Lecture 14
Satisfiability Modulo Theories

1. Motivation: Path Sensitivity Analysis
2. A Basic SMT Solver
3. Optimizing the SMT Solver

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What is Satisfiability Modulo Theories (SMT)?

- Satisfiability
  - the problem of determining whether a formula has a model (an assignment that makes the formula true)

- SAT: Satisfiability of **propositional formulas**
  - A model is a truth assignment to Boolean variables
  - SAT solvers: check satisfiability of propositional formulas
    - Decidable, NP-complete

- SMT: Satisfiability modulo theories
  - Satisfiability of first-order formulas containing operations from background theories such as arithmetic, arrays, uninterpreted functions, etc.
    E.g. $g(a) = c \land f(g(a)) \neq f(c) \lor g(a) = d \land c \neq d$
  - SMT Solvers:
    - check satisfiability of SMT formulas in a decidable first-order theory

User of SMT for Program Correctness & Test Generation

- Precision: Path sensitivity

- Given an assertion A,
  can we generate an input that triggers an error on a given path p?
  - Let F be the formula representing the execution of p
  - Is the formula $F \wedge \neg A$ satisfiable?
    - Not satisfiable? No error on that path
    - Satisfiable?   Find 1 assignment that satisfies the formula
                          (1 set of test input)

Each Statement is a Logical Clause

Program Assume data array bound is [0, N-1]

One execution path
Static Single Assignment (SSA)

```
 1 void ReadBlocks(int data[], int cookie)
 2 {
 3    int i = 0;
 4    while (true)
 5    {
 6       int next;
 7       next = data[i];
 8       if (!(i < next && next < N)) return;
 9       i= i + 1;
10       for (; i<next; i = i + 1){
11          if (data[i] == cookie)
12             i = i + 1;
13          else
14             Process(data[i]);
15       }
16    }
17 }
```

```
 3 i_1 = 0;



 7 next_1 = data_0 [i_1];
 8 i_1 < next_1 && next_1 < N_0
 9 i_2 = i_1 + 1;
10 i_2 < next_1;
11 data_0 [i_2] = cookie_0;
12 i_3 = i_2 + 1;

10 i_4 = i_3 + 1;
10 !(i_4 < next_1);
 7 next_2 = data_0 [i_4];
```

Line 3: $i_1 = 0;$

Line 7: $next_1 = data_0 [i_1];$

Line 8: $i_1 < next_1 \;\&\&\; next_1 < N_0$

Line 9: $i_2 = i_1 + 1;$

Line 10: $i_2 < next_1;$

Line 11: $data_0 [i_2] = cookie_0;$

Line 12: $i_3 = i_2 + 1;$

Line 10: $i_4 = i_3 + 1;$

Line 10: $!(i_4 < next_1);$

Line 7: $next_2 = data_0 [i_4];$

An Execution Path as a Logic Formula

Program Assume data array bound is [0, N-1]

One execution path (SSA)

```
1 void ReadBlocks(int data[], int cookie)
2 {
3    int i = 0;
4    while (true)
5    {
6       int next;
7       next = data[i];
8       if (!(i < next && next < N)) return;
9       i= i + 1;
10      for (; i<next; i = i + 1){
11         if (data[i] == cookie)
12            i = i + 1;
13         else
14            Process(data[i]);
15      }
16   }
17 }
```

$F = \bigwedge$

$$3 \; i_1 = 0;$$

$$7 \; next_1 = data_0 \, [i_1];$$
$$8 \; i_1 < next_1 \; \&\& \; next_1 < N_0$$
$$9 \; i_2 = i_1 + 1;$$
$$10 \; i_2 < next_1;$$
$$11 \; data_0 \, [i_2] = cookie_0;$$
$$12 \; i_3 = i_2 + 1;$$

$$10 \; i_4 = i_3 + 1;$$
$$10 \; !(i_4 < next_1);$$
$$7 \; next_2 = data_0 \, [i_4];$$

Checking for Out-of-Bound Array Access (Line 7, iteration 1)

Program Assume data array bound is [0, N-1]

```
1 void ReadBlocks(int data[], int cookie)
2 {
3    int i = 0;
4    while (true)
5    {
6        int next;
7        next = data[i];
8        if (!(i < next && next < N)) return;
9        i= i + 1;
10       for (; i<next; i = i + 1){
11           if (data[i] == cookie)
12               i = i + 1;
13           else
14               Process(data[i]);
15       }
16   }
17 }
```

Line 7: Array bound assertion A:

$(0 \leq i_1 \wedge i_1 < N_0)$

One execution path (SSA)

$F = \wedge$ {

3 $i_1 = 0;$

7 $next_1 = data_0 [i_1];$
8 $i_1 < next_1$ && $next_1 < N_0$
9 $i_2 = i_1 + 1;$
10 $i_2 < next_1;$
11 $data_0 [i_2] = cookie_0;$
12 $i_3 = i_2 + 1;$

10 $i_4 = i_3 + 1;$
10 $!(i_4 < next_1);$
7 $next_2 = data_0 [i_4];$

}

Check: Is $F \wedge \neg A$ satisfiable?

$i_1 = 0 \wedge \neg(0 \leq i_1 \wedge i_1 < N_0)$

Answer for Out-of-Bound Array Access (Line 7, iteration 1)

Program Assume data array bound is [0, N-1]

```
1 void ReadBlocks(int data[], int cookie)
2 {
3    int i = 0;
4    while (true)
5    {
6       int next;
7       next = data[i];
8       if (!(i < next && next < N)) return;
9       i= i + 1;
10      for (; i<next; i = i + 1){
11         if (data[i] == cookie)
12            i = i + 1;
13         else
14            Process(data[i]);
15      }
16   }
17 }
```

One execution path (SSA)

$$F = \bigwedge \begin{cases} 3 \; i_1 = 0; \\ \\ 7 \; next_1 = data_0 [i_1]; \\ 8 \; i_1 < next_1 \; \&\& \; next_1 < N_0 \\ 9 \; i_2 = i_1 + 1; \\ 10 \; i_2 < next_1; \\ 11 \; data_0 [i_2] = cookie_0; \\ 12 \; i_3 = i_2 + 1; \\ \\ 10 \; i_4 = i_3 + 1; \\ 10 \; !(i_4 < next_1); \\ 7 \; next_2 = data_0 [i_4]; \end{cases}$$

Line 7: Array bound assertion A:

$$(0 \le i_1 \wedge i_1 < N_0)$$

$\mapsto$ maps to

Check: Is $F \wedge \neg A$ satisfiable?

$$i_1 = 0 \wedge \neg(0 \le i_1 \wedge i_1 < N_0)$$

Yes! $\{i_1 \mapsto 0, \; N_0 \mapsto 0\}^{BUG!!}$

Checking for Out-of-Bound Array Access (Line 7, iteration 2)

Program Assume data array bound is [0, N-1]

```
1 void ReadBlocks(int data[], int cookie)
2 {
3    int i = 0;
4    while (true)
5    {
6       int next;
7       next = data[i];
8       if (!(i < next && next < N)) return;
9       i= i + 1;
10      for (; i<next; i = i + 1){
11         if (data[i] == cookie)
12            i = i + 1;
13         else
14            Process(data[i]);
15      }
16   }
17 }
```

One execution path (SSA)

$F = \bigwedge$

$$3\ i_1 = 0;$$

$$7\ next_1 = data_0\ [i_1];$$
$$8\ i_1 < next_1\ \&\&\ next_1 < N_0$$
$$9\ i_2 = i_1 + 1;$$
$$10\ i_2 < next_1;$$
$$11\ data_0\ [i_2] = cookie_0;$$
$$12\ i_3 = i_2 + 1;$$

$$10\ i_4 = i_3 + 1;$$
$$10\ !(i_4 < next_1);$$
$$7\ next_2 = data_0\ [i_4];$$

Line 7: Array bound assertion A:

$$(0 \leq i_4\ \wedge\ i_4 < N_0)$$

Check: Is $F \wedge \neg A$ satisfiable?

$$F\ \wedge\ \neg(0 \leq i_4\ \wedge\ i_4 < N_0)$$

Answer for Out-of-Bound Array Access (Line 7, iteration 2)

Program Assume data array bound is [0, N-1]

```
1 void ReadBlocks(int data[], int cookie)
2 {
3    int i = 0;
4    while (true)
5    {
6       int next;
7       next = data[i];
8       if (!(i < next && next < N)) return;
9       i= i + 1;
10      for (; i<next; i = i + 1){
11         if (data[i] == cookie)
12            i = i + 1;
13         else
14            Process(data[i]);
15      }
16   }
17 }
```

Line 7: Array bound assertion A:

$$(0 \leq i_4 \wedge i_4 < N_0)$$

One execution path (SSA)

$F = \wedge$

```
3 i_1 = 0;



7 next_1 = data_0 [i_1];
8 i_1 < next_1 && next_1 < N_0
9 i_2 = i_1 + 1;
10 i_2 < next_1;
11 data_0 [i_2] = cookie_0;
12 i_3 = i_2 + 1;

10 i_4 = i_3 + 1;
10 !(i_4 < next_1);
7 next_2 = data_0 [i_4];
```

Where:
- 3: $i_1 = 0;$
- 7: $next_1 = data_0 [i_1];$
- 8: $i_1 < next_1 \ \&\& \ next_1 < N_0$
- 9: $i_2 = i_1 + 1;$
- 10: $i_2 < next_1;$
- 11: $data_0 [i_2] = cookie_0;$
- 12: $i_3 = i_2 + 1;$
- 10: $i_4 = i_3 + 1;$
- 10: $!(i_4 < next_1);$
- 7: $next_2 = data_0 [i_4];$

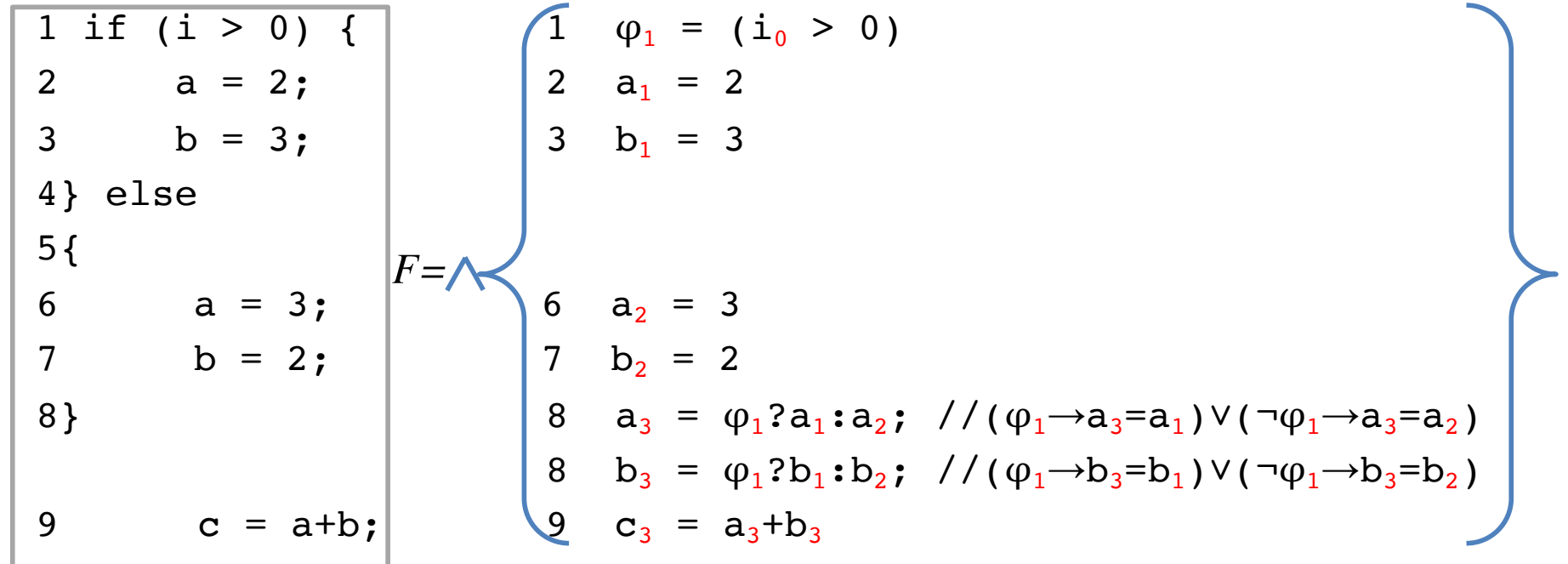| Var	$\mapsto$
N_0	3
i_1	0
i_2	1
i_3	2
i_4	3
$next_1$	2
$data_0$	<2,6,5>
$cookie_0$	6 *BUG!!*

Checking the Whole Program All at Once

- A program has many execution paths
- Conditional statements
  - Represent alternative paths symbolically with one formula using SSA
- Loops
  - Optimistically: Unroll a few times
  - Catches many errors, but not all errors

Conditional Statements

- Conditional statements: φ functions in SSA

```
1 if (i > 0) {
2       a = 2;
3       b = 3;
4} else
5{
6       a = 3;
7       b = 2;
8}

9       c = a+b;
```

$F = \bigwedge$

```
1   φ₁ = (i₀ > 0)
2   a₁ = 2
3   b₁ = 3


6   a₂ = 3
7   b₂ = 2
8   a₃ = φ₁?a₁:a₂;  //(φ₁→a₃=a₁)∨(¬φ₁→a₃=a₂)
8   b₃ = φ₁?b₁:b₂;  //(φ₁→b₃=b₁)∨(¬φ₁→b₃=b₂)
9   c₃ = a₃+b₃
```

- Assert A: $c_3 = 5$
- Is $F \wedge \neg A$ satisfiable?

$$\varphi_1 = (i_0 > 0) \wedge (\varphi_1 \rightarrow c_3 = 5) \wedge (\neg\varphi_1 \rightarrow c_3 = 5) \wedge (c_3 \neq 5)$$

Applying the Resolution Rule to Example

- A resolution rule in propositional logic:

	Given $p \vee A$ and $\neg p \vee B$, add the resolvent $A \vee B$
Resolve	$\dfrac{p \vee A \qquad \neg p \vee B}{A \vee B}$

- Is $F \wedge \neg A$ satisfiable?

  $\varphi_1 = (i_0 > 0) \ \wedge \ (\varphi_1 \rightarrow c_3 = 5) \ \wedge \ (\neg\varphi_1 \rightarrow c_3 = 5) \ \wedge \ (c_3 \neq 5)$

- Recall: $p \rightarrow q \ \equiv \ \neg p \vee q$

  $\varphi_1 = (i_0 > 0) \ \wedge \ (\neg\varphi_1 \vee c_3 = 5) \ \wedge \ (\varphi_1 \vee c_3 = 5) \ \wedge \ (c_3 \neq 5)$

  $\varphi_1 = (i_0 > 0) \ \wedge \ (c_3 = 5) \ \wedge \ (c_3 \neq 5)$

- $F \wedge \neg A$ is not satisfiable
- The assertion A is true.

Loops

- Optimistically: Unroll two times

```
1   for (; i<next; i = i + 1){
2       if (data[i] == cookie)
3           i = i + 1;
4       else
5           Process(data[i]);
6   }
```

```
1  if (i < next) {
2    if (data[i] == cookie)
3        i = i + 1;
4    else
5       Process(data[i]);
6
7    i = i + 1;
8
9    if (i < next) {
10      if (data[i] == cookie)
11          i = i + 1;
12      else
13          Process(data[i]);
14
15      i= i + 1;
16    }
17 }
```

Loops: Apply SSA

```
 1 if (i < next) {
 2    if (data[i] == cookie)
 3       i = i + 1;
 4    else
 5       Process(data[i]);
 6
 7    i = i + 1;
 8
 9    if (i < next) {
10       if (data[i] == cookie)
11          i = i + 1;
12       else
13          Process(data[i]);
14
15       i= i + 1;
16    }
17 }
```

$1 \quad \varphi_1 = (i_0 < next_0);$
$2 \quad \varphi_2 = (data_0[i_0] == cookie_0);$
$3 \quad i_1 = i_0 + 1;$
4
5
$6 \quad i_2 = \varphi_2 \,?\, i_1 \,:\, i_0;$
$7 \quad i_3 = i_2 + 1;$
8
$9 \quad \varphi_3 = (i_3 < next_0);$
$10 \quad \varphi_4 = (data_0[i_3] == cookie_0);$
$11 \quad i_4 = i_3 + 1;$
12
13
$14 \quad i_5 = \varphi_4 \,?\, i_4 \,:\, i_3;$
$15 \quad i_6 = i_5 + 1;$
$16 \quad i_7 = \varphi_3 \,?\, i_6 \,:\, i_3;$
$17 \quad i_8 = \varphi_1 \,?\, i_7 \,:\, i_0;$

Major Categories of Program Analysis Tools

	Static Property Based	Dynamic Execution Based

Complete (Small programs)	**Verification**
	Prove a property in a program
	Floyd-Hoare logic:
	  {pre-condition} s {post-condition}
	Applicable to small programs
		(Symbolic) Model Checking (SMT/BDD)
		Given a system model (sw/hw),
		  simulate the execution
		  to check if a property is true
		  for all possible inputs.
		Symbolic: many states all at once

Incomplete (Large programs)	**Static Analysis (Data flow)** Abstract the program conservatively Check a property Sound: no false-negatives--find all bugs False-positives: false warnings Too imprecise is useless	**Test case generation (SMT/BDD)** Check a property opportunistically (e.g. unroll loops twice) Use analysis to generate test inputs No false-positives: generate a test False-negatives: cannot find all bugs No correctness/security guarantees

2. A Basic SMT Solver

- SMT: Satisfiability modulo theories
  - Satisfiability of first-order formulas
    containing operations from background theories
    such as arithmetic, arrays, uninterpreted functions, etc.

- SMT Solvers:
  - check satisfiability of SMT formulas in a decidable first-order theory

SMT with Linear Inequalities & Function Theories

Uninterpreted function theory:

   Functions assumed to be pure:

   A function always returns the same value for a given input

Example: $x \geq 0 \wedge f(x) \geq 0 \wedge f(y) \geq 0 \wedge x \neq y$

This formula is satisfiable

An example model satisfying the formula

$$x \mapsto 1$$
$$y \mapsto 2$$
$$f(1) \mapsto 0$$
$$f(2) \mapsto 1$$

SMT with Array Theory

Notation: $\text{write}(v, i, x)$ means   v[i] := x;

   $\text{read}(v, i)$ means   returns v[i]

Array theory axiom, $\text{read}(\text{write}(v, i, x), i) = x$

Example: $b + 2 = c \,\wedge\, f(\text{read}(\text{write}(a, b, 3), c - 2)) \neq f(c - b + 1)$

By arithmetic theory, this is equivalent to

$\qquad b + 2 = c \,\wedge\, f(\text{read}(\text{write}(a, b, 3), b)) \neq f(3)$

By array theory, $b + 2 = c \,\wedge\, f(3) \neq f(3)$

By the theory of uninterpreted functions, $f(3) \neq f(3)$ is not true

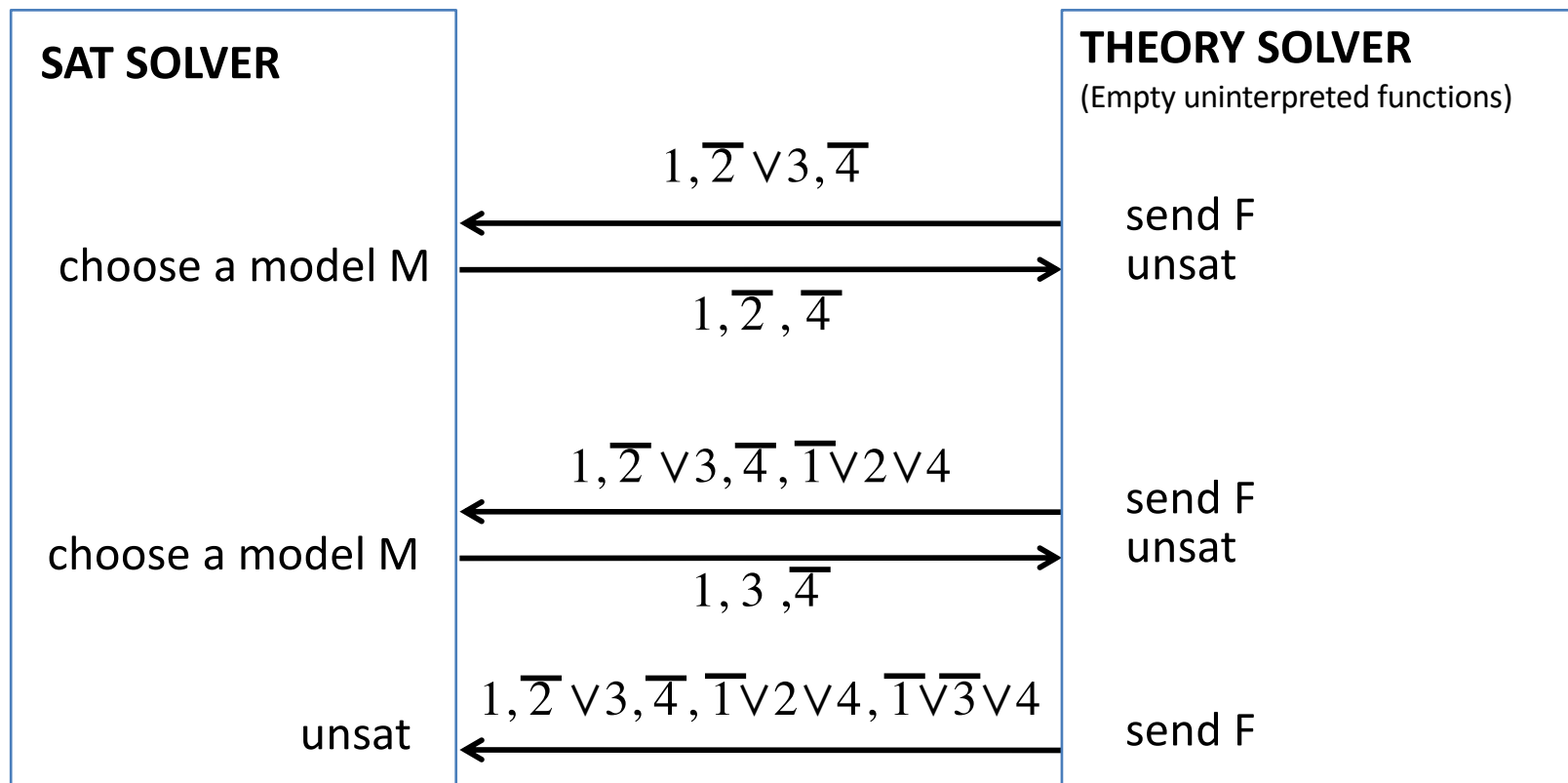Therefore, this formula is not satisfiable

SMT Solvers

- Input: a first-order formula F
- Output
  - F is satisfiable, optionally: a model M
  - F is unsatisfiable, optionally: a proof of unsatisfiability
- Which is easier?
- Main issues
  - formula size (e.g. thousands of atoms or more)
  - formulas with complex Boolean structure
  - combination of theories

Overview of a SMT Solver

- SMT Solver = SAT Solver + Theory Solver
  - Given a formula F,
    the SAT solver enumerates possible truth assignments (M)
  - The theory solver is a decision procedure that checks
    whether the truth assignments are satisfiable in the theories

Example of a Basic Algorithm

$$F: \quad \underbrace{g(a) = c}_{1} \land \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \lor \underbrace{g(a) = d}_{3} \land \underbrace{c \neq d}_{\overline{4}}$$

SAT SOLVER

THEORY SOLVER
(Empty uninterpreted functions)

$1, \overline{2} \lor 3, \overline{4}$

⟵ send F
unsat

choose a model M ⟶

$1, \overline{2}, \overline{4}$

$1, \overline{2} \lor 3, \overline{4}, \overline{1} \lor 2 \lor 4$

⟵ send F
unsat

choose a model M ⟶

$1, 3, \overline{4}$

$1, \overline{2} \lor 3, \overline{4}, \overline{1} \lor 2 \lor 4, \overline{1} \lor 3 \lor 4$

unsat ⟵ send F

Basic Algorithm

- DEFINITION
  T-conflict: check for conflicts with respect to theory T

- DESIGN
  Two independent solvers:
  SAT solver that is independent of theory
  Theory solver that checks for T-conflicts clause by clause

- ALGORITHM

  Repeat

  SAT Solver: propose a full propositional model M for formula F
  if no M is found, F is unsatisfiable.

  Theory Solver:
  Check for T-conflict on model M
  If M is satisfiable: F is satisfiable
  If M has a T-conflict, add constraint to F

3. Improvements (Example, Algorithm, Rules)

A. Incremental model decision:

Don't just guess the entire model (all the assignments)
Check each assignment incrementally, not all at once.
Two kinds of assignments
  Propagated: deduced from the Boolean expression
  Arbitrary <u>decisions</u> (marked with •)
If unsatisfiable, backtrack on decisions
(Propagate, Decide, T-Conflict, Learn, Restart)

B. Use the theory to propagate and learn (T-Propagate)

C. Backtrack to conflicting decision (Conflict, Explain, Backjump)

A. Incremental: Example

$$F: \quad \underbrace{g(a) = c}_{1} \wedge \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \wedge \underbrace{c \neq d}_{\overline{4}}$$

M	F	C	Rule
	$1, \overline{2} \vee 3, \overline{4}$		
$1\,\overline{4}$	$1, \overline{2} \vee 3, \overline{4}$		Propagate+, OK
$1\,\overline{4} \cdot \overline{2}$	$1, \overline{2} \vee 3, \overline{4}$		Decide
$1\,\overline{4} \cdot \overline{2}$	$1, \overline{2} \vee 3, \overline{4}$	$\overline{1} \vee 2 \vee 4$	T-Conflict
$1\,\overline{4} \cdot \overline{2}$	$1, \overline{2} \vee 3, \overline{4}, \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	Learn
$1\,\overline{4}$	$1, \overline{2} \vee 3, \overline{4}, \overline{1} \vee 2 \vee 4$		Restart
$1\,\overline{4}\,2\,3$	$1, \overline{2} \vee 3, \overline{4}, \overline{1} \vee 2 \vee 4$		Propagate+
$1\,\overline{4}\,2\,3$	$1, \overline{2} \vee 3, \overline{4}, \overline{1} \vee 2 \vee 4, \overline{1} \vee \overline{3} \vee 4 \vee \overline{2}$	$\overline{1} \vee \overline{3} \vee 4 \vee \overline{2}$	T-Conflict, Learn
fail			Fail

A. Incremental: Algorithm

- Build incrementally a satisfying truth assignment M for a CNF formula F
  - CNF: conjunction of disjunctions of literals

- Algorithm

  Apply rules until there is a satisfying model or Fail,
  in decreasing priority

  T-conflict: if all the literals $l_1, \ldots, l_n$ in M cannot be satisfied by T,
  set the conflict clause $\quad C := \overline{l_1} \vee \ldots \vee \overline{l_n}$

  Learn: add the new conflict constraint to F

  Restart: Restart the SAT server after learning a new constraint

  Propagate: deduce the truth value of a literal from M and F

  Decide: guess a truth value

  Fail: if there is no decision to roll back

A. Incremental: Rules

Propagate	Deduce the truth value of a literal from M and F $$\frac{l_1 \vee \ldots \vee l_n \vee l \in \mathsf{F} \qquad \overline{T}_1, \ldots, \overline{T}_n \in \mathsf{M} \qquad l, \overline{T} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\, l}$$
Decide	Guess a truth value $$\frac{l \in \mathrm{Lit}(\mathsf{F}) \qquad l, \overline{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M} \bullet l}$$
T-Conflict	If all the literals $l_1, \ldots, l_n$ in M cannot be satisfied by T, set the conflict clause $\mathsf{C} := \overline{l_1} \vee \ldots \vee \overline{l_n}$ $$\frac{\mathsf{C} = \mathsf{no} \quad l_1, \ldots, l_n \in \mathsf{M} \quad l_1, \ldots, l_n \models_T \in \bot}{\mathsf{C} := \overline{l_1} \vee \ldots \vee \overline{l_n}}$$
Learn	Add the new learned constraint to formula F $$\frac{\mathsf{F} \models_{\mathrm{P}} C \qquad C \notin \mathsf{F}}{\mathsf{F} := \mathsf{F} \cup \{C\}}$$
Restart	Restart the SAT solver $$\frac{}{\mathsf{M} := \mathsf{M}^{[0]} \quad \mathsf{C} := \mathsf{no}}$$ Each Decide defines a new level $\mathsf{M}^{[i]}$ means Model M up to level i

A. Incremental: Rules

Fail

Fail if there is no decision to roll back

$$\frac{l_1 \vee \ldots \vee l_n \in F \qquad \overline{l_1}, \ldots, \overline{l_n} \in M \qquad \bullet \notin M}{fail}$$

Improvements (Example, Algorithm, Rules)

A. Incremental model decision
   (Propagate, Decide, T-Conflict, Learn, Restart)

B. Use the theory to propagate and learn (T-Propagate)
   In A, propagation is based only on the Boolean expression;
   Here, we add propagation due to the Theories

C. Backtrack to conflicting decision (Conflict, Explain, Backjump)

B: T-Propagate: Example

$$g(a) = c \wedge f(g(a)) \neq f(c) \vee g(a) = d \wedge c \neq d$$

$$\underbrace{g(a) = c}_{1} \wedge \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \wedge \underbrace{c \neq d}_{\overline{4}}$$

M	F	C	Rule
	$1, \overline{2} \vee 3, \overline{4}$		
$1\,\overline{4}$	$1, \overline{2} \vee 3, \overline{4}$		Propagate+
$1\,\overline{4}\,2$	$1, \overline{2} \vee 3, \overline{4}$		T-Propagate ($1 \models_T 2$)
$1\,\overline{4}\,2\,\overline{3}$	$1, \overline{2} \vee 3, \overline{4}$		T-Propagate ($1, \overline{4} \models_T \overline{3}$)
Fail	$1, \overline{2} \vee 3, \overline{4}$		

Notation:
$1 \models_T 2$: predicate 1 entails predicate 2 under theory T
  If predicate 1 is true, predicate 2 is true under theory T

B. T-Propagate: Algorithm

- Add T-Propagate to increase deduced values using theory T

- Algorithm

  Apply rules until there is a satisfying model or Fail,
  in decreasing priority

  T-conflict: if all the literals $l_1,\ldots, l_n$ in M cannot be satisfied by T,
  set the conflict clause $C := \overline{l_1} \vee \ldots \vee \overline{l_n}$

  Learn: add the new conflict constraint to F

  Restart: Restart the SAT server after learning a new constraint

  Propagate: deduce the truth value of a literal from M and F

  T-Propagate: deduce the truth value of a literal using theory T

  Decide: guess a truth value

  Fail: if there is no decision to roll back

B. T-Propagate: Rules

Deduce the truth value of a literal using theory T

T-Propagate $\dfrac{l \in \mathrm{Lit}(F) \quad M \models_T l \quad l, \overline{l} \notin M}{M := M\, l}$

Improvements (Example, Algorithm, Rules)

A. Incremental model decision
    (Propagate, Decide, T-Conflict, Learn, Restart)

B. Use the theory to propagate and learn (T-Propagate)

C. Backtrack to conflicting decision (Conflict, Explain, Backjump)
    Find the root cause that causes the conflict
    Backtrack by skipping decisions immaterial to the conflict

C. Backjumping: Example

$$F := \{1, \overline{1}\vee 2, \overline{3}\vee 4, \overline{5}\vee \overline{6}, \overline{1}\vee \overline{3}\vee 7, \overline{2}\vee \overline{5}\vee 6\vee \overline{7}\}$$

M	F	C	Rule
	F		
1	F		Propagate
12	F		Propagate
12•3	F		Decide
12•34	F		Propagate
12•34•5	F		Decide
12•34•5$\overline{6}$	F		Propagate
12•34•5$\overline{6}$7	F		Propagate
12•34•5$\overline{6}$7	F	$\overline{2}\vee \overline{5}\vee 6\vee \overline{7}$	Conflict

C. Backjumping: Example Details

$F := \{1, \overline{1} \vee 2, \overline{3} \vee 4, \overline{5} \vee \overline{6}, \overline{1} \vee \overline{5} \vee 7, \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$

$M := 12 \bullet 34 \bullet \overline{5}67$

$C := \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$

> Given $p \vee A$ and $\neg p \vee B$, add the resolvent $A \vee B$
>
> Resolve $\quad \dfrac{p \vee A \quad \neg p \vee B}{A \vee B}$

- Conflict: $\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$ last literal choice is 7
- Explain: Choice of 7 is due to $\overline{1} \vee \overline{5} \vee 7$
- Learn: $\overline{1} \vee \overline{2} \vee \overline{5} \vee 6$ = resolvent of $\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$ and $\overline{1} \vee \overline{5} \vee 7$
- Conflict: $\overline{1} \vee \overline{2} \vee \overline{5} \vee 6$ last literal choice is 6
- Explain: Choice of $\overline{6}$ is due to $\overline{5} \vee \overline{6}$
- Learn: $\overline{1} \vee \overline{2} \vee \overline{5}$ = resolvent of $\overline{1} \vee \overline{2} \vee \overline{5} \vee 6$ and $\overline{5} \vee \overline{6}$
- Conflict: $\overline{1} \vee \overline{2} \vee \overline{5}$
- Backjump: Choice of 5 was a decision
  - Conflict involves literals 1, 2, 5, the decision of 5 is at level 2
  - 1, 2 are both level 0
  - Back jump to level 0, propagate 1,2 and choose $\overline{5}$

C. Backjumping: Example

$$F := \{1, \overline{1}\vee 2, \overline{3}\vee 4, \overline{5}\vee\overline{6}, \overline{1}\vee\overline{3}\vee 7, \overline{2}\vee 5\vee 6\vee 7\}$$

M	F	C	Rule
	F		
1	F		Propagate
12	F		Propagate
12•3	F		Decide
12•34	F		Propagate
12•34•5	F		Decide
12•34•5$\overline{6}$	F		Propagate
12•34•5$\overline{6}$7	F		Propagate
12•34•5$\overline{6}$7	F	$\overline{2}\vee 5\vee 6\vee 7$	Conflict
12•34•5$\overline{6}$7	F	$\overline{1}\vee 2\vee 5\vee 6$	Explain with $\overline{1}\vee\overline{5}\vee 7$
12•34•5$\overline{6}$7	F	$\overline{1}\vee 2\vee 5$	Explain with $\overline{5}\vee 6$
12$\overline{5}$	F		Backjump
12$\overline{5}$•3	F		Decide
12$\overline{5}$•3$\overline{4}$	F		Propagate (SAT)

C. Backjumping: Algorithm

- If M is T-unsatisfiable,
  backtrack to some point where the assignment was still T-satisfiable

- Trace back to the decision that causes the conflict C
  - Let l be the last literal choice that causes conflict C,
    if l is a decision, proceed to the next step
  - Explain:
    if $\bar{l}$ is chosen due to clause C_1 in F (explanation),
    new conflict C = resolvent of C and C_1 (eliminating l)
  - Repeat the above

- Backtrack by skipping decisions immaterial to conflict C
  - Backjump: Keep model up to level i,
    (highest level of satisfiable decisions involved in C);
    add the latest literal l in C

C. Backjumping Rules

	If one of the literals $\overline{l_1}, \ldots, \overline{l_n}$ in M must be inverted in F, set the conflict clause $C := l_1 \vee \ldots \vee l_n$
Conflict	$$\frac{C = no \qquad l_1 \vee \ldots \vee l_n \in F \qquad \overline{l_1}, \ldots, \overline{l_n} \in M}{C := l_1 \vee \ldots \vee l_n}$$

	Given conflict C involving latest l, chosen due to a clause in F, their resolvent is the new conflict
Explain	$$\frac{C = l \vee D \qquad l_1 \vee \ldots \vee l_n \vee \overline{l} \in F \qquad \overline{l_1}, \ldots, \overline{l_n} <_M \overline{l}}{C := l_1 \vee \ldots \vee l_n \vee D}$$

	Keep model up to level i (highest level of sat. decisions involved in C); add latest l in C
Backjump	$$\frac{C = l_1 \vee \ldots \vee l_n \vee l \qquad \text{lev } \overline{l_1}, \ldots, \text{lev } \overline{l_n} \leq i < \text{lev } \overline{l}}{C := no \qquad M := M^{[i]} \, l}$$

$l <_M l'$ if l occurs before l' in M
$M^{[i]}$ means Model M up to level i
lev $l = i$ iff l occurs in decision level i of l

C. Backjumping Rules (cont.)

Replace

Fail if there is no decision to roll back

Fail

$$\frac{l_1 \vee \ldots \vee l_n \in F \qquad \overline{l_1}, \ldots, \overline{l_n} \in M \qquad \bullet \notin M}{fail}$$

with

Fail if there is a conflict and there is no decision to roll back

Fail

$$\frac{C \neq no \qquad \bullet \notin M}{fail}$$

Putting it All Together

Apply rules until there is a satisfying model or Fail,
in decreasing priority

T-conflict: if all the literals $l_1,\ldots, l_n$ in M cannot be satisfied by T,
set the conflict clause $\quad C := \overline{l}_1 \vee \ldots \vee \overline{l}_n$

Explain: If the last literal l in conflict C is not a decision,
If $\overline{l}$ chosen due to clause C_1 in F (explanation),
new conflict = resolvent of C and C_1

Backjump: Keep model up to level i,
(highest level of satisfiable decisions involved in C);
add the latest literal l in C

Learn: add the new conflict constraint to F

Propagate: deduce the truth value of a literal from M and F

T-Propagate: deduce the truth value of a literal using theory T

Decide: guess a truth value

Fail: if there is no decision to roll back

Restart: Restart on the learned F if too many conflicts have been found

Summary

- Use of SMT to handle path sensitivity
  in test generation & static analysis

- Basic optimizations in SMT Solver

  – Incremental model decision
    (Propagate, Decide, T-Conflict, Learn, Restart)

  – Use the theory to propagate and learn (T-Propagate)

  – Smart backtracking (Conflict, Explain, Backjump)

- Many more optimizations to handle combinations of theory etc

- Practical tool: Z3 SMT solver

  – A widely used, open-source project from Microsoft

Further Readings

- "Satisfiability Modulo Theories"
  Clark Barrett and Cesare Tinelli.
  In *Handbook of Model Checking*,
  (Ed Clarke, Thomas Henzinger, and Helmut Veith, eds.), 2016.
  In preparation.
  http://theory.stanford.edu/~barrett/pubs/BT16-abstract.html

- "Satisfiability Modulo Theories"
  Clark Barrett, Roberto Sebastiani, Sanjit Seshia, and Cesare Tinelli.
  In *Handbook of Satisfiability*,
  vol. 185 of Frontiers in Artificial Intelligence and Applications,
  (Armin Biere, Marijn J. H. Heule, Hans van Maaren, and Toby Walsh, eds.),
  Feb. 2009, pp. 825-885. http://theory.stanford.edu/~barrett/pubs/BSST09-abstract.html

- Satisfiability Modulo Theories: Introduction and Applications
  Leonardo De Moura, Nikolaj Bjørner
  Communications of the ACM, Vol. 54 No. 9, Pages 69-77
  Sept 2011