BUDT 730

Data, Models and Decisions

Lecture 18

Decision Tree (1)

Prof. Sujin Kim

Probability Theory

Properties of Probability

Given that we can express probabilities for individual events, there are several important properties that describe how some events relate to others

- Rule of Complements
- Addition Rule
- Conditional Probability and Multiplication Rule
- Law of Total Probability

Rules of Probability

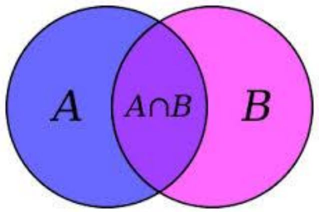
- **Rule of Complements**:

$$P(\bar{A}) = 1 - P(A)$$

- **Addition Rule**:

For any two events *A* and *B*,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$



If *A* and *B* are mutually exclusive,

$$P(A \cup B) = P(A) + P(B).$$

Example

- The following three events involving a company's annual revenue for the coming year:
  - A = Revenue is less than $1 million
  - B = Revenue is at least $1 million but less than $2 million, and
  - C = Revenue is at least $2 million
- Note that the events A, B and C are mutually exclusive and collectively exhaustive.
- Suppose that

$$P(A) = 0.5, P(B) = 0.3\ and\ P(C) = 0.2$$

- Find the following probabilities:
  - $P(\text{revenue is at least \$1 million}) = ?$
  - $P(\text{revenue is less than \$2 million}) = ?$

  - $P(\text{revenue is at least \$1 million}) = P(B \cup C) = P(B) + P(C) = 0.5$
  - $P(\text{revenue is less than \$2 million}) = P(A \cup B) = P(A) + P(B) = 0.8$

Conditional Probability

- Probabilities are always assessed relative to the information currently available
- A formal way to revise probabilities based on new information is to use conditional probabilities.
- Let A and B be any events with probabilities $P(A)$ and $P(\mathrm{B})$. If you are told that B has occurred, then the probability of A might change.
  - The new probability of A is called the **conditional probability** of A given B, or $P(A|B)$.

Example: Assessing Uncertainty at the Bender Company

- Bender Company supplies contractors with materials for the construction of houses
- Bender has a contract to fill an order by the **end of July**, but there is uncertainty as to whether it will receive the materials by the **middle of July**.
- Right now it is **July 1**.
- How can the uncertainty in the situation be assessed?
  - What is the probability that Bender will meet the deadline?

  -> Unconditional probability

  - What is the probability that Bender meets the deadline, given the information the company has at the beginning of July?

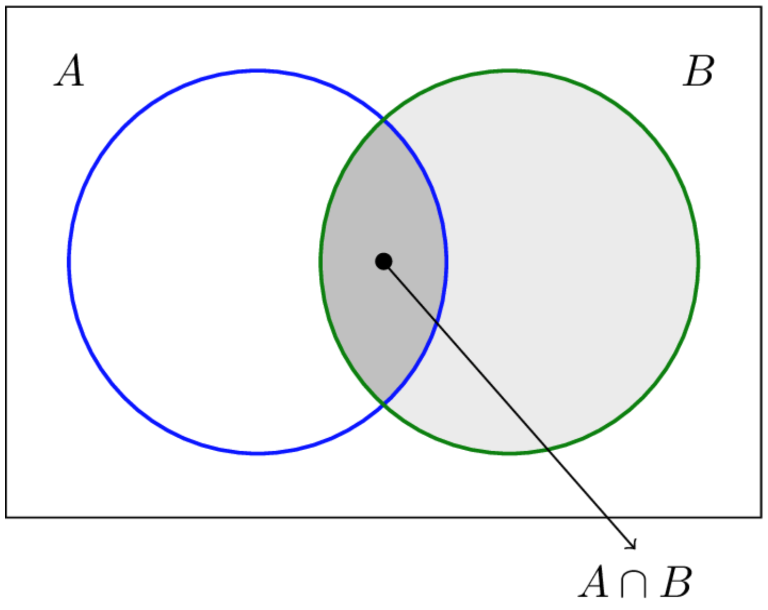
  -> Conditional probability

Conditional Probability

- A conditional probability can be calculated with the following formula:

$$\boxed{P(A \mid B) = \frac{P(A \cap B)}{P(B)}}$$

- The *given* event *B* is assumed to have occurred.
- If P(*B*) = 0, then P(*A* | *B*) is undefined.



$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Multiplication Rule

marginal probability

- $P(A \cap B)$ must be known to find $P(A|B)$.
- However, in some applications, $P(A|B)$ and $\boxed{P(B)}$ (or P(A)) are known. Then you can multiply both sides of the equation by $P(B)$ to obtain the **multiplication rule** for $P(A \cap B)$:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow \boxed{P(A \cap B) \;=\; P(A|B)P(B) \;=\; P(B|A)P(A)}$$

Example:
Assessing Uncertainty at the Bender Company

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- Bender has a contract to fill an order by the **end of July**, but there is uncertainty as to whether it will receive the materials by the **middle of July**.
- Right now it is **July 1**.
- How can the uncertainty in the situation be assessed?

Objective: To apply probability rules to calculate the probability that Bender will meet its end-of -July deadline, given the information the company has at the beginning of July.

Example:
Assessing Uncertainty at the Bender Company

- Two Successive Experiments
  - First: Does Bender receive its materials by mid-July?
  - Second: Does Bender meet its deadline?
- Let A = the event that Bender meets its deadline and
  B = the event that Bender receives the materials by mid-July
- The probabilities that Bender is able to assess on July 1 are probably $P(B)$ and $P(A|B)$
  - Bender estimates that the chances of getting the materials on time are 2 out of 3
  - Bender estimates that if it receives the required materials on time, the chances of meeting the deadline are 3 out of 4
  - Bender estimates that if it does not receive the required materials on time, the chances of meeting the deadline are 1 out of 5

Q: Write the above estimated probabilities in terms of A and B

Example:
Assessing Uncertainty at the Bender Company

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$$P(B) = \frac{2}{3}, P(A|B) = \frac{3}{4}, P(A|\bar{B}) = \frac{1}{5}$$

Example:
Assessing Uncertainty at the Bender Company

True or False:

$$P(A|B) = 1 - P(A|\bar{B})$$

$$P(A|B) = 1 - P(\bar{A}|B)$$

Example:
Assessing Uncertainty at the Bender Company

- Given this information, what is the probability that the materials are received by mid-July **AND** meets its end-of-July deadline?

- What about the probability of NOT meeting the deadline if the materials are not received by mid-July?

- What about the probability of meeting the deadline ?

Example:
Assessing Uncertainty at the Bender Company

- Given this information, what is the probability that the materials are received by mid-July **AND** meets its end-of-July deadline?

$$P(A \cap B) = P(A|B)P(B) = \frac{3}{4} * \frac{2}{3} = \frac{1}{2}$$

- What about the probability of NOT meeting the deadline if the materials are not received by mid-July?

$$P(\bar{A}|\bar{B}) = 1 - P(A|\bar{B}) = 1 - \frac{1}{5} = \frac{4}{5}$$

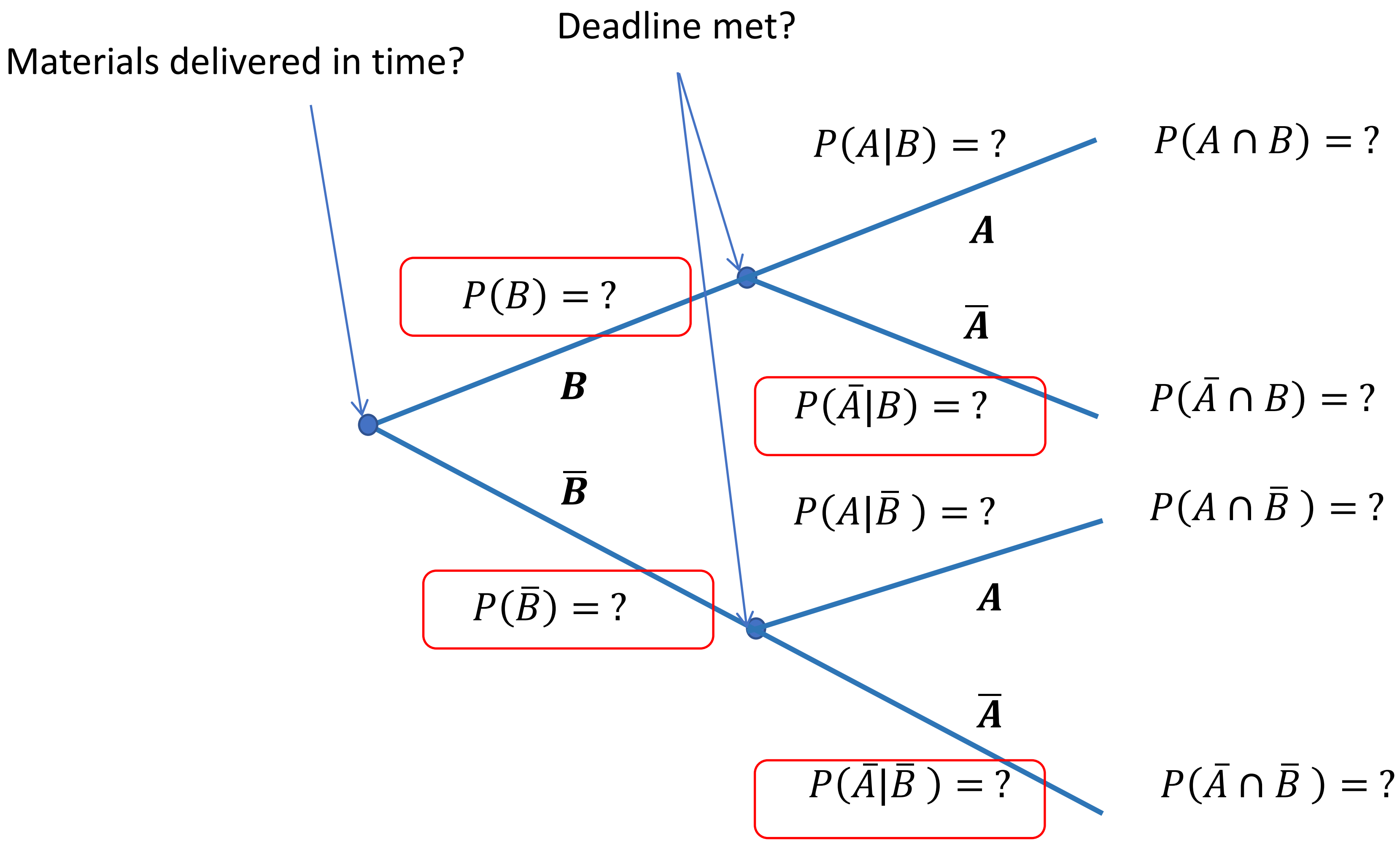
- What about the probability of meeting the deadline ?

$$P(A)$$

Probability Tree

The uncertain situation is depicted graphically in the form of a **probability tree**.

Compute the probabilities in the probability tree.



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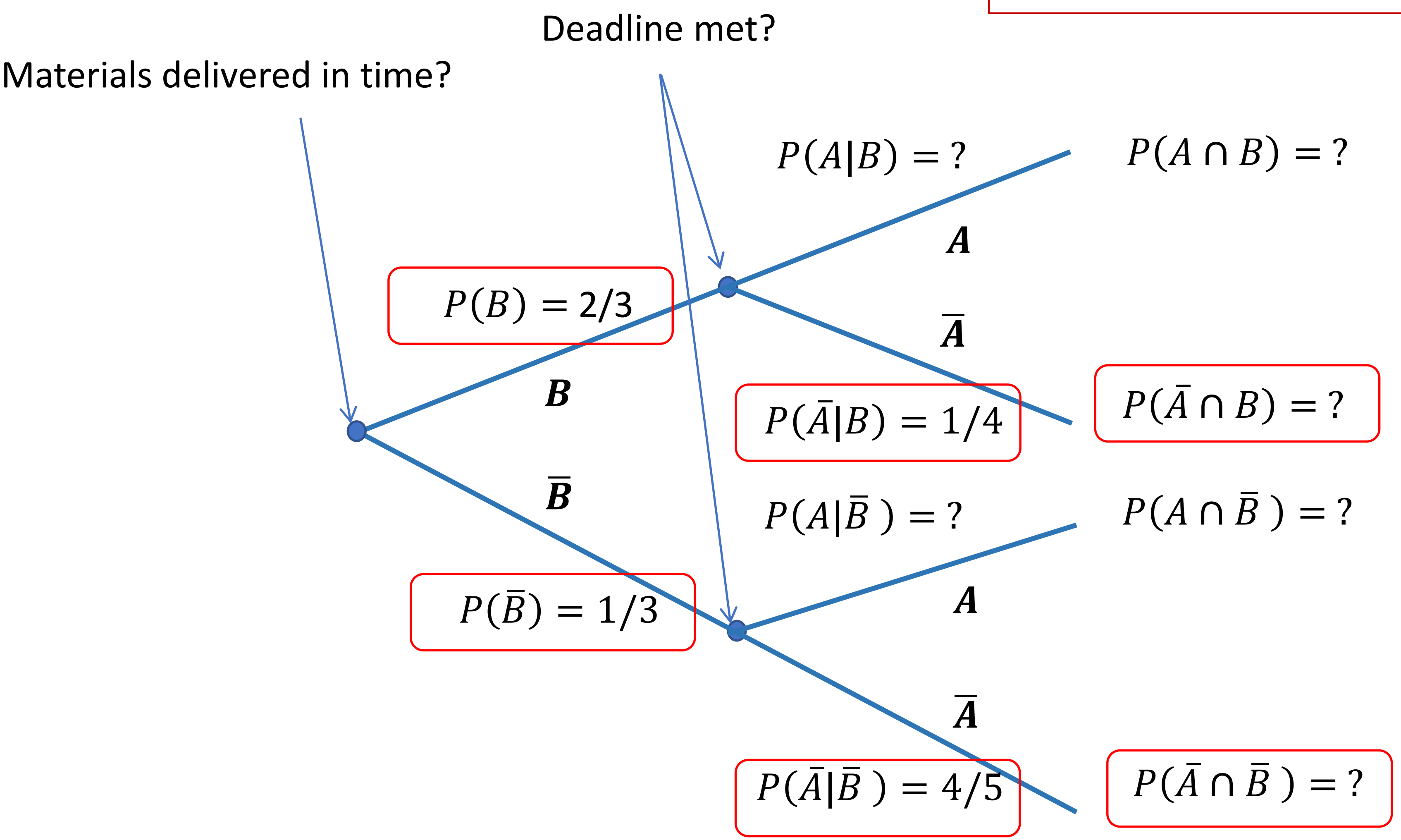

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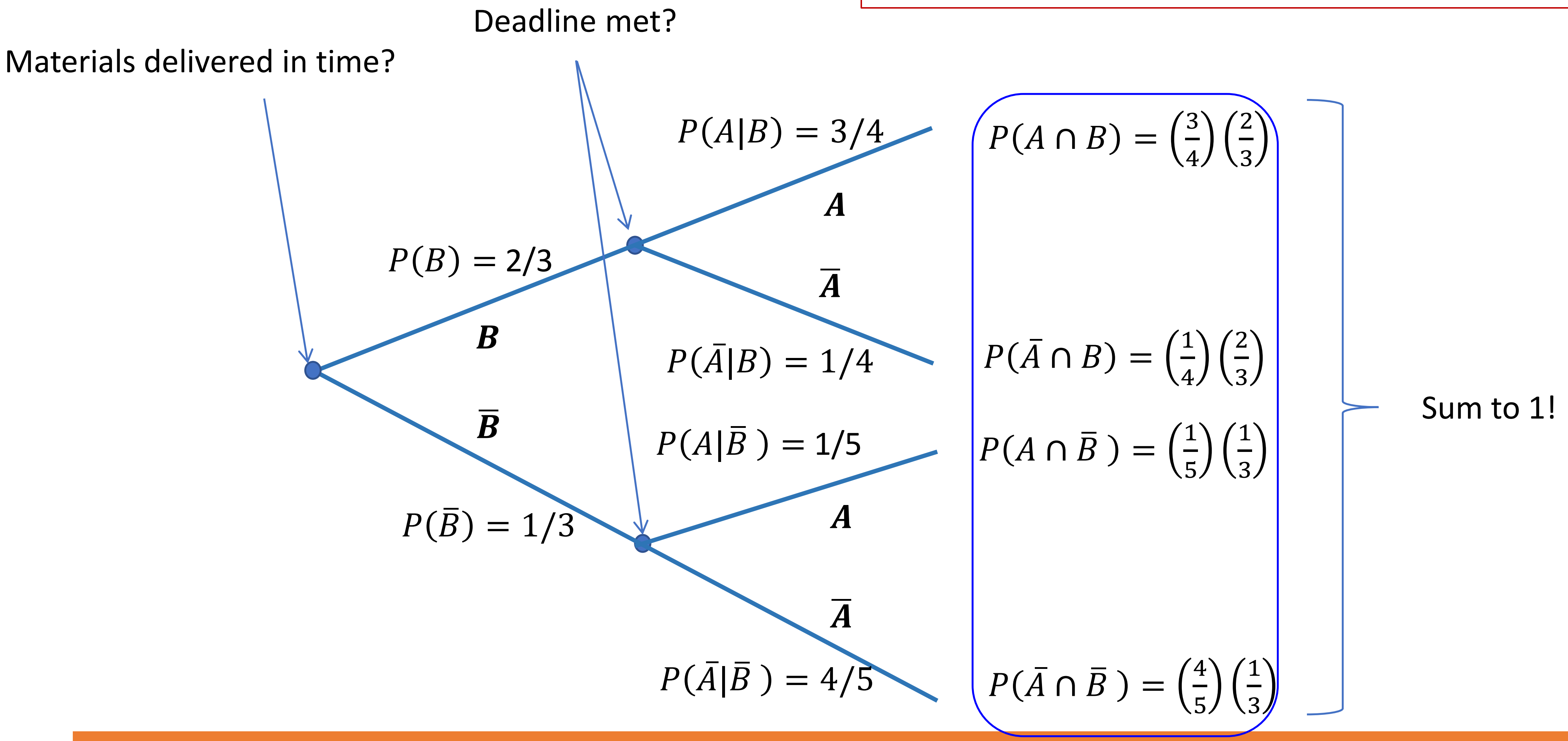


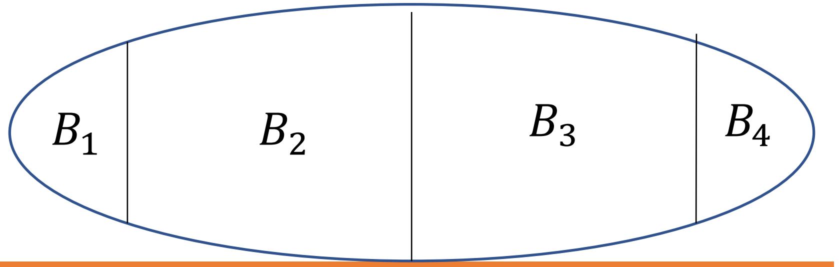
Law of Total Probability

- How about P(A)?
- Determining the (marginal) probability of an event is more complex when conditional probabilities are involved
- The **Law of Total Probability** can help us calculate the dependent event probability:

$$P(A) = P(A \cap B) + P(A \cap \bar{B})$$
$$= \ P(A|B) \ * \ P(B) \ + \ P(A|\bar{B}) \ * \ P(\bar{B})$$

  - In the previous example, the probability of meeting the deadline is _____?_____

  - More generally, for any event *A*,
    $P(A) = \sum_{i=1}^{n} P(A \cap B_i) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$



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  - In the previous example, the probability of meeting the deadline is
    $P(A) = \frac{3}{4} * \frac{2}{3} + \frac{1}{5} * \frac{1}{3} = \frac{1}{2} + \frac{1}{15} = \frac{17}{30} = 0.5667$

  - More generally, for any event *A*,
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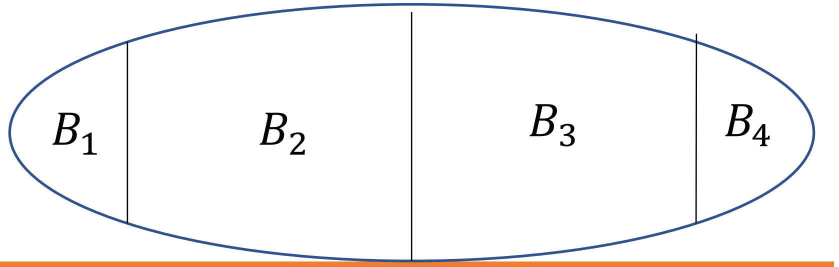


Decision Making under Uncertainty

Learning Objectives

- Decision Making under Uncertainty
  - Learn the concept of expected monetary value (EMV).
  - Understand the concept of risk preference/attitude
- Decision Tree
  - Understand how to construct a single stage decision tree
  - Learn how probabilities are used in the decision-making process

Decision Making under Uncertainty

- Given our coverage of probability, we would like to develop a formal framework for **analyzing decision problems involving uncertainty**
- Our discussion includes:
  - What criteria are used for choosing among alternative decisions
  - How probabilities are used in the decision-making process
  - How early decisions affect decisions made at a later stage
- A powerful graphical tool—a **decision tree**—guides the analysis.
  - A decision tree enables a decision maker to view all important aspects of the problem at once.

Simple Optimization Problem

- An entrepreneur is deciding between two locations to open a new restaurant
- She needs help with choosing between locations A and B
- The net present value cost of opening a store at location A is $1M, while at B is $2M
- By setting up a store at A she will have access to an average flow of 2000 customers each of whose lifetime value is $600
- On the other hand, the more popular location B will have access to an average flow of 3000 customers each of whose lifetime value is $700
- <u>Where should she locate the store?</u>

Simple Optimization Problem: Solution

Compute the long run profit

- Location A: 2000*600 – 1,000,000 = 200,000
- Location B: 3000*700 – 2,000,000 = 100,000

Decision Making under Uncertainty

- In the previous example, we made (optimal) decisions completely ignoring uncertainty
  - We have assumed that we precisely knew cost, demands, etc., or used expected values as input
  - This is a strong assumption!!
  - What if demand fluctuates, what if costs vary?
  - Specifically, there may be outcomes that are not under our control, for example the economy, the weather, our competitors' actions.
- Next:
  - We'll take into account the uncertainty in a decision-making process
  - How? Using the concept of probability and random variables!

Expected Monetary Value

The **expected monetary value** (EMV) for any decision is an average of the possible payoffs (or costs(-)), weighted by the probabilities of the outcomes:

$$\boxed{EMV = \sum_{i=1}^{k} v_i p_i}$$

p_i =probability of outcome i

v_i = payoff with outcome i

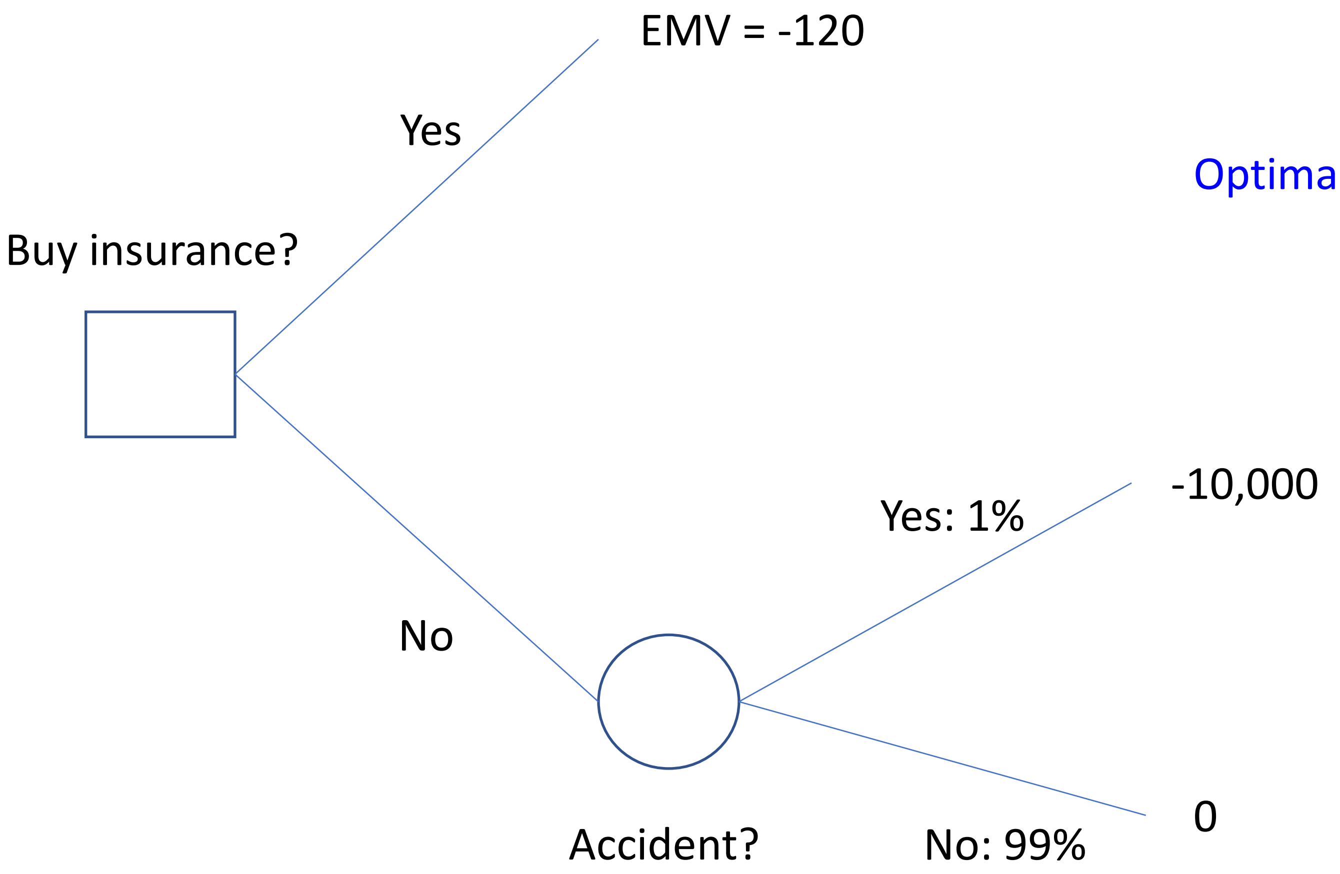
- Equivalent to the **expected value of a discrete random variable**, for which the values are the possible payoffs of the decision
- Represents **the long-run average**, as if you are faced with the decision many time

Example: Ann's Auto Insurance (Part A)

- Ann's wealth is approximately $50,000.
- Ann has a 1% chance of being in an automobile accident during the year that will cost $10,000
- She is offered an insurance policy for $120 (no deductible)
- **If Ann is an EMV maximizer, will she purchase the policy?**

- This is a single stage (one-stage) decision problem: one stage decision is made, right now.

Example: Ann's Auto Insurance (Part A)



Optimal Decision: Do not buy the insurance

EMV
= 1%*10,000 + 99%*0 = -100

In-Class Exercise

New Product Decision at ACME (Example 6-1 in Text)

New Product Decision at ACME

- ACME’s cost accountants estimate the monetary inputs: the fixed costs ($6,000) and the unit margin ($18).
- The uncertain sales volume is really a continuous variable but, as in many decision problems, Acme has replaced the continuum by three representative possibilities: great (45%), fair (35%) and awful (20%)
- The company estimates that the corresponding sales volumes (in thousands of units sold) are 600, 300, and 90, respectively.
- Each sales volume is multiplied by the unit margin to obtain the net revenues.

New Product Decision at ACME

Question:

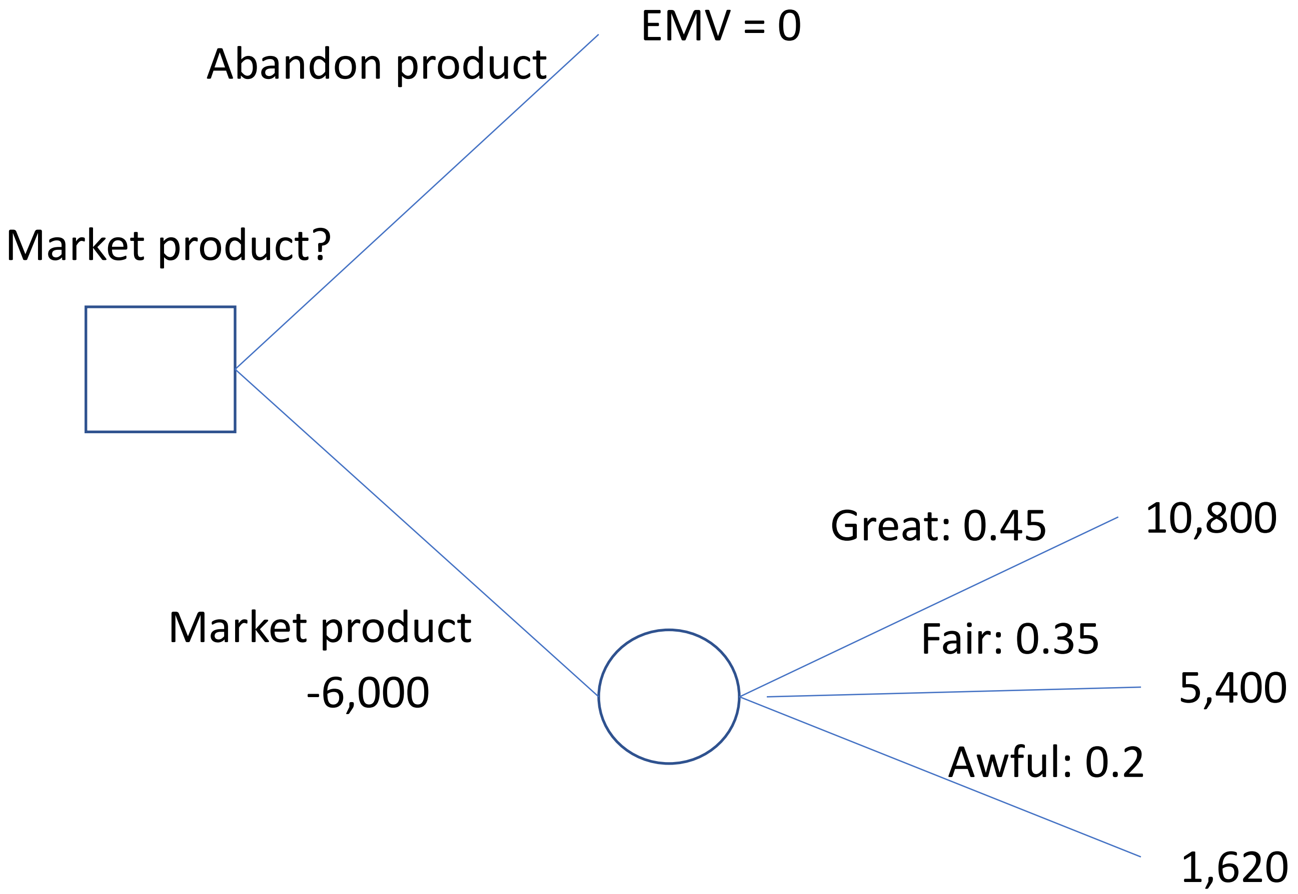
Assuming that ACME is an EMV maximizer, should it finish development and then market the product, or should it stop development at this point and abandon the product?

New Product Decision at ACME

Question: Assuming that ACME is an EMV maximizer, should it finish development and then market the product, or should it stop development at this point and abandon the product?

Inputs			
Fixed cost	$6,000		
Unit margin	$18		
Market	Probability	Sales volume	Net revenue
Great	0.45	600	$10,800
Fair	0.35	300	$5,400
Awful	0.20	90	$1,620

Example: New Product Decision at ACME



Optimal Decision: Market product

EMV

$= -6{,}000 + [\,0.45 * 10{,}800 + 0.35 * 5{,}400 + 0.2 * 1{,}620]$

$= 1{,}074$

Next …

- Ch6 Decision Making Under Uncertainty
  - Risk Analysis
  - Decision Tree
  - Precision Tree
  - Sensitivity Analysis
  - Multi-stage problem