矢量分析

吴粮宇

2022 年 12 月 16 日

目录

1 Einstein求和约定

① 同一项中，对于重复出现的指标，可以省略求和符号，遍历其取值范围求和。

② 上述成对出现的指标叫做哑指标，简称哑标。表示哑标的小写字母可以用另一对小写字母替换，只要其取值范围不变。

③ 当两个求和式相乘时，两个求和式的哑标不能使用相同的小写字母。

2 Kronecker符号与Levi-Cevita符号

2.1 Kronecker符号

2.1.1 定义

任意两个正交单位向量点积用δ_{ij}表示,即

$$\boldsymbol{e_i} \cdot \boldsymbol{e_j} = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{2.1.1}$$

2.1.2 性质

① 对称性：

$$\delta_{ij} = \delta_{ji} \tag{2.1.2}$$

② 定义单位向量/矩阵：

$$\vec{e}_i = \begin{pmatrix} \delta_{i1} \\ \delta_{i2} \\ \delta_{i3} \end{pmatrix} \tag{2.1.3}$$

$$\boldsymbol{I} = \begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix} \tag{2.1.4}$$

③

$$\delta_{im}\delta_{mj} = \delta_{ij} \tag{2.1.5}$$

2.2 Levi-Cevita符号

2.3 定义

三维情况：

$$\varepsilon_{ijk} = \begin{cases} 1 & ijk = 123, 231, 312 \\ -1 & ijk = 321, 213, 132 \\ 0 & \text{otherwise} \end{cases} \tag{2.3.1}$$

2.3.1 性质

① 定义叉乘：

$$\vec{a}\times\vec{b}=\varepsilon_{ijk}a_ib_j\vec{e}_k \tag{2.3.2}$$

② 与混合积对应：

$$\varepsilon_{ijk}=\vec{e}_i\cdot(\vec{e}_j\times\vec{e}_k)=\begin{vmatrix}\delta_{1i}&\delta_{2i}&\delta_{3i}\\\delta_{1j}&\delta_{2j}&\delta_{3j}\\\delta_{1k}&\delta_{2k}&\delta_{3k}\end{vmatrix} \tag{2.3.3}$$

③ 交换任意两个指标，元素变号：

$$\varepsilon_{ijk}=-\varepsilon_{jik} \tag{2.3.4}$$

④ 符号相乘：

$$\varepsilon_{ijk}\varepsilon_{lmn}=\begin{vmatrix}\delta_{il}&\delta_{im}&\delta_{in}\\\delta_{jl}&\delta_{jm}&\delta_{jn}\\\delta_{kl}&\delta_{km}&\delta_{kn}\end{vmatrix} \tag{2.3.5}$$

Proof:

$$\varepsilon_{lmn}=\begin{vmatrix}\delta_{1l}&\delta_{2l}&\delta_{3l}\\\delta_{1m}&\delta_{2m}&\delta_{3m}\\\delta_{1n}&\delta_{2n}&\delta_{3n}\end{vmatrix}=\begin{vmatrix}\delta_{1l}&\delta_{1m}&\delta_{1n}\\\delta_{2l}&\delta_{2m}&\delta_{2n}\\\delta_{3l}&\delta_{3m}&\delta_{3n}\end{vmatrix} \tag{2.3.6}$$

$$\Longrightarrow\varepsilon_{ijk}\varepsilon_{lmn}=\begin{vmatrix}\delta_{1i}&\delta_{2i}&\delta_{3i}\\\delta_{1j}&\delta_{2j}&\delta_{3j}\\\delta_{1k}&\delta_{2k}&\delta_{3k}\end{vmatrix}\begin{vmatrix}\delta_{1l}&\delta_{1m}&\delta_{1n}\\\delta_{2l}&\delta_{2m}&\delta_{2n}\\\delta_{3l}&\delta_{3m}&\delta_{3n}\end{vmatrix}=\begin{vmatrix}\delta_{il}&\delta_{im}&\delta_{in}\\\delta_{jl}&\delta_{jm}&\delta_{jn}\\\delta_{kl}&\delta_{km}&\delta_{kn}\end{vmatrix} \tag{2.3.7}$$

③推论：

a) 一个指标相同：

$$\varepsilon_{ijk}\varepsilon_{mnk}=\begin{vmatrix}0&\delta_{im}&\delta_{in}\\0&\delta_{jm}&\delta_{jn}\\1&0&0\end{vmatrix}=\delta_{im}\delta_{jn}-\delta_{jm}\delta_{in} \tag{2.3.8}$$

b) 两个指标相同：

$$\varepsilon_{ijk}\varepsilon_{mjk}=\sum_{n=1}^{3}(\delta_{im}\delta_{nn}-\delta_{nm}\delta_{in})=3\delta_{im}-\delta_{im}=2\delta_{im} \tag{2.3.9}$$

c) 三个指标相同：

$$\varepsilon_{ijk}\varepsilon_{ijk}=6 \tag{2.3.10}$$

④ 单位矢量叉乘：

$$\vec{e}_i\times\vec{e}_j=\varepsilon_{ijk}\vec{e}_k \tag{2.3.11}$$

3 δ函数

3.1 定义

$$\delta(\vec{r}) = \begin{cases} 0, & \vec{r} \neq 0 \\ \infty, & \vec{r} = 0 \end{cases} \quad ; \quad \int_{-\infty}^{\infty} \delta(\vec{r}) d\tau = 1 \tag{3.1.1}$$

3.2 两个恒等式

$$\delta(\vec{k}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} e^{i\vec{k}\cdot\vec{r}} d\tau \tag{3.2.1}$$

$$\delta(\vec{r}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} e^{i\vec{k}\cdot\vec{r}} d\tau_k \tag{3.2.2}$$

Proof:

利用函数$f(\vec{r})$傅里叶变换：

$$f(\vec{r}) = \int_{-\infty}^{\infty} f(\vec{k}) e^{-i\vec{k}\cdot\vec{r}} d\tau_k \tag{3.2.3}$$

$$f(\vec{k}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} f(\vec{r}) e^{i\vec{k}\cdot\vec{r}} d\tau \tag{3.2.4}$$

令$f(\vec{k}) = \delta(\vec{k})$，代入上述两个式子即可得到上面的恒等式。

3.3 性质

- $\delta(\vec{r}) = \delta(-\vec{r})$
- $\delta(a\vec{r}) = \dfrac{1}{|a|^3}\delta(\vec{r})$
- $\displaystyle\int_{-\infty}^{\infty} \delta(\vec{r}) f(\vec{r}) d\tau = f(0)$

4 矢量的代数运算

4.1 加法与减法

① 满足平行四边形法则。

② 数学定义：

$$\vec{a} + \vec{b} = a_i \vec{e}_i + b_i \vec{e}_i = (a_i + b_i)\, \vec{e}_i \tag{4.1.1}$$

4.2 数乘

$$\lambda \vec{a} = \lambda\,(a_i \vec{e}_i) = (\lambda a_i)\, \vec{e}_i \tag{4.2.1}$$

4.3 数量积(内积)

$$\vec{a}\cdot\vec{b}=\delta_{ij}a_ib_j=a_ib_i=|\vec{a}||\vec{b}|\cos(\vec{a},\vec{b}) \tag{4.3.1}$$

4.4 矢量积(叉积)

$$\vec{a}\times\vec{b}=\varepsilon_{ijk}a_ib_j\vec{e}_k=\begin{vmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix} \tag{4.4.1}$$

性质：

① 反交换律：

$$\vec{a}\times\vec{b}=-\vec{b}\times\vec{a} \tag{4.4.2}$$

② 图像：

$$\left|\vec{a}\times\vec{b}\right|=|\vec{a}||\vec{b}|\sin(\vec{a},\vec{b}) \tag{4.4.3}$$

4.5 并积

$$\vec{a}\vec{b}=a_ib_j\vec{e}_i\vec{e}_j=\begin{pmatrix}a_1b_1 & a_1b_2 & a_1b_3\\ a_2b_1 & a_2b_2 & a_2b_3\\ a_3b_1 & a_3b_2 & a_3b_3\end{pmatrix} \tag{4.5.1}$$

对于单位矢量：

$$\vec{e}_i\vec{e}_j=\begin{pmatrix}\delta_{i1}\\ \delta_{i2}\\ \delta_{i3}\end{pmatrix}\begin{pmatrix}\delta_{j1} & \delta_{j2} & \delta_{j3}\end{pmatrix}=\begin{pmatrix}\delta_{i1}\delta_{j1} & \delta_{i1}\delta_{j2} & \delta_{i1}\delta_{j3}\\ \delta_{i2}\delta_{j1} & \delta_{i2}\delta_{j2} & \delta_{i2}\delta_{j3}\\ \delta_{i3}\delta_{j1} & \delta_{i3}\delta_{j2} & \delta_{i3}\delta_{j3}\end{pmatrix} \tag{4.5.2}$$

4.6 三重标积(混合积)

$$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=a_l\vec{e}_l\cdot\varepsilon_{ijk}b_ic_j\vec{e}_k=\varepsilon_{ijk}a_lb_ic_j\delta_{lk}=\varepsilon_{ijk}a_kb_ic_j=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix} \tag{4.6.1}$$

性质：

$$\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=\vec{b}\cdot(\vec{c}\times\vec{a})=\vec{c}\cdot\left(\vec{a}\times\vec{b}\right) \tag{4.6.2}$$

4.7 三重矢积

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=(\vec{a}\cdot\vec{c})\,\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c} \tag{4.7.1}$$

Proof:

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\varepsilon_{ijk}a_i\left(\vec{b}\times\vec{c}\right)_j\vec{e}_k=\varepsilon_{ijk}a_i\left(\varepsilon_{mnj}b_mc_n\right)\vec{e}_k \tag{4.7.2}$$

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=-\varepsilon_{ikj}\varepsilon_{mnj}a_ib_mc_n\vec{e}_k \tag{4.7.3}$$

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\delta_{in}\delta_{km}a_ib_mc_n\vec{e}_k-\delta_{im}\delta_{kn}a_ib_mc_n\vec{e}_k \tag{4.7.4}$$

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=a_ic_ib_k\vec{e}_k-a_ib_ic_k\vec{e}_k=(\vec{a}\cdot\vec{c})\,\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c} \tag{4.7.5}$$

5 矢量的微积分

5.1 微分

5.1.1 一元情况

微分：

$$d\vec{A}=\vec{A}\,(t+dt)-\vec{A}(t) \tag{5.1.1}$$

导数：

$$\vec{A}'(t)=\frac{d\vec{A}}{dt}=\lim_{\Delta t\to 0}\frac{\vec{A}\,(t+\Delta t)-\vec{A}(t)}{\Delta t} \tag{5.1.2}$$

将$\vec{A}$写作$a_i\vec{e}_i$，当$\vec{e}_i$是常矢量时：

$$\frac{d\vec{A}}{dt}=\frac{d}{dt}\left(a_i\vec{e}_i\right)=\frac{da_i}{dt}\vec{e}_i \tag{5.1.3}$$

5.1.2 多元情况

对于$\vec{A}(x_1,x_2,x_3)$

$$\frac{\partial\vec{A}}{\partial x_1}=\lim_{\Delta x_1\to 0}\frac{\vec{A}\,(x_1+\Delta x_1,x_2,x_3)-\vec{A}(x_1,x_2,x_3)}{\Delta x_1} \tag{5.1.4}$$

$$\frac{\partial\vec{A}}{\partial x_j}=\frac{\partial}{\partial x_j}\left(a_i\vec{e}_i\right)=\frac{\partial a_i}{\partial x_j}\vec{e}_i \tag{5.1.5}$$

全微分：

$$d\vec{A}=\frac{\partial a_i}{\partial x_j}dx_j\vec{e}_i \tag{5.1.6}$$

5.1.3 求导法则

- $\frac{d}{dt}\boldsymbol{C}=\boldsymbol{0}$
- $\frac{d}{dt}(\boldsymbol{A}\pm\boldsymbol{B})=\frac{d\boldsymbol{A}}{dt}\pm\frac{d\boldsymbol{B}}{dt}$
- $\frac{d}{dt}(k\boldsymbol{A})=k\frac{d\boldsymbol{A}}{dt}$
- $\frac{d}{dt}(u\boldsymbol{A})=\frac{du}{dt}\boldsymbol{A}+u\frac{d\boldsymbol{A}}{dt}$
- $\frac{d}{dt}(\boldsymbol{A}\cdot\boldsymbol{B})=\frac{d\boldsymbol{A}}{dt}\cdot\boldsymbol{B}+\boldsymbol{A}\cdot\frac{d\boldsymbol{B}}{dt}$
- $\frac{d}{dt}(\boldsymbol{A}\times\boldsymbol{B})=\frac{d\boldsymbol{A}}{dt}\times\boldsymbol{B}+\boldsymbol{A}\times\frac{d\boldsymbol{B}}{dt}$
- $\frac{d\boldsymbol{A}}{dt}=\frac{d\boldsymbol{A}}{du}\frac{du}{dt}$

5.2 积分

5.2.1 定积分

$$\int_a^b \vec{A}(t)dt=\lim_{max\{\Delta t_i\}\to 0}\sum_{i=1}^n \vec{A}(t_i)\Delta t_i \tag{5.2.1}$$

$$\int_a^b \vec{A}(t)dt=\int_a^b (a_i\vec{e}_i)\,dt=\left(\int_a^b a_i dt\right)\vec{e}_i \tag{5.2.2}$$

5.2.2 推论

$$\vec{A}\cdot d\vec{A}=\left|\vec{A}\right| d\left|\vec{A}\right| \tag{5.2.3}$$

Proof:

$$\begin{aligned}\vec{A}\cdot d\vec{A}&=(a_i\vec{e}_i)\cdot d\,(a_j\vec{e}_j)\\&=a_i da_i\,(\vec{e}_i\cdot\vec{e}_i)\\&=\frac{1}{2}d\,(a_i\cdot a_i)\\&=\frac{1}{2}d\left|\vec{A}\right|^2=\left|\vec{A}\right| d\left|\vec{A}\right|\end{aligned} \tag{5.2.4}$$

5.2.3 分部求导

① $\mathrm{d}(\vec{u}\cdot\vec{v})=\mathrm{d}\vec{u}\cdot\vec{v}+\vec{u}\cdot\mathrm{d}\vec{v}$

② $\mathrm{d}(\vec{u}\times\vec{v})=\mathrm{d}\vec{u}\times\vec{v}+\vec{u}\times\mathrm{d}\vec{v}$

③ $\mathrm{d}(f\vec{u})=\mathrm{d}f\vec{u}+f\mathrm{d}\vec{u}$

6 梯度、旋度与散度

6.1 梯度

6.1.1 定义

$$grad\, f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix} f = \partial_i f \vec{e}_i \tag{6.1.1}$$

P.S.:

Hamilton算符：

$$\nabla \equiv \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix} = \partial_i \vec{e}_i \tag{6.1.2}$$

$$grad\, f = \nabla f \tag{6.1.3}$$

Note:

该算符同时具有矢量性与微分性

6.1.2 物理意义

标量场的变化速度。

6.1.3 性质

① $\mathrm{d}f = \partial_i f \mathrm{d}x_i = \nabla f \cdot \mathrm{d}\vec{r}$

② $\nabla f \cdot \hat{n} = \frac{\partial f}{\partial n}$; $\nabla f \cdot \vec{e}_i = \frac{\partial f}{\partial x_i}$

③ $\nabla(\alpha f + \beta g) = \alpha \nabla f + \beta \nabla g$

④ $\nabla(fg) = f \nabla g + g \nabla f$

⑤ $\nabla f(r) = \frac{\partial f}{\partial r} \nabla r$

⑥ $\oint \nabla f \cdot d\vec{r} = \oint df = 0$

⑦ $\nabla r = \frac{\vec{r}}{r} = \hat{r}$

⑧ $\nabla(\vec{a} \cdot \vec{r}) = \vec{a}$ （$\vec{a}$为常矢量）

6.2 散度

6.2.1 定义

$$div\, \vec{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} = \partial_i F_i = \nabla \cdot \vec{F} \tag{6.2.1}$$

P.S.

Gauss公式：

$$\oiint \vec{F}\cdot d\vec{S} = \iiint \left(div\,\vec{F}\right) dV \tag{6.2.2}$$

6.2.2 物理意义

判断矢量场的奇性(有源/无源)。

6.2.3 性质

① $\nabla\cdot\left(\alpha\vec{F}+\beta\vec{G}\right) = \alpha\nabla\cdot\vec{F}+\beta\nabla\cdot\vec{G}$

② $\nabla\cdot\left(f\vec{F}\right) = \nabla f\cdot\vec{F}+f\nabla\cdot\vec{F}$

③ $\nabla\cdot\vec{F}(r) = \nabla r\cdot\dfrac{\partial\vec{F}}{\partial r}$

④ $\nabla\cdot\vec{r} = 3$

6.3 旋度

6.3.1 定义

$$rot\,\vec{F} = curl\,\vec{F} = \begin{pmatrix} \dfrac{\partial F_3}{\partial x_2}-\dfrac{\partial F_2}{\partial x_3} \\ \dfrac{\partial F_1}{\partial x_3}-\dfrac{\partial F_3}{\partial x_1} \\ \dfrac{\partial F_2}{\partial x_1}-\dfrac{\partial F_1}{\partial x_2} \end{pmatrix} = \nabla\times\vec{F} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ F_1 & F_2 & F_3 \end{vmatrix} = \varepsilon_{ijk}\partial_i F_j \vec{e}_k \tag{6.3.1}$$

P.S.:

Stokes公式：

$$\oiint \vec{F}\cdot d\vec{r} = \iint \left(rot\,\vec{F}\right)\cdot d\vec{S} \tag{6.3.2}$$

6.3.2 物理意义

当空间位置改变时，场方向“扭曲”的速率，即矢量场旋转的趋势。

6.3.3 性质

① $rot\,(\boldsymbol{A}\pm\boldsymbol{B}) = rot\,\boldsymbol{A}\pm rot\,\boldsymbol{B}$

② $\nabla\times\left(f\vec{F}\right) = \nabla f\times\vec{F}+f\nabla\times\vec{F}$

③ $\nabla\times\vec{F}(r) = \nabla r\times\dfrac{\partial\vec{F}}{\partial r}$

④ $\nabla\times\vec{r} = 0$

6.4 算符与矢量的并积

$$\nabla\vec{F}=\begin{pmatrix}\partial_1\\ \partial_2\\ \partial_3\end{pmatrix}\begin{pmatrix}F_1 & F_2 & F_3\end{pmatrix}=\begin{pmatrix}\partial_1F_1 & \partial_1F_2 & \partial_1F_3\\ \partial_2F_1 & \partial_2F_2 & \partial_2F_3\\ \partial_3F_1 & \partial_3F_2 & \partial_3F_3\end{pmatrix}=\partial_iF_j\vec{e}_i\vec{e}_j \tag{6.4.1}$$

6.5 Laplace算符

$$\Delta=\nabla^2=\nabla\cdot\nabla \tag{6.5.1}$$

$$\nabla^2=\delta_{ij}\partial_i\partial_j=\partial_i\partial_i \tag{6.5.2}$$

$$\nabla^2f=\nabla\cdot(\nabla f)\qquad;\qquad\nabla^2\vec{F}=\nabla\cdot\left(\nabla\vec{F}\right) \tag{6.5.3}$$

6.6 其他矢量公式

6.6.1 梯度场无旋

$$\nabla\times(\nabla f)\equiv 0 \tag{6.6.1}$$

6.6.2 旋度场无源

$$\nabla\cdot\left(\nabla\times\vec{F}\right)\equiv 0 \tag{6.6.2}$$

6.6.3 旋度的旋度

$$\nabla\times\left(\nabla\times\vec{F}\right)=\nabla\left(\nabla\cdot\vec{F}\right)-\nabla^2\vec{F} \tag{6.6.3}$$

6.6.4 点乘的梯度

$$\nabla\left(\vec{a}\cdot\vec{b}\right)=(\vec{a}\cdot\nabla)\,\vec{b}+\left(\vec{b}\cdot\nabla\right)\vec{a}+\vec{a}\times\left(\nabla\times\vec{b}\right)+\vec{b}\times(\nabla\times\vec{a}) \tag{6.6.4}$$

6.6.5 叉乘的散度

$$\nabla\cdot\left(\vec{a}\times\vec{b}\right)=\vec{b}\cdot(\nabla\times\vec{a})-\vec{a}\cdot\left(\nabla\times\vec{b}\right) \tag{6.6.5}$$

6.6.6 叉乘的旋度

$$\nabla \times \left(\vec{a} \times \vec{b}\right) = \vec{a}\left(\nabla \cdot \vec{b}\right) - \vec{b}\,(\nabla \cdot \vec{a}) + \left(\vec{b} \cdot \nabla\right)\vec{a} - (\vec{a} \cdot \nabla)\,\vec{b} \tag{6.6.6}$$

7 并矢的几个运算法则

7.1 矢量点乘并矢

- $\vec{A}\cdot\left(\vec{B}\vec{C}\right) = \left(\vec{A} \cdot \vec{B}\right)\vec{C}$
- $\left(\vec{B}\vec{C}\right) \cdot \vec{A} = \vec{B}\left(\vec{C} \cdot \vec{A}\right)$

7.2 双点积

$$\left(\vec{A}\vec{B}\right) : \left(\vec{C}\vec{D}\right) = \left(\vec{B} \cdot \vec{C}\right)\left(\vec{A} \cdot \vec{D}\right) \tag{7.2.1}$$

7.3 叉积

$$\vec{A} \times \vec{e}_i\vec{e}_j = \left(\vec{A} \times \vec{e}_i\right)\vec{e}_j \tag{7.3.1}$$

$$\vec{e}_i\vec{e}_j \times \vec{A} = \vec{e}_i\left(\vec{e}_j \times \vec{A}\right) \tag{7.3.2}$$

7.4 旋度与散度

$$\nabla \times \left(\vec{A}\vec{B}\right) = \left(\nabla \times \vec{A}\right)\vec{B} - \left(\vec{A} \times \nabla\right)\vec{B} \tag{7.4.1}$$

$$\nabla \cdot \left(\vec{A}\vec{B}\right) = \left(\nabla \cdot \vec{A}\right)\vec{B} - \left(\vec{A} \cdot \nabla\right)\vec{B} \tag{7.4.2}$$

7.5 梯度与散度

$$\nabla\varphi = \nabla \cdot (\varphi \boldsymbol{I}) \tag{7.5.1}$$

8 常用正交曲线坐标系中的梯度算符

8.1 直角坐标系

- $\nabla = \vec{e}_x\dfrac{\partial}{\partial x} + \vec{e}_y\dfrac{\partial}{\partial y} + \vec{e}_z\dfrac{\partial}{\partial z}$

对于任意矢量函数$\vec{a}=a_x\vec{e}_x+a_y\vec{e}_y+a_z\vec{e}_z$和任意标量函数$\Psi$:

- $\nabla\cdot\vec{a}=\frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}$
- $\nabla\times\vec{a}=\vec{e}_x\left(\frac{\partial a_z}{\partial y}-\frac{\partial a_y}{\partial z}\right)+\vec{e}_y\left(\frac{\partial a_x}{\partial z}-\frac{\partial a_z}{\partial x}\right)+\vec{e}_z\left(\frac{\partial a_y}{\partial x}-\frac{\partial a_x}{\partial y}\right)$
- $\nabla^2\Psi=\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}+\frac{\partial^2\Psi}{\partial z^2}$

8.2 柱坐标系

- $\nabla=\vec{e}_\rho\frac{\partial}{\partial\rho}+\vec{e}_\phi\frac{\partial}{\rho\partial\phi}+\vec{e}_z\frac{\partial}{\partial z}$

单位矢量变换关系：

- $\vec{e}_\rho=cos\phi\vec{e}_x+sin\phi\vec{e}_y$
- $\vec{e}_\phi=-sin\phi\vec{e}_x+cos\phi\vec{e}_y$
- $\vec{e}_x=cos\phi\vec{e}_\rho-sin\phi\vec{e}_\phi$
- $\vec{e}_y=sin\phi\vec{e}_\rho+cos\phi\vec{e}_\phi$

对于任意矢量函数$\vec{a}$和任意标量函数Ψ:

- $\nabla\cdot\vec{a}=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho a_\rho)+\frac{1}{\rho}\frac{\partial a_\phi}{\partial\phi}+\frac{\partial a_z}{\partial z}$
- $\nabla\times\vec{a}=\vec{e}_\rho\left(\frac{1}{\rho}\frac{\partial a_z}{\partial\phi}-\frac{\partial a_\phi}{\partial z}\right)+\vec{e}_\phi\left(\frac{\partial a_\rho}{\partial z}-\frac{\partial a_z}{\partial\rho}\right)+\vec{e}_z\frac{1}{\rho}\left(\frac{\partial(\rho a_\phi)}{\partial\rho}-\frac{\partial a_\rho}{\partial\phi}\right)$
- $\nabla^2\Psi=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Psi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Psi}{\partial\phi^2}+\frac{\partial^2\Psi}{\partial z^2}$

8.3 球坐标系

- $\nabla=\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+\vec{e}_\phi\frac{1}{rsin\theta}\frac{\partial}{\partial\phi}$

单位矢量变换关系：

- $\vec{e}_r=sin\theta cos\phi\vec{e}_x+sin\theta sin\phi\vec{e}_y+cos\theta\vec{e}_z$
- $\vec{e}_\theta=cos\theta cos\phi\vec{e}_x+cos\theta sin\phi\vec{e}_y-sin\theta\vec{e}_z$
- $\vec{e}_\phi=-sin\phi\vec{e}_x+cos\phi\vec{e}_y$

对于任意矢量函数$\vec{a}$和任意标量函数Ψ:

- $\nabla\cdot\vec{a}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2a_r\right)+\frac{1}{rsin\theta}\frac{\partial\left(sin\theta a_\theta\right)}{\partial\theta}+\frac{1}{rsin\theta}\frac{\partial a_\phi}{\partial\phi}$

- $\nabla \times \vec{a} = \frac{\vec{e}_r}{rsin\theta}\left(\frac{\partial\left(sin\theta a_\phi\right)}{\partial\theta} - \frac{\partial a_\theta}{\partial\phi}\right) + \frac{\vec{e}_\theta}{r}\left(\frac{1}{sin\theta}\frac{\partial a_r}{\partial\phi} - \frac{\partial\left(ra_\phi\right)}{\partial r}\right) + \frac{\vec{e}_\phi}{r}\left(\frac{\partial(ra_\theta)}{\partial r} - \frac{\partial a_r}{\partial\theta}\right)$
- $\nabla^2\Psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}$