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Note of Knot Theory and TQFT

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Abstract: The main purpose of this lecture is to shed light on the relationship between the knot theory and TQFT. But actually the connection between them and the application of them are widely used in physics, so I try to find a path to connect them, though just a way to view the connection(maybe the mainstream), we can get a lot of inspirations. Now we start!

Contents

1 Knot and Knot Invariant

Definition of Knot: A knot is a particular way S^1 sit inside ordinary $\mathbb{R}^3$, or precisely, a knot is a submanifold of $\mathbb{R}^3$ that is differential to S^1.

definition of Link: A *Link* is a submanifold of $\mathbb{R}^3$ that is diffeomorphic to a disjoint union of circles. The circles themselves are called *components* of the link.

(Here are some pic of the unknot, trefoil knot, etc as well as Hopf link, Borromean rings)

1.1 Isotopic and Reidemeister Moves

Two knots or links in three dimension space can be transformed or deformed continuously one into the other (the usual notion of ambient isotopy) if and only if any diagram (obtained by projection to a plane) of one link can be transformed into the other link via a sequence of Reidemeister Moves.

1.2 Linking Number and Knot Invariant

Oriented: A knot or link is said to be oriented if each arc in its diagram is assigned a direction so that at each crossing the orientations appear either as.... (picture: the + refers to the crossing where the arrow from the top left corner is on top of the arrow from the lower left corner and directed to the top right corner)

Linking number: Let $L = \{\alpha, \beta\}$ be a link of two components α and β, then the linking number

$$\ell k(\alpha, \beta) = \frac{1}{2} \sum_{p \in \alpha \sqcap \beta} \epsilon(p) \tag{1.1}$$

Some examples on the blackboard...

writhe or twist number: Let the writhe of K (or twist number) be defined by the formula:

$$\omega(K) = \sum_{p \in c(K)} \epsilon(p) \tag{1.2}$$

where $C(K)$ denotes the set of crossings in the diagram K.

The above are some examples of the knot invariant, actually the simplest invariant of the knot. I will talk about other kinds of the knot invariant at the end of the lecture if I still have time, such as the bracket polynomial and the Jones Polynomial, etc. All of this stuff is easy to handled while interesting at the same time. But the key point of this lecture should focused on the physics , so I will show you what 's the relation between Knot Invariant and the observable in the next section.

Actually, right now!

2 Relation Between knot Invariant And The Observable In Physics System

We consider the level-κ chern-simons theory with an action $\int \frac{\kappa}{2\pi} BdA$ in d dimension(not 4 π because of the wedge product form,if we expand it we will get a factor 1/2), where κ is quantized to be an integer. Consider the following action on any closed $d - dim$ manifold M^d.

$$S[A, B] = \int_{M^d} \frac{\kappa}{2\pi} B \wedge dA \tag{2.1}$$

where A is a 1-form gauge field on M and B is a $(d-2)$-form gauge field on M. The partition function is

$$\begin{aligned} Z &= \int DADBexp[iS[A, B]] \\ &= \int DADBexp[i \int_{M^d} \frac{\kappa}{2\pi} B \wedge dA] \end{aligned} \tag{2.2}$$

Let Φ be a gauge invariant functional $\Phi(A, B)$, specially , an observable of the fields A and B. The expectation of it is

$$\begin{aligned} < \Phi > &= \frac{1}{Z} \int DADB\Phi(A, B)exp[iS[A, B]] \\ &= \frac{1}{Z} \int DADB\Phi(A, B)exp[i \int_{M^d} \frac{\kappa}{2\pi} B \wedge dA] \end{aligned} \tag{2.3}$$

If the observable function is a product of the wilson loops(gauge invariant thus I believe it is an observable hhhh.... A long story but I am not going to explain in details) around the one-dimensional loops $\{\gamma_n^1\}$ separarte and disjoint from $\{S_m^{d-2}\}$ such that

$$\Phi_0(A) = \prod_n exp[ie_n \int_{\gamma_n^1} A] \tag{2.4}$$

with the electric charge $e_n \in \mathbb{Z}$ associated to each loop, then

$$\begin{aligned} < \Phi > &= \frac{1}{Z} \int DADBexp[iS[A, B]]exp[i \sum_n e_n \int_{\gamma_n^1} A] \\ & exp[i \sum_m q_m \int_{S_m^{d-2}} B] \\ &= ... \\ &= exp[-\frac{2\pi i}{N} \sum_{m,n} q_m e_n LK(S_m^{d-2}, \gamma_n^1)] \end{aligned} \tag{2.5}$$

(Finish the prof process!)

Where we can see that, the expectation of the observable is depend on the Linking number or in other words, the spacetime -braiding process of the anyons!

Actually, I can told you that the main purpose of this lecture has been achieved—to explain how the knot theory connect to physics?

Wait! You might ask: What's the physics meaning of the action? How do we know that it carry the essence of the system we considered and why on earth the integral will obtain the linking number?

OK, don't worry. It is a big deal to connect this two stuff so I show you the answer firstly, which stimulate you passion for the process(to be honest, it was the amazing answer aspire me learn it fast to give you this lecture).

In other words, we are now goal-directed: we have to figure out why the action is meaningful and useful, and why the integral is the linking number, or generally speaking, the knot invariant).

I will introduce the chern-simons action as the generalization of the Yang-mills action in the electromagnetic field. As we know that the Maxwell equations encode all the information of the EM field, and the Yang-mills action is equivalent to the Maxwell equations, thus as the generalization $(F = *F)$, we believe that the chern-simons term contains some important information!

The calculation of the integral is just the math stuff, but it is well worth to mention that, the integral involves the Lebesgue measure even the Hausdorff measure since it is an integral defined on the manifold ! Well, this is a troublesome stuff. Optimistically, we believe that, the integral is an invariant, which is much easier to calculate and with much more fun!

Actually the chern-simons term is the effective theory in the 2-d condensed matter system, where we have factional statistics and anyon. Therefore, I will explain C-S theroy from the perspective of fractional statistics and anyon!

Finally, I will give you some concrete physics systems (cyons, QHE) where we do find some evidence of the fractional statistics, in other words, the chern-simons theroy and the existence of the knot invariant in physics system are proved preliminarily.

3 Yang-Mills Equation and Maxwell Equation

3.1 The first pair

$$\nabla \cdot \mathbf{B} = 0 \tag{3.1}$$

$$\nabla \times \mathbf{E} + \frac{\partial B}{\partial t} = 0 \tag{3.2}$$

And we know that

$$\mathbf{B} = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy \tag{3.3}$$

$$\mathbf{E} = E_x dx + E_y dy + E_z dz \tag{3.4}$$

The first pair of the Maxwell equations then become

$$dE = 0, dB = 0 \tag{3.5}$$

3.1.1 Utility form

We can define a unified electromagnetic field F, a 2-form on $\mathbb{R}^4$,as follows:

$$F = B + E \wedge dt \tag{3.6}$$

In view of the components,

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \tag{3.7}$$

We can write them as a matrix:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \tag{3.8}$$

We can also separate the time and space part: To see this, ...

3.2 The Second pair

$$*d * F = J \tag{3.9}$$

where we use the Minkowski metric,

$$J = j - \rho dt \tag{3.10}$$

3.3 Curvature

In order to avoid introducing Fiber Bundle in the speech, we try to introduce Berry phase and Berry curvature to have a first glance at these concept.

Considering an adiabatic process, the Hamiltonian is $H(R(t))$,where $R(t)$ is the parameter vetor in the configuration space (varied slowly), which satisfied

$$H(R)|n(R) >= E_n(R)|n(R) > \tag{3.11}$$

Since from the Schrodinger equation, we can determine $|n(R) >$ up to a phase, ...

Then we have the berry phase as

$$\gamma_n = i \int_C < n(R)|\nabla_R|n(R) > \cdot dR \tag{3.12}$$

Where the part in the integration is the berry connection

$$A_n(R) = i < n(R)|\nabla_R|n(R) > \tag{3.13}$$

$$\begin{aligned} \gamma_n &= -Im \int_C < n(R)|\nabla_R|n(R) > \cdot dR \\ &= -Im \int ds \cdot (< \nabla n(R) \times \nabla n(R) >) \end{aligned} \tag{3.14}$$

where $\nabla n(R) \times \nabla n(R)$ is the Berry connection.

We can also regard the Berry coonection as the magnetic potential **A** and the berry curvature is the **B**

We can also rewrite the berry curvature as

$$F_{vw} = \partial_v A_w - \partial_w A_v \tag{3.15}$$

which is exactly the definition of the curvature of sections of E (E is a G bundle and the connection A is a G-connection, plus, G=U(1)).

I will use berry phase to explain the quantum hall effect at the last part.

3.4 Bianchi identity and Yang-Mills Equation

$$d_D F = 0 \tag{3.16}$$

(Sorry for the brutalness that I put so the conclusion out here without the demonstration, because the main task is to gain the Chern-Simons term!!) By which, we can rewrite the two pairs of the Maxwell equation as

$$d_D F = 0, *d_D * F = J \tag{3.17}$$

which is the famous Yang-Mills equation,where d_D is the exterior covariant derivative of E-valued differential forms.

$$d_D s(v) = D_v s \tag{3.18}$$

From the perspective of the least action principle and path integral, we can(actually I can't, I really don't know how genius Yang and Mills are that they can create such a beautiful action!)

$$S_{YM}(A) = \frac{1}{2}\int_M tr(F \wedge *F) \tag{3.19}$$

which is equivalent to the equation(3.17), that means, the action is the most fundamental thing of the electromagnetic field !It is much more compact than the Maxwell equation!

4 Chern-Simons Action

Considering the Yang-mills action involve $*$, which means that it has relation with metric. If we consider A is self-dual, that is ,$F = *F$,we then have the chern form action

$$S(A) = \int_M tr(F \wedge F) \tag{4.1}$$

$$\begin{aligned} tr(F \wedge F) &= \int_0^1 \frac{d}{ds} tr(F_s \wedge F_s) ds \\ &= 2\int_0^1 tr(\frac{dF_s}{ds} \wedge F_s) ds \\ &= 2d\int_0^1 tr(A \wedge F_s) ds \\ &= 2d\int_0^1 tr(sA \wedge dA + s^2 A \wedge A \wedge A) ds \\ &= dtr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \wedge A) \end{aligned} \tag{4.2}$$

we call the 3-form

$$tr(A \wedge dA + \frac{2}{3}A \wedge A \wedge A \wedge A) \tag{4.3}$$

the chern-simons form.

5 Fractional Statistics

Chern Simon theory is actually the low energy effective theory in the condensed matter system that involved anyons and fractional statistic, which finally will shed light on the relation between knot theroy and chern-simons theroy.

In order to have a better understanding of the physics behind the chern-simon thoery, which has a deep relation with the fractional statistics when being used as an effective theory for low energy condensed matter system, or in other words, chern-simons term implied that there are anyons in the 2-dimensional system. Therefore, I am going to introduce the fractional statistics.

Consider a system of two identical particles and we suppose that there is a short range, infinitely strong repulse force of two identical particles, or simply thinking that we are in a low energy system!

The definition of Statistics: let$\psi(1,2)$ be the wavefunction describing two identical hard-core particles with definite angular momentum , and let us assume that when we move particle 2 around particle 1 by an azimuthal angle $\Delta\phi$,the wave function changes according to

$$\psi(1,2) \to \psi'(1,2) = e^{i\nu\Delta\phi}\psi(1,2) \tag{5.1}$$

The phase acquired by the wavefunction depends on a parameter which is usually called *statistics*.

5.1 A Coarse Picture

Considering two ways of exchange the particles:

1. Moving particle 2 around particle 1 by an angle $\Delta\phi = \pi$
2. Moving particle 2 around particle 1 by an angle $\Delta\phi = -\pi$

We will view these process in the center of momentum frame. As we mentioned above, the frame is a $\mathbb{R}^d - \{0\}$ space.

5.1.1 d≥3

In the first case, the wave function gets a phase $exp(i\pi\nu)$,while for the second case, the wave function gets a phase $exp(-i\pi\nu)$.

Since in $d \geq 3$, there is no intrinsic difference between the two ways above. We can lift the path of the first case into the third dimension, then fold it back onto the plane, and finally superpose it to the path of the way 2, vice versa.

So the two ways are the same physical operation, which means that in $d \geq 3$ one must have

$$e^{i\pi\nu} = e^{-i\pi\nu} \tag{5.2}$$

from which we obtain that $\nu = 0, 1$. If $\nu = 0$, then the particles subject to the bosonic statistics while $\nu = 1$ fermionic statistics.

5.1.2 2-d

The situation changes drastically in two dimensions where the first way can not deform continuously to the second way.

Hence in $d = 2$, two ways above are topologically and physically distinct operations - a reflection of the fact that the eq5.2 does not necessarily hold anymore and the statistical parameter ν can be arbitrary. (we know that SO(2) is abelian ,therefore there are no commutation to quantize....and no restrictions on the possible eigenvalues of the spin come from the algebra. If we believe in some connection between spin and statistics, and arbitrariness of the spin leads us to conceive the idea that the statistics may be arbitrary in 2-d...)

So it is not enough for us to specify the initial and final configurations to completely characterize a system, it is also necessary to specify haw the different trajectories wind and braid around each, or precisely, the time-evolution of the particles is important and can not be neglected.

Well, the process counts, means that the representation of the system will change totally.

(We can also notice that the violation of theP, T symmetry.....

5.2 A Rigorous Process

Now we try to be more rigorous to talking about the the anyonic statistics.

Let M_N^d be the configuration space of a collection of N identical hard-core particles in d dimensions, and let q, q' be two arbitrary points in M_N^d.

The amplitude for the system to evolve from the q at time t to q' at time t' is given by the kernel

$$K(q', t'; q, t) =< q', t'|q, t >= \int_{q(t)=q;q(t')=q'} D_q e^{\frac{i}{\hbar}\int_t^{t'} d\tau \mathscr{L}[q(\tau),\dot{q}(\tau)]} \tag{5.3}$$

The kernel$K(q', t'; q, t)$ evolves the single-valued wavefunction $\psi(q, t)$ according to

$$\begin{aligned} \psi(q', t') &= \int_{M_N^d} dq < q', t'|q, t >< q, t|\psi > \\ &= \int_{M_N^d} dq K(q', t'; q, t)\psi(q, t) \end{aligned} \tag{5.4}$$

where we have define a configuration space M_N^d of a collection of N particles.

Without any loss of generality, we choose $q = q'$ and hence describe loop in M_N^d, as we know from the first part, two loops are considered equivalent if one can deform the other by a continuous deformation.

All homotopic loops are grouped into one class and the set of all such classes is called the fundamental group and denoted as π_1.

With this category in mind, we can organize the sum over the all loops in(5.4) into s sum over homotopic classes α into a PI in each class.Thus,

$$K(q,t';q,t)=\sum_{\alpha\in\pi_1(M_N^d)} K_\alpha(q',t';q,t) \tag{5.5}$$

This formula can be interpreted as a decompostion of the amplitude K into a sum of subamplitudes K_α

It is clear that we can assign different weights to different subamplitudes $K_\alpha(q,t';q,t)$

$$K(q,t';q,t)=\sum_{\alpha\in\pi_1(M_N^d)}\chi(\alpha)\int_{q(t)=q;q(t')=q'} D_{q_\alpha} e^{\frac{i}{\hbar}\int_t^{t'} d\tau \mathscr{L}[q(\tau),\dot{q}(\tau)]} \tag{5.6}$$

(we introduce the $\chi(\alpha)$ to describe the influence of different classes of path, maybe we can get some inspiration from the A-B effect. As we know, that was the key part, since in the simplest double spli system, the arrangement of the phase determine which point is bight and which point is dark. The A-B effect tolds us that if the path can not become a point by continuously contract, then there will exist some quantum effect which can not be eliminated by gauge transformation, like the berry phase, which we call global phase!)

we know that the propagators should obey some rules,thus the weight must satisfy

$$\chi(\alpha_1)\chi(\alpha_2)=\chi(\alpha_1\cdot\alpha_2) \tag{5.7}$$

for any $\alpha_1$and α_2. The equation implies that $\chi(\alpha)$ must be a one-dimensional representation of the fundamental group $\pi_1(M_N^d)$.

To see which representations are possible, we have to specify better what are M_N^d and its fundamental group. To this end, we consider a system of N identical hard-core particles moving in the euclidean d-dimensional space $\mathbb{R}^d$.

Remove the overlapped position and consider that the particles are identical and indistinguishable(we should identify configurations which differ only in the ordering of particles), therefore we conclude that the configuration space for our system is

$$M_N^d=\frac{(\mathbb{R}^d)^N-\Delta}{S_N} \tag{5.8}$$

where

$$\Delta=\{\mathbf{r_1},...\mathbf{r_N}\in(\mathbb{R}^d)^N:\mathbf{r_i}=\mathbf{r_j} for \quad i\neq j\} \tag{5.9}$$

To find the fundamental group of such a space is a standard problem in algebraic topology, which was addressed in 1962. Here we simply quote the results: the fundamental group of M_N^d is given by

$$\pi_1(M_N^d)=\begin{cases} S_N\ if \quad d\geq 3\\ B_N\ if \quad d=2\end{cases} \tag{5.10}$$

where B_N is the Braid group and S_N is the permutation group.

(We can go deeper to undertand the equation(5.10) with the blackboard.....)

5.2.1 Braid Group

Definition of the braid group: The braid group B_N is the group whose elements are isotopy classes of n 1-dimensional braids running vertically in 3-dimensional Cartesian space, the group operation being their concatenation.

The braid group of N strands B_N is an infinite group which is generated by $N-1$ elementary moves $\sigma_1, ...\sigma_{N-1}$ satisfying

$$\sigma_I \sigma_{I+1} \sigma_I = \sigma_{I+1} \sigma_I \sigma_{I+1} \tag{5.11}$$

for $I = 1, 2, ...N-2$ and

$$\sigma_I \sigma_J = \sigma_J \sigma_I \tag{5.12}$$

for $|I - J| \geq 2$.

The inverse of $\sigma_I$is denoted by σ_I^{-1}, the identity by $\nVdash$.

(here we can insert some pictures to explian the rule and the difference between B_N and permutation group S_N... It is well worth to mention that in general $\sigma_I^2 \neq 1$, if $\sigma_I^2 = 1$ for all $\nVdash$, then the braid group reduces to S_N)

5.2.2 World-lines as the strands

Considering the starnds are the world-lines of the particles, we can describe the the particles in the configuration space by listing the azimuthal angles of all possible pairs of particles measured with respect to some arbitrary axes.

(Insert a picture label (time t))

From the picture (reftime t),we have

$$\phi_{12}(t) = 0, \phi_{13}(t) = \eta, \phi_{23}(t) = \epsilon \tag{5.13}$$

(Insert a picture label (time t')) Let us suppose that at time t' the particles reach the postions shown in (Fig.), now the winding angles are

$$\phi_{12}(t') = \epsilon + \pi, \phi_{13}(t') = \eta + \pi, \phi_{23}(t')\pi \tag{5.14}$$

It is always true that

$$\sum_{I<J} \phi_{IJ}(t') - \sum_{I<J} \phi_{IJ}(t) = n\pi \tag{5.15}$$

where n is an integer(in our example $n = 3$).We see that indeed there are $n = 3$ generators($\sigma_1\sigma_2\sigma_1$), $i.e \quad n = 3$ exchanges.(The dynamical process is important!)

5.2.3 The propagator and Braid group

It is not hard to demonstrate that one dimensional representation $\chi(\alpha)$ satisfied the definition of the braid group has the form as follows:

$$\chi(\sigma_K) = e^{-i\nu\pi} \tag{5.16}$$

for any $K = 1, ..., N-1$, where ν is a real parameter defined modulo 2.

We can rewrite the formula

$$\chi(\sigma_K) = exp[-i\nu\Delta\phi_{K,K+1}] = exp[-i\nu\sum_{I<J}\Delta\psi_{IJ}^{(K)}] \tag{5.17}$$

where we have

$$\Delta_{IJ}^{(K)} \equiv \phi_{IJ}(t')^{(K)} - \phi_{IJ}(t)^{(K)} = \pi\delta_{I,K}\delta_{J,K+1} \tag{5.18}$$

(remain other particles static and the movement of $K, K+1$ only changes $\phi_{K,K+1}$ by π)

Using this notion, we can generalised the formula 5.17

$$\chi(\alpha) = exp[-i\nu\sum_{I<J}\int_t^{t'} d\tau\frac{d}{d\tau}\phi_{IJ}^{(\alpha)}(\tau)] \tag{5.19}$$

Thus we can rewrite the propagator as follows:

$$\begin{aligned}K(q,t';q,t) = \sum_{\alpha\in\pi_1(M_N^d)}\int_{q(t)=q;q(t')=q} D_{q_\alpha} \\ e^{\frac{i}{\hbar}\int_t^{t'} d\tau\{\mathscr{L}[q(\tau),\dot{q}(\tau)]-\hbar\nu\sum_{I<J}\frac{d}{d\tau}\phi_{IJ}^{(\alpha)}(\tau)\}}\end{aligned} \tag{5.20}$$

Now we define

$$\mathscr{L}' = \mathscr{L} - \hbar\nu\sum_{I<J}\frac{d}{d\tau}\phi_{IJ}^{\alpha}(\tau) \tag{5.21}$$

then we see that the kernel $K(q,t';q,t)$ is decomposed with respect to $\mathbb{L}'$ into subamplitudes each of which is weighted equally as if we are describing bosons, that is

$$K(q',t';q,t) = \int_{q(t)=q;q(t')=q'} D_q e^{\frac{i}{\hbar}\int_t^{t'} d\tau\mathscr{L}'[q(\tau),\dot{q}(\tau)]} \tag{5.22}$$

connection to knot: In mathematics Alexander's theorem states that every knot or link can be represented as a closed braid; that is, a braid in which the corresponding ends of the strings are connected in pairs.

Actually from the example we discuss above where the $n = 3$ means the number of exchanges of the particles, it also sheds light on the path integral has something related to the knot formed by the world lines of the particles.

5.3 Where is the fraction?

As we all know that, a real physics system must be closed at the edge, or we can say the boundary. So we won't have infinite strands, which implies that we should consider a system with a periodical boundary condition!

We first consider the case of the sphere ($\sum = S^2$). The braid group $B_N(S^2)$ is generated by σ_I with $I = 1,..,N-1$ which satify 5.11 and 5.12 plus an additional constraint(prove this!)

$$\sigma_1\sigma_2...\sigma_{N-1}^2...\sigma_2\sigma_1 = 1 \tag{5.23}$$

Then we can obtain

$$e^{-i2(N-1)\nu\pi} = 1 \tag{5.24}$$

which immediately restricts ν to be rational, that is $\nu = \frac{p}{q}$.

6 Concrete Physical Systems

6.1 Cyon

We discuss the dynamics of a charged point-particle interacting with an arbitrary long magnetic solenoid. The system has been called *cyon* and is the prototype for anyons.

The system is governed by the Lagrangian

$$\mathscr{L} = \frac{1}{2}m\mathbf{v}^2 + \frac{e}{c}\mathbf{v} \cdot \mathbf{A}(\mathbf{r}) \tag{6.1}$$

where $\mathbf{r} = (x, y) \in \mathbb{R}^2$, we choose symmetry gauge and $\mathbf{A}$ is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\Phi}{2\pi}(\frac{-y}{x^2+y^2}\hat{\mathbf{i}} + \frac{x}{x^2+y^2}\hat{\mathbf{j}}) \tag{6.2}$$

$$\mathbf{B} = dA = \Phi\delta^{(2)}(\mathbf{r}) \tag{6.3}$$

$$\mathbf{p} = \frac{\partial \mathscr{L}}{\partial v} = m\mathbf{v} + \frac{e}{c}\mathbf{A} \tag{6.4}$$

The Hamiltonian

$$H = p \cdot v - \mathscr{L} = \frac{1}{2}mv^2 \tag{6.5}$$

$$\begin{aligned} J_c &= r \wedge p = r \wedge mv + \frac{e}{c} r \wedge A \\ &= r \wedge mv + \frac{e\Phi}{2\pi c} \\ &= J + \frac{e\Phi}{2\pi c} \end{aligned} \tag{6.6}$$

$$J_c = -i\hbar\frac{\partial}{\partial\phi} = \hbar m \tag{6.7}$$

......

$$J = \hbar(m - \frac{e\Phi}{hc}), m \in \mathbb{Z} \tag{6.8}$$

Following Wilzeck's work in 1982,

$$s = \frac{J(m=0)}{\hbar} = -\frac{e\Phi}{hc} \tag{6.9}$$

7 Chern-Simons Construction of Fractional Statistics

As what we discussed in the Second section, we obtain the chern-simons action.

When $G = U(1)$, then we have

$$\begin{aligned} S_{CS} &= \frac{\kappa}{2}\int_S A \wedge dA \\ &= \frac{\kappa}{2}\int d^3x \epsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma \end{aligned} \tag{7.1}$$

Here I am going to tell you that the κ is just a coefficient and don't worry about that. But later you will find that I am actually tricking on you. I have to admit that, it is not a normal coefficient and it is actually the level of the chern simons term and Witten have proved that it must be an integer.

If we consider the source then we have,

$$\mathscr{L} = \frac{\kappa}{2}\epsilon^{\alpha\beta\gamma}A_\alpha\partial_\beta A_\gamma - j^\alpha A_\alpha \tag{7.2}$$

OK, since we have got that Lagrangian so I am going to calculate the Euler-Lagrangian equation and see what we can obtain.

The Euler Lagrangian is as follows:

$$\frac{\partial L}{\partial A_\alpha} = \partial_\beta(\frac{\partial L}{\partial(\partial_\beta A_\alpha)}) \tag{7.3}$$

$$\frac{\partial \mathscr{L}}{\partial A_\alpha} = 2\cdot\frac{\kappa}{2}\epsilon^{\alpha\beta\gamma}\partial_\beta A_\gamma - j^\alpha \tag{7.4}$$

where we should remember that α go through all the index so we should also consider the condition when $\gamma = \alpha$ (actually the description is not so rigorous,and some tips: we have to use the partial integral) from which we will get a factor 2!

And the right part of the Euler equation is equal to zero so we actuallly have

$$j^\alpha = \kappa\epsilon^{\alpha\beta\gamma}\partial_\beta A_\gamma \tag{7.5}$$

We can check what we actually have :

$$\begin{aligned} j^0 &= \sum_{n=1}^{N} q_n\delta(x-x_n) \\ &= \kappa\epsilon^{0\beta\gamma}\partial_\beta A_\gamma \\ &= \kappa dA \end{aligned} \tag{7.6}$$

where we can see that

$$e = \kappa B \tag{7.7}$$

or in other words, where there is charge, there is magnetic flux!! (here is the picture from Dunne!)

Consider, for example, non relativistic point charged particles moving in the plane, with magnetic flux lines attached to them. The charged density

$$\rho(\mathbf{x},t) = e\sum_{\alpha=1}^{N}\delta(\mathbf{x}-\mathbf{x}_\alpha(t)) \tag{7.8}$$

describes N particles, with the a^{th} particle following the trajectory $\mathbf{x}_\alpha(t)$. The corresponding current density is $\mathbf{j}(\mathbf{x},t) = e\sum_{\alpha=1}^{N}\mathbf{v}_\alpha\delta(\mathbf{x}-\mathbf{x}_\alpha)$.

$$\mathbf{B}(\mathbf{x},t) = \frac{1}{\kappa}e\sum_{\alpha=1}^{N}\delta((\mathbf{x})-\mathbf{x}_\alpha(t)) \tag{7.9}$$

which follows each particle throughout its motion.

If each particle has mass m,the action is

$$\begin{aligned} S &= S_{kinetic} + S_{CS} \\ &= \frac{m}{2}\sum_{\alpha=1}^{N}\int dt\mathbf{v}_\alpha{}^2 \\ &+ \frac{\kappa}{2}\int d^3x\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - \int d^3x A_\mu J^\mu \end{aligned} \tag{7.10}$$

I have to repeat here, the equation 7.5 can be rewritten as,

$$\begin{aligned} j^\alpha &= \kappa\epsilon^{\alpha\beta\gamma}\partial_\beta A_\gamma \\ &= \frac{\kappa}{2}\epsilon^{\alpha\beta\gamma}F_{\beta\gamma} \end{aligned} \tag{7.11}$$

where the factor 2 comes from the definition of F we have mention before.

Thus through a process of complicated but worthwhile calculation, we will have

$$\begin{aligned} A^i(\mathbf{x},t)) &= \frac{1}{2\pi\kappa}\int d^y\epsilon^{ij}\frac{(x^j-y^j)}{|\mathbf{x}-\mathbf{x}|^2}\rho(\mathbf{y},t) \\ &= \frac{e}{2\pi\kappa}\sum_{\alpha=1}^{N}\epsilon^{ij}\frac{(x^j-x^j_\alpha(t))}{|\mathbf{x}-\mathbf{x}_\alpha(\mathbf{t})|^2} \end{aligned} \tag{7.12}$$

As an aside, using the identity $\partial_i arg(\mathbf{x}) = -\epsilon_{ij}x^j/|\mathbf{x}|^2$,where the argument function $arg(\mathbf{x} = arctan(\frac{y}{x})$.

$$A_i(\mathbf{x}) = \frac{e}{2\pi\kappa}\sum_{\alpha=1}^{N}\partial_i arg(\mathbf{x}-\mathbf{x}_\alpha) \tag{7.13}$$

Returning to the point-anyon action**??**, then we have the Hamiltonian for the system is

$$H = \frac{m}{2}\sum_{\alpha=1}^{N} = \frac{1}{2m}\sum_{\alpha=1}^{N}[\mathbf{p}_\alpha - e\mathbf{A}(\mathbf{x}_\alpha)]^2 \tag{7.14}$$

where

$$A^i(\mathbf{x}_\alpha) = \frac{e}{2\pi\kappa}\sum_{b\neq a}^{N}\epsilon^{ij}\frac{(x^j_a-x^j_b)}{|\mathbf{x}_a-\mathbf{x}_b|^2} \tag{7.15}$$

It is interesting to observe that the solution can be written also as

$$A^i_I(\mathbf{r_1},...,\mathbf{r_N}) = -\frac{e}{2\pi\kappa}\frac{\partial}{\partial r^j_I}\sum_{I\neq J}\phi_{IJ} \tag{7.16}$$

where ϕ_{IJ} is the winding angle of particle J with respect to the particle I, such that

$$\phi_{IJ} = \tan^{-1}(\frac{x^2_I - x^2_J}{x^1_I - x^1_J}) \tag{7.17}$$

Finally we managed to have the Lagrangian

$$\mathcal{L}' = \sum_{I=1}^{N}(\frac{1}{2}m\mathbf{v_I^2}) - \frac{e^2}{2\pi\kappa}\sum_{I<J}\frac{d}{d\tau}\phi_{IJ}(\tau) \tag{7.18}$$

So we immediately deduce that our particle are generally anyons of statistics

$$\nu = \frac{e^2}{2\pi\hbar\kappa} \tag{7.19}$$

If we change th

8 Quantum Hall Effect

8.1 Classical QHE

Considering $q_e = -e$ moving in the electric and magnetic field, thus we have

$$q_e\mathbf{E} = q_e\mathbf{v} \times \mathbf{B} \tag{8.1}$$

The current of the system is

$$\mathbf{j} = nq_e\mathbf{v} \tag{8.2}$$

where n is the density of the particle.

Thus we have

$$\mathbf{j} = \frac{nq_e}{B}E \tag{8.3}$$

$$\begin{aligned}\frac{nq_e}{B} &= \frac{n}{B}\cdot\frac{h}{e}\cdot\frac{e}{h}\cdot q_e \\ &= -\nu\frac{e^2}{h}\end{aligned} \tag{8.4}$$

$$\rho_{xy} = \frac{h}{e^2}\frac{1}{\nu} \tag{8.5}$$

If we consider

$$\nu' = \frac{e^2}{2\pi\hbar\kappa} = \textit{number of magnetic flux}(\Phi_0 \textit{ as the units}) \tag{8.6}$$

$$\nu = \tag{8.7}$$

8.2 Integer QHE

I will give the explanation from the perspective of Landau Level, which is actually a harmonic oscillator model.....on blackboared....

8.3 Fractional QHE

Use Laughlin wavefunction to explain firstly.....

Anyon is simillar to the phonon the solid physics, as a quasi particle, it is quite useful to discibre the physics in the solid physics, which also remind us to thinking to the esscence to the particle. I have show you that, the particles are defined by the statistics property, well if we believe that there are some relervant between the statistics and spin.....

Use the CS term to give the answer of the integer or fractional ν, though, the information is actually encoded in the level of the CS term instead of the knot invariant hhhh.....

9 Mathematics

9.1 Wedge product

The Wedge product is the multiplication operation in exterior algebra. The wedge product is always antisymmetric, associative, and anti-commutative.

The result of the wedge product is always antisimmetric, associative and anti- commutative. The result of the wedge product is known as a bivector.

For two vectors $\mathbf{u}, \mathbf{v}$in $\mathbb{R}^3$,the wedge product is defined as

$$\begin{aligned}\mathbf{u} \wedge \mathbf{v} &= \mathbf{u} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{u} \\ &= \begin{bmatrix} 0 & \mathbf{v_1 v_2} - \mathbf{v_2 v_1} & \mathbf{v_1 v_3} - \mathbf{v_3 v_1} \\ \mathbf{v_2 v_1} - \mathbf{v_1 v_2} & 0 & \mathbf{v_2 v_3} - \mathbf{v_3 v_2} \\ \mathbf{v_3 v_1} - \mathbf{v_1 v_3} & \mathbf{v_3 v_2} - \mathbf{v_2 v_3} & 0 \end{bmatrix}\end{aligned} \tag{9.1}$$

If the associated vector is defined as

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} (\mathbf{u} \wedge \mathbf{v})_{23} \\ -(\mathbf{u} \wedge \mathbf{v})_{13} \\ (\mathbf{u} \wedge \mathbf{v})_{32} \end{bmatrix} \tag{9.2}$$

Cross product and wedge product when written as determinant are in the same way,thus are related by the hodge product:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e_1} & \mathbf{e_1} & \mathbf{e_1} \\ a1 & a2 & a3 \\ b1 & b2 & b3 \end{vmatrix} \tag{9.3}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e_{23}} & \mathbf{e_{31}} & \mathbf{e_{12}} \\ a1 & a2 & a3 \\ b1 & b2 & b3 \end{vmatrix} \tag{9.4}$$

$$*(\mathbf{a} \wedge \mathbf{b}) = \mathbf{a} \times \mathbf{b} \tag{9.5}$$

$$*(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \wedge \mathbf{b} \tag{9.6}$$

Wedge product is distributive The formula for the wedge product is :

$$\mathbf{u} \wedge \mathbf{v} = \tag{9.7}$$

9.2 1-form

Our goal is to generalize the concept of gradient of a function on arbitrary manifolds, then we introduce something like *df* supposed to be like the usual ∇f,and it satisfied

$$\nabla f \cdot v = vf \tag{9.8}$$

Definition of 1-forms: we define $1 - forms$ on any manifold M to be a map from Vect(M) to $C^\infty(M)$ that is linear over $C^\infty(M)$, we use $\Omega^1(M)$ to denote the space of all 1-forms on a manifold.

9.3 p-forms

definition of exterior algebra: the exterior algebra over V,denoted $\wedge V$, is the algebra generated by V with the relation (anticommutative rule)

$$v \wedge w = -w \wedge v \tag{9.9}$$

for all $v, w \in V$.

9.4 2-forms

Suppose $M = \mathbb{R}^n$,the 0-forms on $\mathbb{R}^n$ are just functions,like

$$f \tag{9.10}$$

The 1-forms all look like

$$\omega_\mu dx^\mu \tag{9.11}$$

where the coefficients are functions.
The 2-forms look like

$$\frac{1}{2}\omega_{\mu\nu} dx^\mu \wedge dx^\nu \tag{9.12}$$

2-form has a property:$dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$, The 3-forms look like

$$\frac{1}{3!}\omega_{\mu\nu\lambda} dx^\mu \wedge dx^\nu \wedge dx^{lambda} \tag{9.13}$$

9.5 The Exterior Derivative

We can take the differential of a function , or 0-form, to 1-form.

the exterior derivative, or differential

$$d : \Omega^p(M) \to \Omega^{p+1}(M) \tag{9.14}$$

such that the following properties hold: .....

$$d^2 = 0 \tag{9.15}$$

9.6 Note!

On $\mathbb{R}^3$,d acts like the gradient on 0-forms, the curl on 1-forms and the divergence on 2-forms, so the identity $d^2 = 0$ contains within the identities

$$\nabla \times (\nabla f) = 0 \tag{9.16}$$

$$\nabla \cdot (\nabla \times f) = 0 \tag{9.17}$$

we can prove this :

9.6.1 1-forms

$$\omega = \omega_I dx^I \tag{9.18}$$

we have

$$d\omega = d\omega_I \wedge dx^I \tag{9.19}$$

9.7 Hodge star

Hodge star opertaor: we define the Hodge star opertaor

$$* : \Omega^p(M) \to \Omega^{n-p}(M) \tag{9.20}$$

to be the unique linear map from p-forms to (n-p)forms such that for all $\omega, \mu \in \Omega^p(M)$,

$$\omega \wedge *\mu =< \omega, \nu > vol \tag{9.21}$$

We often call $a * \nu$ the dual of ν

Take dx, dy, dz as a basis of 1-forms on $\mathbb{R}^{\nvDash}$ (with its metric and orientation)

$$*dx = dy \wedge dz, etc \tag{9.22}$$

$$*1 = dx \wedge dy \wedge dz \tag{9.23}$$

10 Physics

Since I don't know where to put this fundamental but important, I decided to include it in the appendix.

10.1 Fundamental Group

In the mathematical field of algebraic topology, the fundamental group of a topological space is the group of the equivalence classes under homotopy of the loops contained in the space. It records information about the basic shape, or holes, of the topological space. The fundamental group is the first and simplest homotopy group. The fundamental group is a homotopy invariant—topological spaces that are homotopy equivalent (or the stronger case of homeomorphic) have isomorphic fundamental groups.

References