

# Modeling and Analysis of Time Series Data

## Chapter 10: Introduction to partially observed Markov process models

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# Latent process models

- Uncertainty and variability are common features biological and social systems. Complex physical systems can also be unpredictable: we can only forecast weather reliably in the near future.
- Time series models of deterministic trend plus colored noise imply perfect predictability if the trend function enables extrapolation.
- To model variability and unpredictability in a dynamic system, we can specify a stochastic (i.e., random) model for the system.
- Often times, the full dynamic system is unobserved. We have only noisy or incomplete measurements.
- We model measurements as random variables conditional on the trajectory of the **latent process**. The latent process is also called a **state process** or **hidden process**.

# The Markov property

- A model for a stochastic dynamic system has the **Markov property** if the future evolution of the system depends only on the current state, plus randomness introduced in future.
- A models with the Markov property may be called a **Markov chain** or a **Markov process**.
- We use the term Markov process since the term chain is often reserved for situations where either time or the latent state (or both) take discrete values.
- The Markov property is often used to model the latent process in a time series model.

many stochastic models have this structure:  
we model all the variables we think affect the future dynamics of the system.

# Notation for discrete time Markov processes

- A time series model  $X_{0:N}$  is a **Markov process** model if the conditional densities satisfy the **Markov property** [P1] that

$$[\text{P1}] \quad f_{X_n | X_{1:n-1}}(x_n | x_{1:n-1}) = f_{X_n | X_{n-1}}(x_n | x_{n-1}).$$

for all  $n \in 1 : N$

- We may suppose there is an underlying continuous time,  $t$ , such that  $X_n$  occurs at time  $t_n$ .
- We write  $X(t)$  for the continuous time model, setting  $X_n = X(t_n)$ .
- $t_{1:N}$  are **measurement times**.
- $t_0$  is the **initialization time**.

# Initial conditions

- We **initialize** the Markov process model at a time  $t_0$ , although data are collected only at times  $t_{1:N}$ .
- The initialization model could be deterministic (a fixed value) or a random variable.
- We model  $X_0 = X(t_0)$  as a draw from a probability density function

$$f_{X_0}(x_0). \quad (1)$$

- A fixed initial value is a special case of a density corresponding to a point mass with probability one at the fixed value.
- A discrete probability mass function is a special case of a density corresponding to a collection of point masses.

*\* this could be a parameter of the system*

# The process model

- The probability density function  $f_{X_n|X_{n-1}}(x_n | x_{n-1})$  is called the **one-step transition density** of the Markov process.
- The Markov property asserts that the next step taken by a Markov process follows the one-step transition density based on the current state, whatever the previous history of the process.
- For a Markov model, the full joint distribution of the latent process is entirely specified by the one-step transition densities, given the initial value.
- Therefore, we also call  $f_{X_n|X_{n-1}}(x_n | x_{n-1})$  the **process model**.

# The joint distribution in terms of one-step transition densities

**Exercise 10.1.** Use [P1] to derive an expression for the joint distribution of a Markov process as a product of the one-step transition densities. In other words, derive

$$[\text{P2}] \quad f_{X_{0:N}}(x_{0:N}) = f_{X_0}(x_0) \prod_{n=1}^N f_{X_n|X_{n-1}}(x_n | x_{n-1}).$$

**Hint:** This involves elementary rules for manipulation of joint and conditional densities, together with application of the Markov property. It is a good exercise to work through by hand to build familiarity with the model class.



**Question 10.1.** Explain why a causal Gaussian AR(1) process is a Markov process.

Gaussian AR(1) process:  $X_n = \phi X_{n-1} + \varepsilon_n$ ,  
 $\varepsilon_n \sim \text{iid } N(0, \sigma^2)$ . Critically,  $\varepsilon_n$  is independent of  $X_{n-1}$ , since  $X_{n-1}$  is constructed using  $\varepsilon_{n-1}, \varepsilon_{n-2}, \varepsilon_{n-3}, \dots$  assuming we are using a causal representation of the AR model.

Then,  $X_n$  given  $X_0, X_1, \dots, X_{n-1}$  depends only on  $X_{n-1}$  by construction

we need  $\varepsilon_n$  to be independent of

$X_0, X_1, X_{n-2}$ . This rules out general uncorrelated (but not independent) white noise. It seems to rule out non-causal constructions where  $X_0, X_1, X_2, \dots, X_{n-2}$  depend on  $\varepsilon_n$

## Time-homogeneous transitions and stationarity

note: for ARMA models, a causal stationary solution requires

- The one step transition density  $f_{X_n|X_{n-1}}$  for a Markov process  $X_{0:N}$  can depend on  $n$ . *AR puts outside unit circle. Here, we look for a causal solution but not a stationary one.*
- $X_{0:N}$  is **time-homogeneous** if  $f_{X_n|X_{n-1}}$  does not depend on  $n$ , so there is a conditional density  $f(\cdot|\cdot)$  such that, for all  $n \in 1:N$ ,

$$f_{X_n|X_{n-1}}(x_n | x_{n-1}) = f(x_n | x_{n-1}). \quad (2)$$

**Question 10.2.** If  $X_{0:N}$  is strict stationary, it is time-homogeneous. Why?

Strict stationarity requires that all joint & conditional distributions are shift-invariant. So,  $f_{X_n|X_{n-1}} = f_{X_{n-1}|X_{n-2}} = \dots$  i.e., we have time-homogeneous transitions.

**Question 10.3.** Time-homogeneity does not necessarily imply stationarity.

Find a counter-example. *Brownian motion. This is a Gaussian random walk - in discrete time,  $X_n = X_{n-1} + \varepsilon_n$ ,  $\varepsilon_n \sim N(0, \sigma^2)$ . This is a time-homogeneous Markov chain, with  $f_{X_n|X_{n-1}} = a \sim N[a, \sigma^2]$  for all  $n$ . Since  $\text{Var}(X_n) = n\sigma^2$ , non-constant, implying  $X_n$  is not stationary.*

# Partially observed Markov process (POMP) models

- **Partial observation** may mean either or both of (i) measurement noise; (ii) entirely unmeasured latent variables.
- These features are present in many systems.
- A **partially observed Markov process** (POMP) model is defined by putting together a Markov latent process model and a **measurement model**.
- POMP models are a general class, covering many models designed for specific applications.
- Statistical methods for to this general class give us flexibility to develop specific POMP models appropriate to a range of applications.

# The measurement model

- The **measurement process** is a collection of random variables  $Y_{1:N}$  which models the data  $y_{1:N}^*$ .
- $Y_n$  is assumed to depend on the latent process only through its value  $X_n$  at the time of the measurement. Formally, this assumption is:

$$[\text{P3}] \quad f_{Y_n|X_{0:N}, Y_{1:n-1}, Y_{n+1:N}}(y_n | x_{0:N}, y_{1:n-1}, y_{n+1:N}) = f_{Y_n|X_n}(y_n | x_n).$$

- We call  $f_{Y_n|X_n}(y_n | x_n)$  the **measurement model**.

# Time-homogeneous measurement models

- In general, the measurement model can depend on  $n$  or on any covariate time series.
- The measurement model is **time-homogeneous** if there is a conditional probability density function  $g(\cdot | \cdot)$  such that, for all  $n \in 1 : N$ ,

$$f_{Y_n|X_n}(y_n | x_n) = g(y_n | x_n). \quad (3)$$

- Time-inhomogeneous process and measurement models are sufficiently common that we benefit from the extra generality of writing  $f_{X_n|X_{n-1}}(x_n|x_{n-1})$  and  $f_{Y_n|X_n}(y_n|x_n)$  versus  $f(x_n|x_{n-1})$  and  $g(y_n|x_n)$ .

e.g. as soon as there are covariates, the s/system usually becomes non-time-homogeneous.

# Four basic calculations for working with POMP models

Many time series models in science, engineering and industry can be written as POMP models. A reason that POMP models form a useful tool for statistical work is that there are convenient recursive formulas to carry out four basic calculations:

- 1 Prediction
- 2 Filtering
- 3 Smoothing
- 4 Likelihood calculation

e.g. weather  
forecasting,  
(very high  
dimensional)

modeled as a  
physical system.

e.g. rocket control.

The latent process is the  
state of the rocket.

The measurements are the  
sensors.

e.g. epidemic  
forecasting, where latent  
process is the spread of the  
disease.

# Prediction

- **One-step prediction** (also called forecasting) of the latent process at time  $t_{n+1}$  given data up to time  $t_n$  involves finding

$$f_{X_{n+1}|Y_{1:n}}(x_{n+1} | y_{1:n}^*). \quad (4)$$

- We may want to predict more than one time step ahead. However, one-step prediction turns out to be closely related to computing the likelihood function, and therefore central to statistical inference.
- Our prediction is a conditional probability density, not a point estimate. In the context of forecasting, this is called a **probabilistic forecast**. What are the advantages of a probabilistic forecast over a point forecast? Are there any disadvantages?

Probabilistic forecast assesses the full range of possible events. This makes it easy to evaluate: what probability did it put on what actually happened?

Disadvantages: more work, harder to explain.

# Filtering

- The **filtering** calculation at time  $t_n$  is to find the conditional distribution of the latent process  $X_n$  given data  $y_{1:n}^*$  available at time  $t_n$ .
- Filtering involves calculating

$$f_{X_n|Y_{1:n}}(x_n | y_{1:n}^*). \quad (5)$$

- This can be evaluated numerically or algebraically. We will see that Monte Carlo methods can be a good tool.
- The name “filtering” comes from the history of signal processing. A noisy received signal was passed through capacitors and resistors to construct a band pass filter estimating the source signal, just like an optical filter removes unwanted frequencies of light.



# Smoothing

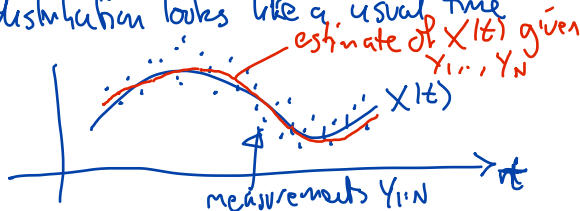
- In the context of a POMP model, smoothing involves finding the conditional distribution of  $X_n$  given all the data,  $y_{1:N}^*$ .

note: smoothing is a "batch" calculation; filtering & prediction are "online".

- So, the smoothing calculation is to find

$$f_{X_n|Y_{1:N}}(x_n | y_{1:N}^*). \quad (6)$$

If the latent state is a smooth approximation to a time series, i.e. a local expected value, then the POMP smoothing distribution looks like a usual time series smoother.



# The likelihood

- The likelihood is the joint density of  $Y_{1:N}$  evaluated at the data,

$$f_{Y_{1:N}}(y_{1:N}^*). \quad (7)$$

- The model may depend on a parameter vector  $\theta$ . We can include  $\theta$  in all the joint and conditional densities above. Then, the **likelihood function** is the likelihood viewed as a function of  $\theta$ . We write

$$\mathcal{L}(\theta) = f_{Y_{1:N}}(y_{1:N}^*; \theta) \quad (8)$$

- If we can compute  $\mathcal{L}(\theta)$  then we can perform numerical optimization to get a maximum likelihood estimate
- Likelihood evaluation and maximization lets us compute profile likelihood confidence intervals, carry out likelihood ratio tests, and make AIC model comparisons.

# The prediction formula

- One-step prediction of the latent process at time  $t_n$  given data up to time  $t_{n-1}$  can be computed recursively in terms of the filtering problem at time  $t_{n-1}$ , via the **prediction formula** for  $n \in 1 : N$ ,

$$[P4] \quad f_{X_n|Y_{1:n-1}}(x_n | y_{1:n-1}^*) = \int f_{X_{n-1}|Y_{1:n-1}}(x_{n-1} | y_{1:n-1}^*) f_{X_n|X_{n-1}}(x_n | x_{n-1}) dx_{n-1}.$$

onestep transition from  
n-1 to n

filter at time n-1

- For the case  $n = 1$ , we let  $1 : k$  is the empty set when  $k = 0$ , so that  $f_{X_0|Y_{1:0}}(x_0 | y_{1:0}^*)$  means  $f_{X_0}(x_0)$ . In other words, the filter distribution at time  $t_0$  is the initial density for the latent process, since at time  $t_0$  we have no data to condition on.

**Exercise 10.2.** Derive [P4] using the definition of a POMP model with elementary properties of joint and conditional densities.

# Hints for deriving the recursion formulas

Any general identity holding for densities must also hold when we condition everything on a new variable.

**Example 1.** From

$$f_{XY}(x, y) = f_X(x) f_{Y|X}(y | x) \quad (9)$$

we can condition on  $Z$  to obtain

$$f_{XY|Z}(x, y | z) = f_{X|Z}(x | z) f_{Y|XZ}(y | x, z). \quad (10)$$

**Example 2.** The prediction formula is a special case of the identity

$$f_{X|Y}(x | y) = \int f_{XZ|Y}(x, z | y) dz. \quad (11)$$

**Example 3.** A conditional form of Bayes' identity is

$$f_{X|YZ}(x | y, z) = \frac{f_{Y|XZ}(y | x, z) f_{X|Z}(x | z)}{f_{Y|Z}(y | z)}. \quad (12)$$

# The filtering formula

- Filtering at time  $t_n$  can be computed by combining the new information in the datapoint  $y_n^*$  with the calculation of the one-step prediction of the latent process at time  $t_n$  given data up to time  $t_{n-1}$ .
- This is carried out via the **filtering formula** for  $n \in 1 : N$ ,

$$[\text{P5}] \quad f_{X_n|Y_{1:n}}(x_n | y_{1:n}^*) = \frac{f_{X_n|Y_{1:n-1}}(x_n | y_{1:n-1}^*) f_{Y_n|X_n}(y_n^* | x_n)}{f_{Y_n|Y_{1:n-1}}(y_n^* | y_{1:n-1}^*)}.$$

**Exercise 10.3.** Derive [P5] using the definition of a POMP model with elementary properties of joint and conditional densities.

- The prediction and filtering formulas are **recursive**. If they can be computed for time  $t_n$  then they enable the computation at time  $t_{n+1}$ .

# The conditional likelihood formula

- The denominator in the filtering formula [P5] is the **conditional likelihood** of  $y_n^*$  given  $y_{1:n-1}^*$ .
- It can be computed in terms of the one-step prediction density, via the **conditional likelihood formula**,

$$[P6] \quad f_{Y_n|Y_{1:n-1}}(y_n^* | y_{1:n-1}^*) = \int f_{X_n|Y_{1:n-1}}(x_n | y_{1:n-1}^*) f_{Y_n|X_n}(y_n^* | x_n) dx_n.$$

- To make this formula work for  $n = 1$ , we take advantage of the convention that  $1 : k$  is the empty set when  $k = 0$ .

# Computation of the likelihood and log likelihood

- The likelihood of the entire dataset,  $y_{1:N}^*$  can be found from [P6], using the identity

$$f_{Y_{1:N}}(y_{1:N}^*) = \prod_{n=1}^N f_{Y_n|Y_{1:n-1}}(y_n^* | y_{1:n-1}^*). \quad (13)$$

- Equation (13) uses the convention that  $1:k$  is the empty set when  $k=0$ , so the first term in the product is

$$f_{Y_1|Y_{1:0}}(y_1^* | y_{1:0}^*) = f_{Y_1}(y_1^*) \quad (14)$$

- If our model has an unknown parameter  $\theta$  then (13) gives the **log likelihood function** as a sum of conditional log likelihoods,

$$\ell(\theta) = \log \mathcal{L}(\theta) = \log f_{Y_{1:N}}(y_{1:N}^*; \theta) = \sum_{n=1}^N \log f_{Y_n|Y_{1:n-1}}(y_n^* | y_{1:n-1}^*; \theta).$$

# The smoothing recursions

- Smoothing is less fundamental for likelihood-based inference than filtering and one-step prediction.
- Nevertheless, sometimes we want to compute the smoothing density, so we develop some necessary formulas.
- The filtering and prediction formulas are recursions forward in time: a solution at time  $t_{n-1}$  is used for the computation at  $t_n$ .
- For smoothing, we have **backwards smoothing recursion formulas**,

$$[\text{P7}] \quad f_{Y_{n:N}|X_n}(y_{n:N}^* | x_n) = f_{Y_n|X_n}(y_n^* | x_n) f_{Y_{n+1:N}|X_n}(y_{n+1:N}^* | x_n).$$

$$\begin{aligned} [\text{P8}] \quad f_{Y_{n+1:N}|X_n}(y_{n+1:N}^* | x_n) \\ = \int f_{Y_{n+1:N}|X_{n+1}}(y_{n+1:N}^* | x_{n+1}) f_{X_{n+1}|X_n}(x_{n+1} | x_n) dx_{n+1}. \end{aligned}$$



# Combining recursions to find the smoothing distribution

The forwards and backwards recursion formulas together allow us to compute the **smoothing formula**,

$$[\text{P9}] \quad f_{X_n|Y_{1:N}}(x_n | y_{1:N}^*) = \frac{f_{X_n|Y_{1:n-1}}(x_n | y_{1:n-1}^*) f_{Y_{n:N}|X_n}(y_{n:N}^* | x_n)}{f_{Y_{n:N}|Y_{1:n-1}}(y_{n:N}^* | y_{1:n-1}^*)}.$$

**Exercise 10.4.** Show how [P7], [P8] and [P9] follow from the basic properties of conditional densities combined with the Markov property.

**Hint:** you can write the left hand side of [P9] as  $f_{X|YZ}$  with  $X = X_n$ ,  $Y = Y_{1:n-1}$ ,  $Z = Y_{n:N}$ .

# Linear Gaussian POMP (LG-POMP) models

- Linear Gaussian partially observed Markov process (LG-POMP) models have many applications across science and engineering.
- Gaussian ARMA models are LG-POMP models. The POMP recursion formulas give a computationally efficient way to obtain the likelihood of a Gaussian ARMA model.
- Smoothing splines (including the Hodrick-Prescott filter, which is a smoothing spline) can be written as an LG-POMP model.
- The **Basic Structural Model** is an LG-POMP used for econometric forecasting. It models a stochastic trend, seasonality, and measurement error, in a framework with econometrically interpretable parameters. This is more interpretable than fitting SARIMA.
- If an LG-POMP model is appropriate, you avoid Monte Carlo computations used for inference in general nonlinear POMP models.

# The general LG-POMP model

Suppose the latent process,  $X_{0:N}$ , and the observation process  $\{Y_n\}$ , takes vector values with dimension  $d_X$  and  $d_Y$ . A general mean zero LG-POMP model is specified by

- A sequence of  $d_X \times d_X$  matrices,  $\mathbb{A}_{1:N}$ ,
- A sequence of  $d_X \times d_X$  covariance matrices,  $\mathbb{U}_{0:N}$ ,
- A sequence of  $d_Y \times d_X$  matrices,  $\mathbb{B}_{1:N}$
- A sequence of  $d_Y \times d_Y$  covariance matrices,  $\mathbb{V}_{1:N}$ .

We initialize with  $X_0 \sim N[0, \mathbb{U}_0]$  and then define the entire LG-POMP model by a recursion for  $n \in 1 : N$ ,

$$[\text{LG1}] \quad X_n = \mathbb{A}_n X_{n-1} + \epsilon_n, \quad \epsilon_n \sim N[0, \mathbb{U}_n],$$

$$[\text{LG2}] \quad Y_n = \mathbb{B}_n X_n + \eta_n, \quad \eta_n \sim N[0, \mathbb{V}_n].$$

Often, but not always, we will have a **time-homogeneous** LG-POMP model, with  $\mathbb{A}_n = \mathbb{A}$ ,  $\mathbb{B}_n = \mathbb{B}$ ,  $\mathbb{U}_n = \mathbb{U}$  and  $\mathbb{V}_n = \mathbb{V}$  for  $n \in 1 : N$ .

# The LG-POMP representation of a Gaussian ARMA

- Let  $\{Y_n\}$  be a Gaussian ARMA( $p, q$ ) model with noise process  $\omega_n \sim \text{normal}[0, \sigma^2]$ , defined by

$$Y_n = \sum_{j=1}^p \phi_j Y_{n-j} + \omega_n + \sum_{k=1}^q \psi_k \omega_{n-k}. \quad (15)$$

- We look for a time-homogeneous LG-POMP defined by [LG1] and [LG2] where  $Y_n$  is the first component of  $X_n$  with no measurement error.
- To do this, we define  $d_X = r = \max(p, q + 1)$  and

$$\mathbb{B} = (1, 0, 0, \dots, 0), \quad (16)$$

$$\mathbb{V} = 0. \quad (17)$$

- We require  $\mathbb{A}$  and  $\mathbb{U}$  such that  $Y_n$  satisfies equation (15).

We state a solution and see if it works out. Consider

$$X_n = \begin{pmatrix} Y_n \\ \phi_2 Y_{n-1} + \cdots + \phi_r Y_{n-r+1} + \psi_1 \omega_n + \cdots + \psi_{r-1} \omega_{n-r+2} \\ \phi_3 Y_{n-1} + \cdots + \phi_r Y_{n-r+1} + \psi_2 \omega_n + \cdots + \psi_{r-1} \omega_{n-r+3} \\ \vdots \\ \phi_r Y_{n-1} + \psi_{r-1} \omega_t \end{pmatrix}$$

We can check that the ARMA equation (15) matches the matrix equation

$$X_n = \mathbb{A} X_{n-1} + \begin{pmatrix} 1 \\ \psi_1 \\ \psi_2 \\ \vdots \\ \psi_{r-1} \end{pmatrix} \omega_n. \text{ where } \mathbb{A} = \begin{pmatrix} \phi_1 & 1 & 0 & \cdots & 0 \\ \phi_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \phi_{r-1} & 0 & \cdots & 0 & 1 \\ \phi_r & 0 & \cdots & 0 & 0 \end{pmatrix}$$

This is a time-homogenous LG-POMP, with  $\mathbb{A}$ ,  $\mathbb{B}$  and  $\mathbb{V}$  as above and

$$\mathbb{U} = \sigma^2(1, \psi_1, \psi_2, \dots, \psi_{r-1})^T(1, \psi_1, \psi_2, \dots, \psi_{r-1}).$$

## Different POMP models can give the same model for $Y_{1:N}$

- There are other LG-POMP representations giving rise to the same ARMA model.
- When only one component of a latent process is observed, any model giving rise to the same observed component is indistinguishable from the data.
- Here, the LG-POMP model has order  $d_X^2 = r^2 = \max(p, q + 1)^2$  parameters. The ARMA model has order  $r$  parameters, so we expect many ways to parameterize the ARMA model as a special case of the much larger LG-POMP model.
- This unidentifiability can also arise for non-Gaussian POMP models, but it is easier to see in the Gaussian case.

# The basic structural model

- The **basic structural model** was developed for econometric analysis.
- It decomposes an observable process  $Y_{1:N}$  as the sum of a **level** ( $L_n$ ), a **trend** ( $T_n$ ) describing the rate of change of the level, and a monthly **seasonal component** ( $S_n$ ).
- The model supposes that the level, trend and seasonality are perturbed with Gaussian white noise at each time point,

$$[\text{BSM1}] \quad Y_n = L_n + S_n + \epsilon_n$$

$$[\text{BSM2}] \quad L_n = L_{n-1} + T_{n-1} + \xi_n$$

$$[\text{BSM3}] \quad T_n = T_{n-1} + \zeta_n$$

$$[\text{BSM4}] \quad S_n = -\sum_{k=1}^{11} S_{n-k} + \eta_n$$

where  $\epsilon_n \sim \text{normal}[0, \sigma_\epsilon^2]$ ,  $\xi_n \sim \text{normal}[0, \sigma_\xi^2]$ ,  $\zeta_n \sim \text{normal}[0, \sigma_\zeta^2]$ ,  
and  $\eta_n \sim \text{normal}[0, \sigma_\eta^2]$ .

## Two common special cases of the basic structural model

- The **local linear trend** model is the basic structural model without the seasonal component,  $\{S_n\}$
- The **local level model** is the basic structural model without either the seasonal component,  $\{S_n\}$ , or the trend component,  $\{T_n\}$ . The local level model is therefore a random walk observed with measurement error.



# Initial values for the basic structural model

- To complete the model, we need to specify initial values.
- We have an example of the common problem of failing to specify initial values: these are not explained in the documentation of the R implementation of the basic structural model, `StructTS`. We could go through the source code to find out what it does.
- Incidentally, `?StructTS` does give some advice which resonates with our experience earlier in the course that optimization for ARMA models is often imperfect.

“Optimization of structural models is a lot harder than many of the references admit. For example, the ‘AirPassengers’ data are considered in Brockwell & Davis (1996): their solution appears to be a local maximum, but nowhere near as good a fit as that produced by ‘StructTS’. It is quite common to find fits with one or more variances zero, and this can include  $\sigma_{\text{eps}}^2$ .”

# The basic structural model is an LG-POMP model

[BSM1–4] can be put in matrix form,

$$\begin{pmatrix} L_n \\ T_n \\ S_n \\ S_{n-1} \\ S_{n-2} \\ \vdots \\ S_{n-10} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & -1 & -1 & -1 & \dots & -1 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} L_{n-1} \\ T_{n-1} \\ S_{n-1} \\ S_{n-2} \\ S_{n-3} \\ \vdots \\ S_{n-11} \end{pmatrix} + \begin{pmatrix} \xi_n \\ \zeta_n \\ \eta_n \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Now, set

$$X_n = (L_n, T_n, S_n, S_{n-1}, S_{n-2}, \dots, S_{n-10})^T, \quad (18)$$

$$Y_n = (1, 0, 1, 0, 0, \dots, 0)X_n + \epsilon_n. \quad (19)$$

We can identify matrices  $\mathbb{A}$ ,  $\mathbb{B}$ ,  $\mathbb{U}$  and  $\mathbb{V}$  giving a time-homogeneous LG-POMP representation [LG1, LG2] for the basic structural model.

# Spline smoothing and its LG-POMP representation

- Spline smoothing is a standard method to smooth scatter plots and time plots. For example, `smooth.spline` and `hpfilter` in R.
- A **smoothing spline** for an equally spaced time series  $y_{1:N}^*$  collected at times  $t_{1:N}$  is the sequence  $x_{1:N}$  minimizing the **penalized sum of squares (PSS)**, which is defined as

$$[\text{SS1}] \quad \text{PSS}(x_{1:N}; \lambda) = \sum_{n=1}^N (y_n^* - x_n)^2 + \lambda \sum_{n=3}^N (\Delta^2 x_n)^2.$$

- The spline is defined for all times, but here we are only concerned with its value at the times  $t_{1:N}$ .
- Here,  $\Delta x_n = (1 - B)x_n = x_n - x_{n-1}$ .

- The **smoothing parameter**,  $\lambda$ , penalizes  $x_{1:N}$  to prevent the spline from interpolating the data.
- If  $\lambda = 0$ , the spline will go through each data point, i.e,  $x_{1:N}$  will interpolate  $y_{1:N}^*$ .
- If  $\lambda = \infty$ , the spline will be the ordinary least squares regression fit,

$$x_n = \alpha + \beta n, \quad (20)$$

since  $\Delta^2(\alpha + \beta n) = 0$ .

- Now consider the linear Gaussian model,

$$[\text{SS2}] \quad X_n = 2X_{n-1} - X_{n-2} + \epsilon_n, \quad \epsilon_n \sim \text{iid } N[0, \sigma^2/\lambda]$$

$$[\text{SS3}] \quad Y_n = X_n + \eta_n, \quad \eta_n \sim \text{iid } N[0, \sigma^2]$$

- Note that  $\Delta^2 X_n = \epsilon_n$ .
- We will show that [SS1] is equivalent to [SS2,SS3].

## Constructing a linear Gaussian POMP (LG-POMP) model matching [SS2] and [SS3]

**Question 10.4.**  $\{X_n, Y_n\}$  defined in [SS2] and [SS3] is not quite an LG-POMP model. However, we can use  $\{X_n\}$  and  $\{Y_n\}$  to build an LG-POMP model. How?

# Deriving the penalized spline from the LG-POMP

- The joint density of  $X_{1:N}$  and  $Y_{1:N}$  in [SS2,SS3] is

$$f_{X_{1:N}Y_{1:N}}(x_{1:N}, y_{1:N}) = f_{X_{1:N}}(x_{1:N}) f_{Y_{1:N}|X_{1:N}}(y_{1:N} | x_{1:N}). \quad (21)$$

Taking logs of (21) we get

$$\log f_{X_{1:N}Y_{1:N}}(x_{1:N}, y_{1:N}) = \log f_{X_{1:N}}(x_{1:N}) + \log f_{Y_{1:N}|X_{1:N}}(y_{1:N} | x_{1:N}).$$

- [SS2,SS3] tell us that  $\{\Delta^2 X_n, n \in 1 : N\}$  and  $\{Y_n - X_n, n \in 1 : N\}$  are independent normal $[0, \sigma^2/\lambda]$  and normal $[0, \sigma^2]$ . Thus,

$$\begin{aligned} \log f_{X_{1:N}Y_{1:N}}(x_{1:N}, y_{1:N}; \sigma, \lambda) = \\ \frac{-1}{2\sigma^2} \sum_{n=1}^N (y_n - x_n)^2 + \frac{-\lambda}{2\sigma^2} \sum_{n=3}^N (\Delta^2 x_n)^2 + C. \end{aligned} \quad (22)$$

- Here,  $C$  depends on  $\sigma$  and  $\lambda$  but not on  $y_{1:N}$ .  $C$  depends on the initial terms  $x_0$  and  $x_{-1}$ , but we suppose these can be ignored, for example by modeling them with an improper uniform density.

- Comparing (22) with [SS1], we see that maximizing the density  $f_{X_{1:N}Y_{1:N}}(x_{1:N}, y_{1:N}^*; \sigma, \lambda)$  as a function of  $x_{1:N}$  is the same problem as finding the smoothing spline by minimizing the penalized sum of squares.
- For a Gaussian density, the mode (i.e., the maximum of the density) is equal to the expected value. Therefore, we have

$$\begin{aligned}
 \arg \min_{x_{1:N}} \text{PSS}(x_{1:N}; \lambda), &= \arg \max_{x_{1:N}} f_{X_{1:N}Y_{1:N}}(x_{1:N}, y_{1:N}^*; \sigma, \lambda), \\
 &= \arg \max_{x_{1:N}} \frac{f_{X_{1:N}Y_{1:N}}(x_{1:N}, y_{1:N}^*; \sigma, \lambda)}{f_{Y_{1:N}}(y_{1:N}^*; \sigma, \lambda)}, \\
 &= \arg \max_{x_{1:N}} f_{X_{1:N}|Y_{1:N}}(x_{1:N} | y_{1:N}^*; \sigma, \lambda), \\
 &= \mathbb{E}[X_{1:N} | Y_{1:N} = y_{1:N}^*; \sigma, \lambda].
 \end{aligned}$$

- Because a (conditional) normal distribution is characterized by its (conditional) mean and variance, the smoothing calculation for an LG-POMP model involves finding the conditional mean and variance of  $X_n$  given  $Y_{1:N} = y_{1:N}^*$ .
- We conclude that the smoothing problem for this LG-POMP model is the same as the spline smoothing problem defined by [SS1].
- If you have experience using smoothing splines, this connection may help you transfer that experience to POMP models.
- Once you have experience with POMP models, this connection helps you understand spline smoothers that are commonly used in many applications.
- For example, the smoothing parameter  $\lambda$  could be selected by maximum likelihood for the POMP model.



Why do we penalize by  $\sum_n (\Delta^2 X_n)^2$  when smoothing?

**Question 10.5.** We found that the smoothing spline corresponds to a particular choice of LG-POMP model given by [SS2, SS3], Why do we choose that penalty, rather than the equivalent penalty from some other LG-POMP model?

**Note:** This LG-POMP model is sometimes reasonable, but presumably there are other occasions when a different LG-POMP model would lead to superior performance.

# The Kalman filter

- The **Kalman filter** is the name given to the prediction, filtering and smoothing formulas [P4–P9] for the linear Gaussian partially observed Markov process (LG-POMP) model.
- Linear Gaussian models have Gaussian conditional distributions.
- The integrals in the general POMP formulas can be found exactly for the Gaussian distribution, leading to linear algebra calculations of conditional means and variances.
- The R function `arima()` uses a Kalman filter to evaluate the likelihood of an ARMA model (or ARIMA, SARMA, SARIMA).

# Review of the multivariate normal distribution

- A random variable  $X$  taking values in  $\mathbb{R}^{d_X}$  is **multivariate normal** with mean  $\mu_X$  and variance  $\Sigma_X$  if we can write

$$X = \mathbb{H}Z + \mu_X,$$

where  $Z$  is a vector of  $d_X$  independent identically distributed  $\text{normal}[0, 1]$  random variables and  $\mathbb{H}$  is a  $d_X \times d_X$  matrix square root of  $\Sigma_X$ , i.e.,

$$\mathbb{H}\mathbb{H}^T = \Sigma_X.$$

- A matrix square root of this type exists for any covariance matrix, though the choice of  $\mathbb{H}$  is not unique.
- We write  $X \sim \text{normal}[\mu_X, \Sigma_X]$ . If  $\Sigma_X$  is invertible,  $X$  has a probability density function,

$$f_X(x) = \frac{1}{(2\pi)^{d_X/2} |\Sigma_X|} \exp \left\{ -\frac{(x - \mu_X)^T [\Sigma_X]^{-1} (x - \mu_X)}{2} \right\}.$$

## Joint multivariate normal vectors

$X$  and  $Y$  are **joint multivariate normal** if the combined vector

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix}$$

is multivariate normal. In this case, we write

$$\mu_Z = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \quad \Sigma_Z = \begin{pmatrix} \Sigma_X & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_Y \end{pmatrix},$$

where

$$\Sigma_{XY} = \text{Cov}(X, Y) = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)^T].$$

- For joint multivariate normal random variables  $X$  and  $Y$ , we have the useful property that the conditional distribution of  $X$  given  $Y = y$  is multivariate normal, with conditional mean and variance

$$[\text{KF1}] \quad \mu_{X|Y}(y) = \mu_X + \Sigma_{XY} \Sigma_Y^{-1} (y - \mu_Y),$$

$$[\text{KF2}] \quad \Sigma_{X|Y} = \Sigma_X - \Sigma_{XY} \Sigma_Y^{-1} \Sigma_{YX}.$$

- We write this as

$$X | Y = y \sim \text{normal}[\mu_{X|Y}(y), \Sigma_{X|Y}].$$

- The joint multivariate normal has a special property that the conditional variance of  $X$  given  $Y = y$  does not depend on the value of  $y$ . In non-Gaussian situations, it will usually depend on  $y$ .
- If  $\Sigma_Y$  is not invertible, we can interpret  $\Sigma_Y^{-1}$  as a generalized inverse.

# Notation for the Kalman filter recursions

We define the conditional means and variances for the filtering, prediction and smoothing distributions:

$$[\text{KF3}] \quad X_n \mid Y_{1:n-1} = y_{1:n-1} \sim \text{normal}[\mu_n^P(y_{1:n-1}), \Sigma_n^P],$$

$$[\text{KF4}] \quad X_n \mid Y_{1:n} = y_{1:n} \sim \text{normal}[\mu_n^F(y_{1:n}), \Sigma_n^F],$$

$$[\text{KF5}] \quad X_n \mid Y_{1:N} = y_{1:N} \sim \text{normal}[\mu_n^S(y_{1:N}), \Sigma_n^S].$$

- For data  $y_{1:N}^*$ , we call  $\mu_n^P = \mu_n^P(y_{1:n-1}^*) = \mathbb{E}[X_n \mid Y_{1:n-1} = y_{1:n-1}^*]$  the **prediction mean**, and  $\Sigma_n^P$  the **prediction variance**.
- $\mu_n^F = \mu_n^F(y_{1:n}^*) = \mathbb{E}[X_n \mid Y_{1:n} = y_{1:n}^*]$  is the **filter mean** and  $\Sigma_n^F$  the **filter variance**.
- $\mu_n^S = \mu_n^S(y_{1:N}^*) = \mathbb{E}[X_n \mid Y_{1:N} = y_{1:N}^*]$  is the **smoothed mean** and  $\Sigma_n^S$  the **smoothed variance**.

# The Kalman matrix recursions

- Applying the properties of linear combinations of Normal random variables, we get the Kalman filter and prediction recursions:

$$[\text{KF6}] \quad \mu_{n+1}^P(y_{1:n}) = \mathbb{A}_{n+1} \mu_n^F(y_{1:n})$$

$$[\text{KF7}] \quad \Sigma_{n+1}^P = \mathbb{A}_{n+1} \Sigma_n^F \mathbb{A}_{n+1}^T + \mathbb{U}_{n+1},$$

$$[\text{KF8}] \quad \Sigma_n^F = ([\Sigma_n^P]^{-1} + \mathbb{B}_n^T \mathbb{V}_n^{-1} \mathbb{B}_n)^{-1},$$

$$[\text{KF9}] \quad \mu_n^F(y_{1:n}) = \mu_n^P(y_{1:n-1}) + \Sigma_n^F \mathbb{B}_n^T \mathbb{V}_n^{-1} \{y_n - \mathbb{B}_n \mu_n^P(y_{1:n-1})\}.$$

# Outline of a derivation of the Kalman matrix recursions

- The prediction recursions [KF6] and [KF7] follow from the property that if  $X$  is a  $d$ -dimensional multivariate normal,  $X \sim \text{normal}(\mu, \Sigma)$ , then  $\mathbb{A}X + b \sim \text{normal}(\mathbb{A}\mu + b, \mathbb{A}\Sigma\mathbb{A}^T)$ .
- Note that the multivariate normal identities [KF1,KF2] also hold when all variables are conditioned on some additional joint Gaussian variable, in this case  $Y_{1:n-1}$ .
- [KF8] and [KF9] can be deduced by writing out the joint density,

$$f_{X_n Y_n | Y_{1:n-1}}(x_n, y_n | y_{1:n-1}) \quad (23)$$

and completing the square in the exponent. The conditional density of  $X_n$  given  $Y_{1:n}$  is proportional to this joint density, with proportionality constant allowing integration to one.

**Exercise 10.5.** The derivation of the Kalman filter is not central to this course. However, working through the algebra to your own satisfaction is a good exercise.



- The Kalman filter matrix equations are easy to code, and quick to compute unless the dimension of the latent space is very large.
- In numerical weather forecasting, with careful programming, they are solved with latent variables having dimension  $d_X \approx 10^7$ .
- A similar computation gives backward Kalman recursions. Putting the forward and backward Kalman recursions together, as in [P9], is called **Kalman smoothing**.

## Further reading

- The approach in this chapter is aligned with King *et al.* (2016)
- Chapter 6 of Shumway and Stoffer (2017) gives an approach emphasizing linear Gaussian state space models.

# References and Acknowledgements

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