Reinforcement Learning
Lecture 3b:

Intro to Reinforcement Learning
[SutBar] Sec. 5.1-5.3, 6.1-6.3, 6.5,
[Sze] Sec. 3.1, 4.3, [SigBuf] Sec. 2.1-2.5,
[RusNor] Sec. 21.1-21.3,

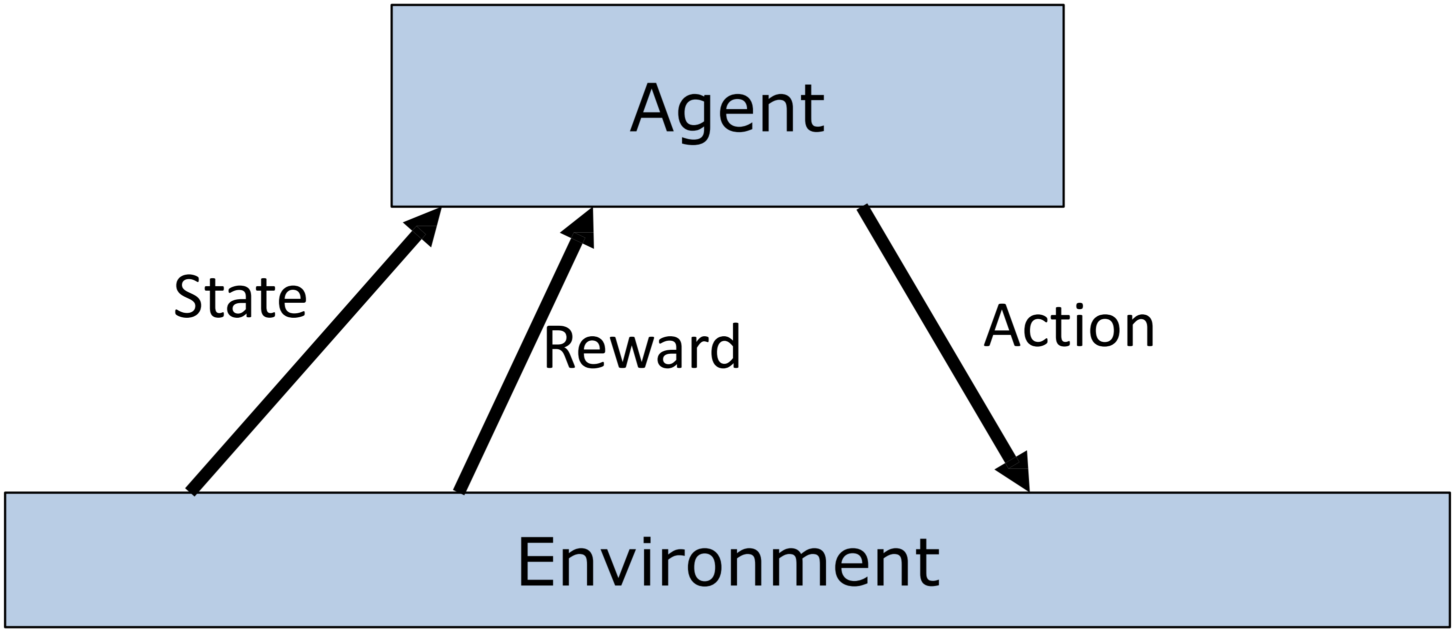
Markov Decision Process

- Definition
  - States: $s \in S$
  - Actions: $a \in A$
  - Rewards: $r \in \mathbb{R}$
  - Transition model: $\Pr(s_t | s_{t-1}, a_{t-1})$
  - Reward model: $\Pr(r_t | s_t, a_t)$
  - Discount factor: $0 \leq \gamma \leq 1$
    - discounted: $\gamma < 1$ undiscounted: $\gamma = 1$
  - Horizon (i.e., # of time steps): h
    - Finite horizon: $h \in \mathbb{N}$ infinite horizon: $h = \infty$

- Goal: find optimal policy π^* such that

$$\pi^* = argmax_\pi \sum_{t=0}^{h} \gamma^t E_\pi[r_t]$$

Reinforcement Learning Problem



Goal: Learn to choose actions that maximize rewards

Reinforcement Learning

- Definition
  - States: $s \in S$
  - Actions: $a \in A$
  - Rewards: $r \in \mathbb{R}$
  - Transition model: $\Pr(s_t | s_{t-1}, a_{t-1})$
  - Reward model: $\Pr(r_t | s_t, a_t)$

  unknown model

  - Discount factor: $0 \leq \gamma \leq 1$
    - discounted: $\gamma < 1$ undiscounted: $\gamma = 1$
  - Horizon (i.e., # of time steps): h
    - Finite horizon: $h \in \mathbb{N}$ infinite horizon: $h = \infty$

- Goal: find optimal policy π^* such that

$$\pi^* = argmax_\pi \sum_{t=0}^{h} \gamma^t E_\pi[r_t]$$

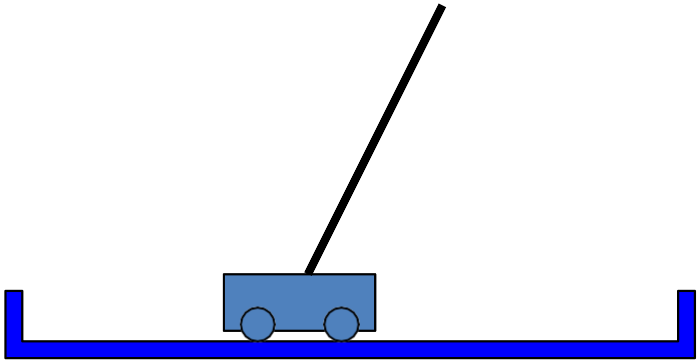
Policy optimization

- Markov Decision Process:
  - Find optimal policy given transition and reward model
  - Execute policy found

- Reinforcement learning:
  - Learn an optimal policy while interacting with the environment

Example: Inverted Pendulum

- State:
  $x(t), x'(t), \theta(t), \theta'(t)$
- Action: Force F
- Reward: 1 for any step where pole balanced



Problem: Find $\pi: S \to A$ that maximizes rewards

Important Components in RL

RL agents may or may not include the following components:

- Model: $\Pr(s'|s,a), \Pr(r|s,a)$
  - Environment dynamics and rewards
- Policy: $\pi(s)$
  - Agent action choices
- Value function: $V(s)$
  - Expected total rewards of the agent policy

Categorizing RL agents

Value based

- No policy (implicit)
- Value function

Policy based

- Policy
- No value function

Actor critic

- Policy
- Value function

Model based

- Transition and reward model

Model free

- No transition and no reward model (implicit)

Toy Maze Example

3	r	r	r	+1
2	u	■	u	-1
1	u	l	l	l
	1	2	3	4

Start state: (1,1)
Terminal states: (4,2), (4,3)
No discount: $\gamma = 1$

Reward is -0.04 for non-terminal states

Four actions: up (u), left (l), right (r), down (d)
Do not know the transition probabilities

What is the value $V(s)$ of being in state s?

Model free evaluation

- Given a policy π, estimate $V^{\pi}(s)$ without any transition or reward model

- **Monte Carlo** evaluation

$$V^{\pi}(s) = E_{\pi}[\sum_{t} \gamma^{t} r_{t}]$$

$$\approx \frac{1}{n(s)} \sum_{k=1}^{n(s)} [\sum_{t} \gamma^{t} r_{t}^{(k)}]$$ (sample approximation)

- **Temporal difference (TD)** evaluation

$$V^{\pi}(s) = \mathrm{E}[\mathrm{r}|\mathrm{s}, \pi(s)] + \gamma \sum_{s'} \Pr(s'|s, \pi(s))\, V^{\pi}(s')$$

$$\approx \mathrm{r} + \gamma V^{\pi}(s')$$ (one sample approximation)

Monte Carlo Evaluation

- Let G_k be a one-trajectory Monte Carlo target

$$G_k = \sum_t \gamma^t r_t^{(k)}$$

- Approximate value function

$$\begin{aligned} V_n^\pi(s) &\approx \frac{1}{n(s)} \sum_{k=1}^{n(s)} G_k \\ &= \frac{1}{n(s)} \left(G_{n(s)} + \sum_{k=1}^{n(s)-1} G_k\right) \\ &= \frac{1}{n(s)} \left(G_{n(s)} + (n(s) - 1) V_{n-1}^\pi(s)\right) \\ &= V_{n-1}^\pi(s) + \frac{1}{n(s)} \left(G_{n(s)} - V_{n-1}^\pi(s)\right) \end{aligned}$$

- **Incremental update**

$$V_n^\pi(s) \leftarrow V_{n-1}^\pi(s) + \alpha_n \left(G_{n(s)} - V_{n-1}^\pi(s)\right)$$

learning rate $1/n(s)$

Temporal Difference Evaluation

- Approximate value function:$V^{\pi}(s) \approx r + \gamma V^{\pi}(s')$
- **Incremental update**

$$V_n^{\pi}(s) \leftarrow V_{n-1}^{\pi}(s) + \alpha_n(r + \gamma V_{n-1}^{\pi}(s') - V_{n-1}^{\pi}(s))$$

- **Theorem:** If α_n is appropriately decreased with number of times a state is visited then $V_n^{\pi}(s)$ converges to correct value
- **Sufficient conditions** for α_n:

  (1) $\sum_n \alpha_n \rightarrow \infty$ (2)$\sum_n (\alpha_n)^2 < \infty$

- Often $\alpha_n \;\; s \;\; = 1/n(s)$
  - Where $n(s) \;=$ # of times s is visited

Temporal Difference (TD) evaluation

```
TDevaluation($\pi, V^\pi$)
    Repeat
        Execute $\pi(s)$
        Observe $s$’ and $r$
        Update counts: $n(s) \leftarrow n(s) + 1$
        Learning rate: $\alpha \leftarrow 1/n(s)$
        Update value: $V^\pi(\mathrm{s}) \leftarrow V^\pi(\mathrm{s}) + \alpha(r + \gamma V^\pi(s') - V^\pi(s))$
        $s \leftarrow s'$
        Until convergence of $V^\pi$
    Return $V^\pi$
```

Comparison

- Monte Carlo evaluation:
  - Unbiased estimate
  - High variance
  - Needs many trajectories

- Temporal difference evaluation:
  - Biased estimate
  - Lower variance
  - Needs less trajectories

Model Free Control

- Instead of evaluating the state value fn, $V^{\pi}(s)$, evaluate the state-action value fn, $Q^{\pi}(s,a)$

$Q^{\pi}(s,a)$: value of executing a followed by π

$$Q^{\pi}(s,a) = E[r|s,a] + \gamma \sum_{s'} \Pr(s'|s,a) V^{\pi}(s')$$

- Greedy policy π'

$$\pi'(s) = argmax_a Q^{\pi}(s,a)$$

Bellman’s Equation

- Optimal state value function $V^*(s)$

$$V^*(s) = \max_a E[r|s,a] + \gamma \sum_{s'} \Pr(s'|s,a) V^*(s')$$

- Optimal state-action value function 7 $^*(s,a)$

$$Q^*(s,a) = E[r|s,a] + \gamma \sum_{s'} \Pr(s'|s,a) \max_{a'} Q^*(s',a')$$

where $V^*(s) = max_a Q^*(s,a)$

$\pi^*(s) = argmax_a Q^*(s,a)$

Monte Carlo Control

- Let G_k^a be a one-trajectory Monte Carlo target

$$G_k^a = \underbrace{\mathrm{r}_0^{(k)}}_{\text{a}} + \underbrace{\sum_{t=1} \gamma^t r_t^{(k)}}_{\pi}$$

- Alternate between
  - **Policy evaluation**

$$Q_n^\pi(s,a) \leftarrow Q_{n-1}^\pi(s,a) + \alpha_n(G_n^a - Q_{n-1}^\pi(s,a))$$

  - **Policy improvement**

$$\pi'(s) \leftarrow argmax_a Q^\pi(s,a)$$

Temporal Difference Control

- Approximate Q-function:

$$Q^*(s,a) = E[r|s,a] + \gamma \sum_{s\prime} \Pr(s'|s,a) \max_{a\prime} Q^*(s',a')$$

$$\approx r + \gamma max_{a'} Q^*(s',a')$$

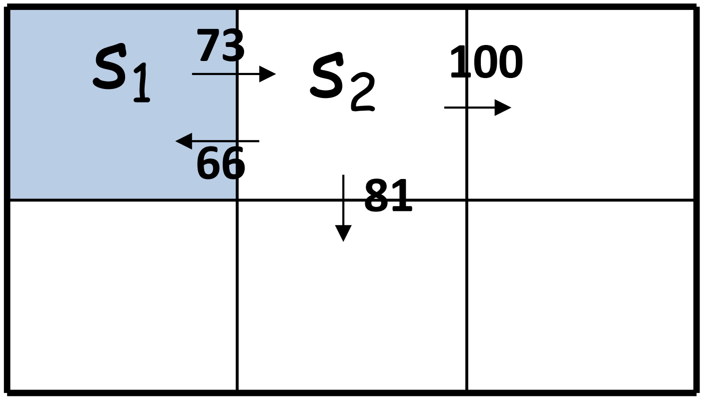
- **Incremental update**

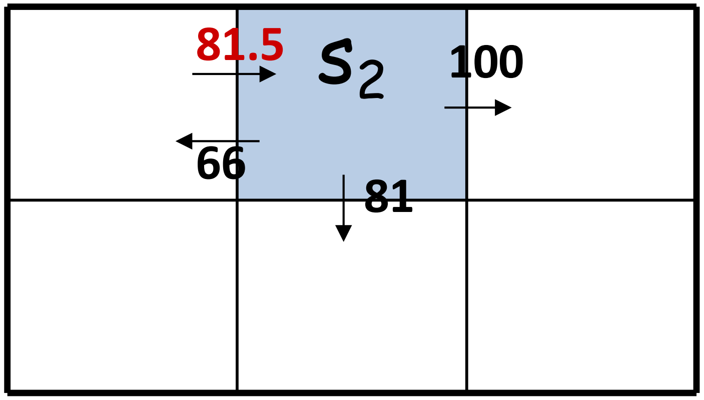
$$Q_n^*(s,a) \leftarrow Q_{n-1}^*(s,a) + \alpha_n(r + \gamma max_{a'} Q_{n-1}^*(s',a') - Q_{n-1}^*(s,a))$$

Q-Learning

```
Qlearning($s, Q^*$)
    Repeat
        Select and execute $a$
        Observe $s$’ and $r$
        Update counts: $n(s,a) \leftarrow n(s,a) + 1$
        Learning rate: $\alpha \leftarrow 1/n(s,a)$
        Update Q-value:
            $Q^*(s,a) \leftarrow Q^*(s,a) + \alpha(r + \gamma \max_{a\prime} Q^*(s',a') - Q^*(s,a))$
            $s \leftarrow s'$
    Until convergence of $Q^*$
Return $Q^*$
```

Q-learning example





$\gamma = 0.9, \;\; \alpha = 0.5, \;\; r = 0$ for non-terminal states

$$
\begin{aligned}
Q(s_1, right) &= Q(s_1, right) + \alpha(r + \gamma \max_{a\prime} Q(s_2, a') - Q(s_1, right)) \\
&= 73 + 0.5(0 + 0.9 \max\{66,81,100\} - 73) \\
&= 73 + 0.5(17) \\
&= 81.5
\end{aligned}
$$

Q-Learning

```
Qlearning($s$,$Q^*$)
      Repeat
        Select and execute $a$
        Observe $s$’ and $r$
        Update counts: $n(s,a) \leftarrow n(s,a) + 1$
        Learning rate: $\alpha \leftarrow 1/n(s,a)$
        Update Q-value:
            $Q^*(s,a) \leftarrow Q^*(s,a) + \alpha(r + \gamma \max_{a'} Q^*(s',a') - Q^*(s,a))$
       $s \leftarrow s'$
   Until convergence of $Q^*$
Return $Q^*$
```

Exploration vs Exploitation

- If an agent always chooses the action with the highest value then it is exploiting
  - The learned model is not the real model
  - Leads to suboptimal results
- By taking random actions (pure exploration) an agent may learn the model
  - But what is the use of learning a complete model if parts of it are never used?
- Need a balance between exploitation and exploration

Common exploration methods

- ε-greedy:
  - With probability ϵ execute random action
  - Otherwise execute best action a^*

$$a^* = argmax_a Q(s,a)$$

- Boltzmann exploration

$$\Pr(a) = \frac{e^{\frac{Q(s,a)}{T}}}{\sum_a e^{\frac{Q(s,a)}{T}}}$$

Exploration and Q-learning

- Q-learning converges to optimal Q-values if
  - Every state is visited infinitely often (due to exploration)
  - The action selection becomes greedy as time approaches infinity
  - The learning rate α is decreased fast enough, but not too fast (sufficient conditions for α):

$$(1)\ \sum_n \alpha_n \rightarrow \infty \qquad (2)\ \sum_n (\alpha_n)^2 < \infty$$

Summary

- We can optimize a policy by RL when the transition and reward functions are unknown

- Model free, value based agent:
  - Monte Carlo learning (unbiased, but lots of data)
  - Temporal difference learning (low variance, less data)

- Active learning:
  - Exploration/exploitation dilemma