Summary of Liang Kong's Research

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In 1996, after some years in studying condensed matter physics, for lack of new ideas in solving strong coupling problems (e.g., high Tc superconductivity) and phase transitions, I decided to look for new ideas from other fields, such as high energy physics and mathematical physics. In the fall of 1997, I transferred to Rutgers to study string theory at the height of the second string revolution. Although I was excited by the new developments, such as mirror symmetry, D-branes, Seiberg-Witten theory, M-theory, AdS/CFT correspondence, the microscopic origin of black hole entropy, non-commutative geometry, I was also overwhelmed and unable to tell which topics is important or necessary. After some intensive studies of various frontier topics and some careful evaluations, partly influenced by my old interests in phase transitions and an old lesson that simple systems nourish new methodologies, I decided to step back and restudy 2D rational conformal field theories (RCFT), a subject which was the base for many new developments but was regarded by most working physicists as something understood and outdated. Since RCFT's were actively pursued only by mathematicians in 90's, I decided to pursue a Ph.D degree in mathematics. This is the beginning of a long and personal journey of understanding QFT's and a journey of simplification from 2D RCFT's to TQFT's then to topological orders, which are the simplest quantum phases or QFT's (as TQFT's defined on an open disk). This simplification leads to revolutionary new insights, languages and methodologies in the study of quantum many-body systems or QFT's, including a unified mathematical theory of gapped/gapless quantum liquids[1] in all dimensions [KZ22a, KZ24, KZ22b]. This journey naturally splits into eight deeply intertwined topics. My contributions to these topics are comparable. We use nD to represent the spacetime dimension.

(1) **Mathematical Foundation of Open-closed Rational Conformal Field Theories**: Together with my Ph.D advisor Yi-Zhi Huang, we started the program to establish the mathematical foundation of open-closed RCFT based on the mathematical theory of vertex operator algebra (VOA) and its representation theory[2]. One motivation of this program is to understand the mathematical meaning of a D-brane and open-closed duality. This program was successfully accomplished in a series of papers [HK04, HK07, Kon07, HK10, Kon08a, Kon08b] (see [Kon11] for a review).

   (a) In [HK04], we introduced the notion of an open-string vertex operator algebra V_{op}, which is the mathematical definition of an open (or boundary) CFT. Under some natural assumptions, we proved that V_{op} is an algebra over the so-called disk partial operad. We also give V_{op} a tensor-categorical description, as an algebra in the category Mod_V of V-modules when V_{op} contains a rational VOA V in its meromorphic center and Mod_V is a modular tensor category.

   (b) In [HK07], we introduced the notion of a full field algebra, which is a non-chiral analogue of a VOA, and gave some rigorous constructions of full field algebras. In [Kon07], we proved

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[1] A gapped/gapless quantum liquid can be viewed as a fully dualizable gapped/gapless QFT.

[2] There are another two successful approaches towards 2D RCFT's: (1) Conformal-net approach, in which a CFT is defined by a conformal net; (2) Fuchs-Runkel-Schweigert approach, in which a 2D RCFT is formulated as a natural transformation from the trivial functor to a non-trivial one obtained by restricting the Reshetikhin-Turaev TQFT functor to the 1-category of surfaces (as objects) and their self-sewings as 1-morphisms [FFRS08].

that a full field algebra over $V\otimes_{\mathbb{C}}\overline{V}$, where V is a rational VOA, is equivalent to a commutative algebra in $\mathrm{Mod}_V \boxtimes \overline{\mathrm{Mod}_V}$, and gave some categorical construction of such full field algebras. In [HK10], we prove that full field algebras constructed in [HK07] are modular invariant, thus are the candidates of modular-invariant closed (or bulk) CFT's.

(c) In [Kon08a], by combining the open parts with the closed parts, we introduced the notion of an open-closed field algebra, and proved that it gives an algebra over Swiss-Cheese partial operad. We also gave a tensor-categorical formulation and construction of such an algebra when it satisfies a V-symmetric boundary condition for a rational VOA V.

(d) In [Kon08b], we gave a precise mathematical formulation of the modular-invariant condition and the Cardy condition for an open-closed field algebra. When it satisfies a V-symmetric boundary condition for a rational VOA V, we obtain rigorously a categorical reformulation of these conditions. Up to certain convergence and modular-invariance assumptions in higher genus surfaces, we obtained a complete tensor-categorical reformulation of an open-closed rational CFT in terms of generators and relations. It was slightly refined in [KR09]. This result automatically includes Moore-Segal's theorem on open-closed TFT's in terms of generators and relations as a special case. We also gave rigorous constructions.

(e) In [KLR14], we proved rigorously that the tensor-categorical reformulation of an open-closed RCFT obtained in our approach in terms of generators and relations [Kon08b, KR09] is equivalent to Fuch-Runkel-Schweigert's state-sum construction of the categorical correlators in an open-closed RCFT [FRS02, FFRS08].

(2) **Mathematical Theory of Open-closed Duality**: In a series of papers [KR08, KR09, DKR11a, DKR11b, DKR15, KYZ21], joint with a few different collaborators, we formulated precisely, proved rigorously and further developed the open-closed duality (or boundary-bulk relation) in open-closed rational CFT's. This development went through three different phases.

(a) In the first phase, the open-closed duality, i.e., bulk is the so-called 'full center' of a boundary, is limited to a single bulk CFT without topological defect lines. In [KR08], we proved, within the framework of RCFT's, that two open CFT's are Morita equivalence if[3] and only if their full centers (or equivalently, their bulk CFT's) are isomorphic. In [KR09], we show that a simple modular-invariant closed (or bulk) CFT is necessarily equipped with consistent open (or boundary) CFT's. Moreover, we also proved that a modular-invariant closed CFT is precisely a Lagrangian algebra[4] in $\mathrm{Mod}_V \boxtimes \overline{\mathrm{Mod}_V}$.

(b) In the second phase, hidden structure as a higher layer of open-closed duality was first observed in [DKR11a]; then we introduced and proved in [DKR15] the functoriality of the open-closed duality by including topological defect lines in a single bulk CFT. In [DKR11b], we applied this functoriality to 2D TQFT's by including topological defects lines between different bulk 2D TQFT's evaluated in a 2-category[5].

(c) In the third phase, based on some new developments in the boundary-bulk relation in the field of 2+1D topological orders [KWZ15, KZ18a], we introduced and proved in [KYZ21] the functoriality of the open-closed duality by including domain walls between different bulk CFT's.

The success of above two programs led to a surprising new insight, which appeared in [Kon11] as a proposal of a new theory of quantum gravity or a new foundation of geometry (called *stringy algebraic geometry*) based on 2D conformal field theories and D-branes. Moreover, open-closed duality

[3] The 'if' part was obtained earlier by Frölich, Fuchs, Runkel and Schweigert in [FFRS07].

[4] The name of 'Lagrangian algebra' was coined later in [DMNO13].

[5] The axiomatic study of 2D TQFT with topological defect lines in this work [DKR11b] was partly rediscovered in 2017-2019 under a new name: a 2D TQFT with a fusion category symmetry [BT17, TW19].

(or boundary-bulk duality), i.e., "bulk = $\mathfrak{Z}$(boundary)", plays the role of equivalence principle in this new theory of quantum gravity [Kon11].

(3) **Boundary-bulk relation in Topological orders**: It is natural to ask if the boundary-bulk relation (i.e., "bulk = $\mathfrak{Z}$(boundary)") holds for other quantum field theories. This question led to a series of works on the boundary-bulk relation for topological orders in all dimensions [KK12, Kon14a, KWZ15, KZ18a, KZ24].

  (a) In [KK12], we proposed and proved (via explicit lattice model constructions) the boundary-bulk relation for all 2+1D topological orders with gapped boundaries. We also provided lattice model constructions of all gapped defects of codimension 1 and higher in all 2+1D non-chiral topological orders. These defects form monoidal 2-categories, which turn out to be the first non-trivial examples of fusion 2-categories [DR18].

  (b) In [KWZ15], we provided a proof of the boundary-bulk relation (i.e., "bulk = $\mathfrak{Z}$(boundary)") for topological orders in all dimensions based on the key observation that the notion of a center is determined by its universal property expressed in terms of morphisms between two mathematical objects of the same type. Therefore, it is enough to introduce a physical definition of a morphism between two (potentially anomalous) topological orders then check the universal property of center directly. Indeed, in [KWZ15], we introduced the notion of a morphism between two topological orders, which is obviously generalizable to other quantum field theories (see [KWZ17, KZ22a]), and checked that the universal property of center holds tautologically. We proved later that this notion of a morphism coincides with the notion of a monoidal functor between two fusion n-categories [KZ18a, KZ24]. Moreover, we conjectured the functoriality of boundary-bulk relation in [KWZ15].

  (c) Similar to the functoriality of the open-closed duality in RCFT's or TQFT's [DKR15, DKR11b], in [KZ18a], we rigorously proved the functoriality of the boundary-bulk relation for 2+1D topological orders by showing that the Drinfeld center can be extended to a center functor, which can be formulated and proved to be a symmetric monoidal equivalence. Moreover, this functoriality provides a fusion formula that is a powerful tool to compute factorization homology on surfaces [AKZ17]. This functoriality was later generalized to higher dimensions in [KZ24] with a precise mathematical characterization of its non-surjectivity, non-fullness and non-faithfulness (see also [KZZZ24]). I believe that this result will remain to be one of the most important guiding principle in the study of higher dimensional topological orders or TQFT's.

(4) **A Unified Mathematical Theory of All Gapped Quantum Liquids**: A gapped quantum liquid is a notion that summarizes all topological orders, SPT/SET orders and spontaneous symmetry breaking (SSB) orders [ZW15, KLWZZ20a]. We have successfully found a unified mathematical characterization of a gapped quantum liquid with a finite internal symmetry in any dimensions based on three guiding principles: remotely detectable principle [KW14], boundary-bulk relation [KWZ15, KWZ17] and condensation completion principle [KTZ20, GJF19, JF22, KLWZZ20a].

  (a) The first mathematical characterization of 2+1D bosonic SPT/SET orders was obtained in [BBCW19] in terms of G-crossed braided fusion categories. We developed a unified mathematical characterization of 2+1D bosonic/fermionic SPT/SET orders based on minimal modular extensions [LKW17a, LKW17b] such that it is equivalent to results in [BBCW19] for bosonic SPT/SET orders. Mathematically, we proved that minimal modular extensions of a symmetry fusion category $\mathcal{E}$ form a finite abelian group $\mathrm{Mext}(\mathcal{E})$; and for a braided fusion category with Müger center given by $\mathcal{E}$, the set of its minimal modular extensions is either empty or a $\mathrm{Mext}(\mathcal{E})$-torsor. This result generated a lot of interests among mathematicians and initiated the program of computing $\mathrm{Mext}(\mathcal{E})$ for $\mathcal{E} = \mathrm{Rep}(G, z)$.

(b) In [KW14, KWZ15], we initiated a program of studying topological orders in all dimensions as a whole, or more precisely, studying the categories of all topological orders. Many questions and conjectures are raised in these works. In particular, the mathematical characterization of a topological order (up to invertible ones) were correctly conjectured in [KWZ15] without knowing how to define a multi-fusion n-category. The most surprising and influential result along this line was the complete classification of all 3+1D TQFT's obtained in [LKW18, LW19]. In particular, the main result of [LKW18] says that a 3+1D topological orders with only bosonic particles is necessarily a twist finite gauge theory, whose modular 2-category of topological defects of codimension 2 and higher was rigorously computed later in [KTZ20]. This result was proved at a physical level of rigor in [LKW18] and was later rigorously proved by Johnson-Freyd in [JF20].

(c) In spite of above success, [KW14, KWZ15] miss some fundamental ingredients. One of the missing ingredients is the mathematical theory of condensation completion first developed for 2-categories by Carqueville and Runkel in [CR16] and further developed by Douglas and Reutter in [DR18] then developed for n-categories by Gaiotto and Johnson-Freyd in [GJF19]. This theory leads to a mathematical theory of multi-fusion n-category developed in [DR18] for $n = 2$ and in [JF22] for all n. A rather complete mathematical characterization of topological orders (up to invertible ones) was fully established by Johnson-Freyd in [JF22]. Main conjectures in [KWZ15] were proved in [JF22]. Another paper [KLWZZ20a], which appeared in the same week as [JF22], developed the mathematical characterization of SPT/SET orders. It contains that of topological orders as a special case, which was largely taken from granted in [KLWZZ20a]. Johnson-Freyd's theory of multi-fusion n-category was later simplified in [KZ22a] via the theory of separable n-categories, which is also based on [GJF19].

(d) In [KLWZZ20a, KLWZZ20b], we found a unified mathematical characterization of a gapped quantum liquid (including topological/SPT/SET/SSB orders) with a finite internal symmetry. Among many other things, this theory implies a topological holography between an n+1D topological order together with a fixed gapped boundary and an nD SPT/SET/SSB orders.

(5) **A Unified Mathematical Theory of Gapped/Gapless Boundaries of 2+1D Topological Orders**: A surprising observation in [KWZ17] is that the proof of "bulk $= \mathfrak{Z}$(boundary)" [KWZ15, KWZ17] applies to gapless boundaries of topological orders. This immediately suggests that there is a unified mathematical theory of gapped and gapless boundaries of (potentially trivial) topological orders in all dimensions.

(a) In a chiral 2+1D topological order, we know its anyons form a modular fusion 1-category $\mathcal{C}$ and its boundaries are topologically protected gapless. By "bulk $= \mathfrak{Z}$(boundary)", something on the gapless boundary should form a categorical structure such that its monoidal center is $\mathcal{C}$. In [KZ18c], we found this categorical structure (i.e., a fusion 1-category enriched in a modular fusion 1-category) through a careful analysis of all possible macroscopic observables on the 1+1D world sheet of the boundary. As a consequence, we established for the first time a unified mathematical theory of gapped and gapless boundaries of 2+1D topological orders. This theory was based on all my previous works on 2D RCFT's [Kon08b, KR08, DKR15] and those on topological orders [KK12, KWZ15, KWZ17, KZ18a]. It automatically recovers all the previous results on anomaly-free 1+1D RCFT's. As a byproduct, we obtain a mysterious holography called 'Topological Wick Rotation', which is a holography between (3D topological order + a gapped boundary) and the topological data of a 2D RCFT's.

(b) In [KZ20, KZ21], we provide a complete 'physical proof' of this theory by providing all necessary mathematical details and minimizing non-mathematical arguments to either a no-go theorem or a physical principle. Moreover, we proposed in [KZ20, KZ21] that the topological Wick rotation naturally generalizes to all dimensions with 2D RCFT's replaced by gapless quantum liquids.

These works provide a solid evidence that there are a single mathematical theory for topological orders and their phase transitions (which close the energy gap). In [CJKYZ20], we gave an illustrating example of a purely boundary phase transition between the smooth boundary and the rough boundary of the 2+1D toric code model, where we have a complete mathematical characterization of the critical point regarded as a special case of gapless boundaries of toric code model.

(6) **A New Theory of Enriched Categories**: The notion of a fusion category enriched in a braided fusion category naturally arises in the study of gapless boundaries of 2+1D topological orders [KZ18c]. This notion was independently introduced by mathematicians only a few months earlier [MP19]. The classical theory of enriched categories is restricted to the 2-category of enriched categories with a fixed ambient category $\mathcal{A}$ (often assumed to be symmetric monoidal), $\mathcal{A}$-functors and $\mathcal{A}$-natural transformations [Kel82]. The physics of gapless phases [KZ18c, KZ20, KZ21, KZ22a], however, predicts a surprising and far richer theory living in the 2-category of enriched categories with arbitrary ambient categories, generalized enriched functors and generalized enriched natural transformations. An ongoing program of ours is to fully developed this new theory.

(a) In [KZ18c], we introduced a constructive definition of the monoidal center of a monoidal category enriched in a braided monoidal category, and proved some general theorems for computing such center. Among many other things, this paper showed that the monoidal center is often enriched in a different braided monoidal category. For example, for a modular fusion category $\mathcal{C}$, by replacing its hom space by internal homs, we obtain an $\mathcal{C}$-enriched fusion category ${}^{\mathcal{C}}\mathcal{C}$, where $\mathrm{hom}_{{}^{\mathcal{C}}\mathcal{C}}(a, b) = [a, b] = b \otimes a^*$ for $a, b \in \mathcal{C}$. In this case, we have $\mathfrak{Z}_1({}^{\mathcal{C}}\mathcal{C}) \simeq \mathcal{C}$, where $\mathcal{C}$ is enriched in the category Vec of finite dimensional vector spaces over $\mathbb{C}$. This fact has a deep physical meaning and also a mathematical meaning because it suggests how to extend a Reshetikhin-Turaev TQFT down to a point, i.e., by assigning a point to ${}^{\mathcal{C}}\mathcal{C}$ [Zhe17].

(b) Although above work [KZ18c] is enough to demonstrate our point, it does not reveal the deeper reason for such interesting phenomena to occur. In [KYZZ24], we reveal that there is a natural 2-category of enriched categories, where the notion of a functor is defined for two enriched categories with different ambient categories. This notion largely generalizes the usual notion of a functor used in the literature or textbook on enriched categories. This work shows that the theory of enriched categories should be developed within this 2-category. This work [KYZZ24] is only the first in a series, where we develop a new theory of (braided monoidal) enriched categories. The second in this series [KYZZ25] is on the representation theory of enriched monoidal categories and was partly predicted in our physics paper [KZ21].

(c) In [KZ20, KZ21], we propose that topological Wick rotations naturally generalize to all dimensions. This demands a yet-unknown mathematical theory of enriched higher categories, a notion which still does not have a mathematical definition. This theory should be an indispensable ingredient of a complete new paradigm of phases and phase transitions.

(7) **A Unified Mathematical Theory of Gapped/Gapless Quantum Liquids**: Based on the mathematical theory of condensation completion [CR16, DR18, GJF19, JF22], by combining [KZ18c, KZ18b, KZ20, KZ21] with [KLWZZ20a, KLWZZ20b], we developed a unified mathematical theory of all gapped/gapless quantum liquids in a trilogy [KZ22a, KZ24, KZ22b].

(a) In the first of the trilogy [KZ22a], we refine Johnson-Freyd's theory on multi-fusion n-categories [JF22] by developing the theory from a slightly different perspective. More precisely, we first developed a theory of separable n-categories, then build that of E_k-fusion n-categories on the top of it. We recover many results in [JF22] and prove many new results on E_k-fusion n-categories and E_k-centers. Moreover, we use it to build a unified mathematical theory of all quantum liquids. In particular, an nD quantum liquid X^n can be completely characterized by a pair $(\mathsf{X}_{\mathrm{lqs}}, \mathsf{X}_{\mathrm{top}})$, where $\mathsf{X}_{\mathrm{lqs}}$ summarizes a minimal dynamical data and is called local quantum symmetry, and $\mathsf{X}_{\mathrm{top}}$ summarizes all the topological data and is called the topological skeleton of X^n. Moreover, $\mathsf{X}_{\mathrm{top}}$ is completely determined via topological Wick rotation by a one-dimension-higher topological order, together with a gapped boundary. We compute explicitly the categories of topological skeletons of all nD quantum liquids, and show that these categories can be obtained from the one for $n = 1$ by iterated delooping. This work also proposed for the first time that all macroscopic observables in a 2D SPT/SET/SSB order form a fusion category enriched in a modular fusion category. This proposal was later validated in Ising chain and Kitaev chain [KWZ22] and in other 2D lattice models in [XZ24].

(b) In the second of the trilogy [KZ24], we proved the functoriality of the E_0-centers of separable n-categories and derived that of the E_1-centers of fusion $(n-1)$-categories as a consequence. This new approach dramatically simplified the proof of the functoriality of the Drinfeld centers of fusion 1-categories obtained in [KZ18c]. We further study E_k-fusion n-categories and develop a theory of E_k-centralizers, and use it to prove some fundamental theorems on minimal modular extensions. We also establish some basic facts on group-theoretical higher categories. In the end, we develop a mathematical theory of SPT/SET order with a finite spacetime symmetry and provide a categorical proof to the crystalline equivalence principle conjecture [TE18], which claims that space symmetries can be dualized to internal symmetries. We also predicts some new SPT orders.

(c) In the third of the trilogy [KZ24], we developed the mathematical theory of local quantum symmetries by generalizing the theory of conformal nets in 2D to that of topological nets in higher dimensions. As a consequence, we obtained a rather complete theory of all gapped/gapless quantum liquids. It allows us study dualities, holography, dimension reductions, symmetry-breaking/condensation/gauging, boundary-bulk duality and anomalies all within a single theoretical framework. It is rather surprising that such a theory is even possible. We are still in a beginning stage to understand what is really going on.

It is important to remark that such a powerful framework is still not enough to cover a complete mathematical theory of phase transitions, which demands us to go beyond quantum liquids to include certain non-liquid phases. This will be the subject to study in the next ten years.

(8) **A Unified Mathematical Theory of Defect Condensations**: A complete mathematical theory of phase transitions among gapped quantum liquids demands a new calculus, in which we can barely understand integers. It means that there are a lot of new mathematics needed in order to achieve this goal. It was known that many phase transitions among topological orders can driven by the condensation of topological defects. The bootstrap study of defect condensations can be accomplished within the framework of quantum liquids.

(a) The first example of defect condensation was known as the so-called 'anyon condensation' in 2+1D topological order. Originated from Moore and Seiberg's work on the extension of chiral algebra in RCFT's [MS89b, MS89c], the physical theory of anyon condensation was officially introduced by Bais, Schroers, Slingerland in 2002 [BSS02, BSS03]. After some success in the special case of finite abelian gauge theories [KS11, Lev13, BJQ13], a complete categorical theory of anyon condensations in all 3D topological orders was established in my work [Kon14b], which was based on many earlier mathematical works [KO02].

(b) In [KZZZ24], we develop a unified mathematical theory of defect condensations for topological orders in all dimensions based on higher categories, higher algebras and higher representations. A k-codimensional topological defect A in an n+1D (potentially anomalous) topological order C^{n+1} is condensable if it is equipped with the structure of a condensable E_k-algebra. Condensing such a defect A amounts to a k-step process. In the first step, we condense the defect A along one of its transversal directions, thus obtaining a $(k-1)$-codimensional defect ΣA, which is naturally equipped with the structure of a condensable E_{k-1}-algebra. In the second step, we condense the defect ΣA in one of the remaining transversal directions, thus obtaining a $(k-2)$-codimensional defect $\Sigma^2 A$, so on and so forth. In the k-th step, we condense the 1-codimensional defect $\Sigma^{k-1}A$ along the only transversal direction, thus defining a phase transition from C^{n+1} to a new n+1D topological order D^{n+1}. We give precise mathematical descriptions of each step in above process, including the precise mathematical characterization of the condensed phase D^{n+1}. We also briefly predicted some results in higher Morita theory, a theory of integrals (generalizing that of factorization homology) and a theory of the condensations of gapless liquid-like defects.

The bootstrap results of defect condensations tell us the relation between the initial phase and the condensed phase. But this theory is not enough to determine the critical point. A more complete theory of phase transitions will be our focus of study in the next a few years.

Remark 0.0.1. Since 2022, some results obtained in (3), (4) and (5) have been gradually rediscovered under the name "sandwich construction", "topological symmetry" [FMT22] or "SymTFT" [ABGE+23] and "topological holography" [MFT23], and have become one of the hottest topics in theoretical physics. After hundreds research papers published in this direction since 2022, as far as I can tell, key results in (4), (5) and (7) are still largely unknown to physics community. It is worth pointing out that these results (e.g.,"SymTFT" and "topological holography") appeared in this new wave of developments after 2022 as new proposals or new ideas but appeared in our works as the consequences of a much more complete theory [KZ18c, KZ20, KZ21]. ♢

I have summarized all my previous works. Although the unified mathematical theory of all quantum liquids (including SPT/SET/SSB order and gapless liquids) and that of defect condensations are breakthroughs that are hard to imagine ten years ago, they are only the beginning of an ongoing journey towards to a revolutionary new calculus of quantum many-body theory or QFT's [Kon24]. This new calculus should automatically include a new mathematical theory of phases and phase transitions. I believe that quantum gravity is impossible without this new calculus [Kon20]. My long term goal is to establish this new calculus.

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