1 History, Pre-QFT

1.1 Difficulties under classical theory

Relativistic quantum mechanics with a fixed particle number, although somewhat successful in its early days, presents three major difficulties:

- Negative energy solutions:

  The relativistic wave equation has negative energy solutions, implying that the system's energy can be arbitrarily low, leading to vacuum instability.

- Negative probability problem:

  The solution of the Klein-Gordon (K-G) equation corresponds to a probability density that is no longer positive definite, necessitating the abandonment of the probabilistic interpretation of the wave function.

- Violation of causality:

  According to relativity, there is no causality between events separated by spacelike intervals. In relativistic wave dynamics, causality is violated.

1.2 Early History

1.2.1 Einstein's photon hypothesis (1905)

$$E = h\nu = \hbar\omega \tag{1.2.1}$$

1.2.2 De Broglie's matter waves (1923)

Influenced by special relativity and the photon hypothesis, De Broglie derived the wave function of a free electron.

The momentum of the free electron:

$$P^\mu = (E, \vec{p}) \tag{1.2.2}$$

Wave function of matter waves (plane wave):

$$\psi = e^{-i\vec{k}\cdot\vec{x}+i\omega t} = e^{-ik^\mu x_\mu} = e^{-ik\cdot x} \tag{1.2.3}$$

Referring to the photon hypothesis, there is a proportional relationship:

$$P^\mu = \hbar k^\mu \Longrightarrow \begin{cases} E = \hbar\omega = h\nu \\ \vec{p} = \hbar\vec{k} = \frac{h}{\lambda} \end{cases} \tag{1.2.4}$$

$$\left.\begin{array}{l} P^2 = P^\mu P_\mu = m^2 \\ P^2 = E^2 - \vec{p}^2 \end{array}\right\} \Longrightarrow E^2 = \vec{p}^2 + m^2 \tag{1.2.5}$$

Therefore, the angular frequency and the wavevector are also not independent:

$$\hbar^2\omega^2 = \hbar^2\vec{k}^2 + m^2 \Longrightarrow \omega^2 = \vec{k}^2 + \left(\frac{m}{\hbar}\right)^2 \tag{1.2.6}$$

1.3 Derivation of the Klein-Gordon (K-G) Equation

1.3.1 Review: Derivation of the Schrödinger Equation

For non-relativistic free particles:

$$E = \frac{\vec{p}^2}{2m} \tag{1.3.1}$$

We can get:

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2\nabla^2}{2m}\psi; \qquad \left(E \to i\hbar\frac{\partial}{\partial t},\ \vec{p} \to -i\hbar\nabla\right) \tag{1.3.2}$$

1.3.2 Attempt at Relativization: The K-G Equation

Relativistic dispersion relation:

$$E^2 = \vec{p}^2 + m^2 \Longrightarrow \left[-\hbar^2 \frac{\partial^2}{\partial t^2} + \hbar^2 \nabla^2 - m^2\right] \psi(\vec{x}, t) = 0 \tag{1.3.3}$$

D'Alembert operator:

$$\Box = g^{\mu\nu} \partial_\mu \partial_\nu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \tag{1.3.4}$$

The K-G equation:

$$\left[\Box + \left(\frac{m}{\hbar}\right)^2\right] \psi(\vec{x}, t) = 0 \tag{1.3.5}$$

The plain-wave solution:

$$\psi(\vec{x}, t) = e^{\frac{i}{\hbar} p \cdot x} \tag{1.3.6}$$

Substituting into the K-G equation, we again obtain the mass-energy relation:

$$E^2 = \vec{p}^2 + m^2 \Longrightarrow E = \pm\sqrt{\vec{p}^2 + m^2} \tag{1.3.7}$$

Negative energy solutions appear, which is not a problem in classical SR since energy cannot continuously change from positive to negative.

In the quantum mechanics world: quantum transitions can directly transition from a positive energy state to a negative energy state. This implies that there is no stable ground state $\rightarrow$ the vacuum becomes unstable.

In NRQM, the continuity equation (probability current conservation) can be derived:

$$\frac{\partial}{\partial t} |\psi|^2 + \nabla \cdot \left[\frac{i\hbar}{2m} \left(\psi^* \nabla \psi - \psi \nabla \psi^*\right)\right] = 0 \tag{1.3.8}$$

Define:

$$\rho = |\psi|^2 ; \quad \vec{j} = \frac{i\hbar}{2m} \left(\psi^* \nabla \psi - \psi \nabla \psi^*\right) \Longrightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0 \tag{1.3.9}$$

where $\rho = |\psi|^2$ is positive definite.

For K-G equation:

$$\left.\begin{array}{l} \rho = NIm\left(\psi^* \frac{\partial}{\partial t} \psi\right) \\ \vec{j} = Nc^2 Im\left(\psi^* \nabla \psi\right) \end{array}\right\} \Longrightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0 \tag{1.3.10}$$

Now, since $\frac{\partial^2}{\partial t^2}$ is included in the K-G equation, ρ is not necessarily positive definite, which means negative probabilities may arise.

1.3.3 The Dirac Equation

By introducing matrices, the wave function ψ is expanded into a column vector.

$$i\hbar \frac{\partial}{\partial t} \psi = H\psi = \left[-i\hbar c \vec{\alpha} \cdot \nabla + \beta m c^2\right] \psi \tag{1.3.11}$$

where $\vec{\alpha} = \alpha^i$ $(i = 1, 2, 3)$ and β are all $n \times n$ matrices to be determined.

Expect that $\left(i\hbar\frac{\partial}{\partial t}\right)^2\psi = H^2\psi$ will come back to K-G equation.

$$\begin{aligned}
H^2 &= \left(-i\hbar c\vec{\alpha}\cdot\nabla + \beta mc^2\right)^2 \\
&= \left(-i\hbar c\sum_i \alpha^i\partial_i + \beta mc^2\right)^2 \\
&= \left(-\hbar^2c^2\sum_{i,j}\frac{1}{2}\{\alpha^i,\alpha^j\}\partial_i\partial_j - i\hbar mc^3\sum_i\{\alpha^i,\beta\}\partial_i + m^2c^4\beta^2\right)\psi
\end{aligned} \tag{1.3.12}$$

where $\{A,B\} = AB + BA$.

To compare with the K-G equation:

$$\{\alpha^i,\alpha^j\} = 2\delta^{ij}I; \quad \sum_i\{\alpha^i,\beta\} = 0; \quad \beta^2 = I \tag{1.3.13}$$

We find that $n = 4$ and define Dirac-Gamma-Matrix:

$$\gamma^\mu = \left(\gamma^0,\vec{\gamma}\right) = (\beta,\beta\vec{\alpha})\,; \quad \text{where}\,\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}I \tag{1.3.14}$$

Therefore, the Dirac equation comes:

$$\left(i\gamma^\mu\partial_\mu - \frac{mc}{\hbar}\right)\psi = 0 \tag{1.3.15}$$

Now,

$$\vec{j} = c\psi^\dagger\vec{\alpha}\psi; \quad \rho = \psi^\dagger\psi = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2 \geq 0 \tag{1.3.16}$$

which shows that ρ is positive definite.

Dirac Sea:

Electrons in negative energy states fill the Dirac Sea. According to Pauli's exclusion principle, electrons in positive energy states cannot transition to negative energy states, thus preventing the problem of vacuum instability.

1.4 Causality Violation

The amplitude for a free particle to propagate from $\vec{x}_0$ to $\vec{x}$:

$$U(t) = \langle\vec{x}|\,e^{-iHt}\,|\vec{x}_0\rangle \tag{1.4.1}$$

1.4.1 NRQM

$$\begin{aligned}
E = \frac{\vec{p}^2}{2m} \Longrightarrow U(t) &= \langle\vec{x}|\,e^{-i\frac{\vec{p}^2}{2m}t}\,|\vec{x}_0\rangle \\
&= \int\frac{d^3p}{(2\pi)^3}\langle\vec{x}|\,e^{-i\frac{\vec{p}^2}{2m}t}\,|\vec{p}\rangle\langle\vec{p}|\vec{x}_0\rangle \\
&= \frac{1}{(2\pi)^3}\int d^3p e^{-i\frac{\vec{p}^2}{2m}t}e^{i\vec{p}\cdot(\vec{x}-\vec{x}_0)} \\
&= \left(\frac{m}{2\pi it}\right)^{\frac{3}{2}} e^{\frac{im}{2t}(\vec{x}-\vec{x}_0)^2}
\end{aligned} \tag{1.4.2}$$

For all $\vec{x}$ and t this expression is nonzero, which indicates that a particle can propagate between any two points in an arbitrarily short time.

1.4.2 Relativistic-QM

$$
\begin{aligned}
E = \sqrt{\vec{p}^2 + m^2} \Longrightarrow U(t) &= \langle \vec{x} | \, e^{-it\sqrt{\vec{p}^2+m^2}} \, | \vec{x}_0 \rangle \\
&= \frac{1}{(2\pi)^3} \int d^3p e^{-it\sqrt{\vec{p}^2+m^2}} e^{i\vec{p}\cdot(\vec{x}-\vec{x}_0)} \\
&= \frac{1}{2\pi^2 |\vec{x}-\vec{x}_0|} \int_0^\infty dppsin(p|\vec{x}-\vec{x}_0|) e^{-it\sqrt{p^2+m^2}}
\end{aligned}
\tag{1.4.3}
$$

For $|\vec{x}-\vec{x}_0|^2 \gg t^2$:

$$
U(t) \sim e^{-m\sqrt{|\vec{x}-\vec{x}_0|^2 - t^2}} \tag{1.4.4}
$$

The amplitude is small but still nonzero outside the light-cone.

1.4.3 QFT

More details later, but QFT does slove the causality problem. (particle and antiparticle)