Info Retrieval Assignment 2

Task 1

a) Provide $\hat{s} \in \mathbb{R}^p$ s.t. $\hat{s} = \underset{s \text{ st } \|s\|=1}{\arg\max} \; s^T \Sigma \Sigma^T s$

$$s^T \Sigma \Sigma s = s^T \Sigma_p^2 s \qquad ①$$

$$= s^T \begin{bmatrix} \sigma_{1,1}^2 & & \\ & \ddots & \\ & & \sigma_{p,p}^2 \end{bmatrix} s$$

$$= \sum_{i=1}^{p} (s_i \sigma_{i,i})^2$$

Note: $\cancel{\sqrt{\sum_{i=1} s_i^2} = 1}$

$\cancel{\Rightarrow \sum_{i=1} s_i^2 = 1}$

$\cancel{\text{So } s_i \leq 1}$

Since $\sigma_{1,1} > \cdots > \sigma_{p,p}$, we maximise ① by setting s_1 to be highest value possible, i.e. $s_1 = 1$, and ~~$s_1 \neq$~~ $s_i = 0$ for $i \neq 1$

So $\hat{s} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^p$ maximises ①

b) Show $\frac{1}{N} \sum_{i=1}^{N} (a x_i)^2 = \frac{1}{N} a^T X X^T a$ maximised when $a = u_1$ ($\|a\| = 1$)

$$\frac{1}{N} a^T X X^T a = \frac{1}{N} a^T U \Sigma V^T V \Sigma^T U^T a \qquad (X = U \Sigma V^T)$$

$$= \frac{1}{N} a^T U \Sigma \Sigma^T U^T a \qquad (V^T V = I_N)$$

$$= \frac{1}{N} a^T U \Sigma_p^2 U^T a$$

Let $s^T = a^T U$

$\Rightarrow \frac{1}{N} a^T X X^T a = \frac{1}{N} s^T \Sigma_p^2 s \qquad ②$

② maximised when $s = \hat{s} = [1 0 \cdots 0]^T$

$\therefore \hat{s}^T = a^T U \Rightarrow \hat{s} = U^T a$

So $a = u_1$ since $U^T u_1 = \begin{bmatrix} u_1 \cdot u_1 \\ u_2 \cdot u_1 \\ \vdots \\ u_n \cdot u_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \hat{s}$