Stats531 Homework2

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Question 2.1

We investigate two ways to calculate the autocovariance function for AR and MA models. These ways are different from the two ways already demonstrated in the notes, but there is some overlap. The instructions below help you work through the case of a causal AR(1) model,

$$X_n = \varphi X_{n-1} + \epsilon_n.$$

where $\{\epsilon_n\}$ is white noise with variance σ^2, and $-1 < \phi < 1$. Assume the process is stationary, i.e., it is initialized with a random draw from its stationary distribution. Show your working for both the approaches A and B explained below. If you want an additional challenge, you can work through the AR(2) or ARMA(1,1) case instead.

A.

Using the stochastic difference equation to obtain a difference equation for the autocovariance function (ACF). Start by writing the ACF as

$$\gamma_h = \text{Cov}(X_n, X_{n+h}) = \text{Cov}(X_n, \varphi X_{n+h-1} + \epsilon_{n+h}), \text{ for } h > 0.$$

Writing the right hand side in terms of γ_{h-1} leads to an equation which is formally a first order linear homogeneous recurrence relation with constant coefficients. To solve such an equation, we look for solutions of the form

$$\gamma_h = A\lambda^h.$$

Substituting this general solution into the recurrence relation, together with an initial condition derived from explicitly computing γ_0, provides an approach to finding two equations that can be solved for the two unknowns, A and λ.

Answer:

$$
\begin{aligned}
\gamma_h &= \mathrm{Cov}(X_n, X_{n+h}) \\
&= \mathrm{Cov}(X_n, \varphi X_{n+h-1} + \epsilon_{n+h}) \\
&= \varphi \mathrm{Cov}(X_n, X_{n+h-1}) + \mathrm{Cov}(X_n, \epsilon_{n+h}) \\
&= \varphi \gamma_{h-1}
\end{aligned}
$$

It's a linear recurrence relations with constant coefficients (cite: Wikipedia, Recurrence relation, Definition), and its solution takes the form:(cite: Wikipedia, linear recurrence with constant coefficients, Solution_example_for_small_orders)

$$\gamma_h = A\lambda^h$$

Then we choose special case $h = 0$ and $h = 1$:

$$
\begin{cases}
\gamma_0 = A\lambda^0 \\
\gamma_1 = A\lambda = \varphi\gamma_0
\end{cases}
$$

Solve for A and λ we can get:

$$
\begin{cases}
A = \gamma_0 \\
\lambda = \varphi
\end{cases}
$$

Specifically, we can find that the form is a first order linear homogeneous recurrence relation with constant coefficients, then, cite Wikipedia, Wikipedia, linear recurrence with constant coefficients, Solution_example_for_small_orders, order 1, we know that the solution takes the form:

$$\gamma_h = \gamma_0 \varphi^h$$

This further proves the solution we get. Therefore the ACF (autocovariance function) is:

$$\gamma_h = \mathrm{Var}(X_0)\varphi^h$$

Since we assume the time series is stationary:

$$\mathrm{Var}(X_0) = \mathrm{Var}(X_1) = \mathrm{Var}(\varphi X_0 + \epsilon_1) = \varphi^2 \mathrm{Var}(X_0) + \sigma^2 \mathrm{Var}(X_0) = \frac{\sigma^2}{1 - \varphi^2}$$

Thus

$$\gamma_h = \mathrm{Var}(X_0)\varphi^h = \frac{\sigma^2\varphi^h}{1-\varphi^2}$$

B.

Via a Taylor series calculation of the MA(∞) representation. Construct a Taylor series expansion of $g(x) = (1-\phi x)^{-1}$ of the form

$$g(x) = g_0 + g_1 x + g_2 x^2 + g_3 x^3 + \ldots$$

Do this either by hand or using your favorite math software (if you use software, please say what software you used and what you entered to get the output). Use this Taylor series to write down the MA(∞) representation of an AR(1) model. Then, apply the general formula for the autocovariance function of an MA(∞) process.

Answer:

$$\begin{aligned}
X_n &= \varphi X_{n-1} + \epsilon_n \\
(1-\varphi B)X_n &= \epsilon_n \\
X_n &= \frac{1}{1-\varphi B}\epsilon_n
\end{aligned}$$

Define:

$$g(x) = \frac{1}{1-\varphi x}$$

Then: (cite Wikipedia, taylor series)

$$\begin{aligned}
g(x) &= g(0) + g'(0)x + \frac{1}{2!}g''(0)x^2 + \cdots + \frac{1}{n!}g^{(n)}(0)x^n + \cdots \\
&= 1 + (-1)(-\varphi)x + \frac{1}{2!}(-1)(-2)(-\varphi)^2 + \cdots + \frac{1}{n!}(-1)^n \cdot n!(-\varphi)^n x^n + \cdots \\
&= 1 + \varphi x + \varphi^2 x^2 + \cdots + \varphi^n x^n + \cdots
\end{aligned}$$

Then: (cite lecture slides4 page 14-15 steps) we can use taylor expansion on the function of operator:

$$
\begin{aligned}
X_n &= g(B)\epsilon_n \\
&= (1 + \varphi B + \varphi^2 B^2 + \cdots)\epsilon_n \\
&= \epsilon_n + \varphi\epsilon_{n-1} + \varphi^2\epsilon_{n-2} + \cdots + \varphi^n\epsilon_0 + \cdots
\end{aligned}
$$

Then: (here I didn't use the ACF of MA(∞) directly, I computed it by hand)

$$
\begin{aligned}
\gamma_h &= \mathrm{Cov}(X_n, X_{n+h}) \\
&= \mathrm{Cov}(\epsilon_n + \varphi\epsilon_{n-1} + \cdots, \epsilon_{n+h} + \varphi\epsilon_{n+h-1} + \cdots) \\
&= \varphi^h \mathrm{Var}(\epsilon_n) + \varphi^{h+2}\mathrm{Var}(\epsilon_{n-1}) + \cdots \\
&= \varphi^h (1 + \varphi^2 + \cdots)\sigma^2 \\
&= \lim_{n\to\infty} \varphi^h \frac{1-\varphi^{2n}}{1-\varphi^2}\sigma^2 \quad (\text{since } \varphi^2 \in (0,1)) \\
&= \frac{\sigma^2\varphi^h}{1-\varphi^2}
\end{aligned}
$$

It's the same as the result we get in **A**.

C.

Check your work for the specific case of an AR(1) model with $\varphi_1 = 0.8$ by comparing your formula with the result of the R function `ARMAacf`.

Answer:

Use the codes below (cite: `help(ARMAacf)`) to generate the ACF of the AR(1) model with $\varphi_1 = 0.8$ ($h \leq 20$):

```r
# Here since ARMAacf gives the correlation not the covariance,
# we need to times Var(X_0) given sigma = 1
ARMAacf(0.8, lag.max = 20)/(1-0.8^2)
```

```
         0          1          2          3          4          5          6 
2.77777778 2.22222222 1.77777778 1.42222222 1.13777778 0.91022222 0.72817778 
         7          8          9         10         11         12         13 
0.58254222 0.46603378 0.37282702 0.29826162 0.23860929 0.19088744 0.15270995 
        14         15         16         17         18         19         20 
0.12216796 0.09773437 0.07818749 0.06254999 0.05004000 0.04003200 0.03202560 
```

Below I compute the ACF using the result in **A** and **B**:

```
f <- function(h){
  return(0.8^h/(1-0.8^2))
}

f(c(0:20))
```

```
 [1] 2.77777778 2.22222222 1.77777778 1.42222222 1.13777778 0.91022222
 [7] 0.72817778 0.58254222 0.46603378 0.37282702 0.29826162 0.23860929
[13] 0.19088744 0.15270995 0.12216796 0.09773437 0.07818749 0.06254999
[19] 0.05004000 0.04003200 0.03202560
```

As we can see, the results are the same.

Question 2.2

Compute the autocovariance function (ACF) of the random walk model. Specifically, find the ACF, $\gamma_{mn} = \text{Cov}(X_m, X_n)$, for the random walk model specified by

$$X_n = X_{n-1} + \epsilon_n,$$

where $\{\epsilon_n\}$ is white noise with variance σ^2, and we use the initial value $X_0 = 0$.

Answer:

(I solve this question independently and I didn't refer to any resources. However, I've read the text book and seen the result of the question on *Time Series Analysis and Its Applications*, Chapter1 Characteristic of time series, section 1.3 Measure of dependence, Example 1.18)

Since:

$$\begin{cases} X_n = X_{n-1} + \epsilon_n \\ X_0 = 0 \end{cases}$$

Then:

$$
\begin{aligned}
X_n &= X_{n-1} + \epsilon_n \\
&= X_{n-2} + \epsilon_{n-1} + \epsilon_n \\
&\cdots \\
&= X_0 + \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n \\
&= \epsilon_1 + \cdots + \epsilon_n
\end{aligned}
$$

Thus:

$$
\begin{aligned}
\gamma_{mn} &= \mathrm{Cov}(X_m, X_n) \\
&= \mathrm{Cov}(\epsilon_1 + \cdots + \epsilon_m, \epsilon_1 + \cdots + \epsilon_n) \\
&= \sum_{i=1}^{\min\{m,n\}} \mathrm{Var}(\epsilon_i) \\
&= \min\{m, n\}\sigma^2
\end{aligned}
$$

Reference

1. first order linear homogeneous recurrence relation with constant coefficients.
2. Wikipedia, Recurrence relation, Definition
3. Wikipedia, linear recurrence with constant coefficients, Solution_example_for_small_orders)
4. Wikipedia, Wikipedia, linear recurrence with constant coefficients, Solution_example_for_small_orders, order 1
5. Wikipedia, taylor series
6. `ARMAacf` RDocument
7. Lecture slides4 page 14-15 steps
8. Shumway, R.H., and Stoffer, D.S., 2017. Time series analysis and its applications (4th edition). New York: Springer.