2 Formalism of Classical Field Theory

2.1 Lagrangian Field Theory

The action in classical mechanics:

$$S = \int L dt = \int \mathscr{L}(\phi, \partial_\mu \phi) d^4x \tag{2.1.1}$$

The principle of least action:

$$\begin{aligned} 0 &= \delta S \\ &= \int d^4x \left\{ \frac{\partial \mathscr{L}}{\partial \phi} \delta \phi + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right\} \\ &= \int d^4x \left\{ \frac{\partial \mathscr{L}}{\partial \phi} \delta \phi - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \right) \delta \phi + \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \delta \phi \right) \right\} \end{aligned} \tag{2.1.2}$$

P.S.

$$[\partial_\mu, \delta] = 0 \tag{2.1.3}$$

$\delta\phi = 0$ at the temporal beginning and end of the region $\Longrightarrow$ the last term is zero.

$$\int d^4x \left\{ \frac{\partial \mathscr{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \right) \right\} \delta\phi = 0 \tag{2.1.4}$$

The integral must vanish for arbitrary $\delta\phi \Longrightarrow$ Euler-Lagrange equation of motion:

$$\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathscr{L}}{\partial \phi} = 0 \tag{2.1.5}$$

2.2 Hamiltonian Field Theory

The conjugate momentum for a discrete system:

$$p \equiv \frac{\partial L}{\partial \dot{q}} \tag{2.2.1}$$

Take the total differential of the Lagrangian function:

$$dL = \sum_i \frac{\partial L}{\partial q_i} dq_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i \tag{2.2.2}$$

According to the Euler-Lagrange equation of motion:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \dot{p} = \frac{\partial L}{\partial q} \tag{2.2.3}$$

Therefore, we get:

$$dL = \sum_i \dot{p}_i dq_i + \sum_i p_i d\dot{q}_i = \sum_i \dot{p}_i dq_i + d\left(\sum_i p_i \dot{q}_i \right) - \sum_i \dot{q}_i dp_i \tag{2.2.4}$$

With some transformation:

$$d\left(\sum_i p_i \dot{q}_i - L \right) = -\sum_i \dot{p}_i dq_i + \sum_i \dot{q}_i dp_i \tag{2.2.5}$$

The Hamiltonian is:

$$H \equiv \sum p\dot{q} - L \tag{2.2.6}$$

Hamilton's canonical equations:

$$\frac{\partial H}{\partial p_i} = \dot{q}_i; \quad -\frac{\partial H}{\partial q_i} = \dot{p}_i \tag{2.2.7}$$

For a continuous system, we can define:

$$\begin{aligned} p(\boldsymbol{x}) \equiv \frac{\partial L}{\partial \dot{\phi}(\boldsymbol{x})} &= \frac{\partial}{\partial \dot{\phi}(\boldsymbol{x})} \int \mathscr{L}\left(\phi(\boldsymbol{y}), \dot{\phi}(\boldsymbol{y})\right) d^3y \\ &\sim \frac{\partial}{\partial \dot{\phi}(\boldsymbol{x})} \sum_{\boldsymbol{y}} \mathscr{L}\left(\phi(\boldsymbol{y}), \dot{\phi}(\boldsymbol{y})\right) d^3y \\ &= \pi(\boldsymbol{x}) d^3x \end{aligned} \tag{2.2.8}$$

where the momentum density conjugate to $\phi(\boldsymbol{x})$:

$$\pi(\boldsymbol{x}) \equiv \frac{\partial \mathscr{L}}{\partial \dot{\phi}(\boldsymbol{x})} \tag{2.2.9}$$

Then the Hamiltonian can be written:

$$H = \sum_{\boldsymbol{x}} p(\boldsymbol{x})\dot{\phi}(\boldsymbol{x}) - L = \int d^3x \left[\pi(\boldsymbol{x})\dot{\phi}(\boldsymbol{x}) - \mathscr{L}\right] \equiv \int d^3x \mathscr{H} \tag{2.2.10}$$

Example of a real-valued field:

$$\mathscr{L} = \frac{1}{2}\dot{\phi}^2(x) - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 \tag{2.2.11}$$

Substituting into the Euler-Lagrange equation of motion:

$$\partial_\mu \left(\frac{\partial\left(\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2\right)}{\partial(\partial_\mu\phi)} \right) - \frac{\partial\left(\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2\right)}{\partial\phi} = 0 \Longrightarrow \partial^\mu\partial_\mu\phi + m^2\phi = 0 \tag{2.2.12}$$

which is the K-G equation (in the scalar field):

$$\left(\partial^\mu\partial_\mu + m^2\right)\phi = 0 \quad \text{or} \quad \left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\phi = 0 \tag{2.2.13}$$

Calculation of Hamiltonian:

$$\pi(\boldsymbol{x}) = \frac{\partial \mathscr{L}}{\partial \dot{\phi}(\boldsymbol{x})} = \dot{\phi}(\boldsymbol{x}) \tag{2.2.14}$$

$$\begin{aligned} H = \int d^3x \mathscr{H} = \int d^3x \left[\pi(\boldsymbol{x})\dot{\phi}(\boldsymbol{x}) - \mathscr{L}\right] &= \int d^3x \left[\pi^2 - \left(\frac{1}{2}\pi^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2\right)\right] \\ &= \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right] \end{aligned} \tag{2.2.15}$$

2.3 Noether's Theorem

The relationship between symmetries and conservation laws in classical field theory.

For an infinitesimal continuous transformation on the field ϕ:

$$\phi(x) \to \phi'(x) = \phi(x) + \alpha\Delta\phi(x) \tag{2.3.1}$$

We call this transformation a symmetry if it leaves the equations of motion invariant.

The Lagrangian should be invariant under this transformation:

$$\mathscr{L}(x) \to \mathscr{L}(x) + \alpha\Delta\mathscr{L} = \mathscr{L}(x) + \alpha\partial_\mu \mathscr{J}^\mu(x) \tag{2.3.2}$$

where

$$\begin{aligned}
\alpha\Delta\mathscr{L} &= \frac{\partial\mathscr{L}}{\partial\phi}(\alpha\Delta\phi) + \left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\right)\partial_\mu(\alpha\Delta\phi) \\
&= \alpha\partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) + \alpha\left[\frac{\partial\mathscr{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\right)\right]\Delta\phi \\
&= \alpha\partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) \qquad \text{(According to the E-L equation)}
\end{aligned} \tag{2.3.3}$$

Then:

$$\alpha\Delta\mathscr{L} = \alpha\partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) = \alpha\partial_\mu\mathscr{J}^\mu(x) \Longrightarrow \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\Delta\phi - \mathscr{J}^\mu(x)\right) = 0 \tag{2.3.4}$$

Then we find a conserved current:

$$\partial_\mu j^\mu(x) = 0, \qquad j^\mu(x) = \frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\phi\right)}\Delta\phi - \mathscr{J}^\mu \tag{2.3.5}$$

The conservation law can also be expressed by saying the charge is a constant in time:

$$Q \equiv \int_{\text{all space}} j^0 d^3x \tag{2.3.6}$$

Example 1: Consider the Lagrangian (ϕ is a complex-valued field):

$$\mathscr{L} = |\partial_\mu\phi|^2 - m^2|\phi|^2 \tag{2.3.7}$$

Under the transformation:

$$\phi \to e^{i\alpha}\phi \tag{2.3.8}$$

The Lagrangian is invariant under this transformation:

$$\mathscr{L} \to \mathscr{L}' = \left|\partial_\mu\phi'\right|^2 - m^2\left|\phi'\right|^2 = \mathscr{L} \tag{2.3.9}$$

For an infinitesimal transformation ($e^{i\alpha} \approx 1 + i\alpha$):

$$\left\{\begin{array}{l}
\phi \to \phi' = \phi + i\alpha\phi \Longrightarrow \Delta\phi = i\phi \\
\phi^* \to \phi^{*'} = \phi^* - i\alpha\phi^* \Longrightarrow \Delta\phi^* = -i\phi^*
\end{array}\right. \tag{2.3.10}$$

where we are treating ϕ and ϕ^* as independent fields.

$$\mathscr{L} \to \mathscr{L}' = \mathscr{L} + \alpha\partial_\mu\mathscr{J}^\mu = \mathscr{L} \Longrightarrow \mathscr{J}^\mu = 0 \tag{2.3.11}$$

Therefore, the conserved Noether current is:

$$\begin{aligned}
j^\mu(x) &= \frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\phi^i\right)}\Delta\phi^i - \mathscr{J}^\mu \\
&= \frac{\partial\left[(\partial^\mu\phi^*)(\partial_\mu\phi)\right]}{\partial\left(\partial_\mu\phi\right)}(i\phi) + \frac{\partial\left[(\partial^\mu\phi^*)(\partial_\mu\phi)\right]}{\partial\left(\partial^\mu\phi^*\right)}(-i\phi^*) \\
&= i\left[(\partial^\mu\phi^*)\phi - \phi^*(\partial^\mu\phi)\right]
\end{aligned} \tag{2.3.12}$$

Example 2:

For an infinitesimal spacetime transformation:

$$x^\mu \to x^\mu - a^\mu \tag{2.3.13}$$

The transformation of the field configuration (with Taylor expession at x):

$$\phi(x) \to \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi(x) \Longrightarrow \Delta\phi = \partial_\mu \phi(x) \tag{2.3.14}$$

The scalar Lagrangian must transform in the same way:

$$\mathscr{L} \to \mathscr{L} + a^\mu \partial_\mu \mathscr{L} = \mathscr{L} + a^\nu \partial_\mu \left(\delta^\nu_\mu \mathscr{L}\right), \quad \text{note: } a^\nu \delta^\mu_\nu = a^\mu \tag{2.3.15}$$

$$\mathscr{L} \to \mathscr{L}' = \mathscr{L} + \alpha \partial_\mu \mathscr{J}^\mu \Longrightarrow \mathscr{J}^\mu = \delta^\nu_\mu \mathscr{L} \tag{2.3.16}$$

Then we obtain the four separately conserved currents (the stress-energy tensor/energy-momentum tensor):

$$T^\mu_\nu \equiv \frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \phi\right)} \partial_\nu \phi - \mathscr{L} \delta^\mu_\nu \tag{2.3.17}$$

For $\mu = 0$:

$$\frac{\partial T^\mu_\nu}{\partial x^\nu} = 0 \Longrightarrow \frac{1}{c}\frac{\partial S_i}{\partial x^i} + \frac{1}{c}\frac{\partial \omega}{\partial t} = 0 \Longrightarrow \frac{\partial S_i}{\partial x^i} = -\frac{\partial \omega}{\partial t} \tag{2.3.18}$$