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Formal Languages

(Part 2)

Lecture 05

Day 05/31

CS 154

Formal Languages and Computability

Fall 2019

Agenda of Day 05

- Rollcall Form and Your List Number
- Summary of Lecture 04
- Quiz 1
- Lecture 05: Teaching ...
  - Formal Languages (Part 2)

Summary of Lecture 04: We learned ...

Alphabets & Strings

- Alphabet is ...
  - ... a nonempty and finite set of symbols, denoted by Σ.
- String is ...
  - ... a finite sequence of symbols from the alphabet.
- Length of string w is ...
  - ... the number of symbols in the string, denoted by $|w|$.
- Empty string is ...
  - ... A string with no symbol, denoted by λ
  - $|\lambda| = 0$

Operations on Strings

- Concatenation of u and v is uv.
  - $\lambda w = w\lambda = w$ (neutral element)
- Reverse of w is denoted by w^R. (easy!)
- Substring (easy!)
- Prefix and Suffix
  - $w = uv$, u=prefix, v=suffix
  - λ is suffix and prefix of every string because: $w = \lambda\, w = w\, \lambda$
- Exponent operator
  - $w^n = w\, w\, w \ldots w$
  - $w\, w^n = w^n\, w = w^{n+1}$
  - $w^0 = \lambda$

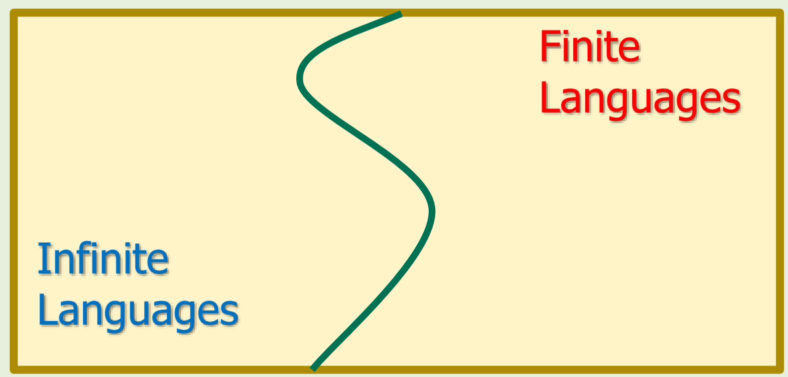
Summary of Lecture 04: We learned ...

Formal Languages

- Star operator: Σ^*
  - The set of all possible strings obtained by concatenating zero or more symbols from Σ.
  - Universal set of all strings over Σ.
- Plus operator: Σ^+
  - The set of all possible strings obtained by concatenating one or more symbols from Σ.
  - $\Sigma^+ = \Sigma^* - \{\lambda\}$
  - $\Sigma^* = \Sigma^+ \cup \{\lambda\}$
- Formal language is ...
  - ... any subset of Σ^*
- Special cases:
  - $\{\ \}$ and $\{\lambda\}$

- Formal languages are sets, so, they have all sets properties.
- Formal languages can be finite or infinite.
- This is the first categorization of languages:

U = All Formal Languages



Any question?

NAME	Alan M. Turing		
SUBJECT	CS 154	TEST NO.	1
DATE	09/05/2019	PERIOD	1 / 2 / 3

TEST RECORD	
PART 1	123
PART 2	
TOTAL	

Your list # goes here!

Take-Home Exam!

Quiz 1

Use Scantron

Formal Languages **Exercises**

Example 21

Given the following languages by set-builder over $\Sigma = \{a, b\}$.

Represent them by using roster method (enumerate the strings):

1. $L_1 = \{a^n b^n : n \geq 0\}$

   This is our celebrity language!

2. $L_2 = \{a^n b^{2n} : n \geq 0\}$
3. $L_3 = \{a^{n+2} b^n : n \geq 0\}$
4. $L_4 = \{a^n b^m : n \geq 0, m \geq 0\}$

Objective of This Lecture

- Last time, we started our journey with "Formal Languages"!
  - We introduced alphabets.
  - We defined strings as mathematical objects.
  - We defined some operations on strings.
  - And finally, we introduced formal languages.

- Now, we have new objects: "Languages"
- So, we need to define some operations on this new objects.
- After that, we put on new glasses through which we look at the strings and languages.
- And finally, we'll examine some familiar sets through our new glasses!

Operations on Languages

Regular Set Operations

- Since languages are sets,
  we can apply all regular set operations on them.

Union

$\{a, aa, ab\} \cup \{a, ab, bbb, bba, b\} = \{a, aa, ab, bbb, bba, b\}$

Intersection

$\{a, aa, ab\} \cap \{a, ab, bbb, bba, b\} = \{a, ab\}$

Minus

$\{a, aa, ab\} - \{a, ab, bbb, bba, b\} = \{aa\}$

- Note that the result of operations on languages is always a language.

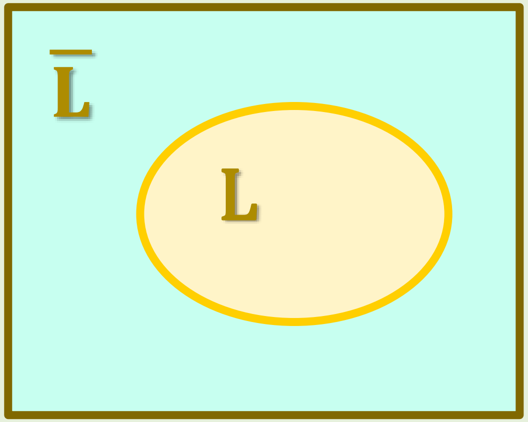
Complement of Languages

Definition

- Let L be a language over Σ.
- Complement of L, denoted by $\overline{L}$, is defined as:

$$\overline{L} = U - L = \Sigma^* - L$$

$U = \Sigma^*$



Complement of Languages

Example 22

Let $L = \{\lambda, b, aa, aab\}$ over $\Sigma = \{a, b\}$; $\overline{L} = ?$

Solution

$\overline{L} = \Sigma^* - L$

$\Sigma^* = \{\cancel{\lambda}, a, \cancel{b}, \cancel{aa}, ab, ba, bb, aaa, \cancel{aab}, \ldots\}$

$\overline{L} = \{\lambda, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$

$= \{a, ab, ba, bb, aaa, \ldots\}$

Homework

- Given following languages over $\Sigma = \{a, b\}$,
  a. Represent L by set builder
  b. Represent $\overline{L}$ by set builder

1. Set of all strings that contains at least one a
2. Set of all strings that contains more than one a
3. Set of all strings that contains exactly one a

Reverse of Languages

Definition

- Reverse of language L is obtained by reversing all of its strings.
- It is denoted by L^R. (pronounced L-reverse)
- Set builder definition:

$$L^R = \{w : w^R \in L\}$$

Example 23

Let $L = \{b, ab, aab, abab\}$; $L^R = ?$

$L^R = \{b, ba, baa, baba\}$

Example 24

Let $L = \{a^n b^n : n \geq 0\}$; $L^R = ?$

$L^R = \{b^n a^n : n \geq 0\}$

Concatenation of Languages

Definition

- Let L_1 and L_2 be two languages over Σ_1 and Σ_2.
- The concatenation of L_1 and L_2, denoted by L_1L_2, is a language defined as:

$$L = L_1L_2 = \{xy : x \in L_1, y \in L_2\} \text{ over } \Sigma = \Sigma_1 \cup \Sigma_2$$

Example 25

Let $L_1 = \{a, ab\}$ over $\Sigma_1 = \{a, b\}$ and
$L_2 = \{c, ca, caa\}$ over $\Sigma_2 = \{a, c\}$; $L_1L_2 = ?$

$L = L_1L_2 = \{a, ab\} \{c, ca, caa\}$

$= \{ac, aca, acaa, abc, abca, abcaa\}$

Over $\Sigma = \Sigma_1 \cup \Sigma_2 = \{a, b\} \cup \{a, c\} = \{a, b, c\}$

Concatenation of Languages: Notes

1. The concatenation of two languages looks like Cartesian product of two sets.
   - Instead of ordered-pair, we concatenate two strings.
2. $\phi L = L \phi = \phi$ (prove it!)
   - ϕ has the same role as 0 (zero) for multiplication.

3. $\{\lambda\} L = L \{\lambda\} = L$
   - $\{\lambda\}$ is the neutral language for concatenation operation.
   - $\{\lambda\}$ has the same role as number 1 (one) for multiplication.
4. Properties of concatenation:

$$L (L_1 \cup L_2) = L L_1 \cup L L_2$$
$$(L_1 \cup L_2) L = L_1 L \cup L_2 L$$
$$L (L_1 \cap L_2) = L L_1 \cap L L_2$$
$$(L_1 \cap L_2) L = L_1 L \cap L_2 L$$

Exponential Operator

Definition

- Let L be a language and n be a natural number.
- L^n is defined as concatenation of n copies of L's.

$$L^n = \underbrace{L\ L\ L\ \ldots\ L}_{n \text{ times}}$$

Example 26

Let $L = \{a, ab\}$; $L^2 = ?$; $L^3 = ?$

$L^2 = \{a, ab\}\ \{a, ab\}$

$= \{aa, aab, aba, abab\}$

$L^3 = L\ L^2 = \{a, ab\}\ \{aa, aab, aba, abab\}$

$= \{aaa, aaab, aaba, aabab, abaa, abaab, ababa, ababab\}$

Exponential Operator

- In general $L\, L^n = L^n\, L = L^{n+1}$
  where $n \in \mathbb{N}$ (natural numbers).

Example 27

Let $L = \{a^n b^n : n \geq 0\}$; $L^2 = ?$

$L^2 = \{a^n b^n\, a^m b^m : n \geq 0, m \geq 0\}$

- Note that n and m are independent.
- For example abaabb (n=1, m=2) belongs to L^2.
- What would be $L^3 = ?$

Homework

Special cases

- $L^0 = ?$
- $L^0 = \{\lambda\}$ (prove it!)
- $\phi^0 = ?$

Homework

- Enumerate at least 5 elements of the following languages:

1. $L_1 = \{w \in \{a, b\}^+\}$
2. $L_2 = \{w \in \{a, b\}^+ : |w| = 2k, K \geq 0\}$
3. $L_3 = \{w \in \{a, b\}^+ : |w| = 2k+1, K \geq 0\}$
4. $L_4 = \{1^{2k} : k \geq 1\}$ over $\Sigma = \{1\}$
5. $L_5 = \{w \in \{a, b\}^+ : n_a(w) = n_b(w)\}$ //number of a's = number of b's
6. $L_6 = \{a^n b^n c^n : n \geq 1\}$
7. $L_7 = \{a^n b^m c^{nm} : n, m \geq 1\}$
8. $L_8 = \{w\#w : w \in \{a, b\}^+\}$; What is the alphabet of L_8?
9. $L_9 = \{w \in \{a, b\}^+ : |w| = 2k+1, K \geq 0, w$ contains at least one a$\}$
10. $L_{10} = \{ww : w \in \{a, b\}^+\}$

Familiar Sets Through Our New Glasses

Objective of This Section

- In this section, first we'll look at some familiar sets through our new glasses.
  - We were familiar with these sets for a long time.

- Then, we'll introduce a new set of numbers called "Unary Numbers".

Familiar Sets Through Our New Glasses

Example 28 : Natural Numbers

Consider the set of natural numbers:

$\mathbb{N} = \{0, 1, 2, 3, 4, ..., 123, ..., 456, ..., 5908764, ...\}$

Can we consider $\mathbb{N}$ as a formal language over $\Sigma = \{0, 1, ..., 9\}$?

Yes, numbers are just sequence of digits (symbols) that can be considered as strings!

Familiar Sets Through Our New Glasses

Example 29 : Binary$^+$ Numbers

Consider the set of positive binary numbers:

B = {0, 1, ..., 1010, ..., 10000001, ..., 111100001, ...}

Can we consider B as a formal language over $\Sigma = \{0, 1\}$?

Yes, for the same reason we saw for natural numbers!

Familiar Sets Through Our New Glasses

- How about the following sets?

Example 30: Prime Numbers

$\Sigma = \{0, 1, 2, ..., 9\}$

$L = \{2, 3, 5, 7, 11, 13, 17, ...\}$

Example 31: Even and Odd Numbers

$\Sigma = \{0, 1, 2, ..., 9\}$

$L_1 = \{0, 2, 4, 6, 8, ...\}$

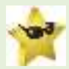
$L_2 = \{1, 3, 5, 7, 9, ...\}$

- Yes, for the same reasons!

Introducing Unary Numbers

Definition

- Let $\Sigma = \{1\}$.
- We define the following set as "unary numbers":

 $A = \{1, 11, 111, 1111, 11111, \ldots\}$

This is our celebrity numbers!

- Equivalent natural numbers: $\{1, 2, 3, 4, 5, \ldots\}$
- How can we represent the unary numbers by set builder?

Unary Numbers: Operations

Example 32: Addition of Unary Numbers

$L = \{1^n+1^m=1^{n+m} : n \geq 1, m \geq 1\}$

Over $\Sigma = \{1, +, =\}$

In computer science,
all data are strings!

Membership: L contains strings like:

1+11=111

11+111=11111

…

Not Membership: L doesn't contain strings like:

1+11=1

11+111=11

Homework

Square of Unary Numbers

$L = \{1^n\#{=}1^k : k = n^2, n \geq 1\}$

Over $\Sigma = \{1, \#, =\}$

Membership: L contains strings like:

??

Not Membership: L doesn't contain strings like:

??

Final Words ...

- This week we introduced formal languages.
- This was our first round of lectures about formal languages.
- We'll get back to this topic several times during semester.

References

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