4 Lorentz Invariance in Wave Equations

4.1 Definition of relativistically invariant

① If the field ϕ satisfies:

$$D\phi(x) = 0 \tag{4.1.1}$$

If we perform a rotation/boost to a different frame of reference, the transformed field satisfies the same equation:

$$D'\phi'(x') = 0 \tag{4.1.2}$$

Then the equation of motion for the field $\phi(x)$ is relativistically invariant.

② An equation of motion is automatically Lorentz invariant if it follows from a Lagrangian that is a Lorentz scalar. If the $\mathscr{L}$ of a system is a Lorentz scalar, then $\mathscr{L}$ satisfies:

$$\mathscr{L}\left[\phi'(x')\right] = \mathscr{L}\left[\phi(x)\right] \tag{4.1.3}$$

Correspondingly,

$$S[\phi'] = S[\phi] \tag{4.1.4}$$

From the $\delta S[\phi] = 0$, we can derive that the equation of motion for the field is relativistically.

Example 1: Consider the K-G theory.
Write an arbitrary Lorentz transformation as:

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{4.1.5}$$

The corresponding transformation of the field:

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x) \tag{4.1.6}$$

Note:

$$\left(\Lambda^{-1}\right)^\mu{}_\nu = \Lambda_\nu{}^\mu; \quad \left(\Lambda^{-1}\right)^\mu{}_\rho \Lambda^\rho{}_\nu = \delta^\mu_\nu \tag{4.1.7}$$

$$g_{\mu\nu}\left(\Lambda^{-1}\right)^\mu{}_\alpha \left(\Lambda^{-1}\right)^\nu{}_\beta = g_{\alpha\beta}; \quad \left(\Lambda^{-1}\right)^\rho{}_\mu \left(\Lambda^{-1}\right)^\sigma{}_\nu g^{\mu\nu} = g^{\rho\sigma} \tag{4.1.8}$$

Check the transformation leaves the form of the Klein-Gordon Lagrangian unchanged.

$$\mathscr{L}_{K-G} = \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}m^2\phi^2 \tag{4.1.9}$$

$$\partial_\mu\phi(x) \to \partial_\mu(\phi(\Lambda^{-1}x)) = \left(\Lambda^{-1}\right)^\nu{}_\mu \left(\partial_\nu\phi\right)\left(\Lambda^{-1}x\right) \tag{4.1.10}$$

Compute the transformation law of the kinetic term:

$$\begin{aligned}
(\partial_\mu\phi(x))^2 &\to g^{\mu\nu}\left(\partial_\mu\phi'(x)\right)\left(\partial_\nu\phi'(x)\right) \\
&= g^{\mu\nu}\left[\left(\Lambda^{-1}\right)^\rho{}_\mu \partial_\rho\phi\right]\left[\left(\Lambda^{-1}\right)^\sigma{}_\nu \partial_\sigma\phi\right]\left(\Lambda^{-1}x\right) \\
&= g^{\rho\sigma}\left(\partial_\rho\phi\right)\left(\partial_\sigma\phi\right)\left(\Lambda^{-1}x\right) \\
&= \left(\partial_\mu\phi\right)^2\left(\Lambda^{-1}x\right)
\end{aligned} \tag{4.1.11}$$

The mass term is simply shifted to the point $(\Lambda^{-1}x)$.
Thus, the whole Lagrangian is transformed as a scalar:

$$\mathscr{L}(x) \to \mathscr{L}\left(\Lambda^{-1}x\right) \tag{4.1.12}$$

The action S is also Lorentz invatiant because it is formed by integrating $\mathscr{L}$ over spacetime.

The equation of motion:

$$\begin{aligned}\left(\partial^2+m^2\right)\phi'(x) &= \left[\left(\Lambda^{-1}\right)^{\mu}{}_{\nu}\partial_\nu\left(\Lambda^{-1}\right)^{\sigma\mu}\partial_\sigma+m^2\right]\phi(\Lambda^{-1}x)\\ &= \left(g^{\nu\sigma}\partial_\nu\partial_\sigma+m^2\right)\phi(\Lambda^{-1}x)\\ &= 0\end{aligned} \tag{4.1.13}$$

Example 2: Vector field and Tensor field.

$$V^\mu(x)\to\Lambda^\mu{}_\nu V^\nu(\Lambda^{-1}x) \tag{4.1.14}$$

$$T^{\mu\nu}\to T'^{\mu\nu}=\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta T^{\alpha\beta} \tag{4.1.15}$$

In general, any equation in which each term has the same set of uncontracted Lorentz indices will naturally be invariant under Lorentz transformations.

Note: For $\Lambda^\mu{}_\nu$:

① 3 rotation parameters $(\theta_x,\theta_y,\theta_z)$:

$$\Lambda^\mu{}_\nu=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&cos\theta_x&sin\theta_x\\0&0&-sin\theta_x&cos\theta_x\end{pmatrix};\quad\begin{pmatrix}1&0&0&0\\0&cos\theta_y&0&-sin\theta_y\\0&0&1&0\\0&sin\theta_y&0&cos\theta_y\end{pmatrix};\quad\begin{pmatrix}1&0&0&0\\0&cos\theta_z&sin\theta_z&0\\0&-sin\theta_z&cos\theta_z&0\\0&0&0&1\end{pmatrix} \tag{4.1.16}$$

② 3 boost parameters $(\beta_x,\beta_y,\beta_z)$:

$$\Lambda^\mu{}_\nu=\begin{pmatrix}cosh\beta_x&sinh\beta_x&0&0\\sinh\beta_x&cosh\beta_x&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix};\quad\begin{pmatrix}cosh\beta_y&0&sinh\beta_y&0\\0&1&0&0\\sinh\beta_y&0&cosh\beta_y&0\\0&0&0&1\end{pmatrix};\quad\begin{pmatrix}cosh\beta_z&0&0&sinh\beta_z\\0&1&0&0\\0&0&1&0\\sinh\beta_z&0&0&cosh\beta_z\end{pmatrix} \tag{4.1.17}$$

where the rapidity:

$$\beta=\frac{1}{2}ln\frac{1+v}{1-v},\quad \beta\approx v \text{ when } v\ll 1 \tag{4.1.18}$$

4.2 Representation of the Lorentz group

If Φ_a is an n component multiplet, the Lorentz transformation law is given by:

$$\Phi_a(x)\to M_{ab}(\Lambda)\Phi_b(\Lambda^{-1}x) \tag{4.2.1}$$

In the following discussion, we will write:

$$\Phi\to M(\Lambda)\Phi \tag{4.2.2}$$

Two successive transformations give a new Lorentz transformation $(\Lambda''=\Lambda'\Lambda)$:

$$\Phi\to M(\Lambda')M(\Lambda)\Phi=M(\Lambda'')\Phi \tag{4.2.3}$$

Then, the matrices M must form an n-dimensional representation of the Lorentz group.

For the 2-dimensional rotation group ($s=\frac{1}{2}$), the matrices are unitary matrices:

$$U=e^{-i\theta^i\sigma^i/2} \tag{4.2.4}$$

where θ^i are three arbitrary parameters and σ^i are the Pauli sigma matrices.

For any continuous group, the transformations that lie infinitesimally close to the identity define a vector space, called the Lie algebra of the group. The basis vectors for this vector space are called the generators of the Lie algebra, or of the group.

For the rotation group, the generators are the angular momentum operators J^i, which satisfy:

$$\left[J^i, J^j\right] = i\epsilon^{ijk}J^k \tag{4.2.5}$$

The finite rotation operations:

$$R = exp\left[-i\theta^i J^i\right] \tag{4.2.6}$$

In the representation of the angular momentum operators:

$$J^i \rightarrow \frac{\sigma^i}{2} \tag{4.2.7}$$

One can find matrix representations of a continuous group by finding matrix representations of the generators of the group, then exponentiating these infinitesimal transformations.

To know the commutation relations of the generators of the group of Lorentz transformations:

For rotation group, in 3-dimensional space:

$$\boldsymbol{J} = \boldsymbol{x} \times \boldsymbol{p} = \boldsymbol{x} \times (-i\nabla) \tag{4.2.8}$$

Write the operators as an antisymmetric tensor:

$$J^i = \frac{1}{2}\epsilon^{ijk}J^{ik} = \frac{1}{2}\epsilon^{ijk}\left[-i\left(x^j\partial^k - x^k\partial^j\right)\right] \tag{4.2.9}$$

which defines:

$$J^{ij} = i\left(x^i\nabla^j - x^j\nabla^i\right) \tag{4.2.10}$$

so that $J^3 = J^{12}$. The generalization to 4-dimensional Lorentz transformations:

$$J^{\mu\nu} = i\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right) \tag{4.2.11}$$

The six operators generate the three boosts and three rotations of the Lorentz group.

The commutators of the differential operators:

$$\left[J^{\mu\nu}, J^{\rho\sigma}\right] = i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right) \tag{4.2.12}$$

For a infinitesimal transformation:

$$\Lambda^\mu_{\ \nu} = \delta^\mu_{\ \nu} + \omega^\mu_{\ \nu}; \quad \omega_{\mu\nu} = g_{\mu\sigma}\omega^\sigma_{\ \nu} \tag{4.2.13}$$

Consider the 4×4 matrices:

$$\left(J^{\mu\nu}\right)_{\alpha\beta} = i\left(\delta^\mu_{\ \alpha}\delta^\nu_{\ \beta} - \delta^\mu_{\ \beta}\delta^\nu_{\ \alpha}\right) \tag{4.2.14}$$

Then for a Lorentz 4-vector:

$$V^\alpha \rightarrow \left(\delta^\alpha_\beta - \frac{i}{2}\omega_{\mu\nu}\left(J^{\mu\nu}\right)^\alpha_{\ \beta}\right)V^\beta \tag{4.2.15}$$