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Formal Languages

(Part 1)

Lecture 04

Day 04/31

CS 154

Formal Languages and Computability

Fall 2019

Agenda of Day 04

- Waiting List Enrollment ...
- Summary of Lecture 03
- Lecture 04: Teaching ...
  - Formal Languages (Part 1)

Summary of Lecture 03: We learned ...

Cartesian Products

- We use Cartesian product when we need ordered collections.
- The Cartesian product of two sets A and B is ...
  - ... the set of all ordered pairs (a , b), where $a \in A$ and $b \in B$.

$$A \times B = \{(a , b) : a \in A , b \in B\}$$

- It does NOT have commutative property.
- Special cases: $(A = B) \lor (A = \phi) \lor (B = \phi)$

- We extended the Cartesian product to n sets to produce n-tuple.

$$S_1 \times S_2 \times \ldots \times S_n = \{(x_1, x_2, \ldots , x_n) : x_1 \in S_1 , \ldots , x_n \in S_n\}$$

Any question?

Summary of Lecture 03: We learned ...

Functions

- We use Functions when we need a relation between two sets.
- A function f from D to R is ...
  - ... a rule that assigns to some elements of D (domain) a unique element of R (range).
  - Denoted by: $f : D \rightarrow R$
- A total function is ...
  - ... a function that all of its domain elements are defined.
- A partial function is ...
  - ... a function that at least one member of its domain is "undefined".

Any question?

Summary of Lecture 03: We learned ...

Graphs

- A graph is a mathematical construct consisting of two sets:
  - A non-empty and finite set of vertices $V = \{v_1 , v_2 , \dots , v_n\}$
  - A finite set of edges $E = \{e_1 , e_2 , \dots , e_m\}$
- A walk is ...
  - ... a sequence of edges from v_i to v_n. $(v_i , v_j) , (v_j , v_k) , \dots , (v_m , v_n)$
- The length of a walk is ...
  - ... the number of edges traversed.
- A path is ...
  - ... a walk that no edge is repeated.
- A simple path is ...
  - ... a path that no vertices is repeated.
- A loop is ...
  - ... an edge from a vertex to itself.
- A cycle is ...
  - ... a path from a vertex (called base) to itself.
- A simple cycle is ...
  - ... a cycle that no vertices other than base is repeated.

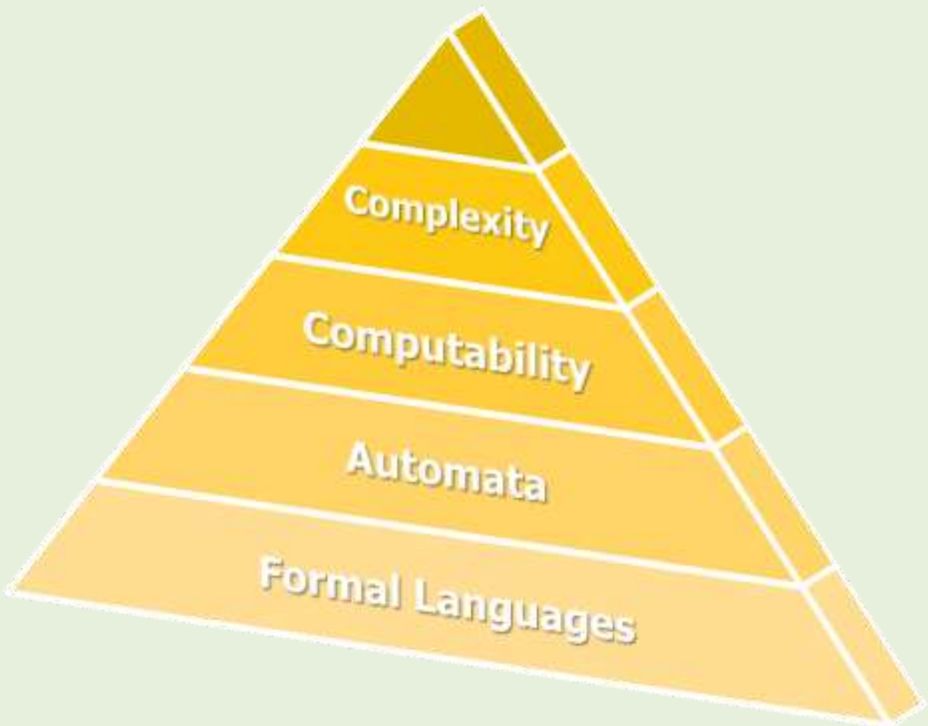
Any question?

The Big Picture of the Course

- The foundation of the computer science is called:

  "Theory of Computation".

- This theory is divided into four branches:



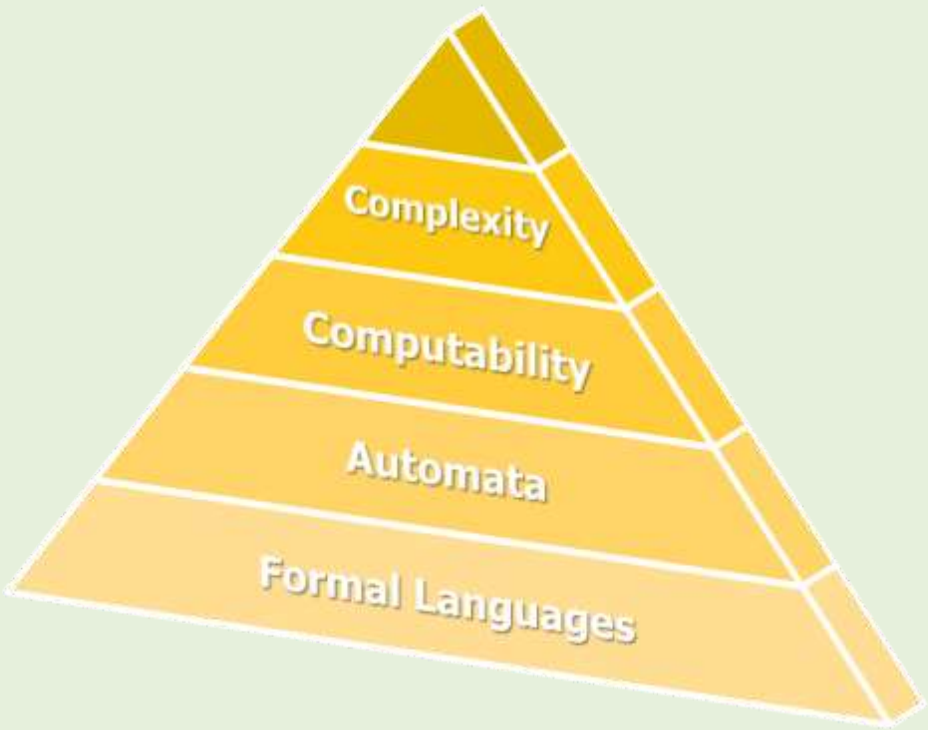
The Big Picture of the Course

- The first three from the bottom show:

  "What can be done with computers?"

- The forth one, complexity, shows:

  "What can be done in practice?"



Objective of This Lecture

- We start our journey with "Formal Languages"!
- But every language consists of "sentences" (aka "strings").
- And every string consists of "alphabets".

- So first, we'll introduce alphabets.
- Then, we'll define strings as mathematical objects.
- After that, we'll see some operations on strings.
- And finally, we'll introduce formal languages.

Alphabets & Strings

Alphabets

Definition

- An alphabet is a nonempty and finite set of "symbols".
- It is denoted by Σ.
  - Symbols are assumed to be indivisible.
- In this course, we usually use lowercase letters a, b, c, ... for alphabet symbols.
- In some cases, we might use digits like 0, and 1 or other symbols as well.

Alphabets Example

Example 1

- $\Sigma = \{a, b\}$ This is our celebrity alphabet!
- $\Sigma = \{0, 1\}$
- $\Sigma = \{\varepsilon, \alpha, \beta\}$

- Can the following set be an alphabet?
  Σ = {ЯE , Ѭ , پ , Γ }

Strings

Definition

- A string is a finite sequence of symbols.
- So, we do NOT have a string of INFINITE symbols.

Example 2

Let $\Sigma = \{a, c, d, e, g, l, o, p, t\}$.

The following strings are valid strings over Σ:

cat , dog , apple

Strings Examples

Example 3

- Let Σ = {a, b}.
- Are the following strings valid strings over Σ?
- baba , aabb , bbbbbbbbbbbba , ...

Solution

- Not all of them!
  - "..." is not a valid string because "." is not in the alphabet!

- Note that in formal languages arena,
  we don't care whether the strings are meaningful or not!

Strings Variable

- We usually use lowercase letters w, u, v, ... for "string variables".
  - If we need more variable, then x, y, z, or other symbols are used.

Example 4

- The following strings can be assigned to w as a variable:
- w = ab
- w = aabb
- w = aaabbb
- ...

Strings Size (aka Length)

Definition

- The size of a string w is the number of its symbols.
- It is denoted by |w|.

Example 5

|aaa| = 3

|babba| = 5

|aaba| = 4

- In general:

$$|a_1\ a_2\ \ldots\ a_n| = n$$

Empty String

Definition

- An empty string is a string with no symbol.
  - In other words: A sequence of zero symbols
- It is denoted by λ (pronounced lambda).

- What is the length of λ?

  $|\lambda| = 0$

Notes

1. Some authors might show empty string as ε (pronounced epsilon).
2. λ cannot be used as a symbol in alphabet.

Operations on Strings

Concatenation of Strings

Definition

- Concatenation of two strings u and v is the string uv.

Example 6

Let u = aaba and v = bb ; uv = ?

uv = aababb

- The length of concatenation:

$$|uv| = |u| + |v|$$

- λ is the neutral element for concatenation:

$$\lambda w = w\lambda = w$$

Example 7

λaabb = aabλb = aλabb = aλabbλ = aabb

Reverse of Strings

Definition

- Reverse of a string w is obtained by writing the symbols in reverse order.
- It is denoted by w^R. (pronounced w-reverse)
- If $w = a_1 a_2 \dots a_{n-1} a_n$, then $w^R = a_n a_{n-1} \dots a_2 a_1$

Example 8

Let $w = aaba$; $w^R = ?$

$w^R = abaa$

A Side Note: Palindrome

- The string w is called palindrome if w reads the same from left to right as from right to left.

Example 9

- radar, reviver, rotator

Some Funny Palindromes (Ignore spaces, apostrophes, commas)

- MADAM I'M ADAM
- STEP NOT ON PETS
- NO LEMONS, NO MELON
- DENNIS AND EDNA SINNED
- A MAN, A PLAN, A CANAL, PANAMA

Homework

- Prove that $(uv)^R = v^R u^R$

Substring

Definition

- Substring of a string w is any string of consecutive symbols of w.

Example 10

String	Substring
aababb	aa
aababb	ab
aababb	bab
aababb	b
aababb	λ
aababb	aababb

Prefix and Suffix

Definition

- Let w be a string. If w = uv, then ...

  u is called "prefix".

  v is called "suffix".

Example 11

Let w = aababb

If we consider u = aa as a prefix of w,
then the rest, v = babb, would be the suffix.

- Are these the only prefix and suffix?

Prefix and Suffix

Example 11 (cont'd)

- The complete list of all possible prefixes and suffixes of w are:

Prefix = u	Suffix = v
λ	aababb
a	ababb
aa	babb
aab	abb
aaba	bb
aabab	b
aababb	λ

- So, λ is prefix and suffix of every string (NOT at the same time).
  because: w = λ w = w λ

Exponential Operator

Definition

- Let w be a string and n be a natural number.
- w^n is defined as the concatenation of n copies of w.

$$w^n = \underbrace{w\ w\ w\ \dots\ w}_{\text{n times}}$$

Example 12

Let $w = a$; $w^2 = ?$; $w^3 = ?$

$w^2 = w\ w = aa = a^2$

$w^3 = w\ w\ w = aaa = a^3$

- Concatenation in formal languages looks like multiplication in elementary algebra.

Exponential Operator

Example 13

Let $w = aaba$; $w^2 = ?$; $w^3 = ?$

$w^2 = w\,w = aabaaaba = a^2ba^3ba$

$w^3 = w\,w\,w = aabaaabaaaba = a^2ba^3ba^3ba$

- In general: $w\,w^n = w^n\,w = w^{n+1}$
  where $n \in \mathbb{N}$ (natural numbers)

Example 14

Let $w = a^m b^m$ where m is a constant.

$|w| = ?$

$|w| = |a^m b^m| = 2m$

Exponential Operator

Special case

- $w^0 = ?$
- $w^0 = \lambda$
- How can you prove this?
  - Hint: use $w\ w^n = w^n\ w = w^{n+1}$ and $w = \lambda\ w = w\ \lambda$

Example 15

$(aaba)^0 = \lambda$

- Note that $aaba^0 = aab$

Formal Languages

Introduction

- Before introducing formal languages, we need to introduce two new operations on alphabets.

- We did not mention them before because we needed the concept of concatenation.

Star Operator on Alphabets

Definition

- Let Σ be an alphabet.
- Σ^* is the set of "all possible strings" obtained by concatenating "ZERO or more" symbols from Σ.

Example 16

Let $\Sigma = \{a\}$; $\Sigma^* = ?$

$\Sigma^* = \{a\}^* = \{\lambda, a, aa, aaa, aaaa, ...\}$

- Note that Σ^* is an infinite set.

Star Operator on Alphabets

Example 17

Let $\Sigma = \{a, b\}$; $\Sigma^* = ?$

$\Sigma^* = \{a, b\}^* = \{\lambda, a, b, aa, ab, ba, bb, aaa, aab, ...\}$

- Note what strategy we used to enumerate all combinations.

- Let $\Sigma = \{a, b, c\}$; $\Sigma^* = ?$

Plus Operator on Alphabets

Definition

- Let Σ be an alphabet.
- Σ^+ is the set of "all possible strings" obtained by concatenating "ONE or more" symbols from Σ.

Example 18

Let $\Sigma = \{a\}$; $\Sigma^+ = ?$

$\Sigma^+ = \{a\}^+ = \{a, aa, aaa, aaaa, \ldots\}$

- Note that Σ^+ is an infinite set.
- Also, note that the only difference between Σ^+ and Σ^* is that Σ^+ does NOT contain λ.

Plus Operator on Alphabets

Example 19

Let $\Sigma = \{a, b\}$; $\Sigma^+ = ?$

$\Sigma^+ = \{a, b\}^+ = \{a, b, aa, ab, ba, bb, aaa, aab, ...\}$

- Since the only difference between Σ^+ and Σ^* is that Σ^+ does NOT contain λ, hence:

$$\Sigma^+ = \Sigma^* - \{\lambda\}$$

$$\Sigma^* = \Sigma^+ \cup \{\lambda\}$$

Formal Languages

Definition

- Let Σ be an alphabet.
- Any subset of Σ^* is called a "formal language" over Σ.
- Σ^* contains all possible strings that can be made by the symbols of Σ.
- That's why it's called the "universal formal language" over Σ.
  - Recall the definition of "universal set".

Formal Languages: Examples

Example 20

Let $\Sigma = \{a, b\}$ be an alphabet ; $\Sigma^* = ?$

$\Sigma^* = \{a, b\}^* = \{\lambda, a, b, aa, ab, ba, bb, aaa, aab, ...\}$

- The following subsets of Σ^* are examples of formal languages over Σ:

- $L_1 = \{a, b, aa, aab\}$ because $L_1 \subseteq \Sigma^*$
- $L_2 = \{\lambda, ba, bb, bbb, aaa, aab\}$ because $L_2 \subseteq \Sigma^*$
- $L_3 = \{a^n : n \geq 0\}$ because $L_3 \subseteq \Sigma^*$
  - Can you enumerate L_3?

Formal Languages: Special Cases

Example 20 (cont'd)

How about the following sets? Are they formal languages? Why?

$L_4 = \phi = \{\ \}$

$L_5 = \{\lambda\}$

Yes they are because both are subsets of Σ^*.

Two Special Formal Languages

- Empty language : $\{\ \}$ or ϕ
- λ-Language : $\{\lambda\}$

Formal Languages: Notes

1. For simplicity, from now on, we use "language" to refer to the formal language.

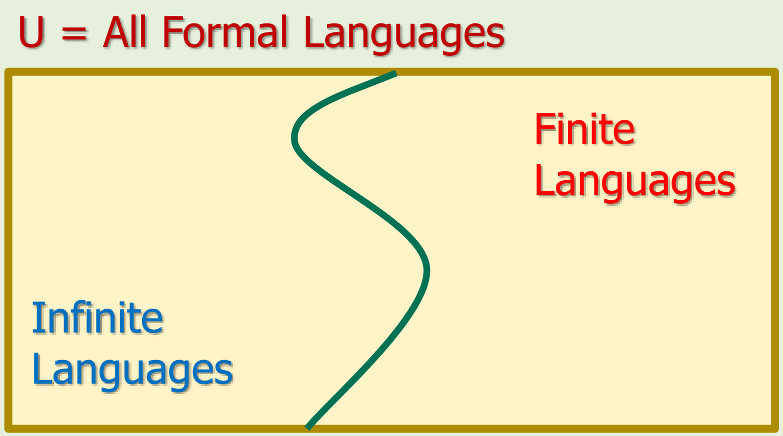
2. A language is a "set".
   So, it has all properties of sets.

3. $\{\lambda\}$ is a language while λ is a string.
   - Language is a set while string is a sequence of symbols.

4. Some authors prefer to call strings as "sentences" to analogize the formal languages with the natural languages.
   - In this course, we mostly use strings!

Languages Categorization: Finite and Infinite

- Like sets, we have both "finite" and "infinite" languages.



- We'll categorize languages from different angles.
- This is our first categorization that is from the size point of view.

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