Reinforcement Learning
Lecture 2b:

Value Iteration

[SutBar] Sec. 4.1, 4.1, [Sze] Sec. 2.2, 2.3,
[Put] Sec. 6.1-6.3, [SigBuf] Chap. 1

Outline

- Convergence properties of
  - Policy evaluation
  - Value iteration

Value Iteration Algorithm

valueIteration(MDP)

$V_0^*(s) \leftarrow \max_a R(s,a) \;\; \forall s$

For $t = 1$ to h do

$V_t^*(s) \leftarrow \max_a R(s,a) + \gamma \sum_{s'} \Pr(s'|s,a)\, V_{t-1}^*(s') \; \forall s$

Return V^*

Optimal policy π^*

$t = 0$: $\pi_0^*(s) \leftarrow \operatorname{argmax}_a R(s,a) \;\; \forall s$

$t > 0$: $\pi_t^*(s) \leftarrow \operatorname{argmax}_a R(s,a) + \gamma \sum_{s'} \Pr(s'|s,a) V_{t-1}^*(s') \;\; \forall s$

NB: t indicates the # of time steps to go (till end of process)

π^* is non stationary (i.e., time dependent)

Value Iteration

- Matrix form:

  R^a: $|S| \times 1$ column vector of rewards for a

  V_t^*: $|S| \times 1$ column vector of state values

  T^a: $|S| \times |S|$ matrix of transition prob. for a

valueIteration(MDP)

$V_0^* \leftarrow \max_a R^a$

For $t = 1$ to h do

$V_t^* \leftarrow \max_a R^a + \gamma T^a V_{t-1}^*$

Return V^*

Example: MDP |A|=|S|=2

2 states, 2-action MDP

$$T^{a_1} = \begin{array}{c c} & \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{array}{c} s_1 \\ s_2 \end{array} & \begin{pmatrix} 0.3 & 0.7 \\ 0.8 & 0.2 \end{pmatrix} \end{array}$$

$$T^{a_2} = \begin{array}{c c} & \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{array}{c} s_1 \\ s_2 \end{array} & \begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix} \end{array}$$

$$R^{a_1} = \begin{array}{c c} \begin{array}{c} s_1 \\ s_2 \end{array} & \begin{pmatrix} 0 \\ 10 \end{pmatrix} \end{array} \qquad R^{a_2} = \begin{array}{c c} \begin{array}{c} s_1 \\ s_2 \end{array} & \begin{pmatrix} -5 \\ +5 \end{pmatrix} \end{array}$$

$$R^{a_1} + \gamma T^{a_1} V^*$$

$$\max = \left\{ \begin{pmatrix} 0 \\ 10 \end{pmatrix} + 0.9 * \begin{pmatrix} 0.3 & 0.7 \\ 0.8 & 0.2 \end{pmatrix} \begin{pmatrix} V^*(s_1) \\ V^*(s_2) \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} -5 \\ +5 \end{pmatrix} + 0.9 * \begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix} \begin{pmatrix} V^*(s_1) \\ V^*(s_2) \end{pmatrix} \right\}$$

<u>element - max</u>.

Infinite Horizon

- Let $h \to \infty$
- Then $V_h^\pi \to V_\infty^\pi$ and $V_{h-1}^\pi \to V_\infty^\pi$

- **Policy evaluation:**

$$V_\infty^\pi(s) = R\big(s, \pi_\infty(s)\big) + \gamma \sum_{s'} \Pr(s'|s, \pi_\infty(s))\, V_\infty^\pi(s') \quad \forall s$$

- **Bellman's equation:**

$$V_\infty^\pi(s) = \max_a R(s, a) + \gamma \sum_{s'} \Pr(s'|s, a)\, V_\infty^\pi(s')$$

Policy evaluation

- Linear system of equations

$$V_\infty^\pi(s) = R\big(s, \pi_\infty(s)\big) + \gamma \sum_{s'} \Pr(s' | s, \pi_\infty(s))\, V_\infty^\pi(s') \; \forall s$$

- Matrix form:

  R: $|S| \times 1$ column vector of sate rewards for π

  V: $|S| \times 1$ column vector of state values for π

  T: $|S| \times |S|$ matrix of transition prob for π

$$V = R + \gamma T V$$

$\pi(s_1) = a_1$, $\pi(s_2) = a_2$ $\longrightarrow$ given fixed policy

$$R^\pi = \begin{pmatrix} 0 \\ 5 \end{pmatrix} \qquad T^\pi = \begin{array}{c} s_1 \\ s_2 \end{array} \begin{pmatrix} 0.3 & 0.7 \\ 0.2 & 0.8 \end{pmatrix} \begin{array}{l} \leftarrow a_1 \\ \rightarrow a_2 \end{array}$$

Solving linear equations

- Linear system: $V = R + \gamma TV$

- Gaussian elimination: $(I - \gamma T)V = R$
- Compute inverse: $V = (I - \gamma T)^{-1}R$
- Iterative methods
    - Value iteration (a.k.a. Richardson iteration)
    - Repeat $V \leftarrow R + \gamma TV$

Contraction

- Let $H(V) \stackrel{\text{def}}{=} R + \gamma T V$ be the policy eval operator
- **Lemma 1**: H is a contraction mapping.

$$\left|\left|H(\tilde{V}) - H(V)\right|\right|_\infty \leq \gamma \left|\left|\tilde{V} - V\right|\right|_\infty$$

- Proof $\left|\left|H(\tilde{V}) - H(V)\right|\right|_\infty$

$$= \left|\left|R + \gamma T \tilde{V} - R - \gamma T V\right|\right|_\infty \quad \text{(by definition)}$$

$$= \left|\left|\gamma T(\tilde{V} - V)\right|\right|_\infty \quad \text{(simplification)}$$

$$= \gamma \left|\left|T\right|\right|_\infty \left|\left|\tilde{V} - V\right|\right|_\infty \quad \text{(since } \left|\left|AB\right|\right| \leq \left|\left|A\right|\right| \left|\left|B\right|\right|\text{)}$$

$$= \gamma \left|\left|\tilde{V} - V\right|\right|_\infty \quad \text{(since } \max_s \textstyle\sum_{s'} T(s, s') = 1\text{)}$$

Convergence

- **Theorem 2**: Policy evaluation converges to V^{π} for any initial estimate V

$$\lim_{n \to \infty} H^{(n)}(V) = V^{\pi} \quad \forall V$$

- Proof
  - By definition $V^{\pi} = H^{(\infty)}(0)$, but policy evaluation computes $H^{(\infty)}(V)$ for any initial V
  - By Lemma 1, $\left|\left|H^{(n)}(V) - H^{(n)}(\tilde{V})\right|\right|_{\infty} \leq \gamma^{n} \left|\left|V - \tilde{V}\right|\right|_{\infty}$
  - Hence, when $n \to \infty$, then $\left|\left|H^{(n)}(\tilde{V}) - H^{(n)}(V)\right|\right|_{\infty} \to 0$ and $H^{(\infty)}(V) = V^{\pi} \quad \forall V$

WHAT IS THE DIFFERENCE BETWEEN A THEOREM, A LEMMA, AND A COROLLARY?

PROF. DAVE RICHESON

(1) **Definition**—a precise and unambiguous description of the meaning of a mathematical term. It characterizes the meaning of a word by giving all the properties and only those properties that must be true.
(2) **Theorem**—a mathematical statement that is proved using rigorous mathematical reasoning. In a mathematical paper, the term theorem is often reserved for the most important results.
(3) **Lemma**—a minor result whose sole purpose is to help in proving a theorem. It is a stepping stone on the path to proving a theorem. Very occasionally lemmas can take on a life of their own (Zorn's lemma, Urysohn's lemma, Burnside's lemma, Sperner's lemma).
(4) **Corollary**—a result in which the (usually short) proof relies heavily on a given theorem (we often say that "this is a corollary of Theorem A").
(5) **Proposition**—a proved and often interesting result, but generally less important than a theorem.
(6) **Conjecture**—a statement that is unproved, but is believed to be true (Collatz conjecture, Goldbach conjecture, twin prime conjecture).
(7) **Claim**—an assertion that is then proved. It is often used like an informal lemma.
(8) **Axiom/Postulate**—a statement that is assumed to be true without proof. These are the basic building blocks from which all theorems are proved (Euclid's five postulates, Zermelo-Frankel axioms, Peano axioms).
(9) **Identity**—a mathematical expression giving the equality of two (often variable)

Approximate Policy Evaluation

- In practice, we can't perform an infinite number of iterations.

- Suppose that we perform value iteration for n steps and $\left|\left|H^{(n)}(V) - H^{(n-1)}(V)\right|\right|_\infty = \epsilon$, how far is $H^{(n)}(V)$ from V^π?

Approximate Policy Evaluation

- **Theorem 3**: If $\left|\left|H^{(n)}(V) - H^{(n-1)}(V)\right|\right|_\infty \le \epsilon$ then

$$\left|\left|V^\pi - H^{(n)}(V)\right|\right|_\infty \le \frac{\epsilon}{1-\gamma}$$

- Proof $\left|\left|V^\pi - H^{(n)}(V)\right|\right|_\infty$

$$= \left|\left|H^{(\infty)}(V) - H^{(n)}(V)\right|\right|_\infty \quad \text{(by Theorem 2)}$$

$$= \left|\left|\sum_{t=1}^{\infty} H^{(t+n)}(V) - H^{(t+n-1)}(V)\right|\right|_\infty$$

$$\le \sum_{t=1}^{\infty} \left|\left|H^{(t+n)}(V) - H^{(t+n-1)}(V)\right|\right|_\infty \quad \left(\left|\left|A+B\right|\right| \le \left|\left|A\right|\right| + \left|\left|B\right|\right|\right)$$

$$= \sum_{t=1}^{\infty} \gamma^t \epsilon = \frac{\epsilon}{1-\gamma} \quad \text{(by Lemma 1)}$$

Optimal Value Function

- Non-linear system of equations

$$V_\infty^*(s) = \max_a R(s,a) + \gamma \sum_{s'} \Pr(s'|\mathrm{s},a)\, V_\infty^*(s') \;\forall \mathrm{s}$$

- Matrix form:

  R^a: $|S|\times 1$ column vector of rewards for a

  V^*: $|S|\times 1$ column vector of optimal values

  T^a: $|S|\times|S|$ matrix of transition prob for a

$$V^* = \max_a R^a + \gamma T^a V^*$$

Contraction

- Let $H^*(V) \stackrel{\text{def}}{=} \max_a R^a + \gamma T^a V$ be the operator in value iteration
- **Lemma 4**: H^* is a contraction mapping.

$$\left|\left|H^*(\tilde{V}) - H^*(V)\right|\right|_\infty \leq \gamma \left|\left|\tilde{V} - V\right|\right|_\infty$$

- Proof: without loss of generality,

  let $H^*\left(\tilde{V}\right)(s) \geq H^*(V)(s)$ and

  let $a_s^* = \underset{a}{\operatorname{argmax}}\, R(s,a) + \gamma \sum_{s'} \Pr(s'|s,a)\, V(s')$

  $\tilde{a}_s^* = \underset{a}{\operatorname{argmax}}\, R(s,a) + \gamma \sum_{s'} \Pr(s'|s,a)\, \tilde{V}(s')$

Contraction

- Proof continued:

- Then $0 \leq H(\tilde{V})(s) - H^*(V)(s)$ (by assumption)

$= R(s, \tilde{a}_s^*) + \gamma \sum_{s'} \Pr(s'|s, \tilde{a}_s^*) \tilde{V}(s')$ (by definition)

$\quad -R(s, a_s^*) + \gamma \sum_{s'} \Pr(s'|s, a_s^*) V(s')$

$\leq R(s, \tilde{a}_s^*) + \gamma \sum_{s'} \Pr(s'|s, \tilde{a}_s^*) \tilde{V}(s')$ (since $\tilde{a}_s^*$ suboptimal for V)

$\quad -R(s, \tilde{a}_s^*) + \gamma \sum_{s'} \Pr(s'|s, \tilde{a}_s^*) V(s')$

$= \gamma \sum_{s'} \Pr(s'|s, \tilde{a}_s^*) [\tilde{V}(s') - V(s')]$

$\leq \gamma \sum_{s'} \Pr(s'|s, \tilde{a}_s^*) \left|\left|\tilde{V} - V\right|\right|_\infty$ (maxnorm upper bound)

$= \gamma \left|\left|\tilde{V} - V\right|\right|_\infty$ (since $\sum_{s'} Pr\,(s'|s, \tilde{a}_s^*) = 1$)

- Repeat the same argument for $H^*(V)(s) \geq H^*(\tilde{V})(s)$ and for each s

Convergence

- **Theorem 5**: Value iteration converges to V^* for any initial estimate V

$$\lim_{n\to\infty} H^{*(n)}(V) = V^* \quad \forall V$$

- Proof
  - By definition $V^* = H^{*(\infty)}(0)$, but value iteration computes $H^{*(\infty)}(V)$ for some initial V
  - By Lemma 4, $\left||H^{*(n)}(V) - H^{*(n)}(\tilde{V})|\right|_\infty \le \gamma^n \left||V - \tilde{V}|\right|_\infty$
  - Hence, when $n \to \infty$, then $\left||H^{*(n)}(V) - H^{*(n)}(0)|\right|_\infty \to 0$ and $H^{*(\infty)}(V) = V^* \;\; \forall V$

Value Iteration

- Even when horizon is infinite, perform finitely many iterations
- Stop when $||V_n - V_{n-1}|| \leq \epsilon$

valueIteration(MDP)

$V_0^* \leftarrow \max_a R^a; \quad n \leftarrow 0$

Repeat

$n \leftarrow n + 1$

$V_n \leftarrow \max_a R^a + \gamma T^a V_{n-1}$

Until $||V_n - V_{n-1}||_\infty \leq \epsilon$

Return V_n

Induced Policy

- Since $||V_n - V_{n-1}||_\infty \leq \epsilon$, by Theorem 5: we know that $||V_n - V^*||_\infty \leq \frac{\epsilon}{1-\gamma}$

- But, how good is the stationary policy $\pi_n(s)$ extracted based on V_n?

$$\pi_n(s) = \underset{a}{\operatorname{argmax}}\, R(s,a) + \gamma \sum_{s'} \Pr(s'|s,a) V_n(s')$$

- How far is V^{π_n} from V^*?

Induced Policy

- **Theorem 6**: $\left|\left|V^{\pi_n} - V^*\right|\right|_\infty \leq \frac{2\epsilon}{1-\gamma}$
- Proof

$$\left|\left|V^{\pi_n} - V^*\right|\right|_\infty = \left|\left|V^{\pi_n} - V_n + V_n - V^*\right|\right|_\infty$$

$$\leq \left|\left|V^{\pi_n} - V_n\right|\right|_\infty + \left|\left|V_n - V^*\right|\right|_\infty \quad \left(\left|\left|A+B\right|\right| \leq \left|\left|A\right|\right| + \left|\left|B\right|\right|\right)$$

$$= \left|\left|H^{\pi_n(\infty)}(V_n) - V_n\right|\right|_\infty + \left|\left|V_n - H^{*(\infty)}(V_n)\right|\right|_\infty$$

$$\leq \frac{\epsilon}{1-\gamma} + \frac{\epsilon}{1-\gamma} \quad \text{(by Theorems 3 and 5)}$$

$$= \frac{2\epsilon}{1-\gamma}$$

Summary

- Value iteration
  - Simple dynamic programming algorithm
  - Complexity: $O(n|A||S|^2)$ (see P16)
    - Here n is the number of iterations

- Can we optimize the policy directly instead of optimizing the value function and then inducing a policy?
  - Yes: by policy iteration