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# Note of Topological Quantum Field Theory(CS,$BF$)

**Lianghong Mo**[1]

[1] *School of Physics, Sun Yat-sen University, Guangzhou, 510275, China*

*E-mail:* molh3@mail2.sysu.edu.cn

Abstract: Abstract...

## Contents

# 1 Chern-Simons Theory

## 1.1 Pure Chern Siomons term

$$\mathcal{L}_{\rfloor\int} = \frac{k}{4\pi}AdA - A_\mu J^\mu \tag{1.1}$$

### 1.1.1 Gauge Transformation

Leave out a total derivative

### 1.1.2 Euler Lgg equation

$$J^\mu = \frac{k}{4\pi}\epsilon^{\mu\nu\rho}F_{\nu\rho} = \frac{k}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu A_\rho \tag{1.2}$$

Pf:

$$\mathcal{L} = \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda - A_\mu J^\mu \tag{1.3}$$

$$\delta S = \int d^3x \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}(\delta A_\mu \partial_\nu A_\lambda + A_\mu\partial_\nu\delta A_\lambda) - J^\mu\delta A_\mu \tag{1.4}$$

$$= \int d^3x \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}(\delta A_\mu \partial_\nu A_\lambda - (\partial_\nu A_\mu)\delta A_\lambda) - J^\mu\delta A_\mu \tag{1.5}$$

$$= \int d^3x [\frac{k}{4\pi}\epsilon^{\mu\nu\lambda}(\partial_\mu A_\nu - \partial_\nu A_\mu) - J^\lambda]\delta A_\lambda \tag{1.6}$$

Thus the equation of motion is

$$\frac{k}{4\pi}\epsilon^{\mu\nu\lambda}(\partial_\mu A_\nu - \partial_\nu A_\mu) - J^\lambda = 0 \tag{1.7}$$

In terms of component:

$$\rho = J^0 = \frac{k}{2\pi}B \tag{1.8}$$

$$J^i = \frac{k}{2\pi}\epsilon^{ij}E_j \tag{1.9}$$

Thus

$$\nu = \frac{\rho}{\frac{B}{\Phi_0}} = \frac{k}{2\pi}\Phi_0 \tag{1.10}$$

$$\sigma_{xy} = \frac{J^i}{E^j} = \frac{k}{2\pi} \tag{1.11}$$

$$\tag{1.12}$$

which require

$$k = \frac{e^2\nu}{\hbar} \tag{1.13}$$

Ignoring the constant, we can find that the coefficient k is the the filling factor.

### 1.1.3 Statistics

Consider nonrelativistic point charged particle moving in the palne with magnetic flux attach to them, the density is

$$\rho(\mathbf{x}, t) = e\sum_{a=1}^{N}\delta(\mathbf{x} - \mathbf{x_a}(\mathbf{t})) \tag{1.14}$$

describes N such particles, and the$\mathbf{x_a}$ refers to the trajectory.

(It is easy to obtain that $\Phi_I = \int_{CI} d^2xB = \frac{e}{2\pi k}$)

The corresponding $\mathbf{j}(\mathbf{x}, \mathbf{t}) = e\sum_{a=1}^{N}\dot{\mathbf{x}}_\mathbf{a}\delta(\mathbf{x} - \mathbf{x_a}(\mathbf{t}))$ . The attached magnetic flux is

$$\mathbf{B}(\mathbf{x}, t) = \frac{2\pi}{\kappa}e\sum_{a=1}^{N}\delta(\mathbf{x} - \mathbf{x_a}(\mathbf{t})) \tag{1.15}$$

Choosing the gauge as follows and we can solve out the magnetic vetor potential of the system:

$$A_0 = 0 \tag{1.16}$$
$$\nabla \cdot A = 0 \tag{1.17}$$
$$\tag{1.18}$$

Thus

$$A = \nabla \times \psi = \epsilon^{ijk}\partial_i\psi_j g_k(i, j, k = 1, 2, 3) \tag{1.19}$$

With1.15, and B is in one direction, thus we have:

$$\mathbf{B} = \epsilon^{ij}\partial_i A_j(i, j = 1, 2) \tag{1.20}$$
$$= \partial_1(\partial^3\psi_1 - \partial^1\psi_3) - \partial_2(\partial^2\psi_3 - \partial^3\psi_2) \tag{1.21}$$
$$= -(\partial_1\partial^1 + \partial_2\partial^2)\psi_3 \tag{1.22}$$
$$\tag{1.23}$$

Notice that in 2-d the derivative of the third dimension is equal to zero, then we have the pde problem as follows:

$$\partial_i\partial^i\psi_3 = -\frac{2\pi\rho}{k} \tag{1.24}$$
$$\tag{1.25}$$

The corresponding green function satiesfied:

$$\nabla^2 G(x, x') = \delta(x - x') \tag{1.26}$$
$$G(x, x') = \frac{1}{2\pi}\ln r \tag{1.27}$$

Pf:Using the rotation symmetry of delta function :

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial G}{\partial r}) = \delta(r) \tag{1.28}$$

condsider $r > 0$,

$$G = c \ln r \tag{1.29}$$

Integrate both sides of the equation 1.28, we get $c = \frac{1}{2\pi}$.

Integrate the source we obtain the solution of $\psi_3$

$$\psi_3(\mathbf{x}) = -\int dy^2 \frac{\rho(\mathbf{y})}{k} \ln r \tag{1.30}$$

$$\begin{aligned} A_j(\mathbf{x}) &= \epsilon^{ji}\partial_i\psi_3 & (1.31)\\ &= -\epsilon^{ji}\int dy^2 \frac{\rho(\mathbf{y})}{k}\partial_i \ln r & (1.32)\\ &= -\epsilon^{ji}\int dy^2 \frac{\rho(\mathbf{y})}{k}\frac{\partial_i r}{r} & (1.33)\\ &= -\epsilon^{ji}\int dy^2 \frac{\rho(\mathbf{y})}{k}\frac{x_i - y_i}{r^2} & (1.34)\\ & & (1.35) \end{aligned}$$

Thus generally

$$\begin{aligned} A_I^i(\vec{x}_1, \vec{x}_I, ..., \vec{x}_N) &= -\epsilon^{ij}\int d^2y \frac{x_i - y_j}{|\vec{x}-\vec{y}|^2}\rho(\vec{y}, t) & (1.36)\\ &= \frac{e}{k}\sum_{a=1}^{N}\epsilon_{ij}\frac{x_I^j - x_J^j(t)}{|\vec{x_I} - \vec{x_J}|^2} & (1.37) \end{aligned}$$

At first sight the non-local vector potential seems badly divergent but it is not the case, because the coincident points does not belong to the configuration space.

Using the identity:

$$\partial_i \arg(\vec{x}) = -\epsilon_{ij}x^j/|\vec{x}|^2 \tag{1.38}$$

Then

$$A_i(\vec{x}) = \frac{e}{k}\sum_{a=1}^{N}\partial_i \arg(\vec{x} - \vec{x}_a) \tag{1.39}$$

The gauge transformation of the wavefunction is

$$\psi(\vec{x}) \to \psi(\tilde{\vec{x}}) = \exp\left(-i\frac{e^2}{k}\sum_{a=1}^{N} arg(\vec{x} - \vec{x}_a)\right)\psi(\vec{x}) \tag{1.40}$$

This lack of single-valueness is the nontrivial remnant of the CS term.

When one such particle moves adiabatically around one another, the wavefunction require an AB phase as:

$$\exp\left(ie\oint_C \vec{A}\cdot dx\right) = \exp\left(i2\pi\frac{e^2}{k}\right) \tag{1.41}$$

### 1.1.4 Generally form and LGG

$$A_I^i(\vec{x}_1, ..., \vec{x}_I, \vec{x}_N) = \frac{e}{k} \sum_{J \neq I} \partial_i \tan^{-1}(\frac{y_I - y_J}{x_I - x_J}) \tag{1.42}$$

The lgg:

$$\mathcal{L}_{matter} + \mathcal{L}_{int} = \sum_{I=1}^{N} (\frac{1}{2} m v_I^2 - e \vec{v}_I \cdot \vec{A}_I(..., \vec{x}_I, ...)) \tag{1.43}$$

$$= T - \frac{e^2}{k} \sum_I \sum_{I \neq J} v_I^i \frac{\partial}{\partial r_I^i} \tan^{-1}(\frac{y_I - y_J}{x_I - x_J}) \tag{1.44}$$

$$= T - \frac{e^2}{k} \sum_I \sum_{I \neq J} v_I^i \frac{\partial}{\partial r_I^i} \tan^{-1}(\frac{y_I - y_J}{x_I - x_J}) \tag{1.45}$$

$$= T - \frac{e^2}{k} \sum_{I<J} (v_I^i - v_J^i) \frac{\partial}{\partial r_I^i} \tan^{-1}(\frac{y_I - y_J}{x_I - x_J}) \tag{1.46}$$

$$\tag{1.47}$$

where we used the property

$$\frac{\partial}{\partial r_I^i} tan^{-1}(\frac{y_I - y_J}{x_I - x_J}) = -\frac{\partial}{\partial r_J^i} \tan^{-1}(\frac{y_I - y_J}{x_I - x_J}) \tag{1.48}$$

what's more,

$$\frac{d}{dt} \tan^{-1}(\frac{y_I - y_J}{x_I - x_J}) = (v_I^i \frac{\partial}{\partial r_I^i} + v_J^i \frac{\partial}{\partial r_J^i}) \tan^{-1}(\frac{y_I - y_J}{x_I - x_J}) \tag{1.49}$$

$$= (v_I^i - v_J^i) \frac{\partial}{\partial r_I^i} \tan^{-1}(\frac{y_I - y_J}{x_I - x_J}) \tag{1.50}$$

Thus, we can rewrite the Lgg

$$\mathcal{L}_{int} = -\frac{e^2}{k} \sum_{I<J} \frac{d}{dt} \tan^{-1}(\frac{y_I - y_J}{x_I - x_J}) \tag{1.51}$$

$$\exp(iS_{int}) = \exp i[\int_t^{t'} d\tau \mathcal{L}_{int}] = \exp\left[-i\frac{e^2}{k} \sum_{I<J} \frac{d}{d\tau} \tan^{-1}(\frac{y_I - y_J}{x_I - x_J})\right] \tag{1.52}$$

The statistics of the Anyons is$\frac{e^2}{k}$ which is in accordance with the landau level filling factor:

$$\nu = \frac{e^2}{k} \tag{1.53}$$

But why the coefficient is statistics, we can have a pictorial understanding!

From the former lecture ( Knot and topological quantum field theory,mo,2021), we know that, the statistics factor or precisely the weight of different braiding process of the propagator is

$$\chi(\alpha) = \exp\left[-i\nu \sum_{I<J} \int_t^{t'} d\tau \frac{d}{d\tau} \tan^{-1}(\frac{y_I - y_J}{x_I - x_J})\right] \tag{1.54}$$

### 1.2 Why the coefficent is quantized?

Consider a compact 3d manifold $M^3$ , which is the boundary of a 4d manifold , that is $\partial M^4 = M^3$. Stokes theory:

$$\frac{k}{4\pi}\int_{M^3} a\wedge da = \frac{k}{4\pi}\int_{M^4} da\wedge da \tag{1.55}$$

$$\frac{k}{4\pi}\int_{M^3} a\wedge da = -\frac{k}{4\pi}\int_{M'^4} da\wedge da \tag{1.56}$$

where

$$\partial(M^4 + M'^4) = 0 \tag{1.57}$$

It is easy to figure out that manifold $X^4 = M^4 + M'^4$ is a closed manifold.

Thus the 2+1 d CS action can be reformulated in 3+1 d in two different way:

$$S_{3+1} = \frac{k}{4\pi}\int_{M^4} da\wedge da \tag{1.58}$$

$$S'_{3+1} = -\frac{k}{4\pi}\int_{M'^4} da\wedge da \tag{1.59}$$

$$\tag{1.60}$$

In order to maintain the partition function, the two actions must :

$$S_{3+1} - S'_{3+1} = \frac{k}{4\pi}\int_{M^4} da\wedge da - [-\frac{k}{4\pi}\int_{M'^4} da\wedge da] = 2\pi n, n\in \mathrm{z} \tag{1.61}$$

which is equivalent to

$$\frac{k}{4\pi}\int_{X^4} da\wedge da = 2\pi n \tag{1.62}$$

According to Atiya-Singer index theorem in even dimension(pf?), $\int_{X^4} da\wedge da$ is quantized :

$$\int_{X^4} da\wedge da = 2(2\pi)^2\cdot m = 8\pi^2 m \tag{1.63}$$

Thus

$$km = n \tag{1.64}$$

k must be an integer.

#### 1.2.1 The coefficient of different BF term

#### 1.2.2 The coefficient of Nonabelian CS

### 1.3 Effective theory of Laughlin wavefunction

Since the hall conductance is $\sigma_{xy} = \frac{\nu e^2}{2\pi\hbar}$, the response of the current to the electromagnetic field is :

$$J^\mu = \frac{e^2}{2\pi\hbar}\epsilon^{\mu\nu\rho}\partial_\nu A_\rho \tag{1.65}$$

We can use a new variable(a gauge field) to substitute the current(which means that the new gauge field contain and maintain all the observable information ) and bring in much more algebra structure. Such a U(1) gauge field $a_\mu$ can be introduced as follows:

$$J^\mu = \frac{1}{2\pi}\partial_\nu a_\lambda \epsilon^{\mu\nu\lambda} \tag{1.66}$$

Surprisingly, we can obtain the above equation of motion from the blow Lgg analogous to the CS term:

$$\mathcal{L} = -\frac{m}{4\pi} a_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda} + \frac{e^2}{2\pi\hbar}\epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda \tag{1.67}$$

## 1.4 K Matrix

### 1.4.1 K matrix CS

$$\mathcal{L} = [-\frac{1}{4\pi} K^{IJ} a_{I\mu}\partial_\nu a_{J\lambda}\epsilon^{\mu\nu\lambda} + \frac{1}{2\pi} q^I A_\mu \partial_\nu a_{I\lambda}\epsilon^{\mu\nu\lambda}] + l_I a^I_\mu j^\mu \tag{1.68}$$

Integrate out $a^I_\mu$, we obtain the effective action:

$$S_{eff} = \int d^3x \frac{q^I q^J K^{-1}_{IJ}}{4\pi} A_\mu \partial_\nu A_\lambda \epsilon^{\mu\nu\lambda} \tag{1.69}$$

$$+ l^I K^{-1}_{IJ} q^J A_\mu j^\mu \tag{1.70}$$

where $j^\mu$ is

$$j^0 = \sum_{\alpha=0}^{N} \delta(\mathbf{x} - \mathbf{x}_\alpha) \tag{1.71}$$

$$\mathbf{j} = \sum_{\alpha=0}^{N} \dot{\mathbf{x}_\alpha} \delta(\mathbf{x} - \mathbf{x}_\alpha) \tag{1.72}$$

Pf:

$$S = \int d^3x [-\frac{1}{4\pi} K^{IJ} a_{I\mu}\partial_\nu a_{J\lambda}\epsilon^{\mu\nu\lambda} \tag{1.73}$$

$$+ \frac{1}{2\pi} q^I A_\mu \partial_\nu a_{I\lambda}\epsilon^{\mu\nu\lambda}] + \kappa \frac{K_{IJ}}{4\pi} a^I_\mu(x)\partial^\mu\partial^\lambda a_\lambda \tag{1.74}$$

$$= \int d^3x [-\frac{1}{4\pi} K^{IJ} a_{I\mu}\partial_\nu a_{J\lambda}\epsilon^{\mu\nu\lambda} \tag{1.75}$$

$$+ \frac{1}{2\pi} q^I A_\mu \partial_\nu a_{I\lambda}\epsilon^{\mu\nu\lambda}] + \kappa \frac{K_{IJ}}{4\pi} a^I_\mu(x)\partial^\mu\partial^\lambda a_\lambda \tag{1.76}$$

$$= \int d^3x d^3y \frac{1}{4\pi} a^I_\mu(x)[K_{IJ}\delta(x-y)(\epsilon^{\mu\nu\lambda}\partial_\nu + \kappa\partial^\mu\partial^\lambda)] a^J_\lambda(y) \tag{1.77}$$

$$+ \int d^3x \frac{q^I}{2\pi} \partial_\mu A_\nu(x)\epsilon^{\mu\nu\lambda} a^I_\lambda(x) \tag{1.78}$$

$$= \frac{1}{2\pi}\int d^3x d^3y \frac{1}{2} a^I_\mu(x)[K_{IJ}\delta(x-y)(\epsilon^{\mu\nu\lambda}\partial_\nu + \kappa\partial^\mu\partial^\lambda)] a^J_\lambda(y) \tag{1.79}$$

$$+ \frac{1}{2\pi}\int d^3x q^I \partial_\mu A_\nu(x)\epsilon^{\mu\nu\lambda} a^I_\lambda(x) \tag{1.80}$$

Integrate out the gauge field $a_\mu$ to obtain the effective action, which is defined as follows;

$$S[A_\mu] = \int D[a] \exp(iS) \propto \exp(iS_{eff}) \tag{1.81}$$

From the appendix, we know that:

$$\int_{-\infty}^{\infty} d^n x \exp i(\frac{1}{2}\mathbf{x}^T \cdot K \cdot \mathbf{x} + \mathbf{J}^T \cdot \mathbf{x}) = \sqrt{\frac{(2\pi i)^n}{det\ K}} \exp\left(-i\frac{1}{2}\mathbf{J}^T \cdot K^{-1} \cdot \mathbf{J}\right) \tag{1.82}$$

Compare the coefficient and variables, we can know that:

$$x \leftrightarrow a \tag{1.83}$$

$$K \leftrightarrow [K_{IJ}\delta(x-y)(\epsilon^{\mu\nu\lambda}\partial_\nu + l\partial^\mu\partial^\lambda)] \tag{1.84}$$

$$\mathbf{J} \leftrightarrow q^I \partial_\mu A_\nu(x)\epsilon^{\mu\nu\lambda} \tag{1.85}$$

Then it can be seen that:

$$S_{eff} = -\frac{1}{2}\int d^3xd^3yq^Iq^JK_{IJ}^{-1}\partial_\alpha A_\beta(x)\epsilon^{\alpha\beta\lambda}D_{\lambda\rho}\partial_\mu A_\nu\epsilon^{\mu\nu\rho} \tag{1.86}$$

where $D_{\lambda\rho}$ satiesfied:

$$(\epsilon^{\mu\nu\gamma}\partial_\gamma + \kappa\partial^\mu\partial^\nu)D_{\lambda\rho}(x-y) = \delta^\mu_\lambda\delta^\nu_\rho\delta(x-y) \tag{1.87}$$

Thus

$$\sigma_{xy} = q^Iq^JK_{IJ}^{-1} \tag{1.88}$$

$$Q_q = l^IK_{IJ}^{-1}q^J \tag{1.89}$$

braiding phase:

$$\exp\{il_1^IK_{IJ}^{-1}l_2^J\}2\pi \tag{1.90}$$

filing fraction :

$$\nu = \mathbf{q^T}K^{-1}\mathbf{q} = \frac{1}{m \pm \frac{1}{m_1 \pm \frac{1}{m_2 \pm \cdots}}} \tag{1.91}$$

...

### 1.4.2 Physical meaning of K matrix from the perspective of Laugnlin wavefucntion

The multilayer FQH states have the form

$$\prod_{a,b,i,j}(z_{ai} - z_{bj})^{k_{ab}/2}Exp(-\sum|z_{ai}|^2) \tag{1.92}$$

where the $z_{ai}$ is the coordinate of the ith electron in the ath layer. The K matrix describes the patterns of zeros in the wavefunction which determines the dancing mode of the electrons. The zeros can also be interpreted as the number of flux quanta attached to each electron

## 2 Different Ways to Compute GSD

### 2.1 Wave function method

#### 2.1.1 Abelian FQH states

Considering the CS term and Maxwell term:

$$S = \int d^3x \frac{K_{IJ}}{4\pi}\epsilon^{\mu\nu\lambda}a^I_\mu \partial_\nu a^J_\lambda + \tag{2.1}$$

$$\sqrt{g}g^{\mu\lambda}g^{\mu\sigma}\frac{1}{f^2}F_{\mu\nu}F_{\lambda\sigma} \tag{2.2}$$

$$D = (det\ K)^g \tag{2.3}$$

Assume K=k,g=1

On a torus, the ground state properties are determined by constant gauge potentials(independent of space but depend on time):

$$a_0(x_1, x_2, t) = 0 \tag{2.4}$$

$$a_1(x_1, x_2, t) = \frac{2\pi x(t)}{L_1} \tag{2.5}$$

$$a_2(x_1, x_2, t) = \frac{2\pi y(t)}{L_2} \tag{2.6}$$

$$\tag{2.7}$$

where $L_i$ is the size of the torus.Insert the above relations into equation **??**,we have:

$$L = \pi k(y\dot{x} - x\dot{y}) + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \tag{2.8}$$

large transformation of a ($V = e^{i2\pi x_1/L_1}$ or $V = e^{i2\pi x_2/L_2}$))

$$a_\mu \to a_\mu - iV^{-1}\partial_\mu V \tag{2.9}$$

is equivalent to

$$(x, y) \to (x + 1, y) or (x, y) \to (x, y + 1) \tag{2.10}$$

Now we focus on the GSD( actually we know the answer from the Landau level, k-fold): Firstly, set $m = 0$,

$$S = \int dt\pi k(y\dot{x} - x\dot{y}) = \int dt 2\pi k x\dot{y} \tag{2.11}$$

(where we have throw away a total derivative)

$$p_y = \frac{\delta L}{\delta \dot{y}} = 2\pi k x \tag{2.12}$$

Commutation relation :

$$[x, y] = \frac{i}{2\pi k} \tag{2.13}$$

(easy to check that the hamiltonian is zero).

**wave function**:

The periodicity condition2.10 implies that

$$\psi(y) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi n y} \tag{2.14}$$

In momentum space,we have

$$\psi'(p_y) = \sum c_n \delta(p_y - 2\pi n) \tag{2.15}$$

We can substitute the $p_y$ into the above equation and obtain:

$$\phi(x) = \sum c_n \delta(kx - n) = \sum \frac{c_n}{k}\delta(x - \frac{n}{k}) \tag{2.16}$$

The periodicity condition of x requies that:

$$\begin{aligned}
\phi(x+1) &= \sum \frac{c_n}{k}\delta(x+1-\frac{n}{k}) && (2.17)\\
&= \sum \frac{c_n}{k}\delta(x - \frac{n-k}{k}) && (2.18)\\
&= \sum \frac{c_{n'+k}}{k}\delta(x - \frac{n'}{k}) && (2.19)\\
&= \sum \frac{c_{n+k}}{k}\delta(x - \frac{n}{k}) && (2.20)\\
&= \phi(x) && (2.21)
\end{aligned}$$

Thus:

$$c_n = c_{n+k} \tag{2.22}$$

the GSD is k.

Generally,

$$L = 2\pi K_{IJ} x^I \dot{y}^J \tag{2.23}$$

Commutation relation:

$$[x_I, y_J] = \frac{i}{2\pi}(K^{-1})_{IJ} \tag{2.24}$$

wave function:

$$\begin{aligned}
\psi(\mathbf{y}) &= \sum_{\mathbf{n}} c_{\mathbf{n}} e^{i2\pi \mathbf{n}\cdot\mathbf{y}} && (2.25)\\
\psi(\mathbf{x}) &= \sum_{\mathbf{n}} c_{\mathbf{n}} \delta(x_I - (K^{-1}_{IJ} n_J) && (2.26)\\
& && (2.27)
\end{aligned}$$

### 2.1.2 SU(2) non-abelian FQH

Effective Lgg:

$$L = \frac{1e_0^2}{} \tag{2.28}$$

## 2.2 Wilson Loop

For abelian FQH on the torus, we have make the assumption that a are in the form of 2.7, then, the wilson loops on the two cycles become:

$$W[\gamma_1] = exp(i \int_0^{L_1} a_1) = exp\{i2\pi x(t)\} \tag{2.29}$$

$$W[\gamma_2] = exp(i \int_0^{L_2} a_2) = exp\{i2\pi y(t)\} \tag{2.30}$$

$$\tag{2.31}$$

$$W[\gamma_1]W[\gamma_2] = exp(-\frac{i2\pi}{k})W[\gamma_2]W[\gamma_1] \tag{2.32}$$

Suppose that $|0>$ is the eigenstate of $W[\gamma_1]$ with eigenvalue 1,

$$W[\gamma_1]W[\gamma_2]|0> = exp(-\frac{i2\pi}{k})W[\gamma_2]W[\gamma_1]|0> \tag{2.33}$$

$$= exp(-\frac{i2\pi}{k})W[\gamma_2]|0> \tag{2.34}$$

$$\tag{2.35}$$

More generally,

$$W[\gamma_1]W^p[\gamma_2]|0> = exp(-\frac{i2\pi p}{k})W^p[\gamma_2]|0> \tag{2.36}$$

$$\tag{2.37}$$

Thus the ground state degeneracy must be k-fold.

### 2.2.1 Fractional Statistics

Let's compute the expectation value of a product of two wilson loop operators on two positively oriented closed contours $\gamma_1$ and $\gamma_2$

$$W[\gamma_1 \cup \gamma 2] =< \exp\left(i \oint_{\gamma_1 \cup \gamma 2} dx_\mu A_\mu\right) > \tag{2.38}$$

It is obvious that

$$< \exp\left(i \oint_{\gamma_1 \cup \gamma 2} dx_\mu A_\mu\right) >_{cs}= \tag{2.39}$$

!!! finished the detail

# 3 BF Theory

## 3.1 BdA term

$$S = \int_{M^4} \frac{k}{2\pi} B \wedge dA \tag{3.1}$$

### 3.1.1 Gauge Transformation

$$A \to A + \chi \tag{3.2}$$
$$B \to B + \xi \tag{3.3}$$
$$\tag{3.4}$$

where $\chi$ is a closed 1-form,$d\chi = 0$; $\xi$ is a closed 2-form,$d\xi$=0;

Without source any closed forms are allowed, but when coupled to the sources we require gauge invariance of Wilson operators:

$$W[L] = exp(i \int_L A) \tag{3.5}$$
$$W[\Sigma] = exp(i \int_\Sigma B) \tag{3.6}$$
$$\tag{3.7}$$

for any oriented loop L and any compact oriented surface $\Sigma$ in $M^4$.

Gauge invariant require that:

$$\int_L \chi = 2\pi n \tag{3.8}$$
$$\int_\Sigma \xi = 2\pi m \tag{3.9}$$

### 3.1.2 Partition Function and Expectation of Wilson Operator

$$Z = \int DA\ DB \exp\Big(i \int_{M^4} \frac{k}{2\pi} B \wedge dA\Big) \tag{3.10}$$

And the expectation of the wilson loop and surface variables is given by the path integral with sources:

$$< W[L], W[\Sigma] >= \frac{\int DA\ DB \exp\big(i \int_{M^4} \frac{k}{2\pi} B \wedge dA\big) + i \int_L A + i \int_\Sigma B}{\int DA\ DB \exp\big(i \int_{M^4} \frac{k}{2\pi} B \wedge dA\big)} \tag{3.11}$$

(remember to add calculation detail !

$$< W[L], W[\Sigma] >= \exp\Big(-\frac{2i}{\pi k} \int_\Sigma d\Sigma_{\mu\nu}(x) \int_L dl_\sigma(y) \epsilon^{\mu\nu\sigma\rho} \frac{(x-y)^\rho}{|x-y|^4}\Big) = \exp\Big[-\frac{2\pi i}{k} I(\Sigma, L)\Big] \tag{3.12}$$

where the I refers to the linking number of a contour L and a surface $\Sigma$.

Similar calculation can be applied to different BF term, which we focus on the physics picture here and later I will add the calculation details in the note.

Consider the K matrix CS term,

$$S[A] = \int_{M^3} \frac{K_{IJ}}{4\pi} A^I \wedge dA^J \tag{3.13}$$

Then

$$W_e[\{\gamma_I\}] = \exp\Big(i \sum_{I=1}^{s} e_I \int_{\gamma_I} A_I\Big) = \exp\Big(i \sum_{I=1}^{s} \int A_I \wedge \delta^\perp(\gamma_I)\Big) \tag{3.14}$$

$$< W_e[\{\gamma_I\}] >= \exp\big[-\pi i (K_{IJ}^{-1} e_I e_J LK(\gamma_I, \gamma_J)\big] \tag{3.15}$$

### 3.1.3 Canonical quantization

The space time is the product manifold $\mathbb{R}^1 \times M^3$, the action is

$$S = \int dt \int_{M^3} d^3x (\frac{k}{4\pi} \epsilon^{\mu\nu\rho\sigma} B_{\mu\nu} \partial_\rho A_\sigma + A_\mu j^\mu + \frac{1}{2} B_{\mu\nu} \Sigma^{\mu\nu}) \tag{3.16}$$

We can obtain the equation motion as follows(Considering the 0-component), which implies that in a compact manifold, the total charge and flux both vanish.

$$\frac{k}{2\pi} \partial_i B^i + j^0 = 0 \tag{3.17}$$

$$\frac{k}{2\pi} \epsilon^{ijk} \partial_j A^k + \Sigma^{0i} = 0 \tag{3.18}$$

The second constraint confines electromagnetic flux to the string world sheets and gives the analog of the Meissner effect in a BCS superconductor!

The first constraint couples the flux to the particle charge, similar to the CS term.

Substitute the constraints back into the action, we have

$$S = \int dt \int_{M^3} d^3x (\frac{k}{2\pi} B^i \dot{A}^i + a_i j^i + \frac{1}{2} B_{ij} \Sigma^{ij} \tag{3.19}$$

Commutator :

$$[A_i(x), B^j(y)] = \frac{2\pi i}{k} \delta^{(3)}(x, y) \tag{3.20}$$

In the temperal gauge,$A_0 = 0, B_{0i} = 0$,

$$H = \int_{M_3} d^3x (-A_i j^i - \frac{1}{2} B_{ij} \Sigma^{ij}) \tag{3.21}$$

wave function can be solved out as

$$\Phi[\theta, K; t] = \tag{3.22}$$

...

### 3.1.4 Linking Matrix Form

Similar to the K matrix, the BF term can be generalized into matrix form using linking matrix. Consider the excitation $\{L^m\}_{m=1}^p$ and $\{\Sigma^m\}_{m=1}^p$, of whom the linming matrix is

$$M^{nm} = I_{M_3}(\Sigma^m, L^n) = \sum_{I_{mn}} sgn(I_{mn}) \tag{3.23}$$

which counts the signed intersection $I_{mn}$ of the two types excitation.

And (...) the action are as follows:

$$S = \int dt [\int_{M_3} (Aj + B\Sigma + \frac{k}{2\pi} ...) + \frac{k}{2\pi} \dot{a}^l M_{lm} b^m] \tag{3.24}$$

where ...means that I haven't figure out the calculation detail.

We can also obtain the wave function and the gsd.

### 3.2 AAdA

$$S[A,B] = \int_{M^4} \sum_{I=1}^{3} B^I \wedge dA^I + \frac{\bar{p}}{(2\pi)^2} A^1 \wedge A^2 \wedge dA^3 \tag{3.25}$$

where $\bar{p} = \frac{N_1 N_2 p}{N_{12}}$ with $p \in \mathbf{Z_{N_{123}}}$

## 4 Some questions

How to obtain the commutation from the expectation and thus the calculation of the gsd return to the calculation of statistics. Besides the wavefunction method, it seems that the types of the excitation also reflect the gsd and we can obtain it from the wilson loop algebra.

How to understand the K matrix of IQHE? It seems that the diagonal of the K matrix is used to described the excitation, while there is no need to introduce such a freedom? And Tong supposed that the background term is enough to determined the IQHE.

Wilson Loop is not unique, then how to judge whether different types of definition have the same statistics?

The magic of KCS is that we can obtain any state of fqhe since any fraction can be expressed in the form of continue fraction and what we need to do is to add the term to form that specific fraction but how to do (quantum phase transition

# 5 Appendix

## 5.1 Gauss Integration

### 5.1.1 One variable Gauss Integration

$$\int_{-\infty}^{\infty} dx \exp\left(-\frac{1}{2}x^2\right) = \sqrt{2\pi} \tag{5.1}$$

**Proof:**

$$G^2 = \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}x^2} \int_{-\infty}^{\infty} dy e^{-\frac{1}{2}y^2} = 2\pi \int_0^{\infty} dr\, r e^{-\frac{1}{2}r^2} \tag{5.2}$$

$$= 2\pi \int_0^{\infty} d\omega e^{-\omega} = 2\pi \tag{5.3}$$

$$\int_{-\infty}^{\infty} dx \exp\left(-\frac{1}{2}ax^2\right) = \sqrt{\frac{2\pi}{a}} \tag{5.4}$$

where we can guess this by using the dimensional analysis.($ax^2 \sim 1, x \sim \frac{1}{\sqrt{a}}$)

Generally(derived from partial derivative on the parameter)

$$\int_{-\infty}^{\infty} dx x^{2n} \exp\left(-\frac{1}{2}ax^2\right) = (\frac{2\pi}{a})^{1/2}\frac{1}{a^n}(2n-1)!! \tag{5.5}$$

equivalently,

$$\langle x^{2n} \rangle \equiv \frac{\int_{-\infty}^{\infty} dx x^{2n} \exp\left(-\frac{1}{2}ax^2\right)}{\int_{-\infty}^{\infty} dx \exp\left(-\frac{1}{2}ax^2\right)} \tag{5.6}$$

$$= \frac{1}{a^n}(2n-1)!! \tag{5.7}$$

the factor $\frac{1}{a^n}$ can also be derived from dimensional analysis $ax^2 \sim 1, x^{2n} \sim \frac{1}{a^n}$. To remember the factor $(2n-1)$!!imagine 2n points function and connect them in pairs. The first one has $2n-1$ choices, the second one has $2n-3$ choices......

For instance,

$$\langle x^6 \rangle = \langle xxxxxx \rangle \tag{5.8}$$

$$\tag{5.9}$$

### 5.1.2 Gauss integration with source

$$\int_{-\infty}^{\infty} dx \exp\left(-\frac{1}{2}ax^2 + Jx\right) = \frac{2\pi}{a} \exp\left(-\frac{J^2}{2a}\right) \tag{5.10}$$

$$\int_{-\infty}^{\infty} dx \exp\left(-\frac{1}{2}ax^2 + iJx\right) = \frac{2\pi}{a} \exp\left(-\frac{-J^2}{2a}\right) \tag{5.11}$$

$$\int_{-\infty}^{\infty} dx \exp i(\frac{1}{2}ax^2 + Jx) = \frac{2\pi}{a} \exp\left(-i\frac{J^2}{2a}\right) \tag{5.12}$$

$$\tag{5.13}$$

**proof of the first one:**

$$-\frac{1}{2}ax^2 + Jx = -\frac{1}{2}a(x - J/a)^2 + \frac{J^2}{2a} \tag{5.14}$$

Thus, the integral over x can be shifted to $x + \frac{J}{a}$.

#### 5.1.3 Multi-variables Guass Integration

$$\int_{-\infty}^{\infty} d^n x \exp\left(-\frac{1}{2}\mathbf{x} \cdot K\mathbf{x}\right) = \int_{\infty}^{\infty} d^n x \exp\left(-\frac{1}{2}x^i K_{ij} K x^j\right) \tag{5.15}$$

Assume K is a symmetry matrix

$$K = S^T S, \quad det\ S = \sqrt{det\ K} \tag{5.16}$$

$$\int_{-\infty}^{\infty} d^n x \exp\left(-\frac{1}{2}\mathbf{x} \cdot K \cdot \mathbf{x}\right) = \int_{-\infty}^{\infty} d^n x \exp\left(-\frac{1}{2}x^i K_{ij} K x^j\right) \tag{5.17}$$

$$= \int_{-\infty}^{\infty} \frac{d^n y}{det\ S} \exp\left(-\frac{1}{2}y_i^2\right) \tag{5.18}$$

$$= \sqrt{\frac{(2\pi)^n}{det\ K}} \tag{5.19}$$

$$\tag{5.20}$$

#### 5.1.4 Multi-variables Gauss integration with source

$$\int_{-\infty}^{\infty} d^n x \exp\left(-\frac{1}{2}\mathbf{x} \cdot K \cdot \mathbf{x} + \mathbf{J} \cdot \mathbf{x}\right) = \sqrt{\frac{(2\pi)^n}{det\ K}} \exp\left(\frac{1}{2}\mathbf{J} \cdot K^{-1} \cdot \mathbf{J}\right) \tag{5.21}$$

**proof:** Diagonalize K with an orthogonal transformation O so that

$$D = OKO^{-1} = OKO^T \tag{5.22}$$

Suppose that

$$y_i = O_{ij} x_j, \mathbf{y} = O\mathbf{x} \tag{5.23}$$

The exponential in the integrand then becomes

$$-\frac{1}{2}\mathbf{x} \cdot K \cdot \mathbf{x} + \mathbf{J} \cdot \mathbf{x} = -\frac{1}{2}\mathbf{x}^T K \mathbf{x} + \mathbf{J}^T \mathbf{x} \tag{5.24}$$

$$= -\frac{1}{2}\mathbf{x}^T O^T OKO^T O\mathbf{x} + \mathbf{J}^T O^T O\mathbf{x} \tag{5.25}$$

$$= -\frac{1}{2}\mathbf{y}^T D\mathbf{y} + (\mathbf{OJ})^T \mathbf{y} \tag{5.26}$$

$$= \sum_i [-\frac{1}{2}D_{ii} y_i^2 + (OJ)_i y_i] \tag{5.27}$$

And

$$\int_{-\infty}^{\infty} d^n x = \int_{-\infty}^{\infty} d^n y \tag{5.28}$$

$$\int_{-\infty}^{\infty} d^n x \exp\left(-\frac{1}{2}\mathbf{y_i}\cdot K\cdot\mathbf{x} + \mathbf{J}\cdot\mathbf{x}\right) = \int_{-\infty}^{\infty} d^n y_i \prod_i \exp\left(-\frac{1}{2}D_{ii}y_i^2 + (OJ)_i y_i\right) \tag{5.29}$$

$$= \prod_i \int_{-\infty}^{\infty} dy_i \exp\left(-\frac{1}{2}D_{ii}y_i^2 + (OJ)_i y_i\right) \tag{5.30}$$

$$= \prod_i (\frac{2\pi}{D_{ii}})^{\frac{1}{2}} \exp\left[(OJ)_i^2/2D_{ii}\right] \tag{5.31}$$

$$= \frac{(2\pi)^N}{\det K} \exp\left[\frac{1}{2}J^T K^{-1} J\right] \tag{5.32}$$

where we used

$$(OJ)^T D^{-1}(OJ) = J^T O^T D^{-1} OJ \tag{5.33}$$

$$= J^T O^{-1} D^{-1} OJ \tag{5.34}$$

$$= J^T K^{-1} J \tag{5.35}$$

Differentiate equation**??** p times with repsect to $J_i, J_j, ...J_k, J_l$ and then set $J = 0$. For example, for $p = 1$, the integrand becomes

$$\exp\left[-\frac{1}{2}\mathbf{x}^T K\mathbf{x}\right] x_i \tag{5.36}$$

since the integrand is odd in $x_i$, theintegrand vanishes.

For $p = 2$the integrand becomes

$$\tag{5.37}$$

$$exp[-\frac{1}{2}\mathbf{x}^T K\mathbf{x}](x_i x_j) \tag{5.38}$$

thus we will bring down the factor $A_{ij}^{-1}$, the exponential can also be derived from the dimensional analysis.

That is

$$\langle x_i x_j\rangle = \frac{\int_{-\infty}^{\infty} d^n x \exp\left(-\frac{1}{2}\mathbf{x}\cdot K\cdot\mathbf{x}\right) x_i x_j}{\int_{-\infty}^{\infty} d^n x \exp\left(-\frac{1}{2}\mathbf{x}\cdot K\cdot\mathbf{x}\right)} \tag{5.39}$$

$$= A_{ij}^{-1} \tag{5.40}$$

$$\langle x_i x_j ... x_k x_l\rangle = \sum_{Wick} (A^{-1})_{ab} ... (A^{-1})_{cd} \tag{5.41}$$

### 5.1.5 Guass Integration comes across with hbar

$$I = \int_{-\infty}^{\infty} dq\, e^{-f(q)/\hbar} = e^{-\frac{1}{\hbar}f(a)}(\frac{2\pi\hbar}{f''(a)})^{\frac{1}{2}} e^{-O(\hbar^{1/2})} \tag{5.42}$$

When $f(q)$ arrives at its minimum when $q = a$, $eg$, $f(q) = f(a) + \frac{1}{2}f''(a)(q-a)^2 + O[(q-a)^3]$

When $f(q)$ is a function of many variables $q_1, q_2, ...q_N$ and with a minimum at $q_j = a_j$, we generalize immediately

$$I = e^{-\frac{1}{\hbar}f(a)}(\frac{2\pi\hbar}{f''(a)})^{\frac{1}{2}} e^{-O(\hbar^{1/2})} \tag{5.43}$$

## References