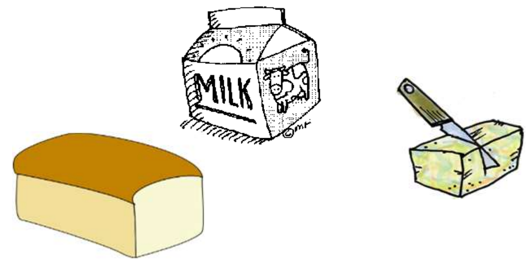
CS 580: Topics on AGT

Lec 2: Fair Division of Divisibles

Instructor: Ruta Mehta

Divisible goods



Goal: Find *fair* and *efficient* allocation

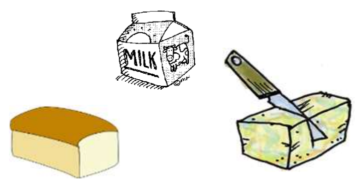

ASSETS
LIABILITIES

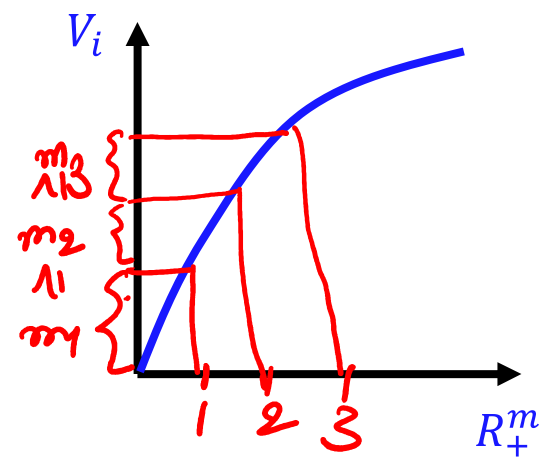

U.S.ARMY
U.S.AIR FORCE

Model

- A: set of n agents

- M: set of m divisible goods (manna)





- Each agent i has
  - Concave valuation function $V_i: R_+^m \to R_+$ over bundles of items
  - Captures ***decreasing marginal returns***.

Goal: Find *fair* and *efficient* allocation

Agreeable (Fair)

Non-wasteful (Efficient)

Allocation: Bundle $X_i \in R_+^m$ to agent i

Envy-free: **No agent *envies* other's allocation over her own.**

For each agent i,
$V_i(X_i) \geq V_i(X_j), \forall j \in [n]$

Proportional: **Each agent *i* gets value at least** $\frac{V_i(M)}{n}$

For each agent $i, V_i(X_i) \geq \frac{V_i(M)}{n}$

Pareto-optimal: **No other allocation is better for all.**

There is no Y, s. t.
$V_i(Y_i) \geq V_i(X_i), \forall i \in [n]$

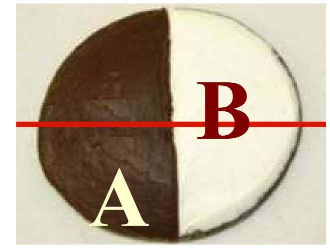
$>$ $\exists i.$

Welfare Maximizing

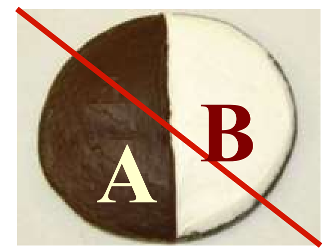
$(max: \sum_i V_i)$

Example: Half moon cookie

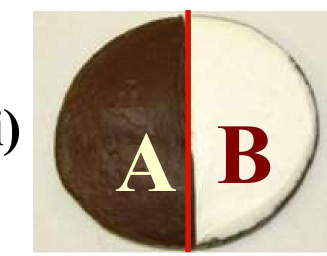
(i)



(ii)



(iii)



Agreeable (Fair)

Non-wasteful (Efficient)

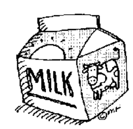
Envy-free: No agent *envies* other's allocation over her own.

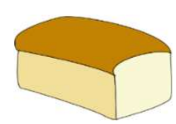
Proportional: Each agent i gets value at least $\frac{V_i(M)}{n}$

Allocation in red

[3, 2, 2]
[0, 0, 0]

[20, 20, 30]
[0, 0, 0]





Agreeable (Fair)

Envy-free: No agent *envies* other's allocation over her own.

Proportional: Each agent i gets value at least $\frac{V_i(M)}{n}$

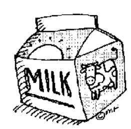
[3, 2, 2] $7/2$
[1/2, 1/2, 1/2]

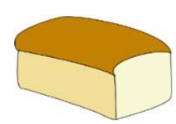
Allocation in red

[20, 20, 30] $\frac{70}{2}$
[1/2, 1/2, 1/2]

Non-wasteful (Efficient)

Pareto-optimal: No other allocation is better for all.





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Non-wasteful (Efficient)

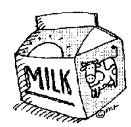
Pareto-optimal: No other allocation is better for all.

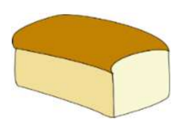
Welfare Maximizing $(max{:}\sum_i V_i)$

Allocation in red

[3, 2, 2]
[1, 1/2, 0]

[20, 20, 30]
[0, 1/2, 1]





Agreeable (Fair)

Envy-free: No agent *envies* other's allocation over her own.

Proportional: Each agent i gets value at least $\frac{V_i(M)}{n}$

Non-wasteful (Efficient)

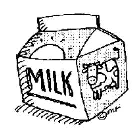
Pareto-optimal: No other allocation is better for all.

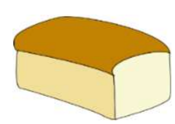
Welfare Maximizing $(max\colon \sum_i V_i)$

Allocation in red

[3, 2, 2] ×100
[0, 0, 0]

[20, 20, 30]
[1, 1, 1]





Agreeable (Fair)

Envy-free: No agent *envies* other's allocation over her own.

Proportional: Eac agent i gets value at least $\frac{V_i(M)}{n}$

Non-wasteful (Efficient)

Pareto-optimal: No other allocation is better for all.

(Nash) Welfare Maximizing ($\Pi_i V_i$)

Allocation in red

[3, 2, 2]

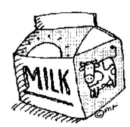
[1, 1/2, 0]

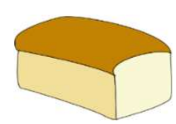
[0, 0, 0]x

[20, 20, 30]

[0, 1/2, 1]

[1, 1, 1]x





$V_1 \rightarrow V_1 \times 100$

$V_2 \quad \text{argmax}(V_1 \cdot V_2)$

$= \text{argmax}\ 100(V_1 \cdot V_2)$

Agreeable (Fair)	Non-wasteful (Efficient)
Envy-free	Pareto-optimal
Proportional	(Nash) Welfare Maximizing

Competitive Equilibrium
(with equal income)

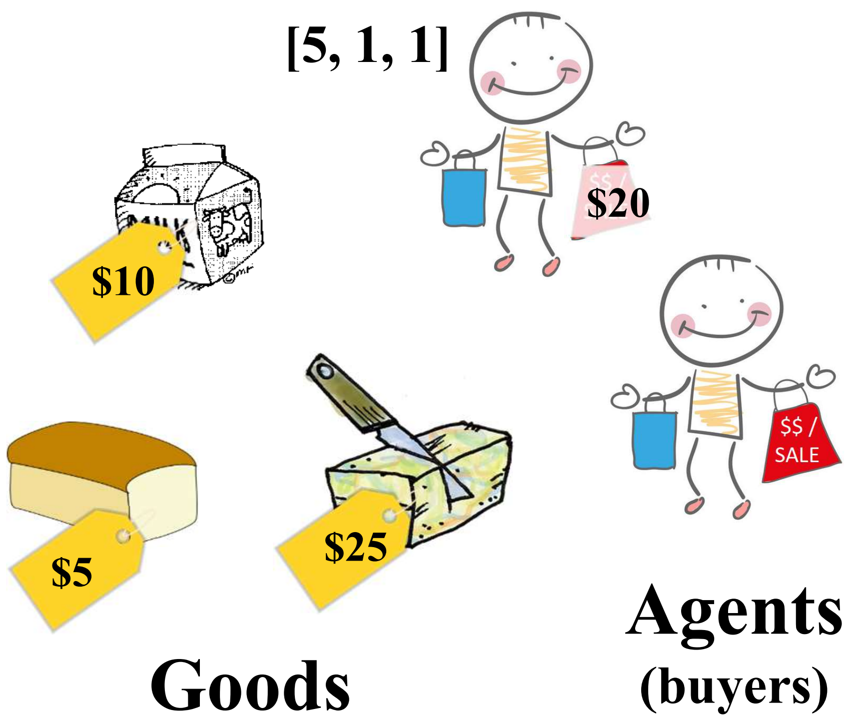
Beginning of Competitive Equilibrium

Adam Smith (1776)

Invisible hand

"Economic concept that describes the unintended greater social benefits and public good brought about by individuals acting in their own self-interests.[1][2] The concept was first introduced by Adam Smith in *The Theory of Moral Sentiments*, written in 1759. According to Smith, it is literally divine providence, that is the hand of God, that works to make this happen."

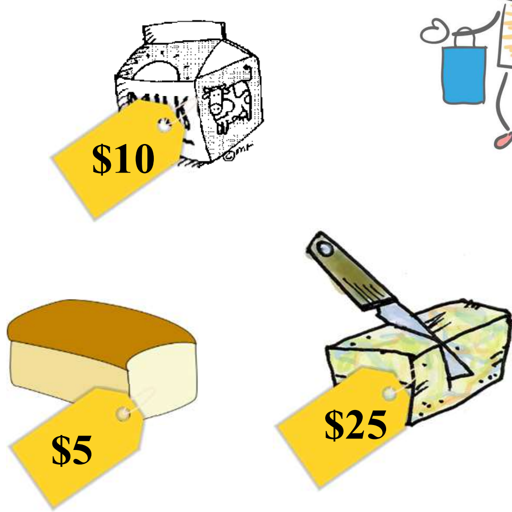
Competitive (market) Equilibrium (CE)



Demand optimal bundle

$$argmax_{\{X \text{ affordable}\}} V_i(X)$$

Competitive (market) Equilibrium (CE)

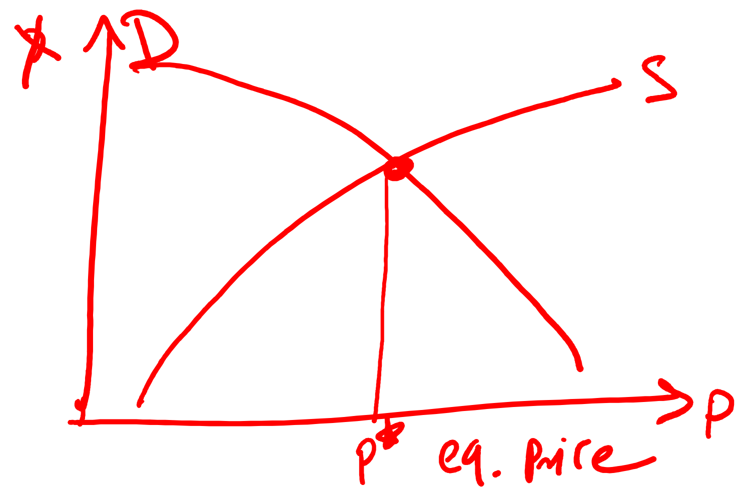


Goods

Agents
(buyers)

Buy optimal bundle ⟶ **Demand**

Competitive Equilibrium:
Demand = Supply



CE Example

[2, 0]

Demand

Supply

[5, 1]

1

2 > 1

Demand > Supply

[1, 4]

1

1

[0, 1]

CE Example

[1, 0]

Demand

[5, 1]

1

Demand = Supply
⇒ CE

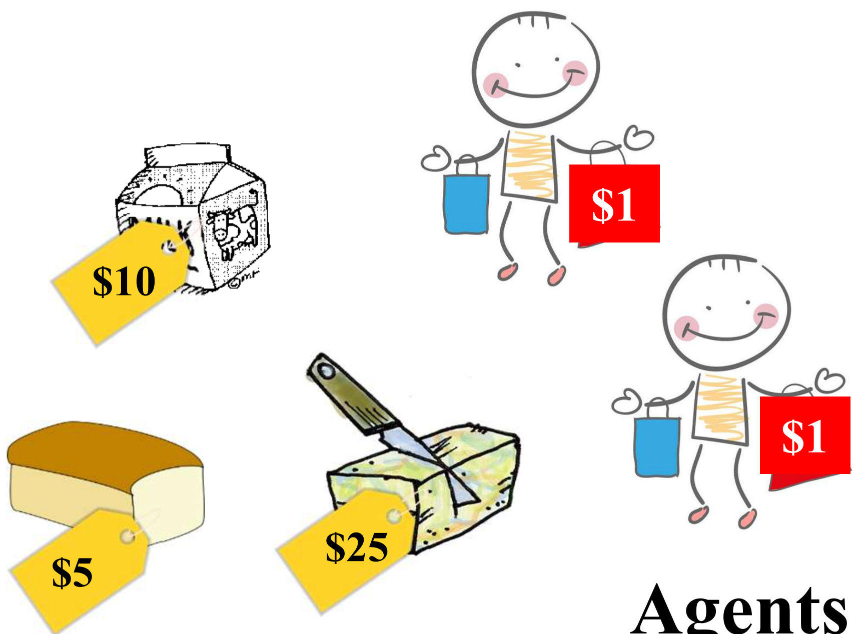
[1, 4]

1

[0, 1]

w/ equal income (CEEI):
Agents have the same amount of money

CEEI: Properties



Goods

Agents
(buyers)

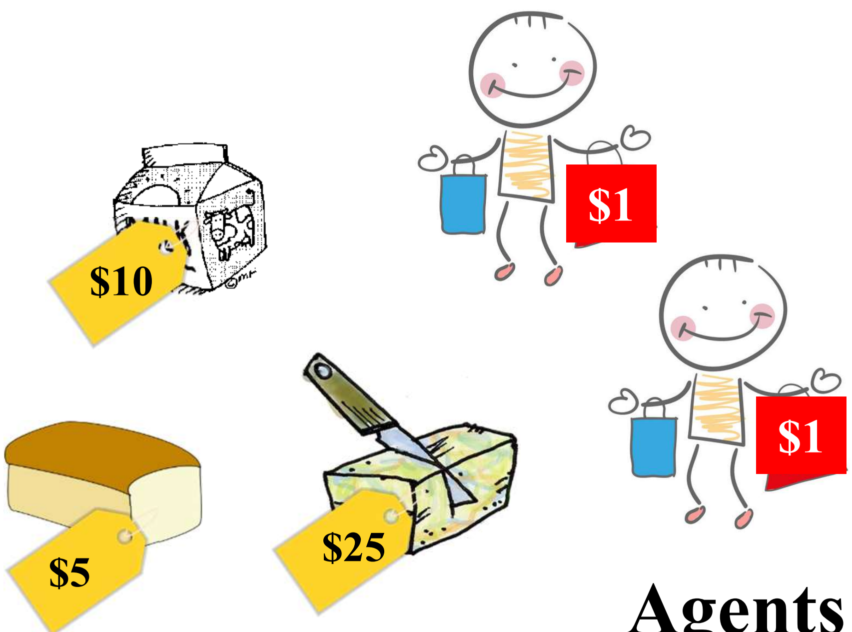
Demand optimal bundle

Competitive Equilibrium:
Demand = Supply

An agent can afford anyone else's bundle, but demands her own
⇒ Envy-free

1^{st} welfare theorem
⇒ Pareto-optimal

CEEI: Properties



Goods

Agents
(buyers)

Demand optimal bundle

Competitive Equilibrium:
Demand = Supply

Envy-free & "Demand=Supply"
⇒ Proportional

Proof. $X_j \in \mathbb{R}^m_+$

Envyfree

$\Rightarrow V_i(X_i) \geq V_i(X_j), \forall j \in [n]$

$\Rightarrow nV_i(X_i) \geq \sum_{j\in[n]} V_i(X_j)$

X_j := amt of good j bought

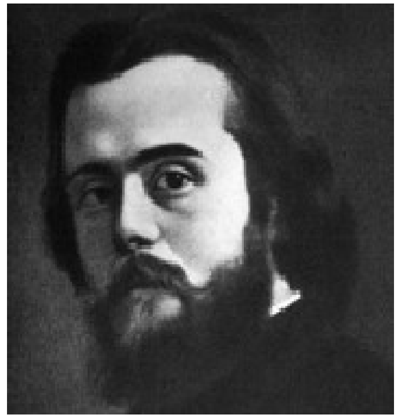
"Demand = Supply"

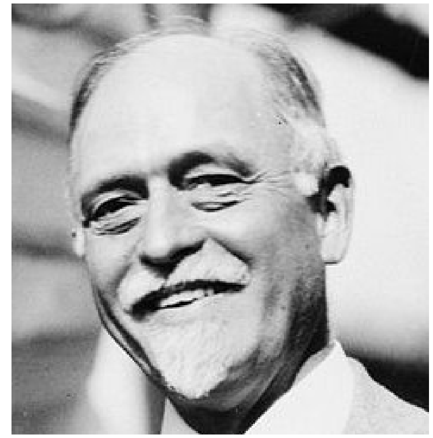
$$\Rightarrow \sum_{j\in[n]} V_i(X_j) \geq V_i(M) \ (\because V_i \text{ concave})$$

$$\Rightarrow V_i(X_i) \geq \frac{V_i(M)}{n}$$

CE History


Adam Smith (1776)


Leon Walras (1880s)


Irving Fisher (1891)


Arrow-Debreu (1954)

(Nobel prize)

(Existence of CE in the exchange model w/ firms)

...

Computation of CE (w/ goods)

Algorithms

- Convex programming formulations
  - Eisenberg-Gale (1959): CEEI w/ 1-homogeneous valuations
  - Shmyrev (2009), DGV (2013), CDGJMVY (2017) …
- (Strongly) Poly-time algorithms (linear valuations)
  - DPSV (2002), Orlin (2010), DM (2015), GV (2019) …
- Simplex-like algorithms: Eaves (1976), GM.SV (2011), GM.V (2014), …

Complexity

- PPAD: Papadimitrou'92, CDDT'09, VY'11, CPY'17, Rubinstein'18, …
- FIXP: EY'09, GM.VY'17, F-RHHH'21 …

Learning: RZ'12, BDM.UV'14, …, FPR'22, …

Matching/mechanisms: BLNPL'14, …, KKT'15, …, FGL'16, …, AJT'17, …, BGH'19, BNT-C'19, …

***Alaei, Bei, Branzei, Chen, Cole, Daskalakis, Deng, Devanur, Duan, Dai, Etessami, Feldman, Fiat, Filos-Ratsikas, Garg, Gkatzelis, Hansen, Hogh, Hollender, Jain, Jalaly, Hoefer, Kleinberg, Lucier, Mai, Mehlhorn, Mehta, Mansour, Morgenstern, Nisan, Paes, Lee, Leme, Papadimitriou, Paparas, Parkes, Roth, Saberi, Sohoni, Talgam-Cohen, Tardos, Vazirani, Vegh, Yazdanbod, Yannakakis, Zhang,… … …**

Simple Tatonnement Procedure (Algo)

Increase prices of the over demanded goods.

Theorem. Tatonnement process Converges to a CE if $V_i s$ are *weak gross substitutes (WGS)*.

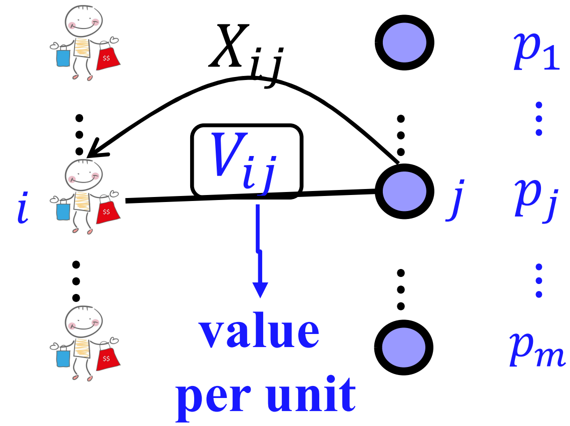
WGS: Increase in price of a good does not decrease demand of any other good.

Example: Linear $V_i s$

$$V_i(X_i) = \sum_{j \in [m]} V_{ij} X_{ij}$$

Linear Valuations: CEEI

$$V_i(X_i) = \sum_{j \in M} V_{ij} X_{ij}$$



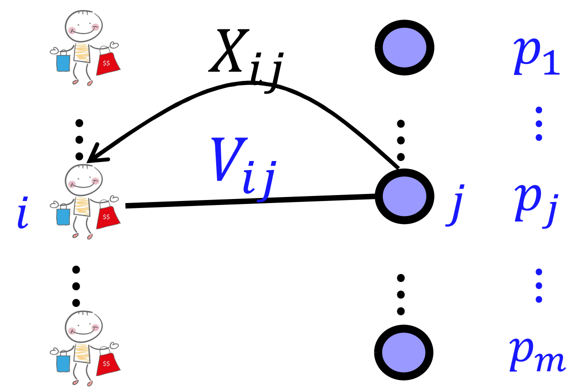
Optimal bundle: can spend at most *one* dollar.

Intuitition

spend wisely: on goods that gives maximum value-per-dollar $\frac{V_{ij}}{p_j}$

Linear Valuations: CEEI

$$V_i(X_i) = \sum_{j \in M} V_{ij} X_{ij}$$



Optimal bundle: can spend at most *one* dollar.

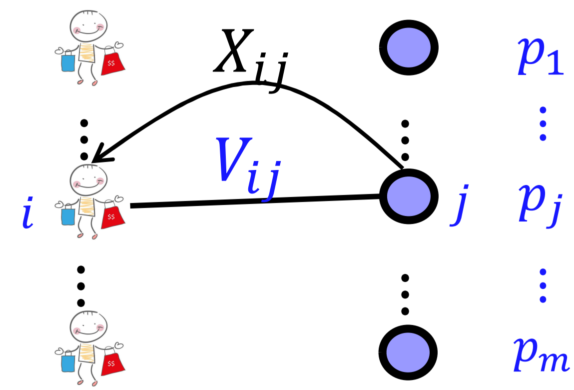
$$\sum_{j \in M} V_{ij} X_{ij} = \sum_{j} \boxed{\frac{V_{ij}}{p_j}} \underbrace{(p_j X_{ij})}_{\text{(\$ spent)}} \leq \left(\max_{k \in G} \frac{V_{ik}}{p_k}\right) \sum_{j} p_j X_{ij} \leq \boxed{\left(\max_{k \in G} \frac{V_{ik}}{p_k}\right)} 1$$

value per dollar spent
(bang-per-buck)

MBB
Maximum
bang-per-buck

Linear Valuations: CEEI

$$V_i(X_i) = \sum_{j\in M} V_{ij} X_{ij}$$



Optimal bundle: can spend at most *one* dollar.

$$\sum_{j\in M} V_{ij}x_{ij} = \sum_j \boxed{\frac{V_{ij}}{p_j}}(p_j X_{ij}) \le \left(\max_{k\in G}\frac{V_{ik}}{p_k}\right)\sum_j p_j x_{ij} \le \boxed{\left(\max_{k\in G}\frac{V_{ik}}{p_k}\right)}\, 1$$

value per dollar spent
(bang-per-buck)

=
iff

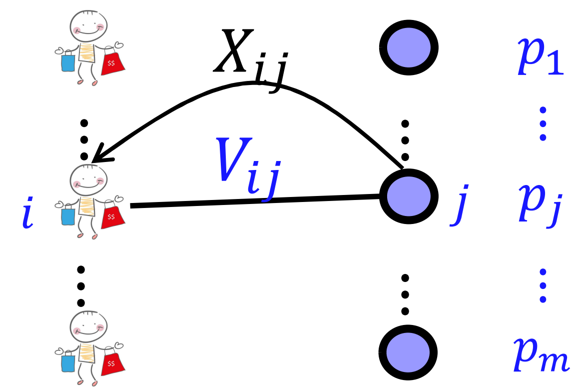
MBB
Maximum
bang-per-buck

Buy only MBB goods.

$$X_{ij} > 0 \;\Rightarrow\; \frac{V_{ij}}{p_j} = MBB$$

Linear Valuations: CEEI

$$V_i(X_i) = \sum_{j\in M} V_{ij} X_{ij}$$



Optimal bundle: can spend at most *one* dollar.

$$\sum_{j\in M} V_{ij}x_{ij} = \sum_j \boxed{\frac{V_{ij}}{p_j}}(p_j X_{ij}) \le \left(\max_{k\in G}\frac{V_{ik}}{p_k}\right)\sum_j p_j x_{ij} \le \boxed{\left(\max_{k\in G}\frac{V_{ik}}{p_k}\right)}\, 1$$

value per dollar spent
(bang-per-buck)

= iff

= iff

MBB
Maximum bang-per-buck

Buy only MBB goods.

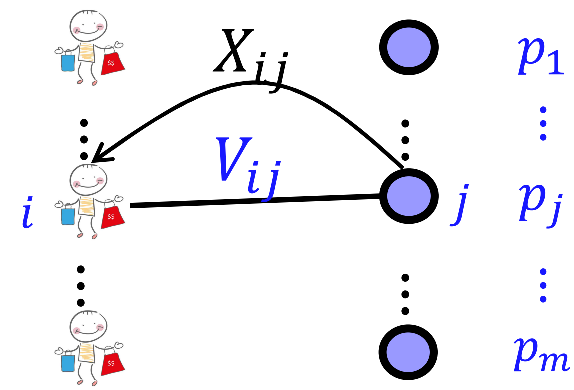
$$X_{ij} > 0 \Rightarrow \frac{V_{ij}}{p_j} = MBB$$

Spends all of 1 dollar.

$$\sum_j p_j X_{ij} = 1$$

Linear Valuations: CEEI

$$V_i(X_i) = \sum_{j \in M} V_{ij} X_{ij}$$



Optimal bundle: can spend at most *one* dollars.

$$\sum_{j \in M} V_{ij} x_{ij} \underset{=}{\leq} \left(\max_{k \in G} \frac{V_{ik}}{p_k} \right) 1$$

iff

1. Buy only MBB goods.

$$X_{ij} > 0 \Rightarrow \frac{V_{ij}}{p_j} = MBB$$

2. Spends all of 1 dollar.

$$\sum_j p_j X_{ij} = 1$$

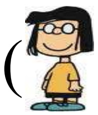
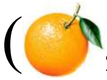
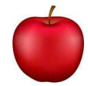
Linear $V_i s$: CEEI Characterization

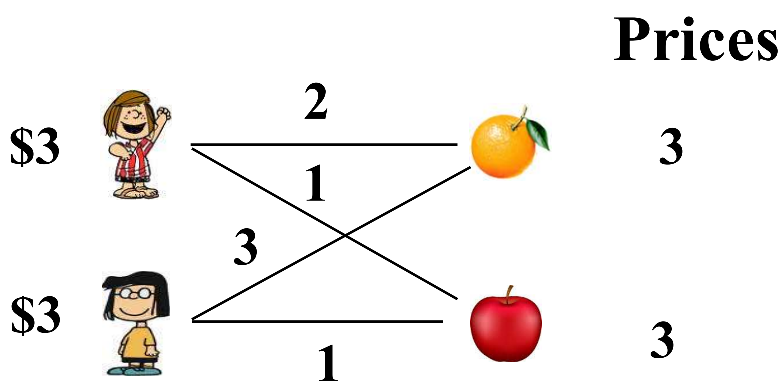
Pirces $p = (p_1, \ldots, p_m)$ and allocation $X = (X_1, \ldots, X_n)$ are at equilibrium iff

- Optimal bundle (OB): For each agent i
  - $\sum_j p_j X_{ij} = 1$
  - $X_{ij} > 0 \Rightarrow \frac{V_{ij}}{p_j} = \max_{k \in M} \frac{V_{ik}}{p_k}$, for all good j
- Market clears: For each good j,

$$\sum_i X_{ij} = 1.$$

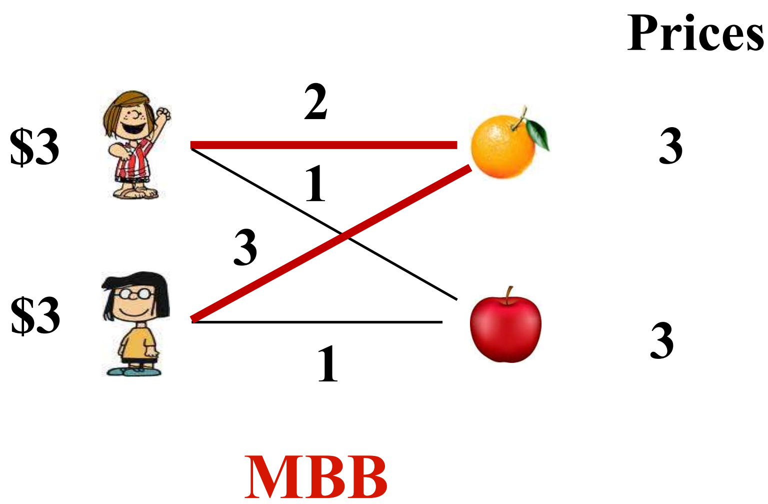
Example

- 2 Buyers (, ), 2 Items (, ) with unit supply
- Each buyer has budget of \$3 and a linear utility function



Example

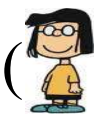
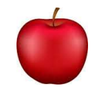
- 2 Buyers (,), 2 Items (,) with unit supply
- Each buyer has budget of $1 and a linear utility function

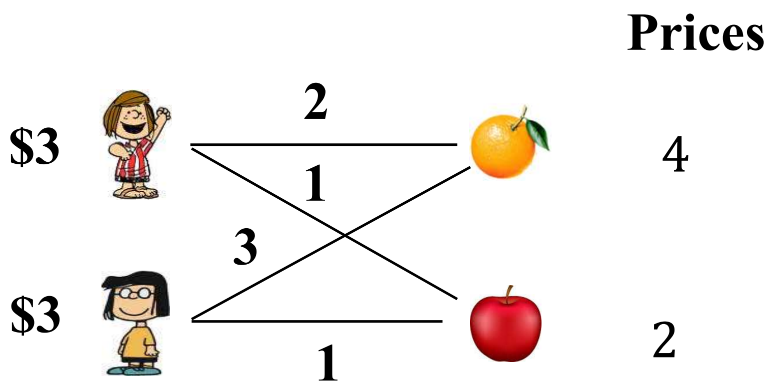


Demand ≠ Supply

Not an Equilibrium!

Example

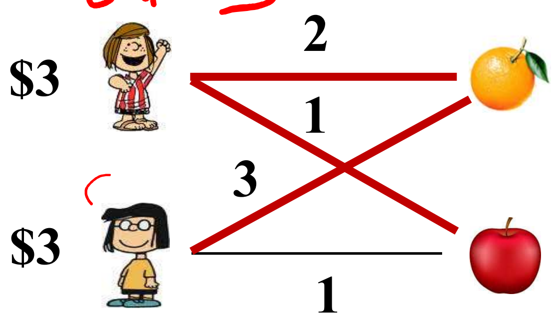
- 2 Buyers (, ), 2 Items (, ) with unit supply
- Each buyer has budget of $1 and a linear utility function



Example

- 2 Buyers (,), 2 Items (,) with unit supply
- Each buyer has budget of $1 and a linear utility function

[1/4, 1]



[3/4, 0]

MBB

	Prices	Demand
	4	1
	2	1

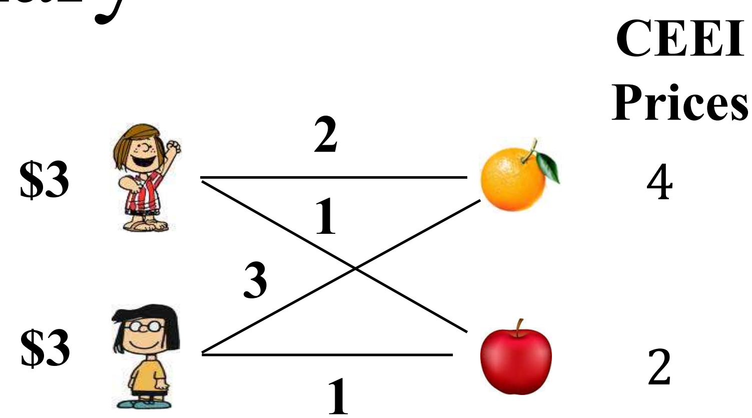
Demand = Supply

Equilibrium!

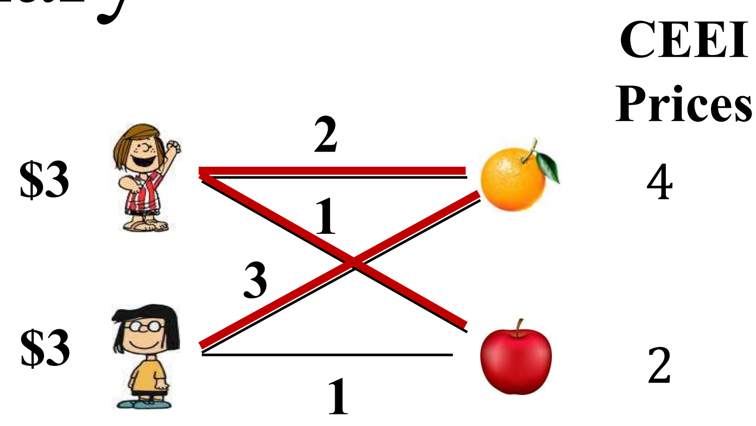
CEEI Properties: Summary

CEEI allocation is

- Pareto optimal (PO)
- Envy-free
- Proportional



CEEI Properties: Summary



CEEI allocation is

- Pareto optimal (PO)
- Envy-free
- Proportional

CEEI Allocation:

$$X_1 = \left(\frac{1}{4}, 1\right), X_2 = \left(\frac{3}{4}, 0\right)$$

$$V_1(X_1) = \frac{3}{2},\ V_2(X_2) = \frac{9}{4} > 2 = \frac{4}{2}$$

$$V_1(X_2) = \frac{3}{2},\ V_2(X_1) = \frac{7}{4}$$

Next…

- **Nash welfare maximizing**

Social Welfare

$$\sum_{i \in A} V_i(X_i)$$

Utilitarian

Issues: May assign 0 value to some agents.
Not scale invariant!

Max Nash Welfare

max: $$\prod_{i \in A} V_i(X_i)$$

s.t. $$\sum_{i \in A} X_{ij} \leq 1, \ \forall j \in G$$
$$X_{ij} \geq 0, \qquad \forall i, \forall j$$

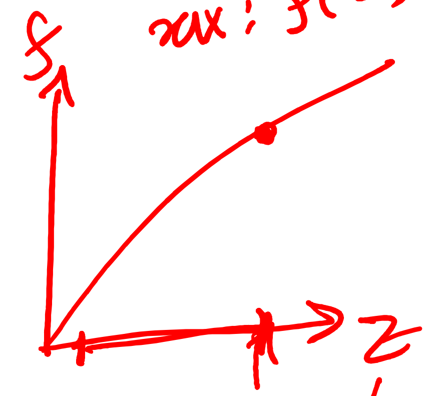
Feasible allocations

Max Nash Welfare (MNW)

max : z

$\downarrow$

max : $f(z)$



$$\textbf{max: } \log\left(\prod_{i\in A} V_i(X_i)\right)$$

$$\begin{aligned} \textbf{s.t.} \quad & \textstyle\sum_{i\in A} X_{ij} \le 1, \ \forall j \in G \\ & X_{ij} \ge 0, \qquad\qquad \forall i, \forall j \end{aligned}$$

Feasible allocations

Max Nash Welfare (MNW)

$$\textbf{max:} \quad \sum_{i \in A} \log V_i(X_i)$$

$$\begin{aligned} \textbf{s.t.} \quad & \textstyle\sum_{i \in A} X_{ij} \le 1, \quad \forall j \in G \\ & X_{ij} \ge 0, \qquad\qquad \forall i, \forall j \end{aligned}$$

Feasible allocations

Eisenberg-Gale Convex Program '59

$$\textbf{max:} \quad \sum_{i \in A} \log V_i(X_i)$$

Dual var.

$$\textbf{s.t.} \quad \sum_{i \in A} X_{ij} \le 1, \ \forall j \in G \longrightarrow p_j$$

$$X_{ij} \ge 0, \qquad \forall i, \forall j$$

Theorem. **Solutions of EG convex program are exactly the CEEI (p, X).**

Proof.

Consequences: CEEI

- **Exists**
- **Forms a convex set**
- **Can be *computed* in polynomial time**
- **Maximizes Nash Welfare**

Theorem. **Solutions of EG convex program are exactly the CEEI (p, X).**

Proof. $\Rightarrow$ **(Using KKT)**

Remember duality

Given a minimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{subject to} \quad & h_i(x) \leq 0, \quad i = 1, \ldots m \\ & \ell_j(x) = 0, \quad j = 1, \ldots r \end{aligned}$$

we defined the **Lagrangian:**

$$L(x, u, v) = f(x) + \sum_{i=1}^{m} u_i h_i(x) + \sum_{j=1}^{r} v_j \ell_j(x)$$

and **Lagrange dual function:**

$$g(u, v) = \min_{x \in \mathbb{R}^n} L(x, u, v)$$

Karush-Kuhn-Tucker conditions

Given general problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{subject to} \quad & h_i(x) \le 0, \;\; i = 1, \ldots m \\ & \ell_j(x) = 0, \;\; j = 1, \ldots r \end{aligned}$$

The **Karush-Kuhn-Tucker conditions** or **KKT conditions** are:

- $0 \in \partial f(x) + \sum_{i=1}^{m} u_i \partial h_i(x) + \sum_{j=1}^{r} v_j \partial \ell_j(x)$ (stationarity)
- $u_i \cdot h_i(x) = 0$ for all i (complementary slackness)
- $h_i(x) \le 0, \; \ell_j(x) = 0$ for all i, j (primal feasibility)
- $u_i \ge 0$ for all i (dual feasibility)

Recall: CEEI Characterization

Pirces $p = (p_1, \ldots, p_m)$ and allocation $X = (X_1, \ldots, X_n)$

- Optimal bundle: For each buyer i
  - $p \cdot X_i = 1$
  - $X_{ij} > 0 \Rightarrow \frac{V_{ij}}{p_j} = \max_{k \in M} \frac{V_{ik}}{p_k}$, for all good j
- Market clears: For each good j,

$$\sum_i X_{ij} = 1.$$

Theorem. **Solutions of EG convex program are exactly the CEE.**

max: $\sum_{i \in A} \log(V_i(X_i))$ $\quad V_i(X_i) = \sum_j V_{ij} X_{ij}$

s.t. $\sum_{i \in A} X_{ij} \leq 1, \ \forall j \in G \longrightarrow p_j \geq 0$ **Dual var.**

$X_{ij} \geq 0, \qquad \forall i, \forall j$

Proof. $\Rightarrow$ (Using KKT)

$$\forall j, \ p_j > 0 \Rightarrow \sum_i X_{ij} = 1$$

Dual condition to X_{ij}:

$$\frac{V_{ij}}{V_i(X_i)} \leq p_j \Rightarrow \frac{V_{ij}}{p_j} \leq V_i(X_i) \Rightarrow p_j > 0 \Rightarrow$$ **market clears**

buy only MBB goods

$$\boxed{X_{ij} > 0 \Rightarrow \frac{V_{ij}}{p_j} = V_i(X_i)}$$

$$\sum_j V_{ij} X_{ij} = \left(\sum_j p_j X_{ij}\right) V_i(X_i)$$

$$\Rightarrow \sum_j p_j X_{ij} = 1$$

$\Rightarrow$ **optimal bundle**

Efficient (Combinatorial) Algorithms

Polynomial time

- Flow based [DPSV'08]
  - General exchange model (barter system) [DM'16, DGM'17, CM'18]
- Scaling + Simplex-like path following [GM.SV'13]

Strongly polynomial time

- Scaling + flow [O'10, V'12]
  - Exchange model (barter system) [GV'19]

We will discuss some of these if there is interest.