Prepared for submission to JHEP

Note of Entanglement

Lianghong Mo[1]

[1] *School of Physics, Sun Yat-sen University, Guangzhou, 510275, China*

E-mail: molh3@mail2.sysu.edu.cn

Abstract: Abstract...

Contents

1 Entanglement Entropy of Quantum Spin Chain

1.1 XY,XX,TFIM

XY model:

$$H = -\sum_{i=1}^{L}[J_x\sigma_x^i\sigma_x^{i+1} + J_y\sigma_y^i\sigma_y^{i+1} + g\sigma_z^i] \tag{1.1}$$

where $J_x = J(\frac{1+\gamma}{2}), J_y = J(\frac{1-\gamma}{2})$

$\gamma = 0$–**XX model:**

$$H = -\sum_{i=1}^{L}[J(\sigma_x^i\sigma_x^{i+1} + \sigma_y^i\sigma_y^{i+1}) + g\sigma_z^i] \tag{1.2}$$

$\gamma = 1$–**TFIM:**

$$H = -\sum_{i=1}^{L}[J\sigma_z^i\sigma_z^{i+1} + g\sigma_x^j] \tag{1.3}$$

1.2 Algebra about Pauli operator

Here we introduce the spin lowering and raising operator $\sigma_\pm$[?]

$$\sigma_\pm^j = \frac{\sigma_x^j \pm i\sigma_y^j}{2} \tag{1.4}$$

based on which we have

$$\sigma_x^j = \sigma_+ + \sigma_-, \sigma_y^j = \frac{\sigma_+ - \sigma_-}{i}, \sigma_z^j = 2\sigma_+^j\sigma_-^j - 1 \tag{1.5}$$

Reexpress the Hamiltonian1.53 using $\sigma_\pm$

$$H = -\sum_{i=1}^{L}\{J(\sigma_+^j\sigma_-^{j+1} + \sigma_+^j\sigma_-^{j+1}) + J\gamma(\sigma_+^j\sigma_+^{j+1} + \sigma_-^j\sigma_-^{j+1}) + 2g\sigma_+^j\sigma_-^j - gL\} \tag{1.6}$$

Here I will neglect the constant term $-gL$ and I will discuss about this term later.

$$H = -\sum_{i=1}^{L}\{J(\sigma_+^j\sigma_-^{j+1} + \sigma_+^j\sigma_-^{j+1}) + J\gamma(\sigma_+^j\sigma_+^{j+1} + \sigma_-^j\sigma_-^{j+1}) + 2g\sigma_+^j\sigma_-^j\} \tag{1.7}$$

1.2.1 Commutation Relation of Spin operator

Bosonic :

$$[a_i, a_j] = 0$$

Fermionic:

$$\{c_i, c_j^\dagger\} = \delta_{ij} \tag{1.8}$$

Pauli:

$$[\sigma_-^i, \sigma_+^i] = 0, i \neq j \tag{1.9}$$
$$\{\sigma_-^i, \sigma_-^i\} = 1, i = j \tag{1.10}$$

Some conclusions:

$$(\sigma_z^i)^2 = 1, (\sigma_\pm)^2 = 0 \tag{1.11}$$

1.3 Jordan-Wigner transformation

There is a way to map Pauli operators into Fermionic operators, called the Jordan wigner transformation.

$$c_i \equiv [\prod_{n=1}^{i-1}(-\sigma_z^n)]\sigma_-^i \tag{1.12}$$

In terms of the tensor structure of Pauli operators, this would read as

$$c_i = (-\sigma_z) \otimes (-\sigma_z) ... \otimes (-\sigma_z) \otimes \sigma_- \otimes 1 \otimes 1 ... \otimes 1 \tag{1.13}$$

The bunch of $(-\sigma_z)$'s is called **Jordan-Wigner string**.

The idea is that, to convert a Pauli operator σ_-^i into a fermionc operator c_i, we must append to it a string of operators $\prod_{n=1}^{i-1}(-\sigma_z^n)$. The conjugate is

$$c_i^\dagger = \left[\prod_{n=1}^{i-1}(-\sigma_z^n)\right]\sigma_+^i \tag{1.14}$$

1.3.1 Mapping between the algebra

Based on Eq.1.12 and Eq.1.14, we have

$$c_i c_i^\dagger = \sigma_-^i \sigma_+^i \tag{1.15}$$
$$c_i^\dagger c_i = \sigma_+^i \sigma_-^i \tag{1.16}$$
$$c_{i+1}^\dagger c_i = \sigma_+^{i+1} \sigma_-^i \tag{1.17}$$
$$c_i^\dagger c_{i+1} = \sigma_+^{i+1} \sigma_-^i \tag{1.18}$$
$$c_{i+1}^\dagger c_i^\dagger = -\sigma_+^{i+1} \sigma_+^i, c_i^\dagger c_{i+1}^\dagger = \sigma_+^{i+1} \sigma_+^i \tag{1.19}$$
$$c_{i+1} c_i = \sigma_-^i \sigma_-^{i+1} \tag{1.20}$$

Before we move on, we have to notice that here are some useful results we will use later

$$-\sigma_z^i = e^{i\pi\sigma_+^i\sigma_-^i} \tag{1.21}$$
$$\tag{1.22}$$

$$c_i = \left[\prod_{n=1}^{i-1} e^{i\pi\sigma_+^n\sigma_-^n}\right]\sigma_-^i \tag{1.23}$$

1.3.2 mapping between states

$$\sigma_z^i = 2\sigma_+^i\sigma_-^i - 1 = 2c_i^\dagger c_i - 1 \tag{1.24}$$

The eigenvalues $n_i = 0, 1$ of $c_i^\dagger c_i$ are therefore related to the σ_i according to

$$n_i = \frac{1+\sigma_i}{2} \tag{1.25}$$

The Pauli basis is therefore equivalent to the Fock basis of the n_j:

$$|n_1, n_2, ...\rangle = |\sigma_1, ...\sigma_L\rangle \tag{1.26}$$

1.4 Fermionic Representation of the spin model

Therefore, the hamiltonian 1.7 becomes

$$H = -\sum_{i=1}^{L}\{J(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) + J\gamma(c_i^\dagger c_{i+1}^\dagger + c_{i+1}c_i) + 2gc_i^\dagger c_i\} \tag{1.27}$$

If $\gamma = 0$, then this Hamiltonian becomes *exactly* the TBD Hamiltonian.

1.4.1 Periodic Boundary Condition

$$H_{PBC} = -J(\sigma_+^L\sigma_-^1 + \sigma_-^L\sigma_+^1) - J\gamma(\sigma_+^L\sigma_+^1 + \sigma_-^L\sigma_-^1) \tag{1.28}$$

It can be verified that

$$c_L^\dagger c_1 = \left[\prod_{n=1}^{L-1}(-\sigma_z^n)\right]\sigma_+^L\sigma_-^1 = -\left[\prod_{n=1}^{L}(-\sigma_z^n)\right]\sigma_+^L\sigma_-^1 \tag{1.29}$$

Inverse the relation

$$\sigma_+^L\sigma_-^1 = -\left[\prod_{n=1}^{L} e^{i\pi c_n^\dagger c_n}\right]c_L^\dagger c_1 = -(-1)^{\hat{\mathcal{N}}}c_L^\dagger c_1 \tag{1.30}$$

Similarly,

$$\sigma_-^L\sigma_+^1 = -(-1)^{\hat{\mathcal{N}}}c_1^\dagger c_L \tag{1.31}$$
$$\sigma_+^L\sigma_+^1 = -(-1)^{\hat{\mathcal{N}}}c_L^\dagger c_1^\dagger \tag{1.32}$$
$$\sigma_-^L\sigma_-^1 = -(-1)^{\hat{\mathcal{N}}}c_1 c_L \tag{1.33}$$
$$\tag{1.34}$$

PBC Hamiltonian:

$$H = -\sum_{i=1}^{L-1}\{J(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) + J\gamma(c_i^\dagger c_{i+1}^\dagger + c_{i+1}c_i)\} - \sum_{i=1}^{L}2gc_i^\dagger c_i \tag{1.35}$$
$$+ (-1)^{\hat{\mathcal{N}}}\{J(c_L^\dagger c_1 + c_1^\dagger c_L) + J\gamma(c_L^\dagger c_1^\dagger + c_1 c_L)\} \tag{1.36}$$

OBC Hamiltonian:

$$H = -\sum_{i=1}^{L-1}\{J(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) + J\gamma(c_i^\dagger c_{i+1}^\dagger + c_{i+1}c_i)\} - \sum_{i=1}^{L}2gc_i^\dagger c_i \tag{1.37}$$

1.4.2 Majorana Representation

The majorana operators $\psi_{i,1}$ and $\psi_{i,2}$ associated to each site i are defined by

$$\psi_{i,1} \equiv \frac{c_i + c_i^\dagger}{\sqrt{2}}, \psi_{i,2} \equiv \frac{c_i - c_i^\dagger}{i\sqrt{2}} \tag{1.38}$$

Equivalently,

$$c_i = \frac{\psi_{i,1} + i\psi_{i,2}}{\sqrt{2}}, c_i^\dagger = \frac{\psi_{i,1} - i\psi_{i,2}}{\sqrt{2}} \tag{1.39}$$

The majorana operator satisfy the following anti-commutation relation

$$\{\psi_{i,\alpha}, \psi_{j,\alpha'}\} = \delta_{i,j}\delta_{\alpha,\alpha'} \tag{1.40}$$

$$c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i = \frac{i}{2}[(\psi_{i,1}\psi_{i+1,2} - \psi_{i+1,2}\psi_{i,1}) + (\psi_{i+1,1}\psi_{i,2} - \psi_{i,2}\psi_{i+1,1})] \tag{1.41}$$

$$c_i^\dagger c_{i+1}^\dagger + c_{i+1}c_i = \frac{i}{2}[(\psi_{i+1,1}\psi_{i,2} - \psi_{i,2}\psi_{i+1,1}) + (\psi_{i+1,2}\psi_{i,1} - \psi_{i,1}\psi_{i+1,2})] \tag{1.42}$$

$$c_i^\dagger c_i = \frac{1}{2}\sum_{\alpha=1,2} \psi_{i,\alpha}\psi_{i,\alpha} + \frac{i}{2}(\psi_{i,1}\psi_{i,2} - \psi_{i,2}\psi_{i,1}) \tag{1.43}$$

$$c_L^\dagger c_1 + c_1^\dagger c_L = \frac{i}{2}[(\psi_{1,1}\psi_{L,2} - \psi_{L,2}\psi_{1,1}) + (\psi_{L,1}\psi_{1,2} - \psi_{1,2}\psi_{L,1})] \tag{1.44}$$

$$c_L^\dagger c_1^\dagger + c_1 c_L = \frac{i}{2}[(\psi_{L,1}\psi_{1,2} - \psi_{1,2}\psi_{L,1}) + (\psi_{L,2}\psi_{1,1} - \psi_{1,1}\psi_{L,2})] \tag{1.45}$$

By inserting above relation into Eq.1.36 and Eq.1.37, the Hamiltonian in the majorana formalism can be written as

$$H = \frac{i}{2}\sum_{i,j}^{L}\sum_{\alpha,\alpha'=1}^{2} \psi_{i,\alpha}A_{i,\alpha;j,\alpha'}\psi_{j,\alpha'} \tag{1.46}$$

1.5 Correlation matrix and Entanglement Entropy

Entanglement entropy of a subregion A is

$$S(A) = \mathrm{tr}\big(f(M^A)\big) \tag{1.47}$$

where

$$f(M^A) = -M^A \log M^A \tag{1.48}$$

Here the $2L \times 2L$ Hermitian matrix M^A is the correlation matrix $\langle\psi_{i,\alpha}\psi_{j,\alpha'}\rangle$. It can be diagonalized as

$$M^A = U(\otimes_{k=1}^{L} \begin{bmatrix} \frac{1+\nu_k}{2} & 0 \\ 0 & \frac{1-\nu_k}{2} \end{bmatrix})U^\dagger \tag{1.49}$$

which implies that

$$S(A) = \sum_k -(\frac{1+\nu_k}{2}\log\frac{1+\nu_k}{2} + \frac{1-\nu_k}{2}\log\frac{1-\nu_k}{2}) \tag{1.50}$$

1.6 Appendix

1.6.1 Provement

1.

$$\begin{aligned}2\sigma_+^j\sigma_-^j - 1 &= 2\cdot\frac{\sigma_x^j + i\sigma_y^j}{2}\frac{\sigma_x^j - i\sigma_y^j}{2} - 1\\ &= 2i\sigma_y^j\sigma_x^j\\ &= \sigma_z\end{aligned}$$

2.

$$\sigma_-^L\sigma_+^1 = \left[\prod_{n=1}^{L-1} e^{i\pi c_n^\dagger c_n}\right] c_L c_1^\dagger \tag{1.51}$$

$$= -(-1)^{\hat{\mathcal{N}}} c_1^\dagger c_L \tag{1.52}$$

1.6.2 Canonical Transformation between XY model and QIC model

Here we will focus on XY model and Quantum Ising model, of which the Hamiltonian are as follows[**?**]

$$\mathcal{H}_{XY} = \sum_{i=1}^{L}(J_i^x S_i^x S_{i+1}^x + J_i^y S_i^y S_{i+1}^y), S_{L+1}^{x,y} = S_1^{x,y} \tag{1.53}$$

$$\mathcal{H}_I = -\frac{1}{2}\sum_{i=1}^{L}\lambda_i\sigma_i^x\sigma_{i+1}^x - \frac{1}{2}\sum_i h_i\sigma_i^z, \sigma_{L+1}^{x,z} = \sigma_1^{x,z} \tag{1.54}$$

These two Hamiltonians can be mapped into each other through a canonical transformation.

Define two set of Pauli operator

$$\sigma_i^x = \prod_{j=1}^{2i-1}(2S_j^x), \sigma_i^z = 4S_{2i-1}^y S_{2i}^y \tag{1.55}$$

$$\tau_i^x = \prod_{j=1}^{2i-1}(2S_j^y), \sigma_i^z = 4S_{2i-1}^x S_{2i}^x \tag{1.56}$$

The inverse transformation is

$$2S_{2i-1}^x = \sigma_i^x\prod_{j=1}^{i-1}\tau_j^z, 2S_{2i}^x = \sigma_i^x\prod_{j=1}^{i}\tau_j^z \tag{1.57}$$

$$2S_{2i-1}^y = \tau_i^x\prod_{j=1}^{i-1}\sigma_j^z, 2S_{2i}^y = \sigma_i^x\prod_{j=1}^{i}\tau_j^z \tag{1.58}$$

$$\tag{1.59}$$

In terms of these Pauli operators the Hamiltonian operator of the XY chain with L spins can be written as the sum of two decoupled quantum Ising chains with L/2 sites:

$$\mathcal{H}_{XY} = \frac{1}{2}[\mathcal{H}_I(\sigma) + H_I(\tau)] \tag{1.60}$$

$$\mathcal{H}_I(\sigma) = -\frac{1}{2}\sum_{i=1}^{L/2} J_{2i}^x \sigma_i^x \sigma_{i+1}^x - \frac{1}{2}\sum_{i=1}^{L/2} J_{2i-1}^y \sigma_i^z \tag{1.61}$$

$$\mathcal{H}_I(\tau) = -\frac{1}{2}\sum_{i=1}^{L/2} J_{2i}^y \tau_i^x \tau_{i+1}^x - \frac{1}{2}\sum_{i=1}^{L/2} J_{2i-1}^x \tau_i^z \tag{1.62}$$

Even though $[\mathcal{H}_I(\sigma), \mathcal{H}_I(\tau)] = 0$, the two hamiltonian are still independent due to the two set of pauli matrix do not commute with each other.

References