Axion Electrodynamics

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Content

Original Maxwell Equations

$$\nabla \cdot B = 0 \tag{1}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{2}$$

$$\nabla \cdot E = 0 \tag{3}$$

$$\nabla \times B = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} \tag{4}$$

Precisely,

$$\partial_\mu F^{\mu\nu} = 0 \tag{5}$$

In the Lagrangian formalism, the corresponding action is

$$S_0 = -\frac{1}{4\mu_0} \int d^4x F_{\mu\nu} F^{\mu\nu} = \int d^4x (\frac{\varepsilon_0}{2} E^2 - \frac{1}{2\mu_0} B^2) \tag{6}$$

Now we introduce the coupling term

$$S_\theta = \int d^4x \frac{\theta e^2}{4\pi^2\hbar}\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} = -\frac{\theta e^2}{4\pi^2\hbar c}\int d^4x E\cdot B \tag{7}$$

where the dual tensor $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$.

EOM

In general, the action governing the electric and magnetic field is given by

$$S = \int d^4x(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{e^2}{16\pi^2\hbar}\theta(x,t)F_{\mu\nu}\tilde{F}^{\mu\nu}) \tag{8}$$

The equation of motion from this action

$$\nabla \cdot E = -\frac{\alpha c}{\pi}\nabla\theta \cdot B \tag{9}$$

$$-\frac{1}{c^2}\frac{\partial E}{\partial t} + \nabla \times B = \frac{\alpha}{\pi c}(\dot{\theta} + \nabla\theta \times E) \tag{10}$$

$$\nabla \cdot B = 0 \tag{11}$$

$$\frac{\partial B}{\partial t} + \nabla \times E = 0 \tag{12}$$

where $\alpha = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{\hbar c}$ is the the fine structure constant.

Induced Charge

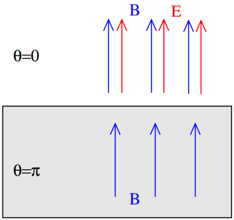

Figure 1: Applying a magnetic field

Consider a topological insulator with $\theta = \pi$ filling the lower half plane, $z < -\varepsilon$ and vacuum filling the upper half, $z > \varepsilon$. In the intermediate region $z \in [-\varepsilon, \varepsilon]$, $\partial_z \theta \neq 0$.

$$\nabla \cdot E = -\frac{\alpha c}{\pi} \nabla \theta \cdot B \tag{13}$$

$$= -\frac{\alpha c}{\pi} (\partial_z \theta) B \equiv \rho \tag{14}$$

The boundary of a topological insulator takes a magnetic field inside and generates an electric field outside.

Induced Current

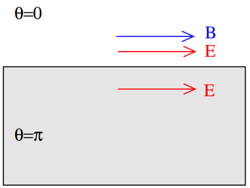

Figure 2: Applying an electric field

The electric field induces a surface current $\mathbf{K}$ perpendicular to E(Hall effect)

$$K_x = \alpha \varepsilon_0 c E_y \tag{15}$$

and the Hall conductivity is

$$\sigma_{xy} = \frac{1}{2} \frac{e^2}{2\pi\hbar} \tag{16}$$

Dirac Hamiltonian of a TI

$$\mathscr{H}_{TI}(k) = \hbar v_F k \cdot \boldsymbol{\alpha} + m_0 \boldsymbol{\alpha}_4 \tag{17}$$

The action of the systems in an external $A_\mu = (A_0, A)$ is[1]

$$S_{TI} = \int d^4 x \boldsymbol{\psi}^\dagger \{ i(\partial_t - ieA_0) - [\mathscr{H}_{TI}(k + eA)] \} \boldsymbol{\psi} \tag{18}$$

$$= \int d^4 x \bar{\boldsymbol{\psi}} [i\gamma^\mu (\partial_\mu - ieA_\mu) - m_0] \boldsymbol{\psi} \tag{19}$$

$$\equiv iS_{TI}^E = \int d\tau d^3 r \bar{\boldsymbol{\psi}} [\gamma_\mu (\partial_\mu - ieA_\mu - m_0 e^{i\pi\gamma_5})] \tag{20}$$

where

- minimal coupling $k \to k + eA$.
- $\boldsymbol{\psi}^\dagger \mathscr{H}_0 \boldsymbol{\psi} \to \boldsymbol{\psi}^\dagger (\mathscr{H}_0 - eA_0) \boldsymbol{\psi}$
- $t \to -i\tau, A_0 \to iA_0, \gamma^j \to i\gamma_j$
- $m_0 = -m_0(\cos\pi + i\gamma_5 \sin\pi) = -m_0 e^{i\pi\gamma_5}$.

Fujikawa method

Consider an infinitesimal chiral transformation

$$\psi \to \psi' = e^{-i\pi d\phi \gamma_5/2}\psi \tag{21}$$

$$\bar{\psi} \to \bar{\psi}' = \bar{\psi} e^{-i\pi d\phi \gamma_5/2} \tag{22}$$

$$\tag{23}$$

Then the partititon function is transformed as

$$Z = \int \mathscr{D}[\psi\bar{\psi}] e^{-S^E_{TI}[\psi,\bar{\psi}]} \to Z' = \int \mathscr{D}[\psi'\bar{\psi}'] e^{-S'^E_{TI}[\psi',\bar{\psi}']} \tag{24}$$

Due to the invariance of partition function, we arrive at

$$S^E_{TI} = \int d^4x \bar{\psi}[\gamma_\mu(\partial_\mu - ieA_\mu) - m_0)]\psi + i\int d^4x \frac{\pi e^2}{32\pi^2\hbar c}\varepsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma} \tag{25}$$

where the θ term emerges.

Hamiltonian of Two-node Weyl Semimetal

$$\mathscr{H}_0(k) = \hbar v_F(\tau_z k\cdot \boldsymbol{\sigma} + b\cdot \boldsymbol{\sigma}) \tag{26}$$

where τ_i and σ_i are Pauli matrix for Weyl nodes and spin degrees of freedom, $b\cdot\boldsymbol{\sigma}$ represent spin-splitting term.
Using Fujikawa method, we obtain the action of the system[1]

$$S = \int d^4x\bar{\Psi}[i\gamma^\mu(\partial_\mu - ieA_\mu)]\Psi + \frac{e^2}{2\pi^2\hbar}\int d^4x\, b\cdot rE\cdot B \tag{27}$$

with $\theta(r,t) = 2b\cdot r$.
Nonzero and non-quantized θ always comes from the time-reversal symmetry breaking or inversion symmetry breaking.

Anomalous Hall Effect

The induced electric density and charge density with $\theta(r,t) = 2b \cdot r$ are given by

$$j(r,t) = \frac{e^2}{4\pi^2\hbar}[\dot{\theta}(r,t)B + \nabla\theta(r,t) \times E] = \frac{e^2}{2\pi^2\hbar} b \times E \tag{28}$$

Set $b = (0,0,b), \Delta = 0$,

$$\mathscr{H}_{\pm}(k) = \hbar v_F(k_x\sigma_x + k_y\sigma_y) + m_{\pm}(k_z)\sigma_z = R(k) \cdot \sigma \tag{29}$$

where $m_{\pm}(k_z) = \hbar v_f(b \pm |k_z|)$. Hall conductivity

$$\sigma_{xy} = -\frac{e^2}{4\pi h}\int dk_x dk_y R \cdot (\frac{\partial R}{\partial k_x} \times \frac{\partial R}{\partial k_y}) = \frac{e^2}{2\pi^2\hbar} b \tag{30}$$

Reference

[1] SEKINE A, NOMURA K. Axion electrodynamics in topological materials[J]. Journal of Applied Physics, 2021, 129(14): 141101.

Thank you!