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Pushdown Automata

(Part 2)

Lecture 14

Day 15/31

CS 154

Formal Languages and Computability

Fall 2019

Agenda of Day 15

- Summary of Lecture 13
- Quiz 5
- Lecture 14: Teaching ...
  - Pushdown Automata (part 2)

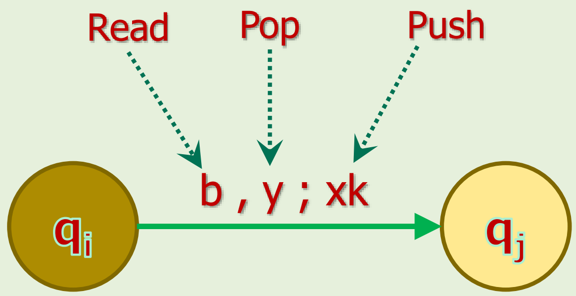
Summary of Lecture 13: We learned ...

PDAs

- NFAs are powerful enough to recognize only regular languages.
- To recognize non-regular languages, we need more powerful machines.
- We noticed that we needed writable memory.
- We added stack to NFAs and ...
- ... introduced PDAs to accept all or at least some of non-regular languages.
- PDA stands for ...

Pushdown Automata.

- PDAs have both deterministic (DPDA) and nondeterministic (NPDA) versions.
- We talked about the structure of PDAs.



- Condition for transition = ...
  - ... input symbol + top of stack
- We learned how to relax these conditions by λ.

Any question?

Summary of Lecture 13: We learned ...

PDAs

- PDAs halt when ...
  - ... the conditions for the next transition are not satisfied. (zero transition)

$$z \leftrightarrow h$$

- The conditions for a string being accepted by a process ...

$$(h \land c \land f) \leftrightarrow a$$

- The conditions for a string being rejected by a process ...

$$(\sim h \lor \sim c \lor \sim f) \leftrightarrow \sim a$$

- The content of the stack does not matter for accepting/rejecting.
- The stack alphabet and the input tape alphabet can be totally different.

Any question?

NAME	Alan M. Turing		
SUBJECT	CS 154	TEST NO.	5
DATE	10/10/2019	PERIOD	1 / 2 / 3

TEST RECORD	
PART 1	123
PART 2	
TOTAL	

Your **list #** goes here!

Take-Home Exam!

Quiz 5

Use Scantron

5. PDAs in Action

Design Examples

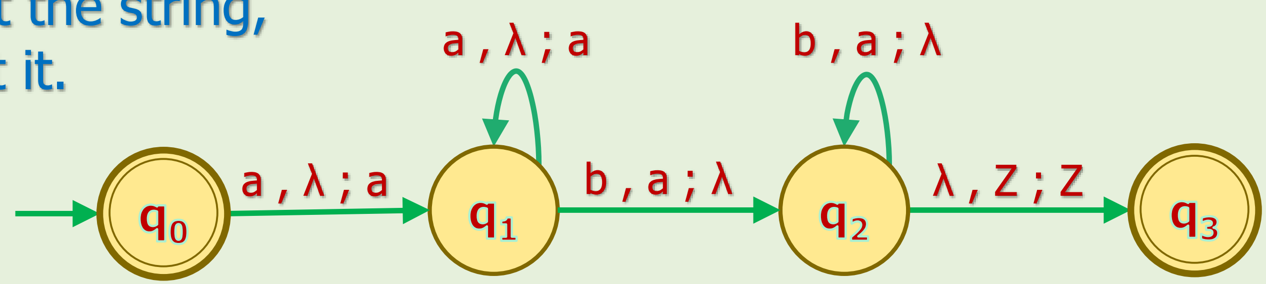
5. PDAs in Action

Example 14

- Design a PDA to accept our famous language $L = \{a^n b^n : n \geq 0\}$.

Solution

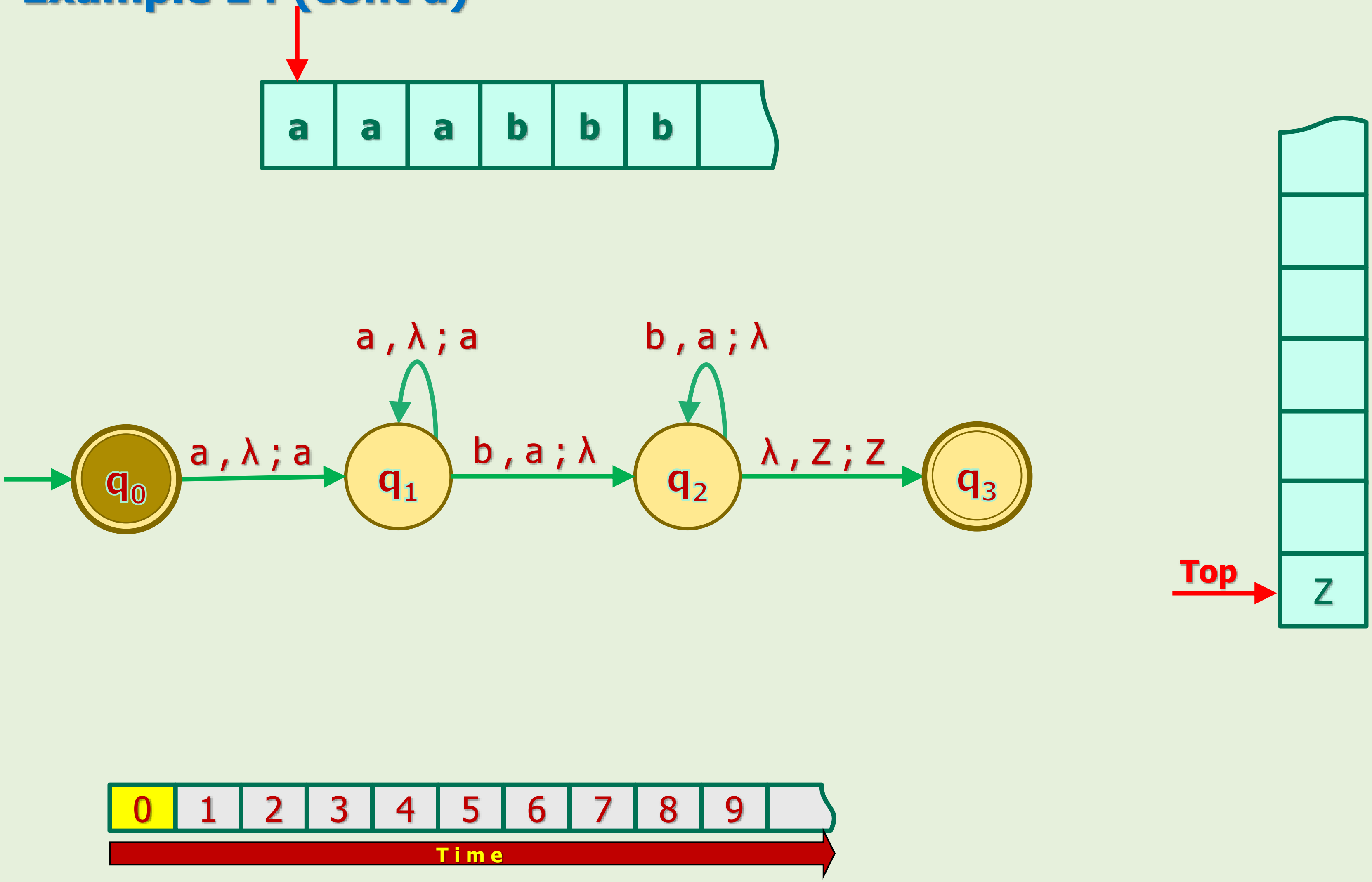
- **Strategy**: read a's and push them in the stack regardless of top of the stack.
- When the first b is sensed, start popping a's to match them with b's.
- Continue popping a's until you are out of b.
- If end of stack is reached, means the number of a's and b's are equal, so, accept the string, otherwise, reject it.



- Let's trace this solution for some strings.

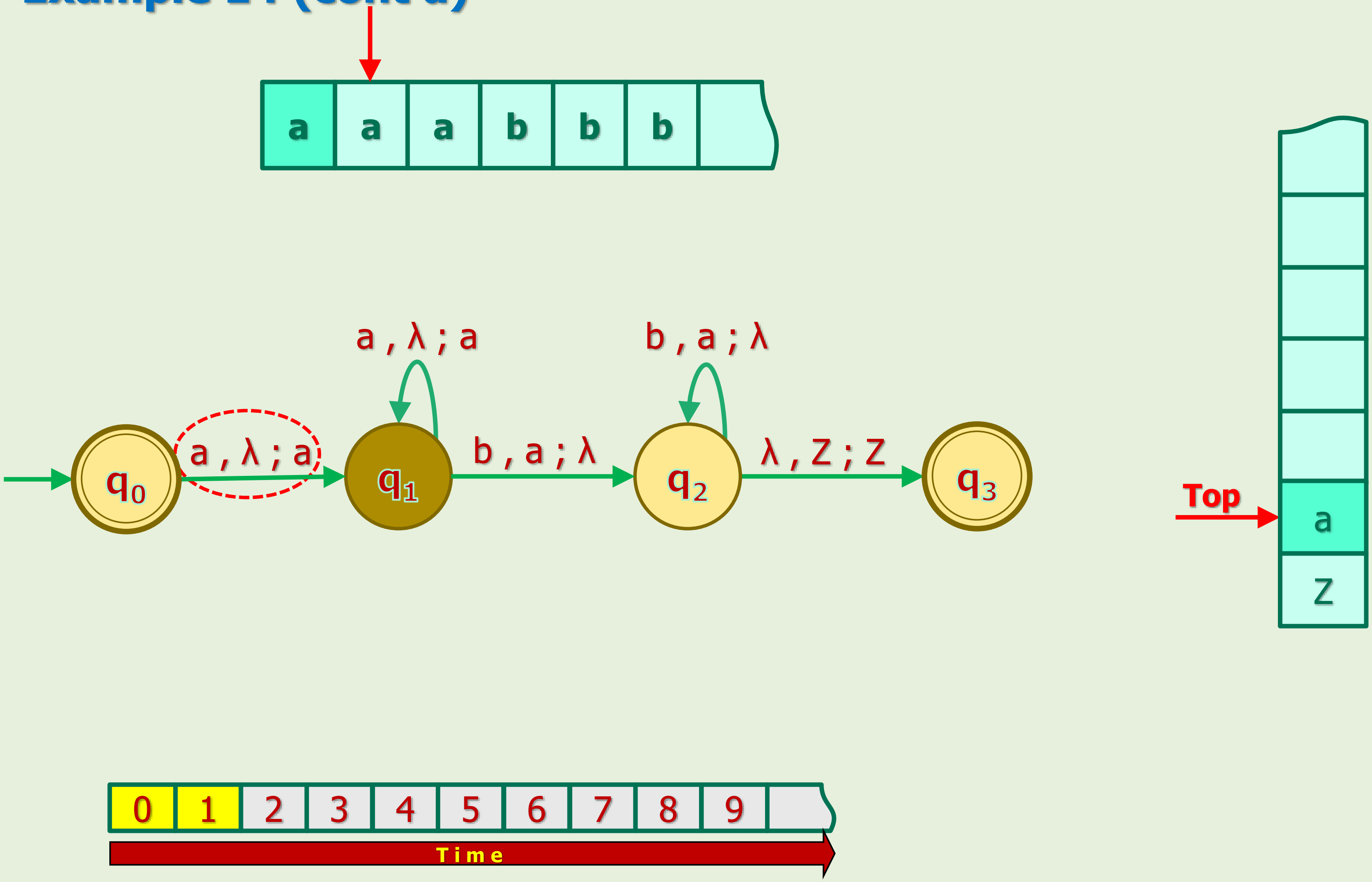
5. PDAs in Action

Example 14 (cont'd)



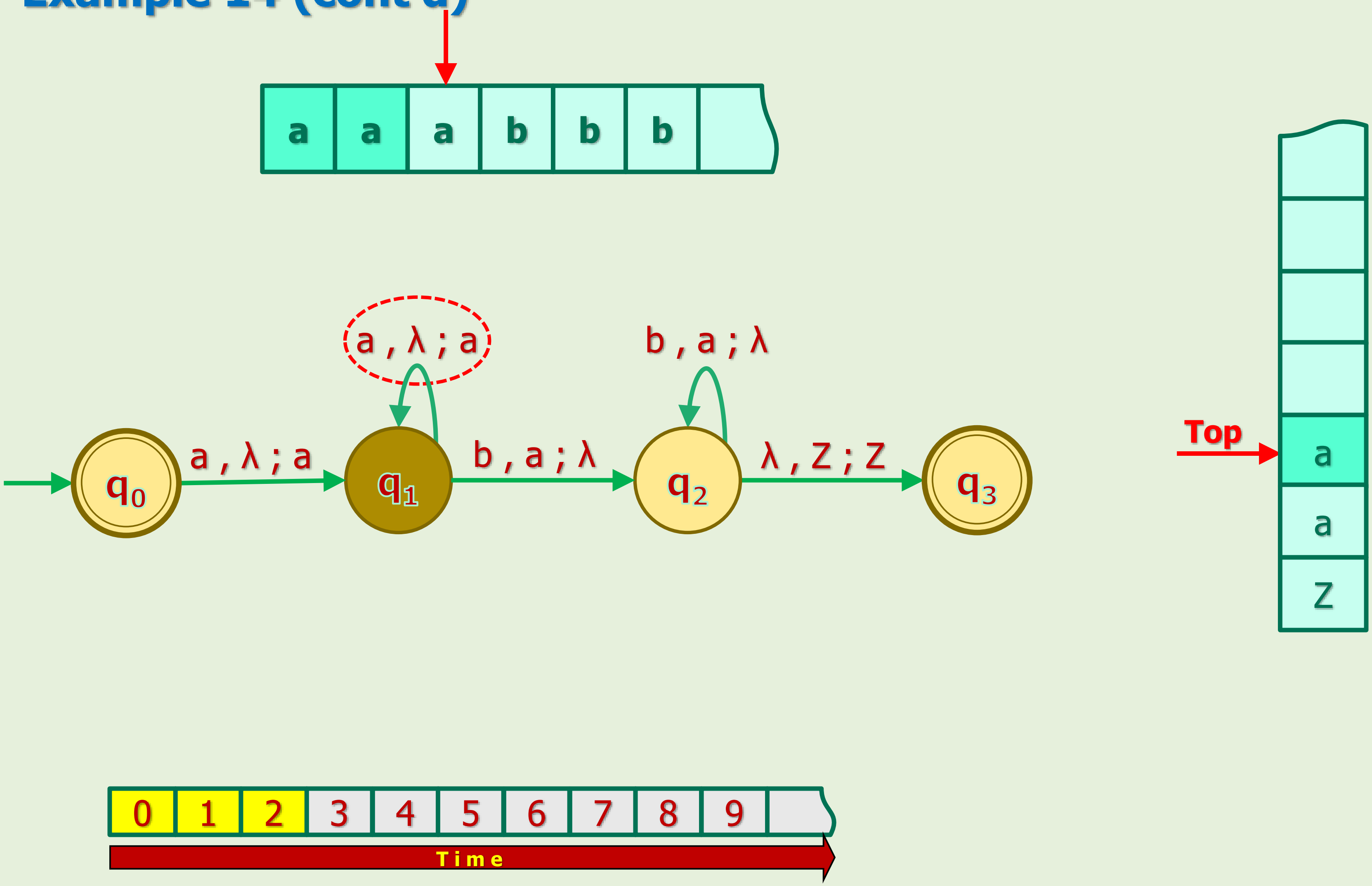
5. PDAs in Action

Example 14 (cont'd)



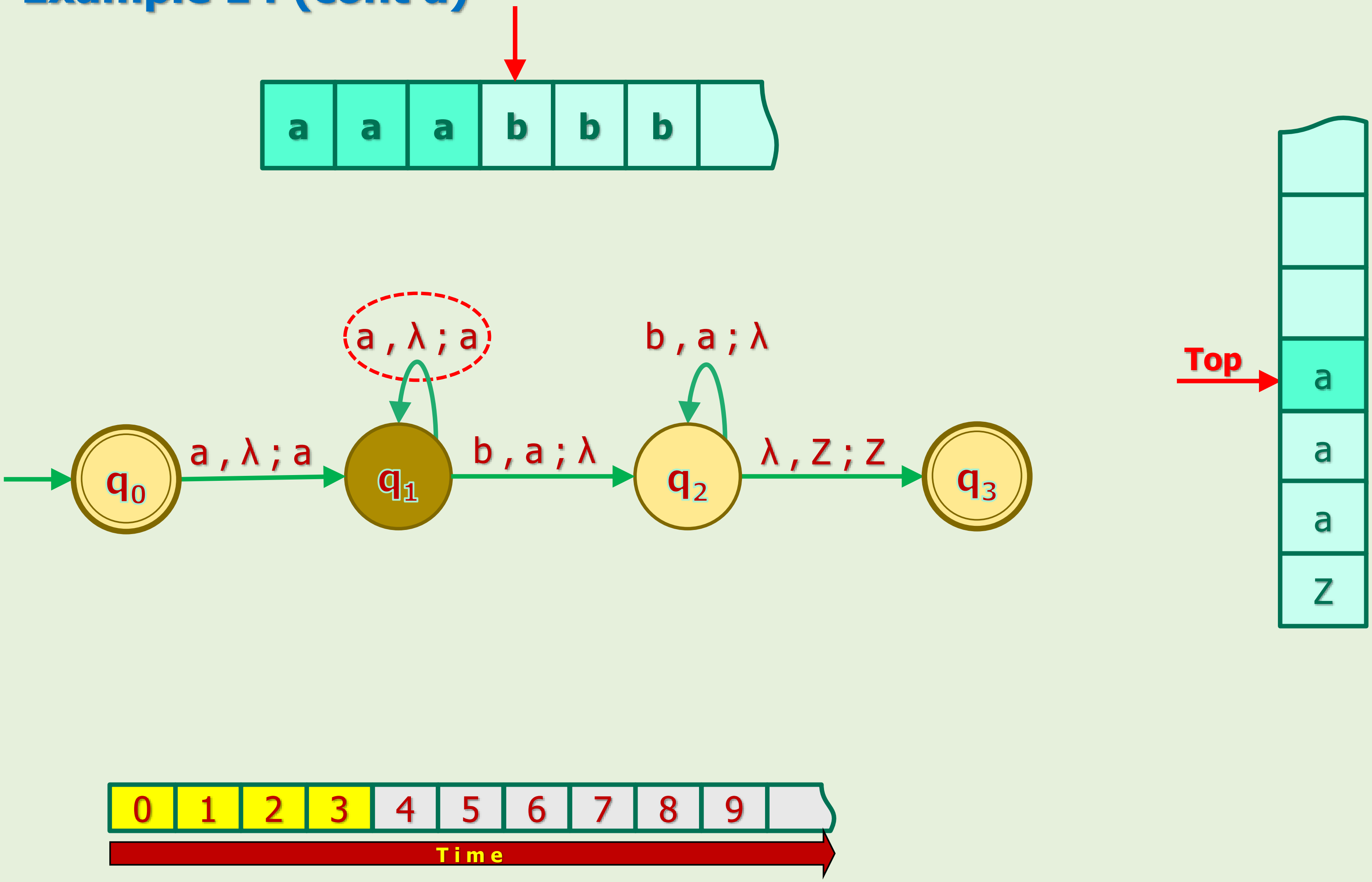
5. PDAs in Action

Example 14 (cont'd)



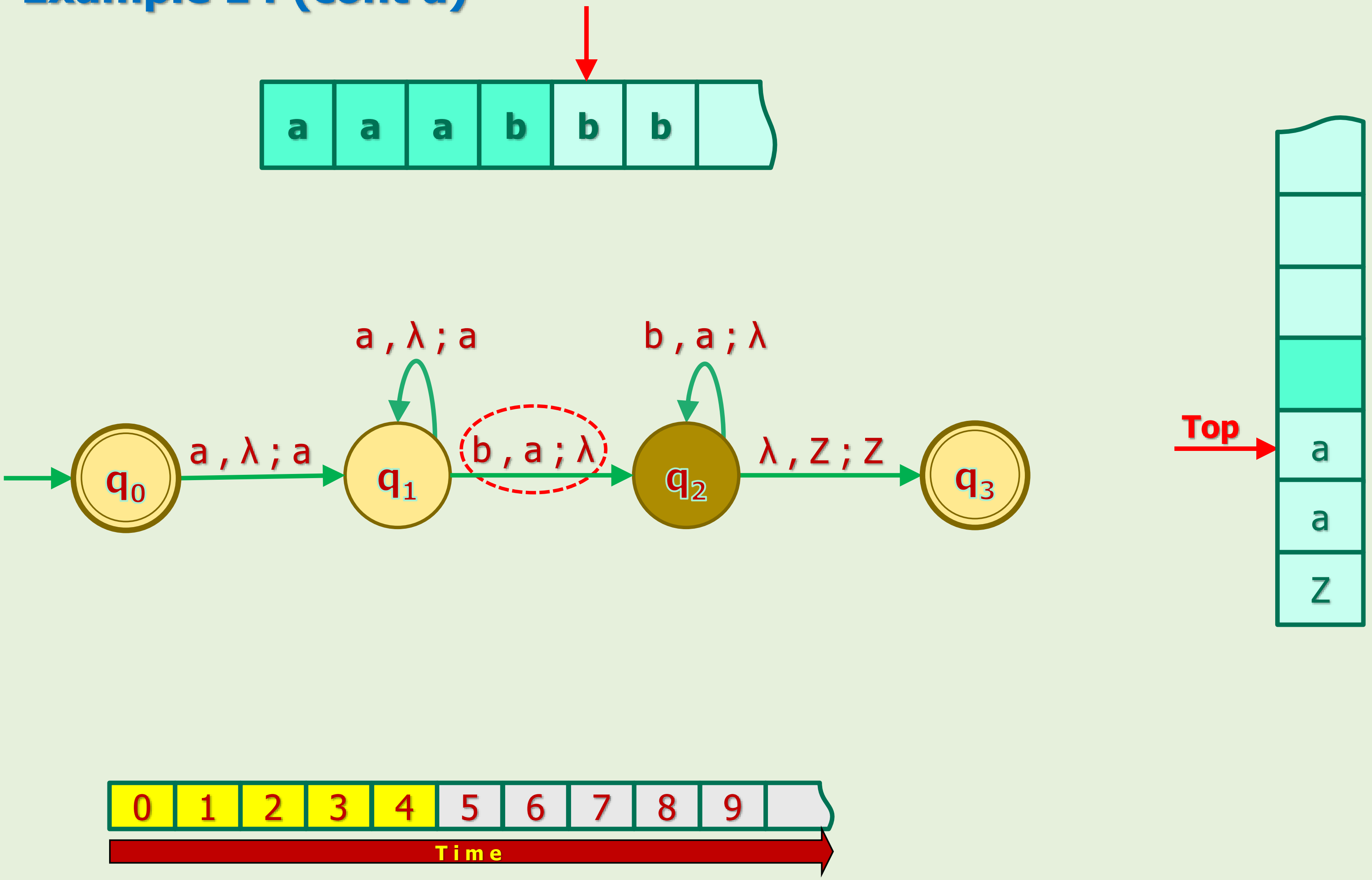
5. PDAs in Action

Example 14 (cont'd)



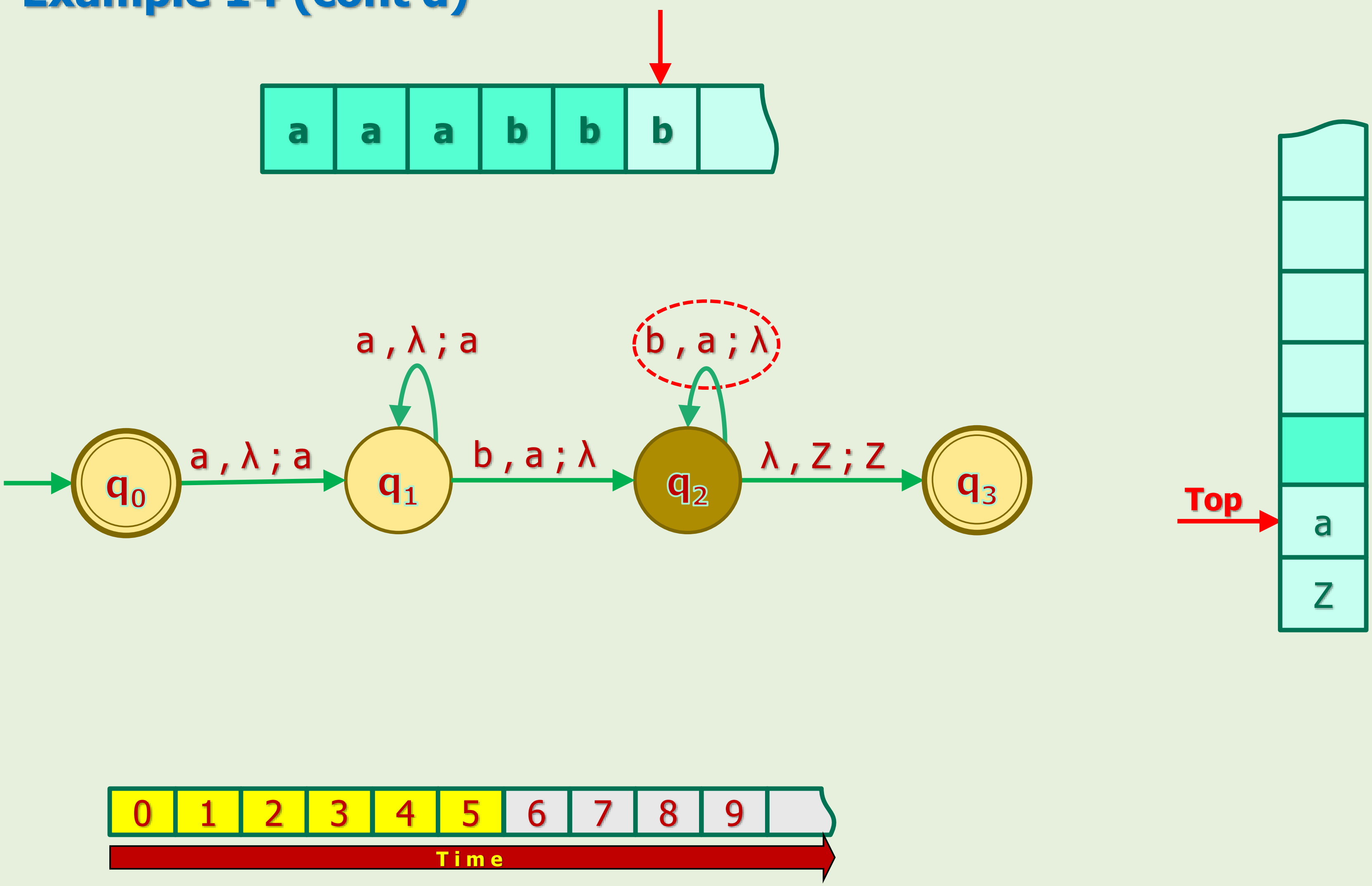
5. PDAs in Action

Example 14 (cont'd)



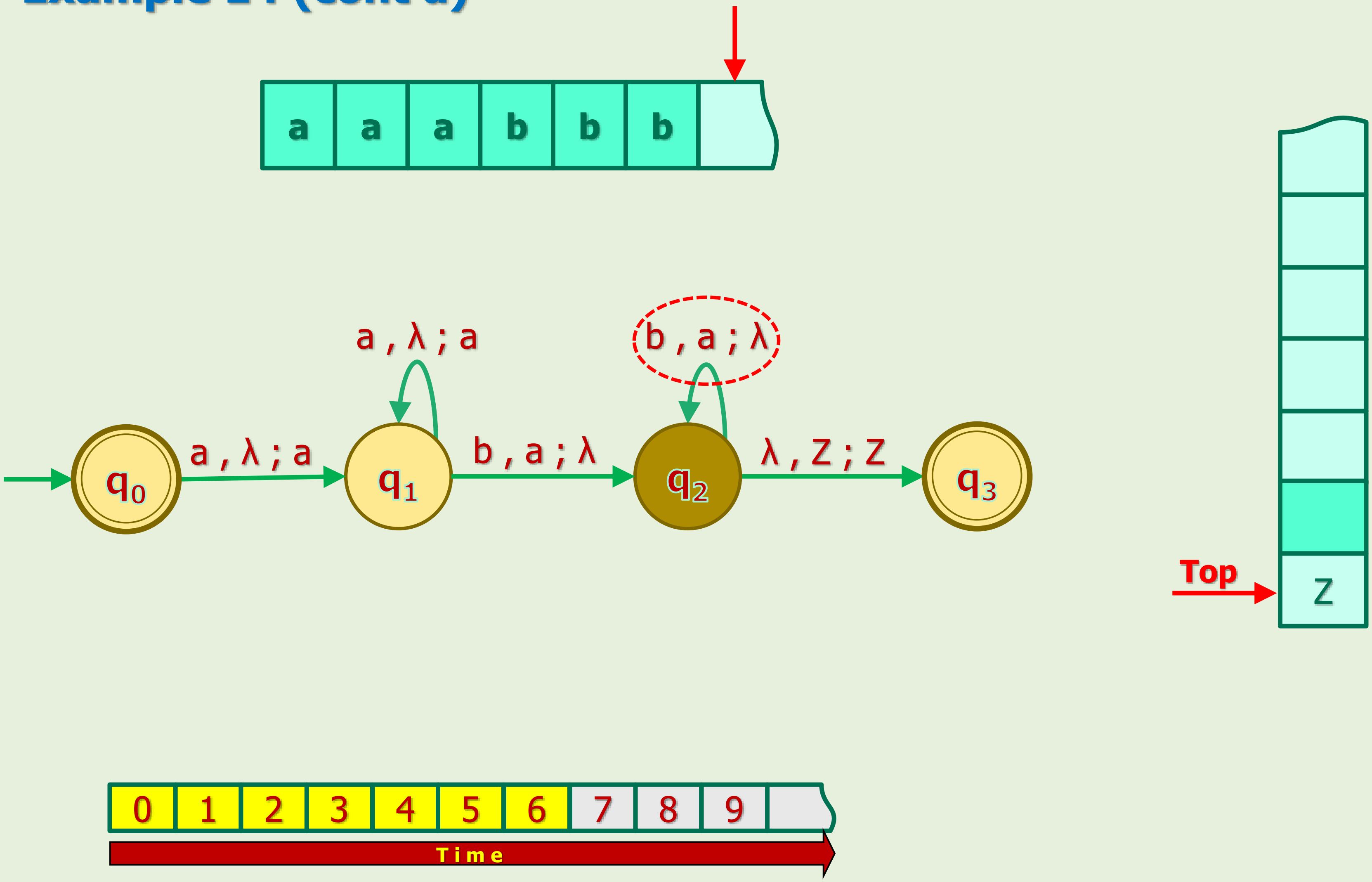
5. PDAs in Action

Example 14 (cont'd)



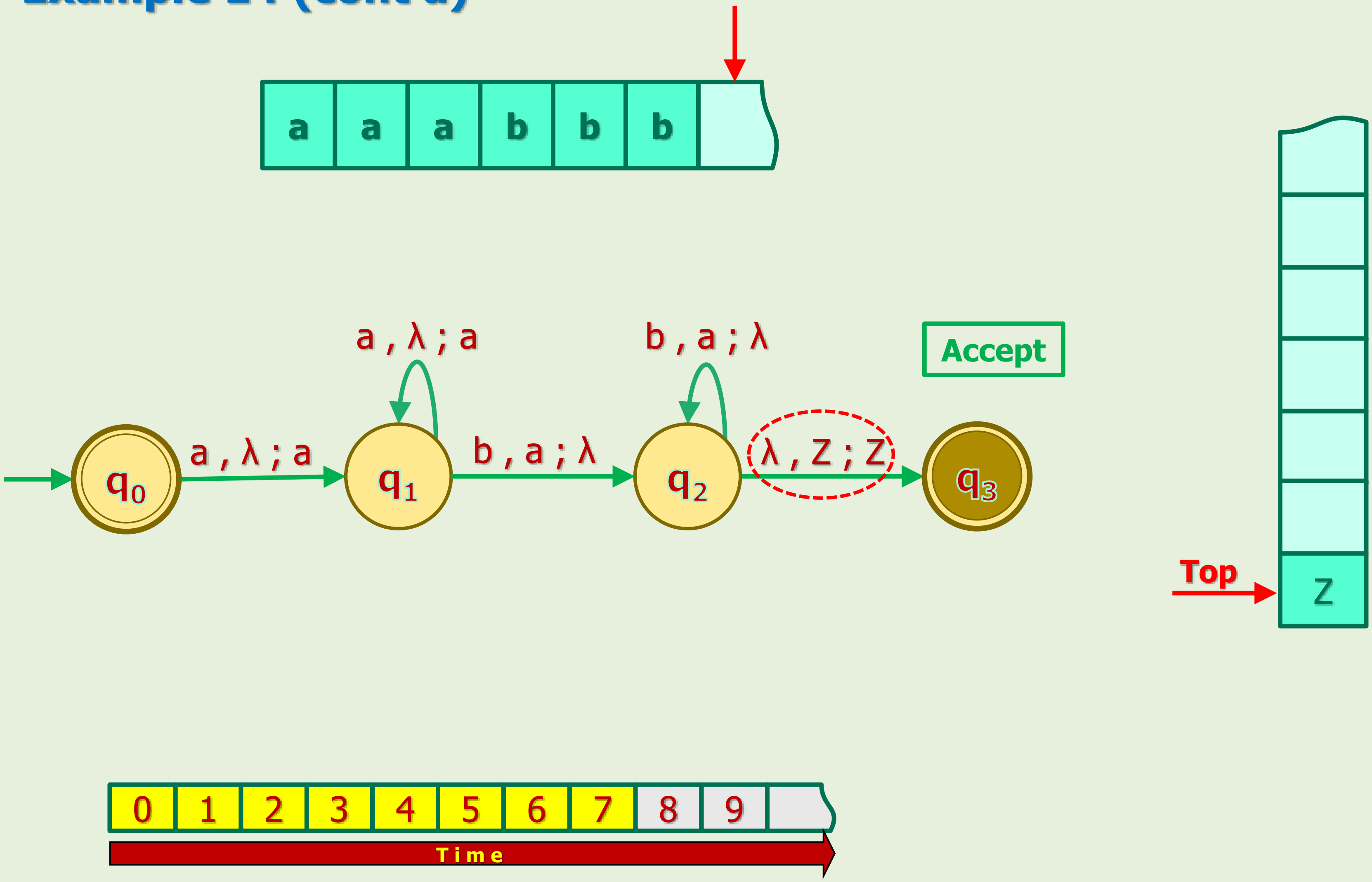
5. PDAs in Action

Example 14 (cont'd)



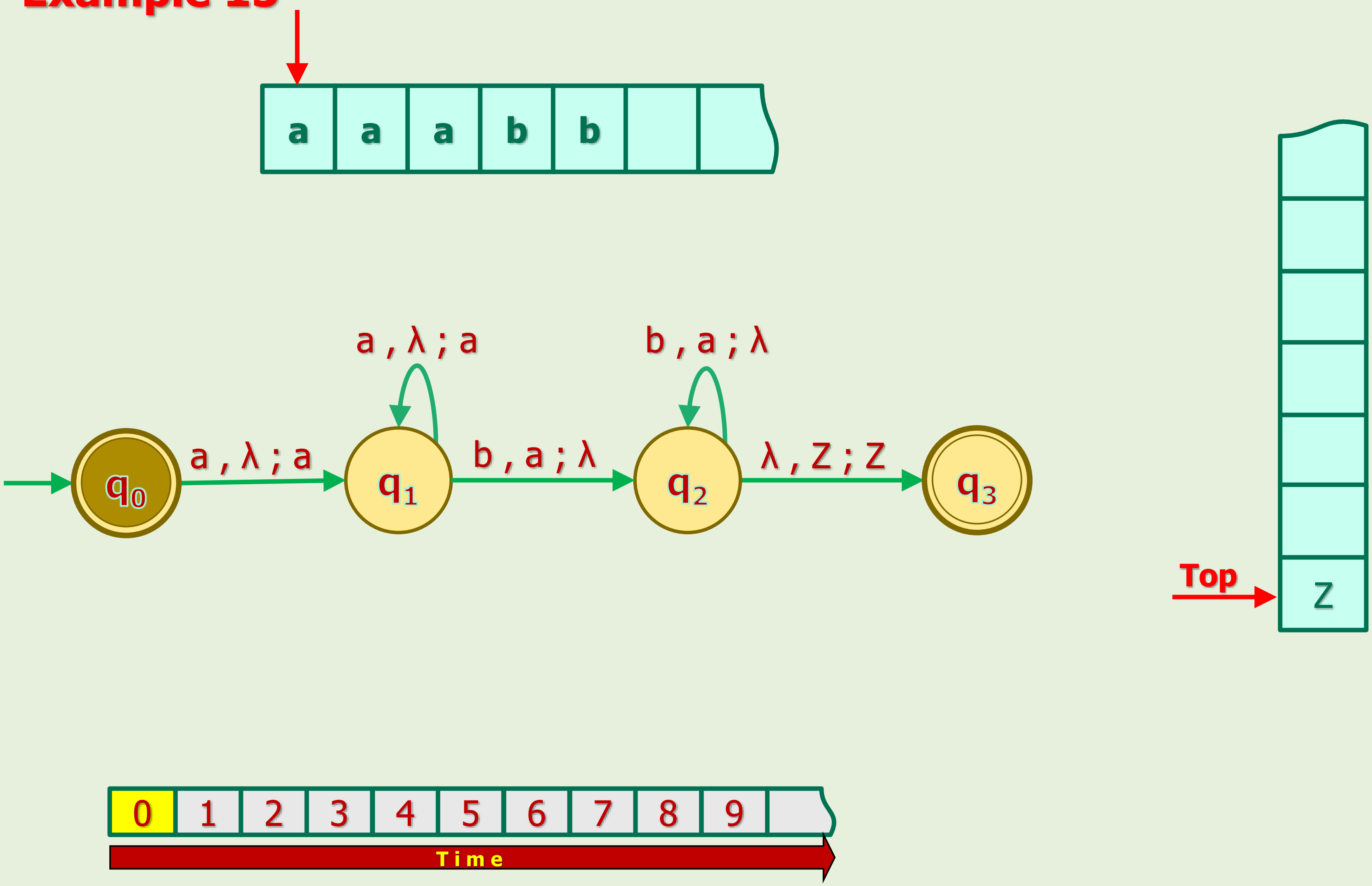
5. PDAs in Action

Example 14 (cont'd)



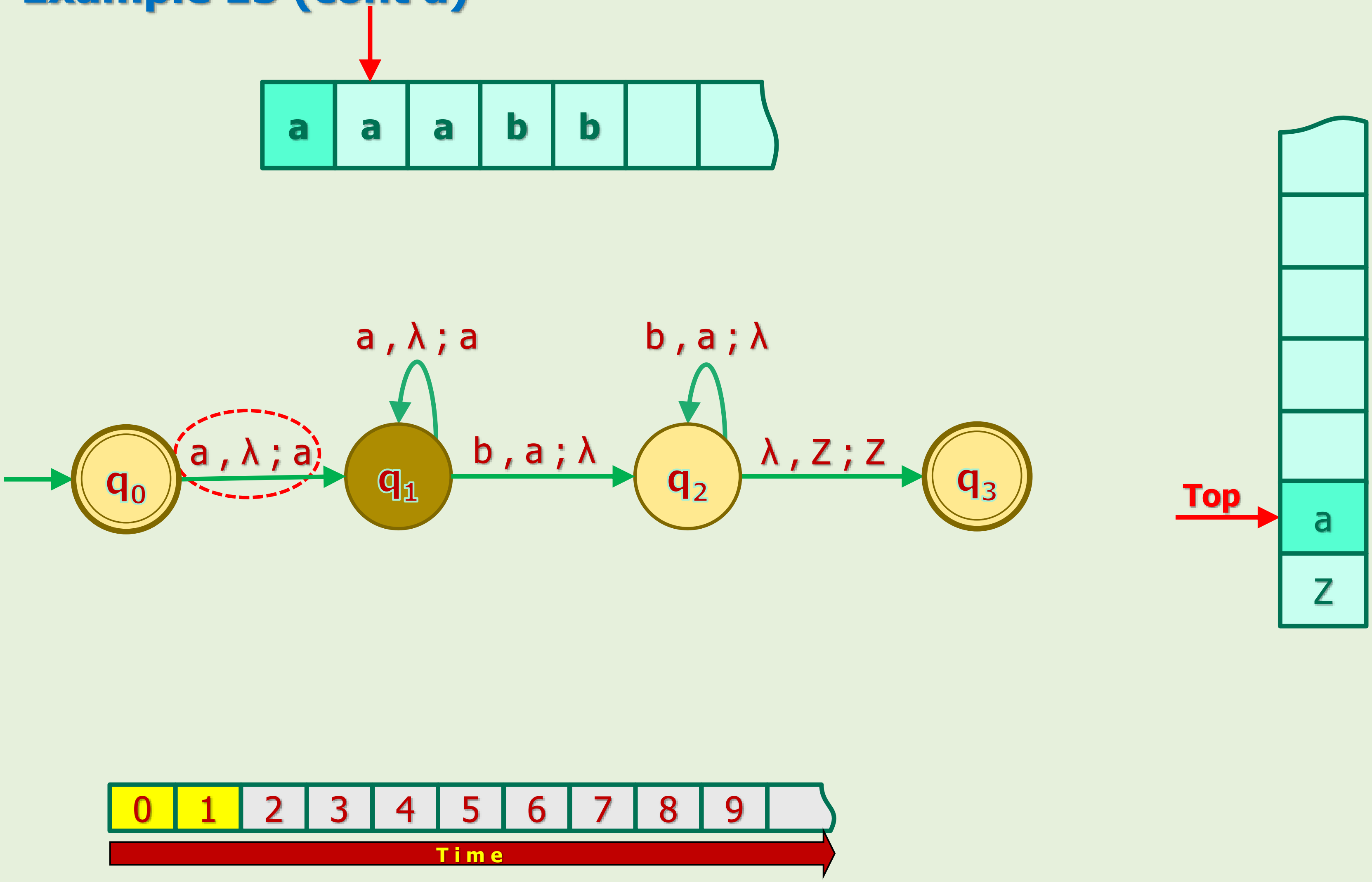
5. PDAs in Action

Example 15



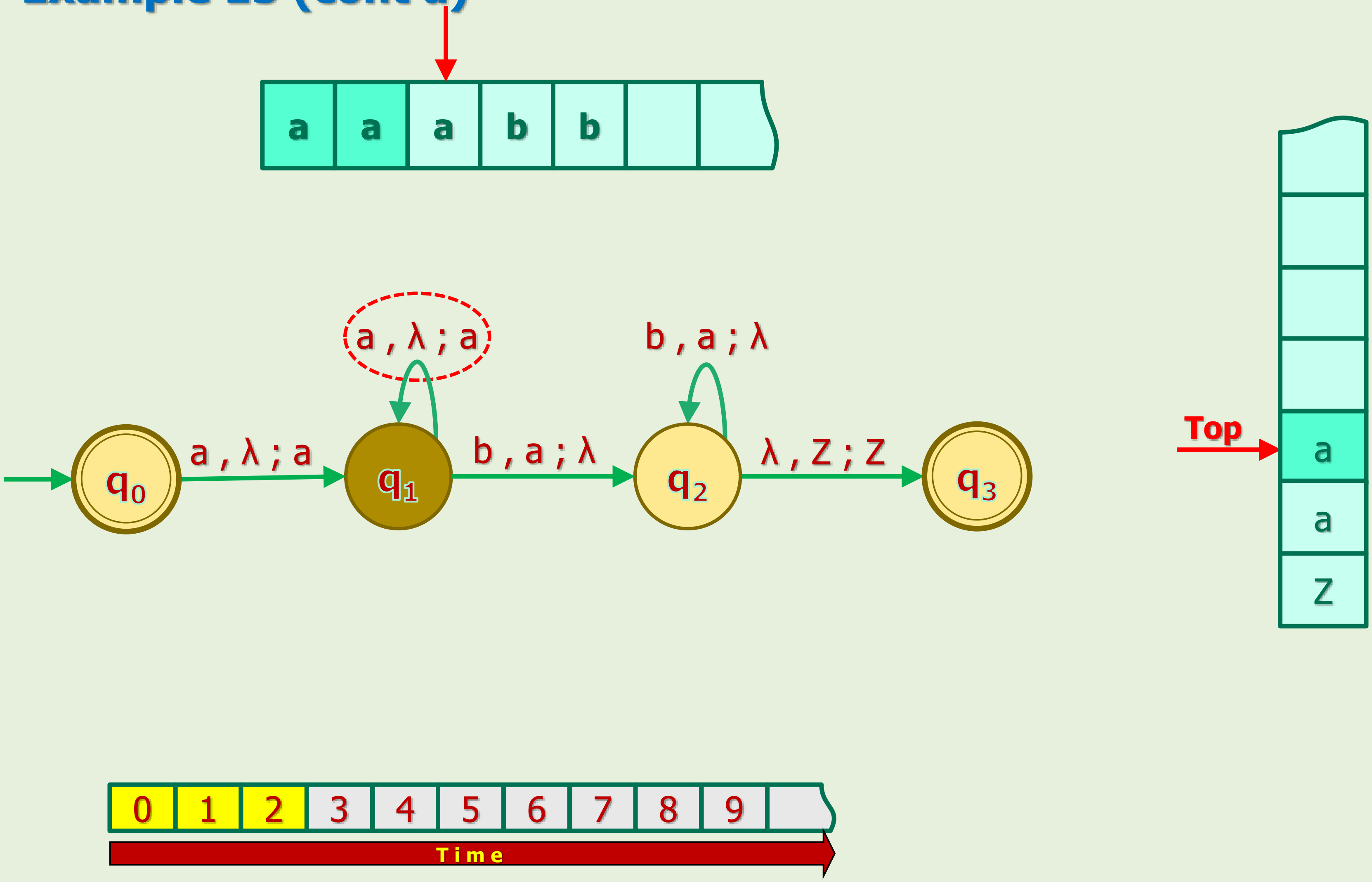
5. PDAs in Action

Example 15 (cont'd)



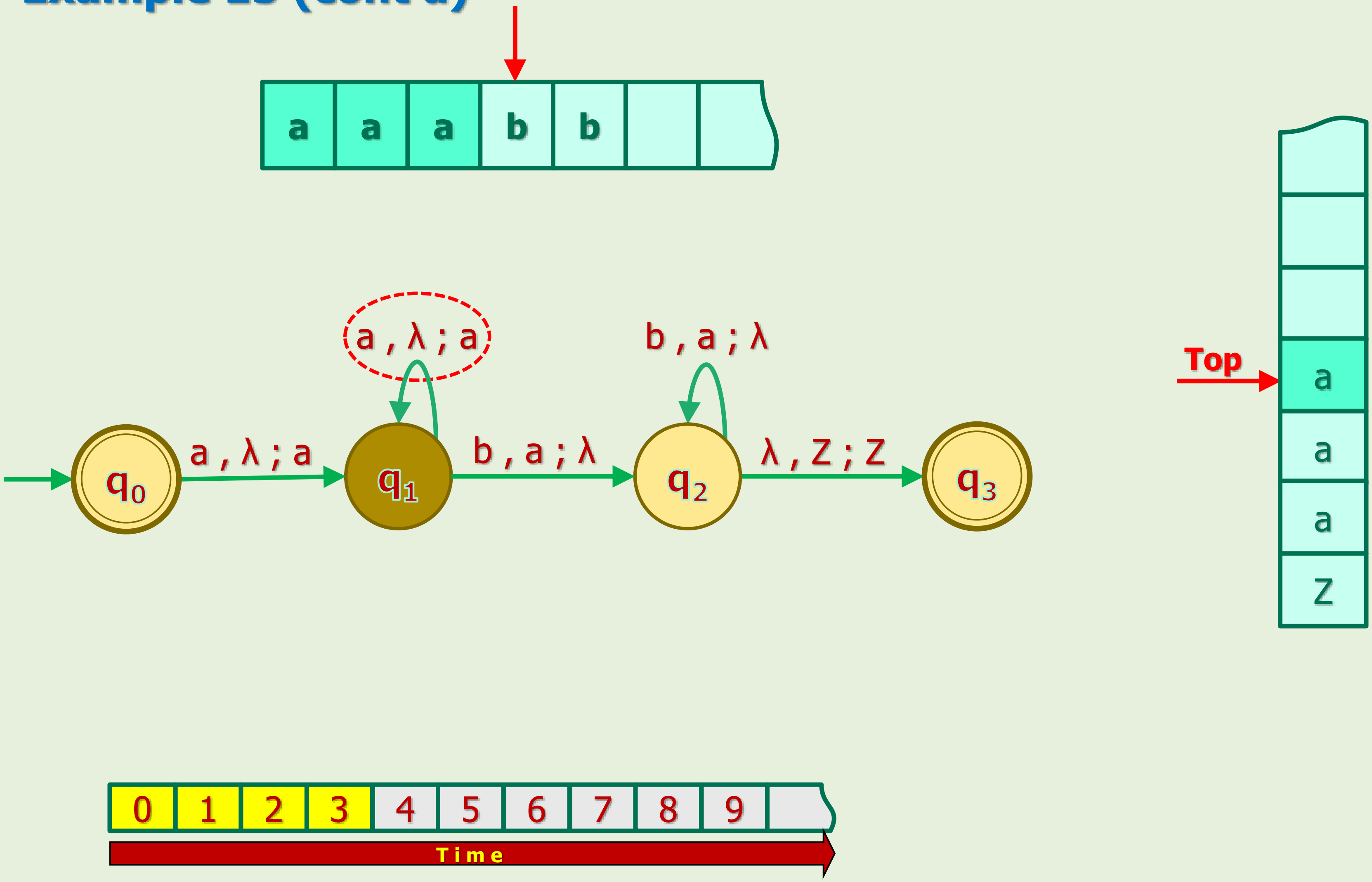
5. PDAs in Action

Example 15 (cont'd)



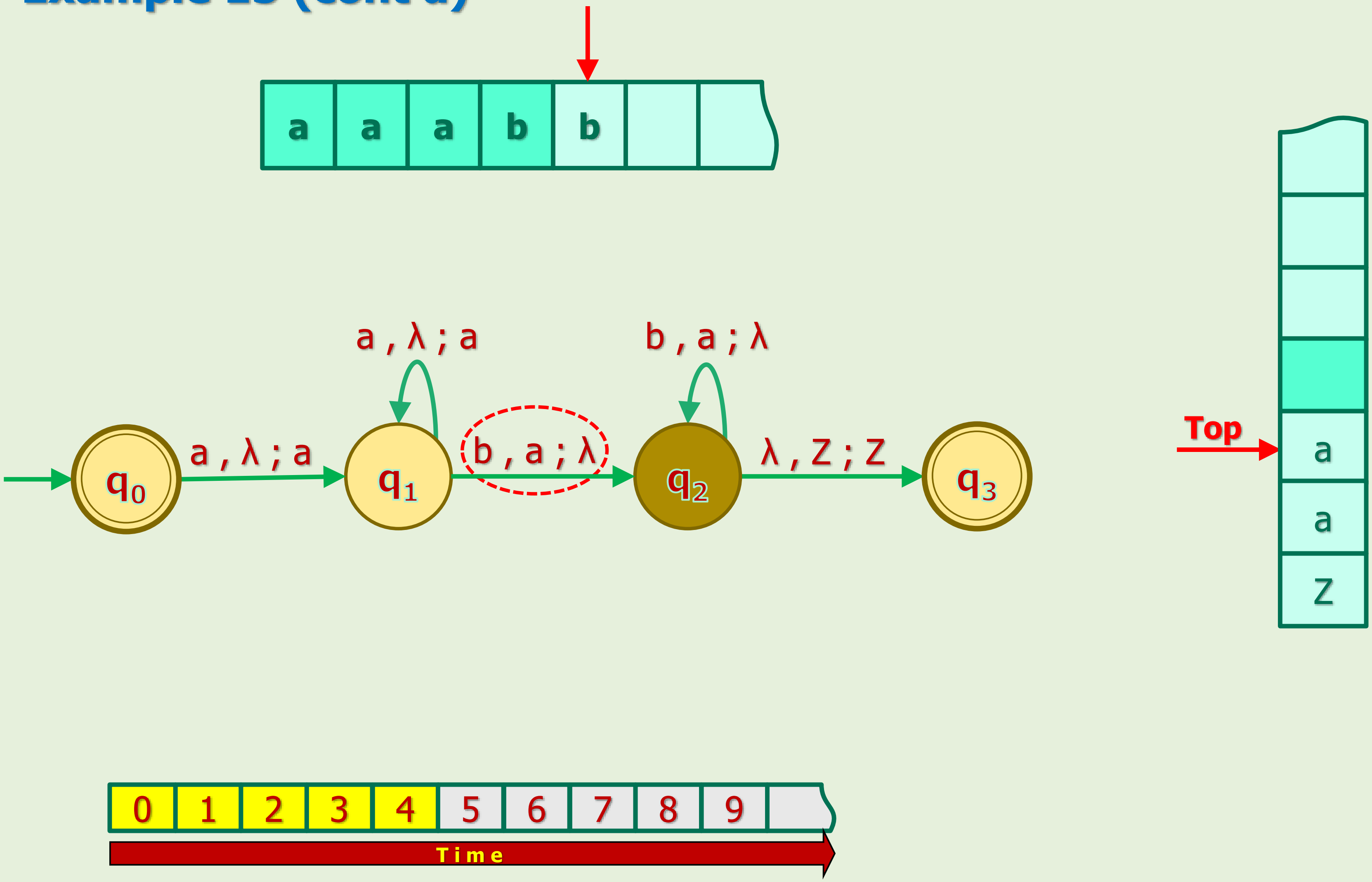
5. PDAs in Action

Example 15 (cont'd)



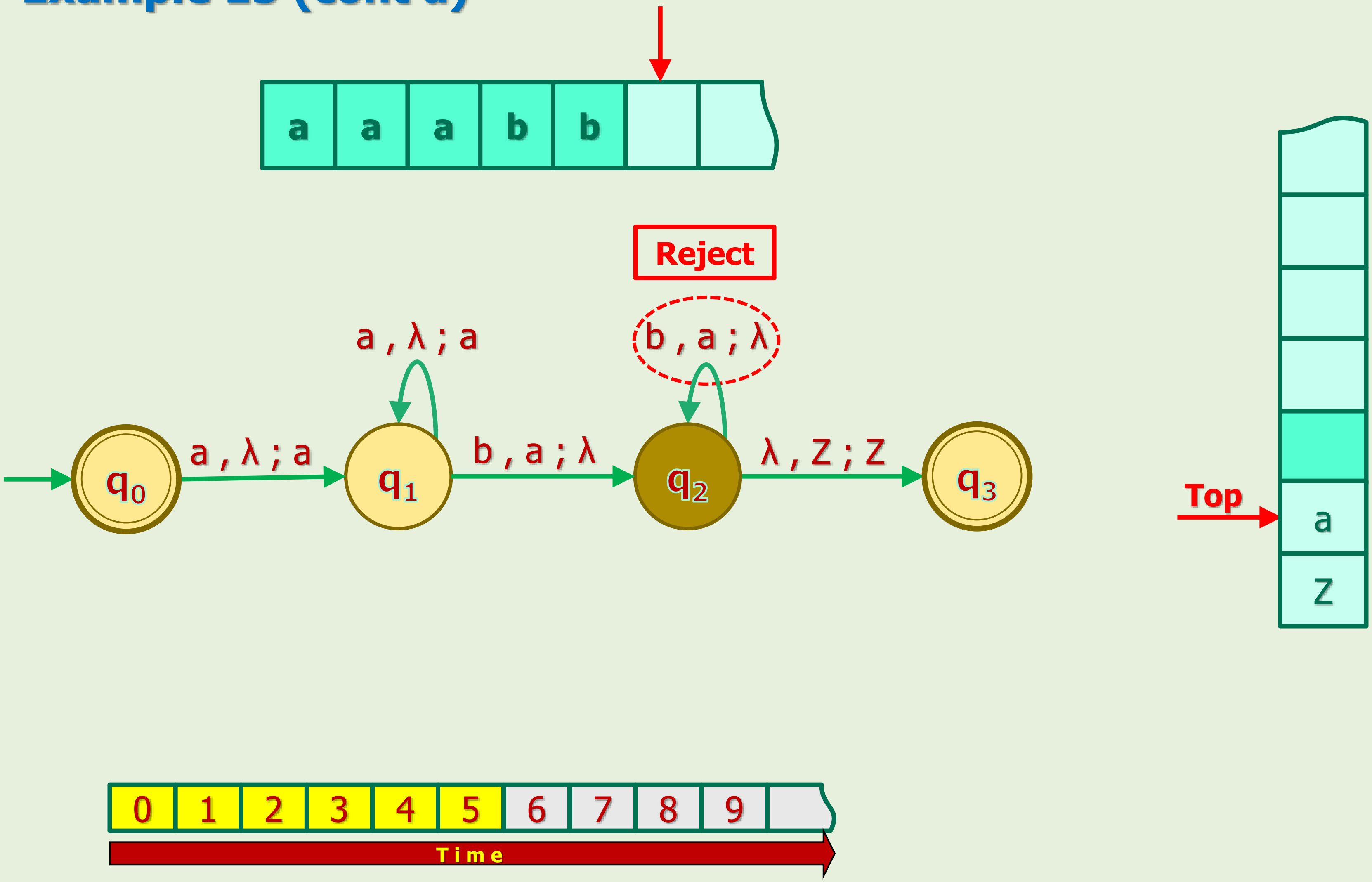
5. PDAs in Action

Example 15 (cont'd)



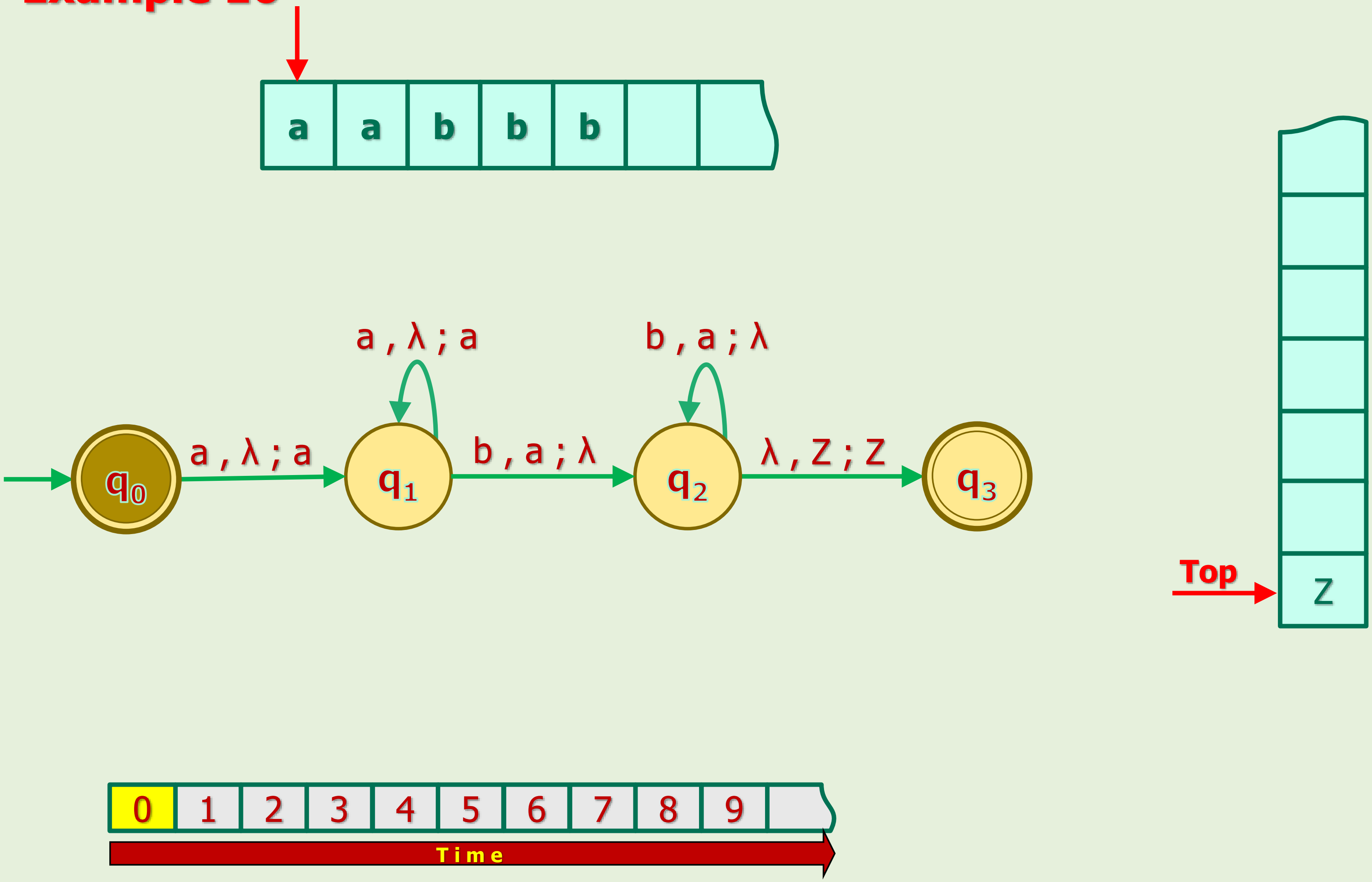
5. PDAs in Action

Example 15 (cont'd)



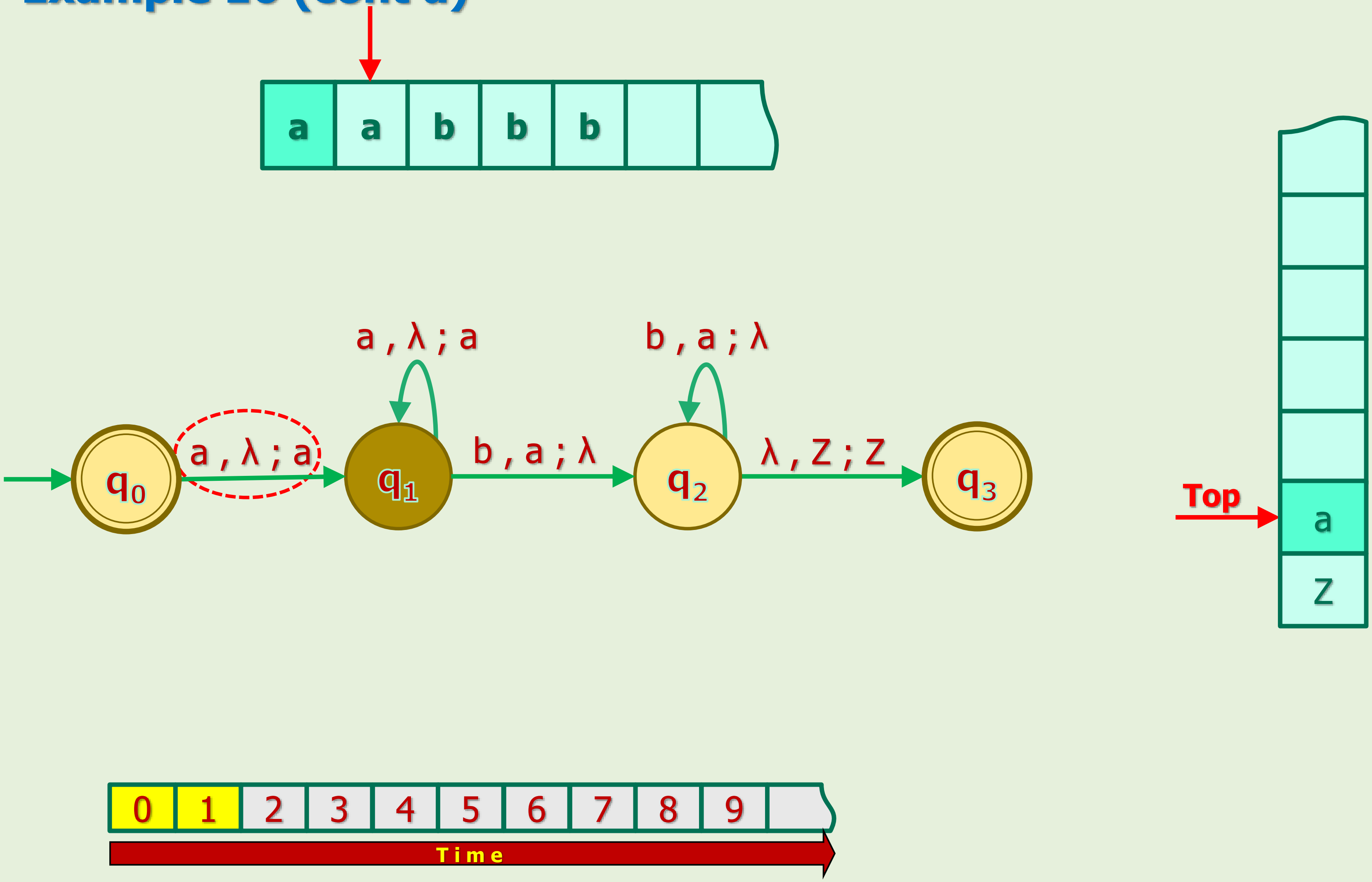
5. PDAs in Action

Example 16



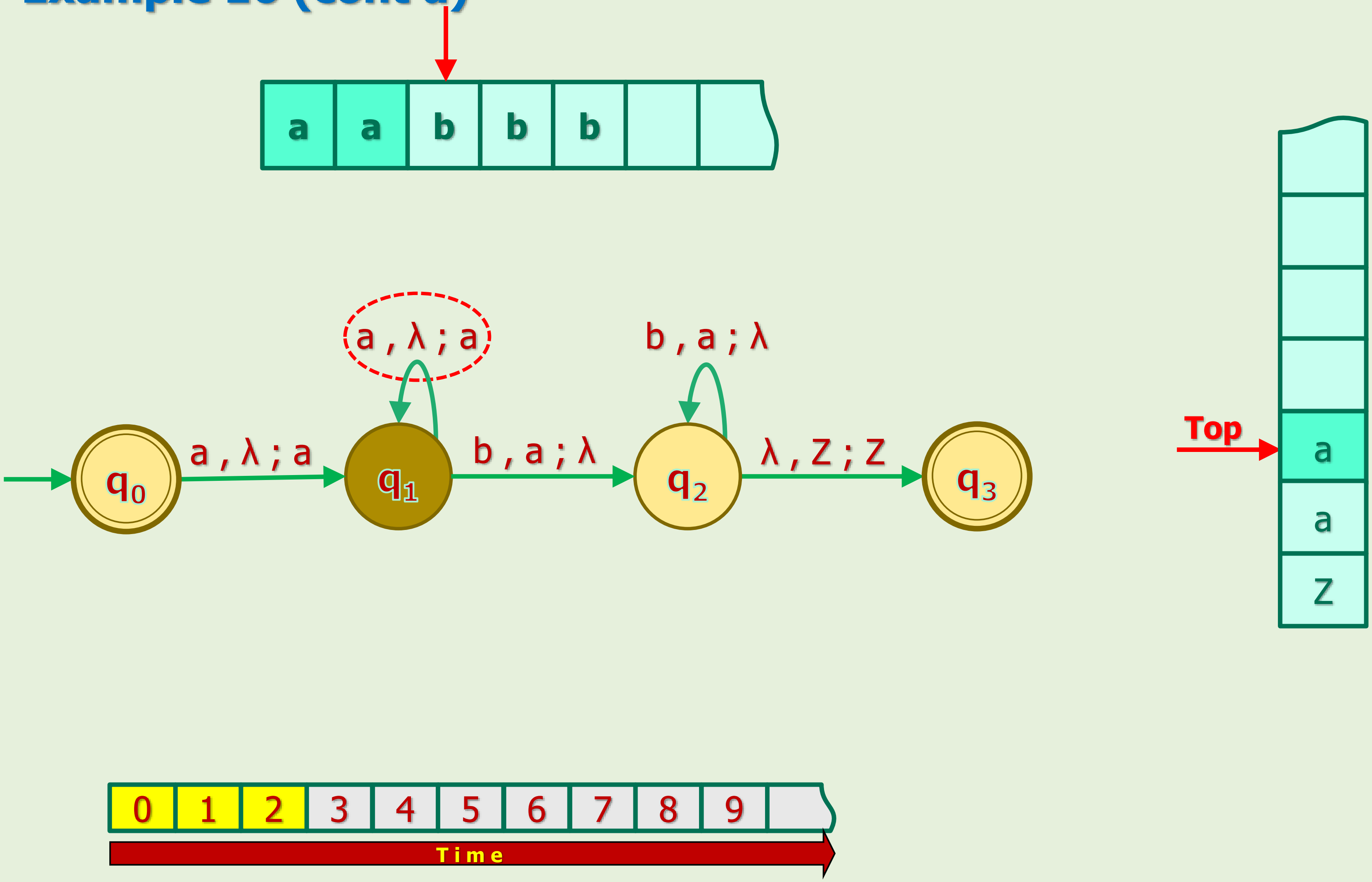
5. PDAs in Action

Example 16 (cont'd)



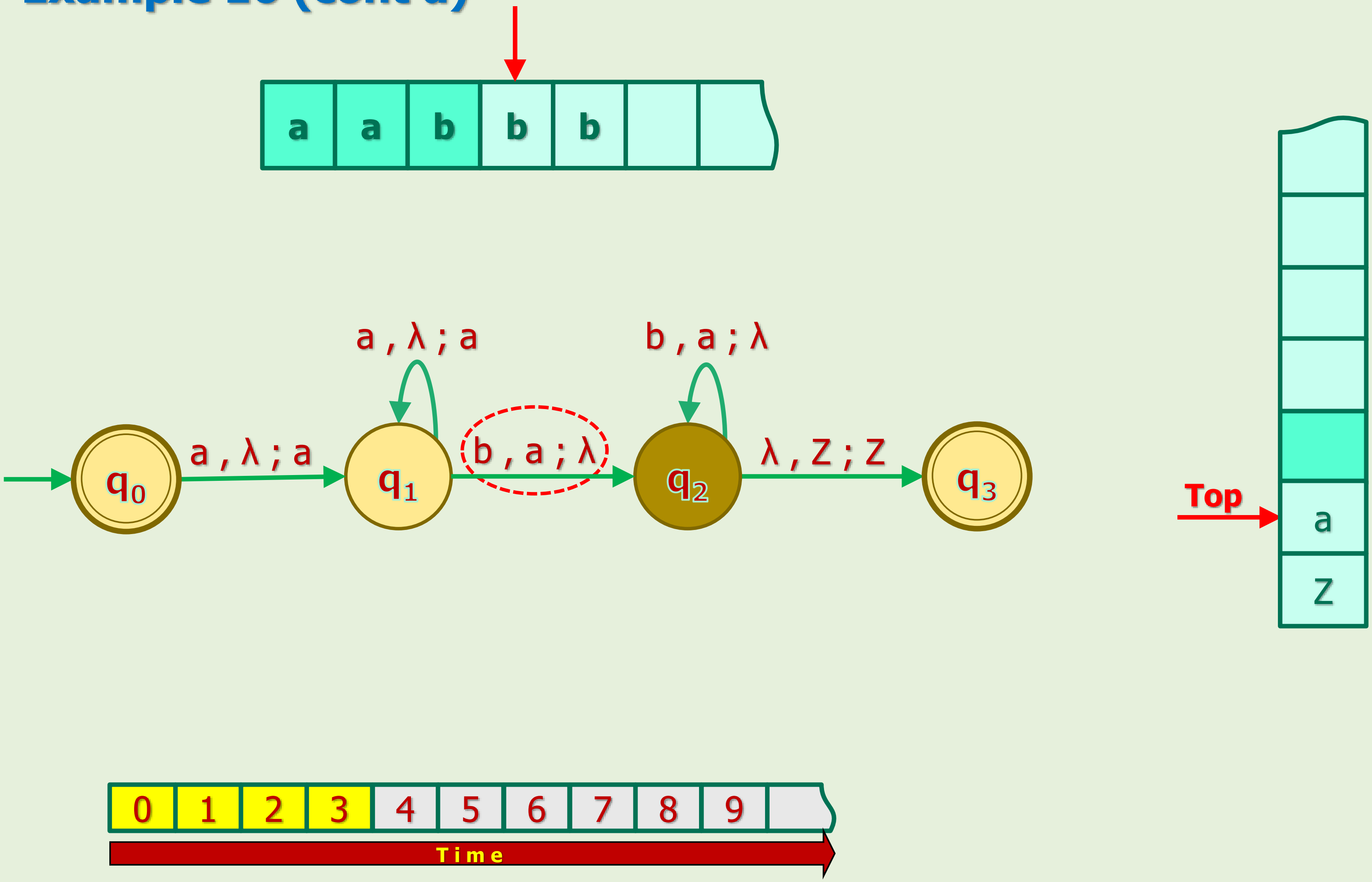
5. PDAs in Action

Example 16 (cont'd)



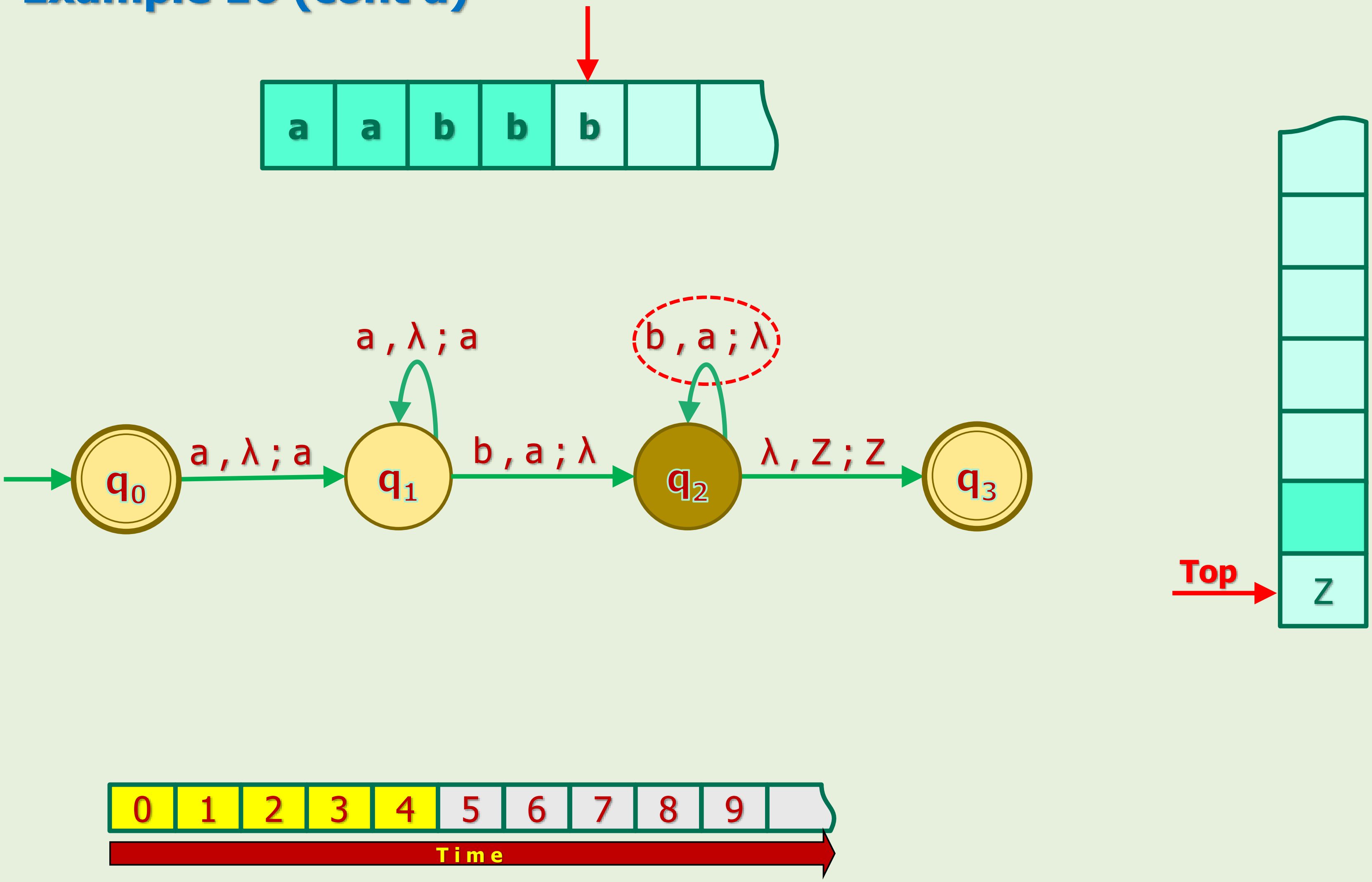
5. PDAs in Action

Example 16 (cont'd)



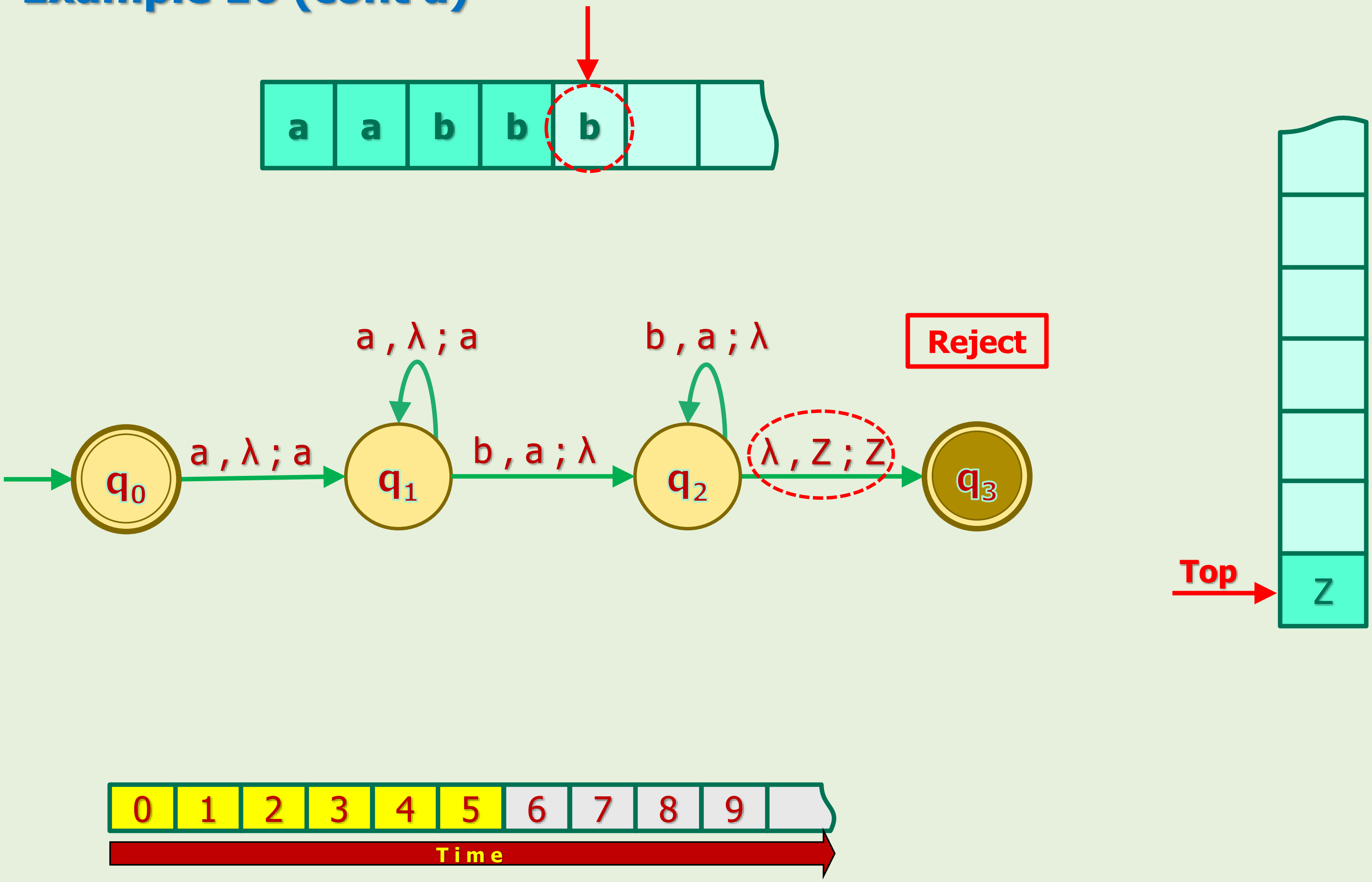
5. PDAs in Action

Example 16 (cont'd)

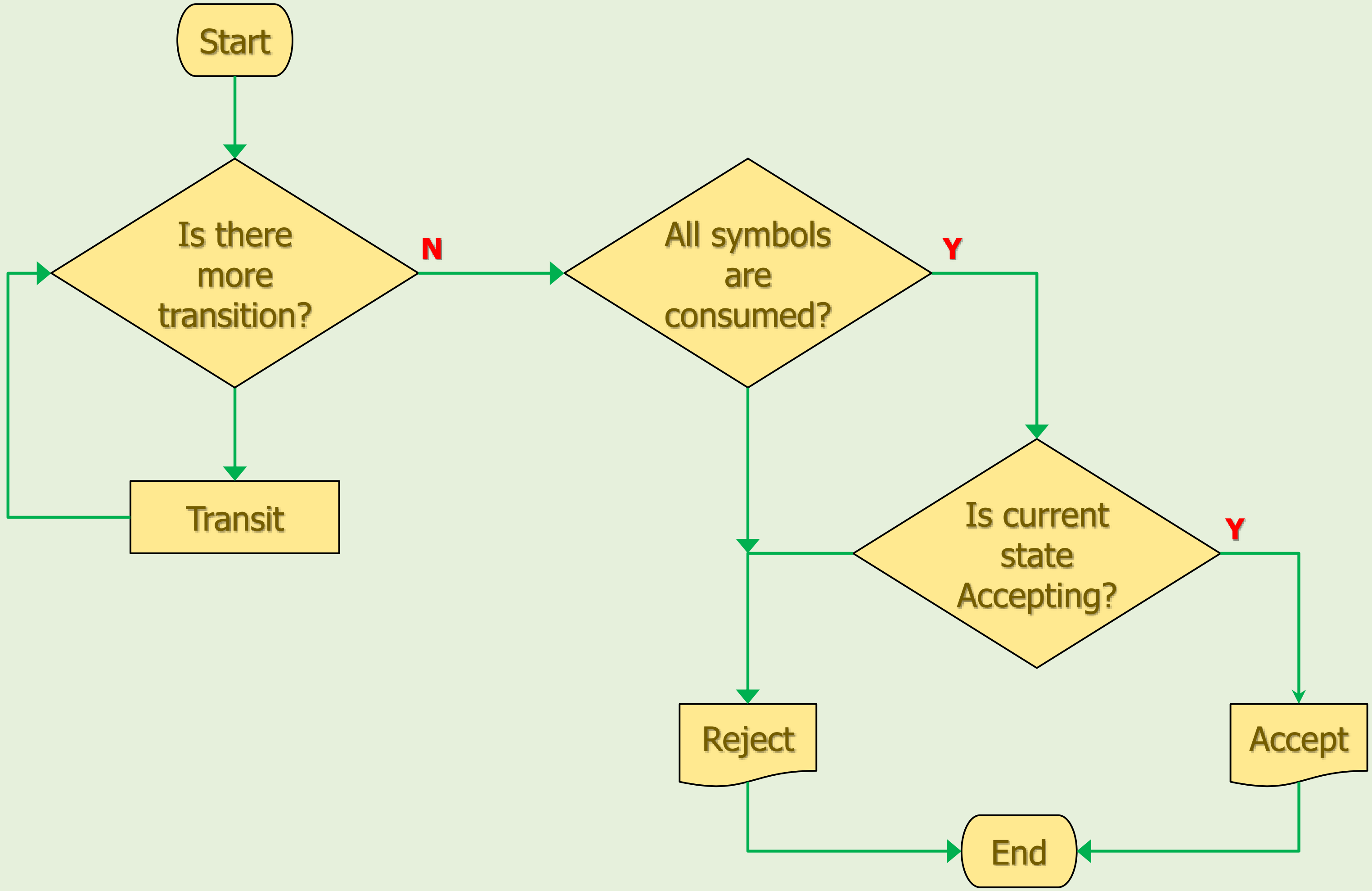


5. PDAs in Action

Example 16 (cont'd)



DPDAs Operation Flowchart



Nondeterministic PDAs (NPDAs)

Nondeterministic PDAs

Determinism:
During any timeframe, there is no more than one transition.

Any violation of this makes a machine nondeterministic.

Recap

- What could be those violations?
  1. λ-transition
  2. When δ is multifunction

- Let's explain each one in detail!

λ-Transitions

λ-transition in automata theory:
The machine can unconditionally transit.

If we put λ in the condition places, we make a λ-transition.

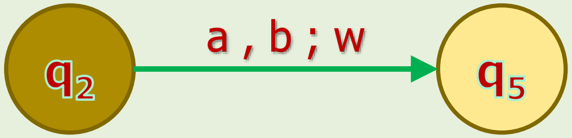
Recap

- This is our knowledge so far:

Automata Class	Transition Condition
DFA/NFA	Input Symbol
NPDA	Input Symbol + Top of stack

NPDAs λ-Transitions

- For example, in the following transition, conditions for transition are:



input symbol = 'a'
AND
top of the stack = 'b'

- So, if we put λ in the conditions parts, we make a λ-transition.

NPDAs λ-Transitions

Definition

- For NPDAs, a transition is called λ-transition if both input and pop parts of the label are λ.

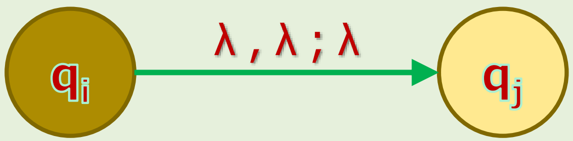
Note

- If the machine initiates a new process starting from q_j, then after replicating the current configuration, it should initialize the stack by pushing w.

NPDAs λ-Transitions

- Moreover, w is a string and can be λ.

- So, the λ-transition that is usually used in practice is:

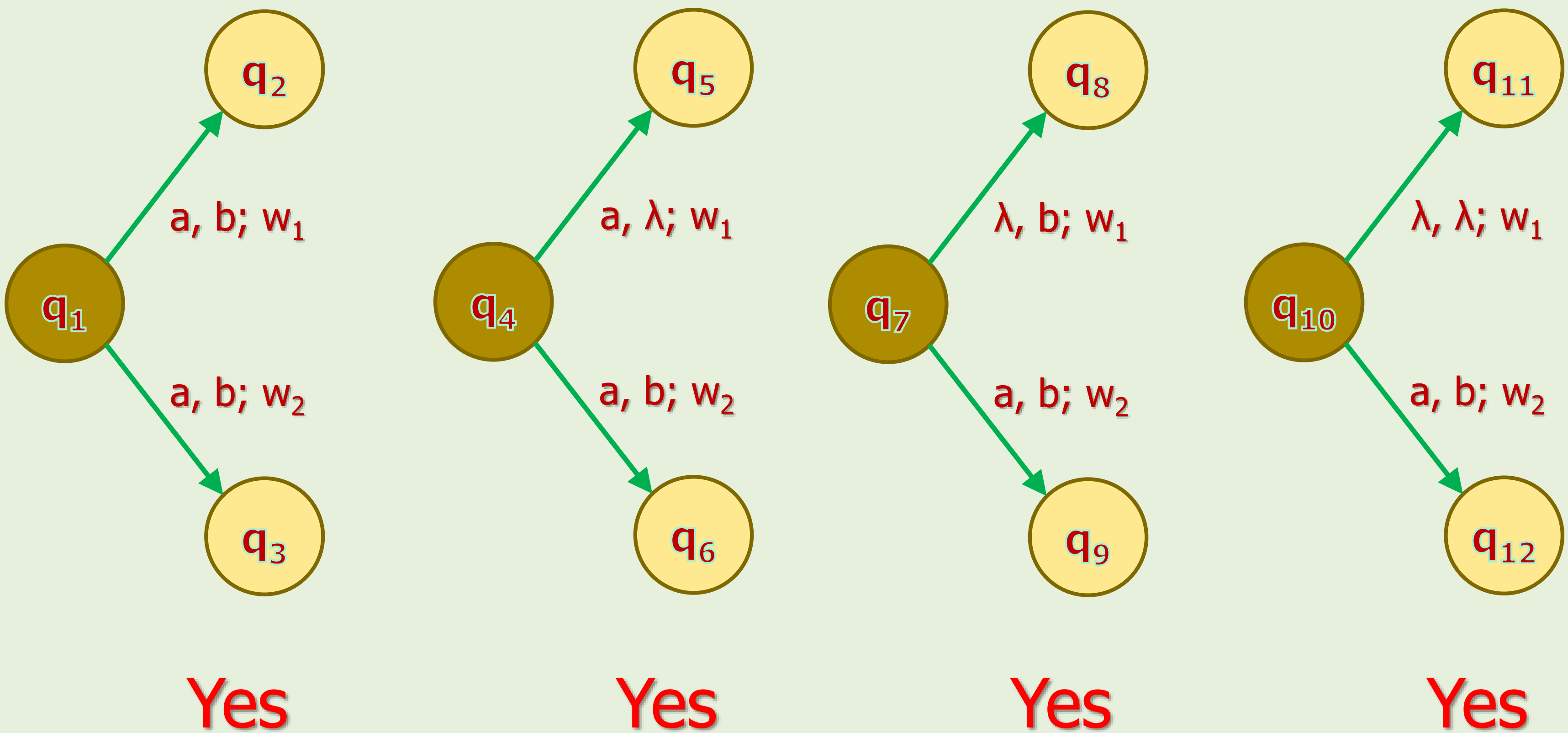


- In this case, the machine does not need to do anything if it transits to q_j.

NPDAs: Multifunction Examples

Example 17

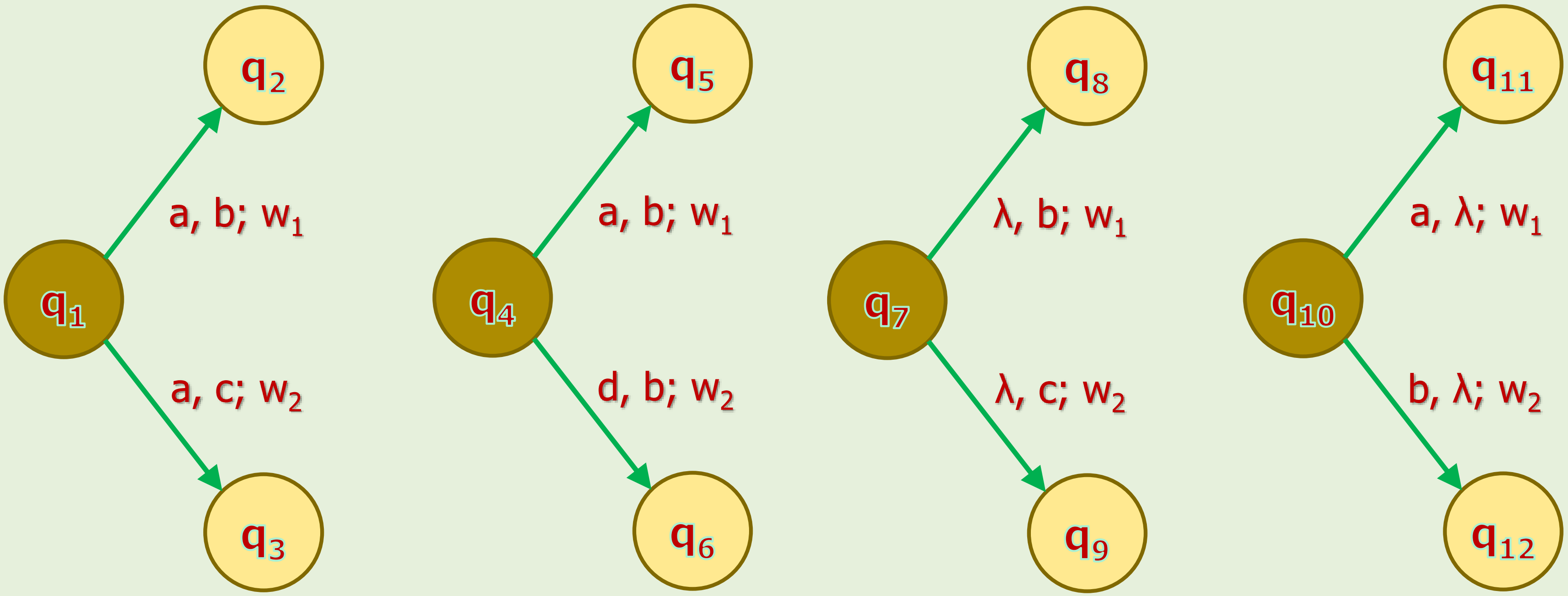
- Are the following transitions violations for determinism?



NPDAs: Multifunction Examples

Example 17 (cont'd)

- Are the following transitions violations for determinism?

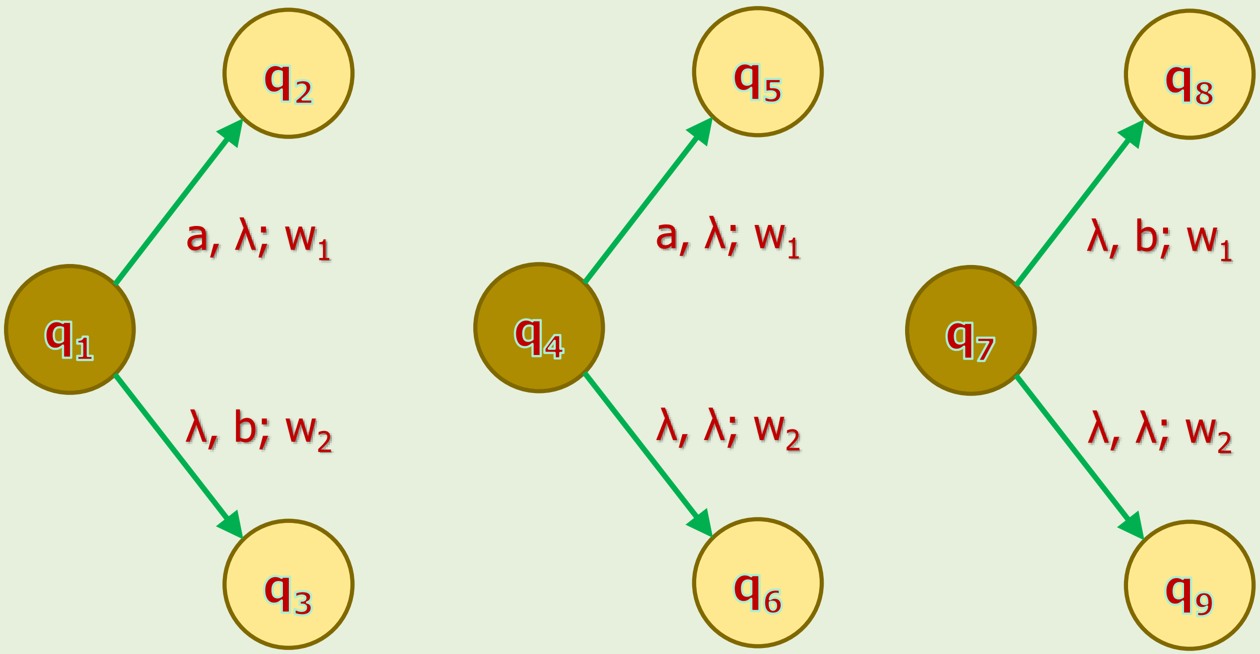


No No No No

NPDAs: Multifunction Examples

Example 17 (cont'd)

- Are the following transitions violations for determinism?



Yes Yes Yes

How NPDAs Behave If They Have Multiple Choices

- We already know the following rule for all types of automata:

  All types of nondeterministic machines start
  parallel processing when they have multiple choices.

- In other words, for every possible choice,
  they create a new process and every process independently
  continues processing the string.

- The procedure of initiating new processes is exactly the same as NFAs.

How NPDAs Behave If They Have Multiple Choices

Procedure of Initiating New Processes

1. It replicates its entire structure
   (transition graph + input tape + stack)
2. It initializes the new process with the current configuration.
3. The new process independently continues processing
   the rest of the input string.

- The only thing we need to know is:

What info do we need for the configuration?

NPDAs' Configuration

1. Current state of the transition graph
2. Input string + Position of the cursor
3. The stack and its content

4.4. How NPDAs Accept/Reject Strings

- We already know that a process of NPDAs accepts a string iff:

$$(h \wedge c \wedge f) \leftrightarrow a$$

- But what if we have multiple processes?
- The rule is the same as NFAs':

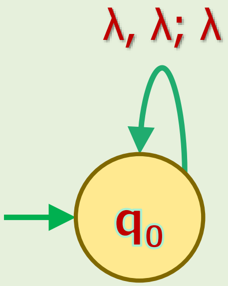
Overall, NPDAs accept a string when at least one process recognize it.

- And for rejection:

Overall, NPDAs reject a string when all processes reject it.

An Interesting NPDA!

- Consider the following NPDA:



- What does it do when we input a string?
- What does it do when we input λ?
- Change the push part to a string like 'b' and answer the above questions.
- Check your answer with JFLAP.

Homework: PDAs Design

- Design a PDA for each of the following languages:
  1. $L = \{a^n b^{2n} : n \geq 0\}$ over $\Sigma = \{a, b\}$
  2. $L = \{a^n b^m c^{n+m} : n \geq 1, m \geq 1\}$ over $\Sigma = \{a, b, c\}$
  3. $L = \{ww^R : w \in \{a, b\}^*\}$
  4. $L = \{w \in \{a, b\}^* : n_a(w) = n_b(w)\}$ //number of a's = number of b's
  5. $L = \{w : n_a(w) > n_b(w)\}$ //number of a's > number of b's
  6. $L = \{1^n+1^m=1^{n+m} : n \geq 1, m \geq 1\}$ over $\Sigma = \{1, +, =\}$ (Unary addition)

6. Definitions

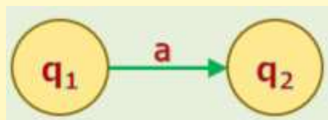
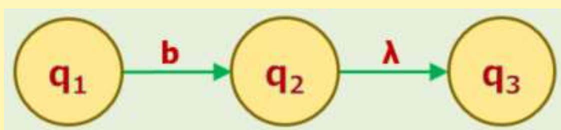
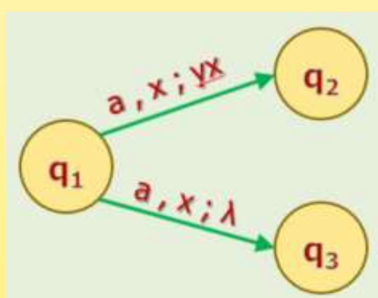
NPDAs Transition Function

- In this section, we are going to formally (mathematically) define the NPDAs.
- As usual, the transition function is the important part of this definition?

- So, let's take some examples on transition functions.
- And try to figure out what the transition functions look like.

- Note that NPDAs' definition is more general than DPDAs'.
- In other words, we can use NPDAs' definition to describe DPDAs.

Transition Function: DFAs, NFAs, NPDAs

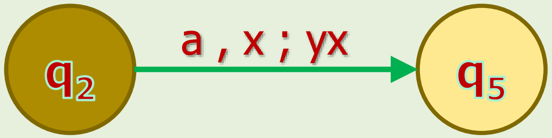
Class	Transition	Sub-Rule Example Transition Function
DFAs		$\delta\ (q_1, a) = q_2$ $\delta : Q \times \Sigma \rightarrow Q$
NFAs		$\delta\ (q_1, b) = \{q_2, q_3\}$ $\delta\ (q_2, a) = \{\ \}$ $\delta : Q \times \Sigma \rightarrow 2^Q$
NPDAs		$\delta\ (q_1\ , a\ , x) = ???$ δ: ???

NPDAs Transition Function: Examples

Example 18

- Write the sub-rule of the following transition over $\Sigma = \{a, b\}$

Solution



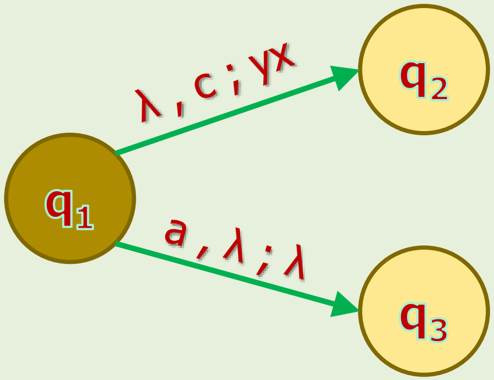
$$\delta\,(q_2\,,\,a\,,\,x) = \{(q_5\,,\,yx)\}$$

$$\delta\,(q_2\,,\,b\,,\,x) = \{\ \}$$

Example 19

- Write the sub-rules of the following transition $\Sigma = \{a, b\}$

Solution



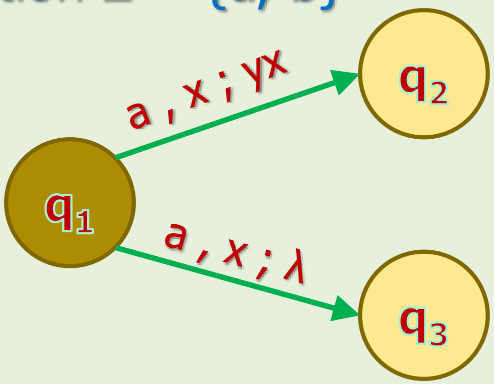
$$\delta\,(q_1, \lambda, c) = \{(q_2, yx)\}$$

$$\delta\,(q_1, a, \lambda) = \{(q_3, \lambda)\}$$

NPDAs Transition Function: Examples

Example 20

- Write the sub-rule of the following transition $\Sigma = \{a, b\}$

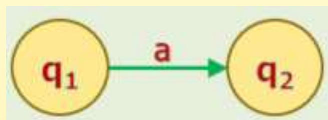
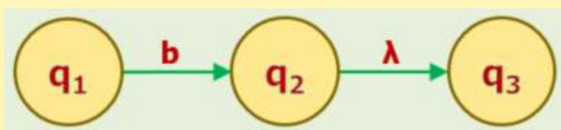
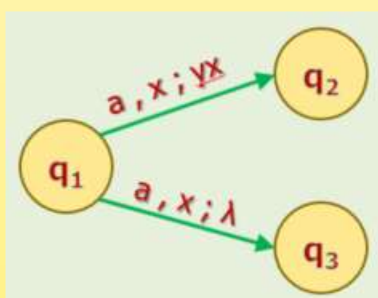


Solution

- The condition for transitions for both edges are the same.
- Therefore, we need only one sub-rule.

$\delta\ (q_1\ ,\ a\ ,\ x) = \{(q_2\ ,\ yx), (q_3\ ,\ \lambda)\}$

Transition Function: DFAs, NFAs, NPDAs

Class	Transition	Sub-Rule Example Transition Function
DFAs		$\delta\ (q_1, a) = q_2$ $\delta : Q \times \Sigma \rightarrow Q$
NFAs		$\delta\ (q_1, b) = \{q_2, q_3\}$ $\delta\ (q_2, a) = \{\ \}$ $\delta : Q \times \Sigma \rightarrow 2^Q$
NPDAs		$\delta\ (q_1\ , a\ , x) = \{(q_2\ , yx), (q_3\ , \lambda)\}$ $\delta: Q \times (\Sigma \cup \{\lambda\}) \times (\Gamma \cup \{\lambda\}) \rightarrow 2^{Q \times \Gamma^*}$

6. Formal Definition of NPDAs

- An NPDA M is defined by the septuple (7-tuple):

$$M = (Q, \Sigma, \Gamma, \delta, q_0, Z, F)$$

- Where:
  - Q is a finite and nonempty set of states of the transition graph.
  - Σ is a finite and nonempty set of symbols called input alphabet.
  - Γ is a finite and nonempty set of symbols called stack alphabet.
  - δ is called transition function and is defined as:

$$\delta: Q \times (\Sigma \cup \{\lambda\}) \times (\Gamma \cup \{\lambda\}) \to 2^{Q \times \Gamma^*}$$

δ is total function.

  - $q_0 \in Q$ is the initial state of the transition graph.
  - $Z \in \Gamma$ is a special symbol called stack start symbol.
  - $F \subseteq Q$ is the set of accepting states of the transition graph.

6. Formal Definition of NPDAs

Set of States

Stack Alphabet

Initial State

Set of Final States

$$M = (Q , \Sigma , \Gamma , \delta , q_0 , Z , F)$$

Input Alphabet

Transition Function

Stack Start Symbol

7. NPDAs vs NFAs

Can NPDAs Simulate NFAs?

- Let's assume that we've constructed an NFA for an arbitrary language L.
- Can we always construct an NPDA for L?
- Yes! Why?
- We should prove that we can always convert an NFA's definition to an NPDA's definition.
- Let's show this through an example first.

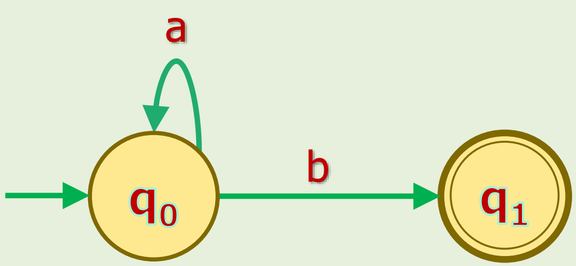
Can NPDAs Simulate NFAs?

Example 21

- Convert the following NFA to an NPDA.

$$\delta: \begin{cases} \delta(q_0, a) = \{q_0\} \\ \delta(q_0, b) = \{q_1\} \end{cases}$$

?



NFA

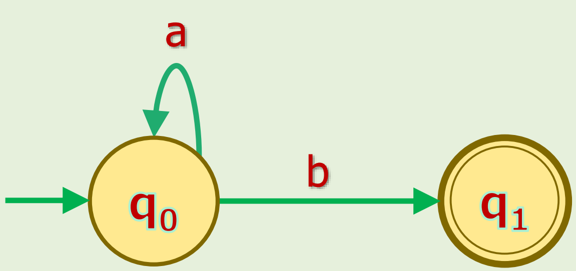
NPDA

Can NPDAs Simulate NFAs?

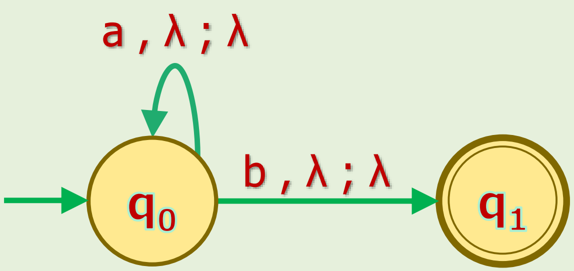
Example 21 (cont'd)

- Convert the following NFA to an NPDA.

$$\delta: \begin{cases} \delta(q_0, a) = \{q_0\} \\ \delta(q_0, b) = \{q_1\} \end{cases} \Rightarrow \delta: \begin{cases} \delta(q_0, a, \lambda) = \{(q_0, \lambda)\} \\ \delta(q_0, b, \lambda) = \{(q_1, \lambda)\} \end{cases}$$



NFA



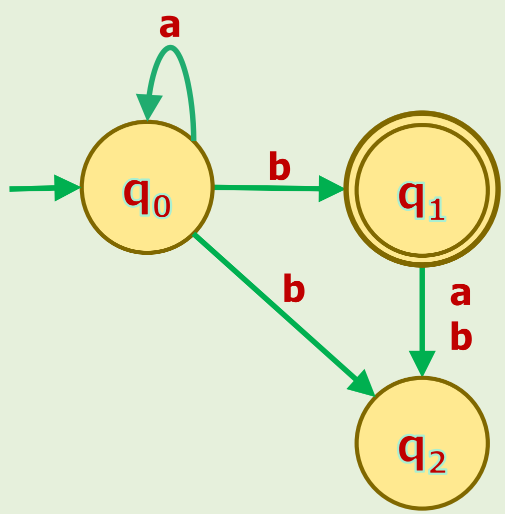
NPDA

Homework

- Convert the following NFA to an NPDA.

$$\delta: \begin{cases} \delta(q_0, a) = \{q_0\} \\ \delta(q_0, b) = \{q_1, q_2\} \\ \delta(q_1, a) = \{q_2\} \\ \delta(q_1, b) = \{q_2\} \end{cases}$$

?



NFA

NPDA

NFAs Can be Converted Directly to NPDAs

	NFA	NPDA
States	$Q = \{q_0, q_1, q_2\}$	$Q = \{q_0, q_1, q_2\}$
Alphabet	$\Sigma = \{a, b\}$	$\Sigma = \{a, b\}$
Stack alphabet	N/A	$\Gamma = \{Z\}$
Sub-rule	$\delta\,(q_i, a) = \{q_j\}$	$\delta\,(q_i\,, x\,, \lambda) = \{(q_j\,, \lambda)\}$
Initial state	q_0	q_0
Stack start symbol	N/A	Z
Final states	$F = \{q_1\}$	$F = \{q_1\}$

Can NPDAs Simulate NFAs?

- As the previous example showed, there is a simple algorithm to convert an NFA to an NPDA.

Algorithm: Converting NFAs' Formal Definition to NPDAs'

- Change all NFAs' sub-rules to NPDAs format by adding λ in the pop and push parts. i.e.:

$$\delta\ (q_i\ ,\ x) = \{q_j\ ,\ q_{j+1}\ ,\ \ldots\ ,\ q_{j+n}\}$$

changes to

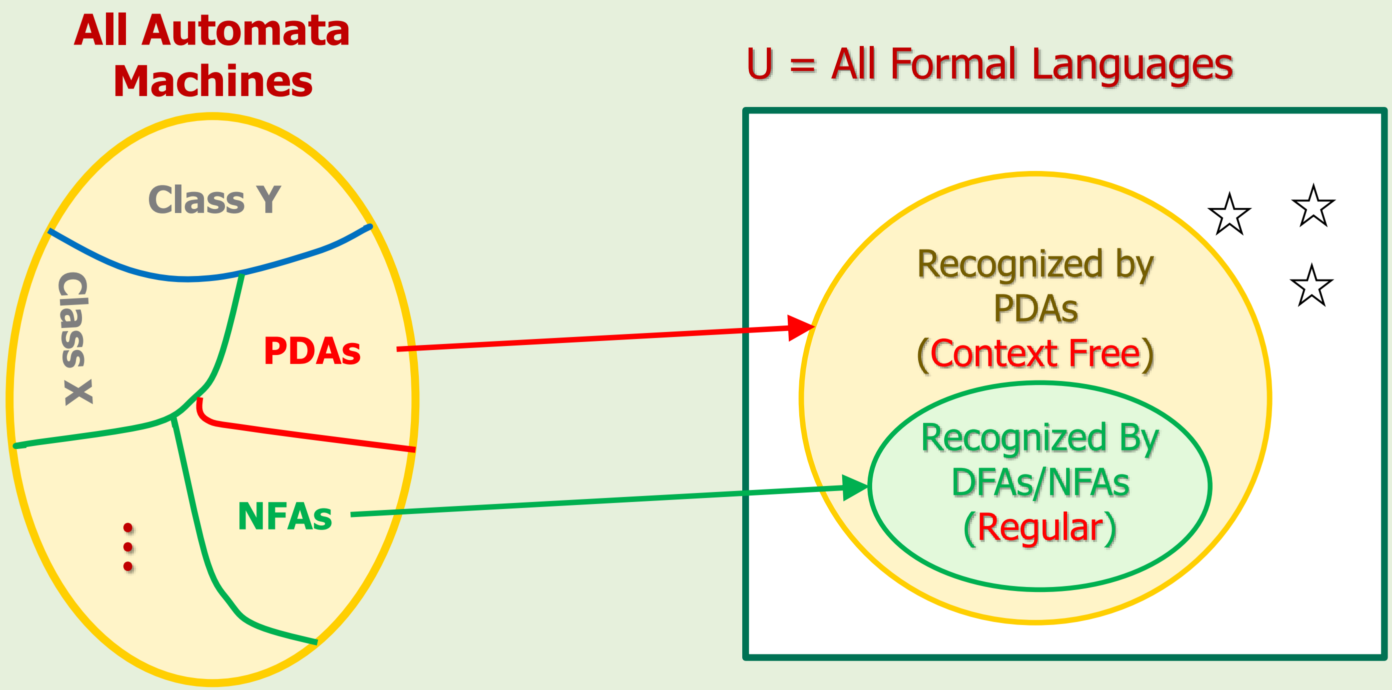
$$\delta\ (q_i\ ,\ x\ ,\ \lambda) = \{(q_j\ ,\ \lambda),\ (q_{j+1}\ ,\ \lambda),\ \ldots\ ,\ (q_{j+n}\ ,\ \lambda)\}$$

- Set $\Gamma = \{Z\}$.
- Set the stack start symbol as Z.
- The rest of the definitions, (i.e. Q, Σ, q_0, F) are the same.

Can NFAs Simulate NPDAs?

- Let's assume that we've constructed an NPDA for an arbitrary language L.
- Can we always construct an NFA for L?
- No! Why?
- There is no way to simulate the stack operations by NFAs!
- It means, there is no way to simulate a read/write memory like stack with a read-only memory.
- Moreover, we know at least the following languages for which we can construct NPDAs but it is impossible to construct NFAs.
  - $L = \{a^n b^n : n \geq 0\}$
  - $L = \{ww^R : w \in \Sigma^*\}$
- Let's summarize our knowledge and figure out what would be the next step.

Machines and Languages Association



- The set of languages that NFAs recognize is a proper subset of the set of languages that PDAs recognize.
  - We'll explain later what the "context free" meaning is.
- So, bottom line, PDAs are more powerful than NFAs.

References

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