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Mathematical Preliminaries

(Part 2)

Lecture 03

Day 03/31

CS 154

Formal Languages and Computability

Fall 2019

Agenda of Day 03

- Waiting List Enrollment ...
- Announcement
- Summary of Lecture 02
- Lecture 03: Teaching ...
  - Covering one slide from the past
  - Mathematical Preliminaries (Part 2)

Announcement

- Our first quiz will be next Thursday!
- Some of the questions are true/false and multiple choice.
- So, please have Scantron 882 E.
- If you forget, no problem at all! I'll sell it at:

$~~20~~ Each

Now Just $19.99!

Summary of Lecture 02: We learned ...

Sets

- A set is ...
  - ... a collection of objects.
- A list is ...
  - ... a collection of ordered objects.
- A set is known when its boundary is clearly defined.
- Three methods to represent sets ...
  - Roster method
  - Venn diagram
  - Set builder
- Universal set of a set is ...
  - ... the set containing all possible elements under consideration.

- The power set of the set S is ...
  - ... the set of all subsets of S.
  - It is denoted by 2^S.
  - $|2^S| = 2^{|S|}$
- A set is called finite if ...
  - ... its size is a natural number.
- A set is called infinite if ...
  - ... we cannot express its size by a natural number.

Any question?

Empty Set Representation by Set Builder

- How to represent empty set by set builder?
- We know: $A - B = \{x : x \in A \text{ AND } x \notin B\}$
- Substitute A for B: $A - A = \{x : x \in A \text{ AND } x \notin A\}$
- $\therefore \quad \phi = \{x : \text{False}\}$

- So, to represent empty set, just put any false statement in the description part of the set builder.

- For example, the following sets represent empty sets:
- $\{x : x$ is the 8^{th} day of week$\}$
- $\{x : x \notin U\}$

Mathematical Preliminaries

Recap from Math 42

Cartesian Products

Motivation

- Recall that in sets, order of elements does NOT matter.
- But in practice, we do need ordered collections.
- As we said before, in computer science we use "Lists" for ordered collections.
- The question is how we can mathematically model lists?

Introduction

- Mathematicians defined a new mathematical structure called "n-tuple".
- It is denoted by $(a_1, a_2, \ldots, a_n)$ where a_i's are objects.
  - A special case of n-tuple is 2-tuple aka ordered-pair (a_1, a_2).
- We use a mathematical operation called Cartesian product to create n-tuples.
- This operation is named after the great French philosopher, mathematician, and physicist René Descartes (1596-1650).

Cartesian Products

Definition

- Let A and B be two sets.
- The Cartesian product of A and B is the set of all ordered-pairs (a , b), where $a \in A$ and $b \in B$.
- Cartesian product of A and B is denoted by A x B.

Set Builder Definition

$$A \times B = \{(a , b) : a \in A , b \in B\}$$

- Note that (a , b) is the pattern of the elements.

Cartesian Products: Examples

Example 25

- Let $A = \{0, 1\}$, $B = \{3, 6, 9\}$; $A \times B = ?$

Solution

- $\{0, 1\} \times \{3, 6, 9\} = \{(0, 3), (0, 6), (0, 9), (1, 3), (1, 6), (1, 9)\}$

Example 26

- Let $Q = \{q_0, q_1\}$, $\Sigma = \{a , b\}$; $Q \times (\Sigma \cup \{\lambda\}) = ?$
  - "λ" is pronounced "lambda".

Solution

Cartesian Products: Properties

- What is the result of the following Cartesian product?

  $A = \{1, 2\}$, $B = \phi$; $A \times B = ?$

  $A \times B = \phi$

  – In fact, the result of Cartesian product would be ϕ if one of the sets is ϕ.
- How can you prove it?

- Does Cartesian product have commutative property?
- Is this a true statement: $A \times B = B \times A$
- In general, No! $A \times B \neq B \times A$

- But in the following special cases, they are equal:

  $A \times B = B \times A$ iff $(A = B) \vee (A = \phi) \vee (B = \phi)$

Cartesian Products Extension

- We can extend the idea to n sets to produce n-tuple:

$$S_1 \times S_2 \times \ldots \times S_n = \{(x_1, x_2, \ldots, x_n) : x_1 \in S_1, \ldots, x_n \in S_n\}$$

Homework

- Let $Q = \{q_0, q_1\}$, $\Sigma = \{a, b\}$, $\Gamma = \{x, y\}$
- $Q \times (\Sigma \cup \{\lambda\}) \times \Gamma = ?$

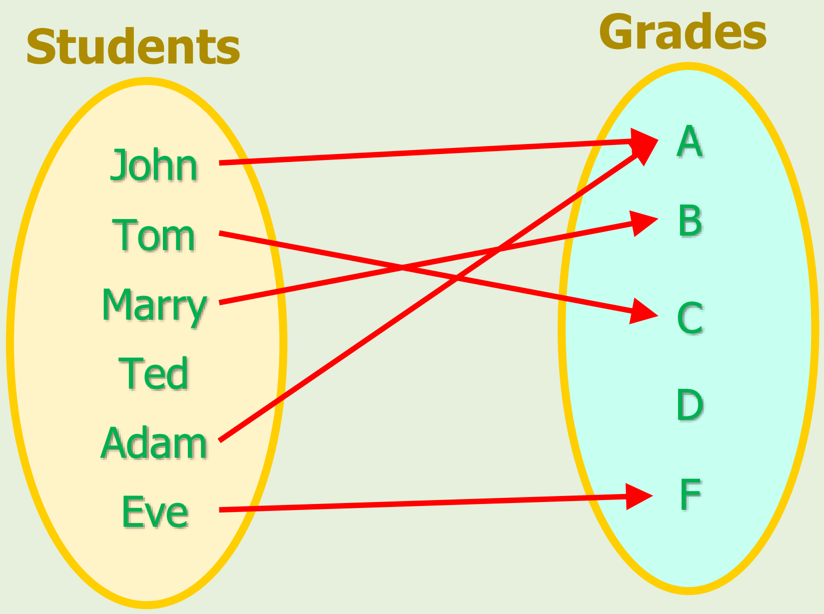
Mathematical Preliminaries

Recap from Math 42

Functions

Introduction

- In many situations in real life,
  there is a relationship between two sets.
- For example, we assign a letter grade to each student of a class.

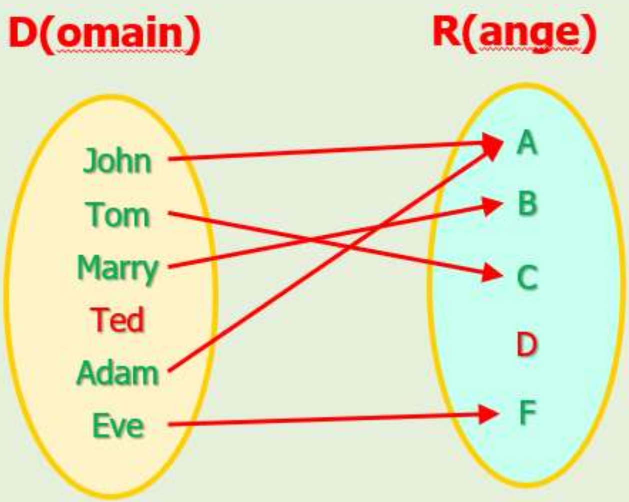


- This relationship is an example of the concept of function.

Functions

Definition

- Let D and R be two sets.
- A function g from D to R is a rule that assigns (or maps) to the elements of D a "unique element" of R.



- The set D is called the "domain" of g.
- The set R is called the "range" of g.

- The function g from D to R is denoted by: $g : D \to R$

Functions: Naming Convention

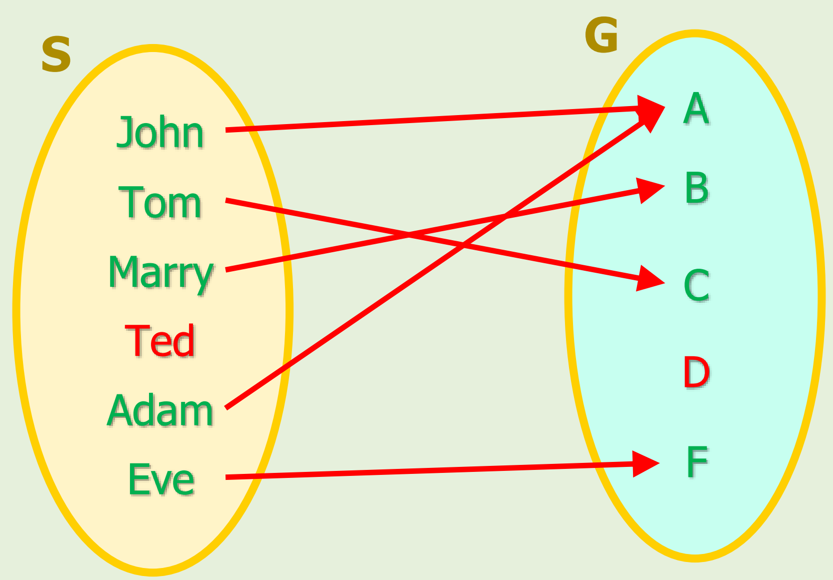
- We usually name a function by lower-case letters (English or Greek) such as f, g, h, δ (pronounce "delta"), etc.

Example 27

S = {John, Tom, Marry, Ted, Adam, Eve}, G = {A, B, C, D, F}

- We can name this function "h".

$h : S \rightarrow G$



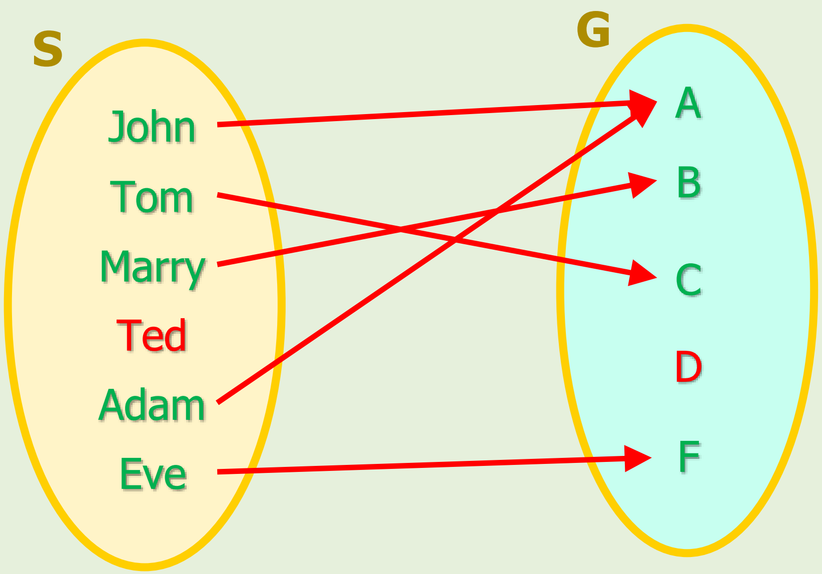
Functions: Algebraic Notation

Example 27 (cont'd)

- What is the rule of this function in algebraic notation?

  S = {John, Tom, Marry, Ted, Adam, Eve}, G = {A, B, C, D, F}

  $h : S \to G$

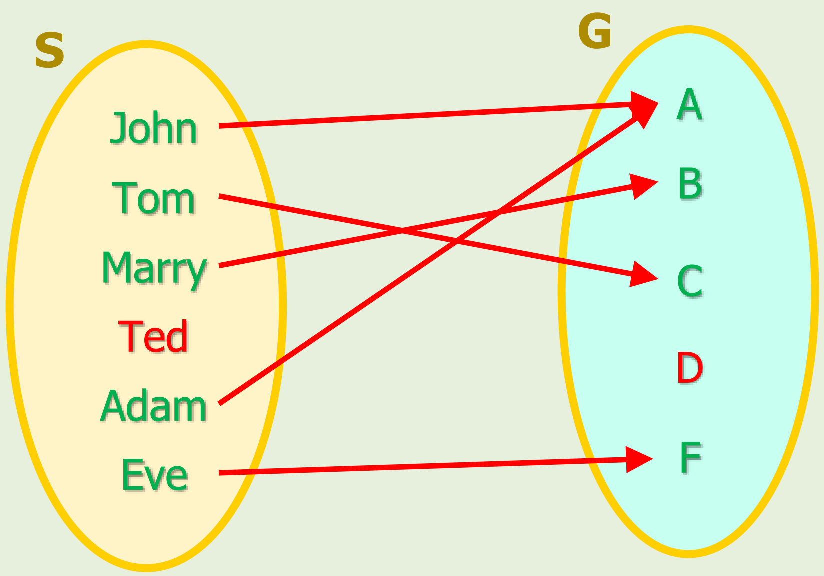


- We can translate the Venn Diagram into algebraic notation like this:

$$\begin{cases} h(\text{John}) = A \\ h(\text{Tom}) = C \\ h(\text{Marry}) = B \\ h(\text{Adam}) = A \\ h(\text{Eve}) = F \end{cases}$$

Functions: Notes

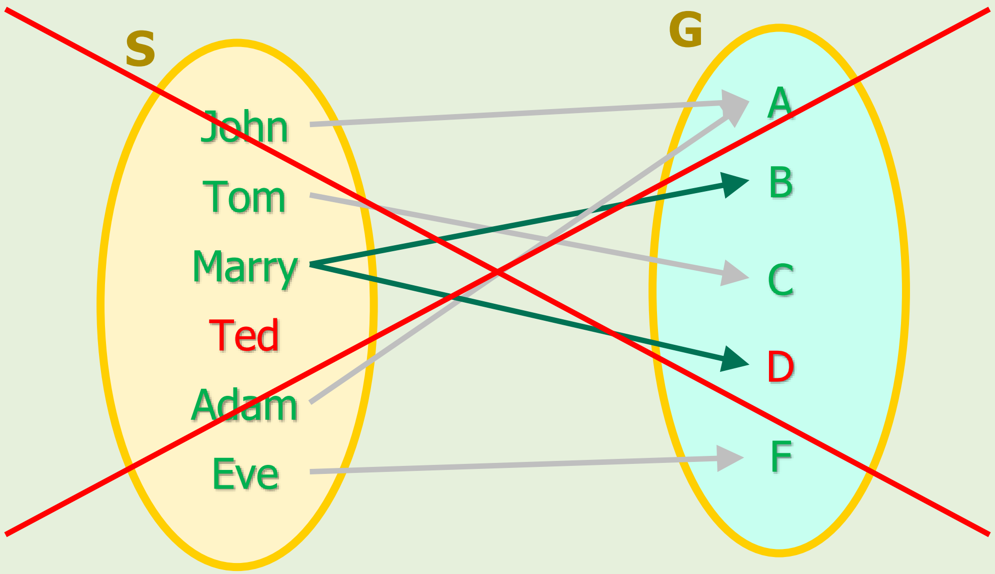
- h(Ted) = ?
- h(Ted) = Undefined



- So, it is possible to have some elements in the domain that is NOT mapped to any value of the range. (e.g. Ted in the domain)
- Also, it is possible to have some elements in the range that is NOT assigned by any value of the domain. (e.g. D in the range)

Functions: Notes

- Is it possible for Marry to have two grades at the same time?



- Absolutely, NO.

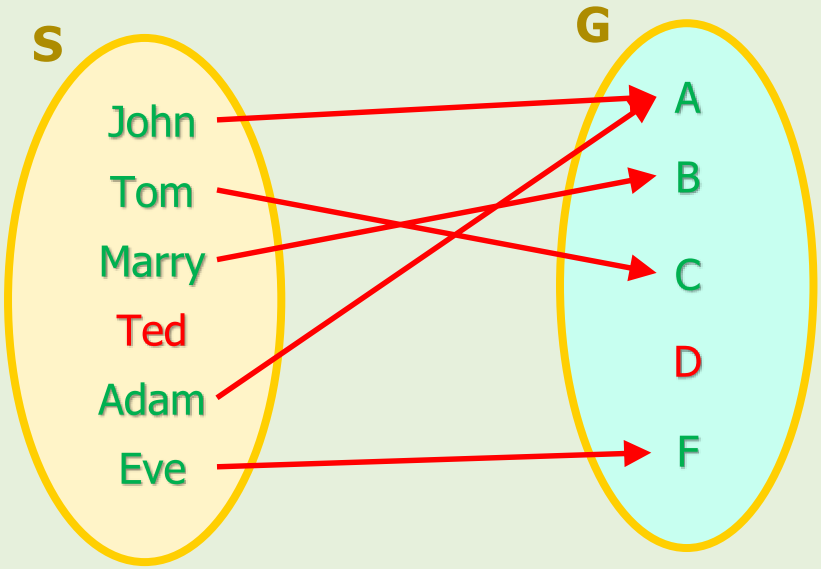
  In this universe,
  it cannot happen.

- That's why, in the definition of function, we said elements of the domain are uniquely mapped to an element of the range.
- In other words, if there is a mapping, it should be unique.

Functions as Sets

Example 27 (cont'd)

$$\begin{cases} h(\text{John}) = A \\ h(\text{Tom}) = C \\ h(\text{Marry}) = B \\ h(\text{Adam}) = A \\ h(\text{Eve}) = F \end{cases}$$



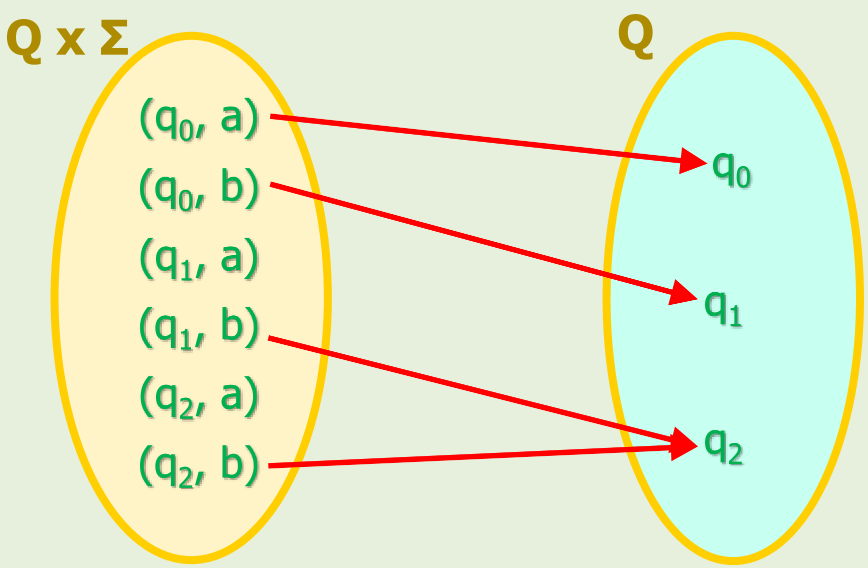
- The above relations can be represented by a set of ordered-pairs:
- H = {(John, A), (Tom, C), (Marry, B), (Adam, A), (Eve, F)}
- And the universal set for H is S x G that contains all possible relations between S's members and G's members.
- U = S x G = {(John, A), (John, B), ..., (John, F), (Tom, A), ... , (Tom, F), ..., (Ted, A), ..., (Ted, F), ..., (Eve, F)}

Functions: Example

Example 28: Mixing Cartesian Product and Function

- Let $Q = \{q_0, q_1, q_2\}$, $\Sigma = \{a , b\}$, $\delta : Q \times \Sigma \rightarrow Q$
- What is the domain and range of δ?
- Domain: $Q \times \Sigma = \{q_0, q_1, q_2\} \times \{a , b\} =$
  $\{(q_0, a), (q_0, b), (q_1, a), (q_1, b), (q_2, a), (q_2, b)\}$
- Range: $\{q_0, q_1, q_2\}$
- The rule of δ is shown in the following Venn diagram.
- Write the rule of δ by using algebraic notation.

$$\begin{cases} \delta(q_0, a) = q_0 \\ \delta(q_0, b) = q_1 \\ \delta(q_1, b) = q_2 \\ \delta(q_2, b) = q_2 \end{cases}$$



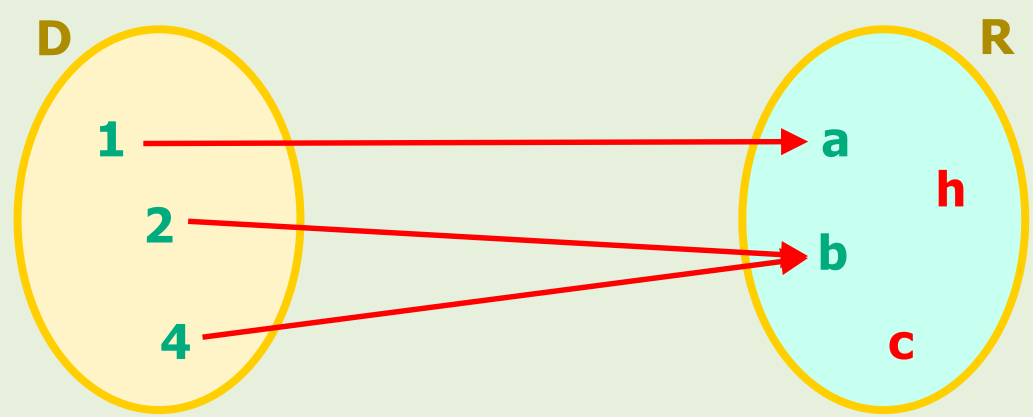
Total Function

Definition

- A function is called total if all of its domain elements are defined.

Example 29

- The following function is total because all domain elements are defined.



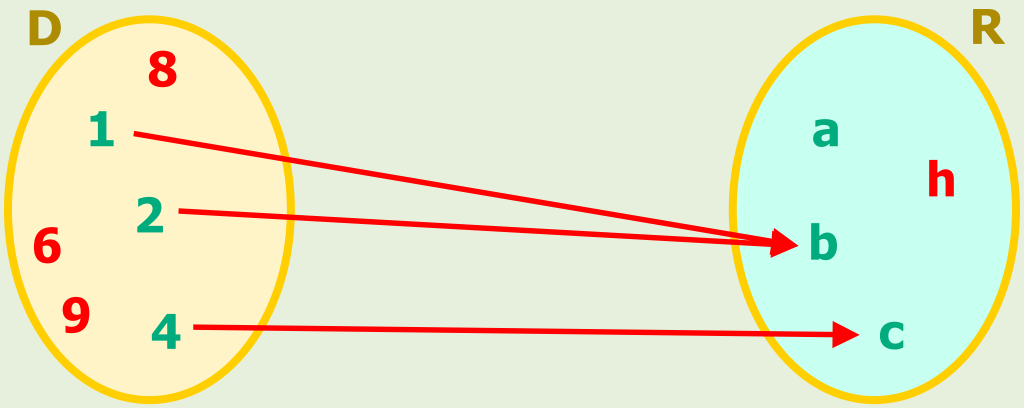
Partial Function

Definition

- A function is called partial if at least one of its domain's elements is undefined.

Example 30

- The following function is partial because f(8) = Undefined
- Note that one example is enough to make a function partial.



Functions

Homework

Functions: Homework #1

- Let $Q = \{q_0, q_1\}$, $\Sigma = \{a\}$, $\Gamma = \{x\}$.
- The function δ is defined as:

$$\delta : Q \times \{\Sigma \cup \{\lambda\}\} \times \Gamma \rightarrow Q$$

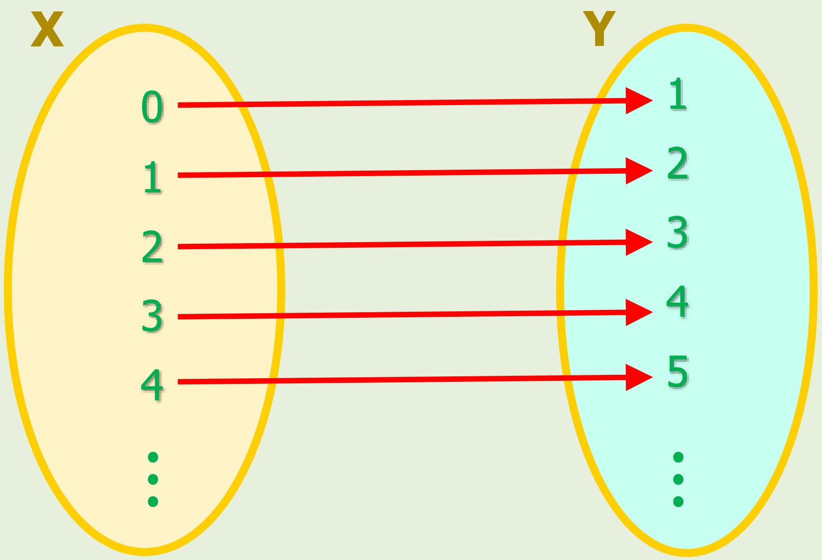
- What is the domain and range of δ?
- Represent the universal set of δ as a set of quadruples (4-tuple).

Functions: Homework #2

- Write the algebraic notation of the following function:

Solution

$$\begin{cases} f(0) = ? \\ f(1) = ? \\ f(2) = ? \\ f(3) = ? \\ \quad \cdots \end{cases}$$



- Sometimes, we can find a pattern between the elements of domain and range.
- What is the pattern of the above example?
- $f(x) = ?$

Functions: Homework #3

- Write a simple Java method to implement the function of HW #2.
- Hint: the argument of the method is the domain and whatever it's returning is range.

Solution

Functions: Homework #4

- What is the range and domain of the following Java code?
- `public int g(int x, int z) { return 2*x + z; }`

Solution

Mathematical Preliminaries

Recap from Math 42

Graphs

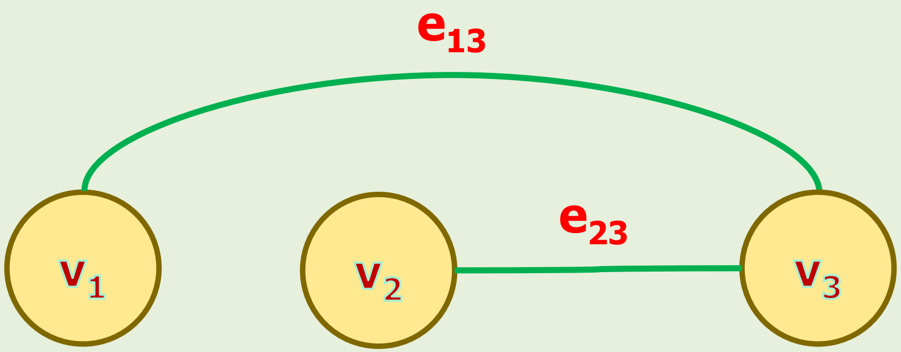
Graphs Definition

Definition

- A graph is a mathematical construct consisting of two sets:
  - A non-empty and finite set of vertices (aka nodes, states) $V = \{v_1, v_2, \ldots, v_n\}$
  - A finite set of edges $E = \{e_1, e_2, \ldots, e_m\}$
  - Each edge connects two vertices.

Example 31

- $V = \{v_1, v_2, v_3\}$
- $E = \{e_{13}, e_{23}\}$



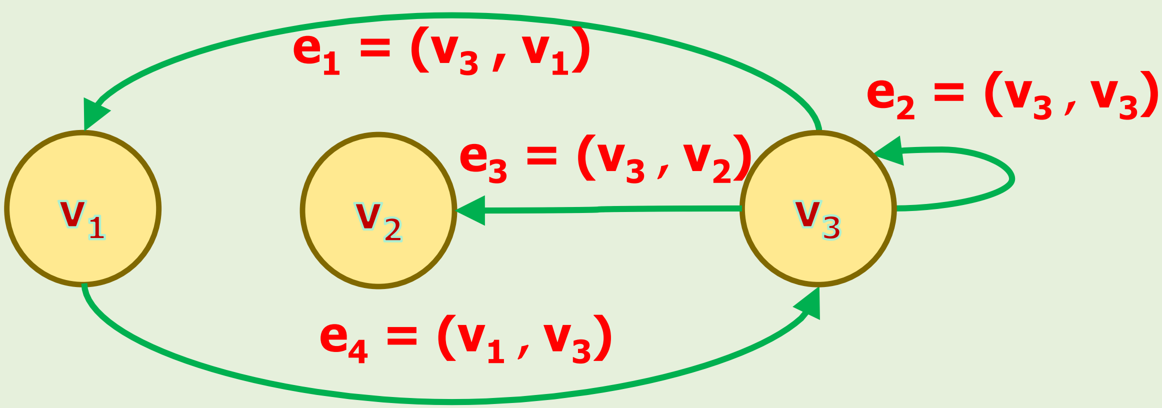
Directed Graphs

- If the direction of the edges matters,
  then we call the graph directed graph (aka digraph).
- The edges are shown by ordered-pair (start-vertex , end-vertex).
  - In this course, we only use directed graphs.

Example 32

- Draw a digraph with the following specifications:

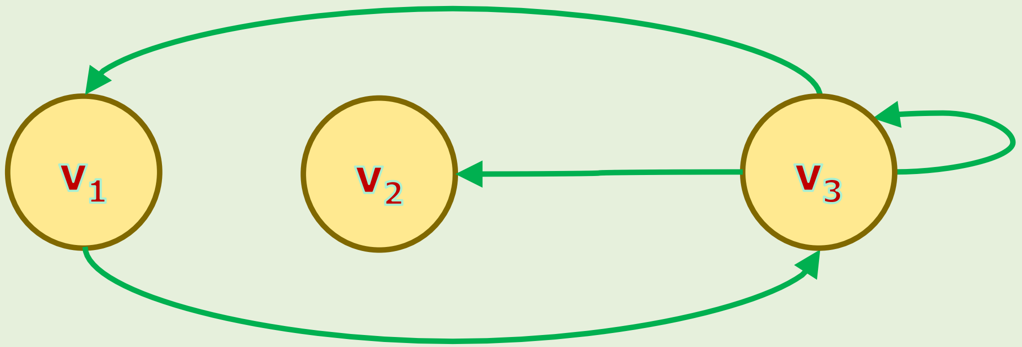
$$V = \{v_1 , v_2 , v_3\} , E = \{(v_1 , v_3), (v_3 , v_1), (v_3 , v_2), (v_3 , v_3)\}$$



Graphs Terminologies

Walk

- A sequence of edges like $(v_i, v_j), (v_j, v_k), \ldots, (v_m, v_n)$, is called a walk from v_i to v_n.
  - Note that the end vertex of e_i is the start vertex of e_{i+1}.
  - In other words, in a walk we cannot jump!

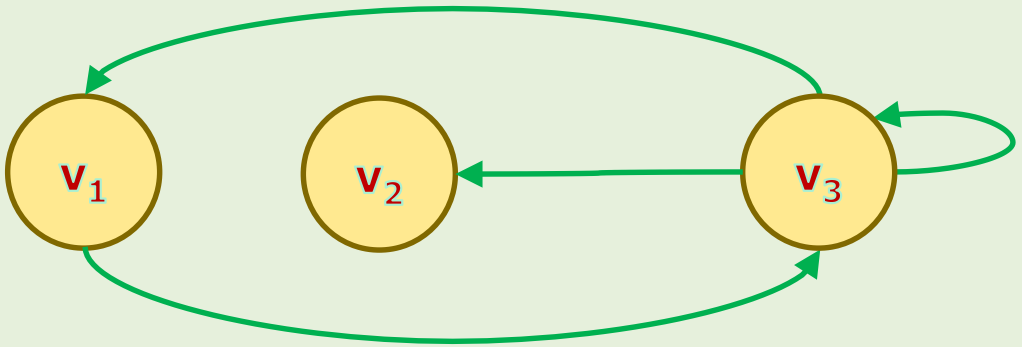


Example 33

- Each of the following sequences are a walk from v_1 to v_3:
  - Walk 1: (v_1, v_3)
  - Walk 2: $(v_1, v_3), (v_3, v_3)$
  - Walk 3: $(v_1, v_3), (v_3, v_1), (v_1, v_3)$
  - …

Length of Walks

- It is the total number of edges traversed.

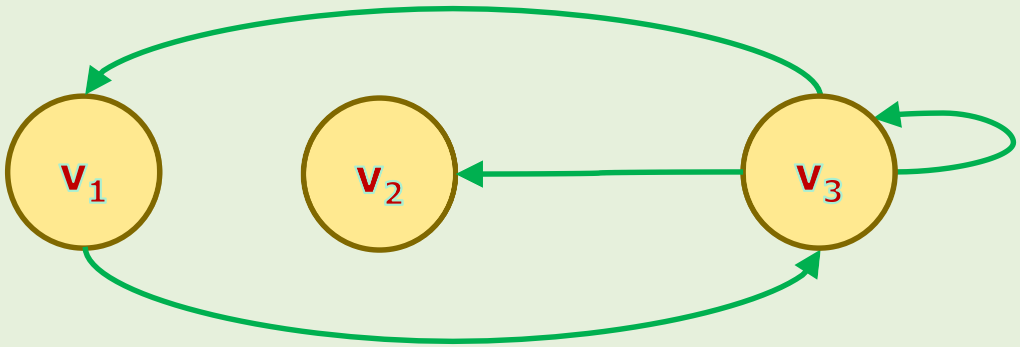


Example 33 (cont'd)

- Walk 1: (v_1, v_3) ; length = 1
- Walk 2: $(v_1, v_3), (v_3, v_3)$; length = 2
- Walk 3: $(v_1, v_3), (v_3, v_1), (v_1, v_3)$; length = 3

Path

- A walk that no edges are repeated.

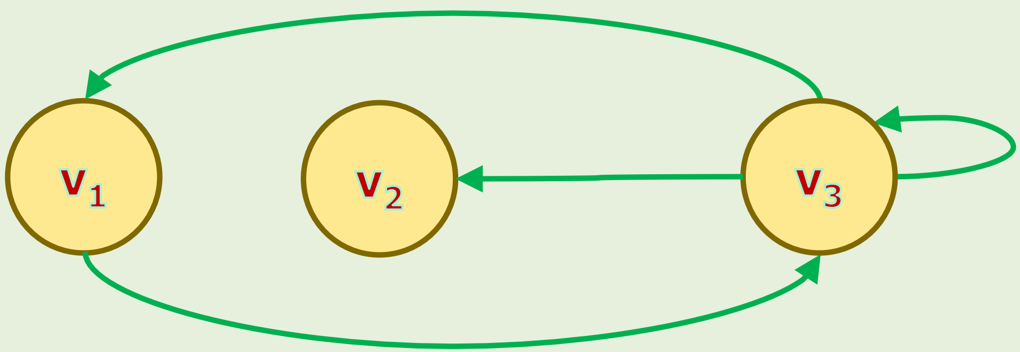


Example 34

- Which one is a path?
  - ✓ Walk 1: (v_1, v_3)
  - ✓ Walk 2: $(v_1, v_3), (v_3, v_3)$
  - ✗ Walk 3: $(v_1, v_3), (v_3, v_1), (v_1, v_3)$

Simple Path

- A path that no vertices are repeated.
  - In other words, no vertex should be visited more than once.

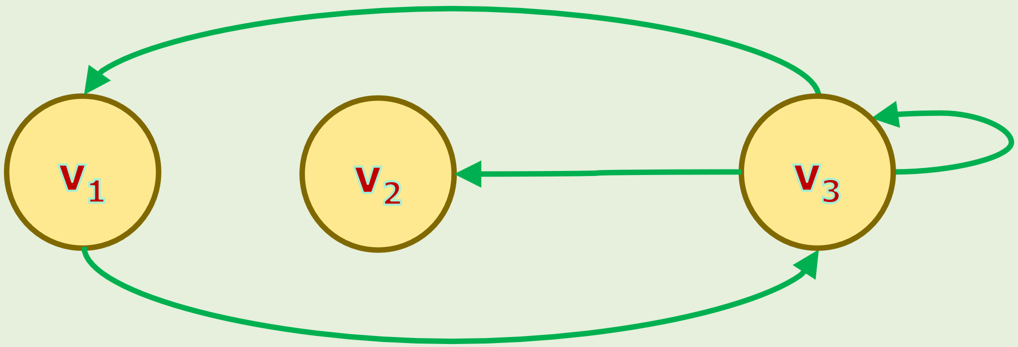


Example 35

- Which one is a simple path?
  - ✓ Walk 1: (v_1, v_3)
  - ✗ Walk 2: $(v_1, v_3), (v_3, v_3)$
  - ✗ Walk 3: $(v_1, v_3), (v_3, v_1), (v_1, v_3)$

Loop

- An edge from a vertex to itself.



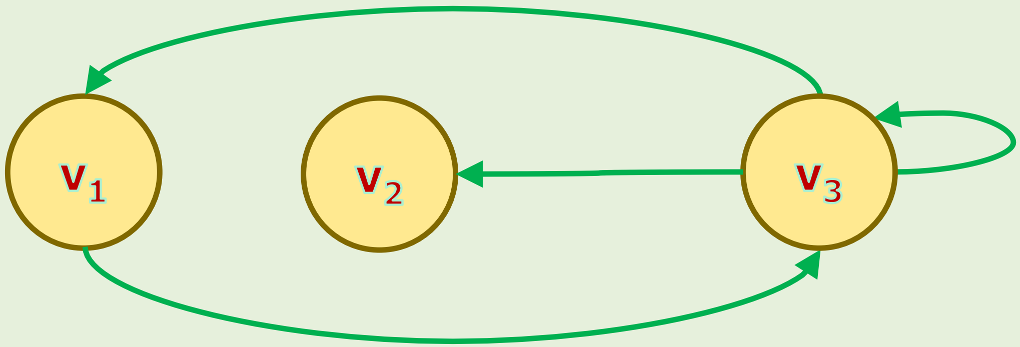
Example 36

- Which one is a loop?
  - ✓ – Walk 1: (v_3 , v_3)

- Is there any other loop in this graph?

Cycle

- A walk from a vertex (called base) to itself with no repeated edges.
- Recall that: Walk + No repeated edges = path
- Rewording: A cycle is a path from a vertex (called base) to itself.



Example 37

- Which one is a cycle?

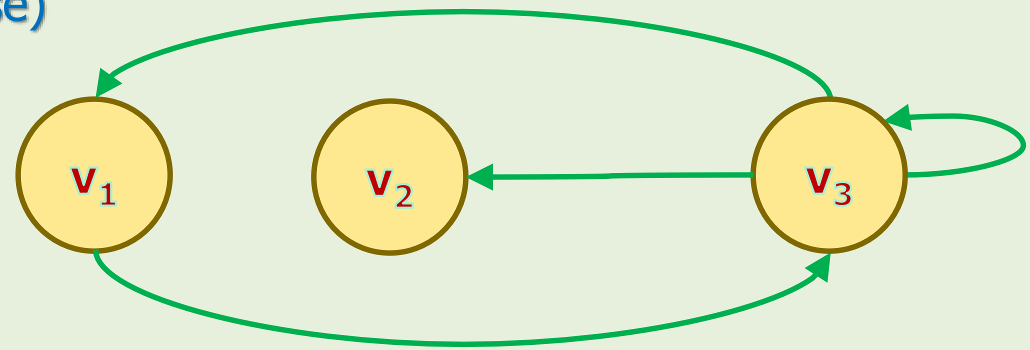
✗ – Walk 1: $(v_1, v_3), (v_3, v_1), (v_1, v_3), (v_3, v_1)$

✓ – Walk 2: $(v_1, v_3), (v_3, v_1)$

✓ – Walk 3: $(v_1, v_3), (v_3, v_3), (v_3, v_1)$

Simple Cycle

- A cycle that no vertices other than the base is repeated.
  - Note that the walk starts from the base and ends to the base.
  - During the walk, the base should not be repeated too.
  - In other words, in a simple cycle, all vertices (except the base) and the edges are visited uniquely.



Example 38

- Which one is a simple cycle?

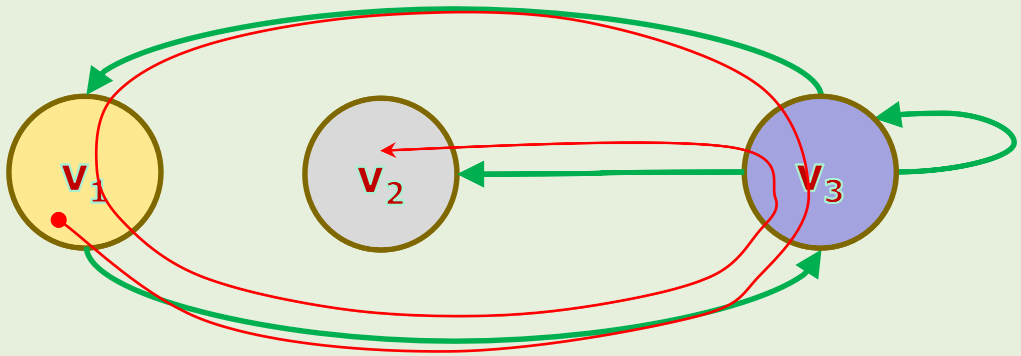
✗ – Walk 1: $(v_1 , v_3) , (v_3 , v_1) , (v_1 , v_3) , (v_3 , v_1)$

✓ – Walk 2: $(v_1 , v_3) , (v_3 , v_1)$

✗ – Walk 3: $(v_1 , v_3) , (v_3 , v_3) , (v_3 , v_1)$

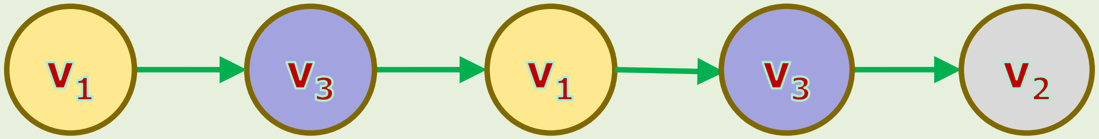
One-Dimensional Projection of a Walk

- One-dimensional projection (or just projection) is another way of representing a walk.



Example 39

- Represent the following walk as a one-dimensional projection.
- Walk from v_1 to v_2: $(v_1 , v_3) , (v_3 , v_1) , (v_1 , v_3) , (v_3 , v_2)$



- The length of this walk (= the number of edges) is clearly shown.

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