0d boundaries.

Theorem^{ph} 4.4.22 (Boundary-bulk relation in 1d). Suppose \mathcal{P} is an anomaly-free 1d topological order. Its topological skeleton is a (unitary) multi-fusion category \mathcal{P} . The category of 0d boundaries of \mathcal{P} is denoted by \mathcal{X} .

- (a) Moving bulk topological defects to boundaries induces a left \mathcal{P} -module structure on \mathcal{X} .
- (b) The monoidal functor $\mathcal{P} \rightarrow \mathfrak{Z}_0(\mathcal{X})$ induced by the module action is an equivalence.

Remark 4.4.23. Consider the direct sum of n copies of the trivial 1d topological order $(1_1)^{\oplus n}$. Clearly the category of its 0d boundaries is the direct sum $\text{Vec}^{\oplus n}$ of n copies of Vec . By Example 4.4.19 and the boundary-bulk relation in 1d (Theorem^{ph} 4.4.22), the topological skeleton of $(1_1)^{\oplus n}$ is $M_n(\text{Vec})$. This is consistent with the boundary-bulk relation in 2d (Theorem^{ph} 4.2.21) because $\mathfrak{Z}_1(M_n(\text{Vec})) \simeq \text{Vec}$.

Conversely, a multi-fusion category \mathcal{C} satisfies $\mathfrak{Z}_1(\mathcal{C}) \simeq \text{Vec}$ if and only if $\mathcal{C} \simeq M_n(\text{Vec})$ for some positive integer n (see for example [KZ18a, Corollary 2.5.3]). Physically, an anomaly-free 1d topological order must be the direct sum of several copies of the trivial 1d topological order [CGW11a]. This also coincides with the classification of 1+1D fully extended TQFT's [FHK94, SP11]. \diamond

The boundary-bulk relation in 1d also helps us to find the relation between topological orders in a more general setting. For example, as depicted in Figure 62, a 0d domain wall x between two 1d boundaries P and Q of a 2d topological order \mathcal{C} can be viewed as a boundary of the anomaly-free 1d topological order $\overline{Q} \boxtimes_{\mathcal{C}} P$, which is obtained by the relative fusion product. Mathematically, we have $\mathfrak{Z}_0(\mathcal{X}) \simeq \mathcal{Q}^{\text{rev}} \boxtimes_{\mathcal{C}} \mathcal{P}$ as multi-fusion categories.

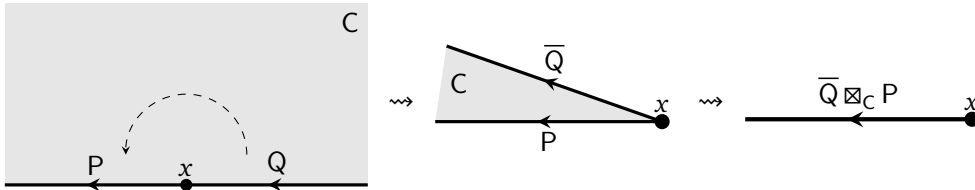


Figure 62: the folding trick

Exercise 4.4.24. Let x be a particle-like topological defect in the toric code model. Its 1d bulk can be obtained via a dimensional reduction process (see Figure 3).

- (1) Put this 1d bulk on a circle. What is the GSD?
- (2) Find some local operators that commute with the Hamiltonian. These local operators should explain the nontrivial GSD of this 1d bulk (defined on an open interval). This is similar to the last example in Section 4.3.3.

- (3) Determine the topological skeleton of this 1d bulk.
- (4) Check the boundary-bulk relation in 1d (Theorem^{ph} 4.4.22) in this example.

4.4.3 Gapped 0d domain walls in the toric code model

All nontrivial 0d domain walls between 1d domain walls in the toric code model can be reduced to the following two cases.

First we consider 0d domain walls between the smooth and rough boundaries as depicted in Figure 63. There is no A_v operator on the vertex on the domain wall. Clearly both e particles and m particles can be created or annihilated by local operators near the domain wall, so there is only the trivial topological excitation $\mathbb{1}$ on the domain wall. In other words, the category \mathcal{X} of 0d domain walls between the smooth and rough boundaries has only one simple object $\mathbb{1}$ and thus is equivalent to Vec . This is consistent with the boundary-bulk relation in 1d and the relative fusion product $\text{Rep}(\mathbb{Z}_2) \boxtimes_{\mathcal{T}\mathcal{C}} \text{Vec}_{\mathbb{Z}_2} \simeq \text{Vec} \simeq \mathfrak{Z}_0(\text{Vec})$ (see Figure 62).

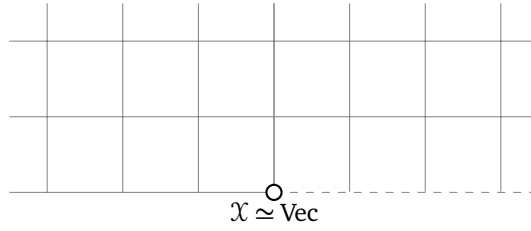


Figure 63: a 0d domain wall between the smooth and rough boundaries of the toric code model

Exercise 4.4.25. The category of 0d domain walls between the smooth boundary and itself is $\text{Rep}(\mathbb{Z}_2)$, which is clearly a $(\text{Rep}(\mathbb{Z}_2), \text{Rep}(\mathbb{Z}_2))$ -bimodule. Find the bulk of such 0d domain walls by the folding trick (see Figure 62) and check the boundary-bulk relation in 1d in this example.

Figure 64 depicts 0d domain walls between the trivial 1d domain wall and the e - m -exchange domain wall. The operator Q is defined as [Bom10, KK12]

$$Q := \sigma_x^1 \sigma_y^2 \sigma_z^3 \sigma_z^4 \sigma_z^5.$$

It can be easily verified that Q commutes with all A_v and B_p operators, including those on the e - m -exchange domain wall.

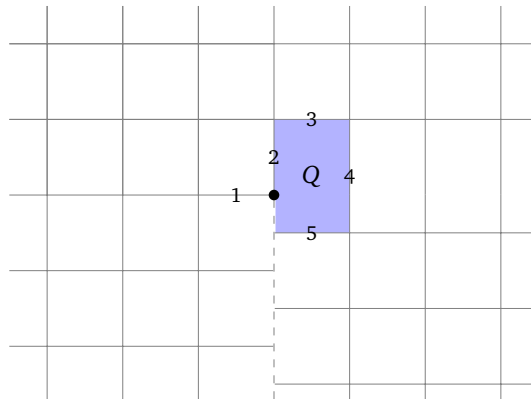


Figure 64: 0d domain walls between the trivial 1d domain wall and the e - m -exchange domain wall

If we take the Hamiltonian to be

$$H = \sum_v (1 - A_v) + \sum_p (1 - B_p) - Q,$$

the ground state subspace is determined by eigenvalues $A_v = B_p = Q = +1$. It generates a topological defect denoted by χ_+ , which is a 0d domain wall between the trivial 1d domain wall and the e - m -exchange domain wall. Similarly, if we take the Hamiltonian to be

$$H = \sum_v (1 - A_v) + \sum_p (1 - B_p) + Q,$$

then the ground state is determined by the eigenvalues $A_v = B_p = +1$ and $Q = -1$. It also generates a topological defect denoted by χ_- . The category \mathcal{X} of 0d domain walls between the trivial 1d domain wall and the e - m -exchange domain wall has two simple objects χ_{\pm} and thus is equivalent to $\text{Vec} \oplus \text{Vec}$.

Remark 4.4.26. It was long believed that the topological defects χ_{\pm} are useful in quantum computation. An experimental realization can be found in [XSWX⁺23]. \diamond

Now we choose the orientation such that \mathcal{X} is a $(\mathcal{TC}, \mathcal{TC}_{\phi})$ -bimodule. It is not hard to compute the module actions in the lattice model. The left \mathcal{TC} -module action on \mathcal{X} is given by

$$e \odot \chi_{\pm} = \chi_{\mp}, \quad m \odot \chi_{\pm} = \chi_{\mp}, \quad f \odot \chi_{\pm} = \chi_{\pm}, \quad (4.4.2)$$

and the right \mathcal{TC}_{ϕ} -module action on \mathcal{X} is given by

$$\chi_{\pm} \odot X = \chi_{\mp}, \quad \chi_{\pm} \odot Y = \chi_{\mp}, \quad \chi_{\pm} \odot Z = \chi_{\pm}. \quad (4.4.3)$$

Exercise 4.4.27. Prove (4.4.2) and (4.4.3) in the lattice model.

Exercise 4.4.28. Find the 1d bulk of χ_{\pm} and check the boundary-bulk relation in 1d in this example. Hint: use the same method in Exercise 4.4.24.

Exercise 4.4.29. The 0d domain walls χ_{\pm} can be fused along the e - m -exchange domain wall. The results should be 0d topological defects in the toric code model.

- (1) Consider the ‘shortest’ e - m -exchange domain wall and two end points as depicted in Figure 65 (a). Prove that

$$Q_1 Q_2 = A_{v_1} A_{v_2} B_p, \quad [Q_i, B_p] = 0, \quad \{Q_i, A_{v_1}\} = \{Q_i, A_{v_2}\} = 0, \quad (i = 1, 2)$$

where the plaquette p and vertices v_1, v_2 are depicted in Figure 65 (b).

- (2) Conclude that the fusion rules of χ_{\pm} along the e - m -exchange domain wall are (see [Bom10, KK12])

$$\chi_{\pm} \otimes \chi_{\pm} = \mathbb{1} \oplus f, \quad \chi_{\pm} \otimes \chi_{\mp} = e \oplus m. \quad (4.4.4)$$

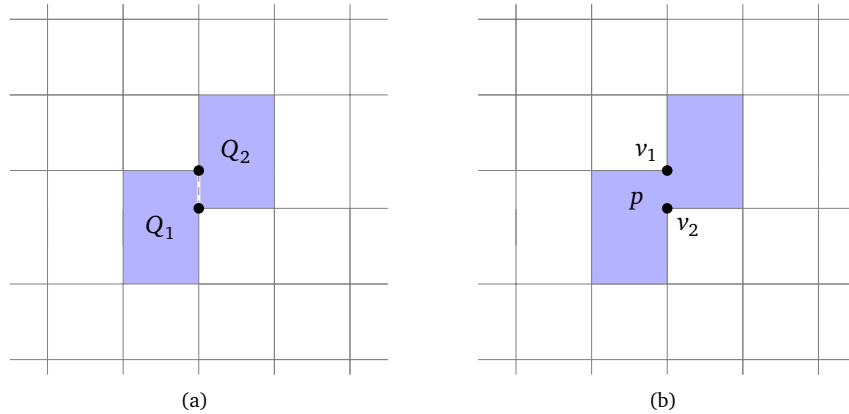


Figure 65: the fusion of χ_{\pm} along the e - m -exchange domain wall

Remark 4.4.30. From (4.4.2) and (4.4.4) we see that the set $\{1, e, m, f, \chi_+, \chi_-\}$ is closed under the fusion. In other words, these topological defects generates a fusion category²¹, denoted by \mathcal{TC}_+ . However, \mathcal{TC}_+ is not a braided fusion category. For example, the ‘braiding’ of e with χ_\pm (see Figure 66) is an instanton

$$\chi_\pm \otimes e \rightarrow m \otimes \chi_\pm$$

because moving an e particle across the e - m -exchange domain wall gives an m particle. These ‘braidings’ do not form an ordinary braiding structure.

Roughly speaking, a G -crossed braided fusion category [Tur00, Kir01, Müg04, DGNO10] (where G is a finite group) is a G -graded fusion category $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ equipped with a compatible G -action and a G -crossed braiding

$$c_{x,y}: x \otimes y \rightarrow g(y) \otimes x, \quad x \in \mathcal{C}_g, y \in \mathcal{C}.$$

A typical example is Vec_G^ω for $\omega \in Z^3(G; \mathbb{C}^\times)$. It has a unique structure of a G -crossed braided fusion category (up to a unique isomorphism) with the obvious G -grading [DGNO10]. In the above example, \mathcal{TC}_+ admits a \mathbb{Z}_2 -grading $\mathcal{TC}_+ = (\mathcal{TC}_+)_0 \oplus (\mathcal{TC}_+)_1$ with $(\mathcal{TC}_+)_0 = \mathcal{TC}$ and $(\mathcal{TC}_+)_1$ generated by χ_\pm . The \mathbb{Z}_2 -action on \mathcal{TC}_+ is given by exchanging e and m . Thus \mathcal{TC}_+ is a \mathbb{Z}_2 -crossed braided fusion category [BBCW19]. \diamond

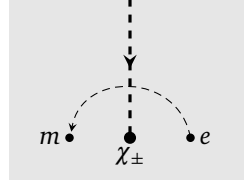


Figure 66: The ‘braiding’ of e or m with χ_\pm

4.4.4 Morita equivalence

Suppose P and Q are gapped 1d boundaries of a 2d topological order C . Let \mathcal{P} and \mathcal{Q} be the topological skeletons of P and Q , respectively. Assume that there exists at least one 0d domain wall x between P and Q . Then the category of 0d domain walls between P and Q is a nonzero finite semisimple $(\mathcal{P}, \mathcal{Q})$ -bimodule \mathcal{X} and $x \in \mathcal{X}$. By reversing the orientation, the opposite category \mathcal{X}^{op} is a finite semisimple $(\mathcal{Q}, \mathcal{P})$ -bimodule (see Example 4.4.9), which is the category of 0d domain walls between Q and P .

The 0d domain walls can be fused along Q (see Figure 67). Thus the category of 0d domain walls between P and P can be obtained by the relative tensor product $\mathcal{X} \boxtimes_{\mathcal{Q}} \mathcal{X}^{\text{op}}$. On the other hand, the category of 0d domain walls between P and itself is just the topological skeleton \mathcal{P} of P . Hence, there is an equivalence $\mathcal{X} \boxtimes_{\mathcal{Q}} \mathcal{X}^{\text{op}} \simeq \mathcal{P}$ of $(\mathcal{P}, \mathcal{P})$ -bimodules. Similarly, there is also an equivalence $\mathcal{X}^{\text{op}} \boxtimes_{\mathcal{P}} \mathcal{X} \simeq \mathcal{Q}$ of $(\mathcal{Q}, \mathcal{Q})$ -bimodules.

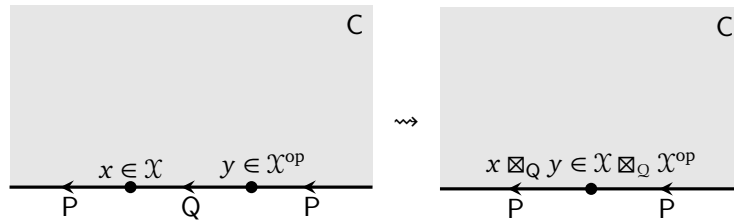


Figure 67: Morita equivalence

²¹Indeed, there are still two possible choices of the associator, which are essentially the same as the two choices of the associator of Ising type fusion categories.

Definition 4.4.31 ([Müg03a, EO04, ENO10]). Let \mathcal{P}, \mathcal{Q} be multi-fusion categories. A finite semisimple $(\mathcal{P}, \mathcal{Q})$ -bimodule \mathcal{X} is called *invertible* if there is an equivalence of $(\mathcal{P}, \mathcal{P})$ -bimodules:

$$\mathcal{X} \boxtimes_{\mathcal{Q}} \mathcal{X}^{\text{op}} \simeq \mathcal{P}$$

and an equivalence of $(\mathcal{Q}, \mathcal{Q})$ -bimodules:

$$\mathcal{X}^{\text{op}} \boxtimes_{\mathcal{P}} \mathcal{X} \simeq \mathcal{Q}$$

If there exists an invertible $(\mathcal{P}, \mathcal{Q})$ -bimodule, we say \mathcal{P} and \mathcal{Q} are *Morita equivalent*. ■

By the above discussion, the topological skeletons \mathcal{P}, \mathcal{Q} of two gapped 1d boundaries P, Q of a 2d topological order \mathcal{C} are Morita equivalent if there exists at least one 0d domain wall between P and Q .

Remark 4.4.32. If \mathcal{C} is an anomaly-free stable 2d topological order, then for every gapped 1d boundaries P, Q of \mathcal{C} , there exists at least one 0d domain wall between them. Figure 68 depicts a proof. By the folding trick, a 0d domain wall between P and Q is the same as a 0d boundary of $\overline{\mathcal{Q}} \boxtimes_{\mathcal{C}} \mathcal{P}$. Since every anomaly-free 1d topological order is the direct sum of several copies of the trivial 1d topological order (see Remark 4.4.23), it has at least one gapped boundary x .

However, if \mathcal{C} is not stable, it is possible that there is no 0d domain wall between two gapped 1d boundaries. Why does the above argument not apply to unstable case? The point is that the dimensional reduction $\overline{\mathcal{Q}} \boxtimes_{\mathcal{C}} \mathcal{P}$ may be zero, and it means that such a dimensional reduction is physically forbidden. \diamond

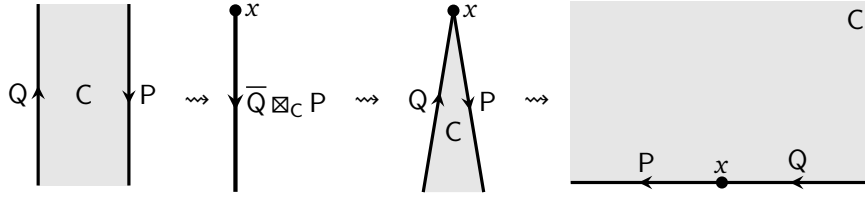


Figure 68: the existence of a 0d domain wall between two boundaries

Example 4.4.33. Suppose \mathcal{M} is a gapped 1d domain walls in an anomaly-free stable 2d topological order \mathcal{C} , then its topological skeleton \mathcal{M} is Morita equivalent to \mathcal{C} [GJF19a, KLWZZ20b]. Indeed, by the folding trick, a gapped 1d domain wall in \mathcal{C} is the same as a gapped 1d boundary of $\mathcal{C} \boxtimes \overline{\mathcal{C}}$ (which is stable), and clearly \mathcal{C}_1 is such a boundary (see Figure 52). ♡

The above discussion, together with the boundary-bulk relation in 2d (Theorem^{ph} 4.2.21), also lead to the following mathematical theorem [Müg03a, NN08, ENO11].

Theorem 4.4.34. Two indecomposable multi-fusion categories \mathcal{P}, \mathcal{Q} are Morita equivalent if and only if their Drinfeld centers $\mathfrak{Z}_1(\mathcal{P}), \mathfrak{Z}_1(\mathcal{Q})$ are equivalent as braided fusion categories.

Example 4.4.35. Let G be a finite group. Recall Example 4.2.32 that $\mathfrak{Z}_1(\text{Rep}(G))$ and $\mathfrak{Z}_1(\text{Vec}_G)$ are equivalent as braided fusion categories. Thus $\text{Rep}(G)$ and Vec_G are Morita equivalent. Indeed, there is an invertible bimodule Vec , where the module action of $\text{Rep}(G)$ and Vec_G on Vec are induced by the forgetful functors $\text{Rep}(G) \rightarrow \text{Vec}$ and $\text{Vec}_G \rightarrow \text{Vec}$, respectively. ♡

Remark 4.4.36. Let \mathcal{P} be an indecomposable multi-fusion category and \mathcal{X} be a nonzero finite semisimple left \mathcal{P} -module. Then all left \mathcal{P} -module functors from \mathcal{X} to itself and left \mathcal{P} -module natural transformations (see Remark 4.4.11) also form a multi-fusion category, denoted by $\text{Fun}_{\mathcal{P}}(\mathcal{X}, \mathcal{X})$. Also \mathcal{X} is a finite semisimple left $\text{Fun}_{\mathcal{P}}(\mathcal{X}, \mathcal{X})$ -module, or equivalently, a finite semisimple right $\text{Fun}_{\mathcal{P}}(\mathcal{X}, \mathcal{X})^{\text{rev}}$ -module. Then \mathcal{P} is Morita equivalent to $\text{Fun}_{\mathcal{P}}(\mathcal{X}, \mathcal{X})^{\text{rev}}$ with an invertible $(\mathcal{P}, \text{Fun}_{\mathcal{P}}(\mathcal{X}, \mathcal{X})^{\text{rev}})$ -bimodule \mathcal{X} .

Conversely, if \mathcal{P}, \mathcal{Q} are multi-fusion categories with an invertible finite semisimple $(\mathcal{P}, \mathcal{Q})$ -bimodule \mathcal{X} , then we have $\mathcal{Q} \simeq \text{Fun}_{\mathcal{P}}(\mathcal{X}, \mathcal{X})^{\text{rev}}$ and $\mathcal{P} \simeq \text{Fun}_{\mathcal{Q}^{\text{rev}}}(\mathcal{X}, \mathcal{X})$ as multi-fusion categories [Müg03a, EO04, ENO10]. ◇

Remark 4.4.37. It is possible that there are different invertible bimodules between two multi-fusion categories. For example, let G be a finite group. Then for each $\beta \in Z^2(G; \mathbb{C}^\times)$ the module category $\text{Rep}(G, \beta)$ is an invertible $(\text{Rep}(G), \text{Rep}(G))$ -bimodule. \diamond

Remark 4.4.38. Suppose P, Q are gapped 1d boundaries of a 2d topological order C . Then Q can be obtained from P by condensing some topological defects of P . Indeed, suppose x is a 0d domain wall between P and Q . By reversing the orientation, x is also a 0d domain wall between Q and P [Kon14a]. Then the fusion $A_x := x \boxtimes_Q x$ is a 0d domain wall between P and P , i.e., $A_x \in \mathcal{P}$. Similarly, $B_x := x \boxtimes_P x \in \mathcal{Q}$ is a 0d domain wall between Q and Q . The critical point of the condensation from P to Q driven by $A_x \in \mathcal{P}$ is depicted in Figure 69. Conversely, it is also the critical point of the condensation from Q to P driven by $B_x \in \mathcal{Q}$.

Mathematically, suppose \mathcal{P}, \mathcal{Q} are fusion categories and \mathcal{M} is an invertible $(\mathcal{P}, \mathcal{Q})$ -bimodule. Then for any nonzero $x \in \mathcal{M}$ the internal hom $A_x := [x, x]_{\mathcal{P}}$ is an algebra $A_x \in \mathcal{P}$. Moreover, \mathcal{Q} is equivalent to the category $\text{BMod}_{A_x|A_x}(\mathcal{P})$ of (A_x, A_x) -bimodules in \mathcal{P} . \diamond

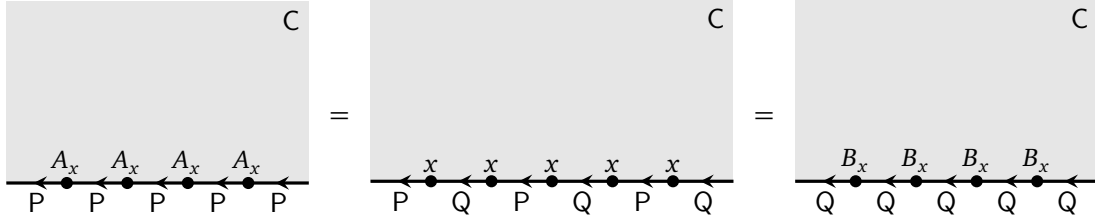


Figure 69: a critical point of the condensation from P to Q

4.4.5 The structure of multi-fusion categories

Consider a (potentially anomalous) 1d topological order P whose ground state subspace is n -fold degenerate. We only consider the case that the bulk of P is stable. In other words, its topological skeleton \mathcal{P} is a multi-fusion category with $\mathbb{1} = \bigoplus_{i=1}^n \mathbb{1}_i$, where each $\mathbb{1}_i$ is generated by a single ground state $|\psi_i\rangle$ and thus is a simple object in \mathcal{P} . The physical intuition that two different ground states $|\psi_i\rangle$ and $|\psi_j\rangle$ can not be smoothly connected implies that $\mathbb{1}_i \otimes \mathbb{1}_i = \mathbb{1}_i$ and $\mathbb{1}_i \otimes \mathbb{1}_j = 0$ for $i \neq j$.

Now we add a perturbation to P such that the ground state falls into a single one $|\psi_i\rangle$. Then we get a stable 1d topological order, denoted by P_i . Its topological skeleton is a fusion category, denoted by \mathcal{P}_{ii} . Moreover, P is the direct sum of these stable topological orders, i.e., $P = \bigoplus_i P_i$.

If we add different perturbations in different regions, every choice of perturbations produces a 0d domain wall between these regions (see Figure 70). Clearly the category of 0d domain walls between P_i and P_j should be a full subcategory of \mathcal{P} defined by

$$\mathcal{P}_{ij} := \mathbb{1}_i \otimes \mathcal{P} \otimes \mathbb{1}_j := \{\mathbb{1}_i \otimes x \otimes \mathbb{1}_j \mid x \in \mathcal{P}\}.$$

When $i = j$, it recovers the topological skeleton \mathcal{P}_{ii} . In particular, the tensor unit of \mathcal{P}_{ii} is $\mathbb{1}_i$.

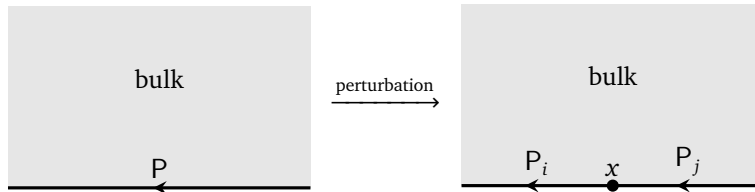


Figure 70: a multi-fusion category \mathcal{P}

Remark 4.4.39. Since the perturbation is on the 1d boundary \mathcal{P} and does not affect the 2d bulk, we immediately know that all \mathcal{P}_i 's share the same bulk with \mathcal{P} . By the boundary-bulk relation in 2d, we have $\mathfrak{Z}_1(\mathcal{P}_{ii}) \simeq \mathfrak{Z}_1(\mathcal{P})$ as braided fusion categories. Also, by Theorem 4.4.34, every \mathcal{P}_{ii} is Morita equivalent to \mathcal{P} . Moreover, \mathcal{P}_{ij} is an invertible $(\mathcal{P}_{ii}, \mathcal{P}_{jj})$ -bimodule. \diamond

It is clear that

$$\mathcal{P} = \mathbb{1} \otimes \mathcal{P} \otimes \mathbb{1} = \left(\bigoplus_i \mathbb{1}_i \right) \otimes \mathcal{P} \otimes \left(\bigoplus_j \mathbb{1}_j \right) = \bigoplus_{i,j} \mathbb{1}_i \otimes \mathcal{P} \otimes \mathbb{1}_j = \bigoplus_{i,j} \mathcal{P}_{ij}.$$

Then we have the following structure theorem of a multi-fusion category [ENO05, EO04, KZ18a].

Theorem 4.4.40. Let \mathcal{P} be a multi-fusion category. Assume that $\mathbb{1} = \bigoplus_i \mathbb{1}_i$ where each $\mathbb{1}_i$ is simple. Define $\mathcal{P}_{ij} := \mathbb{1}_i \otimes \mathcal{P} \otimes \mathbb{1}_j$.

- (a) We have $\mathbb{1}_i \otimes \mathbb{1}_i = \mathbb{1}_i$ and $\mathbb{1}_i \otimes \mathbb{1}_j = 0$ for $i \neq j$.
- (b) $\mathcal{P} = \bigoplus_{i,j} \mathcal{P}_{ij}$.
- (c) The tensor product of \mathcal{P} restricts to a functor $\mathcal{P}_{ij} \times \mathcal{P}_{jk} \simeq \mathcal{P}_{ik}$ for every i, j, k . In other words, $\mathcal{P}_{ij} \otimes \mathcal{P}_{jk} \subseteq \mathcal{P}_{ik}$.
- (d) When \mathcal{P} is indecomposable (see Remark 4.2.26), each component \mathcal{P}_{ij} is non-zero. Moreover, the tensor product induces an equivalence of $(\mathcal{P}_{ii}, \mathcal{P}_{kk})$ -bimodules for every i, j, k :

$$\mathcal{P}_{ij} \boxtimes_{\mathcal{P}_{jj}} \mathcal{P}_{jk} \simeq \mathcal{P}_{ik}.$$

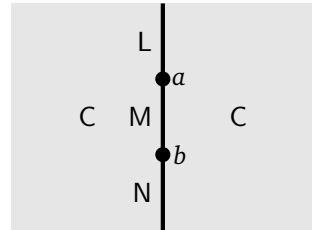
In particular, \mathcal{P}_{ij} is an invertible $(\mathcal{P}_{ii}, \mathcal{P}_{jj})$ -bimodule for all i, j . Hence all fusion categories \mathcal{P}_{ii} are Morita equivalent.

Hence, every multi-fusion category \mathcal{P} can be written as a square matrix (\mathcal{P}_{ij}) , in which diagonal components are fusion categories. It is indecomposable if and only if the matrix is not block-diagonal (up to permutation). The simplest example of an indecomposable multi-fusion category is $M_n(\text{Vec})$, in which every component is Vec .

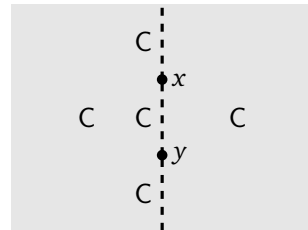
4.5 Condensation completion

4.5.1 Topological skeleton of a 2d topological order

As explained in Section 2.4, the topological skeleton of a 2d topological order \mathcal{C} consists of all topological defects of codimension 1 and higher in \mathcal{C} . There are 1d domain walls in \mathcal{C} , 0d domain walls between 1d domain walls and instantons (see Figure 71). In particular, a particle-like topological defect is a 0d domain wall between two trivial 1d domain walls.



(a) 1d and 0d domain walls in a 2d topological order \mathcal{C}



(b) The particle-like topological defects are 0d domain walls between two trivial domain walls.

Figure 71: the topological skeleton of a 2d topological order \mathcal{C}

We have shown that the 0d domain walls between two 1d domain walls M and N (and instantons between them) form a category, denoted by $\text{Hom}(M, N)$. The main structure of this category is the

fusion of instantons on the world line. Moreover, these 0d domain walls can be fused along 1d domain walls. For example, two 0d domain walls a and b in Figure 71 (a) can be fused along M , and the result is a 0d domain wall between L and N , denoted by $b \boxtimes_M a$ or simply $b \circ a$. In this process, the instantons on the world lines of 0d domain walls are also fused together. So the fusion of 0d domain walls defines a functor

$$\circ: \text{Hom}(M, N) \times \text{Hom}(L, M) \rightarrow \text{Hom}(L, N).$$

These defects and their relationship lead to the following structure.

Definition 4.5.1 (not rigorous). A 2-category \mathbf{C} consists of the following data:

- a set $\text{ob}(\mathbf{C})$, whose elements are called *objects* of \mathbf{C} ;
- a category $\text{Hom}_{\mathbf{C}}(x, y)$ for each $x, y \in \text{ob}(\mathbf{C})$, called the *hom category* from x to y ;
- a functor $\circ: \text{Hom}_{\mathbf{C}}(y, z) \times \text{Hom}_{\mathbf{C}}(x, y) \rightarrow \text{Hom}_{\mathbf{C}}(x, z)$ for every $x, y, z \in \text{ob}(\mathbf{C})$, called the *composition functor*;
- an object $1_x \in \text{Hom}_{\mathbf{C}}(x, x)$ for every $x \in \text{ob}(\mathbf{C})$, called the *identity*;
- some higher coherence data;

These data should satisfy some conditions like (higher) associativity and unity axioms. The objects and morphisms in hom categories are also called *1-morphisms* and *2-morphisms* in \mathbf{C} , respectively. In a fixed hom category $\text{Hom}_{\mathbf{C}}(x, y)$, the composition of two 2-morphisms α, β is denoted by $\beta \cdot \alpha$. ■

Notation 4.5.2. In this work, 2-categories are labeled by $\mathbf{C}, \mathbf{D}, \mathbf{E}, \dots$

Example 4.5.3. We list some 2-categories:

1. Categories, functors and natural transformations form a 2-category.
2. Finite semisimple categories, \mathbb{C} -linear functors and natural transformations form a 2-category, denoted by 2Vec .
3. Monoidal categories, monoidal functors and monoidal natural transformations form a 2-category.
4. Fusion categories, \mathbb{C} -linear monoidal functors and monoidal natural transformations form a 2-category.
5. Braided monoidal categories, braided monoidal functors and monoidal natural transformations form a 2-category.
6. Let \mathcal{C}, \mathcal{D} be a multi-fusion category. Finite semisimple left \mathcal{C} -modules, left \mathcal{C} -module functors and left \mathcal{C} -module natural transformations (see Remark 4.4.11) form a 2-category, denoted by $\text{LMod}_{\mathcal{C}}(2\text{Vec})$. Similarly, finite semisimple right \mathcal{D} -modules, \mathcal{D} -module functors and \mathcal{D} -module natural transformations form a 2-category $\text{RMod}_{\mathcal{D}}(2\text{Vec})$, and finite semisimple $(\mathcal{C}, \mathcal{D})$ -bimodule, $(\mathcal{C}, \mathcal{D})$ -bimodule functors and $(\mathcal{C}, \mathcal{D})$ -bimodule natural transformations form a 2-category $\text{BMod}_{\mathcal{C}|\mathcal{D}}(2\text{Vec})$.
♥

Remark 4.5.4. Two multi-fusion categories \mathcal{P} and \mathcal{Q} are Morita equivalent if and only if $\text{RMod}_{\mathcal{P}}(2\text{Vec})$ and $\text{RMod}_{\mathcal{Q}}(2\text{Vec})$ are equivalent. ◇

The topological skeleton of a 2d topological order \mathcal{C} , denoted by \mathbf{C} , is a 2-category:

- The objects are gapped 1d domain walls (i.e., string-like topological defects) in \mathcal{C} .
- The 1-morphisms between two objects (1d domain walls) are 0d domain walls between them.
- The 2-morphisms between two 1-morphisms (0d domain walls) are instantons between them.
- The composition functors are given by the fusion of 0d domain walls, and the identities are given by trivial 0d domain walls.

In particular, the hom category $\text{Hom}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})$ from the trivial 1d domain wall to itself is the category \mathcal{C} of particle-like topological defects of \mathcal{C} .

Remark 4.5.5. There are different fusion or composition of 2-morphisms in a 2-category. As depicted in Figure 72, instantons (2-morphisms) can be fused vertically. On the other hand, 0d domain walls (1-morphisms) can be fused horizontally along 1d domain walls (objects), and the instantons on the world lines are also horizontally fused together. So there are two ways to fuse four instantons: we can first fuse them vertically to get two instantons, and then do horizontal fusion; we can also horizontally fuse x and y and then do vertical fusion. These two processes should give the same result. In other words, we have

$$(f_2 \cdot f_1) \circ (g_2 \cdot g_1) = (f_2 \circ g_2) \cdot (f_1 \circ g_1).$$

This equation means that the composition $\circ: \text{Hom}_{\mathcal{C}}(M, N) \times \text{Hom}_{\mathcal{C}}(L, M) \rightarrow \text{Hom}_{\mathcal{C}}(L, N)$ is a functor, and is also called the *middle four exchange property* of a 2-category. \diamond

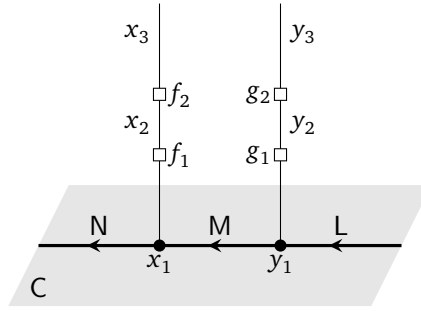
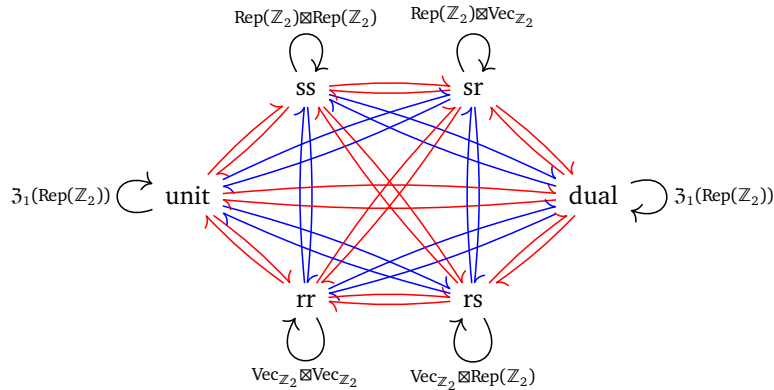


Figure 72: the middle four exchange property

Moreover, the topological skeleton \mathcal{C} of a 2d topological order \mathcal{C} has more structure than a 2-category because 1d domain walls (objects) can also be fused together. This intuition implies that \mathcal{C} is a *monoidal 2-category*. We do not give the precise mathematical definition of a monoidal 2-category here.

4.5.2 Topological skeleton of the toric code model

As explained above, the topological skeleton of the toric code model, including the topological defects of codimension 1 and higher, forms a monoidal 2-category, denoted by TC . Its 2-category structure is depicted in the following diagram:



Here six vertices are simple objects of TC , i.e., simple 1d domain walls in the toric code model. The vertex ‘unit’ is the trivial domain wall, ‘dual’ is the e - m -exchange domain wall, and the other four are non-invertible 1d domain walls (see Section 4.3.2). The arrows between vertices denote hom categories between simple objects, i.e., the categories of 0d domain walls between 1d domain walls. Each black arrow denotes the fusion category of 0d domain walls between a 1d domain wall and itself, i.e., the

topological skeleton of a 1d domain wall, which is computed in Section 4.3.2. The other hom categories are essentially computed in Section 4.4.3. A red line means that the corresponding hom category is equivalent to $\text{Vec} \oplus \text{Vec}$, and a blue line means that the corresponding hom category is equivalent to Vec .

Also, we list the fusion rules of simple objects of \mathbf{TC} in Table 1.

| \otimes | unit | dual | ss | sr | rs | rr |
|-----------|------|------|-----|-----|-----|-----|
| unit | unit | dual | ss | sr | rs | rr |
| dual | dual | unit | rs | rr | ss | sr |
| ss | ss | sr | 2ss | 2sr | ss | sr |
| sr | sr | ss | ss | sr | 2ss | 2sr |
| rs | rs | rr | 2rs | 2rr | rs | rr |
| rr | rr | rs | rs | rr | 2rs | 2rr |

Table 1: the fusion rules of simple string-like topological defects (gapped 1d domain walls) in the toric code model

Exercise 4.5.6. Explicitly calculate the composition functors of the 2-category \mathbf{TC} .

Remark 4.5.7. A topological order admitting a gapped boundary is called a non-chiral topological order. All 2d non-chiral topological order can be realized by the so-called Levin-Wen models [LW05], each of which is constructed from the data of a fusion category \mathcal{A} , and realizes the 2d topological order $(\mathcal{Z}_1(\mathcal{A}), 0)$ [LW05, KK12]. The toric code model can be viewed as an example of Levin-Wen model when $\mathcal{A} = \text{Rep}(\mathbb{Z}_2)$ (or $\text{Vec}_{\mathbb{Z}_2}$). All topological defects in a 2+1D non-chiral topological orders had been constructed explicitly in Levin-Wen models in [KK12]. Moreover, they form a monoidal 2-category $\text{BMod}_{\mathcal{A}|\mathcal{A}}(2\text{Vec})$, which consists of finite semisimple \mathcal{A} - \mathcal{A} -bimodule categories as objects, bimodule functors as 1-morphisms and bimodule natural transformations as 2-morphisms. \diamond

Remark 4.5.8. For each G -crossed braided fusion category there is an associated monoidal 2-category [Cui19]. The monoidal 2-category \mathbf{TC}^* associated to the \mathbb{Z}_2 -crossed braided fusion category \mathcal{TC}_+ (see Remark 4.4.30) is the full sub-2-category of \mathbf{TC} containing the trivial 1d domain wall and the e -m-exchange domain wall (i.e., two invertible objects in \mathbf{TC}). \diamond

4.5.3 Condensation completion

In general, the topological skeleton \mathbf{C} of a 2d topological order \mathcal{C} forms a multi-fusion 2-category, a notion which was introduced by Douglas and Reutter [DR18]. A multi-fusion 2-category is a monoidal 2-category satisfying some properties. The most important property is that a multi-fusion 2-category is *idempotent complete* [DR18]. The idempotent completeness of a 2-category is a generalization of the idempotent completeness of a 1-category defined in Definition 3.3.16. It was first briefly introduced by Carqueville and Runkel under the name of ‘orbifold completion’ in [CR16] and was thoroughly developed later in [DR18]. The theory of idempotent completeness was later generalized to higher categories by Gaiotto and Johnson-Freyd [GJF19a] under the name of *Karoubi completeness* or *condensation completeness* (see also [KLWZZ20b]).

Physically the condensation completeness means that all condensation descendants are included. First we consider the modular tensor category \mathcal{C} of particle-like topological defects of an anomaly-free stable 2d topological order \mathcal{C} . There is a 2-category \mathbf{BC} defined by the following data:

- There is only one object, i.e., $\text{ob}(\mathbf{BC}) := \{*\}$.
- $\text{Hom}_{\mathbf{BC}}(*, *) := \mathcal{C}$.
- The composition and identity are defined by the tensor product and tensor unit of \mathcal{C} .

Moreover, \mathcal{BC} is a monoidal 2-category. This construction is similar to Example 3.3.7 and Example 3.4.25. Since $\mathcal{C} = \text{Hom}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})$, the monoidal 2-category \mathcal{BC} is a sub-2-category of \mathcal{C} , consisting of the trivial 1d domain wall in \mathcal{C} and 0d domain walls on the trivial 1d domain wall. Since all simple 1d domain walls in \mathcal{C} can be obtained from the trivial 1d domain wall by condensing some particle-like topological defects (see Remark 4.4.38), \mathcal{BC} is not condensation complete and the condensation descendants form the 2-category \mathcal{C} . We say that the condensation completion of \mathcal{BC} is \mathcal{C} because \mathcal{C} is the minimal condensation complete 2-category containing \mathcal{BC} .

Example 4.5.9. The condensation completion of $\mathcal{B}(\mathcal{TC})$ is \mathcal{TC} . Similarly, the condensation completion of \mathcal{TC}^{\times} (see Remark 4.5.8) is also \mathcal{TC} [DR18]. \heartsuit

Given a 2d topological order \mathcal{C} (which is not necessarily anomaly-free nor stable), its topological skeleton is a multi-fusion 2-category \mathcal{C} , which can be viewed as a categorical description of \mathcal{C} . Recall Remark 3.5.13 that when \mathcal{C} is anomaly-free and stable, the unitary modular tensor category \mathcal{C} of particle-like topological defects of \mathcal{C} can also be viewed as a categorical description of \mathcal{C} . What is the relation between these two descriptions? Ineed, when \mathcal{C} is anomaly-free and stable, the (unitary) modular tensor category \mathcal{C} and the fusion 2-category \mathcal{C} can determine each other [JF22]. By definition we have $\text{Hom}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}) = \mathcal{C}$. Conversely, \mathcal{C} is equivalent to $\text{RMod}_{\mathcal{C}}(2\text{Vec})$ (see Example 4.5.3) [GJF19a, KLWZZ20b]. This is because that the topological skeleton of each 1d domain wall in \mathcal{C} is Morita equivalent to \mathcal{C} (see Example 4.4.33 and Remark 4.4.36 4.5.4).

By describing a 2d topological order by its topological skeleton, there is another boundary-bulk relation for 2d topological orders and gapped 1d boundaries [KLWZZ20b, KZ22a], similar to Theorem^{ph} 4.4.22.

Theorem^{ph} 4.5.10 (Boundary-bulk relation in 2d, 2nd version). Let \mathcal{C} be an anomaly-free 2d topological order (which is not necessarily stable). Its topological skeleton is a multi-fusion 2-category \mathcal{C} .

- (a) All gapped 1d boundaries of \mathcal{C} and 0d domain walls between them, as well as instantons, form a finite semisimple 2-category, denoted by \mathcal{M} .
- (b) The bulk-to-boundary map induces an equivalence $\mathcal{C} \simeq \mathfrak{Z}_0(\mathcal{M})$ of multi-fusion 2-categories, where $\mathfrak{Z}_0(\mathcal{M}) := \text{Fun}(\mathcal{M}, \mathcal{M})$ is the 2-category of \mathcal{C} -linear 2-functors from \mathcal{M} to itself.

Remark 4.5.11. Suppose \mathcal{C} is an anomaly-free stable 2d topological order and \mathcal{M} is a gapped 1d boundary of \mathcal{C} . By definition \mathcal{M} is an object of the 2-category \mathcal{M} of gapped boundaries of \mathcal{C} , and $\text{Hom}_{\mathcal{M}}(\mathcal{M}, \mathcal{M}) = \mathcal{M}$ is the topological skeleton of \mathcal{M} . On the other hand, \mathcal{M} is the condensation completion of \mathcal{BM} , because the topological skeletons of all gapped 1d boundaries of \mathcal{C} are Morita equivalent. Also \mathcal{M} is equivalent to $\text{RMod}_{\mathcal{M}}(2\text{Vec})$ (see Remark 4.4.36 4.5.4). \diamond

Remark 4.5.12. When \mathcal{C} is an anomaly-free stable 2d topological order, Theorem^{ph} 4.2.21 is a direct corollary of Theorem^{ph} 4.5.10 [KLWZZ20b] by taking the hom category of the tensor unit in the equivalence $\mathcal{C} \simeq \mathfrak{Z}_0(\mathcal{M})$. \diamond

Remark 4.5.13. Recall Theorem^{ph} 3.5.4 that a stable 2d topological order \mathcal{C} is anomaly-free if and only if the (unitary) braided fusion category \mathcal{C} of particle-like topological defects is nondegenerate, i.e., the Müger center $\mathfrak{Z}_2(\mathcal{C}) \simeq \text{Vec}$ is trivial. There is also a boundary-bulk relation for an anomaly-free stable 3d topological order \mathcal{B} and a 2d gapped boundary \mathcal{C} [KWZ15, KWZ17]:

- (a) The bulk topological defects of codimension 2 and higher form a braided fusion 2-category \mathcal{B} .
- (b) The topological skeleton of the boundary \mathcal{C} is a multi-fusion 2-category \mathcal{C} .
- (c) The bulk-to-boundary map induces an equivalence $\mathcal{B} \simeq \mathfrak{Z}_1(\mathcal{C})$ of braided fusion 2-categories.

This boundary-bulk relation has also been checked in a concrete lattice model [KTZ20a]. In particular, a (not necessarily stable) 2d topological order \mathcal{C} is anomaly-free if and only if its topological skeleton \mathcal{C} satisfies $\mathfrak{Z}_1(\mathcal{C}) \simeq 2\text{Vec}$. \diamond

5 Advanced topics

In this section, we recommend a few books on tensor categories and briefly outline a few advanced topics on the interaction between the physics of topological orders and the category theory. We also provide some references so that intrigued readers can explore these topics by themselves. The physical literature on topological orders is gigantic. We only include those directly related to category theory. Many others can be found in Wen’s reviews [Wen17, Wen19] and references therein.

5.1 Historical remarks and books on tensor categories

In previous sections, we have already seen how the mathematical theory of (braided) fusion categories emerges from the physics of 1d and 2d topological orders. Now we give brief historical remarks on fusion category and its application to physics.

In mathematics, the notion of a (multi-)fusion category was studied long ago (see for example [DM82]), but the name was coined in [ENO05]. The subject was revived in 1990s under the influence of physics. In 1989, the mathematical structure of a modular tensor category, originally formulated non-categorically as the so-called Moore-Seiberg data, was first discovered in the study of 1+1D rational conformal field theories by Moore and Seiberg [MS89a, MS90] (see [Hua08b, Hua08a] and references therein for the story on the mathematical side). It was later reformulated categorically by Reshetikhin and Turaev, and was used to construct the so-called Reshetikhin-Turaev TQFT’s [RT91, Tur20]. In 1992, Turaev and Viro introduced the state-sum construction of TQFT’s based on modular tensor categories [TV92]. Barrett and Westbury later generalized Turaev-Viro’s construction to spherical fusion categories [BW96, BW99]. In 2001, Müger proved that the Drinfeld center of a spherical fusion category is a modular tensor category [Müg03b], and Ostrik initiated the study of the representation theory of fusion categories [Ost03a, Ost03b]. Since then, the study of fusion categories has entered its golden age.

In physics, around 1989, it was noticed by mathematical physicists that modular tensor categories can be used to describe anyon systems (see for example [Wit89, MS89a, FRS89, FG90, Reh90, MR91, Wen91]). Condensed matter physicists learned this fact from Kitaev’s review in [Kit06, Appendix E] many years later. In 1997, modular tensor categories reappeared as the categories of anyons in Kitaev’s quantum double models [Kit03]. In 2004, fusion categories were first used by Levin and Wen in their construction of string-net models (also called Levin-Wen models) [LW05]. In 2011, Kitaev and Kong applied the representation theory of fusion categories to the construction of all gapped defects of codimension 1 and 2 in Levin-Wen models, including boundaries, walls and walls between walls [KK12]. In 2013, Kong gave a precise and complete categorical characterization of anyon condensations in 2+1D topological orders in terms of algebras in modular tensor categories [Kon14b]. In 2014, Barkeshli, Bonderson, Cheng and Wang found the precise mathematical characterization of bosonic 2d SPT/SET orders in terms of G-crossed braided fusion category [BBCW19]. Since 2014, we have witnessed a new wave of applying category theory in the study of topological orders (or SPT/SET orders). Some of the references will be provided in later subsections when we discuss advanced topics.

Now we recommend a few books and some references on tensor categories. For elementary notions in category theory, such as categories, functors, natural transformations, adjoint functors, limits/colimits, etc., readers can consult with standard textbooks on category theory (see for example [Mac78, Awo10, Rie17]). For modular tensor categories, readers can consult with Turaev’s book [Tur20] and Bakalov and Kirillov Jr.’s book [BK01]. Both books teach the so-called graphic calculus, which is a very useful tool for explicit calculation. They also discuss the relation between modular tensor categories and TQFT’s. For further study of (braided) fusion categories, the standard references for mathematicians are the book “Tensor categories” written by Etingof, Gelaki, Nikshych and Ostrik [EGNO15] and their influential papers [Ost03a, Ost03b, ENO05, EO04, ENO04, ENO11, ENO10]. The lecture notes [Pen21] written by David Penneys are also friendly to physicists. For applications in topological quantum computing, we recommend Zhenghan Wang’s book [Wan10]. Interestingly, [EGNO15] does not use graphic calculus at all for good and important reasons. Physics oriented readers might find [EGNO15] difficult to follow. In this case, we recommend Turaev and Virelizier’s book [TV17], where the graphic calculus is used. But for those who really want to dig deeper in this subject, we would like to emphasize that non-graphic techniques in [EGNO15] are unavoidable eventually. We also recommend readers to a few important developments [DGNO07, DGNO10, DMNO13, DNO12, DN13, DSPS20, Shi16, KZ18a, FSS19, BJS21, BJSS21, DN21] that are not covered in the book [EGNO15].

There are other important developments of (braided) fusion (higher) categories related to physics. We choose to cover some of them in the later subsections.

5.2 Factorization homology

We have emphasized in Section 2.2 that the notion of a phase is a macroscopic notion defined on a disk-like region in the infinite size limit (i.e., on \mathbb{R}^n). In principle, it can be described by \mathbb{R}^n -observables in the long wave length limit. For example, in 2d, among all such observables, particle-like topological defects and their fusion-braiding properties form a unitary modular tensor category (UMTC) \mathcal{A} , which characterizes the topological order up to invertible ones. The UMTC is an \mathbb{R}^2 -observable but not a global observable defined on any 2-dimensional closed manifolds. For example, the double braiding can not be a global observable on a 2-sphere (see Remark 3.5.14). We should ask the obvious question: what are the global observables on a closed 2-dimensional manifold Σ ?

It turns out that the global observables can be obtained by integrating the \mathbb{R}^2 -observable \mathcal{A} over Σ . We denote the integral by $\int_{\Sigma} \mathcal{A}$. It was explicitly computed in [AKZ17]. The mathematical foundation of this integral is the mathematical theory of *factorization homology* (see [Lur17] or a recent review [AF20] and references therein).

Theorem 5.2.1 ([AKZ17]). Let Σ be a closed 2-dimensional manifold of genus g and \mathcal{A} be a UMTC. Then we have

$$\int_{\Sigma} \mathcal{A} = \left(\text{Vec}, \text{Hom}_{\mathcal{A}}(\mathbb{1}_{\mathcal{A}}, (\bigoplus_i i^* \otimes i)^{\otimes g}) \right),$$

where the vector space $\text{Hom}_{\mathcal{A}}(\mathbb{1}_{\mathcal{A}}, (\bigoplus_i i^* \otimes i)^{\otimes g})$ is an object in Vec and is precisely the ground state degeneracy (GSD) of the associated topological order on Σ .

Remark 5.2.2. Mathematical works on integrating braided monoidal categories on surfaces can also be found in David Ben-Zvi, Adrien Brochier, and David Jordan's works [BZBJ18a, BZBJ18b]. \diamond

Remark 5.2.3. The mathematical theory of factorization homology was originated from Beilinson and Drinfeld's theory of chiral homology [BD04], which was motivated by Geometric Langlands Correspondence and Conformal Field Theory. \diamond

Remark 5.2.4. By the mathematical theory of factorization homology, one can integrate any E_n -algebras, which can be viewed as the local observable algebras of certain quantum field theories, on any n -dimensional closed/open manifolds. For example, a braided monoidal category \mathcal{A} can be viewed as an E_2 -algebra in the symmetric monoidal 2-category \mathcal{Cat} of 1-categories. By integrating \mathcal{A} over a closed surface Σ , we obtain an E_0 -algebra in \mathcal{Cat} , which is defined by a pair (\mathcal{X}, u) , where \mathcal{X} is a 1-category and u is a distinguished object in \mathcal{X} . A special case of this general result is Theorem 5.2.1. This result can also be understood physically as follows. Integrating the UMTC \mathcal{A} over Σ amounts to squeezing the associated topological order to a point, which can be viewed as a particle in the trivial 2d topological order, i.e., (Vec, u) . From this picture, it is easy to see that $\mathcal{X} \neq \text{Vec}$ if \mathcal{A} has degenerate braidings. More discussion of the application of E_n -monoidal categories and E_n -centers in the study of topological orders can be found in [JF22, KZ22a]. \diamond

Remark 5.2.5. In physics, it is also interesting to compute the GSD on 2-dimensional surface decorated by higher codimensional defects, such as gapped boundaries, domain walls and 0d defect junctions [Kap14b, HW15b, LWW15] (see also [HSW12, HWW13, WWH15, HWW17, HLPWW18, BHW17, WLHW18, WHW22] for the computation of the GSD in concrete lattice models in 2d or 3d). This amounts to compute the factorization homology on disk-stratified surfaces [AKZ17], the mathematical theory of which was developed in [AF15, AFT17, AFT16, AFR18] (see also [BZBJ18b]). \diamond

5.3 String-net models

Constructing lattice models to realize topological orders is an important direction in the study of topological orders. It turns out that category theory is also useful in the construction of lattice models and played an important role in the introduction of category theory to the study of topological orders.

It starts from Kitaev's construction of quantum double models (including the toric code model) in [Kit03], where the idea of topological quantum computing was proposed. This work, together with Kitaev's honeycomb model [Kit06], played a crucial role in removing the doubts among theorists on the existence of quantum spin liquids (see [ZKN17] for a review). Kitaev's quantum double models are built from the group algebras of finite groups and are the lattice model realizations of 2d Dijkgraaf-Witten TQFT's [DW90]. The particle-like topological defects in these models can be identified with the UMTC $\mathfrak{Z}_1(\text{Vec}_G)$. Kitaev's quantum double model was later generalized to quantum double models based on (weak) C^* -Hopf algebras [BMCA13, BK12, Cha14] and to twist quantum double models in [HWW13].

Partially motivated by Kitaev's work [Kit03] and Turaev-Viro TQFT's [TV92, BW96], Levin and Wen introduced their string-net models, which are also called Levin-Wen models, in [LW05]. Levin-Wen models can be viewed as the lattice model realizations of Turaev-Viro-Barrett-Westbury TQFT's [KMR10, KKR10, Kir11, BK12]. A Levin-Wen model is built from the data of a unitary fusion category \mathcal{C} . The category of particle-like topological defects was computed in a few examples and was conjectured to be the Drinfeld center of \mathcal{C} in [LW05]. This conjecture was later proved in [KK12] as a byproduct of a systematic construction of the topological defects of all codimensions in Levin-Wen models via the representation theory of unitary fusion categories (see also [Kon13, LW14]). This work further motivated the studies of gapped boundaries in quantum double models [BSW11, BHW17, CCW17]. The relation between Kitaev's quantum double models and Levin-Wen models were clarified in [BA09, BCKA13]. Roughly speaking, quantum double models based on C^* -Hopf algebras is equivalent to (via a Fourier transformation) the so-called extended string-net models constructed from the data of a unitary fusion category \mathcal{C} , together with a fiber functor $\omega: \mathcal{C} \rightarrow \text{Vec}$ [BCKA13]. Another type of extended string-net models and their gapped boundaries (based on Frobenius algebras in unitary fusion categories) were extensively developed by Hu, et al. [HGW18, HWW17, HLPWW18, WHW22].

String-net models in 2d were further generalized to constructions based on mathematical structures different from unitary fusion categories [Run20, HW20, LLB21] and to non-unitary cases in [FGHT⁺12, LVHV20, CLYH22] and to symmetry enriched topological orders [GWW15, CCCH⁺15, GK16, BGK17, HBFL16]. String-net models in 3d were considered in Levin and Wen's original paper [LW05], and were further developed in the so-called Walker-Wang models [WW11], and in a generalization based on G -crossed braided fusion categories [Cui19], and in (higher) gauge theories [BCKMM17, BCKMM19, ZLW19, BD19, BD20, BD21, BD22], and in its full generality based on fusion 2-categories [DR18, XLLC21].

5.4 Anyon condensation and higher algebras

A 2d topological order can be transformed into another one via the so-called anyon condensations (or boson condensations). The anyon condensation theory was originated in the study of the extensions of chiral algebras [MS89b] and was first studied by Moore and Seiberg in a special case in [MS89c]. A systematic study of anyon condensations was initiated by Bais, Schroers, Slingerland in 2002 based on the idea of Hopf symmetry breaking [BSS02, BSS03], and was further developed by Bais and Slingerland in an influential work [BS09], and was followed by many others (see for example [BMD08, BSH09, BW10, BSS11, BR12, BJLP19]). The precise mathematical description of an anyon condensation was first obtained for 2+1D finite abelian gauge theories [KS11, Lev13, BJQ13], and was fully established for all 2+1D topological orders in [Kon14a] based on many earlier mathematical works. In mathematics, without knowing its relation to anyon condensations, the corresponding mathematical theory was developed earlier and independently by Böckenhauer, Evans and Kawahigashi [BEK99, BEK00, BEK01] in 1999-2001 in terms of the language of subfactors and by Kirillov, Jr. and Ostrik [KO02] in 2001 in terms of the language of algebras in modular tensor categories (see also [FFRS06])²², and by Davydov, Müger, Nikshych and Ostrik in [DMNO13]. Ironically, the connection between the mathematical results and the physical problem was unnoticed for many years before it was fully established in 2013 [Kon14a] (see also [ERB14, HW15a] and see [Bur18] for a review and references therein).

The mathematical theory of anyon condensation demands a lot of new mathematical notions. But we decide to summarize this theory below without explaining these notions not only because readers

²²A dictionary between two languages provided by Kawahigashi can be found in the Introduction Section in [Kon14a].

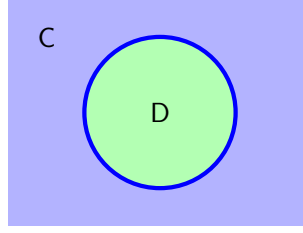


Figure 73: We consider an anyon condensation occurring in a 2d region in a 2d topological phase C . It produces a gapped domain wall between C and the new phase D .

can get some glimpse of this theory and an impression that the theory is rich, precise and completely computable, but also because it reveals its relation to higher algebras and their higher representation theories (see Remark 5.4.2).

Theorem^{ph} 5.4.1 ([Kon14a]). If a 2d topological order $D = (\mathcal{D}, c)$ is obtained from another one $C = (\mathcal{C}, c)$ via a 2d anyon condensation, and if a 1d gapped domain wall is produced by this condensation (see Figure 73), then we must have

1. The vacuum particle in D can be identified with a connected commutative symmetric normalized-special \dagger -Frobenius algebra A in \mathcal{C} . Moreover, \mathcal{D} consists of all deconfined particles and can be identified with the category $\mathcal{C}_A^{\text{loc}}$ of local A -modules in \mathcal{C} , i.e., $\mathcal{D} = \mathcal{C}_A^{\text{loc}}$.
2. Excitations on the wall include all confined and deconfined particles, and can be identified with the unitary fusion category \mathcal{C}_A , which denotes the category of right A -modules in \mathcal{C} .
3. Anyons in C move onto the wall according to the central functor $- \otimes A: \mathcal{C} \rightarrow \mathcal{C}_A$ defined by $x \mapsto x \otimes A$ for all $x \in \mathcal{C}$.
4. Anyons in D move onto the wall according to the embedding $\mathcal{C}_A^{\text{loc}} \hookrightarrow \mathcal{C}_A$, then can move out to the D -side freely.

Remark 5.4.2. In Theorem^{ph} 5.4.1, both of the modular tensor category \mathcal{C} and the commutative algebra A in \mathcal{C} are examples of E_2 -algebras [Lur17] or 2-disk algebras [AF20], which are the 2-dimensional cases of higher dimensional algebras (or simply higher algebras). A local A -module is an A -module equipped with a 2-dimensional A -action. It is an example of higher dimensional representations of an higher algebra. One can also condense particles along a line. This process was called a 1d condensation in [Kon14a], and is controlled by an E_1 -algebra and its bimodules (see Remark 4.4.38). Higher dimensional condensation theory was developed in [KZZZ24] based on the higher representation theories of higher algebras (see for example [Lur17, Hau17] and references therein). \diamond

Remark 5.4.3. More general type of condensations between any two Witt equivalent 2d topological orders was developed in [CRS19]. \diamond

5.5 2d SPT/SET orders

For a quantum many-body system that realizes a trivial topological order, Gu and Wen discovered that, if we impose a symmetry onto the system, then it can realize different and non-trivial phases (e.g. the Haldane phase) that are protected by the symmetry [GW09]. They named such a new phase a symmetry protected topological (or trivial) (SPT) order, which is the trivial topological order if we ignore the symmetry. Chen, Gu and Wen later introduced the notion of a symmetry enriched topological (SET) order [CGW10], which is a non-trivial topological order if we ignore the symmetry. Note that a topological order is simply an SET order with a trivial symmetry, and an SPT order is an SET order with a trivial topological order.

There are a few non-categorical approaches towards SPT orders, such as K-theory [Kit09], (super) group cohomology [CGLW13, GW14, WG18], cobordism groups [Kap14c, KTTW15] and stable homotopy theory [Kit11, Kit13, Kit15, FH21, GJF19b]. We only focus on the categorical approach here. We have seen that the categorical description of a 2d topological order is given by a UMTC. Therefore, it is natural to expect that the categorical description of a 2d SPT/SET order is deeply related to UMTC. For a finite onsite symmetry, there are two approaches.

1. One approach is based on the idea of gauging the symmetry by introducing invertible defects of codimension 1. Based on this idea, Barkeshli, Bonderson, Cheng and Wang developed in [BBCW19] a mathematical theory of bosonic 2d SPT/SET orders and showed that a bosonic 2d bosonic SPT/SET order can be described by a G -crossed braided fusion category (see [Tur00, Kir01, Müg04, DGNO10] for mathematical literature on this notion) and see also Remark 4.4.30). This idea was further studied by Tarantino, Lindner and Fidkowski [TLF16]. This approach can also be applied to fermionic SPT/SET orders [FVM18, ABK21, BB22b, BB22a].
2. The other approach is also based on the idea of gauging the symmetry but in a different way. In [LG12], Levin and Gu gauged the symmetry in a 2d \mathbb{Z}_2 SPT order by adding gauge fluxes and obtained a gauge theory (i.e., 2d topological order) as the result of the gauging. This idea was soon formulated mathematically by Lan, Kong and Wen in [LKW16a, LKW17] as a categorical gauging process, in which additional particles are introduced the system so that each of them has non-trivial double braidings with at least one symmetry charges. After the gauging, we obtain a UMTC. This categorical gauging process has a mathematical name called *minimal modular extension* first appeared in Müger's work [Müg00], and it works for all bosonic and fermionic 2d SPT/SET orders. Moreover, these minimal modular extensions form a finite abelian group [LKW16a], which recovers the classification of bosonic 2d SPT orders by $H^3(G; \mathbb{U}(1))$ [CGLW13] (see Example 4.2.34) and Kitaev's 16-fold way [Kit06] (see Remark 3.6.6). Further developments of minimal modular extensions and its higher dimensional generalizations can be found in [BGHN⁺17, GVR17, VR19, DN21, OY23, JFR23, KZ24, Nik22].

In the bosonic cases, the equivalence between above two approaches were known [Bru00, Kir01, Müg04]. The idea of minimal modular extension can be generalized to higher dimensions (see [KTZ20b, KLWZZ20b, KZ24]).

For SPT orders with spacetime symmetries, there are a lot of physical and non-categorical works (see for example [FK11, SHFH17, WL17, JR17, TE18, BBCJW19, Xio18, SXG18, SHQFH19, ET19, SFQ20, SXH20, GOPWW20, FH20, Deb21] and references therein). There are also works on the real space construction of SET orders with space symmetries (see for example [Lak16, QJW19, MW20, NMLW21] and references therein). Unfortunately, the categorical description of SPT/SET orders with spacetimes symmetries has not yet been developed. This is an interesting direction for future development.

5.6 Boundary-bulk relation and gapless boundaries

We have seen the boundary-bulk relation in the toric code model, which can be stated as the bulk UMTC is the center of the boundary unitary fusion category. We have also mentioned in Theorem^{ph} 4.2.21 that the boundary-bulk relation holds for all anomaly-free stable 2d topological orders and their gapped 1d boundaries. It turns out that it is not an isolated phenomenon. It was known earlier that the boundary-bulk duality (i.e., open-closed duality) in rational conformal field theories (RCFT) also states that the bulk CFT is the center of a compatible boundary CFT [FFRS08, KR08, KR09, Dav10] (see [Kon11] for a review and references therein). If we view the same phenomena occur in different theories as a robust phenomena, it must have a simple reason.

In mathematics, the notion of center can be defined for any algebraic object in any monoidal higher category by its universal property, which is entirely the same for all different notions of center, such as the ordinary center of a group or an algebra, Drinfeld center, Müger center, Hochschild cohomology and E_n -centers, etc. The universal property of the center only depends on how we define the morphisms between objects instead of the set-theoretical definition of an object. Therefore, a formal proof of the boundary-bulk relation for general nd topological orders can be obtained by checking the universal property if we know how to define the right notion of a morphism between two potentially anomalous nd topological orders. This notion was introduced in [KWZ15, KWZ17]. Based on this notion, it is almost tautological to prove the following rather strong result.

Theorem^{ph} 5.6.1 (Boundary-Bulk Relation [KWZ15, KWZ17]). For any anomaly-free nd topological order C and one of its gapped/gapless boundary X , the bulk C is the center of the boundary X , and we denote this relation by $C = \mathfrak{Z}(X)$.

Remark 5.6.2. The proof of the above physical Theorem actually works for even more general systems. More precisely, we can replace the anomaly-free nd topological order by an anomaly-free nd quantum liquid-like gapped/gapless phase [KZ22a]. By ‘liquid-like’, we mean that the phase (and all higher codimensional defects) is ‘soft’ enough so that it does not depend rigidly on the local geometry (e.g., metric) of the phase. In the gapped cases, the phase is topological thus completely independent of the metric. In the gapless cases, the phase is of CFT-type in the sense that it only depends covariantly on the deformations of the metric in a finite and controllable way similar to 2D rational CFT’s. We call such a ‘liquid-like’ quantum phase a quantum liquid. Quantum liquids include all topological orders, SPT/SET orders, gapped symmetry-breaking orders and certain CFT-type gapless phases. \diamond

The significance of Theorem^{ph} 5.6.1 is two-fold. On the one hand, if we know the categorical description \mathcal{X} of the boundary X , by computing the center, we immediately obtain that of the bulk. On the other hand, if we know the categorical description \mathcal{C} of C , we can obtain that of X by solving the equation $\mathcal{C} = \mathfrak{Z}_1(?)$. For example, if \mathcal{C} is the UMTC of a 2d topological order C , then the solutions to the equation $\mathcal{C} = \mathfrak{Z}_1(?)$ provide the mathematical descriptions of its 1d boundaries. When $?$ is a unitary fusion category, then it describes a gapped boundary. When the equation does not have a solution in unitary fusion categories, the boundaries are necessarily gapless. The mathematical theory of gapless boundaries of 2d topological orders was developed in [KZ18c, KZ20, KZ21]. A gapless boundary can be mathematically described by a unitary fusion category \mathcal{X}^\sharp enriched in a UMTC such that $\mathcal{C} = \mathfrak{Z}_1(\mathcal{X}^\sharp)$ [KZ18b, KYZZ24]. This result led to a new and systematic approach towards 1+1D topological phase transitions [CJKYZ20]. The mathematical theory of enriched fusion categories was developed only recently (see [MP17, KZ18b, Zhe17, MPP18, JMPP21, KZ21, KYZZ24]). The gapless boundaries of higher dimensional topological orders demand us to develop the mathematical theory of enriched higher categories (see [KZ21] for more discussions).

Remark 5.6.3. The statement of boundary-bulk relation in Theorem^{ph} 5.6.1 can be further generalized to include domain walls between boundaries and higher codimensional walls between walls. For 2d topological orders with gapped boundaries, this generalized boundary-bulk relation amounts to the fact that the Drinfeld center defines a fully faithful functor [KZ18a]. Generalizing this result to a pointed Drinfeld center functor, we obtain the boundary-bulk relation including domain walls in 1+1D rational CFT’s [KYZ21]. For 2d topological orders with gapped/gapless boundaries, it amounts to the fact that the E_1 -center of certain braided-enriched fusion categories defines a monoidal equivalence [KZ21]. For higher dimensional topological orders with gapped boundaries and gapped higher codimensional walls, the boundary-bulk relation can also be formulated as a center functor [KZ24]. \diamond

5.7 Higher dimensional topological orders and SPT/SET orders

We have seen that an anomaly-free 1d topological order can be described by a unitary fusion 1-category with a trivial E_1 -center (see Section 4.2.4). We have also seen that an anomaly-free 2d topological order can be described either by a UMTC or by a unitary fusion 2-category with a trivial E_1 -center (see Remark 4.5.13). In the study of higher dimensional topological orders, we are forced to enter the mathematical world of higher categories.

In recent years, higher category becomes a very hot topics in mathematics. There are a lot of exciting developments of (∞, n) -categories (see for example [Lur09, Cis19] and references therein). Unfortunately, it is not clear to us how to apply such heavy machinery to the study of topological orders.

Without knowing how to rigorously define a fusion higher category, guided by the so-called remote-detectable principle 3.5.1 and boundary-bulk relation, Kong, Wen and Zheng proposed in [KW14, KWZ15] that topological defects of codimension 2 or higher in an anomaly-free stable nd topological order form a unitary braided fusion $(n-1)$ -category with a trivial E_2 -center, or equivalently, its defects of codimension 1 or higher form a unitary fusion n -category with a trivial E_1 -center. Combining with useful physical intuitions, this proposal led Lan, Kong and Wen to a classification theory of 3d topological orders with only bosonic particles [LKW18], which was later rigorously proved by Johnson-Freyd [JF20], and led Lan and Wen to that of 3d topological orders with fermionic particles [LW19],

which was later slightly corrected and rigorously proved by Johnson-Freyd [JF20] (see also [FHH22] for physical realizations). Further categorical studies of 3d and 4d topological orders can be found in [JFY21a, JFY21b].

Rigorous categorical results related to n d topological orders for $n \geq 3$ start from Douglas and Reutter's definition of a multi-fusion 2-category and their state-sum construction of 3+1D TQFT's based on spherical fusion 2-categories [DR18]. Their definition of a multi-fusion 2-category is based on the idea of the 2-category version of idempotent completion, which was briefly introduced in [CR16] and thoroughly developed in [DR18]. It turns out that this idempotent completion precisely amounts to adding condensation descendants to the categorical description of a topological order [GJF19a, KZWZZ20b]. The necessity of this idempotent completion in the categorical description of topological orders was further confirmed by an explicit computation of $\mathfrak{Z}_1(2\text{Vec}_G^\omega)$ for a finite group and $\omega \in H^4(G; \mathbb{U}(1))$, i.e., the E_1 -center of 2Vec_G^ω , which describes the topological defects of codimension 2 or higher in 3+1D Dijkgraaf-Witten Theory [KTZ20b]. Douglas and Reutter's idempotent completion was later generalized to higher categories by Gaiotto and Johnson-Freyd [GJF19a] under the name of Karoubi completion or condensation completion (see Remark 5.7.1). This generalization is not mathematically complete in the sense that it did not choose any concrete model of higher categories and assumed the not-yet-developed theory of colimits in higher categories. However, it seems that Gaiotto and Johnson-Freyd's work solved the problems that are orthogonal to the higher coherence problems, and should work in any reasonable models of higher categories. This work led to an incomplete-but-workable definition of a multi-fusion n -category [JF22]. Based on this definition, Johnson-Freyd developed a mathematical theory of multi-fusion n -categories and answered many questions in [KW14, KWZ15] (see Remark 5.7.1). This theory was further developed in [KZ22a, KZ24] from a slightly different perspective emphasizing more on separable n -categories.

Remark 5.7.1. All defects of all codimensions in a 2d non-chiral topological order, including all condensation descendants, were explicitly constructed via Levin-Wen models [KK12]. A categorical description of the complete set of these defects was known (see [KW14, Example 16]). The study of condensation completion for topological orders in higher dimensions was initiated in a physical context in [KW14, Section XI.H]. \diamond

Based on the idea of boundary-bulk relation [KWZ15, KWZ17], a unified mathematical theory of all SPT/SET orders in all dimensions was developed in [KLWZZ20b, KLWZZ20a].

Remark 5.7.2. It was noticed in [KZ22a] that a small modification of the main result in [KLWZZ20b] automatically provides the classification of all gapped symmetry-breaking phases. Applying the topological Wick rotation introduced in [KZ20, KZ21] to the results in [KLWZZ20b], Kong and Zheng proposed to use enriched higher categories to give a grand unification of the categorical description of all quantum liquids, including topological orders, SPT/SET orders (with a finite onsite symmetry), gapped symmetry-breaking phases and CFT-type gapless phases [KZ22a]. The validity of this categorical description of all 1d gapped liquids with a finite onsite symmetry was checked for Ising chain and Kitaev chain in [KWZ22]. Moreover, the earlier classification of all 1d gapped quantum phases with a bosonic/fermionic finite onsite symmetry by studying ground state wave functions [CGW11a, SPGC11, CGW11b] was rediscovered in this new categorical approach [KZ22a, KWZ22]. A more complete mathematical theory of all quantum liquids was developed in [KZ22b]. \diamond

References

- [ABCD⁺09] Paul S. Aspinwall, Tom Bridgeland, Alastair Craw, Michael R. Douglas, Mark Gross, Anton Kapustin, Gregory W. Moore, Graeme Segal, Balázs Szendrői, and P.M.H. Wilson. *Dirichlet Branes and Mirror Symmetry*, volume 4 of *Clay Mathematics Monographs*. American Mathematical Society, 2009.
- [ABGEHSN23] Fabio Apruzzi, Federico Bonetti, Iñaki García Etxebarria, Saghar S. Hosseini, and Sakura Schäfer-Nameki. *Symmetry TFTs from string theory*. *Communications in Mathematical Physics*, 402(1):895–949, 2023.
- [ABK21] David Aasen, Parsa Bonderson, and Christina Knapp. *Characterization and classification of fermionic symmetry enriched topological phases*. arXiv preprint, 2021. [arXiv:2109.10911](https://arxiv.org/abs/2109.10911).
- [AC16] Venkitesh Ayyar and Shailesh Chandrasekharan. *Origin of fermion masses without spontaneous symmetry breaking*. *Physical Review D*, 93(8), 2016. [arXiv:1511.09071](https://arxiv.org/abs/1511.09071).
- [AF15] David Ayala and John Francis. *Factorization homology of topological manifolds*. *Journal of Topology*, 8(4):1045–1084, 2015. [arXiv:1206.5522](https://arxiv.org/abs/1206.5522).
- [AF17] David Ayala and John Francis. *The cobordism hypothesis*. arXiv preprint, 2017. [arXiv:1705.02240](https://arxiv.org/abs/1705.02240).
- [AF20] David Ayala and John Francis. *A factorization homology primer*. In *Handbook of Homotopy Theory*, pages 39–101. Chapman and Hall/CRC, 2020. [arXiv:1903.10961](https://arxiv.org/abs/1903.10961).
- [AFR18] David Ayala, John Francis, and Nick Rozenblyum. *Factorization homology I: Higher categories*. *Advances in Mathematics*, 333:1042–1177, 2018. [arXiv:1504.04007](https://arxiv.org/abs/1504.04007).
- [AFT16] David Ayala, John Francis, and Hiro Lee Tanaka. *Factorization homology of stratified spaces*. *Selecta Mathematica*, 23(1):293–362, 2016. [arXiv:1409.0848](https://arxiv.org/abs/1409.0848).
- [AFT17] David Ayala, John Francis, and Hiro Lee Tanaka. *Local structures on stratified spaces*. *Advances in Mathematics*, 307:903–1028, 2017. [arXiv:1409.0501](https://arxiv.org/abs/1409.0501).
- [AKZ17] Yinghua Ai, Liang Kong, and Hao Zheng. *Topological orders and factorization homology*. *Advances in Theoretical and Mathematical Physics*, 21(8):1845–1894, 2017. [arXiv:1607.08422](https://arxiv.org/abs/1607.08422).
- [AM88] Greg Anderson and Greg Moore. *Rationality in conformal field theory*. *Communications in Mathematical Physics*, 117(3):441–450, 1988.
- [And83] P. W. Anderson. *Remarks on the Laughlin theory of the fractionally quantized Hall effect*. *Physical Review B*, 28(4):2264–2265, 1983.
- [ASW84] Daniel Arovas, J. R. Schrieffer, and Frank Wilczek. *Fractional statistics and the quantum Hall effect*. *Physical Review Letters*, 53(7):722–723, 1984.
- [Ati88] Michael Atiyah. *Topological quantum field theories*. *Publications mathématiques de l’IHÉS*, 68(1):175–186, 1988.
- [Awo10] Steve Awodey. *Category Theory*. Oxford University Press, 2010.
- [BA09] Oliver Buerschaper and Miguel Aguado. *Mapping Kitaev’s quantum double lattice models to Levin and Wen’s string-net models*. *Physical Review B*, 80(15):155136, 2009. [arXiv:0907.2670](https://arxiv.org/abs/0907.2670).
- [Bai17] James Bailie. *Semisimple categories*. Preprint online, 2017. <https://jameshbailie.github.io/Papers/papers/Semisimple.pdf>.
- [BB01] A. Beliakova and Ch. Blanchet. *Modular categories of types B, C and D*. *Commentarii Mathematici Helvetici*, 76(3):467–500, 2001. [arXiv:math/0006227](https://arxiv.org/abs/math/0006227).

- [BB22a] Daniel Bulmash and Maissam Barkeshli. *Anomaly cascade in $(2 + 1)$ -dimensional fermionic topological phases*. Physical Review B, 105(15):155126, 2022. [arXiv:2109.10922](#).
- [BB22b] Daniel Bulmash and Maissam Barkeshli. *Fermionic symmetry fractionalization in $(2 + 1)$ dimensions*. Physical Review B, 105(12):125114, 2022. [arXiv:2109.10913](#).
- [BBCJW19] Maissam Barkeshli, Parsa Bonderson, Meng Cheng, Chao-Ming Jian, and Kevin Walker. *Reflection and time reversal symmetry enriched topological phases of matter: Path integrals, non-orientable manifolds, and anomalies*. Communications in Mathematical Physics, 374(2):1021–1124, 2019. [arXiv:1612.07792](#).
- [BBCW19] Maissam Barkeshli, Parsa Bonderson, Meng Cheng, and Zhenghan Wang. *Symmetry fractionalization, defects, and gauging of topological phases*. Physical Review B, 100(11), 2019. [arXiv:1410.4540](#).
- [BCKA13] Oliver Buerschaper, Matthias Christandl, Liang Kong, and Miguel Aguado. *Electric-magnetic duality of lattice systems with topological order*. Nuclear Physics B, 876(2):619–636, 2013. [arXiv:1006.5823](#).
- [BCKMM17] Alex Bullivant, Marcos Calçada, Zoltán Kádár, Paul Martin, and João Faria Martins. *Topological phases from higher gauge symmetry in $3+1D$* . Physical Review B, 95(15):155118, 2017. [arXiv:1606.06639](#).
- [BCKMM19] Alex Bullivant, Marcos Calçada, Zoltán Kádár, João Faria Martins, and Paul Martin. *Higher lattices, discrete two-dimensional holonomy and topological phases in $(3+1)D$ with higher gauge symmetry*. Reviews in Mathematical Physics, 32(04):2050011, 2019. [arXiv:1702.00868](#).
- [BD95] John C. Baez and James Dolan. *Higher-dimensional algebra and topological quantum field theory*. Journal of Mathematical Physics, 36(11):6073–6105, 1995. [arXiv:q-alg/9503002](#).
- [BD04] Alexander Beilinson and Vladimir Drinfeld. *Chiral algebras*. American Mathematical Society, Providence, R.I, 2004.
- [BD19] Alex Bullivant and Clement Delcamp. *Tube algebras, excitations statistics and compactification in gauge models of topological phases*. Journal of High Energy Physics, 2019(10), 2019. [arXiv:1905.08673](#).
- [BD20] Alex Bullivant and Clement Delcamp. *Excitations in strict 2-group higher gauge models of topological phases*. Journal of High Energy Physics, 2020(1), 2020. [arXiv:1909.07937](#).
- [BD21] Alex Bullivant and Clement Delcamp. *Gapped boundaries and string-like excitations in $(3+1)d$ gauge models of topological phases*. Journal of High Energy Physics, 2021(7), 2021. [arXiv:2006.06536](#).
- [BD22] Alex Bullivant and Clement Delcamp. *Crossing with the circle in Dijkgraaf–Witten theory and applications to topological phases of matter*. Journal of Mathematical Physics, 63(8), 2022. [arXiv:2103.12717](#).
- [BEK99] Jens Böckenhauer, David E. Evans, and Yasuyuki Kawahigashi. *On α -induction, chiral generators and modular invariants for subfactors*. Communications in Mathematical Physics, 208(2):429–487, 1999. [arXiv:math/9904109](#).
- [BEK00] Jens Böckenhauer, David E. Evans, and Yasuyuki Kawahigashi. *Chiral structure of modular invariants for subfactors*. Communications in Mathematical Physics, 210(3):733–784, 2000. [arXiv:math/9907149](#).
- [BEK01] Jens Böckenhauer, David E. Evans, and Yasuyuki Kawahigashi. *Longo-Rehren subfactors arising from α -induction*. Publications of the Research Institute for Mathematical Sciences, 37(1):1–35, 2001. [arXiv:math/0002154](#).

- [Bén63] Jean Bénabou. *Catégories avec multiplication*. Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Paris, 260:1887–1890, 1963.
- [Bén65] Jean Bénabou. *Catégories relatives*. Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Paris, 256:3824–3827, 1965.
- [Bez04] Roman Bezrukavnikov. *On tensor categories attached to cells in affine Weyl groups*. In *Representation Theory of Algebraic Groups and Quantum Groups*. Mathematical Society of Japan, 2004. [arXiv:math/0010089](#).
- [BGHN⁺17] Paul Bruillard, César Galindo, Tobias Hagge, Siu-Hung Ng, Julia Yael Plavnik, Eric C. Rowell, and Zhenghan Wang. *Fermionic modular categories and the 16-fold way*. Journal of Mathematical Physics, 58(4):041704, 2017. [arXiv:1603.09294](#).
- [BGK17] Lakshya Bhardwaj, Davide Gaiotto, and Anton Kapustin. *State sum constructions of spin-TFTs and string net constructions of fermionic phases of matter*. Journal of High Energy Physics, 2017(4), 2017. [arXiv:1605.01640](#).
- [BHJL24] Ning Bao, Ling-Yan Hung, Yikun Jiang, and Zhihan Liu. *QG from SymQRG: AdS_3/CFT_2 correspondence as topological symmetry-preserving quantum RG flow*. arXiv preprint, 2024. [arXiv:2412.12045](#).
- [BHW17] Alex Bullivant, Yuting Hu, and Yidun Wan. *Twisted quantum double model of topological order with boundaries*. Physical Review B, 96(16):165138, 2017. [arXiv:1706.03611](#).
- [BJLP19] Marcel Bischoff, Corey Jones, Yuan-Ming Lu, and David Penneys. *Spontaneous symmetry breaking from anyon condensation*. Journal of High Energy Physics, 2019(2), 2019. [arXiv:1811.00434](#).
- [BJQ13] Maissam Barkeshli, Chao-Ming Jian, and Xiao-Liang Qi. *Theory of defects in abelian topological states*. Physical Review B, 88(23), 2013. [arXiv:1305.7203](#).
- [BJS21] Adrien Brochier, David Jordan, and Noah Snyder. *On dualizability of braided tensor categories*. Compositio Mathematica, 157(3):435–483, 2021. [arXiv:1804.07538](#).
- [BJSS21] Adrien Brochier, David Jordan, Pavel Safronov, and Noah Snyder. *Invertible braided tensor categories*. Algebraic & Geometric Topology, 21(4):2107–2140, 2021. [arXiv:2003.13812](#).
- [BK98] S. B. Bravyi and A. Yu. Kitaev. *Quantum codes on a lattice with boundary*. arXiv preprint, 1998. [arXiv:quant-ph/9811052](#).
- [BK01] Bojko Bakalov and Alexander Kirillov, Jr. *Lectures on tensor categories and modular functors*. American Mathematical Society, Providence, RI, 2001.
- [BK12] Benjamin Balsam and Alexander Kirillov, Jr. *Kitaev's lattice model and Turaev-Viro TFTs*. arXiv preprint, 2012. [arXiv:1206.2308](#).
- [BLT20] Lakshya Bhardwaj, Yasunori Lee, and Yuji Tachikawa. *$SL(2, \mathbb{Z})$ action on QFTs with \mathbb{Z}_2 symmetry and the Brown-Kervaire invariants*. Journal of High Energy Physics, 2020(11), 2020. [arXiv:2009.10099](#).
- [BMCA13] Oliver Buerschaper, Juan Martín Mombelli, Matthias Christandl, and Miguel Aguado. *A hierarchy of topological tensor network states*. Journal of Mathematical Physics, 54(1):012201, 2013. [arXiv:1007.5283](#).
- [BMD08] H. Bombin and M. A. Martin-Delgado. *Family of non-abelian Kitaev models on a lattice: Topological condensation and confinement*. Physical Review B, 78(11), 2008. [arXiv:0712.0190](#).
- [BNRW15] Paul Bruillard, Siu-Hung Ng, Eric C. Rowell, and Zhenghan Wang. *Rank-finiteness for modular categories*. Journal of the American Mathematical Society, 29(3):857–881, 2015. [arXiv:1310.7050](#).

- [Bom10] H. Bombin. *Topological order with a twist: Ising anyons from an abelian model*. Physical Review Letters, 105(3), 2010. [arXiv:1004.1838](#).
- [BPZ84] A.A. Belavin, A.M. Polyakov, and A.B. Zamolodchikov. *Infinite conformal symmetry in two-dimensional quantum field theory*. Nuclear Physics B, 241(2):333–380, 1984.
- [BR12] F A Bais and J C Romers. *The modular S-matrix as order parameter for topological phase transitions*. New Journal of Physics, 14(3):035024, 2012. [arXiv:1108.0683](#).
- [Bru00] Alain Bruguières. *Catégories prémodulaires, modularisations et invariants des variétés de dimension 3*. Mathematische Annalen, 316(2):215–236, 2000.
- [BS09] F. A. Bais and J. K. Slingerland. *Condensate-induced transitions between topologically ordered phases*. Physical Review B, 79(4), 2009. [arXiv:0808.0627](#).
- [BSH09] F. A. Bais, J. K. Slingerland, and S. M. Haaker. *Theory of topological edges and domain walls*. Physical Review Letters, 102(22), 2009. [arXiv:0812.4596](#).
- [BSS02] F. A. Bais, B. J. Schroers, and J. K. Slingerland. *Broken quantum symmetry and confinement phases in planar physics*. Physical Review Letters, 89(18), 2002. [arXiv:hep-th/0205117](#).
- [BSS03] Alexander F Bais, Bernd J Schroers, and Joost K Slingerland. *Hopf symmetry breaking and confinement in (2+1)-dimensional gauge theory*. Journal of High Energy Physics, 2003(05):068–068, 2003. [arXiv:hep-th/0205114](#).
- [BSS11] F. J. Burnell, Steven H. Simon, and J. K. Slingerland. *Condensation of achiral simple currents in topological lattice models: Hamiltonian study of topological symmetry breaking*. Physical Review B, 84(12), 2011. [arXiv:1104.1701](#).
- [BSW11] Salman Beigi, Peter W. Shor, and Daniel Whalen. *The quantum double model with boundary: Condensations and symmetries*. Communications in Mathematical Physics, 306(3):663–694, 2011. [arXiv:1006.5479](#).
- [Bur18] F.J. Burnell. *Anyon condensation and its applications*. Annual Review of Condensed Matter Physics, 9(1):307–327, 2018. [arXiv:1706.04940](#).
- [BW90a] B. Blok and X. G. Wen. *Effective theories of the fractional quantum Hall effect at generic filling fractions*. Physical Review B, 42(13):8133–8144, 1990.
- [BW90b] B. Blok and X. G. Wen. *Effective theories of the fractional quantum Hall effect: Hierarchy construction*. Physical Review B, 42(13):8145–8156, 1990.
- [BW91] B. Blok and X. G. Wen. *Structure of the microscopic theory of the hierarchical fractional quantum hall effect*. Physical Review B, 43(10):8337–8349, 1991.
- [BW96] John W. Barrett and Bruce W. Westbury. *Invariants of piecewise-linear 3-manifolds*. Transactions of the American Mathematical Society, 348(10):3997–4022, 1996. [arXiv:hep-th/9311155](#).
- [BW99] John W Barrett and Bruce W Westbury. *Spherical categories*. Advances in Mathematics, 143(2):357–375, 1999. [arXiv:hep-th/9310164](#).
- [BW10] Maissam Barkeshli and Xiao-Gang Wen. *Anyon condensation and continuous topological phase transitions in non-abelian fractional quantum hall states*. Physical Review Letters, 105(21), 2010. [arXiv:1007.2030](#).
- [BZBJ18a] David Ben-Zvi, Adrien Brochier, and David Jordan. *Integrating quantum groups over surfaces*. Journal of Topology, 11(4):874–917, 2018. [arXiv:1501.04652](#).
- [BZBJ18b] David Ben-Zvi, Adrien Brochier, and David Jordan. *Quantum character varieties and braided module categories*. Selecta Mathematica, 24(5):4711–4748, 2018. [arXiv:1606.04769](#).

- [CCCH⁺15] Liang Chang, Meng Cheng, Shawn X Cui, Yuting Hu, Wei Jin, Ramis Movassagh, Pieter Naaijken, Zhenghan Wang, and Amanda Young. *On enriching the Levin-Wen model with symmetry*. Journal of Physics A: Mathematical and Theoretical, 48(12):12FT01, 2015. [arXiv:1412.6589](#).
- [CCW17] Iris Cong, Meng Cheng, and Zhenghan Wang. *Hamiltonian and algebraic theories of gapped boundaries in topological phases of matter*. Communications in Mathematical Physics, 355(2):645–689, 2017. [arXiv:1707.04564](#).
- [CGHP23] Jessica Christian, David Green, Peter Huston, and David Penneys. *A lattice model for condensation in Levin-Wen systems*. Journal of High Energy Physics, 2023(9), 2023. [arXiv:2303.04711](#).
- [CGLW13] Xie Chen, Zheng-Cheng Gu, Zheng-Xin Liu, and Xiao-Gang Wen. *Symmetry protected topological orders and the group cohomology of their symmetry group*. Physical Review B, 87(15), 2013. [arXiv:1106.4772](#).
- [CGW10] Xie Chen, Zheng-Cheng Gu, and Xiao-Gang Wen. *Local unitary transformation, long-range quantum entanglement, wave function renormalization, and topological order*. Physical Review B, 82(15), 2010. [arXiv:1004.3835](#).
- [CGW11a] Xie Chen, Zheng-Cheng Gu, and Xiao-Gang Wen. *Classification of gapped symmetric phases in one-dimensional spin systems*. Physical Review B, 83(3), 2011. [arXiv:1008.3745](#).
- [CGW11b] Xie Chen, Zheng-Cheng Gu, and Xiao-Gang Wen. *Complete classification of one-dimensional gapped quantum phases in interacting spin systems*. Physical Review B, 84(23), 2011. [arXiv:1103.3323](#).
- [Cha05] Claudio Chamon. *Quantum glassiness in strongly correlated clean systems: An example of topological overprotection*. Physical Review Letters, 94(4), 2005. [arXiv:cond-mat/0404182](#).
- [Cha14] Liang Chang. *Kitaev models based on unitary quantum groupoids*. Journal of Mathematical Physics, 55(4):041703, 2014. [arXiv:1309.4181](#).
- [CHZ02] Andrea Cappelli, Marina Huerta, and Guillermo R. Zemba. *Thermal transport in chiral conformal theories and hierarchical quantum hall states*. Nuclear Physics B, 636(3):568–582, 2002. [arXiv:cond-mat/0111437](#).
- [Cis19] Denis-Charles Cisinski. *Higher Categories and Homotopical Algebra*. Cambridge University Press, 2019.
- [CJKYZ20] Wei-Qiang Chen, Chao-Ming Jian, Liang Kong, Yi-Zhuang You, and Hao Zheng. *Topological phase transition on the edge of two-dimensional \mathbb{Z}_2 topological order*. Physical Review B, 102(4), 2020. [arXiv:1903.12334](#).
- [CJW22] Arkya Chatterjee, Wenjie Ji, and Xiao-Gang Wen. *Emergent generalized symmetry and maximal symmetry-topological-order*. arXiv preprint, 2022. [arXiv:2212.14432](#).
- [CLW11] Xie Chen, Zheng-Xin Liu, and Xiao-Gang Wen. *Two-dimensional symmetry-protected topological orders and their protected gapless edge excitations*. Physical Review B, 84(23), 2011. [arXiv:1106.4752](#).
- [CLYH22] Chao-Yi Chen, Bing-Xin Lao, Xin-Yang Yu, and Ling-Yan Hung. *Galois conjugates of string-net model*. Physical Review D, 105(6):065009, 2022. [arXiv:2108.05730](#).
- [CMRSS21] Nils Carqueville, Vincentas Mulevicius, Ingo Runkel, Gregor Schaumann, and Daniel Scherl. *Orbifold graph TQFTs*. arXiv preprints, 2021. [arXiv:2101.02482](#).

- [CMRSS24] Nils Carqueville, Vincentas Mulevicius, Ingo Runkel, Gregor Schaumann, and Daniel Scherl. *Reshetikhin–Turaev TQFTs close under generalised orbifolds*. Communications in Mathematical Physics, 405(242), 2024. [arXiv:2109.04754](#).
- [CR16] Nils Carqueville and Ingo Runkel. *Orbifold completion of defect bicategories*. Quantum Topology, 7(2):203–279, 2016. [arXiv:1210.6363](#).
- [CRS18] Nils Carqueville, Ingo Runkel, and Gregor Schaumann. *Line and surface defects in Reshetikhin–Turaev TQFT*. Quantum Topology, 10(3):399–439, 2018. [arXiv:1710.10214](#).
- [CRS19] Nils Carqueville, Ingo Runkel, and Gregor Schaumann. *Orbifolds of n -dimensional defect TQFTs*. Geometry & Topology, 23(2):781–864, 2019. [arXiv:1705.06085](#).
- [CS17] Simon Catterall and David Schaich. *Novel phases in strongly coupled four-fermion theories*. Physical Review D, 96(3), 2017. [arXiv:1609.08541](#).
- [Cui19] Shawn Cui. *Four dimensional topological quantum field theories from G -crossed braided categories*. Quantum Topology, 10(4):593–676, 2019. [arXiv:1610.07628](#).
- [CW23] Arkya Chatterjee and Xiao-Gang Wen. *Holographic theory for continuous phase transitions: Emergence and symmetry protection of gaplessness*. Physical Review B, 108(7):075105, 2023. [arXiv:2205.06244](#).
- [Dav10] Alexei Davydov. *Centre of an algebra*. Advances in Mathematics, 225(1):319–348, 2010. [arXiv:0908.1250](#).
- [Deb21] Arun Debray. *Invertible phases for mixed spatial symmetries and the fermionic crystalline equivalence principle*. arXiv preprint, 2021. [arXiv:2102.02941](#).
- [Del02] P. Deligne. *Catégories tensorielles*. Moscow Mathematical Journal, 2(2):227–248, 2002.
- [Del07] P. Deligne. *Catégories tannakiennes*. In *The Grothendieck Festschrift*, pages 111–195. Birkhäuser Boston, 2007.
- [DG17] Bianca Dittrich and Marc Geiller. *Quantum gravity kinematics from extended TQFTs*. New Journal of Physics, 19(1):013003, 2017. [arXiv:1604.05195](#).
- [DGNO07] Vladimir Drinfeld, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Group-theoretical properties of nilpotent modular categories*. arXiv preprint, 2007. [arXiv:0704.0195](#).
- [DGNO10] Vladimir Drinfeld, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *On braided fusion categories I*. Selecta Mathematica, 16(1):1–119, 2010. [arXiv:0906.0620](#).
- [DKR11] Alexei Davydov, Liang Kong, and Ingo Runkel. *Field theories with defects and the centre functor*. In *Mathematical Foundations of Quantum Field Theory and Perturbative String Theory*, pages 71–128. American Mathematical Society, 2011. [arXiv:1107.0495](#).
- [DKR15] Alexei Davydov, Liang Kong, and Ingo Runkel. *Functoriality of the center of an algebra*. Advances in Mathematics, 285:811–876, 2015. [arXiv:1307.5956](#).
- [DM82] P. Deligne and J. S. Milne. *Tannakian categories*. In *Lecture Notes in Mathematics*, pages 101–228. Springer Berlin Heidelberg, 1982.
- [DMNO13] Alexei Davydov, Michael Müger, Dmitri Nikshych, and Victor Ostrik. *The Witt group of non-degenerate braided fusion categories*. Journal für die reine und angewandte Mathematik (Crelles Journal), 2013(677), 2013. [arXiv:1009.2117](#).
- [DN13] Alexei Davydov and Dmitri Nikshych. *The Picard crossed module of a braided tensor category*. Algebra & Number Theory, 7(6):1365–1403, 2013. [arXiv:1202.0061](#).
- [DN21] Alexei Davydov and Dmitri Nikshych. *Braided Picard groups and graded extensions of braided tensor categories*. Selecta Mathematica, 27(4), 2021. [arXiv:2006.08022](#).

- [DNO12] Alexei Davydov, Dmitri Nikshych, and Victor Ostrik. *On the structure of the Witt group of braided fusion categories*. Selecta Mathematica, 19(1):237–269, 2012. [arXiv:1109.5558](#).
- [DR18] Christopher L. Douglas and David J. Reutter. *Fusion 2-categories and a state-sum invariant for 4-manifolds*. arXiv preprint, 2018. [arXiv:1812.11933](#).
- [DS17] Alexei Davydov and Darren Simmons. *On Lagrangian algebras in group-theoretical braided fusion categories*. Journal of Algebra, 471:149–175, 2017. [arXiv:1603.04650](#).
- [DSPS19] Christopher L. Douglas, Christopher Schommer-Pries, and Noah Snyder. *The balanced tensor product of module categories*. Kyoto Journal of Mathematics, 59(1), 2019. [arXiv:1406.4204](#).
- [DSPS20] Christopher Douglas, Christopher Schommer-Pries, and Noah Snyder. *Dualizable tensor categories*. Memoirs of the American Mathematical Society, 268(1308):0–0, 2020. [arXiv:1312.7188](#).
- [DW90] Robbert Dijkgraaf and Edward Witten. *Topological gauge theories and group cohomology*. Communications in Mathematical Physics, 129(2):393–429, 1990.
- [EGNO15] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor categories*. American Math. Soc, Providence, RI, 2015.
- [ENO04] Pavel Etingof, Dmitri Nikshych, and Viktor Ostrik. *An analogue of Radford’s S^4 formula for finite tensor categories*. International Mathematics Research Notices, 2004(54):2915, 2004. [arXiv:math/0404504](#).
- [ENO05] Pavel Etingof, Dmitri Nikshych, and Viktor Ostrik. *On fusion categories*. Annals of Mathematics, 162(2):581–642, 2005. [arXiv:math/0203060](#).
- [ENO10] Pavel Etingof, Dmitri Nikshych, and Viktor Ostrik. *Fusion categories and homotopy theory*. Quantum Topology, pages 209–273, 2010. [arXiv:0909.3140](#).
- [ENO11] Pavel Etingof, Dmitri Nikshych, and Victor Ostrik. *Weakly group-theoretical and solvable fusion categories*. Advances in Mathematics, 226(1):176–205, 2011. [arXiv:0809.3031](#).
- [EO04] P. Etingof and V. Ostrik. *Finite tensor categories*. Moscow Mathematical Journal, 4(3):627–654, 2004. [arXiv:math/0301027](#).
- [ERB14] I. S. Eliëns, J. C. Romers, and F. A. Bais. *Diagrammatics for Bose condensation in anyon theories*. Physical Review B, 90(19), 2014. [arXiv:1310.6001](#).
- [ET19] Dominic V. Else and Ryan Thorngren. *Crystalline topological phases as defect networks*. Physical Review B, 99(11), 2019. [arXiv:1810.10539](#).
- [FFFS02] Giovanni Felder, Jürg Fröhlich, Jürgen Fuchs, and Christoph Schweigert. *Correlation functions and boundary conditions in rational conformal field theory and three-dimensional topology*. Compositio Mathematica, 131(2):189–238, 2002. [arXiv:hep-th/9912239](#).
- [FFRS04] Jürg Fröhlich, Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. *Kramers-Wannier duality from conformal defects*. Physical Review Letters, 93(7):070601, 2004. [arXiv:cond-mat/0404051](#).
- [FFRS06] Jürg Fröhlich, Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. *Correspondences of ribbon categories*. Advances in Mathematics, 199(1):192–329, 2006. [arXiv:math/0309465](#).
- [FFRS07] Jürg Fröhlich, Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. *Duality and defects in rational conformal field theory*. Nuclear Physics B, 763(3):354–430, 2007. [arXiv:hep-th/0607247](#).

- [FFRS08] Jens Fjelstad, Jurgen Fuchs, Ingo Runkel, and Christoph Schweigert. *Uniqueness of open/closed rational CFT with given algebra of open states*. Advances in Theoretical and Mathematical Physics, 12(6):1283–1375, 2008. [arXiv:hep-th/0612306](#).
- [FFRS10] Jürg Fröhlich, Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. *Defect lines, dualities, and generalised orbifolds*. In *XVIIth International Congress on Mathematical Physics*, pages 608–613. WORLD SCIENTIFIC, 2010. [arXiv:0909.5013](#).
- [FG90] J. Fröhlich and F. Gabbiani. *Braid statistics in local quantum theory*. Reviews in Mathematical Physics, 02(03):251–353, 1990.
- [FGHT⁺12] M. H. Freedman, J. Gukelberger, M. B. Hastings, S. Trebst, M. Troyer, and Z. Wang. *Galois conjugates of topological phases*. Physical Review B, 85(4):045414, 2012. [arXiv:1106.3267](#).
- [FGM90] J. Fröhlich, F. Gabbiani, and P-A. Marchetti. *Braid Statistics in Three-Dimensional Local Quantum Theory*, In *Physics, Geometry and Topology*, pages 15–79. Springer US, 1990.
- [FH20] Daniel S. Freed and Michael J. Hopkins. *Invertible phases of matter with spatial symmetry*. Advances in Theoretical and Mathematical Physics, 24(7):1773–1788, 2020. [arXiv:1901.06419](#).
- [FH21] Daniel S. Freed and Michael J. Hopkins. *Reflection positivity and invertible topological phases*. Geometry & Topology, 25(3):1165–1330, 2021. [arXiv:1604.06527](#).
- [FHH22] Lukasz Fidkowski, Jeongwan Haah, and Matthew B. Hastings. *Gravitational anomaly of $(3+1)$ -dimensional \mathbb{Z}_2 toric code with fermionic charges and fermionic loop self-statistics*. Physical Review B, 106(16):165135, 2022. [arXiv:2110.14654](#).
- [FHK94] M. Fukuma, S. Hosono, and H. Kawai. *Lattice topological field theory in two dimensions*. Communications in Mathematical Physics, 161(1):157–175, 1994. [arXiv:hep-th/9212154](#).
- [FHL93] Igor B. Frenkel, Yi-Zhi Huang, and James Lepowsky. *On Axiomatic Approaches to Vertex Operator Algebras and Modules*, volume 104 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 1993.
- [FK11] Lukasz Fidkowski and Alexei Kitaev. *Topological phases of fermions in one dimension*. Physical Review B, 83(7), 2011. [arXiv:1008.4138](#).
- [FLW02] Michael H. Freedman, Michael Larsen, and Zhenghan Wang. *A modular functor which is universal for quantum computation*. Communications in Mathematical Physics, 227(3):605–622, 2002. [arXiv:quant-ph/0001108](#).
- [FMT24] Daniel S. Freed, Gregory W. Moore, and Constantin Teleman. *Topological symmetry in quantum field theory*. Quantum Topology, 15(3):779–869, 2024. [arXiv:2209.07471](#).
- [Fre94] Daniel S. Freed. *Higher algebraic structures and quantization*. Communications in Mathematical Physics, 159(2):343–398, 1994. [arXiv:hep-th/9212115](#).
- [Fre99] Daniel S. Freed. *Quantum Groups from Path Integrals*, In *Particles and Fields*, pages 63–107. Springer New York, 1999. [arXiv:q-alg/9501025](#).
- [Fre01] Michael H. Freedman. *Quantum computation and the localization of modular functors*. Foundations of Computational Mathematics, 1(2):183–204, 2001. [arXiv:quant-ph/0003128](#).
- [Fre09] Daniel S. Freed. *Remarks on Chern-Simons theory*. Bulletin of the American Mathematical Society, 46(2):221–254, 2009. [arXiv:0808.2507](#).
- [Fre14] Daniel S. Freed. *Short-range entanglement and invertible field theories*. arXiv preprint, 2014. [arXiv:1406.7278](#).

- [Fre17] Benoit Fresse. *Homotopy of Operads and Grothendieck–Teichmüller Groups: Parts 1 and 2*, volume 217 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2017.
- [Fro12] Georg Frobenius. *Ueber matrizen aus nicht negativen elementen*. Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften, pages 456–477, 1912.
- [Frö76] Jürg Fröhlich. *New super-selection sectors (“soliton-states”) in two dimensional Bose quantum field models*. Communications in Mathematical Physics, 47(3):269–310, 1976.
- [Fro88] Juerg Froehlich. *Statistics and Monodromy in Two- and Three-Dimensional Quantum Field Theory*, In *Differential Geometrical Methods in Theoretical Physics*, pages 173–186. Springer Netherlands, 1988.
- [FRS89] K. Fredenhagen, K. H. Rehren, and B. Schroer. *Superselection sectors with braid group statistics and exchange algebras*. Communications In Mathematical Physics, 125(2):201–226, 1989.
- [FRS02] Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. *TFT construction of RCFT correlators I: partition functions*. Nuclear Physics B, 646(3):353–497, 2002. [arXiv:hep-th/0204148](#).
- [FRS04] Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. *TFT construction of RCFT correlators II: unoriented world sheets*. Nuclear Physics B, 678(3):511–637, 2004. [arXiv:hep-th/0306164](#).
- [FRS05] Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. *TFT construction of RCFT correlators IV: Structure constants and correlation functions*. Nuclear Physics B, 715(3):539–638, 2005. [arXiv:hep-th/0412290](#).
- [FS03] Jürgen Fuchs and Christoph Schweigert. *Category theory for conformal boundary conditions*, In *Vertex Operator Algebras in Mathematics and Physics*, pages 25–71, volume 39 of *Fields Institute Communications*. American Mathematical Society, 2003. [arXiv:math/0106050](#).
- [FS19] Jürgen Fuchs and Christoph Schweigert. *Full logarithmic conformal field theory — an attempt at a status report*. Fortschritte der Physik, 67(8–9), 2019. [arXiv:1903.02838](#).
- [FSS19] Jürgen Fuchs, Gregor Schaumann, and Christoph Schweigert. *Eilenberg-Watts calculus for finite categories and a bimodule Radford S^4 theorem*. Transactions of the American Mathematical Society, 373(1):1–40, 2019. [arXiv:1612.04561](#).
- [FSV13] Jürgen Fuchs, Christoph Schweigert, and Alessandro Valentino. *Bicategories for boundary conditions and for surface defects in 3-d TFT*. Communications in Mathematical Physics, 321(2):543–575, 2013. [arXiv:1203.4568](#).
- [FT14] Daniel S. Freed and Constantin Teleman. *Relative quantum field theory*. Communications in Mathematical Physics, 326(2):459–476, 2014. [arXiv:1212.1692](#).
- [FT21] Daniel S. Freed and Constantin Teleman. *Gapped boundary theories in three dimensions*. Communications in Mathematical Physics, 388(2):845–892, 2021. [arXiv:2006.10200](#).
- [FVM18] Lukasz Fidkowski, Ashvin Vishwanath, and Max A. Metlitski. *Surface topological order and a new ’t Hooft anomaly of interaction enabled 3+1D fermion SPTs*. arXiv preprint, 2018. [arXiv:1804.08628](#).
- [Gal14] César Galindo. *On braided and ribbon unitary fusion categories*. Canadian Mathematical Bulletin, 57(3):506–510, 2014. [arXiv:1209.2022](#).
- [GHKP⁺24] David Green, Peter Huston, Kyle Kawagoe, David Penneys, Anup Poudel, and Sean Sanford. *Enriched string-net models and their excitations*. Quantum, 8:1301, 2024. [arXiv:2305.14068](#).

- [GJ87] James Glimm and Arthur Jaffe. *Quantum Physics*. Springer New York, 1987.
- [GJF19a] Davide Gaiotto and Theo Johnson-Freyd. *Condensations in higher categories*. arXiv preprint, 2019. [arXiv:1905.09566](#).
- [GJF19b] Davide Gaiotto and Theo Johnson-Freyd. *Symmetry protected topological phases and generalized cohomology*. Journal of High Energy Physics, 2019(5), 2019. [arXiv:1712.07950](#).
- [GK16] Davide Gaiotto and Anton Kapustin. *Spin TQFTs and fermionic phases of matter*. International Journal of Modern Physics A, 31(28n29):1645044, 2016. [arXiv:1505.05856](#).
- [GK21] Davide Gaiotto and Justin Kulp. *Orbifold groupoids*. Journal of High Energy Physics, 2021(2), 2021. [arXiv:2008.05960](#).
- [GKSW15] Davide Gaiotto, Anton Kapustin, Nathan Seiberg, and Brian Willett. *Generalized global symmetries*. Journal of High Energy Physics, 2015(2), 2015. [arXiv:1412.5148](#).
- [GMS80] G. A. Goldin, R. Menikoff, and D. H. Sharp. *Particle statistics from induced representations of a local current group*. Journal of Mathematical Physics, 21(4):650–664, 1980.
- [GMS81] G. A. Goldin, R. Menikoff, and D. H. Sharp. *Representations of a local current algebra in nonsimply connected space and the Aharonov–Bohm effect*. Journal of Mathematical Physics, 22(8):1664–1668, 1981.
- [Gol23] Gerald A. Goldin. *The prediction of anyons: Its history and wider implications*. SciPost Physics Proceedings, (14), 2023. [arXiv:2212.12632](#).
- [GOPWW20] Meng Guo, Kantaro Ohmori, Pavel Putrov, Zheyang Wan, and Juven Wang. *Fermionic finite-group gauge theories and interacting symmetric/crystalline orders via cobordisms*. Communications in Mathematical Physics, 376(2):1073–1154, 2020. [arXiv:1812.11959](#).
- [Gre10] Justin Greenough. *Monoidal 2-structure of bimodule categories*. Journal of Algebra, 324(8):1818–1859, 2010. [arXiv:0911.4979](#).
- [GS12] Pinhas Grossman and Noah Snyder. *Quantum subgroups of the Haagerup fusion categories*. Communications in Mathematical Physics, 311(3):617–643, 2012. [arXiv:1102.2631](#).
- [GS15] Pinhas Grossman and Noah Snyder. *The Brauer-Picard group of the Asaeda-Haagerup fusion categories*. Transactions of the American Mathematical Society, 368(4):2289–2331, 2015. [arXiv:1202.4396](#).
- [Gu17] Zheng-Cheng Gu. *The emergence of 3+1D Einstein gravity from topological gravity*. arXiv preprint, 2017. [arXiv:1709.09806](#).
- [GVR17] César Galindo and César F. Venegas-Ramírez. *Categorical fermionic actions and minimal modular extensions*. arXiv preprint, 2017. [arXiv:1712.07097](#).
- [GW09] Zheng-Cheng Gu and Xiao-Gang Wen. *Tensor-entanglement-filtering renormalization approach and symmetry-protected topological order*. Physical Review B, 80(15), 2009. [arXiv:0903.1069](#).
- [GW12] Zheng-Cheng Gu and Xiao-Gang Wen. *Emergence of helicity ± 2 modes (gravitons) from qubit models*. Nuclear Physics B, 863(1):90–129, 2012. [arXiv:0907.1203](#).
- [GW14] Zheng-Cheng Gu and Xiao-Gang Wen. *Symmetry-protected topological orders for interacting fermions: Fermionic topological nonlinear σ -models and a special group supercohomology theory*. Physical Review B, 90(11), 2014. [arXiv:1201.2648](#).

- [GWW15] Zheng-Cheng Gu, Zhenghan Wang, and Xiao-Gang Wen. *Classification of two-dimensional fermionic and bosonic topological orders*. Physical Review B, 91(12):125149, 2015. [arXiv:1010.1517](#).
- [Haa96] Rudolf Haag. *Local Quantum Physics*. Springer Berlin Heidelberg, 1996.
- [Haa11] Jeongwan Haah. *Local stabilizer codes in three dimensions without string logical operators*. Physical Review A, 83(4), 2011. [arXiv:1101.1962](#).
- [Hal84] B. I. Halperin. *Statistics of quasiparticles and the hierarchy of fractional quantized hall states*. Physical Review Letters, 52(18):1583–1586, 1984.
- [Hau17] Rune Haugseng. *The higher Morita category of \mathbb{E}_n -algebras*. Geometry & Topology, 21(3):1631–1730, 2017. [arXiv:1412.8459](#).
- [HBFL16] Chris Heinrich, Fiona Burnell, Lukasz Fidkowski, and Michael Levin. *Symmetry-enriched string nets: Exactly solvable models for SET phases*. Physical Review B, 94(23):235136, 2016. [arXiv:1606.07816](#).
- [HBJP23] Peter Huston, Fiona Burnell, Corey Jones, and David Penneys. *Composing topological domain walls and anyon mobility*. SciPost Physics, 15(3), 2023. [arXiv:2208.14018](#).
- [HGW18] Yuting Hu, Nathan Geer, and Yong-Shi Wu. *Full dyon excitation spectrum in extended Levin-Wen models*. Physical Review B, 97(19):195154, 2018. [arXiv:1502.03433](#).
- [HK64] Rudolf Haag and Daniel Kastler. *An algebraic approach to quantum field theory*. Journal of Mathematical Physics, 5(7):848–861, 1964.
- [HK04] Yi-Zhi Huang and Liang Kong. *Open-string vertex algebras, tensor categories and operads*. Communications in Mathematical Physics, 250(3):433–471, 2004. [arXiv:math.QA/0308248](#).
- [HK07] Yi-Zhi Huang and Liang Kong. *Full field algebras*. Communications in Mathematical Physics, 272(2):345–396, 2007. [arXiv:math/0511328](#).
- [HL94] Yi-Zhi Huang and James Lepowsky. *Tensor products of modules for a vertex operator algebra and vertex tensor categories*. In *Lie Theory and Geometry*, pages 349–383. Birkhäuser Boston, 1994. [arXiv:hep-th/9401119](#).
- [HLPWW18] Yuting Hu, Zhu-Xi Luo, Ren Pankovich, Yidun Wan, and Yong-Shi Wu. *Boundary Hamiltonian theory for gapped topological phases on an open surface*. Journal of High Energy Physics, 2018(1), 2018. [arXiv:1706.03329](#).
- [HPT16] André Henriques, David Penneys, and James Tener. *Categorified trace for module tensor categories over braided tensor categories*. Documenta Mathematica, 21:1089–1149, 2016. [arXiv:1509.02937](#).
- [HSW12] Yuting Hu, Spencer D. Stirling, and Yong-Shi Wu. *Ground-state degeneracy in the Levin-Wen model for topological phases*. Physical Review B, 85(7):075107, 2012. [arXiv:1105.5771](#).
- [Hua95] Yi-Zhi Huang. *A theory of tensor products for modul  categories for a vertex operator algebra, IV*. Journal of Pure and Applied Algebra, 100(1-3):173–216, 1995. [arXiv:q-alg/9505019](#).
- [Hua08a] Yi-Zhi Huang. *Rigidity and modularity of vertex tensor categories*. Communications in Contemporary Mathematics, 10(supp01):871–911, 2008. [arXiv:math/0502533](#).
- [Hua08b] Yi-Zhi Huang. *Vertex operator algebras and the Verlinde conjecture*. Communications in Contemporary Mathematics, 10(01):103–154, 2008. [arXiv:math/0406291](#).
- [Hua17] Yi-Zhi Huang. *On the applicability of logarithmic tensor category theory*. arXiv preprint, 2017. [arXiv:1702.00133](#).

- [HW15a] Ling-Yan Hung and Yidun Wan. *Generalized ADE classification of topological boundaries and anyon condensation*. Journal of High Energy Physics, 2015(7), 2015. [arXiv:1502.02026](#).
- [HW15b] Ling-Yan Hung and Yidun Wan. *Ground-state degeneracy of topological phases on open surfaces*. Physical Review Letters, 114(7), 2015. [arXiv:1408.0014](#).
- [HW20] Alexander Hahn and Ramona Wolf. *Generalized string-net model for unitary fusion categories without tetrahedral symmetry*. Physical Review B, 102(11):115154, 2020. [arXiv:2004.07045](#).
- [HWW13] Yuting Hu, Yidun Wan, and Yong-Shi Wu. *Twisted quantum double model of topological phases in two dimensions*. Physical Review B, 87(12), 2013. [arXiv:1211.3695](#).
- [HWW17] Yuting Hu, Yidun Wan, and Yong-Shi Wu. *Boundary Hamiltonian theory for gapped topological orders*. Chinese Physics Letters, 34(7):077103, 2017. [arXiv:1706.00650](#).
- [HZW05] Alioscia Hamma, Paolo Zanardi, and Xiao-Gang Wen. *String and membrane condensation on three-dimensional lattices*. Physical Review B, 72(3), 2005. [arXiv:cond-mat/0411752](#).
- [HZZK⁺22] Yong-Ju Hai, Ze Zhang, Hao Zheng, Liang Kong, Jiansheng Wu, and Dapeng Yu. *Uniquely identifying topological order based on boundary-bulk duality and anyon condensation*. National Science Review, 10(3), 2022. [arXiv:2005.03236](#).
- [JF20] Theo Johnson-Freyd. *(3+1)D topological orders with only a \mathbb{Z}_2 -charged particle*. arXiv preprint, 2020. [arXiv:2011.11165](#).
- [JF22] Theo Johnson-Freyd. *On the classification of topological orders*. Communications in Mathematical Physics, 2022. [arXiv:2003.06663](#).
- [JFR23] Theo Johnson-Freyd and David Reutter. *Minimal nondegenerate extensions*. Journal of the American Mathematical Society, 2023. [arXiv:2105.15167](#).
- [JFY21a] Theo Johnson-Freyd and Matthew Yu. *Fusion 2-categories with no line operators are grouplike*. Bulletin of the Australian Mathematical Society, 104(3):434–442, 2021. [arXiv:2010.07950](#).
- [JFY21b] Theo Johnson-Freyd and Matthew Yu. *Topological orders in (4+1)-dimensions*. arXiv preprint, 2021. [arXiv:2104.04534](#).
- [JL09] David Jordan and Eric Larson. *On the classification of certain fusion categories*. Journal of Noncommutative Geometry, pages 481–499, 2009. [arXiv:0812.1603](#).
- [JMPP21] Corey Jones, Scott Morrison, David Penneys, and Julia Plavnik. *Extension theory for braided-enriched fusion categories*. International Mathematics Research Notices, 2022(20):15632–15683, 2021. [arXiv:1910.03178](#).
- [JMR14] Shenghan Jiang, Andrej Mesaros, and Ying Ran. *Generalized modular transformations in (3+1)D topologically ordered phases and triple linking invariant of loop braiding*. Physical Review X, 4(3), 2014. [arXiv:1404.1062](#).
- [JR17] Shenghan Jiang and Ying Ran. *Anyon condensation and a generic tensor-network construction for symmetry-protected topological phases*. Physical Review B, 95(12):125107, 2017. [arXiv:1611.07652](#).
- [JS91] André Joyal and Ross Street. *Tortile Yang-Baxter operators in tensor categories*. Journal of Pure and Applied Algebra, 71(1):43–51, 1991.
- [JS93] A. Joyal and R. Street. *Braided tensor categories*. Advances in Mathematics, 102(1):20–78, 1993.

- [JW20] Wenjie Ji and Xiao-Gang Wen. *Categorical symmetry and noninvertible anomaly in symmetry-breaking and topological phase transitions*. Physical Review Research, 2(3):033417, 2020. [arXiv:1912.13492](#).
- [Kap14a] Anton Kapustin. *Bosonic topological insulators and paramagnets: a view from cobordisms*. arXiv preprint, 2014. [arXiv:1404.6659](#).
- [Kap14b] Anton Kapustin. *Ground-state degeneracy for Abelian anyons in the presence of gapped boundaries*. Physical Review B, 89(12), 2014. [arXiv:1306.4254](#).
- [Kap14c] Anton Kapustin. *Symmetry protected topological phases, anomalies, and cobordisms: Beyond group cohomology*. arXiv preprint, 2014. [arXiv:1403.1467](#).
- [Kas95] Christian Kassel. *Quantum groups*. Springer, New York Berlin, 1995.
- [KF97] C. L. Kane and Matthew P. A. Fisher. *Quantized thermal transport in the fractional quantum hall effect*. Physical Review B, 55(23):15832–15837, 1997. [arXiv:cond-mat/9603118](#).
- [Kir01] Alexander Kirillov, Jr. *Modular categories and orbifold models II*. arXiv preprint, 2001. [arXiv:math/0110221](#).
- [Kir11] Alexander Kirillov, Jr. *String-net model of Turaev-Viro invariants*. arXiv preprint, 2011. [arXiv:1106.6033](#).
- [Kit03] Alexei Kitaev. *Fault-tolerant quantum computation by anyons*. Annals of Physics, 303(1):2–30, 2003. [arXiv:quant-ph/9707021](#).
- [Kit06] Alexei Kitaev. *Anyons in an exactly solved model and beyond*. Annals of Physics, 321(1):2–111, 2006. [arXiv:cond-mat/0506438](#).
- [Kit09] Alexei Kitaev. *Periodic table for topological insulators and superconductors*. In *AIP Conference Proceedings*. AIP, 2009. [arXiv:0901.2686](#).
- [Kit11] Kitaev. *Toward topological classification of phases with short-range entanglement*. Talk, 2011. <https://online.kitp.ucsb.edu/online/topomat11/kitaev>.
- [Kit13] Kitaev. *On the classification of short-range entangled state*. Talk given in “Topological Phases of Matter” program, 2013. http://scgp.stonybrook.edu/video_portal/video.php?id=2010.
- [Kit15] Kitaev. *Homotopy-theoretic approach to SPT phases in action: \mathbb{Z}_{16} classification of three-dimensional superconductors*. Talk given in “Symmetry and Topology in Quantum Matter” workshop, 2015. <http://www.ipam.ucla.edu/abstract/?tid=12389&pcode=STQ2015>.
- [KK12] Alexei Kitaev and Liang Kong. *Models for gapped boundaries and domain walls*. Communications in Mathematical Physics, 313(2):351–373, 2012. [arXiv:1104.5047](#).
- [KKR10] Robert Koenig, Greg Kuperberg, and Ben W. Reichardt. *Quantum computation with Turaev-Viro codes*. Annals of Physics, 325(12):2707–2749, 2010. [arXiv:1002.2816](#).
- [KL87] V. Kalmeyer and R. B. Laughlin. *Equivalence of the resonating-valence-bond and fractional quantum Hall states*. Physical Review Letters, 59(18):2095–2098, 1987.
- [KL20] Kyle Kawagoe and Michael Levin. *Microscopic definitions of anyon data*. Physical Review B, 101(11), 2020. [arXiv:1910.11353](#).
- [KLWZZ20a] Liang Kong, Tian Lan, Xiao-Gang Wen, Zhi-Hao Zhang, and Hao Zheng. *Algebraic higher symmetry and categorical symmetry: A holographic and entanglement view of symmetry*. Physical Review Research, 2(4), 2020. [arXiv:2005.14178](#).

- [KLWZZ20b] Liang Kong, Tian Lan, Xiao-Gang Wen, Zhi-Hao Zhang, and Hao Zheng. *Classification of topological phases with finite internal symmetries in all dimensions*. Journal of High Energy Physics, 2020(9), 2020. [arXiv:2003.08898](#).
- [KMR10] Zoltán Kádár, Annalisa Marzuoli, and Mario Rasetti. *Microscopic description of 2d topological phases, duality, and 3d state sums*. Advances in Mathematical Physics, 2010:1–18, 2010. [arXiv:0907.3724](#).
- [KO02] Alexander Kirillov, Jr. and Viktor Ostrik. *On a q -analogue of the McKay correspondence and the ADE classification of $\hat{\mathfrak{sl}}_2$ conformal field theories*. Advances in Mathematics, 171(2):183–227, 2002. [arXiv:math/0101219](#).
- [Kon94] Maxim Kontsevich. *Homological algebra of mirror symmetry*. arXiv preprint, 1994. [arXiv:alg-geom/9411018](#).
- [Kon07] Liang Kong. *Full field algebras, operads and tensor categories*. Advances in Mathematics, 213(1):271–340, 2007. [arXiv:math/0603065](#).
- [Kon08a] Liang Kong. *Cardy condition for open-closed field algebras*. Communications in Mathematical Physics, 283(1):25–92, 2008. [arXiv:math/0612255](#).
- [Kon08b] Liang Kong. *Open-closed field algebras*. Communications in Mathematical Physics, 280(1):207–261, 2008. [arXiv:math/0610293](#).
- [Kon11] Liang Kong. *Conformal field theory and a new geometry*. In *Mathematical Foundations of Quantum Field Theory and Perturbative String Theory*, pages 199–244. American Mathematical Society, 2011. [arXiv:1107.3649](#).
- [Kon13] Liang Kong. *Some universal properties of Levin-Wen models*. In *XVIIth International Congress on Mathematical Physics*, pages 444–455. World Scientific, 2013. [arXiv:1211.4644](#).
- [Kon14a] Liang Kong. *Anyon condensation and tensor categories*. Nuclear Physics B, 886:436–482, 2014. *Erratum and addendum: “Anyon condensation and tensor categories” [Nucl. Phys. B 886 (2014) 436–482]*. Nuclear Physics B, 973:115607, 2021. See a corrected version in, [arXiv:1307.8244](#).
- [Kon14b] Liang Kong. *Anyon condensation and tensor categories*. Nuclear Physics B, 886:436–482, 2014. [arXiv:1307.8244](#).
- [Kon17] Liang Kong. 浅议现代数学物理对数学的影响 (*Impact of modern mathematical physics on mathematics, in Chinese*). Published online in Chinese journal 《数理人文》, 2017. https://jupiter.math.nycu.edu.tw/~mshc/014_201807/014_06.pdf.
- [KPMT20] Justin Kaidi, Julio Parra-Martinez, and Yuji Tachikawa. *Classification of string theories via topological phases*. Physical Review Letters, 124(12):121601, 2020.
- [KR08] Liang Kong and Ingo Runkel. *Morita classes of algebras in modular tensor categories*. Advances in Mathematics, 219(5):1548–1576, 2008. [arXiv:0708.1897](#).
- [KR09] Liang Kong and Ingo Runkel. *Cardy algebras and sewing constraints, I*. Communications in Mathematical Physics, 292(3):871–912, 2009. [arXiv:0807.3356](#).
- [Kre49] Mark Grigorievich Krein. *A principle of duality for bicomact groups and quadratic block algebras*. Doklady Akademii Nauk SSSR, 69:725–728, 1949.
- [KS11] Anton Kapustin and Natalia Saulina. *Topological boundary conditions in abelian Chern-Simons theory*. Nuclear Physics B, 845(3):393–435, 2011. [arXiv:1008.0654](#).
- [KTTW15] Anton Kapustin, Ryan Thorngren, Alex Turzillo, and Zitao Wang. *Fermionic symmetry protected topological phases and cobordisms*. Journal of High Energy Physics, 2015(12):1–21, 2015. [arXiv:1406.7329](#).

- [KTZ20a] Liang Kong, Yin Tian, and Zhi-Hao Zhang. *Defects in the 3-dimensional toric code model form a braided fusion 2-category*. Journal of High Energy Physics, 2020(78), 2020. [arXiv:2009.06564](#).
- [KTZ20b] Liang Kong, Yin Tian, and Shan Zhou. *The center of monoidal 2-categories in 3+1D Dijkgraaf-Witten theory*. Advances in Mathematics, 360:106928, 2020. [arXiv:1905.04644](#).
- [KW14] Liang Kong and Xiao-Gang Wen. *Braided fusion categories, gravitational anomalies, and the mathematical framework for topological orders in any dimensions*. arXiv preprint, 2014. [arXiv:1405.5858](#).
- [KWZ15] Liang Kong, Xiao-Gang Wen, and Hao Zheng. *Boundary-bulk relation for topological orders as the functor mapping higher categories to their centers*. arXiv preprint, 2015. [arXiv:1502.01690](#).
- [KWZ17] Liang Kong, Xiao-Gang Wen, and Hao Zheng. *Boundary-bulk relation in topological orders*. Nuclear Physics B, 922:62–76, 2017. [arXiv:1702.00673](#).
- [KWZ22] Liang Kong, Xiao-Gang Wen, and Hao Zheng. *One dimensional gapped quantum phases and enriched fusion categories*. Journal of High Energy Physics, 2022(3), 2022. [arXiv:2108.08835](#).
- [KYZ21] Liang Kong, Wei Yuan, and Hao Zheng. *Pointed Drinfeld center functor*. Communications in Mathematical Physics, 381(3):1409–1443, 2021. [arXiv:1912.13168](#).
- [KYZZ24] Liang Kong, Wei Yuan, Zhi-Hao Zhang, and Hao Zheng. *Enriched monoidal categories I: Centers*. Quantum Topology, 16(2):343–417, 2024. [arXiv:2104.03121](#).
- [KZ18a] Liang Kong and Hao Zheng. *The center functor is fully faithful*. Advances in Mathematics, 339:749–779, 2018. [arXiv:1507.00503](#).
- [KZ18b] Liang Kong and Hao Zheng. *Drinfeld center of enriched monoidal categories*. Advances in Mathematics, 323:411–426, 2018. [arXiv:1704.01447](#).
- [KZ18c] Liang Kong and Hao Zheng. *Gapless edges of 2d topological orders and enriched monoidal categories*. Nuclear Physics B, 927:140–165, 2018. [arXiv:1705.01087](#).
- [KZ20] Liang Kong and Hao Zheng. *A mathematical theory of gapless edges of 2d topological orders. Part I*. Journal of High Energy Physics, 2020(2), 2020. [arXiv:1905.04924](#).
- [KZ21] Liang Kong and Hao Zheng. *A mathematical theory of gapless edges of 2d topological orders. Part II*. Nuclear Physics B, 966:115384, 2021. [arXiv:1912.01760](#).
- [KZ22a] Liang Kong and Hao Zheng. *Categories of quantum liquids I*. Journal of High Energy Physics, 2022(8), 2022. [arXiv:2011.02859](#).
- [KZ22b] Liang Kong and Hao Zheng. *Categories of quantum liquids III*. arXiv preprint, 2022. [arXiv:2201.05726](#).
- [KZ24] Liang Kong and Hao Zheng. *Categories of quantum liquids II*. Communications in Mathematical Physics, 405(9), 2024. [arXiv:2107.03858](#).
- [KZZZ24] Liang Kong, Zhi-Hao Zhang, Jiaheng Zhao, and Hao Zheng. *Higher condensation theory*. arXiv preprint, 2024. [arXiv:2403.07813](#).
- [Lak16] Ethan Lake. *Anomalies and symmetry fractionalization in reflection-symmetric topological order*. Physical Review B, 94(20), 2016. [arXiv:1608.02736](#).
- [Lau83] R. B. Laughlin. *Anomalous quantum Hall effect: An incompressible quantum fluid with fractionally charged excitations*. Physical Review Letters, 50(18):1395–1398, 1983.

- [Law93] R. J. Lawrence. *Triangulations, Categories and Extended Topological Field Theories*, In *Quantum Topology*, pages 191–208. World Scientific, 1993.
- [Lev13] Michael Levin. *Protected edge modes without symmetry*. *Physical Review X*, 3(2), 2013. [arXiv:1301.7355](#).
- [LG12] Michael Levin and Zheng-Cheng Gu. *Braiding statistics approach to symmetry-protected topological phases*. *Physical Review B*, 86(11), 2012. [arXiv:1202.3120](#).
- [LH08] Hui Li and F. D. M. Haldane. *Entanglement spectrum as a generalization of entanglement entropy: Identification of topological order in non-abelian fractional quantum Hall effect states*. *Physical Review Letters*, 101(1), 2008. [arXiv:0805.0332](#).
- [LKW16a] Tian Lan, Liang Kong, and Xiao-Gang Wen. *Modular extensions of unitary braided fusion categories and 2+1D topological/SPT orders with symmetries*. *Communications in Mathematical Physics*, 351(2):709–739, 2016. [arXiv:1602.05936](#).
- [LKW16b] Tian Lan, Liang Kong, and Xiao-Gang Wen. *Theory of (2+1)-dimensional fermionic topological orders and fermionic/bosonic topological orders with symmetries*. *Physical Review B*, 94(15), 2016. [arXiv:1507.04673](#).
- [LKW17] Tian Lan, Liang Kong, and Xiao-Gang Wen. *Classification of (2+1)-dimensional topological order and symmetry-protected topological order for bosonic and fermionic systems with on-site symmetries*. *Physical Review B*, 95(23), 2017. [arXiv:1602.05946](#).
- [LKW18] Tian Lan, Liang Kong, and Xiao-Gang Wen. *Classification of 3+1D bosonic topological orders: The case when pointlike excitations are all bosons*. *Physical Review X*, 8(2), 2018. [arXiv:1704.04221](#).
- [LLB21] Chien-Hung Lin, Michael Levin, and Fiona J. Burnell. *Generalized string-net models: A thorough exposition*. *Physical Review B*, 103(19):195155, 2021. [arXiv:2012.14424](#).
- [LM77] J. M. Leinaas and J. Myrheim. *On the theory of identical particles*. *Il Nuovo Cimento B Series 11*, 37(1):1–23, 1977.
- [LR97] Roberto Longo and John E. Roberts. *A theory of dimension*. *K-Theory*, 11(2):103–159, 1997. [arXiv:funct-an/9604008](#).
- [Lur08] Jacob Lurie. *On the classification of topological field theories*. *Current Developments in Mathematics*, 2008(1):129–280, 2008. [arXiv:0905.0465](#).
- [Lur09] Jacob Lurie. *Higher Topos Theory*. Princeton University Press, 2009. [arXiv:math/0608040](#).
- [Lur17] Jacob Lurie. *Higher Algebra*. Book online, 2017. <http://www.math.harvard.edu/~lurie/papers/HA.pdf>.
- [LVHV20] Laurens Lootens, Robijn Vanhove, Jutho Haegeman, and Frank Verstraete. *Galois conjugated tensor fusion categories and nonunitary conformal field theory*. *Physical Review Letters*, 124(12):120601, 2020. [arXiv:1902.11241](#).
- [LW05] Michael A. Levin and Xiao-Gang Wen. *String-net condensation: A physical mechanism for topological phases*. *Physical Review B*, 71(4), 2005. [arXiv:cond-mat/0404617](#).
- [LW14] Tian Lan and Xiao-Gang Wen. *Topological quasiparticles and the holographic bulk-edge relation in (2+1)-dimensional string-net models*. *Physical Review B*, 90(11), 2014. [arXiv:1311.1784](#).
- [LW19] Tian Lan and Xiao-Gang Wen. *Classification of 3+1D bosonic topological orders (II): The case when some pointlike excitations are fermions*. *Physical Review X*, 9(2), 2019. [arXiv:1801.08530](#).

- [LWW15] Tian Lan, Juven C. Wang, and Xiao-Gang Wen. *Gapped domain walls, gapped boundaries, and topological degeneracy*. Physical Review Letters, 114(7), 2015. [arXiv:1408.6514](#).
- [LY23] Yalei Lu and Holiverse Yang. *The boundary phase transitions of the 2+1D \mathbb{Z}_N topological order via topological Wick rotation*. Journal of High Energy Physics, 2023(10), 2023. [arXiv:2208.01572](#).
- [LYW23] Tian Lan, Gen Yue, and Longye Wang. *Category of SET orders*. arXiv preprint, 2023. [arXiv:2312.15958](#).
- [Mac63] Saunders MacLane. *Natural associativity and commutativity*. Rice University Studies, 49(4):28–46, 1963.
- [Mac78] Saunders MacLane. *Categories for the Working Mathematician*. Springer New York, 1978.
- [Maj91] Shahn Majid. *Representations, duals and quantum doubles of monoidal categories*. In *Proceedings of the Winter School "Geometry and Physics"*, pages 197–206, 1991.
- [Mal99] Juan Maldacena. *The large N limit of superconformal field theories and supergravity*. International Journal of Theoretical Physics, 38(4):1113–1133, 1999. [arXiv:hep-th/9711200](#).
- [Mas98] Heinrich Maschke. *Ueber den arithmetischen charakter der coefficienten der substitutionen endlicher linearer substitutionsgruppen*. Mathematische Annalen, 50(4):492–498, 1898.
- [Mas99] Heinrich Maschke. *Beweis des satzes, dass diejenigen endlichen linearen substitutionsgruppen, in welchen einige durchgehends verschwindende coefficienten auftreten, intransitiv sind*. Mathematische Annalen, 52(2-3):363–368, 1899.
- [McG23] John McGreevy. *Generalized symmetries in condensed matter*. Annual Review of Condensed Matter Physics, 14(1):57–82, 2023. [arXiv:2204.03045](#).
- [MH73] John Milnor and Dale Husemoller. *Symmetric Bilinear Forms*. Springer Berlin Heidelberg, 1973.
- [Mil74] R. James Milgram. *Surgery with coefficients*. The Annals of Mathematics, 100(2):194, 1974.
- [Moo24] Gregory Moore. *From RCFT to MTC*. Online talk given in “Perspectives in Theoretical Physics Colloquium”, 2024. <https://www.youtube.com/watch?v=BfYYVJ9pNug>.
- [MP17] Scott Morrison and David Penneys. *Monoidal categories enriched in braided monoidal categories*. International Mathematics Research Notices, 2019(11):3527–3579, 2017. [arXiv:1701.00567](#).
- [MPP18] Scott Morrison, David Penneys, and Julia Plavnik. *Completion for braided enriched monoidal categories*. arXiv preprint, 2018. [arXiv:1809.09782](#).
- [MR91] Gregory Moore and Nicholas Read. *Nonabelions in the fractional quantum hall effect*. Nuclear Physics B, 360(2–3):362–396, 1991.
- [MR23] Vincentas Mulevičius and Ingo Runkel. *Constructing modular categories from orbifold data*. Quantum Topology, 13(3):459–523, 2023. [arXiv:2002.00663](#).
- [MS88] Gregory Moore and Nathan Seiberg. *Polynomial equations for rational conformal field theories*. Physics Letters B, 212(4):451–460, 1988.
- [MS89a] Gregory Moore and Nathan Seiberg. *Classical and quantum conformal field theory*. Communications in Mathematical Physics, 123(2):177–254, 1989.

- [MS89b] Gregory Moore and Nathan Seiberg. *Naturality in conformal field theory*. Nuclear Physics B, 313(1):16–40, 1989.
- [MS89c] Gregory Moore and Nathan Seiberg. *Taming the conformal zoo*. Physics Letters B, 220(3):422–430, 1989.
- [MS90] Gregory Moore and Nathan Seiberg. *Lectures on RCFT*. In *NATO ASI Series*, pages 263–361. Springer US, 1990.
- [MS21] Michaël Mignard and Peter Schauenburg. *Modular categories are not determined by their modular data*. Letters in Mathematical Physics, 111(3), 2021. [arXiv:1708.02796](#).
- [Müg00] Michael Müger. *Galois theory for braided tensor categories and the modular closure*. Advances in Mathematics, 150(2):151–201, 2000. [arXiv:math/9812040](#).
- [Müg03a] Michael Müger. *From subfactors to categories and topology I: Frobenius algebras in and Morita equivalence of tensor categories*. Journal of Pure and Applied Algebra, 180(1-2):81–157, 2003. [arXiv:math/0111204](#).
- [Müg03b] Michael Müger. *From subfactors to categories and topology II: The quantum double of tensor categories and subfactors*. Journal of Pure and Applied Algebra, 180(1-2):159–219, 2003. [arXiv:math/0111205](#).
- [Müg03c] Michael Müger. *On the structure of modular categories*. Proceedings of the London Mathematical Society, 87(02):291–308, 2003. [arXiv:math/0201017](#).
- [Müg04] Michael Müger. *Galois extensions of braided tensor categories and braided crossed G-categories*. Journal of Algebra, 277(1):256–281, 2004. [arXiv:math/0209093](#).
- [Müg12] Michael Müger. *Modular categories*. arXiv preprint, 2012. [arXiv:1201.6593](#).
- [Mul22] Vincentas Mulevicius. *Condensation inversion and Witt equivalence via generalised orbifolds*. arXiv preprint, 2022. [arXiv:2206.02611](#).
- [MW12] Scott Morrison and Kevin Walker. *Blob homology*. Geometry & Topology, 16(3):1481–1607, 2012. [arXiv:1009.5025](#).
- [MW20] Bin-Bin Mao and Chenjie Wang. *Mirror anomaly in fermionic topological orders*. Physical Review Research, 2(2), 2020. [arXiv:2002.07714](#).
- [Naa11] Pieter Naaijken. *Localized endomorphisms in Kitaev’s toric code on the plane*. Reviews in Mathematical Physics, 23(04):347–373, 2011. [arXiv:1012.3857](#).
- [NC20] Gregor Schaumann Nils Carqueville, Ingo Runkel. *Orbifolds of Reshetikhin-Turaev TQFTs*. Theory and Applications of Categories, 35(15):513–561, 2020. [arXiv:1809.01483](#).
- [Nik22] Dmitri Nikshych. *Computing the group of minimal non-degenerate extensions of a super-Tannakian category*. Communications in Mathematical Physics, 396(2):685–711, 2022. [arXiv:2201.04076](#).
- [NMLW21] Shang-Qiang Ning, Bin-Bin Mao, Zhengqiao Li, and Chenjie Wang. *Anomaly indicators and bulk-boundary correspondences for three-dimensional interacting topological crystalline phases with mirror and continuous symmetries*. Physical Review B, 104(7), 2021. [arXiv:2105.02682](#).
- [NN08] Deepak Naidu and Dmitri Nikshych. *Lagrangian subcategories and braided tensor equivalences of twisted quantum doubles of finite groups*. Communications in Mathematical Physics, 279(3):845–872, 2008. [arXiv:0705.0665](#).
- [NRW24] Siu-Hung Ng, Eric C Rowell, and Xiao-Gang Wen. *Recovering R-symbols from modular data*. arXiv preprint, 2024. [arXiv:2408.02748](#).

- [NTW85] Qian Niu, D. J. Thouless, and Yong-Shi Wu. *Quantized Hall conductance as a topological invariant*. Physical Review B, 31(6):3372–3377, 1985.
- [Ost03a] Victor Ostrik. *Module categories, weak Hopf algebras and modular invariants*. Transformation Groups, 8(2):177–206, 2003. [arXiv:math/0111139](https://arxiv.org/abs/math/0111139).
- [Ost03b] Viktor Ostrik. *Module categories over the Drinfeld double of a finite group*. International Mathematics Research Notices, 2003(27):1507, 2003. [arXiv:math/0202130](https://arxiv.org/abs/math/0202130).
- [OY23] Victor Ostrik and Zhiqiang Yu. *On the minimal extension and structure of braided weakly group-theoretical fusion categories*. Advances in Mathematics, 419:108961, 2023. [arXiv:2105.01814](https://arxiv.org/abs/2105.01814).
- [Pen21] David Penneys. *Topological phases of matter*. Lecture notes online, 2021. <https://people.math.osu.edu/penneys.2/8800//Math8800Spring2021.html>.
- [Per07] Oskar Perron. *Zur theorie der matrices*. Mathematische Annalen, 64(2):248–263, 1907.
- [Qi18] Xiao-Liang Qi. *Does gravity come from quantum information?* Nature Physics, 14(10):984–987, 2018.
- [QJW19] Yang Qi, Chao-Ming Jian, and Chenjie Wang. *Folding approach to topological order enriched by mirror symmetry*. Physical Review B, 99(8), 2019. [arXiv:1710.09391](https://arxiv.org/abs/1710.09391).
- [QKL12] Xiao-Liang Qi, Hosho Katsura, and Andreas W. W. Ludwig. *General relationship between the entanglement spectrum and the edge state spectrum of topological quantum states*. Physical Review Letters, 108(19), 2012. [arXiv:1103.5437](https://arxiv.org/abs/1103.5437).
- [Qui99] Frank Quinn. *Group categories and their field theories*. In *Proceedings of the Kirbyfest*. Mathematical Sciences Publishers, 1999. [arXiv:math/9811047](https://arxiv.org/abs/math/9811047).
- [Rea90] N. Read. *Excitation structure of the hierarchy scheme in the fractional quantum hall effect*. Physical Review Letters, 65(12):1502–1505, 1990.
- [Reh88] Karl-Henning Rehren. *Locality of conformal fields in two dimensions: Exchange algebra on the light-cone*. Communications in Mathematical Physics, 116(4):675–688, 1988.
- [Reh90] Karl-Henning Rehren. *Braid group statistics and their superselection rules*. In *The Algebraic Theory of Superselection Sectors*, pages 333–355. World Scientific, 1990.
- [Rie17] Emily Riehl. *Category Theory in Context*. Guilford Publications, 2017.
- [RM92] N. Read and G. Moore. *Fractional quantum Hall effect and nonabelian statistics*. Progress of Theoretical Physics Supplement, 107:157–166, 1992. [arXiv:hep-th/9202001](https://arxiv.org/abs/hep-th/9202001).
- [RSW09] Eric Rowell, Richard Stong, and Zhenghan Wang. *On classification of modular tensor categories*. Communications in Mathematical Physics, 292(2):343–389, 2009. [arXiv:0712.1377](https://arxiv.org/abs/0712.1377).
- [RT91] N. Reshetikhin and V. G. Turaev. *Invariants of 3-manifolds via link polynomials and quantum groups*. Inventiones Mathematicae, 103(1):547–597, 1991.
- [RT06] Shinsei Ryu and Tadashi Takayanagi. *Holographic derivation of entanglement entropy from the anti-de Sitter space/conformal field theory correspondence*. Physical Review Letters, 96(18), 2006. [arXiv:hep-th/0603001](https://arxiv.org/abs/hep-th/0603001).
- [Run20] Ingo Runkel. *String-net models for nonspherical pivotal fusion categories*. Journal of Knot Theory and Its Ramifications, 29(06):2050035, 2020. [arXiv:1907.12532](https://arxiv.org/abs/1907.12532).
- [Seg88] G. Segal. *The definition of conformal field theory*. In *Differential Geometrical Methods in Theoretical Physics*, pages 165–171. Springer Netherlands, 1988.
- [Ser73] Jean-Pierre Serre. *A Course in Arithmetic*. Springer New York, 1973.

- [SFQ20] Zhida Song, Chen Fang, and Yang Qi. *Real-space recipes for general topological crystalline states*. Nature Communications, 11(1), 2020. [arXiv:1810.11013](#).
- [SH19] Ce Shen and Ling-Yan Hung. *Defect Verlinde formula for edge excitations in topological order*. Physical Review Letters, 123(5):051602, 2019. [arXiv:1901.08285](#).
- [SHFH17] Hao Song, Sheng-Jie Huang, Liang Fu, and Michael Hermele. *Topological phases protected by point group symmetry*. Physical Review X, 7(1), 2017. [arXiv:1604.08151](#).
- [Shi16] Kenichi Shimizu. *On unimodular finite tensor categories*. International Mathematics Research Notices, 2017(1):277–322, 2016. [arXiv:1402.3482](#).
- [SHQFH19] Zhida Song, Sheng-Jie Huang, Yang Qi, Chen Fang, and Michael Hermele. *Topological states from topological crystals*. Science Advances, 5(12), 2019. [arXiv:1810.02330](#).
- [SM16] Brian Swingle and John McGreevy. *Renormalization group constructions of topological quantum liquids and beyond*. Physical Review B, 93(4), 2016. [arXiv:1407.8203](#).
- [SP11] Christopher J. Schommer-Pries. *The classification of two-dimensional extended topological field theories*. arXiv preprint, 2011. [arXiv:1112.1000](#).
- [SPGC11] Norbert Schuch, David Pérez-García, and Ignacio Cirac. *Classifying quantum phases using matrix product states and projected entangled pair states*. Physical Review B, 84(16), 2011. [arXiv:1010.3732](#).
- [SSWW16] Nathan Seiberg, T. Senthil, Chong Wang, and Edward Witten. *A duality web in 2+1 dimensions and condensed matter physics*. Annals of Physics, 374:395–433, 2016. [arXiv:1606.01989](#).
- [SSWX19] T. Senthil, Dam Thanh Son, Chong Wang, and Cenke Xu. *Duality between (2+1)d quantum critical points*. Physics Reports, 827:1–48, 2019. [arXiv:1810.05174](#).
- [Sto92] Michael Stone. *Quantum Hall Effect*. WSPC, 1992.
- [SW03] Paolo Salvatore and Nathalie Wahl. *Framed discs operads and Batalin-Vilkovisky algebras*. The Quarterly Journal of Mathematics, 54(2):213–231, 2003.
- [Swi12] Brian Swingle. *Entanglement renormalization and holography*. Physical Review D, 86(6), 2012. [arXiv:0905.1317](#).
- [SXG18] Ken Shiozaki, Charles Zhaoxi Xiong, and Kiyonori Gomi. *Generalized homology and Atiyah-Hirzebruch spectral sequence in crystalline symmetry protected topological phenomena*. arXiv preprint, 2018. [arXiv:1810.00801](#).
- [SXH20] Hao Song, Charles Zhaoxi Xiong, and Sheng-Jie Huang. *Bosonic crystalline symmetry protected topological phases beyond the group cohomology proposal*. Physical Review B, 101(16):165129, 2020. [arXiv:1811.06558](#).
- [Tam01] D Tambara. *A duality for modules over monoidal categories of representations of semisimple Hopf algebras*. Journal of Algebra, 241(2):515–547, 2001.
- [Tan39] Tadao Tannaka. *Über den dualitätssatz der nichtkommutativen topologischen gruppen*. Tohoku Mathematical Journal, First Series, 45:1–12, 1939.
- [TE18] Ryan Thorngren and Dominic V. Else. *Gauging spatial symmetries and the classification of topological crystalline phases*. Physical Review X, 8(1), 2018. [arXiv:1612.00846](#).
- [tH93] G. 't Hooft. *Dimensional reduction in quantum gravity*. arXiv preprint, 1993. [arXiv:gr-qc/9310026](#).
- [Tho85] D. J. Thouless. *Long-range order and the fractional quantum Hall effect*. Physical Review B, 31(12):8305–8307, 1985.

- [TLF16] Nicolas Tarantino, Netanel H Lindner, and Lukasz Fidkowski. *Symmetry fractionalization and twist defects*. New Journal of Physics, 18(3):035006, 2016. [arXiv:1506.06754](#).
- [TSG82] D. C. Tsui, H. L. Stormer, and A. C. Gossard. *Two-dimensional magnetotransport in the extreme quantum limit*. Physical Review Letters, 48(22):1559–1562, 1982.
- [Tur00] Vladimir Turaev. *Homotopy field theory in dimension 3 and crossed group-categories*. arXiv preprint, 2000. [arXiv:math/0005291](#).
- [Tur20] Vladimir G. Turaev. *Quantum Invariants of Knots and 3-Manifolds*. Walter de Gruyter GmbH & Co.KG, 2020.
- [TV92] V.G. Turaev and O.Y. Viro. *State sum invariants of 3-manifolds and quantum 6j-symbols*. Topology, 31(4):865–902, 1992.
- [TV17] Vladimir Turaev and Alexis Virelizier. *Monoidal Categories and Topological Field Theory*. Springer International Publishing, 2017.
- [TW84] R. Tao and Yong-Shi Wu. *Gauge invariance and fractional quantum Hall effect*. Physical Review B, 30(2):1097–1098, 1984.
- [TW97] Vladimir Turaev and Hans Wenzl. *Semisimple and modular categories from link invariants*. Mathematische Annalen, 309(3):411–461, 1997.
- [TY98] Daisuke Tambara and Shigeru Yamagami. *Tensor categories with fusion rules of self-duality for finite abelian groups*. Journal of Algebra, 209(2):692–707, 1998.
- [TY23] Yuji Tachikawa and Mayuko Yamashita. *Topological modular forms and the absence of all heterotic global anomalies*. Communications in Mathematical Physics, 402(2):1585–1620, 2023. [arXiv:2108.13542](#).
- [Vaf88] Cumrun Vafa. *Toward classification of conformal theories*. Physics Letters B, 206(3):421–426, 1988.
- [Ver88] Erik Verlinde. *Fusion rules and modular transformations in 2D conformal field theory*. Nuclear Physics B, 300:360–376, 1988.
- [VR19] César F. Venegas-Ramírez. *Minimal modular extensions for super-Tannakian categories*. arXiv preprint, 2019. [arXiv:1908.07487](#).
- [Wal63] C.T.C. Wall. *Quadratic forms on finite groups, and related topics*. Topology, 2(4):281–298, 1963.
- [Wal23] Daniel Wallick. *An algebraic quantum field theoretic approach to toric code with gapped boundary*. Journal of Mathematical Physics, 64(10), 2023. [arXiv:2212.01952](#).
- [Wan10] Zhenghan Wang. *Topological Quantum Computation*. American Mathematical Society, 2010.
- [Wan21] Juven Wang. *Unified model beyond grand unification*. Physical Review D, 103(10):105024, 2021. [arXiv:2012.15860](#).
- [Wen89] Xiao-Gang Wen. *Vacuum degeneracy of chiral spin states in compactified space*. Physical Review B, 40(10):7387–7390, 1989.
- [Wen90] Xiao-Gang Wen. *Topological orders in rigid states*. International Journal of Modern Physics B, 04(02):239–271, 1990.
- [Wen91] Xiao-Gang Wen. *Non-Abelian statistics in the fractional quantum Hall states*. Physical Review Letters, 66(6):802–805, 1991.
- [Wen03] Xiao-Gang Wen. *Quantum order from string-net condensations and the origin of light and massless fermions*. Physical Review D, 68(6), 2003. [arXiv:hep-th/0302201](#).

- [Wen13] Xiao-Gang Wen. *A lattice non-perturbative definition of an $SO(10)$ chiral gauge theory and its induced standard model*. Chinese Physics Letters, 30(11):111101, 2013. [arXiv:1305.1045](#).
- [Wen17] Xiao-Gang Wen. *Colloquium: Zoo of quantum-topological phases of matter*. Reviews of Modern Physics, 89(4), 2017. [arXiv:1610.03911](#).
- [Wen19] Xiao-Gang Wen. *Choreographed entanglement dances: Topological states of quantum matter*. Science, 363(6429):eaal3099, 2019. [arXiv:1906.05983](#).
- [WG18] Qing-Rui Wang and Zheng-Cheng Gu. *Towards a complete classification of symmetry-protected topological phases for interacting fermions in three dimensions and a general group supercohomology theory*. Physical Review X, 8(1), 2018. [arXiv:1703.10937](#).
- [WHW22] Hongyu Wang, Yuting Hu, and Yidun Wan. *Extend the Levin-Wen model to two-dimensional topological orders with gapped boundary junctions*. Journal of High Energy Physics, 2022(7), 2022. [arXiv:2201.04072](#).
- [Wig60] Eugene P. Wigner. *The unreasonable effectiveness of mathematics in the natural sciences*. Communications on Pure and Applied Mathematics, 13(1):1–14, 1960.
- [Wil82a] Frank Wilczek. *Magnetic flux, angular momentum, and statistics*. Physical Review Letters, 48(17):1144–1146, 1982.
- [Wil82b] Frank Wilczek. *Quantum mechanics of fractional-spin particles*. Physical Review Letters, 49(14):957–959, 1982.
- [Wit89] Edward Witten. *Quantum field theory and the Jones polynomial*. Communications in Mathematical Physics, 121(3):351–399, 1989.
- [WL14] Chenjie Wang and Michael Levin. *Braiding statistics of loop excitations in three dimensions*. Physical Review Letters, 113(8), 2014. [arXiv:1403.7437](#).
- [WL17] Chenjie Wang and Michael Levin. *Anomaly indicators for time-reversal symmetric topological orders*. Physical Review Letters, 119(13):136801, 2017. [arXiv:1610.04624](#).
- [WLHW18] Hongyu Wang, Yingcheng Li, Yuting Hu, and Yidun Wan. *Gapped boundary theory of the twisted gauge theory model of three-dimensional topological orders*. Journal of High Energy Physics, 2018(10), 2018. [arXiv:1807.11083](#).
- [WN90] Xiao-Gang Wen and Qian Niu. *Ground-state degeneracy of the fractional quantum Hall states in the presence of a random potential and on high-genus Riemann surfaces*. Physical Review B, 41(13):9377–9396, 1990.
- [Wu84] Yong-Shi Wu. *General theory for quantum statistics in two dimensions*. Physical Review Letters, 52(24):2103–2106, 1984.
- [WW11] Kevin Walker and Zhenghan Wang. *$(3+1)$ -TQFTs and topological insulators*. Frontiers of Physics, 7(2):150–159, 2011. [arXiv:1104.2632](#).
- [WW15] Juven C. Wang and Xiao-Gang Wen. *Boundary degeneracy of topological order*. Physical Review B, 91(12), 2015. [arXiv:1212.4863](#).
- [WWH15] Yidun Wan, Juven C. Wang, and Huan He. *Twisted gauge theory model of topological phases in three dimensions*. Physical Review B, 92(4), 2015. [arXiv:1409.3216](#).
- [WWZ89] Xiao-Gang Wen, Frank Wilczek, and A. Zee. *Chiral spin states and superconductivity*. Physical Review B, 39(16):11413–11423, 1989.
- [WZ92] X. G. Wen and A. Zee. *Classification of Abelian quantum Hall states and matrix formulation of topological fluids*. Physical Review B, 46(4):2290–2301, 1992.

- [Xio18] Charles Zhaoxi Xiong. *Minimalist approach to the classification of symmetry protected topological phases*. Journal of Physics A: Mathematical and Theoretical, 51(44):445001, 2018. [arXiv:1701.00004](#).
- [XLLC21] Wenjie Xi, Ya-Lei Lu, Tian Lan, and Wei-Qiang Chen. *A lattice realization of general three-dimensional topological order*. arXiv preprint, 2021. [arXiv:2110.06079](#).
- [XSWX⁺23] Shibo Xu, Zheng-Zhi Sun, Ke Wang, Liang Xiang, Zehang Bao, Zitian Zhu, Fanhao Shen, Zixuan Song, Pengfei Zhang, Wenhui Ren, Xu Zhang, Hang Dong, Jinfeng Deng, Jiachen Chen, Yaozu Wu, Ziqi Tan, Yu Gao, Feitong Jin, Xuhao Zhu, Chuanyu Zhang, Ning Wang, Yiren Zou, Jiarun Zhong, Aosai Zhang, Weikang Li, Wenjie Jiang, Li-Wei Yu, Yunyan Yao, Zhen Wang, Hekang Li, Qiujiang Guo, Chao Song, H. Wang, and Dong-Ling Deng. *Digital simulation of projective non-abelian anyons with 68 superconducting qubits*. Chinese Physics Letters, 40(6):060301, 2023.
- [Xu06] Cenke Xu. *Novel algebraic boson liquid phase with soft graviton excitations*. arXiv preprint, 2006. [arXiv:cond-mat/0602443](#).
- [XZ24] Rongge Xu and Zhi-Hao Zhang. *Categorical descriptions of one-dimensional gapped phases with Abelian onsite symmetries*. Physical Review B, 110(15):155106, 2024. [arXiv:2205.09656](#).
- [Yam04] Shigeru Yamagami. *Frobenius duality in C^* -tensor categories*. Journal of Operator Theory, 52(1):3–20, 2004.
- [Yet92] David N. Yetter. *Framed tangles and a theorem of Deligne on braided deformations of Tannakian categories*. In *Deformation Theory and Quantum Groups with Applications to Mathematical Physics*, pages 325–349. American Mathematical Society, 1992.
- [YX15] Yi-Zhuang You and Cenke Xu. *Interacting topological insulator and emergent grand unified theory*. Physical Review B, 91(12), 2015. [arXiv:1412.4784](#).
- [Zhe17] Hao Zheng. *Extended TQFT arising from enriched multi-fusion categories*. arXiv preprint, 2017. [arXiv:1704.05956](#).
- [ZHK89] S. C. Zhang, T. H. Hansson, and S. Kivelson. *Effective-field-theory model for the fractional quantum Hall effect*. Physical Review Letters, 62(1):82–85, 1989.
- [ZKN17] Yi Zhou, Kazushi Kanoda, and Tai-Kai Ng. *Quantum spin liquid states*. Reviews of Modern Physics, 89(2):025003, 2017. [arXiv:1607.03228](#).
- [ZLW19] Chenchang Zhu, Tian Lan, and Xiao-Gang Wen. *Topological nonlinear σ -model, higher gauge theory, and a systematic construction of 3+1D topological orders for boson systems*. Physical Review B, 100(4), 2019. [arXiv:1808.09394](#).
- [ZLZH⁺23] Jiaheng Zhao, Jia-Qi Lou, Zhi-Hao Zhang, Ling-Yan Hung, Liang Kong, and Yin Tian. *String condensations in 3+1D and Lagrangian algebras*. Advances in Theoretical and Mathematical Physics, 27(2):583–622, 2023. [arXiv:2208.07865](#).
- [Zuo17] Fen Zuo. *Simplicity condition and boundary-bulk duality*. arXiv preprint, 2017. [arXiv:1706.00312](#).
- [ZW15] Bei Zeng and Xiao-Gang Wen. *Gapped quantum liquids and topological order, stochastic local transformations and emergence of unitarity*. Physical Review B, 91(12), 2015. [arXiv:1406.5090](#).