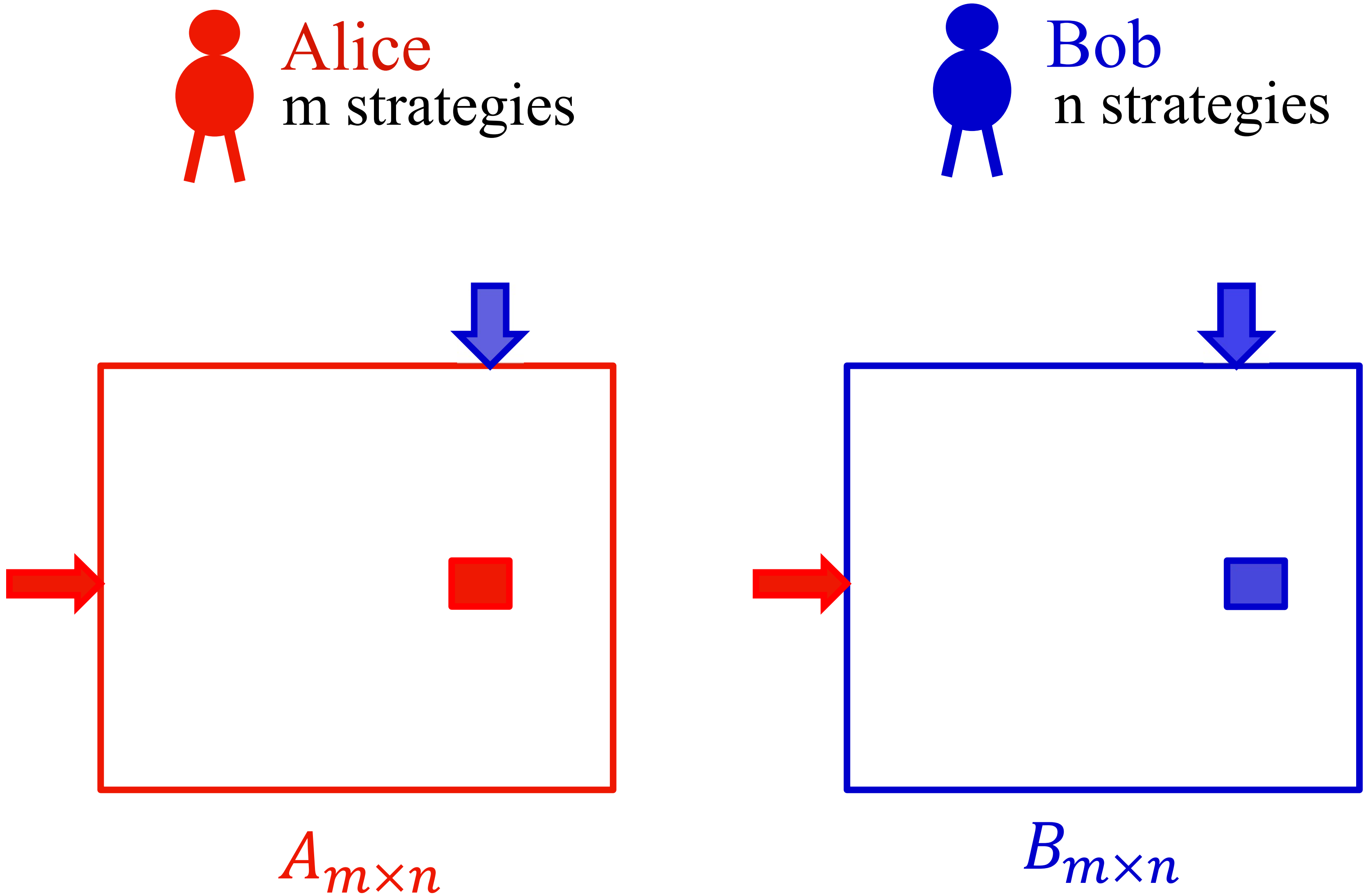
Two-Player Game & Nash Equilibrium

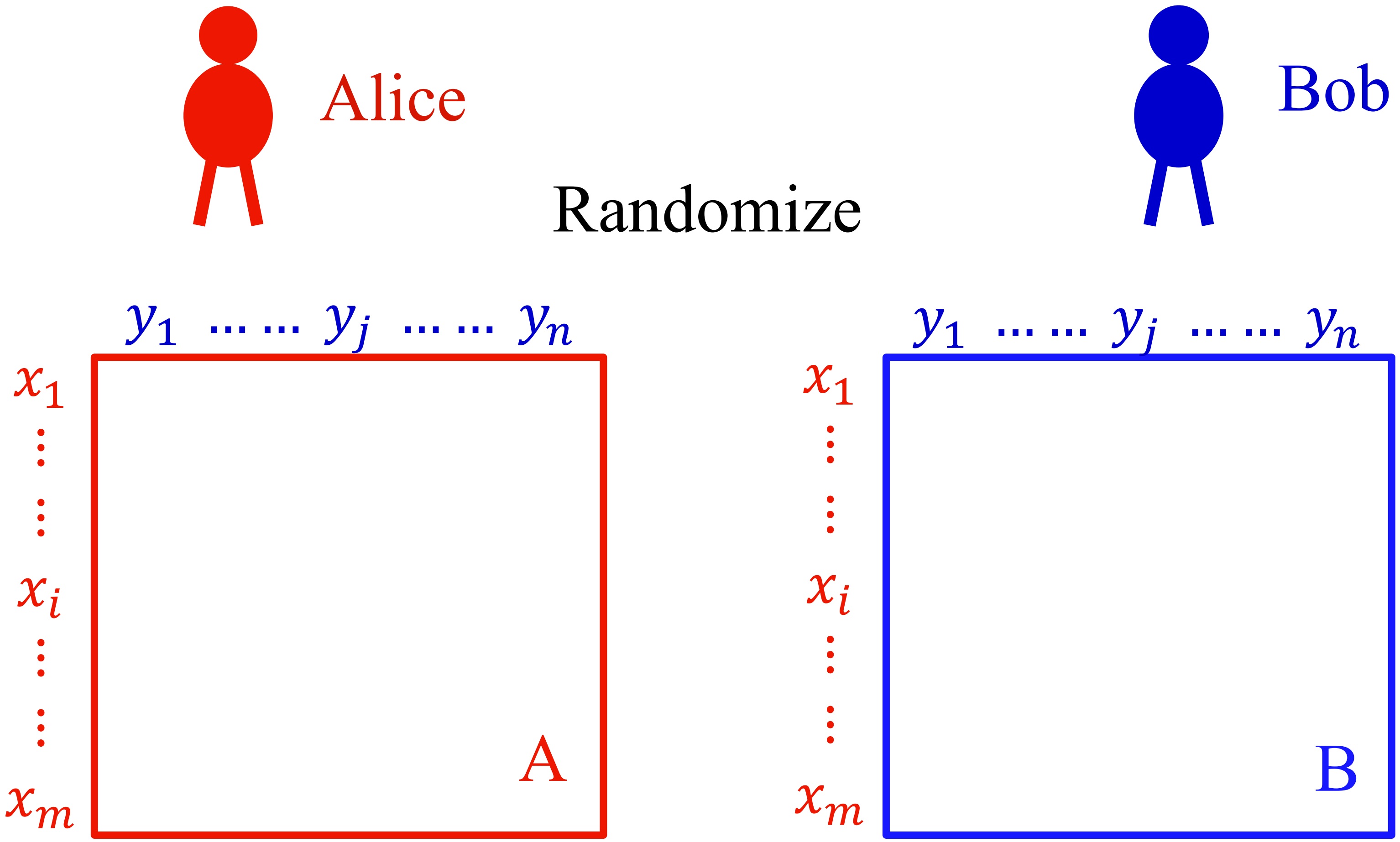
Zhengyang Liu, BIT

Nov 3, 2022

	S	P	R
S	0,0	1,-1	-1,1
P	-1,1	0,0	1,-1
R	1,-1	-1,1	0,0

Our focus: Two-player games





$x_1, \ldots, x_m \geq 0$

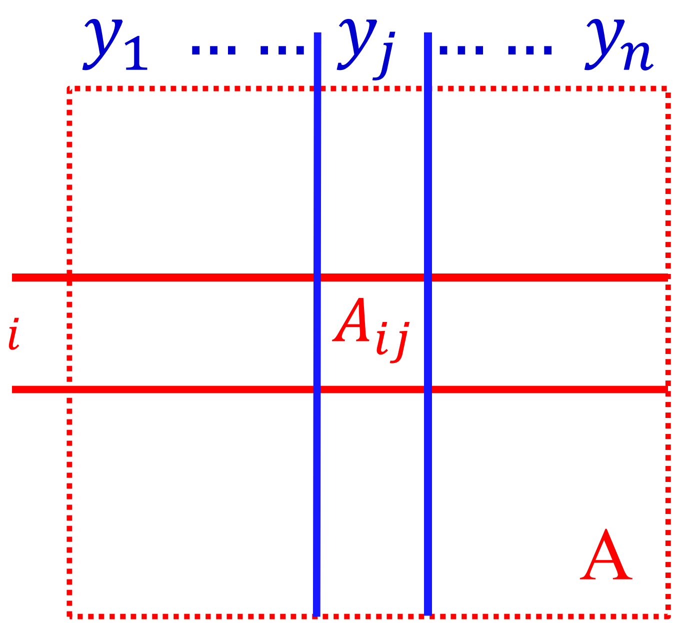
$\sum_{i=1}^{m} x_i = 1$

$y_1, \ldots, y_m \geq 0$

$\sum_{j=1}^{n} y_j = 1$

2-Nash Characterization

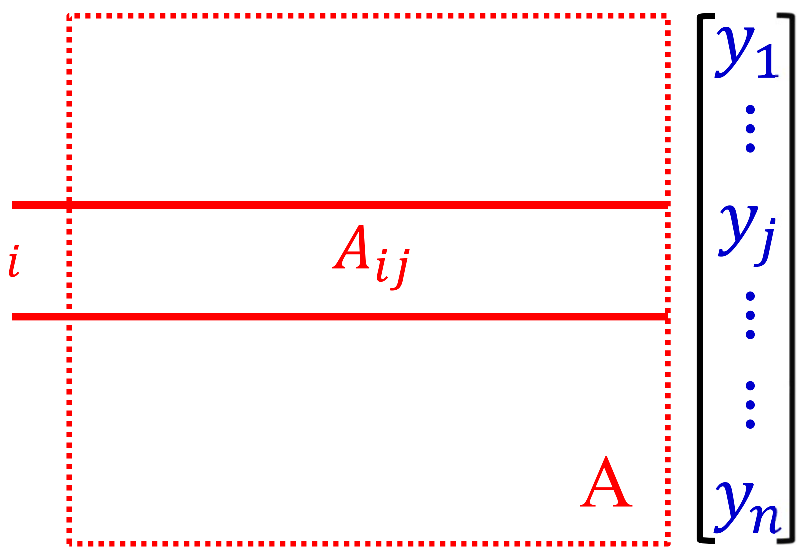
- For Alice, i^{th} strategy gives



$$\longrightarrow \sum_j A_{ij} y_j$$

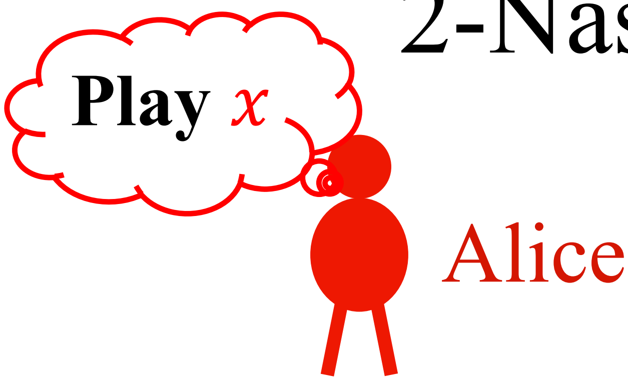
2-Nash Characterization

- For Alice, i^{th} strategy gives

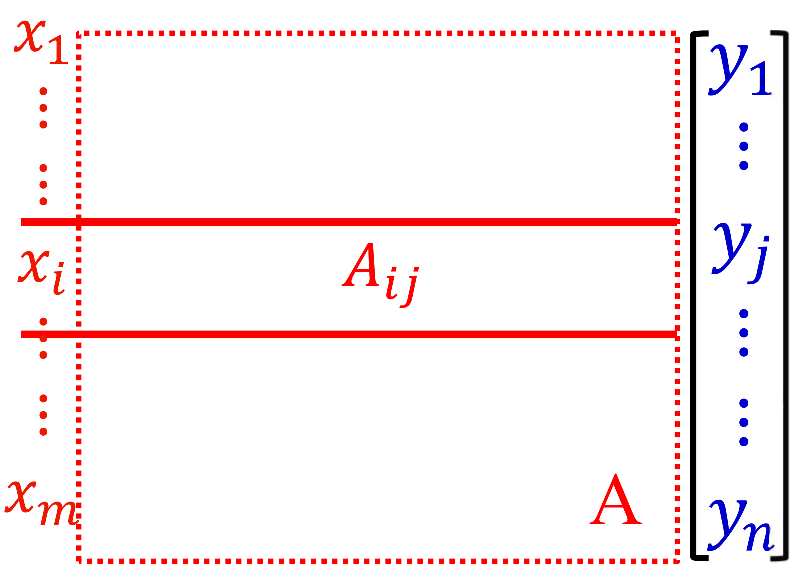


$$\rightarrow \sum_j A_{ij} y_j = (Ay)_i$$

2-Nash Characterization

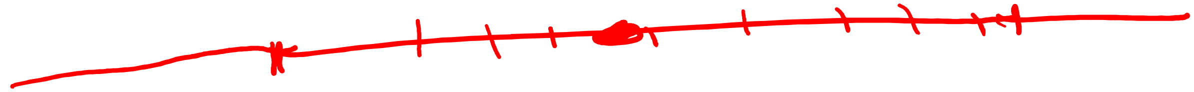


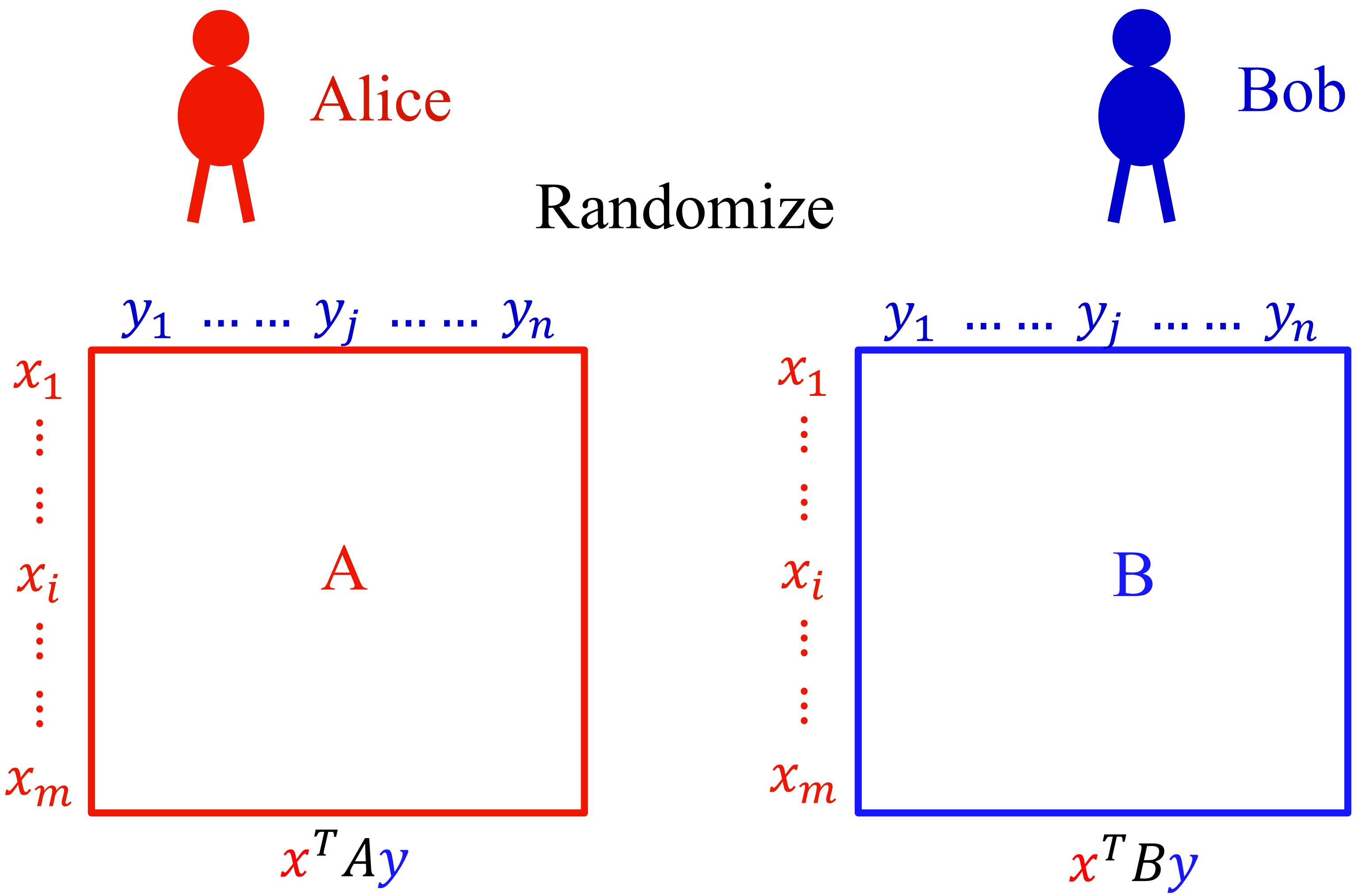
- Alice's expected payoff is



$$\sum_i x_i (Ay)_i = x^T A y$$

$$\min_i (Ay)_i \leq \sum_i x_i (Ay)_i \leq \max_i (Ay)_i$$





NE: No unilateral deviation is beneficial

$$x^T A y \geq z^T A y, \qquad \forall z \in \Delta_m$$
$$x^T B y \geq x^T B z, \qquad \forall z \in \Delta_n$$

Two-Player Games

- A pair of payoff matrices (R, C) of size $m \times n$, where Row player has m actions and Column player has n actions. (action $\iff$ pure strategy)

- So the meaning of $R_{i,j}$ and $C_{i,j}$?

- Mixed strategy: a distribution over pure strategies. Denote by Δ_n the set of all mixed strategies over n actions. That is,

$$\Delta_n := \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i \in [n]} x_i = 1, x_i \geq 0\}\ .$$

- Expected payoff: given $\mathbf{x} \in \Delta_m, \mathbf{y} \in \Delta_n$, they are $\mathbf{x}^T R \mathbf{y}$ and $\mathbf{x}^T C \mathbf{y}$, just calculation…

Nash Equilibrium

Two-player version

- A pair of strategies $(\mathbf{x}, \mathbf{y})$ is NE iff neither can increase her payoff by deviating from her strategy unilaterally. That is

$$\mathbf{x}^T R \mathbf{y} \geq \mathbf{x}'^T R \mathbf{y}, \ \forall \mathbf{x}' \in \Delta_m;$$
$$\mathbf{x}^T C \mathbf{y} \geq \mathbf{x}^T C \mathbf{y}', \ \forall \mathbf{y}' \in \Delta_n .$$

- Or an equivalent definition
  - Support of $\mathbf{x}$: $\mathrm{supp}(\mathbf{x}) := \{i \in [n] \mid x_i \neq 0\}$.
  - Each action in the support of $\mathbf{x}$ (or $\mathbf{y}$) should be the best response to the other.

$$x_i > 0 \Rightarrow \mathbf{e}_i^T R \mathbf{y} \geq \mathbf{e}_k^T R \mathbf{y}, \forall k \in [m]$$
$$y_j > 0 \Rightarrow \mathbf{x}^T C \mathbf{e}_j \geq \mathbf{x}^T C \mathbf{e}_k, \forall k \in [n]$$

Zero-Sum Games

The game with absolute conflict…

	M	T
E	3, -3	-1, 1
S	-2, 2	1, -1

- Zero-Sum iff $R + C = 0$, that is $R_{i,j} + C_{i,j} = 0$.
- Given row player using (x_1, x_2), we have
  - Column has $\mathbb{E}[M] = -3x_1 + 2x_2$, $\mathbb{E}[T] = x_1 - x_2$ and gets the better one.
  - Since zero-sum, row will choose $(x_1, x_2) \in \arg\max_{x_1, x_2} \min(3x_1 - 2x_2, -x_1 + x_2)$.

Some Observations

$$\begin{aligned} &\max z \\ \text{s.t.} \quad 3x_1 - 2x_2 &\geq z \\ -x_1 + x_2 &\geq z \\ x_1 + x_2 &= 1 \\ x_1, x_2 &\geq 0. \end{aligned}$$

$$\begin{aligned} &\max w \\ \text{s.t.} \quad -3y_1 + y_2 &\geq w \\ 2y_1 - y_2 &\geq w \\ y_1 + y_2 &= 1 \\ y_1, y_2 &\geq 0, \end{aligned}$$

$x_1 = 3/7, x_2 = 4/7, z = 1/7$

$y_1 = 2/7, y_2 = 5/7, w = -1/7$

Nash equilibrium!

LP & Duality

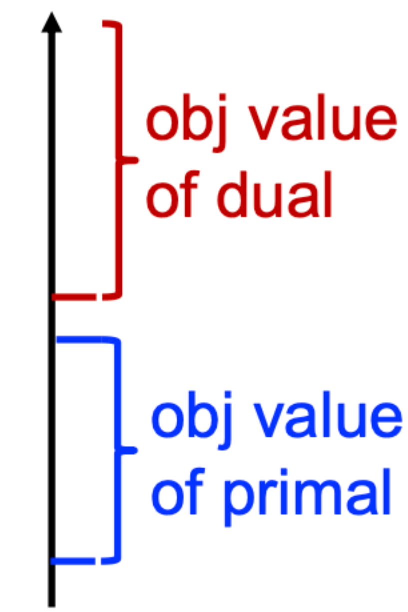
$$A = \begin{bmatrix} A^1 \\ A^2 \\ \vdots \\ A^m \end{bmatrix} = [A_1, \ldots, A_n]$$

Primal LP

max	$c^T \cdot x$
s.t.	$Ax \leq b$
	$x \geq 0$

Dual LP

min	$b^T \cdot y$
s.t.	$A^T y \geq c$
	$y \geq 0$



- $c, x \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}$ and $b, y \in \mathbb{R}^m$
- y_i is the dual variable to the primal constraint $A^i x \leq b_i$
- $A_j^T y \geq c_j$ is the dual constraint to the primal variable x_j

- Weak Duality: $c^T x \leq b^T y$
- Strong Duality: If either the primal or dual is feasible and bounded, then so is the other and OPT(primal) = OPT(dual).

LP for zero-sum games

$z'' = -z', C = -R$

$$\begin{aligned} \max \quad & z \\ \text{s.t.} \quad & \boldsymbol{x}^T R \geq z \mathbf{1}^T \\ & \boldsymbol{x}^T \mathbf{1} = 1 \\ & \forall i, \; x_i \geq 0. \end{aligned}$$

$$\begin{aligned} \min \quad & z' \\ \text{s.t.} \quad & -\boldsymbol{y}^T R^T + z' \mathbf{1}^T \geq \mathbf{0} \\ & \boldsymbol{y}^T \mathbf{1} = 1 \\ & \forall j, \; y_j \geq 0. \end{aligned}$$

$$\begin{aligned} \max \quad & z'' \\ \text{s.t.} \quad & C \boldsymbol{y} \geq z'' \mathbf{1} \\ & \boldsymbol{y}^T \mathbf{1} = 1 \\ & \forall j \; y_j \geq 0. \end{aligned}$$

$$\max_{\boldsymbol{x}} \min_{\boldsymbol{y}} \boldsymbol{x}^T R \boldsymbol{y}.$$

$$\max_{\boldsymbol{y}} \min_{\boldsymbol{x}} \boldsymbol{x}^T C \boldsymbol{y} = -\min_{\boldsymbol{y}} \max_{\boldsymbol{x}} \boldsymbol{x}^T R \boldsymbol{y}.$$

Theorem 1

If $(\mathbf{x}, z)$ is optimal for LP(1), and $(\mathbf{y}, z'')$ is optimal for LP(3), then $(\mathbf{x}, \mathbf{y})$ is a Nash equilibrium of (R, C). Moreover, the payoffs of the row/column player in this Nash equilibrium are z and $z'' = -z$ respectively.

LP $\Longleftrightarrow$ NE

$$\begin{aligned} \max \quad & z \\ \text{s.t.} \quad & \boldsymbol{x}^T R \geq z \mathbf{1}^T \\ & \boldsymbol{x}^T \mathbf{1} = 1 \\ & \forall i, \ x_i \geq 0. \end{aligned} \qquad \begin{aligned} \max \quad & z'' \\ \text{s.t.} \quad & C\boldsymbol{y} \geq z'' \mathbf{1} \\ & \boldsymbol{y}^T \mathbf{1} = 1 \\ & \forall j \ y_j \geq 0. \end{aligned}$$

- By def of NE, it is sufficient to show that $\mathbf{x}^T R \mathbf{y} \geq z \geq \mathbf{x}'^T R \mathbf{y}$.
- There exists a Nash equilibrium in every two-player zero-sum game.
- The Minimax Theorem: $\max_x \min_y x^T R y = \min_y \max_x x^T R y$.

Theorem 2

If $(\mathbf{x}, \mathbf{y})$ is a Nash equilibrium of (R, C), then $(\mathbf{x}, \mathbf{x}^T R \mathbf{y})$ is an optimal solution of LP(1), and $(\mathbf{y}, -\mathbf{x}^T C \mathbf{y})$ is an optimal solution of LP (2).

Normal Form Games

- NFG: $\langle n, (S_p)_{p\in[n]}, (u_p)_{p\in[n]} \rangle$
    - Number of players in the game, $[n] = \{1,\ldots,n\}$
    - A set S_p of pure strategies of player $p \in [n]$
    - A utility function $u_p : \times_{p\in[n]} S_p \to \mathbb{R}$
- Recall RSP game…

More math…

- The set Δ^{S_p} of mixed strategies to player p over S_p

- The set $S := \times_{p\in[n]} S_p$ of all the pure strategy profile.
  $\mathbf{s} = (s_1, \ldots, s_n) \sim S$

- The set $\Delta := \times_{p\in[n]} \Delta^{S_p}$ of all the mixed strategy profile.
  $\mathbf{x} = (\mathbf{x_1}, \ldots, \mathbf{x_n}) \sim \boldsymbol{\Delta}$

- Given $\mathbf{x} \in \Delta$, we define the expected payoff of player p is

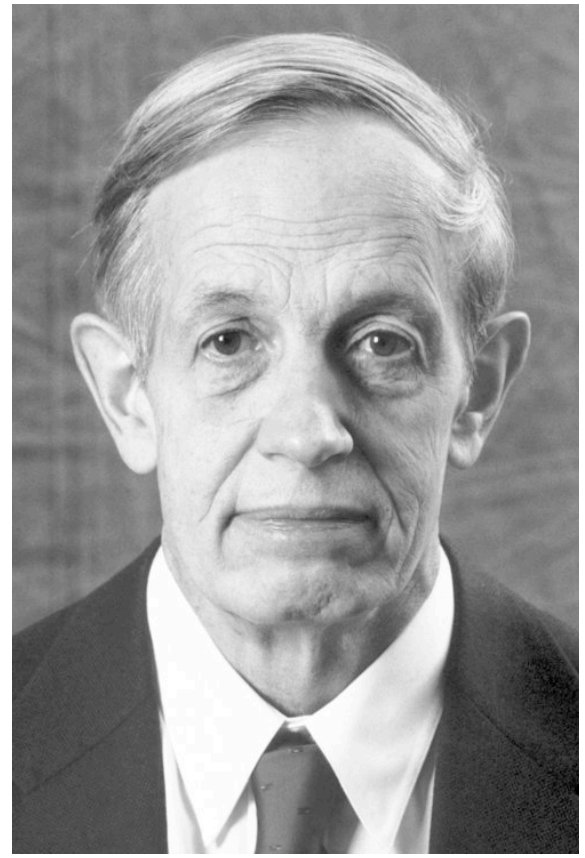
$$u_p(\mathbf{x}) = \sum_{\mathbf{s}\in S} u_p(\mathbf{s}) \prod_{q\in[n]} \mathbf{x}_q(s_q) = \mathbb{E}_{\mathbf{s}\sim\mathbf{x}}\left[u_p(\mathbf{s})\right].$$

- NE $\mathbf{x} \in \Delta$ in multi-player games iff given any $\mathbf{x}'_p \in \Delta^{S_p}$

$$u_p(\mathbf{x}) \geq u_p(\mathbf{x}'_p; \mathbf{x}_{-p})$$

"As far as I can see, there could be no theory of games … without that theorem … I thought there was nothing worth publishing until the Minimax Theorem was proved"

John von Neumann



Nash’s Theorem: “Every (finte) game has a Nash equilibrium.”

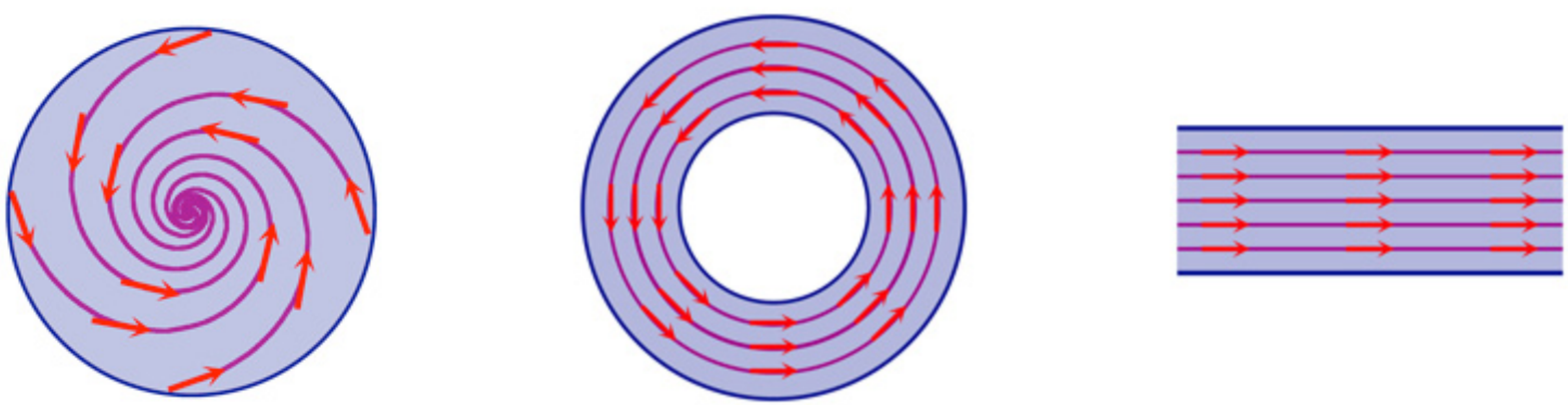
John Forbes Nash Jr.

Proof of Nash's Theorem

An reduction to fixed point.

- The idea is to construct a reduction from the problem of finding an NE in a NFG to the problem of finding a fixed point in a well-defined domain.

- [Brouwer's Fixed Point Thm] Let D be a good (convex, compact) subset of $\mathbf{R}^n$. If a function $f : D \to D$ is continuous, then there exists an $x \in D$ such that $f(x) = x$.



- Let's make a mapping between these two problems.
- So the question is how to construct the continuous function f.
    - It's a good choice to set $f : \Delta \to \Delta$.
- We define a gain function $G_{p,s_p}(\mathbf{x}) := \max\{u_p(s_p; \mathbf{x}_{-p}) - u_p(\mathbf{x}), 0\}$.
    - Can you increase your utility when only using s_p instead of $\mathbf{x}_p$?
- We define $\mathbf{y} = f(\mathbf{x})$, where $y_{p,s_p} := \dfrac{x_{p,s_p} + G_{p,s_p}(\mathbf{x})}{1 + \sum_{s'_p \in S_p} G_{p,s'_p}(\mathbf{x})}$.
- f is well-defined, continuous and Δ is good enough $\Rightarrow$ Bingo!
- Next we will show that any fixed point of f is an NE of the game.

$$y_{p,s_p} := \frac{x_{p,s_p} + G_{p,s_p}(\mathbf{x})}{1 + \sum_{s'_p \in S_p} G_{p,s'_p}(\mathbf{x})}$$

- Given $\mathbf{x} = f(\mathbf{x})$, sufficient to show that $G_{p,s_p}(\mathbf{x}) = 0$, $\forall p, s_p$
- Proof by contradiction!
  - Assume that there exists p, s_p such that $G_{p,s_p}(\mathbf{x}) > 0$
    - $x_{p,s_p} > 0$, otherwise $x_{p,s_p} = 0$ but $y_{p,s_p} > 0$
    - There exists some other pure strategy s'_p such that $x_{p,s'_p} > 0$ and $u_p(s'_p; \mathbf{x}_{-p}) - u_p(\mathbf{x}) < 0$
      - By $u_p(\mathbf{x}) = \sum_{s \in S_p} x_{p,s} \cdot u_p(s; \mathbf{x}_{-p})$
    - We have $y_{p,s'_p} < x_{p,s'_p}$, so $\mathbf{x}$ is not a fixed point!

Algorithms for 2-player NE

We will never discuss the multi-player case in the future…

Overview

- Support Enumeration Algorithm
- The Lipton-Markakis-Mehta (LMM) Approximation Algorithm
- The Lemke-Howson (LH) Algorithm (Not mentioned)

Support Enumeration Algorithm

What if we know the supports of an NE?

- Let (R, C) be a two-player game, where $R, C \in \mathbb{R}^{m \times n}$.
- Someone tells us the supports S_R and S_C of their NE $(\mathbf{x}, \mathbf{y})$, that is $S_R = \mathrm{supp}(\mathbf{x})$ and $S_C = \mathrm{supp}(\mathbf{y})$.
- How many possible pairs of S_R and S_C? So the running time is not good…

$$
\begin{aligned}
& \max 0 \\
\text{s.t.} \quad & \mathbf{e}_i^T R\mathbf{y} \geq \mathbf{e}_j^T R\mathbf{y}, \forall i \in S_R, j \in [m] \\
& \mathbf{x}^T C\mathbf{e}_i \geq \mathbf{x}^T C\mathbf{e}_j, \forall i \in S_C, j \in [n] \\
& \mathbf{x}^T \mathbf{1} = 1, \mathbf{y}^T \mathbf{1} = 1 \\
& x_i = 0, \forall i \notin S_R, y_j = 0, \forall j \notin S_C
\end{aligned}
$$

Relax the goal!

Approximate NE

- In the literature, we have two different definition of approximate NE. (We assume that $R, C \in [0,1]^{n \times n}$)

- "ϵ-Approximate" NE: given any $\epsilon > 0$,

$$\mathbf{x}^T R\mathbf{y} \geq \mathbf{x}'^T R\mathbf{y} - \epsilon, \ \forall \mathbf{x}' \in \Delta_n;$$
$$\mathbf{x}^T C\mathbf{y} \geq \mathbf{x}^T C\mathbf{y}' - \epsilon, \ \forall \mathbf{y}' \in \Delta_n.$$

- "ϵ-Well-Supported" NE: given any $\epsilon > 0$,

$$x_i > 0 \Rightarrow \mathbf{e}_i^T R\mathbf{y} \geq \mathbf{e}_k^T R\mathbf{y} - \epsilon, \forall k \in [n]$$
$$y_j > 0 \Rightarrow \mathbf{x}^T C\mathbf{e}_j \geq \mathbf{x}^T C\mathbf{e}_k - \epsilon, \forall k \in [n]$$

- ϵ-WSNE $\Rightarrow$ ϵ-ANE; one can prove that $\epsilon^2/8$-ANE $\Rightarrow$ ϵ-WSNE

The LMM Algorithm

Lipton, R. J., Markakis, E., and Mehta, A. (2003). Playing large games using simple strategies. (EC'03)

Theorem 1 (Liption et al.)
For any $\epsilon \in (0,1)$, there exists an ϵ-ANE where each player plays only $k = O(\log n/\epsilon^2)$ actions with positive probability.

- We use probabilistic method, that is, $\Pr[A] > 0 \Rightarrow A$ exists.
- Idea: approximating the original NE $(\mathbf{x}, \mathbf{y})$ with large enough samples from $\mathbf{x}, \mathbf{y}$. What is the sample complexity?

Theorem 2 (Chernoff Bound)
Let $X_1, \ldots, X_m$ be m random variables over $[0,1]$. For any $\epsilon > 0$ and X be the mean of $\{X_i\}_{i\in[m]}$, we have $\Pr\left[\,|X - \mathbb{E}[X]| \geq \epsilon\right] \leq 2\exp\left(-2m\epsilon^2\right)$.

Proof of Theorem 1

- Let $(\mathbf{x}, \mathbf{y})$ be any NE of our instance.
- Take k i.i.d. samples (actions) $(r_1, \ldots, r_k)$ from the distribution $\mathbf{x}$.
- Let $\tilde{\mathbf{x}}$ be the "empirical" strategy which plays r_i uniformly at random. Similarly with $\tilde{\mathbf{y}}$.
- We will show, when k is large enough, below could happen: $|\mathbf{e}_i^T R\mathbf{y} - \mathbf{e}_i^T R\tilde{\mathbf{y}}| \le \epsilon/2$ and $|\mathbf{x}^T C\mathbf{e}_j - \tilde{\mathbf{x}}^T C\mathbf{e}_j| \le \epsilon/2$ where $i, j \in [n]$.
- If so, we have

$$\mathbf{e}_i^T R\tilde{\mathbf{y}} \le \mathbf{e}_i^T R\mathbf{y} + \epsilon/2 \le \frac{1}{k}\sum_{j=1}^{k} \mathbf{e}_{r_j}^T R\mathbf{y} + \epsilon/2 \le \frac{1}{k}\sum_{j} \mathbf{e}_{r_j}^T R\tilde{\mathbf{y}} + \epsilon = \tilde{\mathbf{x}}^T R\tilde{\mathbf{y}} + \epsilon$$

■

Proving that $|\mathbf{e}_i^T R\mathbf{y} - \mathbf{e}_i^T R\tilde{\mathbf{y}}| \le \epsilon/2$

Theorem 3 (The Union Bound)
$\Pr[A_1 \cup A_2] \le \Pr[A_1] + \Pr[A_2]$

- We focus on a bad case that $|\mathbf{e}_i^T R\mathbf{y} - \mathbf{e}_i^T R\tilde{\mathbf{y}}| > \epsilon/2$ for fixed i
- By Chernoff bound, we have (by setting $X_j = \mathbf{e}_i^T R\mathbf{e}_{r_j}$)
  $\Pr[|\mathbf{e}_i^T R\mathbf{y} - \mathbf{e}_i^T R\tilde{\mathbf{y}}| > \epsilon/2] \le 2\exp(-k\epsilon^2/2)$.
- By the union bound, we have $2n$ bad cases, so the probability that any of the bad cases happens is at most $4n\exp(-k\epsilon^2/2)$.
- For $k > 2\log(4n)/\epsilon^2$, the probability above is less than 1!

Remark for LMM algo

- One can approximate any NE w.r.t. ANE
- The running time is $\binom{n}{k}^2 = n^{O(\frac{\log n}{\epsilon^2})}$.
- With reasonable assumption (ETH for PPAD), Rubinstein (FOCS'16) proved that LMM is optimal, that is, finding an ϵ-ANE needs at least $n^{\log^{1-o(1)} n}$.

Q&A?

zhengyang@bit.edu.cn