Differential Equations

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2022 年 2 月 15 日

目录

1 Chapter 1. Introduction to Ordinary Differential Equations

1.1 Basic concepts of ODE

1.1.1 n-th order ODE

$$F(x, y, y', y'', \cdots, y^{(n)}) = 0$$

n represents the highest order of derivative.

1.1.2 linear or non-linear

E.g.

$$y'' + xy = 0, \quad \text{2-nd order linear ODE.}$$
$$y' + e^x y^2 = 0, \quad \text{1-st order non-linear ODE.}$$

1.1.3 solutions of n-th order ODE $F(x, y, \cdots, y^{(n)}) = 0$

general remarks:

①if exist it's a n-times differentiable function of x.

②explicit solution.

③implicit solution.

④general solution and particular solution.

1.1.4 additional constraints in physics problem

1.1.4.1 IVP e.g. $y(x_0) = y_0, y'(x_0) = y_1, \cdots, y^{(n-1)}(x_0) = y_{n-1}$

1.1.4.2 BVP e.g.

$$y(x_0) + h_{01}y'(x_0) + \cdots + h_{0(n-1)}y^{(n-1)}(x_0) = u_0$$
$$y(x_1) + h_{11}y'(x_1) + \cdots + h_{1(n-1)}y^{(n-1)}(x_1) = u_1$$

1.1.5 Typical ODE

a) first-order ODE

b) linear ODE with constant coefficients

c) linear ODE with non-constant coefficients

1.2 First-order ODEs $(F(x, y, y') = 0)$

1.2.1 Autonomous

$$\frac{dy}{dx} = f(y) \qquad \text{(c is called a critical point if it is zero of } f(y)\text{,namely } f(c) = 0)$$

Obviously then $y(x) = c$ is a constant solution of the automonous ODE.
$X = A + \int \frac{dy}{f(y)}$, with A be an arbitrary constant
c: attractor/repeller/semistable

1.2.2 Seperable variable

1.2.2.1 $\frac{dy}{dx} = g(x)h(y) \implies \frac{dy}{h(y)} = g(x)dx$
$\implies \int \frac{dy}{h(y)} = \int g(x)dx + C$
(note if r is the zero of h(y),then there also exist constant solution of y(x)=r)

1.2.2.2 $\frac{dy}{dx} = f(ax + by)$ changing variable,first,z=ax+by.
$\implies \frac{dy}{dx} = \frac{\frac{dz}{dx} - a}{b} = f(z)$
$\implies \frac{dz}{dx} = bf(z) + a$
$\implies x = \int \frac{dz}{bf(z) + a} + C$

1.2.2.3 $\frac{dy}{dx} = f(\frac{y}{x})$ Assuming z=$\frac{y}{x}$
$\implies \frac{dy}{dx} = z + x\frac{dz}{dx} = f(z)$
again using seperable variables:

$$x=C\exp(\int \frac{dz}{f(z) - z})$$

1.2.2.4 $\frac{dy}{dx} = f(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2})$ using substitution $X = x - x_1, Y = y - y_1$
with x_1, y_1satisfy

$$\begin{cases} a_1x_1 + b_1y_1 + c_1 = 0 \\ a_2x_1 + b_2y_1 + c_2 = 0 \end{cases}$$

then with the new variables
$\frac{dy}{dx} = \frac{dY}{dX} = f(\frac{a_1X + b_1Y}{a_2X + b_2Y})$
$\implies \frac{dY}{dX} = f(\frac{a_1 + b_1\frac{Y}{X}}{a_2 + b_2\frac{Y}{X}}) = F(\frac{Y}{X})$ -case 3.

1.2.3 Solution of linear first-order ODE

$y'(x) + p(x)y(x) = g(x)$, standard form (coefficient of $y'(x) = 1$)

suppose p(x) and g(x) are continuous on I.

$\Longrightarrow y(x) = e^{-\int p(x)dx}(C + \int g(x)e^{\int p(x)dx}dx)$

Proof:

first let $\mu(x) = exp(\int p(x)dx)$, note $\mu(x) > 0$, $\mu'(x) = p(x)\mu(x)$.

$$y'(x) + p(x)y(x) = g(x) \Longleftrightarrow \mu(x)y'(x) + \mu(x)p(x)y(x) = \mu(x)g(x)$$

by the product rule: $(\mu(x)y(x))' = \mu(x)g(x)$

$\Longrightarrow \mu(x)y(x) = \int \mu(x)g(x)dx + C$

$\Longrightarrow y(x) = e^{-\int p(x)dx}(C + \int g(x)e^{\int p(x)dx}dx)$

1.2.4 Equation $F(x, y, y') = 0$ can not be solved wrt. y'

1.2.4.1 $F(y') = 0 \quad \Longrightarrow y' = k_i$ (zeros of F(x)),

thus $y = k_i x + C$

1.2.4.2 $F(x, y') = 0$ Assuming parametric solution of $F(p, q) \equiv 0$ being $p = \varphi(t), q = \Psi(t)$,

thus $x(t) = \varphi(t), y(t) = \int \Psi(t)\varphi'(t)dt + C$

E.g.

$x^2 + (y')^2 = 1 \qquad$ let $x = cost, \quad y' = sint$

thus $dy = sintdx = -sin^2tdt \Longrightarrow y(t) = \frac{1}{4}sin2t - \frac{t}{2} + C$

1.2.4.3 $F(y, y') = 0$ Assuming parametric solution of $F(p, q) \equiv 0$ being $p = \varphi(t), q = \Psi(t)$,

then $y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \varphi'(t)\frac{dt}{dx} = \Psi(t) \Longrightarrow \frac{dx}{dt} = \frac{\varphi'(t)}{\Psi(t)}$

and $x(t) = \int \frac{\varphi'(t)}{\Psi(t)}dt + C, y(t) = \varphi(t)$

E.g.

$y = (y')^4 - (y')^3 + 2 \qquad$ let $y' = t, \quad y = t^4 - t^3 + 2$

then $x(t) = \int \frac{4t^3 - 3t^2}{t}dt + C = \frac{4}{3}t^3 - \frac{3}{2}t^2 + C$

1.2.4.4 $y = f(x, y')$ let $y' = p$, then $y = f(x, p)$, $p = \frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}\frac{dp}{dx}$

$\Longrightarrow p = p(x) \Longrightarrow y = \int p(x)dx + C$

1.2.4.5 $x = f(y, y')$ let $y' = p$, then $x = f(y, p)$, $\frac{1}{p} = \frac{dx}{dy} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial p}\frac{dp}{dy}$

$\Longrightarrow p = p(y) \Longrightarrow x = \int \frac{dy}{p(y)} + C$

1.2.4.6 Bernoulli equation: $y' + P(x)y = Q(x)y^n (n \neq 0, 1)$

$$\Longrightarrow \frac{d(y^{1-n})}{dx} + (1-n)P(x)y^{1-n} = (1-n)Q(x).$$

1.2.5 Exact differential

Considering ODE, $M(x, y)dx + N(x, y)dy = 0$,if left side can be written as

$$df(x, y) = M(x, y)dx + N(x, y)dy,$$

thus solution of ODE can be written as $f(x, y) = C$

(equivalence $\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$)

E.g.

$(12x + 5y - 9)dx + (5x + 2y - 3)dy = 0$

$M(x, y) = \frac{\partial f}{\partial x} = 12x + 5y - 9$

$f(x, y) = \int M(x, y)dx + g(y) = 6x^2 + (5y - 9)x + g(y)$

$\frac{\partial f}{\partial y} = N \Longrightarrow 5x + g'(y) = 5x + 2y - 3 \Longrightarrow g(y) = y^2 - 3y + C$

finally $6x^2 + (5y - 9)x + y^2 - 3y = C$

1.3 Second-order and Higher-order ODEs

1.3.1 method of order reduction

1.3.1.1 $F(x, y^{(k)}, y^{(k-1)}, \cdots, y^{(n)}) = 0$

using substitution $p = y^{(k)}$,$F(x, p, p', \cdots, p^{(n-k)}) = 0$

order is lower by k

1.3.1.2 $F(y, y', y'', \cdots, y^{(n)}) = 0$

define $y' = p$, then $y'' = \frac{dp}{dx} = \frac{dp}{dy}p$, similar for $y^{(n)}$

original ODE reduces to $F(y, p, p', \cdots, p^{(n-1)}) = 0$, order reduced by one

after getting $p = p(y)$

$\Longrightarrow dx = \frac{dy}{p(y)} \Longrightarrow x = \int \frac{dy}{p(y)} + C$

1.3.1.3 $F(x, y, y', \cdots, y^{(n)}) = 0$

and left side is an exact differential of a function

$\Phi(x, y, y', y'', \cdots, y^{(n-1)})$,namely $\dfrac{d\Phi}{dx} = F$

Thus the original ODE is equivalent to $\Phi(x, y, y', y'', \cdots, y^{(n-1)}) = C$

the order of ODE is reduced by one

1.3.1.4 $F(x, y, y', \cdots, y^{(n)}) = 0$

and F is homogeneous wrt. arguments $y, y', \cdots, y^{(n)}$,namely

$F(x, ky, ky', \cdots, ky^{(n)}) = k^p F(x, y, y', \cdots, y^{(n)})$

Then let $y = e^{\int z(x)dx}$,with an unknown function z(x),(assume y>0)

Thus $y' = zy$,$y'' = y(z' + z^2)$,$\cdots$

$F(x, y, y', \cdots, y^{(n)}) = e^{\int z(x)dx} f(x, z, z', \cdots, z^{(n-1)}) = 0$

$\Longrightarrow f(x, z, z', \cdots, z^{(n-1)}) = 0$, order reduced by one

1.3.2 Linear ODEs

standard form:

$y^{(n)} + P_{n-1}(x)y^{(n-1)} + \cdots + P_1(x)y' + P_0(x)y = g(x)$ (leading coefficient of $y^{(n)}$ is 1)

$$linear \quad ODEs \begin{cases} homogeneous, if\ g(x) = 0 \\ nonhomogeneous, otherwise \end{cases}$$

1.3.2.1 The Wronskian of homogeneous linear ODE

Let $y_1, \cdots, y_n$ be any n solutions to the n-th order homogeneous linear ODE as above.

The Wronskian W($y_1, \cdots, y_n$) of these solutions is defined by the following determinant:

$$W(y_1, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

1.3.2.2 THEOREM 1

W(x) satisfy the first-order ODE:

$W'(x) + P_{n-1}(x)W(x) = 0$,for x in I

$\Longrightarrow W(x) = Ce^{-\int P_{n-1}(x)dx}$

[As a concequence, either W(x)≠0 for all x in I, or W(x)≡0 on I.]

Proof:

$$W(x)=\begin{vmatrix} y_1(x) & \cdots & y_n(x) \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix} = \sum_{\sigma} y_{\sigma_0}^{(0)} y_{\sigma_1}^{(1)} \cdots y_{\sigma_{n-1}}^{(n-1)} \cdot (-1)^t$$

easy to show:

$$W'(x)=\sum_{\sigma}\left\{ y_{\sigma_0}^{(1)} y_{\sigma_1}^{(1)} \cdots y_{\sigma_{n-1}}^{(n-1)} + \cdots + y_{\sigma_0}^{(0)} y_{\sigma_1}^{(1)} \cdots y_{\sigma_{n-2}}^{(n-1)} y_{\sigma_{n-1}}^{(n-1)} + y_{\sigma_0}^{(0)} y_{\sigma_1}^{(1)} \cdots y_{\sigma_{n-2}}^{(n-2)} y_{\sigma_{n-1}}^{(n)} \right\} \cdot (-1)^t$$

$$= \begin{vmatrix} y_1(x) & \cdots & y_n(x) \\ \vdots & \ddots & \vdots \\ y_1^{(n-2)}(x) & \cdots & y_n^{(n-2)}(x) \\ y_1^{(n)}(x) & \cdots & y_n^{(n)}(x) \end{vmatrix} = -P_{n-1} W(x)$$

1.3.2.3 THEOREM 2

The homogeneous linear ODE

$y^{(n)} + P_{n-1}(x) y^{(n-1)} + \cdots + P_1(x) y' + P_0(x) y = 0,$

where the coefficient functions $P_j(x)$ are all continous on an interval I, has n solutions $y_1, y_2, \cdots, y_n$ with non-vanishing Wronskian on I.

Furthermore, given any such set $y_1, y_2, \cdots, y_n$ and any solution y,then $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ for a unique choice of constants $c_1, c_2, \cdots, c_n$.

The set of solutions $y_1, y_2, \cdots, y_n$ with nonvanishing Wronskians is called a fundamental set of solutions.

1.3.2.4 THEOREM 3

Suppose that u(x) and v(x) are solutions of the linear homogeneous ODE, and let c and d be any two numbers. Then the linear combination $cu(x) + dv(x)$ is also a solution of the ODE.(also called superposition principle)

1.3.2.5 THEOREM 4

Any solution y of the NH linear ODE has the form: $y = y_h + y_p$.

1.3.2.6 THEOREM 5

IVP with the initial conditions:

$y(x_0) = y_0, y'(x_0) = y_0', \cdots, y^{(n-1)}(x_0) = y_0^{(n-1)}$

Then this problem has a unique solution y on the interval I. The solution exists and is unique.

Proof:

first from THEOREM 4, we know $y_h + y_p = y_p + c_1 y_1 + \cdots + c_n y_n$ is a solution.

thus we just need to solve: $\begin{cases} y_p(x_0) + c_1 y_1(x_0) + \cdots + c_n y_n(x_0) = y_0 \\ \vdots \\ y_p^{(n-1)}(x_0) + c_1 y_1^{(n-1)}(x_0) + \cdots + c_n y_n^{(n-1)}(x_0) = y_0^{(n-1)} \end{cases}$

$$\Leftrightarrow \begin{pmatrix} y_1(x_0) & \cdots & y_n(x_0) \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_0 - y_p(x_0) \\ \vdots \\ y_0^{(n-1)} - y_p^{(n-1)}(x_0) \end{pmatrix}$$

since $\begin{vmatrix} y_1(x_0) & \cdots & y_n(x_0) \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{vmatrix} = W(x_0) \neq 0$

thus the solution exist and is unique to be $\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_1(x_0) & \cdots & y_n(x_0) \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{pmatrix}^{-1} \begin{pmatrix} y_0 - y_p(x_0) \\ \vdots \\ y_0^{(n-1)} - y_p^{(n-1)}(x_0) \end{pmatrix}$

1.3.2.7 THEOREM 6

Let y_1,y_2,$\cdots$,y_n be any n solutions to the n-th order homogeneous linear ODE with coefficients continuous on an interval I. The following are equivalent:

(i)$y_1, y_2, \cdots, y_n$ are linearly independent on I;

(ii)$y_1, y_2, \cdots, y_n$ form a fundamental set of the ODE.

(iii)W($y_1, y_2, \cdots, y_n$)(x_0)$\neq$ 0 for some x_0 in I;

(iv)W($y_1, y_2, \cdots, y_n$)(x_0)$\neq$ 0 for all x in I;

1.4 Linear ODEs with constant coefficients

1.4.1 n-th order homogeneous linear ODEs with constant coefficients

general form : $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$,where each a_j is a constant and $a_n \neq 0$.

Guess a solution like $y = e^{\lambda x}$, substituting into the ODE,

$\Longrightarrow a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0$ (characteristic equation)

Also define $P(\lambda) = a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda + a_0$ as the characteristic polynomial.

1.4.1.1 n distinct real roots

Trivial:

$y_h = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \cdots + c_n e^{\lambda_n x}$

[show linearly independence:

$$W(e^{\lambda_1 x}, e^{\lambda_2 x}, \cdots, e^{\lambda_n x}) = \begin{vmatrix} e^{\lambda_1 x} & \cdots & e^{\lambda_n x} \\ \lambda_1 e^{\lambda_1 x} & \cdots & \lambda_n e^{\lambda_n x} \\ \vdots & \ddots & \vdots \\ \lambda_1^{n-1} e^{\lambda_1 x} & \cdots & \lambda_n^{n-1} e^{\lambda_n x} \end{vmatrix} = (\prod_{i=1}^{n} e^{\lambda_i x}) \begin{vmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_n \\ \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \cdots & \lambda_n^{n-1} \end{vmatrix} = e^{\sum \lambda_i x} \prod_{i>j} (\lambda_i - \lambda_j)$$

$W(e^{\lambda_1 x}, e^{\lambda_2 x}, \cdots, e^{\lambda_n x}) \neq 0$ on interval I.]

1.4.1.2 real roof μ with multiplicity $m \geq 1$

$e^{\mu x}, xe^{\mu x}, \cdots, x^{m-1}e^{\mu x}$ all are solutions and independent.

1.4.1.3 complex root $\mu = \alpha + i\beta$ and $\overline{\mu} = \alpha - i\beta$

$e^{\alpha x}cos\beta x$ and $e^{\alpha x}sin\beta x$ are two linearly independent solutions.

If the paried complex root has m multiplicity, $m \geq 1$,then

$e^{\alpha x}cos\beta x$,$xe^{\alpha x}cos\beta x$,$\cdots$,$x^{m-1}e^{\alpha x}cos\beta x$

$e^{\alpha x}sin\beta x$,$xe^{\alpha x}sin\beta x$,$\cdots$,$x^{m-1}e^{\alpha x}sin\beta x$

all are solutions and independent.(2m in total)

So one solution for each root, we have n in total, forming the fundamental set.

1.4.2 Nonhomogeneous linear ODEs with constant coeffients

$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x)$

according to previous theorem on solutions, we just need to find a particular solution, then $y = y_h + y_p$.

1.4.2.1 The method of undetermined coefficients

To find a particular solution when $g(x) = (b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0)e^{\alpha x}\begin{cases} cos\beta x \\ sin\beta x \end{cases}$

We use $y_p = (A_m x^m + A_{m-1}x^{m-1} + \cdots + A_0)e^{\alpha x}cos\beta x + (B_m x^m + B_{m-1}x^{m-1} + \cdots + B_0)e^{\alpha x}sin\beta x$

providing that no term in the expression of y_p is a solution of the associated homogeneous equation.(If not, we modify above expression by multiplying by x or x^2 or $\cdots$.)

1.4.2.2 The superposition rule

if $g(x) = g_1(x) + g_2(x)$

$F(y_1^{(n)}, \cdots, y_1, x) = g_1(x)$,$F(y_2^{(n)}, \cdots, y_2, x) = g_2(x)$

let $y = y_1 + y_2 \Longrightarrow F(y^{(n)}, \cdots, y, x) = g(x)$

1.5 Linear ODEs with nonconstant coefficients(second-order)

a nonhomogeneous second-order linear ODE in standard form:

$y'' + P(x)y' + Q(x)y = g(x)$

1.5.1 Find solutions for the associated homogeneous ODE with method of Reduction of Order

$$y_2(x) = y_1(x) \int \frac{exp(-\int P(x)dx)}{y_1^2(x)} dx$$

Proof:

Suppose $y_2(x) = y_1(x)v(x)$, substituting to the ODE, we have:

$$v''y_1 + vy_1'' + 2v'y_1' + Pv'y_1 + Pvy_1' + Qy_1v = 0$$

then

$$y_1v'' + 2y_1'v' + y_1Pv' = 0$$

let $z = v'$, we arrive at $z' + \frac{2y_1' + y_1P}{y_1}z = 0$

$\Longrightarrow z = \frac{exp(-\int P(x)dx)}{y_1^2(x)}$

Thus $v(x) = \int \frac{exp(-\int P(x)dx)}{y_1^2(x)}dx$

1.5.2 Find solutions for the original ODE with method of Variation of Parameters

$$y_p = y_1(x)\int \frac{-y_2g(x)}{W(y_1,y_2)}dx + y_2(x)\int \frac{y_1g(x)}{W(y_1,y_2)}dx$$

Proof:

We start with a trial solution of $y_p = u_1(x)y_1 + u_2(x)y_2$

We further assume : $u_1'(x)y_1 + u_2'(x)y_2 = 0$

(That is always allowed since we have two functions to be solved but with only one constraints)

The condition $u_1'(x)y_1 + u_2'(x)y_2 = 0$ also implies:

$$u_1''(x)y_1 + u_2''(x)y_2 = -(u_1'(x)y_1' + u_2'(x)y_2')$$

Now we substituting y_p back to the original ODE.

$y_p'' + P(x)y_p' + Q(x)y_p = y_1'u_1' + y_2'u_2' = g(x)$

Thus we have $\begin{cases} y_1u_1' + y_2u_2' = 0 \\ y_1'u_1' + y_2'u_2' = g(x) \end{cases}$

The determinant of above linear eqautions is just the Wronskian.

Thus it has a unique solution:

$$u_1' = \frac{-y_2g(x)}{W(y_1,y_2)}, \quad u_2' = \frac{y_1g(x)}{W(y_1,y_2)}$$

1.5.3 Euler's Equations

$x^2y'' + \alpha xy' + \beta y = 0$

General solution of Euler's Equation:

Let r_1, r_2 denote the indicial roots, $r^2 + (\alpha - 1)r + \beta = 0$

Then the general solution is given by following cases:

Case I.

If r_1 and r_2 are distinct real roots, then $y = c_1|x|^{r_1} + c_2|x|^{r_2}$

Case II.

If $r_1 = r_2$, then $y = (c_1 + c_2 ln|x|)|x|^{r_1}$

Case III.

If r_1 and r_2 are complex conjugated roots with $r_1 = a + ib$, then $y = |x|^a[c_1 cos(bln|x|) + c_2 sin(bln|x|)]$

And obviously we can drop the absolute values if $x > 0$.

Proof:

Taking $x > 0$ as an example, using change of variables, we define $t = lnx$.

Thus
$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$
$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}$
Substituting back to original ODE, we have:

$$\frac{d^2y}{dt^2} + (\alpha - 1)\frac{dy}{dt} + \beta y(t) = 0$$

That is the linear ODE with constant coefficients.

E.g.

$$(1+x)^2y'' + (1+x)y' + y = 2cos[ln(1+x)]$$

let $t = ln(1+x)$, $\frac{d^2y}{dt^2} + y = 2cost$, $y = tsint + C_1 sint + C_2 cost$
thus $y = ln|1+x|sin(|1+x|) + C_1 sin(|1+x|) + C_2 cos(|1+x|)$

$$x^3y''' + x^2y'' - 2xy' + 2y = 0$$

let $t = lnx$, $y' = \frac{dy}{dt}\frac{1}{x}$, $y'' = \frac{1}{x^2}(\frac{d^2y}{dt^2} - \frac{dy}{dt})$, $y''' = \frac{1}{x^3}(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt})$, thus
$\frac{d^3y}{dt^3} - 2\frac{d^2y}{dt^2} - \frac{dy}{dt} + 2y = 0, \quad \lambda = \pm 1, 2, \quad y = C_1x^2 + C_2|x| + C_3\frac{1}{|x|}$

1.6 System of linear ODEs

$F_j(y_i^{(n)}, y_i^{(n-1)}, \cdots, y_i, x) = 0$

1.6.1 Several methods

A.method of elimination

system of ODEs $\rightarrow$ a single ODE.

B.reversed problem

a single ODE $\rightarrow$ a system.

general case:

$y^{(n)} + P_{n-1}(t)y^{(n-1)} + \cdots + P_1(t)y' + P_0(t)y = g(x)$

let $x_n = y^{(n-1)}, x_{n-1} = y^{(n-2)}, \cdots, x_1 = y.$

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \cdots \\ x_n' = -P_{n-1}(t)x_n - \cdots - P_0(t)x_1 + y \end{cases}$$

C. System of linear first-order ODEs

precise definition:

$$\begin{cases} \dot{x}_1(t) = \sum_{i=1}^{n} a_{1i}(t)x_i(t) + f_1(t) \\ \dot{x}_2(t) = \sum_{i=1}^{n} a_{2i}(t)x_i(t) + f_2(t) \\ \vdots \\ \dot{x}_n(t) = \sum_{i=1}^{n} a_{ni}(t)x_i(t) + f_n(t) \end{cases} \qquad or \quad \dot{X}(t) = A(t)X(t) + F(t) \quad [matrixform]$$

fundamental solution matrix $\Phi(t)$

1.6.2 Special case with constant coefficients

$\dot{X}$(t)=AX (homogeneous)

Theorem: the fundamental solution matrix can be exp[At].

(note matrix exponential exp(B)=I+B+$\frac{1}{2!}B^2 + \cdots$)

Concerning IVP,$X|_{t=0} = X_0$,then

X(t)=exp[At]$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$=exp[At]$X_0$

Case I. A has n distinct real eigenvalues λ_i.

each value with eigenvector v_i,(i=1,$\cdots$,n)

let V=(v_1,v_2,$\cdots$,v_n),

$\Longrightarrow$exp[At]=exp[At]VV^{-1}=$(e^{\lambda_1 t}v_1, \cdots, e^{\lambda_n t}v_n)V^{-1}$

(One may also define $\Phi(t) \equiv exp[At]V$,which is another choice of fundamental solution matrix.)

Case II. A has m($m \leq n$) distinct real eigenvalues λ_i with multiplicity α_i.

From theorem of generalized eigenvectors, solution vectors of $(A-\lambda_i I)^{\alpha_i}u=0$ form a dimentional α_i subspace U_i.

Since $\sum_i \oplus U_i$is complete,

Suppose initial condition $X_0=\sum_{i=1}^{m} u_i$,

then X(t)=exp[At]X_0=$\sum_{i=1}^{m} exp[At]u_i$

$$=\sum_{i=1}^{m} e^{\lambda_i t} exp[(A-\lambda_i I)t]u_i$$

$$=\sum_{i=1}^{m} e^{\lambda_i t}(\sum_{j=0}^{\alpha_j-1}\frac{(A-\lambda_i I)^j t^j}{j!})u_i$$

P.S.:calculate exp[B]:

① diagonal form $UBU^{-1}=\Lambda$,

$\Longrightarrow$ exp[B]=$U^{-1}exp[\Lambda]U$=$U^{-1}\begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n}\end{bmatrix}U$

② Jordan form: $UBU^{-1}=J$

$\Longrightarrow$ exp[B]=$U^{-1}exp[J]U$

exp[J]=exp[λI+J-λI]=exp[λI]exp[J-λI]=$e^{\lambda}[I+(J-\lambda I)+\cdots+\frac{1}{(n-1)!}(J-\lambda I)^{n-1}]$

2 Chapter 2. Series solutions of linear second-order ODEs and special Functions

2.1 Review of Power series

2.1.1 infinite series

①Cauchy's condition;

②d'Alembert's convergence test(ratio test);

③Cauchy's convergence test(root test);

④Gauss test;

2.1.2 power series

①a power series centered at a: $\sum_{n=0}^{\infty} c_n(x-a)^n$,($c_n$ are real numbers);

②Radius of convergence;

③Analytic function:

A function is analytic at point a if it can be represented by a power series centered at a and with a positive or ∞ radius.

Example: take a=0 for simplicity,

$$\sum_{m=0}^{\infty} x^m = \frac{1}{1-x}, \qquad \text{for } |x| < 1$$

$$\sum_{m=0}^{\infty} \frac{x^m}{m!} = e^x, \qquad \text{for } -\infty < x < +\infty$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = sinx, \qquad \text{for } -\infty < x < +\infty$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = cosx, \qquad \text{for } -\infty < x < +\infty$$

$$\sum_{m=1}^{\infty} \frac{(-1)^{m+1} x^m}{m} = ln(1+x), \qquad \text{for } |x| < 1$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2m+1} = arctanx, \qquad \text{for } |x| < 1$$

$$\sum_{m=0}^{\infty} \frac{x^{2m+1}}{(2m+1)!} = sinhx, \qquad \text{for } -\infty < x < +\infty$$

$$\sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!} = coshx, \qquad \text{for } -\infty < x < +\infty$$

④ Operations on power series:

A) linear combinations and products

$f(x) = \sum_{m=0}^{\infty} a_m (x-a)^m$ and $g(x) = \sum_{m=0}^{\infty} b_m (x-a)^m$ centered at a and with radius R_1 and R_2.

Then $\alpha f(x) + \beta g(x) = \sum_{m=0}^{\infty} (\alpha a_m + \beta b_m)(x-a)^m$;

$$f(x) \cdot g(x) = \sum_{m=0}^{\infty} (a_0 b_m + a_1 b_{m-1} + \cdots + a_m b_0)(x-a)^m$$

these new power series have radius at least as large as $\min\{R_1, R_2\}$ and converge to functions on the left.

B) Composition of power series

If $f(x)$ has a power series expansion centered at a, and $g(x)$ has a power series expansion centered at $f(a)$, then $g(f(x))$ has a power series expansion centered at a.

Examples:

$$e^{x^2} = \sum_{m=0}^{\infty} \frac{(x^2)^m}{m!} = \sum_{m=0}^{\infty} \frac{x^{2m}}{m!} \quad (-\infty < x < +\infty);$$

$$\frac{1}{3+2x} = \frac{1}{3}\frac{1}{1+\frac{2x}{3}} = \frac{1}{3}\sum_{m=0}^{\infty}(-\frac{2x}{3})^m = \sum_{m=0}^{\infty} \frac{(-2)^m}{3^{m+1}} x^m \quad (|x| < \frac{3}{2})$$

C) Differentiation term by term

Given a power series and the analytic function it represents: $\quad f(x) = \sum_{m=0}^{\infty} C_m (x-a)^m, \quad |x-a| < R$

then $f(x)$ is differentiable on the same interval, and $f'(x) = \sum_{m=1}^{\infty} m C_m (x-a)^{m-1}, \quad |x-a| < R$

We can repeat it for $f'(x)$, thus $f(x)$ is infinitely differentiable.

D) The Identity Principle

Given $f(x) = \sum_{m=0}^{\infty} a_m (x-a)^m$ and $g(x) = \sum_{m=0}^{\infty} b_m (x-a)^m$ both have a positive radius of convergence.

If $f(x) \equiv g(x)$ on an interval containing a , then $a_m = b_m$ for all m.

(Or, any analytic function at a has a unique power series representation, with coefficients being $\frac{f^{(n)}(a)}{n!}$, thus be its Taylor series.)

E) Integration term by term

$f(x) = \sum_{m=0}^{\infty} C_m (x-a)^m$, for $|x-a| < R$, then

$$\int_a^x f(t)dt = \sum_{m=0}^{\infty} \frac{C_m}{m+1}(x-a)^{m+1}, \text{ for } |x-a| < R$$

F) Shifting index in a power series

$$\sum_{m=s}^{\infty} a_m (x-a)^m = \sum_{m=s-k}^{\infty} a_{m+k}(x-a)^{m+k}$$

$\sum_{m=s}^{\infty} a_m(x-a)^{m+k} = \sum_{m=s+k}^{\infty} a_{m-k}(x-a)^m$

⑤ Smooth function

If a function $f(x)$ is infinitely differentiable (smooth) at an interval containing a, then we can define Taylor series:

$$g(x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(a)}{m!}(x-a)^m$$

not neccessary converge to $f(x)$.

e.g.

$f(x) = \begin{cases} exp(-\frac{1}{x}), & x > 0 \\ 0, & x \leq 0 \end{cases}$

$f^{(n)}(0) \equiv 0$, thus $g(x) \equiv 0 \neq f(x)$ for any small interval containing a.

For analytic function, its Taylor series must converge to the function ifself.

So $f(x)$ is not analytic.

2.2 Series Solution of ODE about ordinary point

Motivation: solving the 2nd-order linear ODE with non-constant coefficients.

$$y'' + P(x)y' + Q(x)y = g(x)$$

(Assuming P(x),Q(x)and g(x) are continuous on interval I.)

Method of power series at ordinary point.

Theorem 1:

Suppose P(x),Q(x) and g(x) have power series expansions at a with non-zero R(analytic at a), then a is called an ordinary point of the ODE, and any solutions of the ODE can be expressed as a power series centered at a (analytic). $y = \sum_{n=0}^{\infty} a_n(x-a)^n$.

We can plug above into ODE to solve a_n and find the general solution.

Moreover, the radius of converge is at least as large as the minimum of those of $P(x)$, $Q(x)$ and $g(x)$.

Advantages of Method of power series:

① can be applied with general case of non-constant coefficients for general solutions

② can find solutions that are not ordinary known functions

③ can be done systematicly in computer programs and also can give approximate numeric solutions.

2.3 Series solution of ODE about singular points

Without loss of generality we assume a=0 in the following:

$$y'' + P(x)y' + Q(x)y = 0$$

Assuming 0 is a singular point of ODE, if both $xP(x)$ and $x^2Q(x)$ are analytic at 0, then we say x=0 is a regular singular point.

We try solutions $y = x^r \sum_{n=0}^{\infty} a_n x^n$, with $a_0 \neq 0$,let $x > 0$, r can be either positive or negative.

We can write $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ in terms of generalized power series.

for power series with negative powers, those operations still valid.

$y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$, $y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$

also rember $xP(x) = \sum_{n=0}^{\infty} p_n x^n$, $x^2Q(x) = \sum_{n=0}^{\infty} q_n x^n$

substitute all above into ODE, we get

$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + (\sum_{n=0}^{\infty} p_n x^n)(\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-2}) + (\sum_{n=0}^{\infty} q_n x^n)(\sum_{n=0}^{\infty} a_n x^{n+r-2}) = 0$

coefficients of all powers must vanish.

power of $x^{r-2} \Longrightarrow (r(r-1) + p_0 r + q_0)a_0 = 0$

power of $x^{r-1} \Longrightarrow r(r+1)a_1 + p_0(r+1)a_1 + p_1 r a_0 + q_0 a_1 + q_1 a_0 = 0$

power of $x^{r-2+m} \Longrightarrow ((m+r)(m+r-1) + p_0(m+r) + q_0)a_m + \cdots$

$\vdots$

since $a_0 \neq 0$, thus

$$r(r-1) + p_0 r + q_0 = 0 \qquad \text{(Indicial equation)}$$

$$\text{(indicial roots, in case of real roots } r_1 \geq r_2\text{)}$$

The strategy,first determine p_0, q_0,$p_0 = xP(x)|_{x=0}, q_0 = x^2Q(x)|_{x=0}$.

Theorem 2(The Frobenius method):

Suppose x=0 is a regular singular point of ODE

$$y'' + P(x)y' + Q(x)y = 0$$

let $r_1 \geq r_2$ denote the two real indicial roots, then the ODE has two linearly independent solutions y_1, y_2 of the form

Case I. If $r_1 - r_2$ is not an integer.

$$y_1 = |x|^{r_1} \sum_{n=0}^{\infty} a_n x^n, \quad y_2 = |x|^{r_2} \sum_{n=0}^{\infty} b_n x^n, \ a_0 \neq 0 \text{ and } b_0 \neq 0$$

Case II. If $r_1 = r_2 = r$,then

$$y_1 = |x|^r \sum_{n=0}^{\infty} a_n x^n, \quad y_2 = y_1 ln|x| + |x|^r \sum_{n=1}^{\infty} b_n x^n, a_0 \neq 0$$

Case III. If $r_1 - r_2$ is a positive integer, with $r_1 > r_2$, then

$$y_1 = |x|^{r_1} \sum_{n=0}^{\infty} a_n x^n, \quad y_2 = k y_1 ln|x| + |x|^{r_2} \sum_{n=0}^{\infty} b_n x^n, \text{where } a_0 \neq 0, b_0 \neq 0,$$

(the undetermined constant k may or may not be 0.)

2.4 Special functions

e.g.: Gamma function:
$\Gamma(x) \equiv \int_0^{\infty} t^{x-1} e^{-t} dt, \quad (\Gamma(1) = 1, \Gamma(\frac{1}{2}) = \sqrt{\pi}, \Gamma(n+1) = n!).$
$\Gamma(x+1) = \int_0^{\infty} t^x e^{-t} dt = x \cdot \Gamma(x).$
further define: $\phi(x) = \frac{d}{dx} ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$as digamma function.

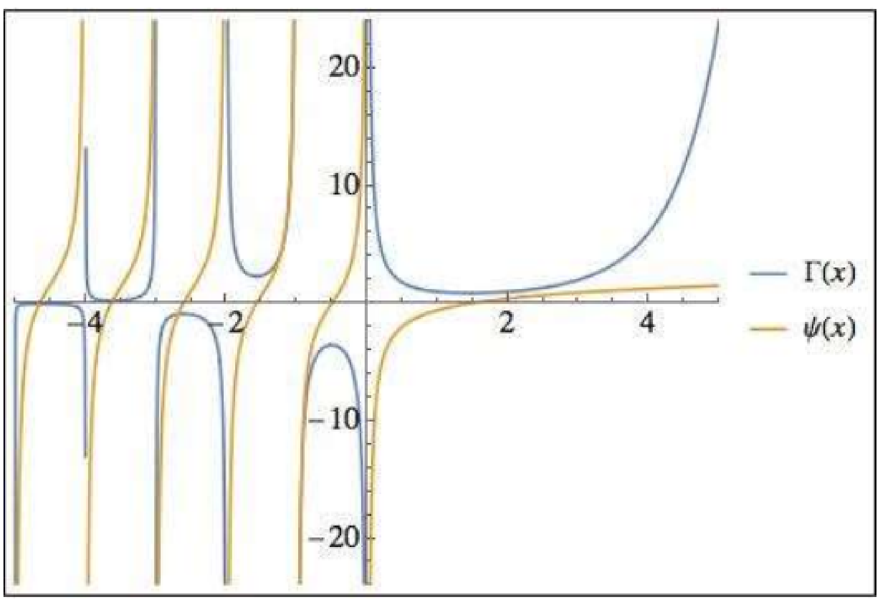


图 1: function pictures of $\Gamma(x)$and$\phi(x)$

2.4.1 Legendre Polynomials/Functions(from power series)

2.4.1.1 Legendre's Differential equations

$$(1 - x^2) y'' - 2xy' + \mu y = 0 \quad , -1 < x < 1. \tag{1}$$

(μ be constant.)

We identify $x=0$ is an ordinary point, thus all solutions$\rightarrow$ $y=\sum_{n=0}^{\infty} a_n x^n$.

plug into ODE:

$$\sum_{n=2}^{\infty} n(n-1)a_n(x^{n-2}-x^n) - 2\sum_{n=1}^{\infty} n a_n x^n + \mu \sum_{n=0}^{\infty} a_n x^n = 0.$$

$\Longrightarrow$ recurrence relation: $(n+1)(n+2)a_{n+2} = (n(n+1)-\mu)a_n, n \geq 0$.

Thus $a_{n+2} = \dfrac{n(n+1)-\mu}{(n+2)(n+1)} a_n$.

Taking either the even or odd terms, we get two independent solutions, known as Legendre functions.

2.4.1.2 Legendre Polynomials

General Legendre functions are not bounded at $x=\pm 1$. Only if $\mu = m(m+1)$,then we get a truncated power series solution of the ODE, and it is a m-th order polynomials.

m is even, $\quad y = a_0 + a_2 x^2 + \cdots + a_m x^m$;

m is odd, $\quad y = a_1 x + a_3 x^3 + \cdots + a_m x^m$;

$(a_n = -\dfrac{(n+2)(n+1)}{(m-n)(m+n+1)} a_{n+2})$

Conventionally we choose/normalize $a_m = \dfrac{(2m)!}{2^m (m!)^2}$.

thus from the recurrence relation:

$a_{m-2n} = (-1)^n \dfrac{(2m-2n)!}{2^m n!(m-n)!(m-2n)!}$

We define the polynomial solution, called the Legendre polynomial of degree m, $P_m(x)$as:

$P_m(x) = \dfrac{1}{2^m} \sum_{n=0}^{M} (-1)^n \dfrac{(2m-2n)!}{n!(m-n)!(m-2n)!} x^{m-2n}$

with $M = \dfrac{m}{2}$ for even m or $M = \dfrac{(m-1)}{2}$ for odd m.

Note that the Legendre equation with $\mu = m(m+1)$ also has a second linearly independent power series solution(unbounded at $x=\pm 1$),we denote as $Q_m(x)$, called a Legendre function of the second kind.

The general solution for $\mu = m(m+1)$: $y_h = c_1 P_m(x) + c_2 Q_m(x)$.

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \dfrac{3x^2-1}{2}$

$P_3(x) = \dfrac{5x^3-3x}{2}$

$P_4(x) = \dfrac{35x^4-30x^2+3}{8}$

$\ldots$

2.4.1.3 Properties of Legendre polynomials

① Rodrigues formula: $P_m(x) = \frac{1}{2^m m!}\frac{d^m}{dx^m}(x^2-1)^m$,m=0,1,2,$\cdots$.
② Bonnet's recurrence relation:
$(m+1)P_{m+1}(x) + mP_{m-1}(x) = (2m+1)xP_m(x)$,m=1,2,$\cdots$.
②(B) $P'_{m+1}(x) = P'_{m-1}(x) + (2m+1)P_m(x)$.
③ $P_m(x)$ is even/odd when m is even/odd.
④ $P_m(1) = 1, P_m(-1) = (-1)^m, P_{2n+1}(0) = 0, P_{2n}(0) = (-1)^n \frac{(2n-1)!!}{(2n)!!}$.
⑤ $|P_m(x)| \leq 1$ for all m and all x in [-1,1].
⑥ $P_m(x)$ has m distinct zeros in [-1,1].
⑦ all relative maxima and minima of $P_m(x)$ occur in [-1,1].
⑧ integral relations,

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \begin{cases} 0, for \quad m \neq n \\ \frac{2}{2n+1}, for \quad m = n \end{cases}$$

⑨generating function:

$$\frac{1}{\sqrt{1-2rx+r^2}} = \sum_{l=0}^{\infty} r^l P_l(x), r<1;$$

$$\frac{1}{\sqrt{1-2rx+r^2}} = \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} P_l(x), r>1.$$

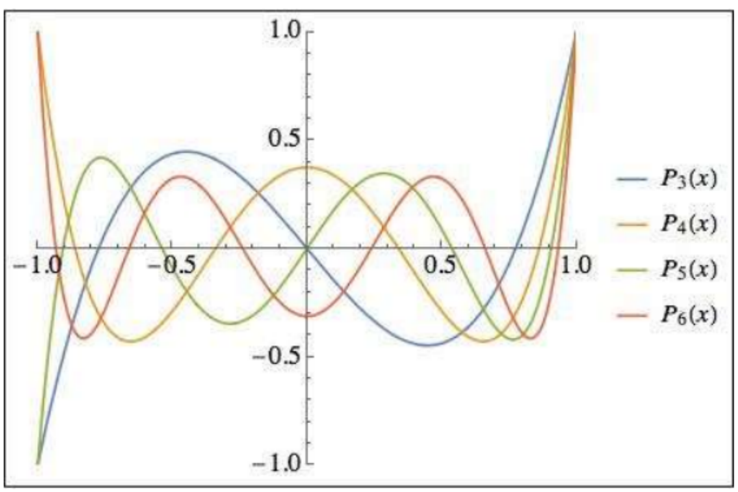


图 2: function pictures of $P_n(x)$

2.4.1.4 Associated Legendre Function

associated Legendre's equation:
$(1-x^2)y'' - 2xy' + [l(l+1) - \frac{m^2}{1-x^2}]y = 0, -1<x<1, 0 \leq m \leq l(m \to integer)$.
One solution of above ODE is :

$P_l^m(x) = (-1)^m(1-x^2)^{\frac{m}{2}}\dfrac{d^m P_l(x)}{dx^m} = \dfrac{(-1)^m}{2^l l!}(1-x^2)^{\frac{m}{2}}\dfrac{d^{l+m}}{dx^{l+m}}(x^2-1)^l.$
called associated Legendre functions of order m.
(at endpoints: $P_l^{m\neq 0}(\pm 1) = 0, P_l^m(x)$ is bounded in [-1,1];$Q_l^m(\pm 1) \to \infty$.)

Proof:
let $y = (1-x^2)^{\frac{m}{2}}v(x)$,thus the ODE comes to
$(1-x^2)v'' - 2(m+1)xv' + [l(l+1) - m(m+1)]v = 0.$
On another hand from Legendre equation:
$(1-x^2)P_l'' - 2xP_l' + l(l+1)P_l = 0.$
differentiating m times:
$(1-x^2)(P_l^{[m]})'' - 2mx(P_l^{[m]})' - 2\frac{m(m-1)}{2}P_l^{[m]} - 2x(P_l^{[m]})' - 2mP^{[m]} + l(l+1)P_l^{[m]} = 0.$
(with $P_l^{[m]} = \dfrac{d^m}{dx^m}P_l(x)$)
Thus it is a solution of v(x) above.
(the other solution:$Q_l^m(x) = (-1)^m(1-x^2)^{\frac{m}{2}}\dfrac{d^m Q_l(x)}{dx^m}$
$\Longrightarrow$ the general solution of the associated Legendre equation: $y_h = C_1P_l^m(x) + C_2Q_l^m(x)$)
We also define:
$P_l^{-m}(x) = (-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x) = \frac{(-1)^{-m}}{2^l l!}(1-x^2)^{\frac{-m}{2}}\frac{d^{l-m}}{dx^{l-m}}(x^2-1)^l.$
$P_l^{-m}(x) \propto P_l^m(x).$

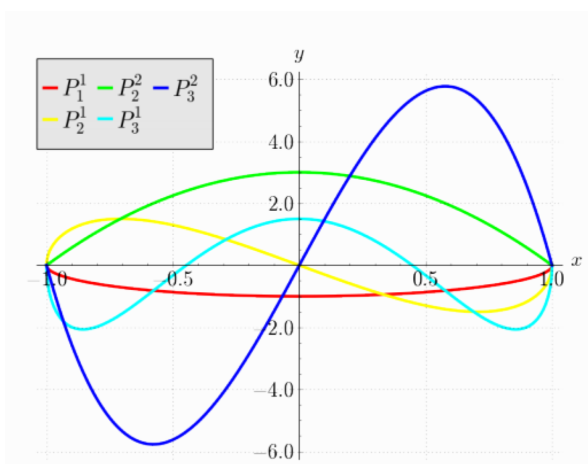


图 3: function pictures of $P_l^m(x)$

2.4.2 Bessel Functions

2.4.2.1 Bessel functions of the First kind

Bessel's equation of order-ν:

$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \nu \geq 0$(not neccessary integer)

$x = 0$ is a regular singular point.

$r(r-1) + r - \nu^2 = 0 \Longrightarrow r = \pm\nu$

Thus we have at least one solution of the form: $y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+\nu}$.

plug back into ODE:

$\Longrightarrow \sum_{n=0}^{\infty}(n^2 + 2n\nu)c_n x^{n+\nu} + \sum_{n=2}^{\infty} c_{n-2}x^{n+\nu} = 0.$

so $c_1 = 0, c_n = -\dfrac{c_{n-2}}{n^2 + 2n\nu}$, for $n \geq 2$

thus $c_{2n+1} = 0, c_{2n} = (-1)^n \dfrac{1}{2^{2n}n!(n+\nu)\cdots(\nu+1)}c_0 = (-1)^n \dfrac{\Gamma(1+\nu)}{2^{2n}n!\Gamma(n+\nu+1)}c_0.$

Conventionally we choose $c_0 = \dfrac{1}{2^\nu\Gamma(1+\nu)}$.

$J_\nu(x) \equiv y_1(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{n!\Gamma(n+\nu+1)}(\dfrac{x}{2})^{2n+\nu}$(check radius of convergence R=∞)

$r_1 - r_2 = 2\nu.(\nu\rightarrow$ 0/positive integer/half positive integer/otherwise)

A) for $\nu \rightarrow$ half positive integer/otherwise:

$J_{-\nu}(x) \equiv y_2(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{n!\Gamma(n-\nu+1)}(\dfrac{x}{2})^{2n-\nu}$ (has negative powers, thus=∞ at $x = 0+$)

thus the general solution:

$y_h(x) = C_1J_\nu(x) + C_2J_{-\nu}(x)$.

B) for ν is a positive integer:

We can also define $J_{-m}(x)$ by using the power series,find $J_{-m}(x) = (-1)^mJ_m(x)$.

(They are not linearly independent!)

asymptotic behavior:

$J_\nu(0) = \begin{cases} 1, \nu = 0 \\ 0, \nu > 0 \end{cases}$;

$J_\nu(x) \sim \sqrt{\dfrac{2}{\pi x}}cos(x - \dfrac{\nu\pi}{2} - \dfrac{\pi}{4}) + O(x^{-\frac{3}{2}})$ (when x$\rightarrow+\infty$) obvious $\lim_{x\rightarrow+\infty} J_\nu(x) = 0$.

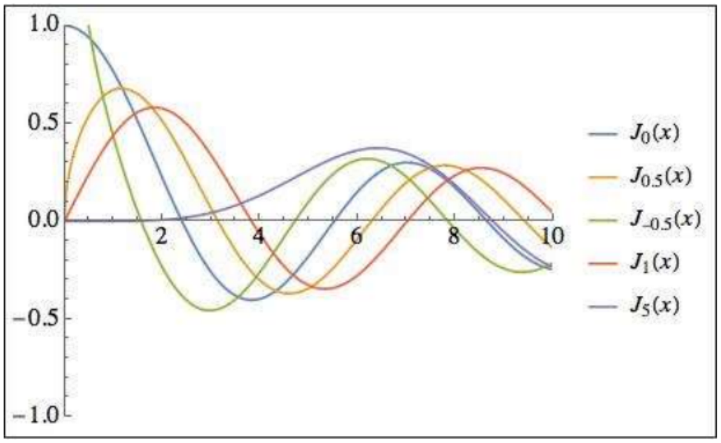


图 4: function pictures of $J_\nu(x)$

2.4.2.2 Bessel function of the second kind

If $\nu \geq 0$ is not an integer,define:
$Y_\nu(x) \equiv \dfrac{cos\nu\pi \cdot J_\nu(x) - J_{-\nu}(x)}{sin\nu\pi}$(another solution linearly independent with $J_\nu(x)$)

when ν is integer (m), define:
$Y_m(x) \equiv \lim\limits_{\nu\to m} Y_\nu(x)$ (One can imagine it is a solution of ODE by taking limit in ODE.)
calculate the limit and find:
$$Y_m(x) = \frac{2}{\pi}J_m(x)ln(\frac{x}{2}) - \frac{1}{\pi}\sum_{n=0}^{m-1}\frac{(m-n-1)!}{n!}(\frac{x}{2})^{-m+2n} - \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(m+n)!}[\phi(m+n+1)+\phi(n+1)](\frac{x}{2})^{m+2n}$$
($Y_m(x)$is linearly independent with $J_m(x)$)
Thus the general solution can be like($\forall\ \nu \geq 0.$):
$y_h = C_1 J_\nu(x) + C_2 Y_\nu(x)$.
We also define: $H_\nu^{(1)}(x) \equiv J_\nu(x) + iY_\nu(x) \to$ Hankel function.

asymptotic behavior:

$$Y_\nu(x) \to \begin{cases} -\infty, x \to 0+ \\ \sqrt{\dfrac{2}{\pi x}} sin(x - \dfrac{\nu\pi}{2} - \dfrac{\pi}{4}), x \to +\infty \end{cases}$$

recurrence relations(for either Z=J/Y/I/K):

e.g. $2Z'_\nu(x) = Z_{\nu-1}(x) - Z_{\nu+1}(x); \qquad [x^\nu Z_\nu(x)]' = x^\nu Z_{\nu-1}(x)$

$$\frac{2\nu Z_\nu(x)}{x} = Z_{\nu-1}(x) + Z_{\nu+1}(x); \qquad [x^{-\nu} Z_\nu(x)]' = -x^{-\nu} Z_{\nu+1}(x)$$

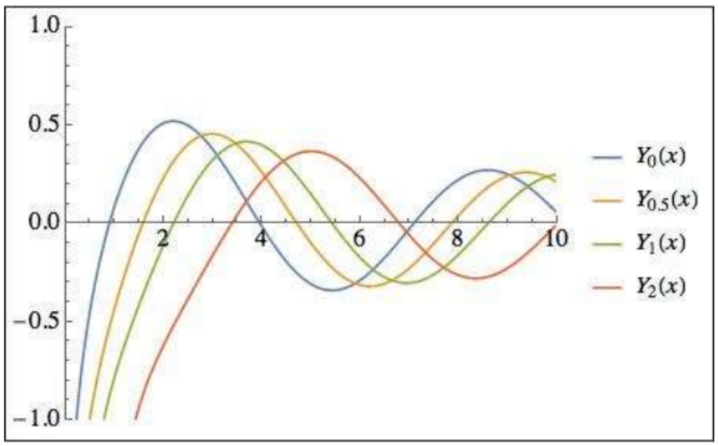


图 5: function pictures of $Y_\nu(x)$

2.4.2.3 Generalization

A) Parametric Bessel equation of order-ν:

$x^2y'' + xy' + (\alpha^2x^2 - \nu^2)y = 0, \alpha > 0.$

let $z = \alpha x$,then find general solution: $y_h = C_1 J_\nu(\alpha x) + C_2 Y_\nu(\alpha x)$.

B) Modified Bessel equation of order-ν:

$x^2y'' + xy' - (x^2 + \nu^2)y = 0.$

let $z = ix$,then find general solution: $y_h = C_1 I_\nu(x) + C_2 K_\nu(x)$.

$$\left(with \begin{cases} I_\nu(x) = i^{-\nu} J_\nu(ix) \\ K_\nu(x) = \dfrac{\pi}{2}\dfrac{I_{-\nu}(x) - I_\nu(x)}{sin\nu\pi} \\ K_n(x) = \lim\limits_{\nu \to n} K_\nu(x) \end{cases} \right) \to \text{called Modified Bessel functions.}$$

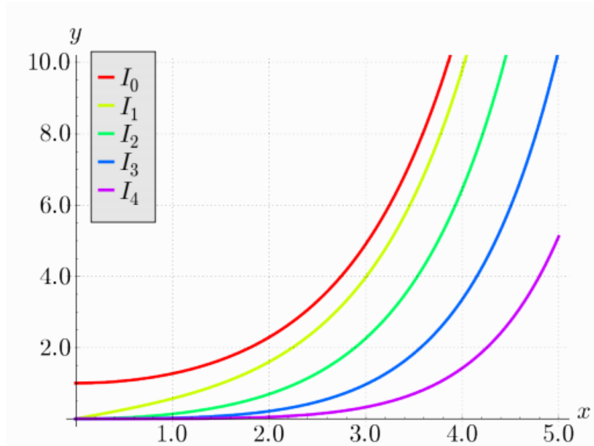


图 6: function pictures of $I_n(x)$

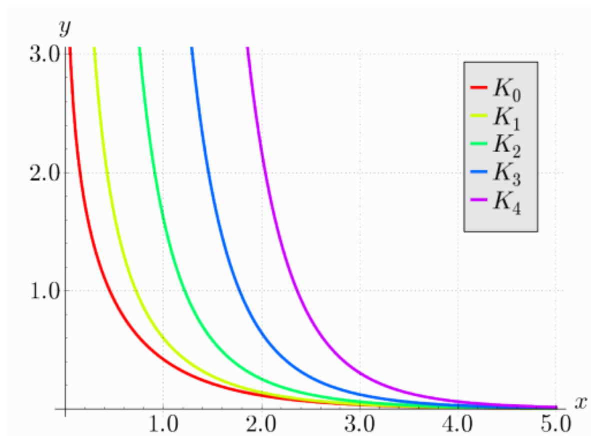


图 7: function pictures of $K_n(x)$

C) Spherical Bessel equation of order-m:
$x^2y'' + 2xy' + (kx^2 - m(m+1))y = 0, k > 0.$

let $w = yx^{\frac{1}{2}}$,plug into ODE:
$\Longrightarrow x^2w'' + xw' + (kx^2 - (m+\frac{1}{2})^2)w = 0.$
define: $j_m(x) = \sqrt{\frac{\pi}{2x}}J_{m+\frac{1}{2}}(x), y_m(x) = \sqrt{\frac{\pi}{2x}}Y_{m+\frac{1}{2}}(x)$ (called Spherical Bessel functions)
thus we find the general solution: $y_h = C_1 j_m(\sqrt{k}x) + C_2 y_m(\sqrt{k}x)$.

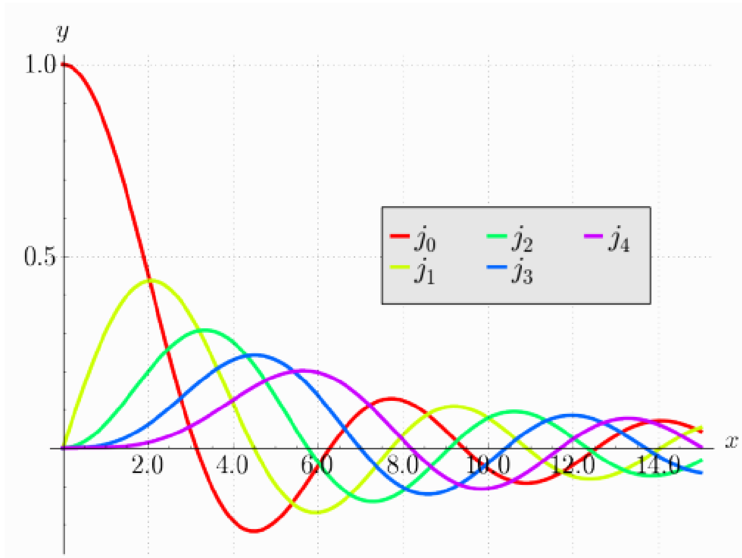


图 8: function pictures of $j_m(x)$

2.4.2.4 Generating function

generating function:

$$exp[\frac{x}{2}(t-\frac{1}{t})] = \sum_{n=-\infty}^{+\infty} J_n(x)t^n, 0 < |t| < \infty.$$

let $t = e^{i\theta}$,$e^{ixsin\theta} = \sum_{n=-\infty}^{+\infty} J_n(x)e^{in\theta}$.

$$\Longrightarrow J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ixsin\theta}(e^{in\theta})^* d\theta = \frac{1}{\pi}\int_0^{\pi} cos(xsin\theta - n\theta)d\theta.$$

let $t = ie^{i\theta}$,$e^{ixcos\theta} = \sum_{n=-\infty}^{+\infty} J_n(x)i^n e^{in\theta} = J_0(x) + 2\sum_{n=1}^{\infty} i^n J_n(x)cosn\theta$.

2.4.2.5 Applications

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}sinx; \qquad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}cosx$$

$$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}[\frac{sinx}{x} - cosx]; \qquad J_{-\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}[-\frac{cosx}{x} - sinx]$$

$\cdots$

2.4.3 Integrations

$$\int_{-1}^{1} P_n(x)dx = \delta_{n_0}\cdot 2;(P_0(x) = 1)$$

$$\int_{-1}^{1} xP_n(x)dx = \delta_{n_1}\cdot\frac{2}{3};$$

$\int_{-1}^{1} x^m P_n(x)dx \to x^m \to x\cdot x^{m-1} \to \cdots$(use the recurrence relation);($\int_{-1}^{1} x^m P_n(x)dx = 0$ when n>m)

$\int_{-1}^{1} xP_n(x)P_m(x)dx = \int_{-1}^{1}(\frac{m+1}{2m+1}P_{m+1}(x) + \frac{m}{2m+1}P_{m-1}(x))P_n(x)dx;$

general case: $\int x^\alpha J_\nu(x)dx, \alpha-\nu = odd$

example: $\int x^4 J_3(x)dx = \int d(x^4 J_4(x)) = x^4 J_4(x) + C$

$\int x^3 J_0(x)dx = \int x^2 d(xJ_1(x)) = x^3 J_1(x) - 2\int x^2 J_1(x)dx = x^3 J_1(x) - 2x^2 J_2(x) + C$

$$\int_0^\infty e^{-ax} J_0(bx)dx = \int_0^\infty e^{-ax}\sum_{n=0}^\infty \frac{(-1)^n}{(n!)^2}(\frac{bx}{2})^{2n}dx$$
$$= \sum_{n=0}^\infty \frac{(-1)^n}{(n!)^2}(\frac{b}{2})^{2n}\int_0^\infty e^{-ax}x^{2n}dx = \sum_{n=0}^\infty \frac{(-1)^n}{(n!)^2}(\frac{b}{2})^{2n}\frac{(2n)!}{a^{2n+1}} = \frac{1}{a}\sum_{n=0}^\infty \frac{(-1)^n(2n-1)!!}{n!}(\frac{b}{a})^{2n} = \frac{1}{\sqrt{a^2+b^2}}.$$

$\int_0^a xJ_p(\lambda_{pm}x)J_p(\lambda_{pn}x)dx$,with $\lambda_{pi} \equiv \frac{\alpha_{pi}}{a}$,$J_p(\alpha_{pi}) = 0, \alpha_{pi} > 0.$

$=a^2\int_0^1 zJ_p(\alpha_{pm}z)J_p(\alpha_{pn}z)dz = a^2\int_0^1 z^{-p}J_p(\alpha_{pm}z)z^{p+1}J_p(\alpha_{pn}z)dz$

$=a^2\int_0^1 z^{-p}J_p(\alpha_{pm}z)\frac{1}{(\alpha_{pn})}d(z^{p+1}J_{p+1}(\alpha_{pn}z))$

$=\frac{\alpha_{pm}}{\alpha_{pn}}\int_0^a xJ_{p+1}(\lambda_{pm}x)J_{p+1}(\lambda_{pn}x)dx$

thus $\int_0^a xJ_p(\lambda_{pm}x)J_p(\lambda_{pn}x)dx \propto \delta_{mn}$,and $y \equiv J_p(\lambda x)$

$x^2y'' + xy' + (\lambda^2x^2 - p^2)y = 0 \to (xy')' + (\lambda^2 x - \frac{p^2}{x})y = 0$

times xy' on both side:

$[(xy')^2]' + (\lambda^2x^2 - p^2)[y^2]' = 0$

if$\lambda = \alpha_{pm}$ and integrate over $0 \to 1$,

$[xJ_p'(\lambda x)]^2|_0^1 + (\lambda^2x^2 - p^2)J_p^2(\lambda x)|_0^1 - 2\lambda^2\int_0^1 xJ_p^2(\lambda x)dx = 0$

with $\int_0^1 xJ_p^2(\alpha_{pm}x)dx = \frac{1}{2\alpha_{pm}^2}\alpha_{pm}^2 J_p'(\alpha_{pm})^2 = \frac{1}{2}(\frac{J_{p-1}(\alpha_{pm}) - J_{p+1}(\alpha_{pm})}{2})^2 = \frac{1}{2}(J_{p+1}(\alpha_{pm}))^2$

3 Chapter 3. Orthogonal functions and Fourier series

3.1 Orthogonal functions

3.1.1 Inner product

$(\vec{u},\vec{v}) = \vec{u}\cdot\vec{v} = \sum_{i=1}^{n} u_i^* v_i$, satisfying:

①$(\vec{u},\vec{v}) = (\vec{v},\vec{u})^*$

②$(k\vec{u},\vec{v}) = k^*(\vec{u},\vec{v})$ (k is a scalar)

③$(\vec{u},\vec{u}) = 0$ if $\vec{u} = \vec{0}$ and$(\vec{u},\vec{u}) > 0$ if $\vec{u} \neq \vec{0}$

④$(\vec{u}+\vec{v},\vec{w}) = (\vec{u},\vec{w}) + (\vec{v}+\vec{w})$

(defined on complex number field C.)

Inner-product space+Complete ≡ Hilbert space

3.1.2 Generalization to functions

Square-integrable C-valued $\longrightarrow L^2[a,b]$:

$f_1(x)$ and $f_2(x)$ defined on an interval $[a,b]$(can be infinite)

Define their inner product: $(f_1, f_2) = \int_a^b f_1^*(x) f_2(x)dx.$

(It is obvious that the definition satisfies all 4 conditions.)

A) $\vec{0} \equiv$ almost vanishing function.

B) $f_1(x)$ and $f_2(x)$ are orthogonal if $(f_1, f_2) = 0$.

C) define the norm of a function:

$||f|| = \sqrt{(f,f)} = (\int_a^b |f^2(x)|dx)^{\frac{1}{2}}$

D) A set of functions $\{f_1, f_2, \cdots\}$ defined on the interval $[a,b]$ is called an orthogonal set if :

$||f_n|| \neq 0$ for all n, and $(f_n, f_m) = 0$ for all $n \neq m$.

(If in addition, $||f_n|| = 1$ for all n, the set is called an orthonormal set.)

E) A function set $\{f_n(x)\}$ is called complete if $\forall g(x)$ in the prescribed function space(can be a subspace of $L^2[a,b]$) can be expressed as : $g(x) = \sum_{n=0}^{\infty} C_n f_n(x)$.

(In case the set is also orthogonal, then $(f_m(x), g(x)) = C_m ||f_m(x)||^2$,

namely $C_m = \dfrac{(f_m(x), g(x))}{||f_m(x)||^2} \longrightarrow$ equivalent to that only zero function can be orthogonal to all $f_n(x)$)

3.1.3 More general definition of inner product

$(f_1, f_2) = \int_a^b f_1^*(x) f_2(x) w(x)dx.$

(where $w(x)$ is a non-negative piecewise continuous function on $[a,b]$ that is not identically 0 on any subsinterval of $[a,b]$)

A)Orthogonality wrt. $w(x)$: $\int_a^b f_1^*(x)f_2(x)w(x) = 0.$

B)norm wrt. the weight function $w(x)$: $||f|| = (\int_a^b |f^2(x)|w(x)dx)^{\frac{1}{2}}$

3.1.4 BVP

2nd-order linear ODE with linear boundary condition:

$$\begin{cases} y'' + p(x)y' + q(x)y = f(x), x \in [a,b] \\ \alpha_1 y(a) + \alpha_2 y'(a) + \alpha_3 y(b) + \alpha_4 y'(b) = A_1, & \equiv U_1[y] = A_1 \\ \beta_3 y(a) + \beta_4 y'(a) + \beta_1 y(b) + \beta_2 y'(b) = A_2, & \equiv U_2[y] = A_2 \end{cases}$$

Robin's BCs: $\alpha_3 = \alpha_4 = \beta_3 = \beta_4 = 0$;(3rd-kind)
Dirichlet's BCs:$\alpha_3 = \alpha_4 = \beta_3 = \beta_4 = 0 \quad + \quad \alpha_2 = \beta_2 = 0$;(1st-kind)
Neuman's BCs:$\alpha_3 = \alpha_4 = \beta_3 = \beta_4 = 0 \quad + \quad \alpha_1 = \beta_1 = 0$.(2nd-kind)

P.S.:
Can always decomposite into:
Problem I: non-homogeneous Eq+ homogeneous BCs
Problem II: homogeneous Eq+ non-homogeneous BCs

Solution I:
general solution: $y(x) = C_1 y_1(x) + C_2 y_2(x) + z(x)$

$$\text{with } z(x) = -y_1(x)\int_a^x \frac{f(s)y_2(s)}{W[y_1,y_2](s)}ds + y_2(x)\int_a^x \frac{f(s)y_1(s)}{W[y_1,y_2](s)}ds.$$

to fullfill BCs,

$$\begin{cases} C_1U_1[y_1] + C_2U_1[y_2] = -U_1[z] \\ C_1U_2[y_1] + C_2U_2[y_2] = -U_2[z] \end{cases}$$

Thus if $(U_1[y_1]U_2[y_2] - U_1[y_2]U_2[y_1]) \neq 0$, exist a unique solution.
(otherwise no solution in general except the two equations are identicle.)

Solution II:

general solution: $y(x) = C_1y_1(x) + C_2y_2(x)$

to fullfill BCs,

$$\begin{cases} C_1U_1[y_1] + C_2U_1[y_2] = A_1 \\ C_1U_2[y_1] + C_2U_2[y_2] = A_2 \end{cases}$$

Thus if $(U_1[y_1]U_2[y_2] - U_1[y_2]U_2[y_1]) \neq 0$ and $A_1^2 + A_2^2 \neq 0$ exist a unique solution.

3.2 Sturm-Liouville Theory

3.2.1 A regular Sturm-Liouville problem

a boundary value problem on a closed finite interval $[a, b]$ of the form:

$[p(x)y']' + [q(x) + \lambda r(x)]y = 0$, $a \leq x \leq b$

$$\begin{cases} c_1y(a) + c_2y'(a) = 0 \\ d_1y(b) + d_2y'(b) = 0 \end{cases}$$

(where $c_1^2 + c_2^2 > 0$,$d_1^2 + d_2^2 > 0$,and λ is a parameter.)

The regularity means requring:

$p(x), p'(x), q(x)$,and $r(x)$ be continuous on $[a, b]$ and with $p(x), r(x) > 0$ on $[a, b]$.

Note a 2-nd order linear ODE with a prescribed constant λ in the following form can always be converted into the SL form.

$a(x)y'' + b(x)y' + (c(x) + \lambda d(x))y = 0.$

Let :

$p(x) = exp(\int \frac{b(x)}{a(x)}dx),$

$q(x) = \frac{c(x)}{a(x)}exp(\int \frac{b(x)}{a(x)}dx),$

$r(x) = \frac{d(x)}{a(x)}exp(\int \frac{b(x)}{a(x)}dx),$

thus y satisfies:

$[p(x)y']' + [q(x) + \lambda r(x)]y = 0$.

Remarks:

For above BVP, obviously $y \equiv 0$ is a trivial solution.

The non-zero solutions of a sturm-Liouville problem are called the eigenfunctions of the problem.

And those values of λ for which non-zero solutions can be found are called the eigenvalues.

Theorem 1:

The eigenvalues of a regular Sturm-Liouville problem are all real and form an increasing sequence.

$\lambda_1 < \lambda_2 < \cdots$, where $\lambda_j \to \infty$ as $j \to \infty$.

Theorem 2:

Each eigenvalue of a regular Sturm-Liouville problem has just one linearly independent eigenfunction corresponding to it.

Proof:

Suppose y_1 and y_2 are eigenfunctions with same λ.

$W(y_1, y_2)(a) = \begin{vmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{vmatrix} = y_1(a)y_2'(a) - y_2(a)y_1'(a)$

from the boundary conditions:

$$\begin{cases} c_1 y_1(a) + c_2 y_1'(a) = 0 \\ c_1 y_2(a) + c_2 y_2'(a) = 0 \end{cases} \quad \text{with } c_1 \text{ or } c_2 \neq 0 \Longrightarrow \begin{vmatrix} y_1(a) & y_1'(a) \\ y_2(a) & y_2'(a) \end{vmatrix} = 0 \Longrightarrow W(y_1, y_2)(a) = 0.$$

Theorem 3:

Eigenfunctions of a regular Sturm-Lionville problem corresponding to different eigenvalues are orthogonal wrt. the weight function $r(x)$.(also valid for the singular SL problems providing conditions:

$\lim_{x \uparrow b} p(x)(y_1(x)y_2'(x) - y_2(x)y_1'(x)) - \lim_{x \downarrow a} p(x)(y_1(x)y_2'(x) - y_2(x)y_1'(x)) = 0$ is satisfied.)

Proof:

Suppose $\lambda_j \neq \lambda_k$ are eigenvalues of a SL problem, and y_j, y_k be the corresponding eigenfunctions.

We have:

$[p(x)y_j']' + [q(x) + \lambda_j r(x)]y_j = 0 \longrightarrow \times y_k$

$[p(x)y_k']' + [q(x) + \lambda_k r(x)]y_k = 0 \longrightarrow \times y_j$

We get

$y_k[p(x)y_j']' - y_i[p(x)y_k']' = (\lambda_k - \lambda_j)y_i y_k r(x)$

$[p(x)(y_k y_j' - y_j y_k')]' = (\lambda_k - \lambda_j)y_i y_k r(x)$

Take integral from a to b:

$$(\lambda_k - \lambda_j)\int_a^b y_j y_k r(x)dx = p(x)(y_k y_j' - y_j y_k')|_a^b = 0 \Longrightarrow (y_k, y_j) = 0$$

Theorem 4:

Let $y_1, y_2, \cdots$ be the set of all eigenfunctions for a regular SL problem on $[a, b]$.

If $f(x)$ and $f'(x)$ are piecewise continuous on $[a, b]$, then we have $f = \sum_{j=1}^{\infty} A_j y_j(x)$,with

$$A_j = \frac{(y_i, f)}{(y_j, y_j)} = [\int_a^b r(x)y_j^2(x)dx]^{-1}\int_a^b r(x)y_j(x)f(x)dx.$$

The series converges to $f(x)$ at points of continuous and to $\dfrac{f(x^+ + f(x^-))}{2}$ otherwise.
(Note this conclusion is also valid for the singular SL problem with Legendre and Bessel's equation.)

3.2.2 Singular Sturm-Liouville problem

Regularity hold on (a, b).
a or b be infinite/conditions not hold on a or b.
(BCs usually are different.)

Classical singular SL problems:
(a) Legendre's equation:
$(1-x^2)y'' - 2xy' + \lambda y = 0$ $(x \in [-1, 1]) \Longrightarrow [(1-x^2)y']' + \lambda y = 0$
BCs: y bounded at ± 1.
$p(1) = p(-1) = 0$.
Solution: $\lambda_n = n(n+1)$, n=0,1,$\cdots$; $\quad y_n(x) = P_n(x)$.

(b) Parametric form of Bessel's equation:
$x^2 y'' + xy' + (\lambda' x^2 - \nu^2)y = 0$ $(x \in [0, a]) \Longrightarrow [xy']' + [-\frac{\nu^2}{x} + \lambda' x]y = 0$
BCs: y bounded at 0, y(a)=0.
Solution:
① $\lambda' = 0$: not eigenvalue.
② $\lambda' < 0$: not eigenvalue.

$\quad y = C_1 I_\nu(\sqrt{-\lambda'}x) + C_2 K_\nu(\sqrt{-\lambda'}x)$.

③ $\lambda' = \lambda^2 > 0$:

$\quad y = C_1 J_\nu(\lambda x) + C_2 Y_\nu(\lambda x)$.

$\Longrightarrow C_2 = 0$, $C_1 J_\nu(\lambda a) = 0$.
$\Longrightarrow \lambda_n = \dfrac{\alpha_{\nu n}}{a}$, n=1,2,$\cdots$.
$\Longrightarrow \lambda_n' = (\dfrac{\alpha_{\nu n}}{a})^2$, $y_n = J_\nu(\dfrac{\alpha_{\nu n}}{a}x)$, n=1,2,$\cdots$.

3.2.3 Formulation of SL problem on Hilbert space

In Hilbert space H:
$\forall \vec{u}, \vec{v}$:

$\quad (\hat{L}\vec{u}, \vec{v}) = (\vec{u}, \hat{M}\vec{v}) \longrightarrow \hat{L}, \hat{M}$ being adjoint.

$\quad (\hat{L}\vec{u}, \vec{v}) = (\vec{u}, \hat{L}\vec{v}) \longrightarrow \hat{L}$ being self-adjoint operator/Hermitian matrix.

find solutions for $\hat{L}\vec{u} = \lambda \vec{u}, \lambda \in C$.

① exist eigenvalues and all eigenvalues are real.

② eigenvectors of different eigenvalues are orthogonal.

③ all eigenvectors form a complete orthogonal basis.

For SL problem:

in real Hilbert space $H = L^2[a,b]$ with inner product $(f,g) \equiv \int_a^b r(x)f(x)g(x)dx$.

considering on the domain:

$D = \{u \in H | u', u'' \in H, U_1[u] = 0, U_2[u] = 0\}$.

define: $\hat{L} = -\frac{1}{r(x)}\left\{\frac{d}{dx}[p(x)\frac{d}{dx}] + q(x)\right\}$ is self-adjoint.

thus, $\hat{L}\vec{y} = \lambda y \iff [p(x)y']' + [q(x) + \lambda r(x)]y = 0$.

further BCs,

$U_1[y] \equiv \alpha_1 y(a) + \alpha_2 y'(a) = 0$

$U_2[y] \equiv \beta_1 y(b) + \beta_2 y'(b) = 0$

Prove self-adjoint:

$(\hat{L}u,v) = \int_a^b r(x)(\hat{L}u)vdx = \int_a^b -v\{[p(x)u']' + q(x)u\}dx = -\int_a^b q(x)uvdx + \int_a^b p(x)u'v'dx - p(x)vu' \mid_a^b$

$(u,\hat{L}v) = \int_a^b r(x)(\hat{L}v)udx = \int_a^b -u\{[p(x)v']' + q(x)v\}dx = -\int_a^b q(x)uvdx + \int_a^b p(x)u'v'dx - p(x)v'u \mid_a^b$

since $\begin{cases} \alpha_1 u(a) + \alpha_2 u'(a) = 0 \\ \alpha_1 v(a) + \alpha_2 v'(a) = 0 \end{cases}$

$\alpha_1^2 + \alpha_2^2 > 0 \Longrightarrow u(a)v'(a) - v(a)u'(a) = 0$.

Similarly $u(b)v'(b) - v(b)u'(b) = 0$.

$\Longrightarrow$ the difference : $p(x)(v'u - u'v) \mid_a^b = 0$.

3.2.4 BVP with non-homogeneous ODE→ Green's function

$$\begin{cases} [p(x)y']' + q(x)y = f(x) & on[a,b] \\ y(a) = 0 \\ y(b) = 0 \end{cases}$$

Green's function $G(x,s)$ on $[a,b] \times [a,b]$:

① continuous on $[a,b] \times [a,b]$.

② $G(a,s) = G(b,s) = 0$.

③ $\forall x \in (a,b)$ but $x \neq s$: $\frac{\partial}{\partial x}[p(x)\frac{\partial G(x,s)}{\partial x}] + q(x)G(x,s) = 0$.

④ at $x = s$: $\left.\frac{\partial G}{\partial x}\right|_{x=s^+} - \left.\frac{\partial G}{\partial x}\right|_{x=s^-} = \frac{1}{p(s)}$.

if exist $G(x, s)$ then,
$y(x) = \int_a^b G(x, s)f(s)ds.$

Supposing $y_1(x)$,$y_2(x)$ are linearly independent solution of homogeneous ODE,
and further $y_1(x)|_{x=a} = 0, y_2(x)|_{x=b} = 0$, then
$$G(x, s) = \begin{cases} \dfrac{y_2(s)y_1(x)}{W[y_1, y_2](s) \cdot p(s)}, a \le x \le s \\ \dfrac{y_1(s)y_2(x)}{W[y_1, y_2](s) \cdot p(s)}, s \le x \le b \end{cases}$$
and original BVP: $y(x) = \int_a^b G(x, s)f(s)ds = y_1(x) \int_x^b \frac{f(s)y_2(s)}{W(s) \cdot p(s)} ds + y_2(x) \int_a^x \frac{f(s)y_1(s)}{W(s) \cdot p(s)} ds.$

Dirac delta function $\delta(x)$:
$\delta(x) = 0, (x \neq 0) \qquad \int_{-\infty}^{+\infty} \delta(x)dx = 1.$
$\Longrightarrow \int_{-\infty}^{+\infty} f(x)\delta(x)dx = f(0).$
$\Longrightarrow f(x) = \int_a^b f(s)\delta(s - x)ds.$
Indeed for every s, $G(x, s)$ is the solution of
$[p(x)y']' + q(x)y = \delta(x - s)$, $y|_{x=a} = y|_{x=b} = 0.$

3.3 Classical orthogonal polynomials

define: on interval $[a, b]$(a,b can be infinite) wrt. weight function $r(x)$
$\{P_n(x)\}, n = 0, 1, \cdots$ $P_n(x)$ being n-th order polynomials,satisfying
$(P_n(x), P_m(x)) \propto \delta_{mn}$.
($P_n(x)$ are determined upto overall normalization c_n)

Schmidt algorithm:
$P_0(x) = c_0$
$P_1(x) = c_1(x + d_1^0 \frac{P_0(x)}{c_0})$
$P_2(x) = c_2(x^2 + d_2^1 \frac{P_1(x)}{c_1} + d_2^0 \frac{P_0(x)}{c_0})$
$\vdots$
$(P_1(x), P_0(x)) = 0 \Longrightarrow d_1^0 = -\frac{(x, P_0)}{(\frac{P_0}{c_0}, P_0)}$
$(P_2(x), P_0(x)) = 0 \Longrightarrow d_0^2$

$(P_2(x), P_1(x)) = 0 \Longrightarrow d_1^2$

$\vdots$

3.3.1 Examples

Laguerre polynomials: $[0, +\infty) \quad r = x^{\alpha}e^{-x} \quad \alpha > -1 \to L_n^{(\alpha)}(x)$

Hermite polynomials: $(-\infty, +\infty) \quad r = e^{-x^2} \to H_n(x)$

Jacob polynomials: $[-1, +1] \quad r = (1-x)^{\alpha}(1+x)^{\beta} \quad \alpha, \beta > -1 \to P_n^{(\alpha,\beta)}(x)$

$\alpha = \beta = 0 \to Legendre's polynomials$

3.3.2 SL problem

In view of SL problem (construct SL $\to$ eigenfunction orthogonal!)

$$\begin{cases} [\sigma(x)r(x)y']' + \lambda r(x)y = 0 & , x \in [a,b] \\ |y(a)| < +\infty, |y(b)| < +\infty & , if\ a, b \quad finite. \quad (or \displaystyle\int r(x)y^2dx < +\infty) \end{cases}$$

ODE can be written as:

$\sigma(x)y'' + \dfrac{[\sigma(x)r(x)]'}{r(x)}y' + \lambda y = 0.$

suppose:

$y = P_0(x) \longrightarrow \lambda_0 P_0(x) = 0.\ \lambda_0 = 0.$

$y = P_1(x) \longrightarrow [\sigma(x)r(x)]' = \tau(x)r(x).$ with $\tau(x) = Ax + B.$

$y = P_2(x) \longrightarrow \sigma(x) = C_2x^2 + C_1x + C_0.$

in standard form:

$y'' + \dfrac{\tau(x)}{\sigma(x)}y' + \dfrac{\lambda}{\sigma(x)}y = 0.$

without of generality,

$$\sigma(x) = \begin{cases} (x-a)(b-x), if \quad a, b \neq \pm\infty & \equiv [-1, 1] \\ x - a, if \quad b = +\infty & \equiv [0, +\infty) \\ b - x, if \quad a = -\infty & \equiv (-\infty, 0] \\ 1, if \quad a = -\infty, b = +\infty & \equiv (-\infty, +\infty) \end{cases}$$

$r(x) = \dfrac{1}{\sigma(x)}exp[\displaystyle\int \frac{\tau(x)}{\sigma(x)}dx]$

independent solution:

① If $a = 0, b = +\infty$, then $A = -1, B = 1 + \alpha$, $r(x) = x^{\alpha}e^{-x}, \alpha > -1.$

$xy'' + (1+\alpha-x)y' + \lambda y = 0$. ($x=0$ is a regular singular point of ODE)
for $y = P_n(x) = a_n x^n + \cdots + a_0, a_n \neq 0$:
term of x^n: $a_n(-n+\lambda)x^n = 0$, thus $\lambda = \lambda_n = n, n = 0, 1, \cdots$. $P_n(x) = L_n^{(\alpha)}(x)$.
In general $\lambda_n = -An - C_2 n(n-1)$ from term of x^n.
② If $a = -\infty, b = +\infty$. then $A = -2, B = 0, r(x) = e^{-x^2}$
$y'' - 2xy' + \lambda y = 0$. ($x = 0$ is an ordinary point of ODE)
$\lambda_n = 2n, n = 0, 1, \cdots . P_n(x) = H_n(x)$.
③ If $a = -1, b = 1$. then $A = -(\alpha+\beta+2), B = \beta-\alpha, \alpha, \beta > -1, r(x) = (1-x)^\alpha(1+x)^\beta$.
$(1-x^2)y'' + (\beta-\alpha-(\alpha+\beta+2)x)y' + \lambda y = 0$.
$\lambda_n = n(\alpha+\beta+n+1), n = 0, 1, \cdots . P_n(x) = P_n^{(\alpha,\beta)}(x)$.
when $\alpha = \beta = 0 \Longrightarrow$ Legendre polynomials.
when $\alpha = \beta = -\frac{1}{2} \Longrightarrow$ Chebyshev's polynomials.

3.3.3 Summary on properties of classical OPs

① Orthogonality, for $n \neq k$,
$\int_a^b P_n(x)P_k(x)r(x)dx = 0, \int_a^b P'_n(x)P'_k(x)r_1(x)dx = 0$
$(r_1(x) \equiv r(x) \cdot \sigma(x).)$
② $P_n(x)$ has n zeros inside the interval $[a, b]$, thus all zeros are real numbers.
③ They can be written as:
$P_n(x) = \frac{c_n}{r(x)} \frac{d^n}{dx^n} \{\sigma^n(x)r(x)\}$
the normalizations are usually chosen as:
$c_n^{Jacobi} = \frac{(-1)^n}{2^n n!}, c_n^{Laguerre} = \frac{1}{n!}, c_n^{Hermite} = (-1)^n$. thus $P_0(x) = 1$.
Proof:
STEP1.
$\frac{d}{dx}(\sigma^n(x)r(x)q_l(x)) = \frac{d}{dx}(\sigma r \sigma^{n-1} q_l) = r\sigma^{n-1}(\tau q_l + (n-1)q_l\sigma' + \sigma q'_l) = r\sigma^{n-1} \cdot (l+1)-th$ polynomials
$\Longrightarrow \frac{c_n}{r(x)} \frac{d^n}{dx^n} \{\sigma^n(x)r(x)\}$ is a n-th order polynomials.

STEP2.
let $y_n(x) = \frac{1}{r(x)} \frac{d^n}{dx^n} \{\sigma^n(x)r(x)\}$
further, $(y_n(x), y_m(x)) = \int_a^b r(x)y_n y_m dx = \int_a^b y_m d(\frac{d^{n-1}}{dx^{n-1}}(\sigma^n r))$
$= y_m \frac{d^{n-1}}{dx^{n-1}}(\sigma^n r)|_a^b - \int_a^b y'_m \frac{d^{n-1}}{dx^{n-1}}(\sigma^n r)dx = \cdots = 0$.
④ the norms of OPs:
$||P_n(x)||^2 = \int_a^b P_n^2(x)r(x)dx = (-1)^n n! a_n C_n \int_a^b \sigma^n(x)r(x)dx$.

$||P_n(x)||^2 = \dfrac{2}{2n+1}$;
$||P_n^{(\alpha,\beta)}(x)||^2 = \dfrac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}$;
$||L_n^{(\alpha)}(x)||^2 = \dfrac{\Gamma(n+\alpha+1)}{n!}$;
$||H_n(x)||^2 = 2^n n!\sqrt{\pi}$.

⑤ generating function

$\Psi(x,z) = \dfrac{r(t_0)}{r(x)}\dfrac{1}{1-z\sigma'(t_0)} = \sum_{n=0}^{\infty}\dfrac{P_n(x)}{C_n n!}z^n$.

with t_0 being root of $t - x - z\sigma(t) = 0$.

⑥ m-th order derivative $P_n^{[m]}(x) \equiv \dfrac{d^m P_n(x)}{dx^m}$ satisfy

$[\sigma^{m+1}(x)r(x)P_n^{[m]}(x)']' + \lambda_{nm}\sigma^m(x)r(x)P_n^{[m]}(x) = 0$.

$(\lambda_{nm} = -(n-m)(A + C_2(n+m-1)).)$

and $\int_a^b \sigma^m(x)r(x)P_n^{[m]}(x)P_k^{[m]}(x)dx = 0$, if $n \neq k$

3.4 Fourier series

Pre class:

Piecewise Continuous function on $[a,b]$:

$f(a+), f(b-)$ exist;

f is defined and continuous on (a,b) except at a finite number of points in (a,b) where the left and right limits exist.

Periodic function: $f(x) = f(x+T)$

According to the Theorem 4 in 3.2.1:

Let $y_1, y_2, \cdots$ be the set of all eigenfunctions for a regular SL problem on $[a,b]$.

If $f(x)$ and $f'(x)$ are piecewise continuous on $[a,b]$, then we have $f = \sum_{j=1}^{\infty} A_j y_j(x)$,with

$A_j = \dfrac{(y_i, f)}{(y_j, y_j)} = [\int_a^b r(x)y_j^2(x)dx]^{-1}\int_a^b r(x)y_j(x)f(x)dx$.

The series converges to $f(x)$ at points of continuous and to $\dfrac{f(x^+ + f(x^-))}{2}$ otherwise.

3.4.1 Fourier series

Orthogonal complete set : $\left\{\dfrac{1}{2}, cos\dfrac{n\pi x}{p}, sin\dfrac{n\pi x}{p}\right\}, n = 1, 2, \cdots$ for $x \in [-p,p]$， wrt. $r(x) = 1$.

for f, f' piecewise continuous on $[-p,p]$:

$f = \dfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n cos\dfrac{n\pi x}{p} + b_n sin\dfrac{n\pi x}{p})$

thus, $a_n = \frac{(f, cos\frac{n\pi x}{p})}{(cos\frac{n\pi x}{p}, cos\frac{n\pi x}{p})} = \frac{1}{p}\int_{-p}^{p} f(x)cos\frac{n\pi x}{p}dx, b_n = \frac{1}{p}\int_{-p}^{p} f(x)sin\frac{n\pi x}{p}dx.$
$\{a_n, b_n\}$ are referred to as Fourier coefficients of $f(x)$.

Theorem 1: Fourier series representation

The Fourier series of $f(x)$ converge to $f(x)$ at continuous points,
$\frac{f(x-)+f(x+)}{2}$ at discontinuous points, $\frac{f(p-)+f(p+)}{2}$ at end points

Fourier series of even or odd functions:
even function on $[-p,p] \Longrightarrow b_n = 0, f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\frac{n\pi x}{p}$.(cosine series)
odd function on $[-p,p] \Longrightarrow a_n = 0, f(x) = \sum_{n=1}^{\infty} b_n sin\frac{n\pi x}{p}$.(sine series)

Fourier series of $f(x)$ defined at $[0,p]$:
using even periodic extension: $f_1(x) = \begin{cases} f(x), if \quad 0 \le x \le p \\ f(-x), if \quad -p < x < 0 \end{cases}$
using odd periodic extension: $f_2(x) = \begin{cases} f(x), if \quad 0 \le x \le p \\ -f(-x), if \quad -p < x < 0 \end{cases}$

Complex form of Fourier series: $\left\{e^{i\frac{n\pi x}{p}}\right\}, n = 0, \pm 1, \pm 2, \cdots$
$f(x) = \sum_{n=-\infty}^{+\infty} C_n e^{i\frac{n\pi x}{p}}$
with $C_n = \frac{1}{2p}\int_{-p}^{p} e^{-i\frac{n\pi x}{p}} f(x)dx, n = 0, \pm 1, \pm 2, \cdots$
$(C_n = (C_{-n})^* = a_n + ib_n)$
then $f(x) = \sum_{n=-\infty}^{+\infty} 2Re[C_n e^{i\frac{n\pi x}{p}}]$

3.4.2 Operations on Fourier series

① linear combinations of Fourier series:
if $u(x)$ and $v(x)$ can be expanded as:
$u(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cos\frac{n\pi x}{p} + b_n sin\frac{n\pi x}{p})$
$v(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty}(c_n cos\frac{n\pi x}{p} + d_n sin\frac{n\pi x}{p})$
Thus obviously $g(x) = \lambda_1 u(x) + \lambda_2 v(x)$ can be expanded as:
$g(x) = \frac{s_0}{2} + \sum_{n=1}^{\infty}(s_n cos\frac{n\pi x}{p} + t_n sin\frac{n\pi x}{p})$
with

$$s_n = (cos\frac{n\pi x}{p}, g(x)) = \lambda_1 a_n + \lambda_2 c_n$$
$$t_n = (sin\frac{n\pi x}{p}, g(x)) = \lambda_1 b_n + \lambda_2 d_n$$

② Term-by-Term differentiation:

Suppose that $f(x), f'(x)$, and $f''(x)$ are all piecewise continuous on $[-p, p]$ and in addition $f(x)$ is continuous and $f(p) = f(-p)$,if

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cos\frac{n\pi x}{p} + b_n sin\frac{n\pi x}{p})$$

then

$$f'(x) = \sum_{n=1}^{\infty}(-na_n\frac{\pi}{p}sin\frac{n\pi x}{p} + nb_n\frac{\pi}{p}cos\frac{n\pi x}{p})$$

Proof:

$f'(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty}(c_n cos\frac{n\pi x}{p} + d_n sin\frac{n\pi x}{p})$

$c_n = \frac{1}{p}\int_{-p}^{p} cos\frac{n\pi x}{p} f'(x)dx = \frac{1}{p}(f(x)cos\frac{n\pi x}{p}\Big|_{-p}^{p} - \int_{-p}^{p}(-\frac{n\pi}{p})sin\frac{n\pi}{p}xf(x)dx)) = \frac{1}{p}(0 + n\pi b_n) = \frac{n\pi b_n}{p}.$

③ Term-by-Term integration:

Suppose that $f(x), f'(x)$ be piecewise continuous, thus $f(x)$ can be expanded :

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cos\frac{n\pi}{p}x + b_n sin\frac{n\pi}{p}x), \quad -p < x < p$

For the integral $F(x) = \int_0^x f(t)dt$, if $\int_{-p}^{p} f(t)dt = 0$, then

$F(x) = A_0 + \sum_{n=1}^{\infty}(\frac{p}{n\pi}a_n sin\frac{n\pi}{p}x - \frac{p}{n\pi}b_n cos\frac{n\pi}{p}x)$, with $A_0 = \frac{p}{\pi}\sum_{n=1}^{\infty}\frac{b_n}{n}$.

3.4.3 Generalized Fourier series

① Fourier-Legendre series:

Orthogonal complete set : $\{P_n(x)\}, n = 0, 1, \cdots$ on $[-1, 1]$ wrt. $r(x) = 1$

$f(x) = \sum_{n=0}^{\infty} c_n P_n(x), \qquad c_n = \frac{(P_n(x), f(x))}{(P_n(x), P_n(x))}$

② Fourier-Bessel series:

Orthogonal complete set : $\{J_p(\lambda_{pn}x)\}, n = 1, 2, \cdots$ on $[0, a]$ wrt. $r(x) = x$

$f(x) = \sum_{n=1}^{\infty} c_n J_p(\lambda_{pn}x), \qquad c_n = \frac{(J_p(\lambda_{pn}x), f(x))}{(J_p(\lambda_{pn}x), J_p(\lambda_{pn}x))}$

SL problem:

$[xy']' + (\lambda^2 x - \frac{p^2}{x})y = 0 \qquad 0 \le x \le a$

BCs: $d_1 y(a) + d_2 y'(a) = 0 \quad (d_1^2 + d_2^2 > 0); y(0)$ being finite.

Solution:

$y = J_p(\lambda x)$.

$d_1 J_p(\lambda a) + d_2 \lambda J_p'(\lambda a) = 0. \Longrightarrow \lambda = \lambda_{pn}, \quad n = 1, 2, \cdots.$

(note:$\lambda J_p'(\lambda a) = \lambda \frac{1}{2}(J_{p-1}(\lambda a) - J_{p+1}(\lambda a)) = \frac{1}{2a}(\lambda a)(\frac{2pJ_p(\lambda a)}{\lambda a} - 2J_{p+1}(\lambda a)) = \frac{p}{a}J_p - \lambda J_{p+1}$

eigenfunctions: $\{J_p(\lambda_{pn} x)\}, n = 1, 2, \cdots$.

Norm of the eigenfunctions:

$[xy']' + (\lambda^2 x - \frac{p^2}{x})y = 0 \Longrightarrow (\times 2xy') \quad \frac{d}{dx}[xy']^2 + (\lambda^2 x^2 - p^2)\frac{d}{dx}y^2 = 0.$

integrating on $[0, a]$,and let $y = J_p(\lambda x)$,

$$2\lambda^2 \int_0^a xy^2 dx = \left\{[xy']^2 + (\lambda^2 x^2 - p^2)y^2\right\}\Big|_0^a$$

thus,

$$2\lambda^2 \int_0^a xJ_p^2(\lambda x)dx = \lambda^2 a^2 [J_p'(\lambda a)]^2 + (\lambda^2 a^2 - p^2)[J_p(\lambda a)]^2.$$

Case I. $d_1 = 1, d_2 = 0$, or $J_p(\lambda_{pn} a) = 0$.

$\lambda_{pn} = \frac{\alpha_{pn}}{a}$; eigenfunctions: $J_p(\lambda_{pn} x)$.

$$\|J_p(\lambda_{pn} x)\|^2 = \int_0^a xJ_p^2(\lambda_{pn} x)dx = \frac{a^2}{2}J_{p+1}^2(\lambda_{pn} a)$$

Case II. $d_1 = 1, d_2 = h$, or $J_p(\lambda_{pn} a) + h\lambda_{pn} J_p'(\lambda_{pn} a) = 0$.

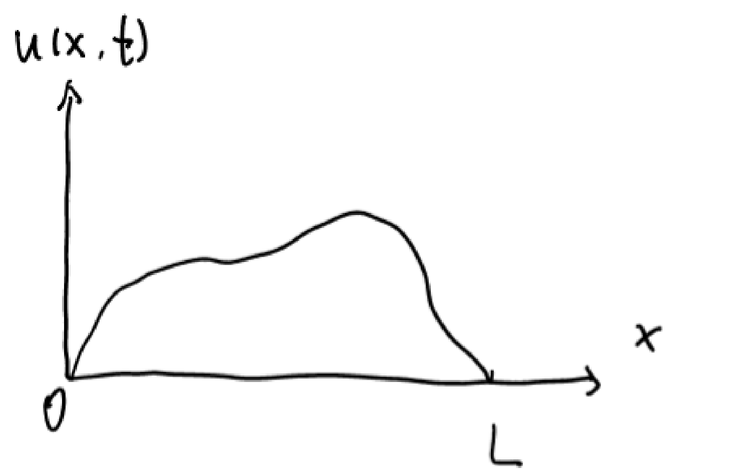
$$\|J_p(\lambda_{pn} x)\|^2 = \frac{1}{2}(a^2 - \frac{p^2}{\lambda_{pn}^2} + \frac{a^2}{\lambda_{pn}^2 + h^2})J_p^2(\lambda_{pn} a)$$

Case III. $d_1 = 0, d_2 = 1$, or $J_p'(\lambda_{pn} a) = 0$.

$\|J_p(\lambda_{pn} x)\|^2 = \frac{1}{2}(a^2 - \frac{p^2}{\lambda_{pn}^2})J_p^2(\lambda_{pn} a)$ (if p=0, $\lambda = 0$ is also an eigenvalue)

3.5 Separation of Variables and origin of the BVP

Example: Vibrating strings and one dimensional wave equation.



① two ends fixed at 0 and L;

② the motion is transverse only and small, at x-u plane.

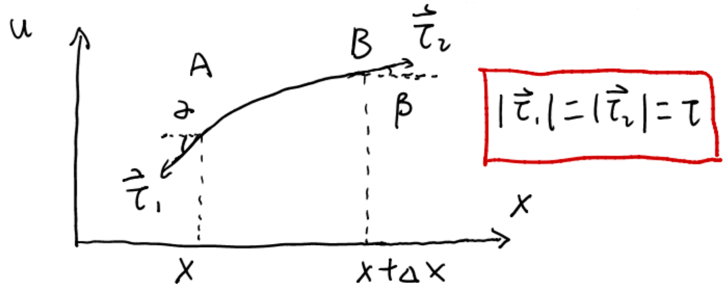


图 9: Force on a small portion of the string

$F_{\perp} = \tau(sin\beta - sin\alpha) = \tau \left(\frac{\partial u}{\partial x}|_{x+\Delta x,t} - \frac{\partial u}{\partial x}|_{x,t} \right)$

$\frac{\partial^2 u}{\partial t^2} = \frac{F_{\perp}}{\rho \Delta x} = \frac{\tau}{\rho} \frac{\left(\frac{\partial u}{\partial x}|_{x+\Delta x,t} - \frac{\partial u}{\partial x}|_{x,t} \right)}{\Delta x} = \frac{\tau}{\rho} \frac{\partial^2 u}{\partial x^2}$. (in the sense of $\Delta x \to 0$)

$\Longrightarrow$ PDE: $\quad \frac{\partial^2 u}{\partial t^2} = \frac{\tau}{\rho} \frac{\partial^2 u}{\partial x^2}$

BCs: $u(0,t) = 0, \quad u(L,t) = 0.$

ICs: $u(x,t)|_{t=0} = f(x), \quad \left. \frac{\partial u(x,t)}{\partial t} \right|_{t=0} = v(x)$

The method of separation of variables:

seeking a solution of the form: $u(x,t) = X(x)T(t)$.

thus, $\frac{\partial^2 u}{\partial t^2} = XT'', \quad \frac{\partial^2 u}{\partial x^2} = X''T$, plug into wave function (let velocity square $c^2 = \frac{\tau}{\rho}$), get :

$\frac{T''}{c^2 T} = \frac{X''}{X}$, must equal constant k $\Longrightarrow \begin{cases} X'' - kX = 0 \\ T'' - kc^2 T = 0 \end{cases}$

from the boundary condition,

$X(0) \cdot T(t) = X(L)T(t) \equiv 0$, for $\forall t$

if $T(t) \equiv 0$, trivial solution. So must be $X(0) = X(L) = 0$.

Thus we arrive at a regular Sturm-Liouville problem:

$X'' - kX = 0, X(0) = X(L) = 0$

from previous we know the eigenvalue and eigenfunctions:

$k_n = -(\frac{n\pi}{L})^2, \quad X_n(x) = sin\frac{n\pi}{L}x, n = 1, 2, \cdots.$

knowing k we can also solve for the time dependent part:

$T'' + (\frac{n\pi}{L})^2 c^2 T = 0$

$\Longrightarrow T_n = a_n cos\frac{cn\pi}{L}t + b_n sin\frac{cn\pi}{L}t$

the full solution with boundary conditions and eigenvalues k_n,

$u_n(x,t) = sin\frac{n\pi}{L}x(a_n cos\frac{cn\pi}{L}t + b_n sin\frac{cn\pi}{L}t), n = 1, 2, \cdots . \leftarrow$ normal mode.
since any of u_n is a solution of original PDE, thus
$u(x,t) = \sum_{n=1}^{\infty} sin\frac{n\pi}{L}x(a_n cos\lambda_n t + b_n sin\lambda_n t), \lambda_n = \frac{n\pi}{L}c$
can be thought as a general solution of PDE with BC.

$\{a_n, b_n\}$ will be eventually determined by the IC.(Uniquely!)

With ICs:

$$\begin{cases} f(x) = \sum_{n=1}^{\infty} c_n sin\frac{n\pi}{L}x = u(x,0) = \sum_{n=1}^{\infty} a_n sin\frac{n\pi}{L}x \\ v(x) = \sum_{n=1}^{\infty} d_n sin\frac{n\pi}{L}x = \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} b_n \lambda_n sin\frac{n\pi}{L}x \end{cases} \Longrightarrow \begin{cases} a_n = c_n \\ b_n = \frac{d_n}{\lambda_n} \end{cases}$$

D'Alembert's Method for Dim.1 Wave equation:

$u(0,t) = u(L,t) = 0$, supposing $u(x,0) = f(x)$, $u(x,0)|_t = g(x)$

We define:

$$f^*(x) = \begin{cases} f(x), 0 < x < L \\ -f(-x), -L < x < 0 \\ f(x + 2mL), otherwise \end{cases} \qquad \text{odd periodic extension, similar for } g^*(x)$$

Thus solution of the wave equation can be written as:

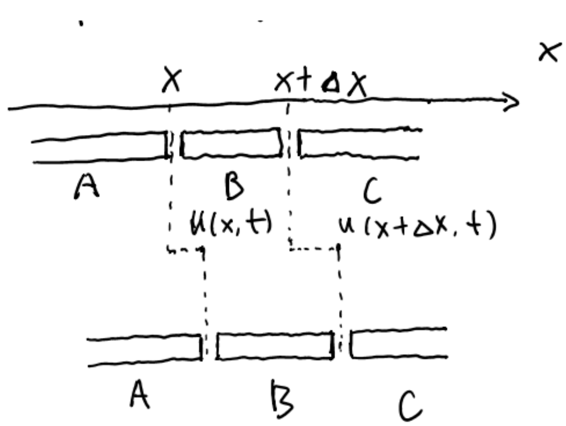
$$u(x,t) = \frac{1}{2}[f^*(x - ct) + f^*(x + ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g^*(s)ds$$

If we further define $G(x) = \int_a^x g^*(z)dz$,thus

$$\int_{x-ct}^{x+ct} g^*(z)dz = G(x + ct) - G(x - ct)$$

So we can rewrite $u(x,t) = \frac{1}{2}(f^*(x + ct) + \frac{1}{c}G(x + ct)) + \frac{1}{2}(f^*(x - ct) - \frac{1}{c}G(x - ct))$

Example: Longitudinal vibrations of elastic bars



$$F_x = -\frac{\partial u(x,t)}{\partial x}ES + \frac{\partial u(x+\Delta x,t)}{\partial x}ES = ES(\frac{\partial u(x+\Delta x,t)}{\partial x} - \frac{\partial u(x,t)}{\partial x})$$
$$\frac{\partial^2 u}{\partial t^2} = \frac{F_x}{\rho S \Delta x} = \frac{E}{\rho}\frac{\frac{\partial u}{\partial x}\Big|_{x+\Delta x,t} - \frac{\partial u}{\partial x}\Big|_{x,t}}{\Delta x} = \frac{E}{\rho}\frac{\partial^2 u}{\partial x^2} \qquad (c^2 = \frac{E}{\rho})$$

Applications:

Seismic Waves

①Body waves:

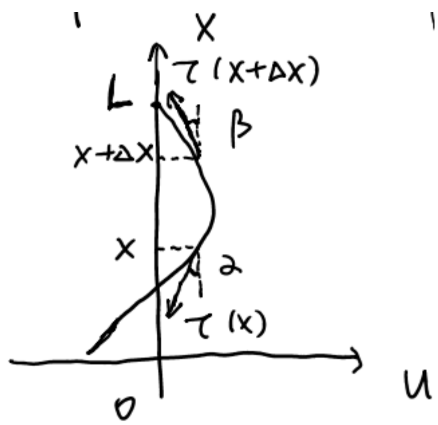
Primary Wave(P-wave) velocity $c_p = \sqrt{\frac{\lambda + 2\mu}{\rho}}$

Secondary Wave(S-wave) velocity $c_s = \sqrt{\frac{\mu}{\rho}}$

(In rocks, $c_p \approx 5000m/s$, $c_s \approx 3000m/s$)

②Surface waves

Example: The hanging chain



The change of tensions can not be neglected:$\tau(x+\Delta x) - \tau(x) \sim \rho g \Delta x \Longrightarrow \tau(x) = \rho g x$

Thus $F_\perp = \tau(x+\Delta x) sin\beta - \tau(x) sin\alpha = \rho \Delta x \frac{\partial^2 u}{\partial t^2}$

$$\Longrightarrow \frac{\partial^2 u}{\partial t^2} = \frac{1}{\rho}\frac{\tau(x+\Delta x)\ \frac{\partial u}{\partial x}\Big|_{x+\Delta x} - \tau(x)\ \frac{\partial u}{\partial x}\Big|_{x}}{\Delta x} = \frac{1}{\rho}\frac{\partial}{\partial x}\left(\tau \frac{\partial u}{\partial x}\right) = g\left(x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\right)$$

$u(L,t) = 0, u(x,0) = f(x), u_t(x,0) = v(x)$

Separation of variables:

$u(x,t) = X(x)T(t) \Longrightarrow XT'' = gT(xX'' + X')$

$$\frac{1}{g}\frac{T''}{T} = \frac{xX'' + X'}{X} \equiv \lambda$$

① The spatial part:

$xX'' + X' - \lambda X = 0, \quad 0 < x < L, X(L) = 0$(singular SL problem)

let $s = 2\sqrt{x}$,

$s^2\dfrac{d^2X}{ds^2} + s\dfrac{dX}{ds} - \lambda s^2 X = 0, \quad 0 < s < 2\sqrt{L}, X(2\sqrt{L}) = 0$ (parametric Bessel's equation of zero'th order)

eigenvalues: $\lambda_n = -\left(\dfrac{\alpha_n}{2\sqrt{L}}\right)^2 = -\dfrac{\alpha_n^2}{4L}, \quad n = 1, 2, \cdots$

eigenfunctions: $X_n(x) = J_0(\alpha_n\sqrt{\dfrac{x}{L}})$

② Time dependent part:

$T'' = \lambda_n g T \Longrightarrow T(t) = a_n cos(\dfrac{\alpha_n}{2}\sqrt{\dfrac{g}{L}}t) + b_n sin(\dfrac{\alpha_n}{2}\sqrt{\dfrac{g}{L}}t)$

$u_n(x,t) = J_0(\alpha_n\sqrt{\dfrac{x}{L}})\left(a_n cos(\dfrac{\alpha_n}{2}\sqrt{\dfrac{g}{L}}t) + b_n sin(\dfrac{\alpha_n}{2}\sqrt{\dfrac{g}{L}}t)\right)$

$u(x,t) = \sum_{n=1}^{\infty} J_0(\alpha_n\sqrt{\dfrac{x}{L}})\left(a_n cos(\dfrac{\alpha_n}{2}\sqrt{\dfrac{g}{L}}t) + b_n sin(\dfrac{\alpha_n}{2}\sqrt{\dfrac{g}{L}}t)\right)$

with $\{a_n, b_n\}$ determined from initial condition

4 Chapter 4. Introduction to Partial Differential Equations

4.1 Partial differential equations

PDE: $\Phi\left(x_1, x_2, \cdots, x_n, u, \dfrac{\partial u}{\partial x_i}(i = 1, \cdots, n), \dfrac{\partial^2 u}{\partial x_i \partial x_j}(i, j = 1, \cdots, n), \cdots\right) = 0$

order-the highest derivative that appears.

Most of problems in physics lead to second-order PDE:

$\Phi\left(x_1, x_2, \cdots, x_n, u, \dfrac{\partial u}{\partial x_i}, \dfrac{\partial^2 u}{\partial x_i \partial x_j}\right) = 0$

quasi-linear (linear wrt. second-order derivatives):

$\sum_{i,j=1}^{n} a_{ij}(x_1, \cdots, x_n)u_{x_i x_j} + F\left(x_1, x_2, \cdots, x_n, u, \dfrac{\partial u}{\partial x_i}\right) = 0$ (a_{ij} can be symmetric)

linear:

$\sum_{i,j=1}^{n} a_{ij}(x_1, \cdots, x_n)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x_1, \cdots, x_n)u_{x_i} + b_0(x_1, \cdots, x_n)u = f(x_1, \cdots, x_n)$

(homogeneous when $f = 0$)

Classification of 2nd-order linear PDE:

only two independent variables (note always assume $u_{xy} = u_{yx}$):

$a_{11}(x,y)u_{xx} + 2a_{12}(x,y)u_{xy} + a_{22}(x,y)u_{yy} + F(x, y, u, u_x, u_y) = 0$

definition:

hyperbolic: $a_{12}^2 - a_{11}a_{22} > 0$

parabolic: $a_{12}^2 - a_{11}a_{22} = 0$

elliptic: $a_{12}^2 - a_{11}a_{22} < 0$

with more independent variables (first turn into a form with all $a_{ij}(i \neq j) = 0$):

$\sum_{i=1}^{n} a_{ii}u_{x_i x_i} + F(x_i, u, u_{x_i}) = 0$

definition:

parabolic: exists $a_{ii}=0$

elliptic: all a_{ii} have same sign

hyperbolic: only one a_{ii} has different sign

super-hyperbolic: more than one a_{ii} have different sign

4.2 Some Examples of Equations of Mathematical Physics

A. The wave equation (hyperbolic)

one-dimensional : $u_{tt}-a^2u_{xx}=f(x,t)$

higher-dimensional : $u_{tt}-a^2\nabla^2u=f(x_1,\cdots,x_n,t)$

Example: acoustic equation

Ideal fluid,$\vec{v}$, pressure p, density ρ

① The continuous equation:

$\Delta x\Delta y\Delta z\Delta\rho=\Delta x\Delta y(v_z\rho|_z-v_z\rho|_{z+\Delta z})\Delta t+\cdots=-\Delta x\Delta y\Delta z\Delta t\dfrac{\partial(v_z\rho)}{\partial z}+\cdots=-\Delta x\Delta y\Delta z\Delta t\vec{\nabla}\cdot(\rho\vec{v})$

$\Longrightarrow \dfrac{\partial\rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{v})=0$

steady flow field: $\vec{\nabla}\cdot(\rho\vec{v})=0$

incompressible fluid: $\vec{\nabla}\cdot\vec{v}=0$

② The equation of motion:

$\rho(\Delta x\Delta y\Delta z)\dfrac{d}{dt}\hat{v}_x=\Delta y\Delta z(P|_x-P|_{x+\Delta x})$

$\Longrightarrow \rho\dfrac{d}{dt}\hat{v}_x=-\dfrac{\partial P}{\partial x},\quad or\quad \rho\dfrac{d}{dt}\vec{v}=-\vec{\nabla}\cdot P$

$\dfrac{d}{dt}\vec{v}=\dfrac{\partial}{\partial t}\vec{v}+(\vec{v}\cdot\vec{\nabla})\vec{v}$ (x→ x+vdt——— v(x,t)→ v(x+vdt,t+dt))

$\Longrightarrow \dfrac{\partial}{\partial t}\vec{v}+(\vec{v}\cdot\vec{\nabla})\vec{v}=-\dfrac{1}{\rho}\vec{\nabla}\cdot P$

steady flow field($\dfrac{\partial}{\partial t}\vec{v}=0$)+incompressible fluid$\Longrightarrow$ $P+\frac{1}{2}\rho v^2=C$ (Bernoulli's principle)

③ The equation of state:

gas with constant temp. : $\dfrac{P}{\rho}=Constant$

gas adiabatic expansion:

$PV=nRT\quad PdV=-dT\cdot C$

$\Longrightarrow PdV=-C\dfrac{PdV+VdP}{nR}$

$\Longrightarrow (nR+C)PdV=-CVdP$ define $\gamma=\dfrac{nR+C}{C}$(adiabatic index)

$\Longrightarrow PV^\gamma=Constant$ or $\dfrac{P}{\rho^\gamma}=Constant$

Now suppose a small oscillation of air:

$v=v_1(\vec{r},t),\quad P(\vec{r},t)=P_0+P_1(\vec{r},t),\quad \rho(\vec{r},t)=\rho_0+\rho_1(\vec{r},t)$

with $\dfrac{P_1}{P_0} \ll 1, \quad \dfrac{\rho_1}{\rho_0} \ll 1$

From equation of state: $\dfrac{P_1}{P_0} = \beta \dfrac{\rho_1}{\rho_0}$

From equation of motion: $\dfrac{\partial}{\partial t}\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\dfrac{1}{\rho}\vec{\nabla}\cdot P$ (where $(\vec{v}\cdot\vec{\nabla})\vec{v}$ is second-order small)

From equation of continuous: $\dfrac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0$

$\Longrightarrow \dfrac{\partial^2 \rho_1}{\partial t^2} = -\vec{\nabla}\cdot(\dfrac{\partial \rho}{\partial t}\vec{v} + \rho(-\dfrac{1}{\rho})\vec{\nabla}P)$

where $\dfrac{\partial \rho}{\partial t}$ is second-order small

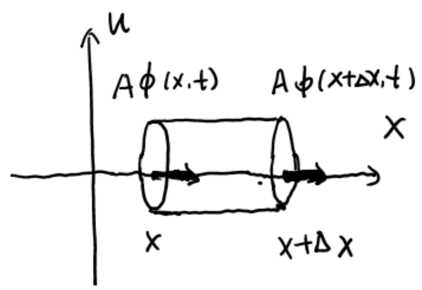
$\Longrightarrow = \nabla^2 P = \beta \dfrac{P_0}{\rho_0}\nabla^2 \rho_1 \qquad (c^2 = \beta \dfrac{P_0}{\rho_0} \Longrightarrow c = \sqrt{\beta}\sqrt{\dfrac{P_0}{\rho_0}})$

Also since $\dfrac{P_0}{\rho_0} \propto \dfrac{T}{m}$, thus velocity increase with temp., and decrease with molecular weight.

B. Heat conduction/diffusion equation (parabolic)

one-dimensional case: $u_t - a^2 u_{xx} = f(x,t)$

Example: heat equation of a uniform insulated bar.



u→ temp.; ϕ → heat flux; A→ area; e → heat-energy density; Q → heat source;

S(x)→ specific heat (heat energy required to raise one degree/mass);

$\rho(x)$→ mass density; k_0→ thermal conductivity

Fourier's law of thermal conductivity: $\phi(x,t) = -k_0 \dfrac{\partial u}{\partial x}$

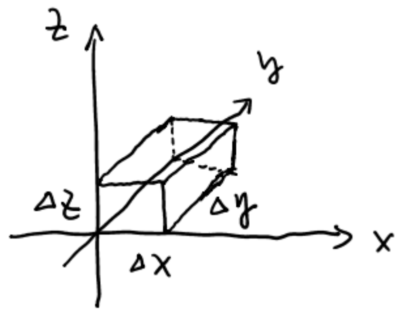
$\dfrac{\partial}{\partial t}(eA\Delta x) = A(\phi(x,t) - \phi(x+\Delta x, t)) + QA\Delta x$

$\Longrightarrow S(x)\rho(x)\dfrac{\partial u}{\partial t} = k_0 \dfrac{\partial^2 u}{\partial x^2} + Q$

so $\dfrac{\partial u}{\partial t} = \dfrac{k_0}{S\rho}\dfrac{\partial^2 u}{\partial x^2} + \dfrac{1}{S\rho}Q$ Constant $\dfrac{k_0}{S\rho}$ is called the thermal diffusivity

higher-dimensional case: $u_t - a^2\nabla^2 u = f(x_1, \cdots, x_n, t)$

Example: diffusion equation in three dimensions



u $\rightarrow$ number density; $\vec{\phi}\rightarrow$ number flux/diffusion flux

Fick's law: $\vec{\phi} = -D\vec{\nabla}u$

Thus $\frac{\partial}{\partial t}(u\Delta x\Delta y\Delta z) = \Delta y\Delta z(\phi_1(x,y,z,t) - \phi_1(x+\Delta x,y,z,t)) + \cdots + \Delta x\Delta y\Delta z\cdot S$

$\Longrightarrow \frac{\partial u}{\partial t} = D\cdot\nabla^2 u + S, \quad D>0$, S be the source

C. Poisson equation (elliptic)

n-dimensional case: $\nabla^2 u = \Delta u = \rho(x_1, x_2, \cdots, x_n)$

if $\rho = 0, \quad \nabla^2 u = 0 \longrightarrow$ Laplace equation

e.g.:

electromagnetism: $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$

gravitational potential : $\nabla^2 V = -4\pi G\rho$

Supposing in $\frac{\partial u}{\partial t} = a^2\nabla^2 u + S(x_1,\cdots,x_n,t)$ having $S(x_1,\cdots,x_n,t) = S(x_1,\cdots,x_n)$

thus giving long enough time, $\frac{\partial u}{\partial t} = 0$

$\Longrightarrow a^2\nabla^2 u + S(x_1,\cdots,x_n) = 0$ turns to poisson/Laplace equation

4.3 Formulation of problems of mathematical physics

A. Initial and boundary conditions

Initial conditionn, E.g.:

in diffusion problem: $u(\vec{x},t_0) = \phi(\vec{x})$, initial concentration

in heat substance: $u(\vec{x},t_0) = T(\vec{x})$, initial temperature

in wave equations: $\begin{cases} u(\vec{x},t_0) = \phi(\vec{x}) & \text{initial position} \\ \frac{\partial u}{\partial t}(\vec{x},t_0) = \psi(\vec{x}) & \text{initial velocity} \end{cases}$

Boundary conditions:

Three kinds of boundary conditions:

Dirichlet condition: $u\big|_{\Sigma} = f_1$

Neumann condition: the normal derivative($\vec{n}\cdot\vec{\nabla}u$) is specified: $\frac{\partial u}{\partial n}\Big|_{\Sigma} = f_2$

Robin condition: $(\frac{\partial u}{\partial n}+hu)\Big|_{\sum}=f_3$

($f_i=0\longrightarrow$ homogeneous BCs)

Example:

heat conduction:

system in a large reservoir (with perfect thermal conduction): $u(\sum,t)=g(t)$

system insulated: $\frac{\partial u(\sum,t)}{\partial n}=0,\qquad$ no heat flow

in a small reservoir: $\frac{\partial u(\sum,t)}{\partial n}=-a[u(\sum,t)-g(t)]$

B. Superposition principle

u_1 and u_2 are solutions of a linear homogeneous PDE$\longrightarrow$ $u=c_1u_1+c_2u_2$ is also a solution.

If in addition u_1 and u_2 satisfy a linear homogeneous boundary conditions, then so will $u=c_1u_1+c_2u_2$.

4.4 Special example for homogeneous PDE with constant coefficients

General: two independent variables:

$L(D_x,D_y)u=[A_0D_x^n+A_1D_x^{n-1}D_y+\cdots+A_nD_y^n+B_0D_x^{n-1}+\cdots+MD_x+ND_y+P]u=0$

① $L(D_x,D_y)$ being homogeneous in D_x,D_y.(only A_i non-vanishing)

the auxiliary equation: $A_0\alpha^n+A_1\alpha^{n-1}+\cdots+A_n=0$

Case I: has distinct n roots, $\alpha_1,\alpha_2,\cdots,\alpha_n\Longrightarrow L=\prod_i(D_x-\alpha_iD_y)$

general solution: $u=\phi_1(y+\alpha_1x)+\cdots+\phi_n(y+\alpha_nx)$

(ϕ_i being arbitrary independent functions)

Case II: if exist double root α, then $u=x\phi_1(y+\alpha x)+\phi_2(y+\alpha x)+\cdots$

or α being m-times root, $u=x^{m-1}\phi_1(y+\alpha x)+x^{m-2}\phi_2(y+\alpha x)+\cdots+\phi_m(y+\alpha x)+\cdots$

Proof:

imaging $L(D_x,D_y)=\prod_i(D_x-\alpha_iD_y), Lu=0$

means $(\frac{\partial}{\partial x}-\alpha_1\frac{\partial}{\partial y})[\prod_{i>1}(D_x-\alpha_iD_y)u]=0$

let $x'=\alpha_1x+y, y'=\alpha_1x-y\Longrightarrow\frac{\partial}{\partial x}-\alpha_1\frac{\partial}{\partial y}=2\alpha_1\frac{\partial}{\partial y'}$

thus $\prod_{i>1}(D_x-\alpha_iD_y)u=Const\,wrt.y'=\phi_1(x')=\phi_1(\alpha_1x+y)$

further,$\cdots$. For double root, $\frac{\partial^2}{\partial y'^2}[\prod_{i>2}(D_x-\alpha_iD_y)u]=0$

$\Longrightarrow\prod_{i>2}(D_x-\alpha_iD_y)u=cy'+Const\,wrt.y'=x\phi_1(x')+\phi_2(x')$

② $L(D_x, D_y)$ not homogeneous in D_x, D_y.

e.g.:

$(D_x - \alpha D_y - \beta)u = 0$

let $x' = \alpha x + y, y' = \alpha x - y \Longrightarrow 2\alpha \cdot \dfrac{\partial u}{\partial y'} = \beta u$, now thinking x' as Constant.(ODE wrt. y')

$u = e^{\frac{\beta}{2\alpha}y'} \cdot \phi(x') = e^{\frac{\beta}{2\alpha}(2\alpha x - x')}\phi(x') = e^{\beta x}\psi(\alpha x + y)$

$(D_x^2 - D_x D_y - 2D_y^2 + 2D_x + 2D_y)u = 0$, or $(D_x + D_y)(D_x - 2D_y + 2)u = 0$,

$u = \phi(x - y) + e^{-2x}\psi(y + 2x)$

$(D_x - \alpha D_y - \beta)^2 u = 0$,then

$u = xe^{\beta x}\phi(y + \alpha x) + e^{\beta x}\psi(y + \alpha x)$

5 Chapter 5. Partial differential equations in rectangular

any physics problems:

PDEs $\bigoplus$ boundary conditions $\bigoplus$ initial conditions

It is always true as in physics:

① the solution exists ② the solution is unique ③ the solution is stable

5.1 Solution of the one dimensional PDEs: separation of variable

A. Homogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, t > 0$$

with initial condition $u(x, 0) = \varphi(x)$ and $u_t(x, 0) = \psi(x)$, and boundary conditions.

Separation of variable $u(x, t) = X(x)T(t)$.

$\Longrightarrow \dfrac{X''(x)}{X(x)} = \dfrac{1}{c^2}\dfrac{T''(t)}{T(t)} = -\lambda$

①1st-kind boundary condition: $u(0, t) = u(L, t) = 0$

$\Longrightarrow X(0)T(t) = X(L)T(t) = 0 \Longrightarrow X(0) = X(L) = 0$

SL problem: $X''(x) = -\lambda X(x), \quad X(0) = X(L) = 0$

$\lambda = 0, \quad X(x) = ax + b \longrightarrow$ zero solution;

$\lambda < 0, \quad X(x) = asinh(\sqrt{-\lambda}x) + bcosh(\sqrt{-\lambda}x) \longrightarrow$ zero solution;

$\lambda > 0, \quad X(x) = asin(\sqrt{\lambda}x) + bcos(\sqrt{\lambda}x) \longrightarrow \lambda = \left(\dfrac{n\pi}{L}\right)^2, n = 1, 2, \cdots$

Thus, $\lambda_n = \left(\dfrac{n\pi}{L}\right)^2, \quad X_n = sin\left(\dfrac{n\pi x}{L}\right), n = 1, 2, \cdots$

The time dependent part:

$T''(t) = -\lambda_n c^2 T(t) \Longrightarrow T(t) = A_n cos\left(\frac{n\pi c}{L}t\right) + B_n sin\left(\frac{n\pi c}{L}t\right)$

The full solution with BVC:

$u_n(x,t) = sin\left(\frac{n\pi x}{L}\right)\left(A_n cos\left(\frac{n\pi c}{L}t\right) + B_n sin\left(\frac{n\pi c}{L}t\right)\right)$

by superposition rule:

$u(x,t) = \sum_{n=1}^{\infty} sin\left(\frac{n\pi x}{L}\right)\left(A_n cos\left(\frac{n\pi c}{L}t\right) + B_n sin\left(\frac{n\pi c}{L}t\right)\right)$ is solution of PDE+BVC as well.

$u(x,0) = \sum_{n=1}^{\infty} A_n sin\left(\frac{n\pi x}{L}\right), \qquad u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n sin\left(\frac{n\pi x}{L}\right)$

Thus if choosing $\begin{cases} A_n = \frac{2}{L}\int_0^L \varphi(x) sin\left(\frac{n\pi x}{L}\right)dx \\ B_n = \frac{2}{n\pi c}\int_0^L \psi(x) sin\left(\frac{n\pi x}{L}\right)dx \end{cases}$

$\Longrightarrow$ solution for PDE+BVC+IVC

② 2nd-kind of boundary condition: $u_x(0,t) = u_x(L,t) = 0$

$\Longrightarrow X'(0)T(t) = X'(L)T(t) = 0 \Longrightarrow X'(0) = X'(L) = 0$

SL problem: $X''(x) = -\lambda X(x), \quad X'(0) = X'(L) = 0$

$\lambda = 0, \quad X(x) = ax + b \longrightarrow X(x) = b$;

$\lambda < 0, \quad X(x) = asinh(\sqrt{-\lambda}x) + bcosh(\sqrt{-\lambda}x) \longrightarrow$ zero solution;

$\lambda > 0, \quad X(x) = asin(\sqrt{\lambda}x) + bcos(\sqrt{\lambda}x) \longrightarrow \lambda = \left(\frac{n\pi}{L}\right)^2, n = 1, 2, \cdots$

Thus, $\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad X_n = cos\left(\frac{n\pi x}{L}\right), n = 0, 1, 2, \cdots$

The time dependent part:

$T''(t) = -\lambda_n c^2 T(t) \Longrightarrow T(t) = \begin{cases} A_0 + B_0 t, n = 0 \\ A_n cos\left(\frac{n\pi c}{L}t\right) + B_n sin\left(\frac{n\pi c}{L}t\right) \end{cases}$

The full solution with BVC:

$u(x,t) = A_0 + B_o t + \sum_{n=1}^{\infty} cos\left(\frac{n\pi x}{L}\right)\left(A_n cos\left(\frac{n\pi c}{L}t\right) + B_n sin\left(\frac{n\pi c}{L}t\right)\right)$

At initial time:

$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n cos\left(\frac{n\pi x}{L}\right) \equiv \varphi(x)$

$u_t(x,0) = B_0 + \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n cos\left(\frac{n\pi x}{L}\right) \equiv \psi(x)$

Get A_n, B_n from Fourier expansion of $\varphi(x)$ and $\psi(x)$:

$A_n = \begin{cases} \frac{2}{L}\int_0^L \varphi(x) cos(\frac{n\pi x}{L})dx, n > 0 \\ \frac{1}{L}\int_0^L \varphi(x) cos(\frac{n\pi x}{L})dx, n = 0 \end{cases}$

$B_n = \begin{cases} \frac{2}{n\pi c}\int_0^L \varphi(x) cos(\frac{n\pi x}{L})dx, n > 0 \\ \frac{1}{L}\int_0^L \varphi(x) cos(\frac{n\pi x}{L})dx, n = 0 \end{cases}$

③ mixed boundary condition: $u(0,t) = u_x(L,t) = 0$

$\Longrightarrow X(0)T(t) = X'(L)T(t) = 0 \Longrightarrow X(0) = X'(L) = 0$

SL problem: $X''(x) = -\lambda X(x), \quad X(0) = X'(L) = 0$

$\lambda = 0, \quad X(x) = ax + b \longrightarrow$ zero solution ;

$\lambda < 0, \quad X(x) = asinh(\sqrt{-\lambda}x) + bcosh(\sqrt{-\lambda}x) \longrightarrow$ zero solution;

$\lambda > 0, \quad X(x) = asin(\sqrt{\lambda}x) + bcos(\sqrt{\lambda}x) \longrightarrow \lambda = \left(\frac{(\frac{1}{2}+n)\pi}{L}\right)^2, n = 0, 1, 2, \cdots$

Thus, $\lambda_n = \left(\frac{(\frac{1}{2}+n)\pi}{L}\right)^2, \quad X_n = sin\left(\frac{(\frac{1}{2}+n)\pi x}{L}\right), n = 0, 1, 2, \cdots$

The time dependent part:

$T''(t) = -\lambda_n c^2 T(t) \Longrightarrow T(t) = A_n cos\left(\frac{(\frac{1}{2}+n)\pi c}{L}t\right) + B_n sin\left(\frac{(\frac{1}{2}+n)\pi c}{L}t\right)$

The full solution with BVC:

$u(x,t) = \sum_{n=0}^{\infty} sin\left(\frac{(\frac{1}{2}+n)\pi x}{L}\right)\left(A_n cos\left(\frac{(\frac{1}{2}+n)\pi c}{L}t\right) + B_n sin\left(\frac{(\frac{1}{2}+n)\pi c}{L}t\right)\right)$

At initial time:

$u(x,0) = \sum_{n=0}^{\infty} A_n sin\left(\frac{(\frac{1}{2}+n)\pi x}{L}\right) \equiv \varphi(x)$

$u_t(x,0) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2}+n)\pi c}{L} B_n sin\left(\frac{(\frac{1}{2}+n)\pi x}{L}\right) \equiv \psi(x)$

satisfying IVC if

$A_n = \frac{2}{L}\int_0^L \varphi(x) sin\left(\frac{(\frac{1}{2}+n)\pi x}{L}\right) dx$

$B_n = \frac{2}{(\frac{1}{2}+n)\pi c}\int_0^L \psi(x) sin\left(\frac{(\frac{1}{2}+n)\pi x}{L}\right) dx$

B. Homogeneous heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, t > 0$$

with initial condition $u(x,0) = \varphi(x)$ and boundary conditions.

Separation of variable: $u(x,t) = X(x)T(t)$

$\Longrightarrow \frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T'(t)}{T(t)} = -\lambda$

① 1st-kind boundary condition: $u(0,t) = u(L,t) = 0$

$\Longrightarrow X(0)T(t) = X(L)T(t) = 0 \Longrightarrow X(0) = X(L) = 0$

SL problem: $X''(x) = -\lambda X(x), X(0) = X(L) = 0$

solution: $\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad X_n = sin\left(\frac{n\pi x}{L}\right), n = 1, 2, \cdots$

time dependent part:

$T'(t) = -\lambda_n c^2 T(t)$

full solution with BVC, $u(x,t) = \sum_{n=1}^{\infty} A_n sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi c}{L}\right)^2 t}$

at initial time: $u(x,t) = \sum_{n=1}^{\infty} A_n sin\left(\frac{n\pi x}{L}\right) \equiv \varphi(x)$

Thus choosing $A_n = \frac{2}{L}\int_0^L \varphi(x) sin\left(\frac{n\pi x}{L}\right) dx$, get solution of BVC+IVC.

② 2nd-kind boundary condition: $u_x(0,t) = u_x(L,t) = 0$

$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi c}{L}\right)^2 t}$

with $A_n = \begin{cases} \frac{2}{L}\int_0^L \varphi(x) cos(\frac{n\pi x}{L}) dx, n > 0 \\ \frac{1}{L}\int_0^L \varphi(x) cos(\frac{n\pi x}{L}) dx, n = 0 \end{cases}$

③ mixed boundary condition: $u(0,t) = u_x(L,t) = 0$

$u(x,t) = \sum_{n=0}^{\infty} A_n sin\left(\frac{(\frac{1}{2}+n)\pi x}{L}\right) e^{-\left(\frac{(\frac{1}{2}+n)\pi c}{L}\right)^2 t}$

with $A_n = \frac{2}{L}\int_0^L \varphi(x) sin\left(\frac{(\frac{1}{2}+n)\pi x}{L}\right) dx$, n=0,1,2,$\cdots$

④ mixed 3rd-kind: $u(0,t) = 0, u_x(L,t) + ku(L,t) = 0,\ k > 0$

$\Longrightarrow X(0)T(t) = (X'(L) + kX(L))T(t) = 0 \Longrightarrow X(0) = X'(L) + kX(L) = 0$

SL problem: $X''(x) = -\lambda X(x), X(0) = 0, X'(L) + kX(L) = 0$

solution: $\lambda_n = (\mu_n)^2, X_n = sin\mu_n x, [\mu_n = -ktg(\mu_n L), n = 1, 2, \cdots]$

thus $u(x,t) = \sum_{n=1}^{\infty} A_n sin(\mu_n x) e^{-c^2\mu_n^2 t}$

choosing $A_n = \frac{\int_0^L \varphi(x) sin(\mu_n x) dx}{\int_0^L sin^2(\mu_n x) dx}$,solution for BVC+IVC

C. Nonhomogeneous equations

e.g.:

non-homogeneous wave equation with homogeneous boundary conditions of 1st-kind.

$$\begin{cases} u_{tt} = c^2 u_{xx} + f(x,t) \\ u(0,t) = u(L,t) = 0 \qquad\qquad t > 0, 0 < x < L \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \end{cases}$$

Assuming a solution with undetermined coefficients:

$u(x,t) = \sum_{n=1}^{\infty} T_n(t) sin\left(\frac{n\pi x}{L}\right)$ (already satisfying boundary conditions)

Plug into PDE, we arrive at:

$\sum_{n=0}^{\infty} T_n''(t) sin\left(\frac{n\pi x}{L}\right) = -c^2 \sum_{n=0}^{\infty} T_n(t)\frac{n^2\pi^2}{L^2} sin\left(\frac{n\pi x}{L}\right) + \sum_{n=0}^{\infty} f_n(t) sin\left(\frac{n\pi x}{L}\right)$

with $f_n(t) = \frac{2}{L}\int_0^L f(x,t) sin\left(\frac{n\pi x}{L}\right) dx$

Thus $T_n(t)$ satisfy: $T_n'' + \frac{n^2\pi^2c^2}{L^2} T_n = f_n(t), n = 0, 1, 2, \cdots$

Similarly from initial conditions: $T_n(0) = \varphi_n, T_n'(0) = \psi_n$

with $\varphi_n = \frac{2}{L}\int_0^L \varphi(x) sin\left(\frac{n\pi x}{L}\right) dx, \psi_n = \frac{2}{L}\int_0^L \psi(x) sin\left(\frac{n\pi x}{L}\right) dx$

Solve $T_n(t)$ ($\omega_n = \dfrac{n\pi c}{L}$):
$y_1(t) = sin\omega_n t, y_2(t) = cos\omega_n t, W(y_1, y_2) = -\omega_n$
thus $y_p = \dfrac{1}{\omega_n}\displaystyle\int_0^t f_n(\tau) sin(\omega_n(t-\tau))d\tau$
using IVC, we get:
$T_n(t) = \dfrac{1}{\omega_n}\displaystyle\int_0^t f_n(\tau) sin(\omega_n(t-\tau))d\tau + \varphi_n cos\omega_n t + \dfrac{\psi_n}{\omega_n} sin\omega_n t$

D. Wave and heat equations with nonhomogeneous BVC

basic idea: let $u(x,t) = v(x,t) + \omega(x,t)$

with $v(x,t)$ satisfying the BVC. (turns to solve a nonhomogeneous PDE with homogeneous BVC)

E.g.:

wave equation: $t > 0, 0 < x < L$

choose $v(x,t) = f_2(t) + (f_2(t) - f_1(t))\dfrac{x-L}{L}$

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = f_1(t), u(L,t) = f_2(t) \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \end{cases} \Longrightarrow \begin{cases} \omega_{tt} = c^2\omega_{xx} - v_{tt} \\ \omega(0,t) = \omega(L,t) = 0 \\ \omega(x,0) = \varphi(x) - v(x,0) \\ \omega_t(x,0) = \psi(x) - v_t(x,0) \end{cases}$$

The heat equation: $t > 0, 0 < x < L$

$$\begin{cases} \dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = T_1, u(L,t) = T_2 \\ u(x,0) = f(x) \end{cases} \Longrightarrow \begin{cases} \dfrac{\partial u_2}{\partial t} = c^2 \dfrac{\partial^2 u_2}{\partial x^2} \\ u_2(0,t) = u_2(L,t) = 0 \\ u_2(x,0) = f(x) - u_1(x) \end{cases}$$

let $u(x,t) = u_1(x) + u_2(x,t)$, with $u_1(x) = \dfrac{T_2 - T_1}{L}x + T_1$ $\quad$ ($t \to \infty, u(x,t) \to u_1(x)$ steady state solution)

5.2 D'Alembert's method

A. Wave equation on infinite intervals (thus no BVC)

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \end{cases} \qquad t > 0, -\infty < x < +\infty$$

Solution: $u(x,t) = \dfrac{1}{2}[\varphi(x+ct) + \varphi(x-ct)] + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct}\psi(x')dx'$

Proof: changing of variable

let $y = x + ct, z = x - ct$, thus
$\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z}, \quad \dfrac{\partial u}{\partial t} = c(\dfrac{\partial u}{\partial y} - \dfrac{\partial u}{\partial z})$
$\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} + 2\dfrac{\partial^2 u}{\partial y \partial z}, \quad \dfrac{\partial^2 u}{\partial t^2} = c^2(\dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} - 2\dfrac{\partial^2 u}{\partial y \partial z})$
So the PDE becomes $\dfrac{\partial^2 u}{\partial y \partial z} = 0$.

$u = \int f(y)dy + g(z) = v(x+ct) + \omega(x-ct)$

apply IVC$\Longrightarrow \begin{cases} v(x) + \omega(x) = \varphi(x) \\ v'(x) - \omega'(x) = \dfrac{\psi(x)}{c} \end{cases}$

solving for v and ω, $\begin{cases} v(x) + \omega(x) = \varphi(x) \\ v(x) - \omega(x) = \displaystyle\int_{x_0}^{x} \frac{\psi(\tau)}{c} d\tau + v(x_0) - \omega(x_0) \end{cases}$

thus $\begin{cases} v(x) = \dfrac{1}{2}\varphi(x) + \dfrac{1}{2}\displaystyle\int_{x_0}^{x} \frac{\psi(\tau)}{c} d\tau + \frac{1}{2}(v(x_0) - \omega(x_0)) \\ \omega(x) = \dfrac{1}{2}\varphi(x) - \dfrac{1}{2}\displaystyle\int_{x_0}^{x} \frac{\psi(\tau)}{c} d\tau - \frac{1}{2}(v(x_0) - \omega(x_0)) \end{cases}$

finally we get $u(x,t) = v(x+ct) + \omega(x-ct) = \dfrac{1}{2}[\varphi(x+ct) + \varphi(x-ct)] + \dfrac{1}{2}\displaystyle\int_{x-ct}^{x+ct} \frac{\psi(\tau)}{c} d\tau$

B. Wave equation on semi-infinite interval

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \end{cases} \qquad t > 0, 0 \leq x < +\infty$$

With BVC:$u(0,t) = 0$.

define odd extension of $\varphi(x)$ and $\psi(x)$: $f^*(x) = \begin{cases} f(x), x \geq 0 \\ -f(-x), x < 0 \end{cases}$

$u(x,t) = \dfrac{1}{2}[\varphi^*(x+ct) + \varphi^*(x-ct)] + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} \psi^*(\tau)d\tau$

With BVC:$u_x(0,t) = 0$.

define even extension of $\varphi(x)$ and $\psi(x)$: $f^+(x) = \begin{cases} f(x), x \geq 0 \\ f(-x), x < 0 \end{cases}$

$u(x,t) = \dfrac{1}{2}[\varphi^+(x+ct) + \varphi^+(x-ct)] + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} \psi^+(\tau)d\tau$

C. Wave equation on finite intervals

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \end{cases} \qquad t > 0, 0 < x < L$$

define the 2L-peoriodic extension $\widetilde{f}(x) = \begin{cases} f(x), 0 < x < L \\ -f(-x), -L < x < 0 \\ f(x - 2pL), otherwise \end{cases}$

thus the solution can be written as:

$u(x,t) = \dfrac{1}{2}[\widetilde{\varphi}(x+ct) + \widetilde{\varphi}(x-ct)] + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} \widetilde{\psi}(\tau)d\tau$

5.3 Two-dimensional wave and heat equation

Suppose that a thin elastic membrane is stretched over a rectangular frame with dimensions a and b, and the edges are fixed.

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} \right), \quad 0 < x < a, 0 < y < b, t > 0 \\ u(0, y, t) = u(a, y, t) = u(x, 0, t) = u(x, b, t) = 0 \\ u(x, y, 0) = \varphi(x, y) \\ u_t(x, y, 0) = \psi(x, y) \end{cases}$$

Assume $u(x, y, t) = X(x)Y(y)T(t)$.

$\Longrightarrow XYT'' = c^2(X''YT + XY''T) \Longrightarrow \dfrac{X''}{X} + \dfrac{Y''}{Y} = \dfrac{1}{c^2}\dfrac{T''}{T}$

thus we must have:

$\dfrac{X''}{X} = -\rho, \ \dfrac{Y''}{Y} = -r, \ \dfrac{T''}{T} = -c^2(\rho + r)$

from the BVC:

$X(0)Y(y)T(t) = X(a)Y(y)T(t) = X(x)Y(0)T(t) = X(x)Y(b)T(t) = 0$

it requires: $X(0) = X(a) = Y(0) = Y(b) = 0$

brings to two SL problems:

$$\begin{cases} \rho_n = \left(\dfrac{n\pi}{a}\right)^2, X_n = sin\left(\dfrac{n\pi x}{a}\right) \\ r_m = \left(\dfrac{m\pi}{b}\right)^2, Y_m = sin\left(\dfrac{m\pi y}{b}\right) \end{cases} \qquad m, n = 1, 2, \cdots$$

for time dependent part: ($\lambda_{nm} = c\pi\sqrt{(\frac{n}{a})^2 + (\frac{m}{b})^2}$)

$T_{nm}(t) = A_{nm}cos\lambda_{nm}t + B_{nm}sin\lambda_{nm}t$

full solution:

$$u(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} sin\frac{n\pi x}{a} sin\frac{m\pi y}{b} \left(A_{nm}cos\lambda_{nm}t + B_{nm}sin\lambda_{nm}t\right)$$

The coefficients can be determined from IVC: (two dimensional Fourier series)

$$\begin{cases} A_{nm} = \dfrac{4}{ab}\displaystyle\int_0^a dx \int_0^b dy\varphi(x, y)sin\frac{n\pi x}{a}sin\frac{m\pi y}{b} \\ B_{nm} = \dfrac{4}{c\pi\sqrt{(nb)^2 + (ma)^2}}\displaystyle\int_0^a dx \int_0^b dy\psi(x, y)sin\frac{n\pi x}{a}sin\frac{m\pi y}{b} \end{cases}$$

Similar for two-dimensional heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = c^2 \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} \right), \quad 0 < x < a, 0 < y < b, t > 0 \\ u(0, y, t) = u(a, y, t) = u(x, 0, t) = u(x, b, t) = 0 \\ u(x, y, 0) = \varphi(x, y) \end{cases}$$

full solution:

$$u(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} sin\frac{n\pi x}{a} sin\frac{m\pi y}{b} e^{-\lambda_{nm}^2 t}$$

with $\lambda_{nm} = c\pi\sqrt{(\frac{n}{a})^2 + (\frac{m}{b})^2}$

$$A_{nm} = \frac{4}{ab}\int_0^a dx \int_0^b dy \varphi(x,y) sin\frac{n\pi x}{a} sin\frac{m\pi y}{b} \qquad m,n = 1,2,\cdots$$

The physics in Microwave oven:

① Generation of microwaves.

② pumping into metal box, form of standing waves.

③ forced rotation of electric dipoles.

④ heat source and redistribution.

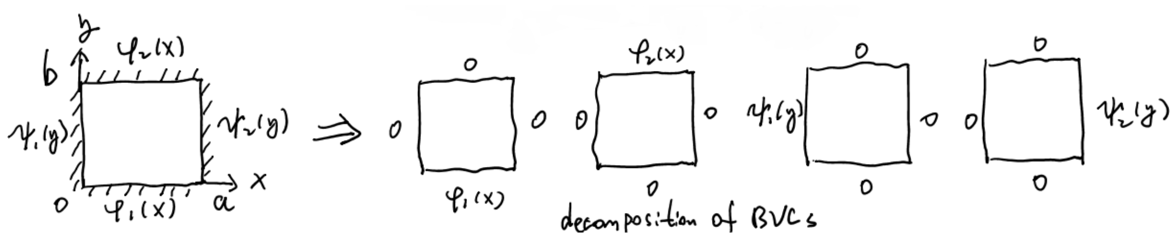
5.4 Laplace's equation in rectangular coordinates

Laplace's equation: $\nabla^2 u = 0$

In one dimension: $\dfrac{\partial^2 u}{\partial x^2} = 0 \Longrightarrow u = ax + b$ (trivial)

Two dimensional case:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, 0 < y < b \\ u(x,0) = \varphi_1(x), u(x,b) = \varphi_2(x) \\ u(0,y) = \psi_1(y), u(a,y) = \psi_2(y) \end{cases} \qquad \text{BVCs (no IVC)}$$



$u = u_1 + u_2 + u_3 + u_4$, each satisfies PDE with one of above BVCs, taking u_1 as an example.

$$\begin{cases} \dfrac{\partial^2 u_1}{\partial x^2} + \dfrac{\partial^2 u_1}{\partial y^2} = 0, \quad 0 < x < a, 0 < y < b \\ u_1(x,0) = \varphi_1(x), u_1(x,b) = u_1(0,y) = u_1(a,y) = 0 \end{cases}$$

Seperation of variables: $u_1(x,y) = X(x)Y(y)$

$$\frac{X''}{X} = \frac{-Y''}{Y} = -\lambda, \quad X(0) = X(a) = 0, Y(b) = 0$$

Solving SL problem:

$$u_1(x,y) = \sum_{n=1}^{\infty} sin\frac{n\pi}{a}x\left(A_n sinh\frac{n\pi}{a}(b-y) + 0\right)$$

with A_n determined by $u_1(x,0) = \varphi_1(x) \Longrightarrow A_n = \dfrac{2}{a sinh\frac{n\pi b}{a}}\displaystyle\int_0^a \varphi_1(x) sin\frac{n\pi x}{a}dx$

Similarly can find u_2, u_3, u_4, finally:

$$u(x,y) = \sum_{n=1}^{\infty} A_n sin\frac{n\pi x}{a} sinh\frac{n\pi}{a}(b-y) + \sum_{n=1}^{\infty} B_n sin\frac{n\pi x}{a} sinh\frac{n\pi}{a}y + \sum_{n=1}^{\infty} C_n sinh\frac{n\pi}{b}(a-x) sin\frac{n\pi y}{b} + \sum_{n=1}^{\infty} D_n sinh\frac{n\pi x}{b} sin\frac{n\pi y}{b}$$

with $B_n = \dfrac{2}{a sinh\frac{n\pi b}{a}} \displaystyle\int_0^a \varphi_2(x) sin\frac{n\pi x}{a} dx;$

$C_n = \dfrac{2}{b sinh\frac{n\pi a}{b}} \displaystyle\int_0^b \psi_1(y) sin\frac{n\pi y}{b} dy;$

$D_n = \dfrac{2}{b sinh\frac{n\pi a}{b}} \displaystyle\int_0^b \psi_2(y) sin\frac{n\pi y}{b} dy, \qquad n = 1, 2, \cdots.$

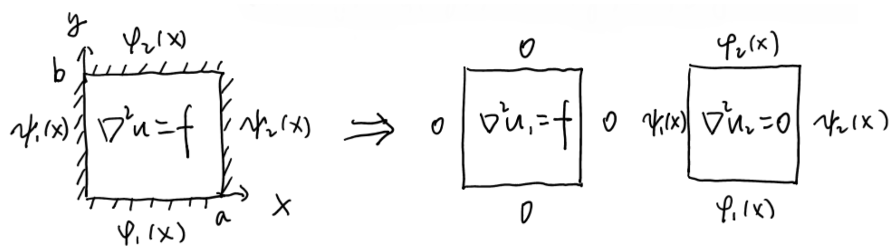
5.5 Poisson equation: method of eigenfunction expansions

Poisson's equation $\nabla^2 u = f$

in two dimensions:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f(x, y), \quad 0 < x < a, 0 < y < b \\ u(x, 0) = \varphi_1(x), u(x, b) = \varphi_2(x) \\ u(0, y) = \psi_1(y), u(a, y) = \psi_2(y) \end{cases}$$

$u = u_1 + u_2$



that requires solving for 0 BVCs:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f(x, y), \quad 0 < x < a, 0 < y < b \\ u(0, y) = u(a, y) = u(x, 0) = u(x, b) = 0 \end{cases}$$

try naive seperation of variables: $u = X(x)Y(y)$, fails.

From before we know $\phi_{nm} = sin\dfrac{n\pi x}{a} sin\dfrac{m\pi y}{b}, n, m = 1, 2, \cdots$

satisfying 0 BVCs, and $\nabla^2 \phi_{nm} = -\pi^2(\dfrac{n^2}{a^2} + \dfrac{m^2}{b^2})\phi_{nm}$

$\Big[\nabla^2\phi = -\lambda\phi,$ Helmholtz equation, one can imagine ϕ_{nm} as eigenfunctions in case of 0 BVCs$\Big]$

the eigenfunctions are complete on $0 \leq x \leq a$ and $0 \leq y \leq b$,

$f(x, y) = \displaystyle\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} sin\frac{n\pi x}{a} sin\frac{m\pi y}{b}$ (double Fourier series).

$\left[\text{or think } f(x, y) = \displaystyle\sum_{n=1}^{\infty} f_n(y) sin\frac{n\pi x}{a} = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\gamma_{nm} sin\frac{m\pi y}{b}\right) sin\frac{n\pi x}{a}\right]$

with $C_{nm} = \dfrac{4}{ab}\displaystyle\int_0^a dx \int_0^b dy f(x, y) sin\frac{n\pi x}{a} sin\frac{m\pi y}{b}.$

Thus assuming $u(x, y) = \displaystyle\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} E_{nm} sin\frac{n\pi x}{a} sin\frac{m\pi y}{b}$, with $E_{nm} = \dfrac{-C_{nm}}{\left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right)}$

then $\nabla^2 u(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm} sin\frac{n\pi x}{a} sin\frac{m\pi y}{b} = f(x,y)$, and $u = 0$ at boundaries.

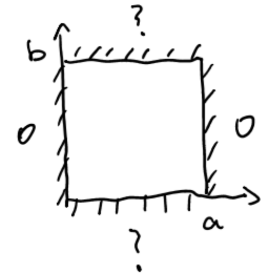
One can use similar trick but only using one-dimensional expansion.
assuming $u(x,y) = \sum_{n=1}^{\infty} g_n(y) sin\frac{n\pi x}{a}$
plug into PDE$\Longrightarrow \sum_{n=1}^{\infty}\left(-\frac{n^2\pi^2}{a^2}g_n(y)\right) sin\frac{n\pi x}{a} + \sum_{n=1}^{\infty} g_n''(y) sin\frac{n\pi x}{a} = \sum_{n=1}^{\infty} f_n(y) sin\frac{n\pi x}{a}$
$\Longrightarrow g_n''(y) - \frac{n^2\pi^2}{a^2}g_n(y) = f_n(y), \quad n = 1, 2, \cdots$
similarly get BVCs: $g_n(0) = g_n(b) = 0$.
solving the non-homogeneous ODE, recall:
$h_1 = sinh\left(\frac{n\pi}{a}(b-y)\right), \quad h_2 = sinh\left(\frac{n\pi}{a}y\right), \quad W(h_1,h_2) = \frac{n\pi}{b} sinh\frac{n\pi b}{a}$
thus particular solution: $g_{n,p} = h_1 \int_{-}^{y} \frac{-h_2 f_n}{W} ds - h_2 \int_{y}^{-} \frac{h_1 f_n}{W} ds$
one can adjust the integration region to satisfy BVC, e.g.:
$g_n = h_1 \int_0^{y} \frac{-h_2 f_n}{W} ds - h_2 \int_{y}^{b} \frac{h_1 f_n}{W} ds, \quad g_n(0) = g_n(b) = 0$
finally the full solution to original problem $u(x,y) = \sum_{n=1}^{\infty} g_n(y) sin\frac{n\pi x}{a}$

Example: $\nabla^2 u = 1$ in a $|x|$ rectangle with 0 BVCs.
in terms of double Fourier series:
$u = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm} sin\left(n\pi x\right) sin\left(m\pi y\right), B_{nm} = -\frac{4}{\pi^2(n^2+m^2)}\int_0^1 dx \int_0^1 dy sin\left(n\pi x\right) sin\left(m\pi y\right)$
thus $B_{nm} = -\frac{16}{\pi^4}\frac{1}{mn(m^2+n^2)}$, both m and n odd; 0, otherwise.
in terms of single Fourier series:
$u = \sum_{n=1}^{\infty} g_n(y) sin\left(n\pi x\right), \quad f_n(y) = 2\int_0^1 sin\left(n\pi x\right) dx = \frac{2}{n\pi}\left(1 - cos n\pi\right)$

$$g_n(y) = \frac{4}{n^2\pi^2 sinh n\pi}\left(-sinh n\pi(1-y)\int_0^{y} sinh n\pi s ds - sinh n\pi y \int_y^1 sinh n\pi(1-s) ds\right)$$
$$= \frac{4}{n^3\pi^3}\frac{sinh\left(n\pi(1-y)\right) + sinh\left(n\pi y\right) - sinh n\pi}{sinh n\pi}$$

5.6 Neumann and Robin conditions

Considering Laplace's equation with more general BVCs, e.g.:
two dimensions: $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(0,y) = u(a,y) = 0, \cdots$



① Dirichlet condition: $u(x,0)=\psi_1(x), u(x,b)=\psi_2(x)$;

② Neumann condition: $u_y(x,0)=g_1(x), u_y(x,b)=g_2(x)$;

③ Robin condition: $u_y(x,0)+\alpha u(x,0)=f_1(x),\cdots$

can also de different on two ends.(mixed)

again using seperation of variables, $u=X(x)Y(y)$,

$u(x,y)=\sum_{n=1}^{\infty} sin\frac{n\pi x}{a}\left(A_n sinh\frac{n\pi y}{a}+B_n cosh\frac{n\pi y}{a}\right)$, satisfying PDE and BVCs on vertical lines,

only need to determine A_n, B_n using BVCs on horizontal lines.

E.g.: Robin $u_y(x,b)=f(x), u_y(x,0)+2u(x,0)=0$

$$\begin{cases} \frac{n\pi}{a}\left(A_n cosh\frac{n\pi b}{a}+B_n sinh\frac{n\pi b}{a}\right)=f_n(x) \\ \frac{n\pi}{a}A_n+2B_n=0 \end{cases} \Longrightarrow A_n, B_n$$

5.7 The maximum principle

A. Maximum principle for the heat equation:

Considering heat equation with nonconstant BVC:

$$\begin{cases} u_t=c^2u_{xx}, \quad 0<x<L, t>0 \\ u(0,t)=g_1(t), u(L,t)=g_2(t),\ t>0 \\ u(x,0)=\varphi(x), \quad 0<x<L \end{cases}$$

Suppose $g_{1,2}(x)$ and $\varphi(x)$ are all bounded, namely exist m and M, $m\le\varphi(x)\le M$, $\quad m\le g_{1,2}(x)\le M$

then the solution satisfies: $m\le u(x,t)\le M$, for $0\le x\le L, t\ge 0$

(show the local min or max must be on the boundaries $x=0,L$ or $t=0$)

①$\Longrightarrow$ uniqueness of the solution:

Supposing both $u_1(x,t)$ and $u_2(x,t)$ are solutions, then $u=u_1(x,t)-u_2(x,t)$ is a solution for heat equation with zero BVC and zero IVC. $\Longrightarrow 0\le u(x,t)\le 0 \Longrightarrow u_1=u_2$

②$\Longrightarrow$ comparable principle:

Supposing $u_1(x,t)$ and $u_2(x,t)$ are two solutions with BVC and IVC , $\{g_1,g_2,\varphi\}$ and $\{g_1^*,g_2^*,\varphi^*\}$

If $g_1\ge g_1^*, g_2\ge g_2^*, \varphi\ge\varphi^*$, then $u_1(x,t)\ge u_2(x,t)$.

B. Maximum principle for Laplace's equation:

Considering Laplace's equation with BVCs:

$$\begin{cases} u_{xx} + u_{yy} = 0, \quad 0 < x < a, 0 < y < b \\ u(x,0) = \varphi_1(x), u(x,b) = \varphi_2(x), 0 < x < a \\ u(0,y) = \psi_1(y), u(a,y) = \psi_2(y), 0 < y < b \end{cases}$$

Suppose $\varphi_{1,2}(x)$ and $\psi_{1,2}(x)$ are bounded, namely existing m and M, $m \leq \varphi_{1,2}(x) \leq M$, $m \leq \psi_{1,2}(x) \leq M$ then the solution satisfying

$m \leq u(x,y) \leq M$ for $0 \leq x \leq a, 0 \leq y \leq b$

Proof:

let $u(x,y)$ be a solution for Laplace's equation,

construct $v(x,y) = u(x,y) + \dfrac{x^2+y^2}{4n}, \quad n \in N$

thus $\nabla^2 v = \nabla^2 u + \dfrac{1}{n} = \dfrac{1}{n} > 0$

suppose $v(x,y)$ reach a maximum for $0 < x_0 < a$, $0 < y_0 < b$,

that requires $\left.\dfrac{\partial v}{\partial x}\right|_{x_0,y_0} = \left.\dfrac{\partial v}{\partial y}\right|_{x_0,y_0} = 0, \qquad \left.\dfrac{\partial^2 v}{\partial x^2}\right|_{(x_0,y_0)} \leq 0, \left.\dfrac{\partial^2 v}{\partial y^2}\right|_{(x_0,y_0)} \leq 0$

$\Longrightarrow \left.\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right|_{(x_0,y_0)} \leq 0 \longrightarrow$ Confliction!

In the closed region $v(x,y)$ must have a maximum,

from above it can only be on the boundary $\Longrightarrow v(x,y) \leq M + \dfrac{a^2+b^2}{4n}$

and$u \leq v \leq M + \dfrac{a^2+b^2}{4n}, \quad n \to +\infty \Longrightarrow u \leq M$

5.8 *Schrödinger's* equation

One dimensional quantum system:

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x,t)\psi(x,t)$$

If $V(x,t) = V(x)$, let $\psi(x,t) = T(t) \cdot \phi(x)$, thus

$$i\hbar \frac{T'}{T} = -\frac{\hbar^2}{2m} \cdot \frac{\phi''}{\phi} + V(x) = E$$

E.g.: infinite well potential: $V(x) = \begin{cases} 0, \quad 0 < x < L \\ \infty, \quad otherwise \end{cases}$

$\Longrightarrow -\dfrac{\hbar^2}{2m} \cdot \dfrac{\phi''}{\phi} = E, \quad 0 < x < L, \qquad BVC: \phi(0) = \phi(L) = 0$

solving for the eigenvalues:

$E_n = \dfrac{n^2}{2m}\left(\dfrac{\pi\hbar}{L}\right)^2, \qquad \phi_n = sin\left(\dfrac{n\pi}{L}x\right), n = 1, 2, \cdots$

time dependence:

$i\hbar\dfrac{T'}{T} = E_n \longrightarrow T_n = exp(-\dfrac{iE_n t}{\hbar})$

6 Chapter 6. Partial differential equations in Polar and Cylindrical Coordinates

6.1 General product solutions of Laplace's and Helmholtz's equations

6.1.1 Laplace's equation, $u(r, \theta, z)$

$\Delta u = \nabla^2 u = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} + \dfrac{\partial^2 u}{\partial z^2}$

let $u = R(r)\Theta(\theta)Z(z)$, $\quad \dfrac{R''}{R} + \dfrac{1}{r}\dfrac{R'}{R} + \dfrac{1}{r^2}\dfrac{\Theta''}{\Theta} + \dfrac{Z''}{Z} = 0$

must have $\dfrac{Z''}{Z} = \mu, \quad \dfrac{\Theta''}{\Theta} = -m^2, \qquad R'' + \dfrac{1}{r}R' + (-\dfrac{m^2}{r^2} + \mu)R = 0$

angular component:

natural/periodic condition: $\Theta(0) = \Theta(2\pi), \quad \Theta'(0) = \Theta'(2\pi)$ (since $(\Theta(0)) \equiv (\Theta(2\pi))$ in physics)

that fixes the eigenvalue to be $m^2 \geq 0$, $m = 0, 1, 2, \cdots$

solutions: $\Theta_m(\theta) = Acosm\theta + Bsinm\theta$

for Z component, general solution $Z(z) = \begin{cases} C + Dz, & \mu = 0 \\ Ce^{-\sqrt{\mu}z} + De^{\sqrt{\mu}z}, & \mu > 0 \\ Ccos\sqrt{-\mu}z + Dsin\sqrt{-\mu}z, & \mu < 0 \end{cases}$

the actual value of μ may be fixed later by BVCs associated with z.

now the radial part: $r^2R'' + rR' + (\mu r^2 - m^2)R = 0 \begin{cases} \mu = 0, \text{ Euler's equation} \\ \mu \neq 0, \text{ Bessel's equations} \end{cases}$

$$R(r) = \begin{cases} E + Flnr, & \mu = 0,\ m = 0 \\ Er^m + Fr^{-m}, & \mu = 0, m > 0 \\ EJ_m(\sqrt{\mu}r) + FY_m(\sqrt{\mu}r), & \mu > 0, m \geq 0 \\ EI_m(\sqrt{-\mu}r) + FK_m(\sqrt{-\mu}r), & \mu < 0, m \geq 0 \end{cases}$$

6.1.2 Helmholtz's equation

$$\Delta u + ku = 0, \qquad \text{(think k as eigenvalues)}$$

In a similar manner,

$\dfrac{\Theta''}{\Theta} = -m^2, \quad \dfrac{Z''}{Z} = \mu, \quad R'' + \dfrac{1}{r}R' + [\dfrac{-m^2}{r^2} + (k + \mu)]R = 0$

let $\nu = k + \mu$, thus the general product solution is similar as for Laplace's equation, with radial part:

$$R(r) = \begin{cases} E + Flnr, & \nu = 0, m = 0 \\ Er^m + Fr^{-m}, & \nu = 0, m > 0 \\ EJ_m(\sqrt{\nu}r) + FY_m(\sqrt{\nu}r), & \nu > 0, m \geq 0 \\ EI_m(\sqrt{-\nu}r) + FK_m(\sqrt{-\nu}r), & \nu < 0, m \geq 0 \end{cases}$$

6.2 Laplace's equation in Circular regions

Removing z from last section, ($\mu = 0$)

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 < \theta < 2\pi$$

thus the general solution becomes,

$$u(r,\theta) = a_0 + c_0 lnr + \sum_{n=1}^{\infty} r^n [a_n cosn\theta + b_n sinn\theta] + \sum_{n=1}^{\infty} r^{-n}[c_n cosn\theta + d_n sinn\theta]$$

The coefficients a_n, b_n, c_n, d_n are determined by BVCs.

If considering the steady-state temperature distribution in a circular plate of radius a, with temperature at boundary: $u(a,\theta) = f(\theta)$. $(0 \le \theta < 2\pi)$

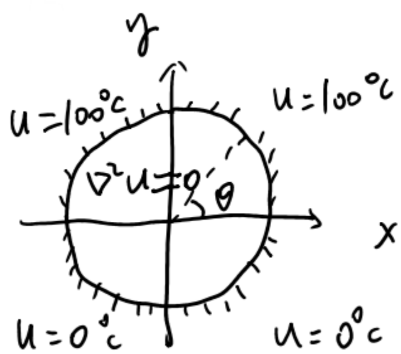
We further require the physical solution to be finite at $r = 0$, thus

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n [a_n cosn\theta + b_n sinn\theta]$$

To satisfy the BVCs, simply take:

$$\begin{cases} a_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(\theta) cosn\theta d\theta \\ b_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(\theta) sinn\theta d\theta \end{cases}$$

Example: A Dirichlet problem on the disk with unit radius.



$$\nabla^2 u(r,\theta) = 0, \quad \text{with BVCs} \quad u(1,\theta) = \begin{cases} 100, & if \quad 0 < \theta < \pi \\ 0, & if \quad \pi < \theta < 2\pi \end{cases}$$

Solution:

$$a_n = \frac{1}{\pi}\int_0^{\pi} 100 cosn\theta d\theta = \begin{cases} 100, & n = 0 \\ 0, & n > 0 \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} 100 sinn\theta d\theta = \frac{100}{n\pi}(1 - cosn\pi)$$

$$\Longrightarrow u(r,\theta) = 50 + \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(1 - cosn\pi) r^n sinn\theta$$

a useful formula: $\sum_{n=1}^{\infty} r^n \frac{sinn\theta}{n} = tan^{-1}\left(\frac{rsin\theta}{1-rcos\theta}\right)$ valid for $0<r<1$, and all θ.

Proof:

Taylor expansion of the right:

$$\frac{dRHS}{dr} = \frac{1}{1+\frac{r^2sin^2\theta}{(1-rcos\theta)^2}}\left(\frac{sin\theta}{1-rcos\theta}+\frac{rsin\theta cos\theta}{(1-rcos\theta)^2}\right) = \frac{sin\theta}{1+r^2-2rcos\theta}$$

$$\frac{1}{1+r^2-2rcos\theta} = \sum_{n=0}^{\infty}(2rcos\theta - r^2)^n = \sum_{n=0}^{\infty}\sum_{m=0}^{n}(-1)^m r^{n+m}(2cos\theta)^{n-m}\frac{n!}{m!(n-m)!}$$

$$= \sum_{n=0}^{\infty} r^n \sum_{m=0}^{M}(-1)^m(2cos\theta)^{n-2m}\frac{(n-m)!}{(n-2m)!m!}, \qquad M=\frac{n}{2} \quad or \frac{n-1}{2}$$

thus $\frac{sin\theta}{1+r^2-2rcos\theta} = \sum_{n=0}^{\infty} r^n sin(n+1)\theta.$

now apply on the example:

$$u(r,\theta) = 50 + \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(1-cosn\pi)r^n sinn\theta$$
$$= 50 + \frac{100}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n}r^n sinn\theta - \frac{1}{n}r^n sinn(\theta-\pi)\right)$$
$$= 50 + \frac{100}{\pi}\left[tan^{-1}\left(\frac{rsin\theta}{1-rcos\theta}\right) + tan^{-1}\left(\frac{rsin\theta}{1+rcos\theta}\right)\right]$$

Isotherms, $u(r,\theta)\equiv T$,

$\frac{\pi}{100}(T-50) = tan^{-1}\frac{x}{1-y} + tan^{-1}\frac{x}{1+y}$, take tan on both side.

$\Longrightarrow x^2+y^2-1-2ytan\left(\frac{\pi T}{100}\right)=0, \quad or \quad x^2+[y-tan\left(\frac{\pi T}{100}\right)]^2 = 1+tan^2\left(\frac{\pi T}{100}\right)$

If restrict to a planar region of wedge:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad 0<\theta<\alpha$$

suppling with BVCs, $u(r,0)=u(r,\alpha)=0$.

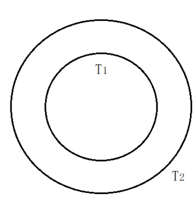
Thus the periodic conditions should be replaced, $\Theta(0)=\Theta(\alpha)=0 \Longrightarrow \Theta_n(\theta) = sin\frac{n\pi\theta}{\alpha}$, integer $n\geq 1$.

and the radial part: $R_n(r) = a_n r^{\frac{n\pi}{\alpha}}$,

general solution $u(r,\theta) = \sum_{n=1}^{\infty} a_n r^{\frac{n\pi}{\alpha}} sin\frac{n\pi}{\alpha}\theta$.

a_n can be determined. (e.g. using BVCs on circular $r=a$).

e.g.



$r_1 \le r \le r_2; \quad 0 \le \theta < 2\pi.$
$u(r,\theta) = v(r) = C_1 + C_2 lnr = a\left(\frac{lnr - lnr_1}{lnr_2 - lnr_1}\right) + b\left(\frac{lnr - lnr_2}{lnr_1 - lnr_2}\right) \Longrightarrow b = T_1, \quad a = T_2$

6.3 Helmholtz's equation and Poisson's equations in circular regions

6.3.1 Helmholtz's equation (think k as eigenvalues to be determined)

$$\nabla^2\phi = -k\phi(r,\theta), \quad 0 < r < a, \quad 0 < \theta < 2\pi$$

with boundary condition $\phi(a,\theta) = 0, \quad 0 < \theta < 2\pi$
The angular part $\Theta_m(\theta) = Acosm\theta + Bsinm\theta, \quad m = 0, 1, 2, \cdots$
The radial part satisfying: $r^2R'' + rR' + (kr^2 - m^2)R = 0$
BVCs: $R(0)$ finite, $R(a) = 0$. [non-trivial solution requires $k > 0$]
$\Longrightarrow k = \lambda_{mn}^2 \equiv (\frac{\alpha_{mn}}{a})^2$, $n = 1, 2, \cdots \qquad R_n(r) = J_m(\lambda_{mn}r)$
thus the eigenvalues: $k = \lambda_{mn}^2 = \left(\frac{\alpha_{mn}}{a}\right)^2, \quad m = 0, 1, \cdots, \infty, \quad n = 1, 2, \cdots, \infty$
corresponding eigenfunctions: $J_m(\lambda_{mn}r)cosm\theta$ and $J_m(\lambda_{mn}r)sinm\theta$

Any function $f(r,\theta)$, $0 < r < a$, $0 < \theta < 2\pi$ can be expanded using the eigenfunctions.

$$f(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn}r)(a_{mn}cosm\theta + b_{mn}sinm\theta)$$

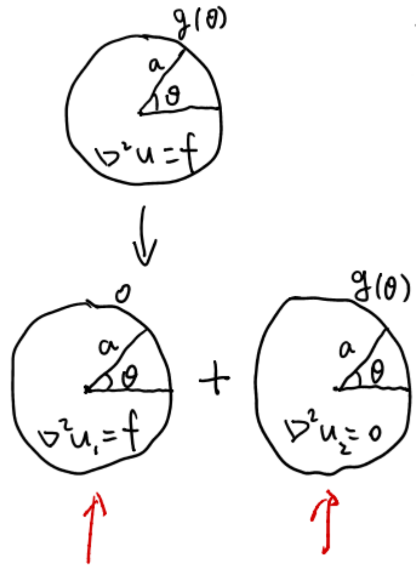
a_{mn} and b_{mn} can be calculated by taking integrals timed by individual eigenfunctions.
$(f,g) \equiv \int_0^a rdr \int_0^{2\pi} d\theta f(r,\theta)g(r,\theta)$
$\|cosm\theta J_m(\lambda_{mn}r)\|^2 = \frac{1}{2}a^2 J_{m+1}^2(\alpha_{mn})\pi \ \ (m > 0)$
[think as Fourier-Bessel+Fourier expansions]

Example: The method of eigenfunction expansions.
solve $\nabla^2 u = u + 3r^2cos2\theta$ in the unit disk, given $u = 0$ on the boundary ($r = 1$).
solution: assuming $u = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\alpha_{mn}r)(A_{mn}cosm\theta + B_{mn}sinm\theta)$
$\Longrightarrow \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\alpha_{mn}r)(-\alpha_{mn}^2 - 1)(A_{mn}cosm\theta + B_{mn}sinm\theta) = 3r^2cos2\theta$
easy to show only $A_{2n} \neq 0$, and
$A_{2n} = \frac{3\int_0^1 r \cdot r^2 J_2(\alpha_{2n}r)dr}{(-\alpha_{2n}^2 - 1)\frac{1}{2}J_3^2(\alpha_{2n})} = \frac{-6}{(1 + \alpha_{2n}^2)}\frac{1}{\alpha_{2n}J_3(\alpha_{2n})}$

6.3.2 Poisson's equation

Considering the Poisson problem in a disk:

$\nabla^2 u = f(r,\theta), \quad 0 < r < a, \quad 0 < \theta < 2\pi, \quad u(a,\theta) = g(\theta)$



solution: $u = u_1 + u_2$ with u_2 from Laplace case, u_1 can be solved based on eigenfunction expansion.

for u_1:

$u(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn} r)(A_{mn} cos m\theta + B_{mn} sin m\theta)$

$f(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn} r)(C_{mn} cos m\theta + D_{mn} sin m\theta)$

substituting into the PDE:

$\nabla^2 u = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} -\left(\frac{\alpha_{mn}}{a}\right)^2 J_m(\lambda_{mn} r)(A_{mn} cos m\theta + B_{mn} sin m\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn} r)(C_{mn} cos m\theta + D_{mn} sin m\theta)$

$\Longrightarrow A_{mn} = \frac{C_{mn}}{-\left(\frac{\alpha_{mn}}{a}\right)^2}, \quad B_{mn} = \frac{D_{mn}}{-\left(\frac{\alpha_{mn}}{a}\right)^2}$

6.4 The wave equations in polar coordinates

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

in region $0 < r < a$, $0 < \theta < 2\pi$, $t > 0$

BVCs: $u(a,\theta,t) = 0$; IVCs: $\begin{cases} u(r,\theta,0) = f(r,\theta) \\ \left.\frac{\partial u(r,\theta,t)}{\partial t}\right|_{t=0} = g(r,\theta) \end{cases}$

using separation of variables, $u = \Theta(\theta)R(r)T(t)$,

$\frac{1}{c^2}\frac{T''}{T} = \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{R''}{R} + \frac{1}{r}\frac{R'}{R}$, thus

$\frac{\Theta''}{\Theta} = k_1, \quad \frac{1}{c^2}\frac{T''}{T} = -k_2, \quad R'' + \frac{R'}{r} + (\frac{k_1}{r^2} + k_2)R = 0$

angular part: $\Theta_m(\theta) = a cos m\theta + b sin m\theta, \quad m = 0, 1, \cdots$ (periodic conditions)

radial part, $R(a)=0$, $R(0)$ finite.
$k_2=\lambda_{mn}^2=(\frac{\alpha_{mn}}{a})^2, \quad R(r)=J_m(\lambda_{mn}r), \quad n=1,2,\cdots$
time-dependent part:
$T(t)=Acos\lambda_{mn}ct+Bsin\lambda_{mn}ct$
thus the full solution:

$$u_{mn}(r,\theta,t)=J_m(\lambda_{mn}r)[(a_{mn}cosm\theta+b_{mn}sinm\theta)cos\lambda_{mn}ct+(c_{mn}cosm\theta+d_{mn}sinm\theta)sin\lambda_{mn}ct],\ m\geq 0, n\geq 1$$

general solution can be expressed as:

$$u(r,\theta,t)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}u_{mn}(r,\theta,t)$$

plug in the IVCs, can determine $a_{mn}, b_{mn}, c_{mn}, d_{mn}$.
thus,

$$\begin{aligned}
a_{mn}&=\frac{2\delta_m}{\pi a^2 J_{m+1}^2(\alpha_{mn})}\int_0^a rdr\int_0^{2\pi}d\theta f(r,\theta)cosm\theta J_m(\lambda_{mn}r);\\
b_{mn}&=\frac{2\delta_m}{\pi a^2 J_{m+1}^2(\alpha_{mn})}\int_0^a rdr\int_0^{2\pi}d\theta f(r,\theta)sinm\theta J_m(\lambda_{mn}r);\\
c_{mn}&=\frac{2\delta_m}{\pi a^2 J_{m+1}^2(\alpha_{mn})}\int_0^a rdr\int_0^{2\pi}d\theta g(r,\theta)cosm\theta\frac{J_m(\lambda_{mn}r)}{\lambda_{mn}c};\\
d_{mn}&=\frac{2\delta_m}{\pi a^2 J_{m+1}^2(\alpha_{mn})}\int_0^a rdr\int_0^{2\pi}d\theta g(r,\theta)sinm\theta\frac{J_m(\lambda_{mn}r)}{\lambda_{mn}c};
\end{aligned}$$

with $\delta_m=\begin{cases}1, & m>0\\ \frac{1}{2}, & m=0\end{cases}$ to account for definition of 0-th coefficients.
In radially symmetric case ($f=f(r), g=g(r)$) only a_{0n}, c_{0n} exist, and dependence on θ drops out.

Example:
① $f(r,\theta)=0, g(r,\theta)=v_0$ (zero displacement and constant velocity)
$u(r,t)=\sum_{n=1}^{\infty}J_0(\lambda_{0n}r)(a_ncos\lambda_{0n}ct+b_nsin\lambda_{0n}ct)$
further $a_n=0, \quad b_n=\frac{2}{a^2J_1^2(\lambda_{0n}n)}\int_0^a rv_0\frac{J_0(\lambda_{0n}r)dr}{\lambda_{0n}c}$
$\Longrightarrow b_n=\frac{2v_0a}{a^2J_1(\lambda_{0n})c\lambda_{0n}^2}=\frac{2av_0}{\alpha_{0n}^2cJ_1(\lambda_{0n})}, n=1,2,\cdots$
② $f(r,\theta)=(1-r^2), g(r,\theta)=0,\ a=1$
$u(r,t)=\sum_{n=1}^{\infty}J_0(\alpha_{0n}r)a_ncos\alpha_{0n}ct$
$\Longrightarrow a_n=\frac{2}{J_1^2(\alpha_{0n})}\int_0^1 r(1-r^2)J_0(\alpha_{0n}r)dr=\frac{2}{J_1^2(\alpha_{0n})}\frac{2J_2(\alpha_{0n})}{\alpha_{0n}^2}=\frac{8}{\alpha_{0n}^3J_1(\alpha_{0n})}$
[using $\frac{2\nu Z_\nu(x)}{x}=Z_{\nu+1}(x)+Z_{\nu-1}(x)$]

③ considering a more general initial conditions:

$f(r,\theta) = (1-r^2)rsin\theta, g(r,\theta) = (1-r^2)r^2sin2\theta,\ a=1$

One can easily verify: $a_{mn}=0$, $b_{1n}\neq 0$, $c_{mn}=0$, $d_{2n}\neq 0$

and

$$
\begin{aligned}
b_{1n} &= \frac{2}{\pi J_2^2(\alpha_{1n})}\int_0^1 rdr\int_0^{2\pi} d\theta(1-r^2)rJ_1(\alpha_{1n}r)sin^2\theta \\
&= \frac{2}{J_2^2(\alpha_{1n})}\int_0^1 r^2(1-r^2)J_1(\alpha_{1n}r)dr = \frac{4J_3(\alpha_{1n})}{\alpha_{1n}^2J_2^2(\alpha_{1n})} = \frac{16}{\alpha_{1n}^3J_2(\alpha_{1n})} \\
d_{2n} &= \frac{2}{\pi J_3^2(\alpha_{2n})}\int_0^1 rdr\int_0^{2\pi} d\theta(1-r^2)r^2\frac{J_2(\alpha_{2n}r)sin^2 2\theta}{c\alpha_{2n}} \\
&= \frac{2}{J_3^2(\alpha_{2n})}\int_0^1 r^3(1-r^2)J_2(\alpha_{2n}r)dr\frac{1}{c\alpha_{2n}} = \frac{4J_4(\alpha_{2n})}{c\alpha_{2n}^3J_3^2(\alpha_{2n})} = \frac{24}{c\alpha_{2n}^4J_3(\alpha_{2n})}
\end{aligned}
$$

6.5 The heat equation in polar coordinates

two dimensional heat equation:

$$\frac{\partial u}{\partial t} = c^2\nabla^2 u = c^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\right)$$

with $0<r<a,\quad 0<\theta<2\pi,\quad t>0.$

BVCs: $u(a,\theta,t)=0;\qquad$ IVCs: $u(r,\theta,0)=g(r,\theta)$

product solution: $u=R(r)\Theta(\theta)T(t)$,

$$\frac{1}{c^2}\frac{T'}{T} = \frac{R''}{R}+\frac{1}{r}\frac{R'}{R}+\frac{1}{r^2}\frac{\Theta''}{\Theta}$$

with separation of variable:

$\frac{\Theta''}{\Theta}=-k_1,\quad \frac{T'}{T}=-c^2k_2,\quad R''+\frac{1}{r}R'+(k_2-\frac{k_1}{r^2})R=0$

periodic conditions on Θ leads to:

$k_1=m^2,\quad m=0,1,2,\cdots;\quad \Theta_m(\theta)=ecosm\theta+bsinm\theta$

for the radial part with $R(0)$ finite, $R(a)=0$ (from BVCs)

$k_2=0 \Longrightarrow f+dlnr,\quad or\ fr^m+dr^{-m}$. no non-zero solution.

$k_2<0 \Longrightarrow fI_m(\sqrt{-k_2}r)+dK_m(\sqrt{-k_2}r)$. no non-zero solution.

$k_2>0 \Longrightarrow fJ_m(\sqrt{k_2}r)+dY_m(\sqrt{k_2}r)$, thus $\sqrt{k_2}=\lambda_{mn}\equiv\frac{\alpha_{mn}}{a}, n=1,2,\cdots; R(r)=J_m(\lambda_{mn}r)$

time dependence:

$\frac{T'}{T}=-c^2k_2\Longrightarrow T(t)=e^{-c^2\lambda_{mn}^2t}$ (overall constant can all be absorbed into definition of e and b in Θ)

putting all possible product solution together:

$u(r,\theta,t)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}J_m(\lambda_{mn}r)(a_{mn}cosm\theta+b_{mn}sinm\theta)e^{-\lambda_{mn}^2c^2t}$

if choosing :

$a_{mn}=\frac{2\delta_m}{\pi a^2J_{m+1}^2(\alpha_{mn})}\int_0^a rdr\int_0^{2\pi}d\theta J_m(\lambda_{mn}r)cosm\theta g(r,\theta)$

$$b_{mn} = \frac{2\delta_m}{\pi a^2 J^2_{m+1}(\alpha_{mn})} \int_0^a rdr \int_0^{2\pi} d\theta J_m(\lambda_{mn} r) sinm\theta g(r,\theta)$$

with $\delta_m = \begin{cases} 1, & m > 0 \\ \frac{1}{2}, & m = 0 \end{cases}$ then:

$u(r,\theta,0) = g(r,\theta)$, satisfying IVCs.

6.6 Laplace's equation in a cylinder

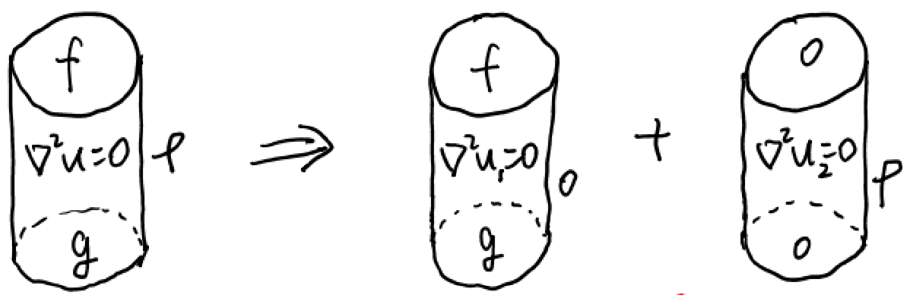
Three dimensional Laplace's equation:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0, \quad 0 < r < a,\ 0 < \theta < 2\pi,\ 0 < z < h$$

With the general Dirichlet BVCs,

$u(a,\theta,z) = \rho(\theta,z), u(r,\theta,0) = g(r,\theta), u(r,\theta,h) = f(r,\theta)$

using superposition principle:



problem 1:

separation of variable , $u = R(r)\Theta(\theta)Z(z)$

$$\frac{\nabla^2 u}{u} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z} = 0$$

thus,

$\frac{\Theta''}{\Theta} = -k_1, \quad \frac{Z''}{Z} = k_2, \quad R'' + \frac{1}{r}R' + (k_2 - \frac{k_1}{r^2})R = 0$

periodic conditions on Θ leads to:

$k_1 = m^2, \quad m = 0, 1, 2, \cdots; \quad \Theta_m(\theta) = Acosm\theta + Bsinm\theta$

for the radial part with $R(0)$ finite, $R(a) = 0$ (from BVCs)

$k_2 = \lambda^2_{mn} \equiv (\frac{\alpha_{mn}}{a})^2, n = 1, 2, \cdots; R(r) = J_m(\lambda_{mn} r)$

and finally $Z(z) = Ccosh\lambda_{mn} z + Dsinh\lambda_{mn} z$

adding all product solutions together:

$$u(r,\theta,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn}r)\left[A_{mn}cosh\lambda_{mn}z + B_{mn}sinh\lambda_{mn}z\right]cosm\theta +$$
$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\lambda_{mn}r)\left[C_{mn}cosh\lambda_{mn}z + D_{mn}sinh\lambda_{mn}z\right]sinm\theta$$

with coefficients determined by:

$A_{mn} = \dfrac{2\delta_m}{\pi a^2 J_{m+1}^2(\alpha_{mn})}\displaystyle\int_0^a rdr\int_0^{2\pi} d\theta g(r,\theta)J_m(\lambda_{mn}r)cosm\theta$

$C_{mn} = \dfrac{2\delta_m}{\pi a^2 J_{m+1}^2(\alpha_{mn})}\displaystyle\int_0^a rdr\int_0^{2\pi} d\theta g(r,\theta)J_m(\lambda_{mn}r)sinm\theta$

problem2:

separation of variable , $u = R(r)\Theta(\theta)Z(z)$

$$\frac{\nabla^2 u}{u} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z} = 0$$

thus,

$\dfrac{\Theta''}{\Theta} = -k_1, \quad \dfrac{Z''}{Z} = k_2, \quad R'' + \dfrac{1}{r}R' + (k_2 - \dfrac{k_1}{r^2})R = 0$

periodic conditions on Θ leads to:

$k_1 = m^2, \quad m = 0,1,2,\cdots; \quad \Theta_m(\theta) = Acosm\theta + Bsinm\theta$

with BVCs: $Z(0) = Z(h) = 0, \quad k_2 = -\left(\dfrac{n\pi}{h}\right)^2, n = 1,2,\cdots; \quad Z(z) = sin\dfrac{n\pi z}{h}$

the radial part (with $R(0)$ be finite): $\quad R(r) = I_m\left(\dfrac{n\pi}{h}r\right)$

adding all product solutions together:

$$u(r,\theta,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} I_m\left(\frac{n\pi}{h}r\right)sin\frac{n\pi z}{h}\left[B_{mn}cosm\theta + D_{mn}sinm\theta\right]$$

from further BVCs: $u(a,\theta,z) = \rho(\theta,z)$ easily derive:

$B_{mn} = \dfrac{2\delta_m}{I_m\left(\frac{n\pi a}{h}\right)\pi h}\displaystyle\int_0^h dz\int_0^{2\pi} d\theta\rho(\theta,z)cosm\theta sin\frac{n\pi z}{h}$

$D_{mn} = \dfrac{2\delta_m}{I_m\left(\frac{n\pi a}{h}\right)\pi h}\displaystyle\int_0^h dz\int_0^{2\pi} d\theta\rho(\theta,z)sinm\theta sin\frac{n\pi z}{h}$

(note in both problem 1 and 2, if BVCs be radially symmetric, then only coefficients with $m = 0$ are non-zero.)

6.7 Wave and heat equation in a cylinder

recall the Helmholtz's equation (think as SL problem of PDEs)

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\partial^2 u}{\partial z^2} = -ku$$

separation of variable , $u = R(r)\Theta(\theta)Z(z)$

$$\frac{\nabla^2 u}{u} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z} = -k$$

thus,
$\frac{\Theta''}{\Theta} = -k_1, \quad \frac{Z''}{Z} = k_2, \quad R'' + \frac{1}{r}R' + (k + k_2 - \frac{k_1}{r^2})R = 0$
with periodic conditions on Θ leads to:
$k_1 = m^2, \quad m = 0, 1, 2, \cdots;$
with BVCs: $Z(0) = Z(h) = 0, \quad k_2 = -\left(\frac{n\pi}{h}\right)^2$, $n = 1, 2, \cdots;$
lastly $R(0)$ finite, $R(a) = 0$ gives $k + k_2 = \lambda_{mp}^2 \equiv \left(\frac{\alpha_{mp}}{a}\right)^2, \quad p = 1, 2, \cdots$ meaning:
$k = \left(\frac{n\pi}{h}\right)^2 + \lambda_{mp}^2, \quad p = 1, 2, \cdots$ for fixed $m \geq 0$ and $n \geq 1$.
thus the eigenfunctions are :
$J_m(\lambda_{mp}r)sin\left(\frac{n\pi}{h}z\right)cosm\theta$ and $J_m(\lambda_{mp}r)sin\left(\frac{n\pi}{h}z\right)sinm\theta$
similar as in SL problem, eigenvalues/functions depend on the BVCs.
e.g. in case of $u|_{z=0} = 0$ and $u_z|_{z=h} = 0$
that leads to $Z(0) = 0$ and $Z'(h) = 0, \quad k_2 = -\left(\frac{(n+\frac{1}{2})\pi}{h}\right)^2$, $n = 0, 1, \cdots$
the eigenvalues: $k = \left(\frac{(n+\frac{1}{2})\pi}{n}\right)^2 + \lambda_{mp}^2, p = 1, 2, \cdots; m = 0, 1, \cdots; n = 0, 1, \cdots$
the eigenfunctions: $J_m(\lambda_{mp}r)sin\left(\frac{(n+\frac{1}{2})\pi}{h}z\right)cosm\theta$ and $J_m(\lambda_{mp}r)sin\left(\frac{(n+\frac{1}{2})\pi}{h}z\right)sinm\theta$

Example: considering the radially symmetric problem.

$$\frac{\partial u}{\partial t} = c^2\nabla^2 u, \quad 0 < r < a,\ 0 < z < h,\ t > 0$$

with BVCs: $u(a, z, t) = u(r, 0, t) = u_0, \quad u_z(r, h, t) = 0$
IVCs: $u(r, z, 0) = u_0 + f_1(r)f_2(z)$
Solution:
with decomposition $u = u_1 + u_2$, and let $u_1 = u_0$
thus $\frac{\partial u_2}{\partial t} = c^2\nabla^2 u_2, \quad u_2(a, z, t) = u_2(r, 0, t) = 0,\ u_{2z}(r, h, t) = 0,\ u_2(r, z, 0) = f_1(r)f_2(z)$
It can be solved either using separation of variable or eigenfunction expansion.

e.g. the solution can be expanded using eigenfunctions of the Helmholtz's equation with same BVCs, namely

$$u_2(r, z, t) = \sum_{n=0}^{\infty}\sum_{p=1}^{\infty} a_{np}(t)J_0(\lambda_{0p}r)sin(\frac{n+\frac{1}{2}}{h}\pi z)$$

plug into PDE,

$$\frac{1}{c^2}\frac{da_{np}(t)}{dt} = -k_{pn}^2 a_{np}(t)$$

thus $a_{np}(t) = A_{np}exp(-c^2k_{pn}^2 t)$
using ICs:

$$u_2(r,z,0)=\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}A_{np}J_0(\lambda_{0p}r)sin(\frac{n+\frac{1}{2}}{h}\pi z)=f_1(r)f_2(z)$$

thus choosing

$$A_{np}=\frac{2}{a^2J_1^2(\lambda_{0p}a)}\int_0^a rf_1(r)J_0(\lambda_{0p}r)dr\frac{2}{h}\int_0^h f_2(z)sin(\frac{n+\frac{1}{2}}{h}\pi z)dz$$

and finally

$$u=u_0+\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}A_{np}exp(-c^2k_{pn}^2t)J_0(\lambda_{0p}r)sin(\frac{n+\frac{1}{2}}{h}\pi z)$$

additional exercises:

① An integral formula for Bessel functions, for any $k\geq 0$ and integer $l\geq 0$

$$\int r^{k+1+2l}J_k(r)dr=\sum_{n=0}^{l}(-1)^n2^n\frac{l!}{(l-n)!}r^{k+1+2l-n}J_{k+n+1}(r)+C$$

proof:

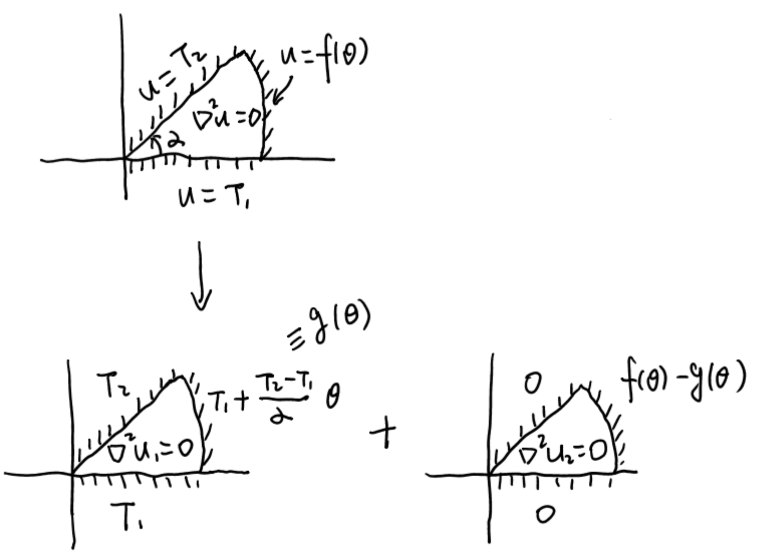
a) for $l=0$ and all k holds: $\displaystyle\int r^{k+1}J_k(r)dr=r^{k+1}J_{k+1}(r)+C$

b) assuming ir holds for $l-1$ and all k.

$$\int r^{k+1+2l}J_k(r)dr=\int r^{2l}d(r^{k+1}J_{k+1}(r))=r^{k+1+2l}J_{k+1}-2l\int r^{k+1+2l-1}J_{k+1}(r)dr$$

$$=r^{k+1+2l}J_{k+1}(r)-2l\sum_{n=0}^{l-1}(-1)^n2^n\frac{(l-1)!}{(l-1-n)!}r^{k+2l-n}J_{k+n+2}(r)+C=\text{RHS}$$

② consider a problem on the wedge as following:



then from superposition principle,

$u=u_1+u_2$

with $a=1,\alpha=\dfrac{\pi}{4},T_1=0,T_2=1,\ f(\theta)=3sin4\theta.$

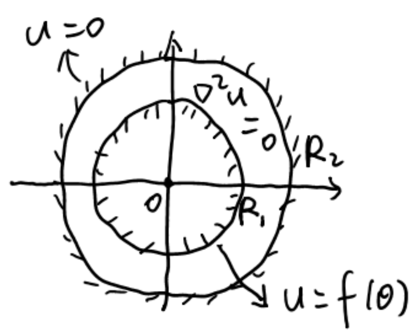
for problem 1, it is easy to guess the solution be: $u_1(r,\theta) = T_1 + \frac{T_2 - T_1}{\alpha}\theta = \frac{4\theta}{\pi}$
for problem 2, from previous sections,

$$u_2(r,\theta) = \sum_{n=1}^{\infty}(\frac{r}{1})^{\frac{n\pi}{\alpha}} a_n sin\frac{n\pi\theta}{\alpha} = \sum_{n=1}^{\infty} a_n r^{4n} sin4n\theta$$

from BVCs, $u_2(1,\theta) = 3sin4\theta - \frac{4\theta}{\pi}$
$a_n = \frac{2}{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}(3sin4\theta - \frac{4\theta}{\pi})sin4n\theta d\theta = 3\delta_{1n} + \frac{2}{n\pi}cosn\pi$
thus $u_2(r,\theta) = 3r^4 sin4\theta + \sum_{n=1}^{\infty}\frac{2}{n\pi}cosn\pi r^{4n} sin4n\theta$

③ A Neumann problem on the disk.
solve the Neumann problem $\nabla^2 u = 0$ for $0 \leq r < a$ with BVCs, $u_r(a,\theta) = f(\theta)$
For this problem to have a solution, we must require the compatibility condition $\int_0^{2\pi} f(\theta)d\theta = 0$.
Proof: Gauss's theorem
$\int_V \vec{\nabla}\cdot(\vec{\nabla}u)dV = \int_{\sum}\vec{\nabla}u\cdot d\vec{s}$ or $\int_{\sum}\vec{\nabla}\cdot(\vec{\nabla}u)ds = \int_L \vec{\nabla}u\cdot d\vec{l}$
thus $0 = \int_0^{2\pi} ad\theta\frac{\partial u(a,\theta)}{\partial r} = a\int_0^{2\pi} f(\theta)d\theta$
(physics intepretation: electricstatic field/ steady state of diffusion)

④ Solve the Dirichlet problem on angular regions.
Find the steady-state solution for problem below.



$0 < \theta < 2\pi$, BVCs: $u(R_1,\theta) = f(\theta), \quad u(R_2,\theta) = 0$
Solution:
Separation of variable: $u = R(r)\Theta(\theta)$
the angular part: $\Theta(\theta) = acosm\theta + bsinm\theta, \quad m = 0, 1, 2, \cdots$
radial part follows Euler's equation:
$m = 0, \;\; R(r) = c + dlnr$

$m > 0, \;\; R(r) = cr^m + dr^{-m}$

with the BVCs $R(R_2) = 0$, thus

$m = 0, \;\; R(r) = c\dfrac{lnr - lnR_2}{lnR_1 - lnR_2}$

$m > 0, \;\; R(r) = c\left(\dfrac{R_1}{r}\right)^m \dfrac{R_2^{2m} - r^{2m}}{R_2^{2m} - R_1^{2m}}$

the full solution:

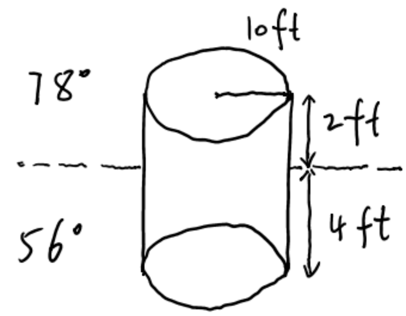
$$u(r,\theta) = \frac{a_0}{2}\frac{lnr - lnR_2}{lnR_1 - lnR_2} + \sum_{m=1}^{\infty}(a_m cosm\theta + b_m sinm\theta)\left(\frac{R_1}{r}\right)^m \frac{R_2^{2m} - r^{2m}}{R_2^{2m} - R_1^{2m}}$$

the coefficients a_n, b_n can be determined via $u(R_1, \theta) = f(\theta)$

$a_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} cosn\theta f(\theta) d\theta$

$b_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} sinn\theta f(\theta) d\theta$

⑤ Find the steady-state temperature in a cylindrical barrel floating in water as shown in below.

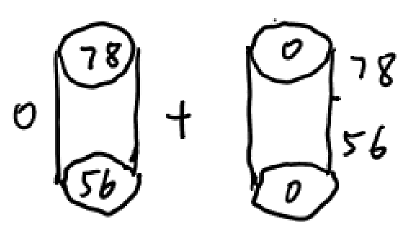


Formulation of the problem:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

with $0 < r < 10$, $0 < z < 6$, $0 < \theta < 2\pi$, $u(r,\theta,0) = 56$, $u(r,\theta,6) = 78$, $\;u(10,\theta,z) = \begin{cases} 78, & z > 4 \\ 56, & z < 4 \end{cases}$

decomposition: $u = u_1 + u_2$



problem 1:

$u_1 = \displaystyle\sum_{n=1}^{\infty} J_0(\lambda_{0n} r)(a_n cosh\lambda_{0n} z + b_n sinh\lambda_{0n} z)$

with $u_1(r,\theta,0)=56$ and $u_1(r,\theta,6)=78$

$$a_n=\frac{2}{a^2J_1^2(\lambda_{0n}a)}\int_0^a rdr56J_0(\lambda_{0n}r)=\frac{112}{\alpha_{0n}J_1(\alpha_{0n})}$$

$$b_n=\frac{\frac{78}{56}a_n-a_n cosh\lambda_{0n}h}{sinh\lambda_{0n}h}=a_n\frac{\frac{39}{28}-cosh\frac{3}{5}\alpha_{0n}}{sinh\frac{3}{5}\alpha_{0n}}$$

problem 2:

$$u_2=\sum_{n=1}^{\infty}I_0(\frac{n\pi}{h}r)a_n sin\frac{n\pi z}{h}$$

with $u_2(a,\theta,z)=\begin{cases}78, & z>4\\ 56, & z<4\end{cases}$

thus $a_n=\dfrac{2}{hI_0(\frac{n\pi a}{h})}\left(\int_0^4 56sin\frac{n\pi z}{6}dz+\int_4^6 78sin\frac{n\pi z}{6}dz\right)=\dfrac{4}{3n\pi I_0(\frac{5n\pi}{3})}\left[84-117cosn\pi+33cos\frac{2n\pi}{3}\right]$

6.8 Orthogonal coordinates ($|R^3$ as example)

For any coordinate system:

$x^1=\xi(x,y,z),\quad x^2=\eta(x,y,z),\quad x^3=\zeta(x,y,z),\quad \dfrac{\partial(x^1,x^2,x^3)}{\partial(x,y,z)}\neq 0.$

① coordinate surface ($x^1\equiv constant,\ x^2\equiv constant,\ x^3\equiv constant$) $\perp$.

② The infinitimal distance: $ds^2=dx^2+dy^2+dz^2=\sum_{i,j=1}^{3}g_{ij}dx^idx^j$

Supposing $g_{ij}=g_{ii}\delta_{ij}$, (or equivalently any two coordinate surfaces are perpendicular), then $\{x^1,x^2,x^3\}$ is called orthogonal coordinate system.

The infinitimal volumn : $dV=\sqrt{g_{11}g_{22}g_{33}}dx^1dx^2dx^3=\sqrt{detG}dx^1dx^2dx^3$

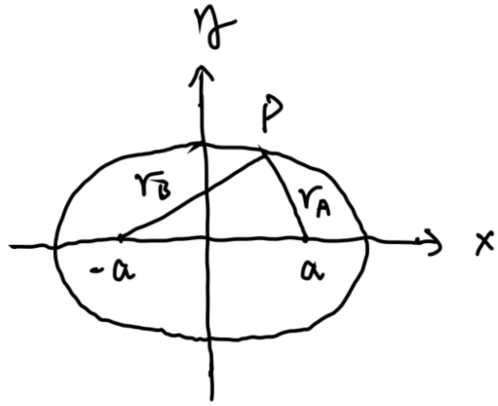
Examples:

rectangle/cartesian: $ds^2=dx^2+dy^2+dz^2$

spherical: $ds^2=dr^2+r^2d\theta^2+r^2sin^2\theta d\varphi^2$

cylindrical: $ds^2=dr^2+r^2d\theta^2+dz^2$

elliptic cylindrical:



$$\xi = \frac{r_B + r_A}{2a} \equiv coshu, \quad \eta = \frac{r_B - r_A}{2a} \equiv cosv, \quad z = z$$
$$ds^2 = a^2(cosh^2 u - cos^2 v)(du^2 + dv^2) + dz^2$$

Laplacian in various coordinates:

Cylindrical: $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$

Spherical: $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta} sin\theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 sin^2\theta}\frac{\partial^2}{\partial \varphi^2}$

Elliptic cylindrical: $\nabla^2 = \frac{1}{a^2(cosh^2 u - cos^2 v)}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + \frac{\partial^2}{\partial z^2}$

7 Chapter 7. Partial differential equations in Spherical Coordinates

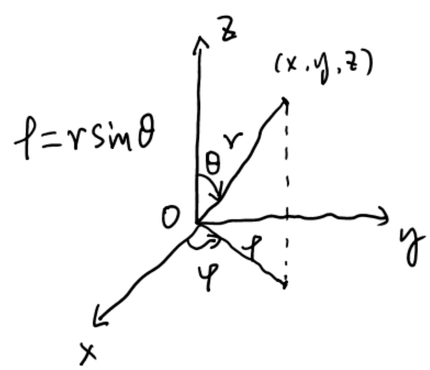
7.1 Preparations

7.1.1 Laplace's operator in spherical coordinates

$$\begin{aligned}\nabla^2 &= \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta} sin\theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 sin^2\theta}\frac{\partial^2}{\partial \varphi^2} \\ &= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial \theta^2} + ctg\theta\frac{\partial}{\partial \theta} + csc^2\theta\frac{\partial^2}{\partial \varphi^2}\right)\end{aligned}$$

coordinate transform:

$x = rsin\theta cos\varphi, \ y = rsin\theta sin\varphi, \ z = rcos\theta \qquad 0 \le \theta \le \pi, \ 0 \le \varphi < 2\pi, \ r > 0$



① first from rectangular to cylindrical: $\quad \{x, y, z\} \longrightarrow \{\rho, \varphi, z\}$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$

Now further replace $\{\rho, z\}$ with $\{r, \theta\}$ $\qquad \begin{cases} \rho = rsin\theta \\ z = rcos\theta \end{cases}$

thus

$$\left.\begin{array}{l}\frac{\partial^2}{\partial\rho^2}+\frac{\partial^2}{\partial z^2}=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\\ \frac{\partial}{\partial\rho}=\frac{\rho}{r}\frac{\partial}{\partial r}+\frac{z\rho}{r^3 sin\theta}\frac{\partial}{\partial\theta}\end{array}\right\}\Longrightarrow \Delta=\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2}+ctg\theta\frac{\partial}{\partial\theta}+csc^2\theta\frac{\partial^2}{\partial\varphi^2}\right)$$

7.1.2 Spherical Bessel's equation

$$x^2y''+2xy'+(kx^2-m(m+1))y=0,\quad \text{arbitrary } m,\ \ k>0$$

with change of variable $y=x^{-\frac{1}{2}}\omega$,

$$x^2\omega''+x\omega'+(kx^2-(m+\tfrac{1}{2})^2)\omega=0$$

thus the solution: $y=\frac{C_1}{\sqrt{x}}J_{m+\frac{1}{2}}(\sqrt{k}x)+\frac{C_2}{\sqrt{x}}Y_{m+\frac{1}{2}}(\sqrt{k}x)$

define spherical Bessel's functions: $\begin{cases} j_n(x)=\sqrt{\frac{\pi}{2x}}J_{n+\frac{1}{2}}(x)\\ y_n(x)=\sqrt{\frac{\pi}{2x}}Y_{n+\frac{1}{2}}(x)\end{cases}$

the solution of spherical Bessel's equation: $y=C_1j_m(\sqrt{k}x)+C_2y_m(\sqrt{k}x)$

useful identities:

$j_0(x)=\frac{sinx}{x},\quad j_{-1}(x)=\frac{cosx}{x},\cdots$

$y_0(x)=-\frac{cosx}{x},\quad y_{-1}(x)=\frac{sinx}{x},\cdots$

asymptotics, for any $m\geq 0$

$j_m(x)|_{x\to 0+}\longrightarrow\begin{cases}1, & m=0\\ 0, & m>0\end{cases}\qquad\qquad y_m(x)|_{x\to 0+}\longrightarrow -\infty$

$j_m(x)|_{x\to+\infty}\longrightarrow\frac{1}{x}cos(x-\frac{m+1}{2}\pi)$

$y_m(x)|_{x\to+\infty}\longrightarrow\frac{1}{x}sin(x-\frac{m+1}{2}\pi)$

SL problem with spherical Bessel's equation:

$$[x^2y']'+(kx^2-m(m+1))y=0,\qquad y(0)\text{finite},\ \ y(a)=0$$

eigenvalues: $k=\lambda_{mn}^2\equiv(\frac{\alpha_{m+\frac{1}{2},n}}{a})^2$

eigenfunctions: $y_n=j_m(\lambda_{mn}x)$

($\alpha_{m+\frac{1}{2},n}$ be n-th zero of $j_m(x)$ or $J_{m+\frac{1}{2}}(x)$)

note the orthogonal relation:

$$\int_0^a x^2 j_m(\lambda_{mn}x) j_m(\lambda_{ml}x)dx = \delta_{nl}\frac{\pi}{2}\frac{1}{\lambda_{mn}}\frac{a^2}{2}J^2_{m+\frac{3}{2}}(\lambda_{mn}a) = \delta_{nl}a^3\frac{j^2_{m+1}(\alpha_{m+\frac{1}{2},n})}{2}$$

Similar we can have expansion: $f(x) = \sum_{n=1}^{\infty} a_n j_m(\lambda_{mn}x)$

with $a_n = \dfrac{\int_0^a x^2 f(x) j_m(\lambda_{mn}x)dx}{a^3 \frac{j^2_{m+1}(\alpha_{m+\frac{1}{2},n})}{2}}$

7.1.3 Associated Legendre's Equation:

$$(1-x^2)y'' - 2xy' + [k - \frac{m^2}{1-x^2}]y = 0, \qquad -1 < x < 1, \quad \text{m be positive integer or 0}$$

using substituting $y = (1-x^2)^{\frac{m}{2}}v(x)$, find

$$v = C_1\frac{d^m}{dx^m}P(x,k) + C_2\frac{d^m}{dx^m}Q(x,k)$$

with $P(x,k)$, $Q(x,k)$ be solutions of general Legendre's equation.

SL problem with $y(\pm 1)$ finite:

eigenvalues: $k = l(l+1), \qquad l = m, m+1, \cdots$

eigenfunctions: $P_l^m(x) = (-1)^m(1-x^2)^{\frac{m}{2}}\dfrac{d^m P_l(x)}{dx^m}$

(further define: $P_l^{-m}(x) = (-1)^m\dfrac{(l-m)!}{(l+m)!}P_l^m(x)$)

Orthogonality and expansions of associated Legendre's functions:

$$\int_{-1}^1 P_n^m(x)P_l^m(x)dx = \delta_{nl}\int_{-1}^1 \left[P_n^m(x)\right]^2 dx$$

We know :

$$\int_{-1}^1 [P_n^0(x)]^2 dx = \int_{-1}^1 [P_n(x)]^2 dx = \frac{2}{2n+1}$$

further:

$$\frac{d[P_n^m(x)]}{dx} = (-1)^m[\frac{m}{2}(1-x^2)^{\frac{m}{2}-1}(-2x)\frac{d^m P_n(x)}{dx^m} + (1-x^2)^{\frac{m}{2}}\frac{d^{m+1}P_n(x)}{dx^{m+1}}]$$

$$\Longrightarrow P_n^{m+1}(x) = -(1-x^2)^{\frac{1}{2}}\frac{d[P_n^m(x)]}{dx} - m(1-x^2)^{-\frac{1}{2}}xP_n^m(x)$$

thus, $\int_{-1}^1 [P_n^{m+1}(x)]^2 dx = \int_{-1}^1 (1-x^2)\left(\frac{d[P_n^m(x)]}{dx}\right)^2 dx + 2m\int_{-1}^1 xP_n^m(x)\frac{d[P_n^m(x)]}{dx}dx + m^2\int_{-1}^1 \frac{x^2}{1-x^2}[P_n^m(x)]^2 dx$

the first term:

$$\int_{-1}^1 (1-x^2)\left(\frac{d[P_n^m(x)]}{dx}\right)^2 dx = (1-x^2)P_n^m(x)\frac{d[P_n^m(x)]}{dx}\bigg|_{-1}^1 - \int_{-1}^1 P_n^m(x)\frac{d}{dx}[(1-x^2)\frac{d[P_n^m(x)]}{dx}]dx$$

the second term:

$$2m\int_{-1}^{1} xP_n^m(x)d[P_n^m(x)] = 2mx[P_n^m(x)]^2\Big|_{-1}^{1} - 2m\left(\int_{-1}^{1}[P_n^m(x)]^2dx + \int_{-1}^{1} xP_n^m(x)d[P_n^m(x)]\right)$$

$$\Longrightarrow 2m\int_{-1}^{1} xP_n^m(x)d[P_n^m(x)] = -m\int_{-1}^{1}[P_n^m(x)]^2dx$$

we know :

$$\frac{d}{dx}[(1-x^2)\frac{d[P_n^m(x)]}{dx}] = -\left(n(n+1) - \frac{m^2}{1-x^2}\right)P_n^m(x)$$

adding all together:

$$\int_{-1}^{1}[P_n^{m+1}(x)]^2dx = (n-m)(n+m+1)\int_{-1}^{1}[P_n^m(x)]^2dx$$

thus

$$\int_{-1}^{1}[P_n^m(x)]^2dx = (n-m+1)(n+m)\cdot(n-m+2)(n+m-1)\cdots n(n+1)\frac{2}{2n+1} = \frac{(n+m)!}{(n-m)!}\frac{2}{2n+1}$$

Supposing $f(x)$ and $f'(x)$ be piecewise continuous on $[-1,1]$, thus
$f(x) = \sum_{n=m}^{\infty} a_nP_n^m(x), \quad with\ \ a_n = \frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}\int_{-1}^{1} f(x)P_n^m(x)dx$

7.1.4 Spherical harmonics $\quad 0 \le \theta \le \pi,\ \ 0 \le \varphi < 2\pi$

$$Y_{n,m}(\theta,\varphi) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}P_n^m(cos\theta)e^{im\varphi}$$
$$n = 0,1,\cdots; \qquad m = 0,\pm1,\cdots,\pm n$$

One can easily check:

$$(Y_{n,m}(\theta,\varphi), Y_{n',m'}(\theta,\varphi)) = \int_0^{\pi} sin\theta d\theta\int_0^{2\pi} d\varphi Y_{n,m}(\theta,\varphi)\overline{Y}_{n',m'}(\theta,\varphi) = \delta_{nn'}\delta_{mm'}\int_0^{\pi} sin\theta d\theta\int_0^{2\pi} d\varphi|Y_{n,m}|^2 = \delta_{nn'}\delta_{mm'}$$

(complex conjugate: $\overline{Y}_{n',m'}(\theta,\varphi) = Y_{n',-m'}(\theta,\varphi)\cdot(-1)^{m'}$)

$\{e^{im\varphi}\}$ are complete for $0 \le \varphi < 2\pi$;
$\{P_n^m(cos\theta)\}$ are complete for any fixed m, $\quad$ on $0 \le \theta \le \pi$
thus,

$$f(\theta,\varphi) = \sum_{m=-\infty}^{+\infty}\sum_{n=|m|}^{+\infty} a_{nm}Y_{n,m}(\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_{nm}Y_{n,m}(\theta,\varphi)$$

added:

complex conjugate:

$c = a + bi, \quad a, b \in R, \bar{c} = a - bi$

$c = re^{i\theta}, \quad r, \theta \in R, \bar{c} = re^{-i\theta}$

for spherical harmonics:

$$\overline{Y}_{n,m}(\theta,\varphi) = \overline{\left(\sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}P_l^m(cos\theta)e^{im\varphi}\right)} = \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}P_l^m(cos\theta)e^{-im\varphi}$$

$$= \sqrt{\frac{2n+1}{4\pi}\frac{(n+m)!}{(n-m)!}}(-1)^m\frac{(n-m)!}{(n+m)!}P_l^m(cos\theta)e^{-im\varphi}(-1)^m = (-1)^m Y_{n,-m}(\theta,\varphi)$$

and the expansion coefficients (complex number in general):

$$a_{nm} = \int_0^\pi sin\theta d\theta \int_0^{2\pi} d\varphi f(\theta,\varphi)\overline{Y}_{n,m}(\theta,\varphi)$$

since $f(\theta,\varphi)$ are real, thus we always have: $a_{nm} = \bar{a}_{n-m}\cdot(-1)^m$

First few spherical harmonics:

$Y_{0,0} = \frac{1}{2\sqrt{\pi}}, \;\; Y_{1,0} = \sqrt{\frac{3}{\pi}}\frac{1}{2}cos\theta$

$Y_{1,1} = -\sqrt{\frac{3}{2\pi}}\frac{1}{2}sin\theta e^{i\varphi}, \;\; Y_{1,-1} = \sqrt{\frac{3}{2\pi}}\frac{1}{2}sin\theta e^{-i\varphi}$

norm of the spherical harmonics: $|Y_{l,m}| \equiv (Y_{l,m}, \overline{Y}_{l,m})^{\frac{1}{2}}$

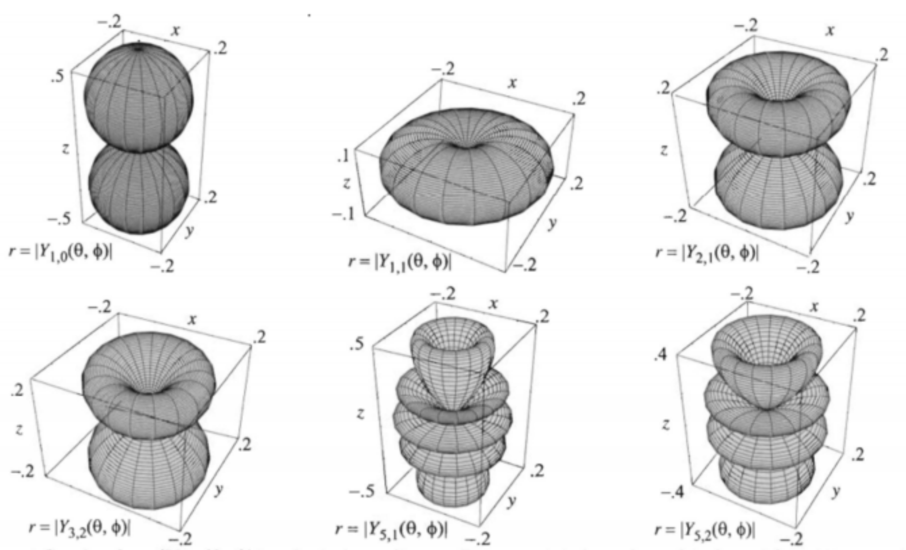


Figure 1 Graphs of $r = |Y_{n,m}(\theta, \phi)|$ in spherical coordinates. Because r is independent of ϕ, the graphs are symmetric with respect to the z-axis.

Example:

spherical harmonic expansion for $f(\theta,\varphi)=\frac{1}{2\pi}\varphi$.

$$f(\theta,\varphi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}a_{nm}Y_{n,m}(\theta,\varphi),\qquad 0\le\theta\le\pi,\quad 0\le\varphi<2\pi$$

We know:

$$a_{nm}=\int_0^{\pi}sin\theta d\theta\int_0^{2\pi}d\varphi f(\theta,\varphi)\overline{Y}_{n,m}(\theta,\varphi)=\int_{-1}^{1}ds\int_0^{2\pi}d\varphi P_n^m(s)e^{-im\varphi}\varphi\frac{1}{2\pi}\sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}$$

$\int_0^{2\pi}d\varphi e^{-im\varphi}\varphi=\begin{cases}\frac{2\pi}{m}i, & m\neq 0\\ 2\pi^2, & m=0\end{cases}$

$\int_{-1}^{1}dsP_n^m(s)=0$, if $n-m$ is odd.

$\int_{-1}^{1}dsP_0^0(s)=2,\quad\int_{-1}^{1}dsP_1^1(s)=-\frac{\pi}{2},\quad\int_{-1}^{1}dsP_1^{-1}(s)=\frac{\pi}{4},\cdots$

$\Longrightarrow a_{00}=\sqrt{\pi},\quad a_{1-1}=-\frac{1}{4}\sqrt{\frac{3\pi}{2}}i,\quad a_{11}=-\bar{a}_{1-1},\cdots$

7.2 General product solutions of Laplace's and Helmholtz's equations in spheratical coordinates

seperation of variable: $u(r,\theta,\varphi)=R(r)Y(\theta,\varphi)$

in case of Laplace's equation: $\Delta u=0$

$\Longrightarrow\frac{1}{R}\frac{d}{dr}(r^2\frac{dR}{dr})+\frac{1}{Y}[\frac{1}{sin\theta}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial}{\partial\theta})+\frac{1}{sin^2\theta}\frac{\partial^2}{\partial\varphi^2}]Y=0$

thus $[\frac{1}{sin\theta}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial}{\partial\theta})+\frac{1}{sin^2\theta}\frac{\partial^2}{\partial\varphi^2}]Y=-\mu Y,\qquad\frac{1}{R}\frac{d}{dr}(r^2\frac{dR}{dr})=\mu$

for the angular part:

it is similar to a SL problem, with further seperation of variable $Y(\theta,\varphi)=\Theta(\theta)\Phi(\varphi)$

we get : $\frac{1}{\Theta}\frac{1}{sin\theta}\frac{d}{d\theta}(sin\theta\frac{d\Theta}{d\theta})+\frac{1}{\Phi}\frac{1}{sin^2\theta}\frac{d^2\Phi}{d\varphi^2}=-\mu$

thus must have: $\qquad\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}=constant=-m^2,\quad m=0,1,\cdots$ (as from periodic BVCs)

$$\Phi=acosm\varphi+bsinm\varphi$$

and $\frac{1}{sin\theta}\frac{d}{d\theta}(sin\theta\frac{d\Theta}{d\theta})=(\frac{m^2}{sin^2\theta}-\mu)\Theta$

changing of variable: $x=cos\theta$ ($[-1,1]$)

$$(1-x^2)\Theta'' - 2x\Theta' + (\mu - \frac{m^2}{1-x^2})\Theta = 0$$

so Θ follows the associated Legendre's equation.

To have finite solution at $x = \pm 1$ ($\theta = 0, \pi$) requries:

$\mu = l(l+1)$, with $l = m, m+1, \cdots$; and $\Theta = P_l^m(x) = P_l^m(cos\theta)$

thus finally:

eigenvalues: $\mu = l(l+1)$, with $l = 0, 1, \cdots$;

eigenfunctions ($2l+1$ in total) : $P_l^m(cos\theta)cosm\varphi$ and $P_l^m(cos\theta)sinm\varphi$ with $m = 0, 1, \cdots, l$

or using another linear combinations:

eigenvalues: $\mu = l(l+1)$, with $l = 0, 1, \cdots$;

eigenfunctions : $Y_{l,m}(\theta, \varphi)$ with $m = 0, \pm 1, \cdots, \pm l$

the radial part:

$$\frac{1}{R}\frac{d}{dr}(r^2\frac{dR}{dr}) = \mu = l(l+1) \Longrightarrow r^2R'' + 2rR' - l(l+1)R = 0$$

again Euler's equation with indicial equation: $\lambda^2 + \lambda - l(l+1) = 0$

so $R(r) = ar^l + br^{-(l+1)}$.

thus the full product solution: $\quad u = (ar^l + br^{-(l+1)})Y_{lm}(\theta, \varphi)$

In case of Helmhotz's equation: $\Delta u = -ku$

the only modification is the radial part:

$$\frac{1}{R}\frac{d}{dr}(r^2\frac{dR}{dr}) = \mu - kr^2 \Longrightarrow r^2R'' + 2rR' + (kr^2 - l(l+1))R = 0$$

So that turns into the spherical Bessel's equation:

$$\begin{cases} k > 0: \quad \text{solution} \quad R(r) = aj_l(\sqrt{k}r) + by_l(\sqrt{k}r) \\ k < 0: \quad \text{solution} \quad R(r) = a\frac{1}{\sqrt{r}}I_{l+\frac{1}{2}}(\sqrt{-k}r) + b\frac{1}{\sqrt{r}}K_{l+\frac{1}{2}}(\sqrt{-k}r) \end{cases}$$

the full solution for Helmhotz's equation:

$$u = \begin{cases} (aj_l(\sqrt{k}r) + by_l(\sqrt{k}r))Y_{lm}(\theta, \varphi), \quad k > 0 \\ (a\frac{1}{\sqrt{r}}I_{l+\frac{1}{2}}(\sqrt{-k}r) + b\frac{1}{\sqrt{r}}K_{l+\frac{1}{2}}(\sqrt{-k}r))Y_{lm}(\theta, \varphi), \quad k < 0 \end{cases}$$

given certain BVCs of r, that represents a SL problem, the eigenvalues k and the associated eigenfunctions will be further determined.

7.3 Solutions of Laplace's equation

7.3.1 Case 1:

considering a region of $r \leq a$, requiring u be finite ar $r = 0$, thus

$$u(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} C_{lm}(\frac{r}{a})^l Y_{l,m}(\theta,\varphi)$$

Now with the BVCs: $u(a,\theta,\varphi) = f(\theta,\varphi)$

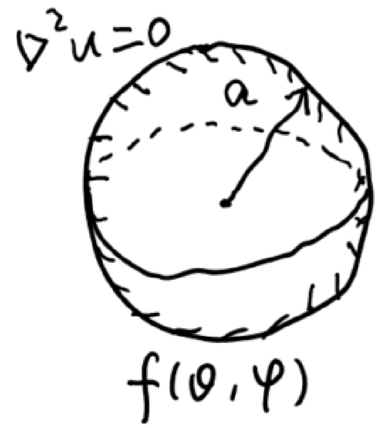
If choosing $C_{lm} = \int_0^{\pi} sin\theta d\theta \int_0^{2\pi} d\varphi f(\theta,\varphi)\overline{Y}_{l,m}(\theta,\varphi)$, satisfy both PDE & BVCs.

In case of axial symmetric, namely $f(\theta,\varphi) = f(\theta)$, thus solution

$$u(r,\theta,\varphi) = \sum_{l=0}^{\infty} C_l(\frac{r}{a})^l P_l(cos\theta) \quad \text{(only have } m = 0 \text{ component)}$$

and choosing $C_l = \dfrac{2l+1}{2}\displaystyle\int_0^{\pi} sin\theta d\theta f(\theta) P_l(cos\theta)$ will give $u(a,\theta,\varphi) = f(\theta)$

7.3.2 Case 2:



considering a region of $r \geq a$,then similarly

$$u(r,\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}C_{lm}(\frac{r}{a})^{-(l+1)}Y_{l,m}(\theta,\varphi)$$

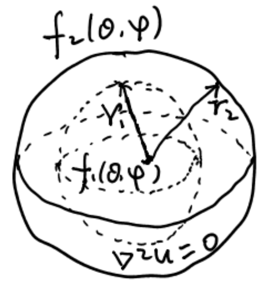
if requiring $u(r\to\infty)=0$.

with BVCs $u(a,\theta,\varphi)=f(\theta,\varphi)$

samely choosing $C_{lm}=\int_0^{\pi}sin\theta d\theta\int_0^{2\pi}d\varphi f(\theta,\varphi)\overline{Y}_{l,m}(\theta,\varphi)$

7.3.3 Case 3:

Considering region between two concentric sphere, $r_1\le r\le r_2$



the general solution:

$$u(r,\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}(a_{lm}r^l+b_{lm}r^{-(l+1)})Y_{lm}(\theta,\varphi)=$$
$$\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}Y_{lm}(\theta,\varphi)r^{-(l+1)}\left\{C_{lm}\left(\frac{r^{2l+1}-r_1^{2l+1}}{r_2^{2l+1}-r_1^{2l+1}}\right)+D_{lm}\left(\frac{r^{2l+1}-r_2^{2l+1}}{r_1^{2l+1}-r_2^{2l+1}}\right)\right\}$$

must keeping both powers of r.

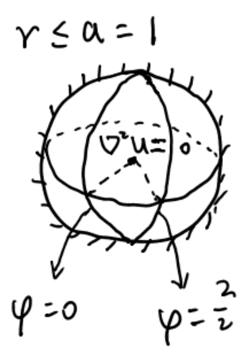
To satisfy $u(r_1,\theta,\varphi)=f_1(\theta,\varphi)$ and $u(r_2,\theta,\varphi)=f_2(\theta,\varphi)$

simply choosing:

$$C_{lm}=r_2^{l+1}\int_0^{\pi}sin\theta d\theta\int_0^{2\pi}d\varphi f_2(\theta,\varphi)\overline{Y}_{lm}(\theta,\varphi)$$
$$D_{lm}=r_1^{l+1}\int_0^{\pi}sin\theta d\theta\int_0^{2\pi}d\varphi f_1(\theta,\varphi)\overline{Y}_{lm}(\theta,\varphi)$$

Example:

solve the steady-state problem inside a unit sphere , with temperature on the boundary.



$$u(1,\theta,\varphi)=\begin{cases}100^\circ, & 0\le\varphi\le\frac{\pi}{2}\\ 0^\circ, & \text{otherwise}\end{cases}$$

Inside the sphere the general solution is : $\quad u=\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}C_{lm}r^lY_{lm}(\theta,\varphi)$

with $C_{lm}=\int_0^\pi sin\theta d\theta\int_0^{2\pi}d\varphi f(\theta,\varphi)\overline{Y}_{lm}(\theta,\varphi)=100\int_0^\pi sin\theta d\theta\int_0^{\frac{\pi}{2}}d\varphi\overline{Y}_{l,m}(\theta,\varphi)$
Recall the expressions for Y_{lm}:

$$C_{00}=100\left(\int_{-1}^1 dx\cdot 1\right)\cdot\left(\int_0^{\frac{\pi}{2}}d\varphi\cdot 1\right)\frac{1}{\sqrt{4\pi}}=50\sqrt{\pi}$$
$$C_{1-1}=100\left(\int_0^\pi sin^2\theta d\theta\right)\cdot\left(\int_0^{\frac{\pi}{2}}d\varphi e^{i\varphi}\right)\left(\sqrt{\frac{3}{2\pi}}\frac{1}{2}\right)=25\sqrt{3\pi}e^{\frac{i\pi}{4}}$$
$$C_{10}=100\left(\int_{-1}^1 xdx\right)\cdot\left(\int_0^{\frac{\pi}{2}}d\varphi\right)\sqrt{\frac{3}{4\pi}}=0$$
$$C_{11}=(-1)^m\overline{C}_{1-1}=-25\sqrt{3\pi}e^{-\frac{i\pi}{4}}$$
$$C_{2-2}=100\left(\int_0^\pi sin^3\theta d\theta\right)\cdot\left(\int_0^{\frac{\pi}{2}}d\varphi e^{i2\varphi}\right)\left(\frac{3}{4}\sqrt{\frac{5}{6\pi}}\right)=100\sqrt{\frac{5}{6\pi}}i$$
$$C_{2-1}=C_{20}=0=C_{21}=0$$
$$C_{22}=(-1)^m\overline{C}_{2-2}=-100\sqrt{\frac{5}{6\pi}}i$$

7.4 The Helmholtz's equation with application to the Poisson, heat, and wave equations

The Helmholtz's equation together with certain BVCs can be think as a SL problem.
$(0<r<a,\ 0<\theta<\pi,\ 0<\varphi<2\pi)$
$$\begin{cases}\nabla^2\psi(r,\theta,\varphi)=-k\psi(r,\theta,\varphi)\\ \psi(r,\theta,\varphi)|_{r=a}=0\end{cases}$$
Recall the general product solution:

$k=0:\qquad Y_{lm}(ar^l+br^{-l-1})\longrightarrow a=b=0$

$k<0:\qquad Y_{lm}(\frac{a}{\sqrt{r}}I_{l+\frac{1}{2}}+\frac{b}{\sqrt{r}}K_{l+\frac{1}{2}})\longrightarrow a=b=0$

Nontrivial solution only if $k=\lambda_{lj}^2\equiv\left(\frac{\alpha_{l+\frac{1}{2},j}}{a}\right)^2,\ \ j=1,2,\cdots$
for each eigenvalue (l and j fixed),
eigenfunctions ($2l+1$ in total): $\Psi_{jlm}=Y_{lm}(\theta,\varphi)j_l(\lambda_{lj}r),\ \ m=0,\pm1,\cdots,\pm l$

Orthogonality of solutions of the Helmholtz's equation:

$$\int_0^a r^2dr\int_0^\pi sin\theta d\theta\int_0^{2\pi}d\varphi\Psi_{jlm}\overline{\Psi}_{j'l'm'}=0,\quad\text{if any } j\neq j',\ l\neq l',\ \text{ or } m\neq m'$$

in analogy to the SL problem of ODE, or explicitly,

$$\int_0^{\pi} sin\theta d\theta \int_0^{2\pi} d\varphi Y_{lm}\overline{Y}_{l'm'} = \delta_{ll'}\delta_{mm'}$$

$$\int_0^a r^2 dr j_l(\lambda_{lj}r) j_{l'}(\lambda_{l'j'}r) \quad \text{if } l = l',\ \frac{a^3}{2} j_{l+1}^2(\alpha_{l+\frac{1}{2},j})\delta_{jj'}$$

thus we also have norms of above eigenfunctions:

$$\int_0^a r^2 dr \int_0^{\pi} sin\theta d\theta \int_0^{2\pi} d\varphi \left|\Psi_{jlm}\right|^2 = \frac{a^3}{2} j_{l+1}^2(\alpha_{l+\frac{1}{2},j}), \qquad \text{independent of } m$$

Series expansions of functions defined in a ball:

Let $f(r,\theta,\varphi)$ be a square integrable function, defined for $0 < r < a,\ \ 0 < \theta < \pi,\ \ 0 < \varphi < 2\pi$

it can be expanded as :

$$f(r,\theta,\varphi) = \sum_{j=1}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l} A_{jlm} j_l(\lambda_{lj}r) Y_{lm}(\theta,\varphi)$$

where $A_{jlm} = \dfrac{2}{a^3 j_{l+1}^2(\alpha_{l+\frac{1}{2},j})}\displaystyle\int_0^a r^2 dr \int_0^{\pi} sin\theta d\theta \int_0^{2\pi} d\varphi f(r,\theta,\varphi) j_l(\lambda_{lj}r)\overline{Y}_{lm}(\theta,\varphi)$

Note above triple series always converge to zero on the sphere $r = a$, and satisfy the periodic condition in φ, thus solution of PDE with zero BVCs, e.g. Poisson, heat equations must can be expressed as above.

Example: function inside an unit ball $f(r,\theta,\varphi) = \begin{cases} 1, & 0 < r < \dfrac{1}{2} \\ 0, & \text{otherwise} \end{cases}$

find the expansion using eigenfunctions of Helmhotz's equation (Ψ_{jlm})

Solution:

One can easily identify only terms with $l = m = 0$, thus $f(r,\theta,\varphi) = \displaystyle\sum_{j=1}^{\infty} A_j j_0(\lambda_{0j}r)$

with $A_j = \dfrac{2}{\frac{1}{2}^3 j_1^2(\alpha_{\frac{1}{2},j})}\displaystyle\int_0^{\frac{1}{2}} r^2 dr j_0(\lambda_{0j}r) = \dfrac{16}{j_1^2(\alpha_{\frac{1}{2},j})}\dfrac{1}{\lambda_{0j}^3}\sqrt{\dfrac{\pi}{2}}\displaystyle\int_0^{\frac{\lambda_{0j}}{2}} x^{\frac{3}{2}} J_{\frac{1}{2}}(x)dx$

$$= \frac{16}{2\alpha_{\frac{1}{2},j}}\frac{j_1(\frac{\alpha_{\frac{1}{2},j}}{2})}{j_1^2(\alpha_{\frac{1}{2},j})} = 8(\frac{2sin\frac{j\pi}{2}}{j\pi} - cos\frac{j\pi}{2})$$

$$\left(j_0(x) = \frac{sinx}{x},\ \ j_1(x) = \frac{sinx}{x^2} - \frac{cosx}{x}, \qquad \alpha_{\frac{1}{2},j} = j\pi\right)$$

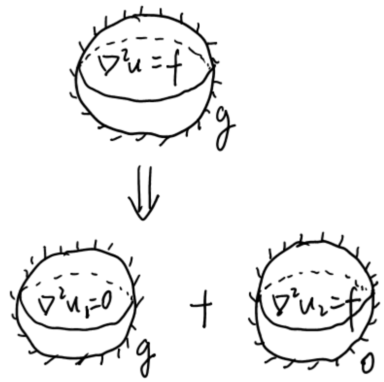
Poisson's equation in a ball:

We consider Dirichlet problem of Poisson's equation:

$0 < r < a,\ \ 0 < \theta < \pi.\ \ 0 < \varphi < 2\pi$

$\nabla^2 u(r,\theta,\varphi) = f(r,\theta,\varphi), \qquad u(r,\theta,\varphi)|_{r=a} = g(\theta,\varphi)$

let $u = u_1 + u_2$



with u_1 know from last section.

$u_2 = \sum_{j=1}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l} B_{jlm} j_l(\lambda_{lj} r) Y_{lm}(\theta,\varphi)$

from Helmholtz's equation, we know:

$$\nabla^2 u_2 = \sum_{j=1}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l} (-\lambda_{lj}^2) B_{jlm} j_l(\lambda_{lj} r) Y_{lm} = f = \sum_{j=1}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l} A_{jlm} j_l(\lambda_{lj} r) Y_{lm} \Longrightarrow B_{jlm} = -\frac{A_{jlm}}{\lambda_{lj}^2}$$

Example: Solve the Poisson problem inside the unit ball with $f = 1$ and $g = \dfrac{\varphi}{2\pi}$

Solution:

with the superposition rule $u = u_1 + u_2$

for u_1 refer to last section.

for u_2 :

first, $f = \sum_{j=1}^{\infty} A_j j_0(\alpha_{\frac{1}{2},j} r)$ with $A_j = \dfrac{2}{j_1^2(\alpha_{\frac{1}{2},j})}\displaystyle\int_0^1 r^2 dr\, j_0(\alpha_{\frac{1}{2},j} r) = \dfrac{2}{\alpha_{\frac{1}{2},j} j_1(\alpha_{\frac{1}{2},j})}$

thus $u_2 = \sum_{j=1}^{\infty} B_j j_0(\alpha_{\frac{1}{2},j} r)$

with $B_j = -\dfrac{A_j}{(\alpha_{\frac{1}{2},j})^2} = -\dfrac{2}{(\alpha_{\frac{1}{2},j})^3 j_1(\alpha_{\frac{1}{2},j})} = \dfrac{2}{(j\pi)^2}(-1)^j$

A nonhomogeneous heat equation:

recall the heat equation: $\dfrac{\partial u}{\partial t} = c^2 \nabla^2 u + q(r,\theta,\varphi,t)$ with $0 < r < a,\ 0 < \theta < \pi,\ 0 < \varphi < 2\pi,\ t > 0$

the BVCs: $u(a,\theta,\varphi,t) = 0$

the IVCs: $u(r,\theta,\varphi,0) = f(r,\theta,\varphi)$

As mentioned earlier, the solution must can be expressed:

$$u(r,\theta,\varphi,t)=\sum_{j=1}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}B_{jlm}(t)j_l(\lambda_{lj}r)Y_{lm}(\theta,\varphi)$$

plug into PDE:

$$\sum_{j=1}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\left(B'_{jlm}(t)+c^2\lambda_{lj}^2B_{jlm}(t)\right)j_l(\lambda_{lj}r)Y_{lm}(\theta,\varphi)=q(r,\theta,\varphi,t)$$

In addition the nonhomogeneous term and IVC can also be expanded:

$$q(r,\theta,\varphi,t)=\sum_{j=1}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}q_{jlm}(t)j_l(\lambda_{lj}r)Y_{lm}(\theta,\varphi)$$
$$f(r,\theta,\varphi)=\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}f_{jlm}j_l(\lambda_{lj}r)Y_{lm}(\theta,\varphi)$$

put q back into PDE:

$$B'_{jlm}(t)+c^2\lambda_{lj}^2B_{jlm}(t)-q_{jlm}(t)=0,\quad \text{for any } j,l,m$$

from the IVCs: $B_{jlm}(0)=f_{jlm}$, thus

$$B_{jlm}(t)=e^{-c^2\lambda_{lj}^2t}\left(f_{jlm}+\int_0^t q_{jlm}(\tau)e^{c^2\lambda_{lj}^2\tau}d\tau\right)$$

in case q has no t dependence:

$$B_{jlm}(t)=e^{-c^2\lambda_{lj}^2t}\left(f_{jlm}-\frac{q_{jlm}}{c^2\lambda_{lj}^2}\right)+\frac{q_{jlm}}{c^2\lambda_{lj}^2}$$

Example: A heat problem with symmetry, a solid ball at 30° with radius $a=1$ placed in a fridge of constant temperature of 0°. Take $c=1$ and determine the temperature inside the ball.

Solution:

using the eigenfunction expansion with the fact that only $m=l=0$ contributes.

$$u(r,\theta,\varphi,t)=\sum_{j=1}^{\infty}B_{j00}(t)j_0(\lambda_{0j}r)$$

and

$$B_{j00}(t)=f_{j00}e^{-\lambda_{0j}^2t},\qquad f_{j00}=\frac{2}{j_1^2(\lambda_{0j})}30\int_0^1 r^2dr\,j_0(\lambda_{0j}r)=\frac{60}{\alpha_{\frac{1}{2},j}j_1(\alpha_{\frac{1}{2},j})}=(-1)^{j+1}\cdot 60$$

in the center: $u(0,\theta,\varphi,t)=\sum_{j=1}^{\infty}60(-1)^{j+1}e^{-j^2\pi^2t}$

A homogeneous wave equation in a ball:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u, \qquad 0 < r < a,\ 0 < \theta < \pi,\ 0 < \varphi < 2\pi,\ t > 0$$

with homogeneous BVCs: $u(a, \theta, \varphi, t) = 0$

and IVCs: $u(r, \theta, \varphi, 0) = f(r, \theta, \varphi), \quad u_t(r, \theta, \varphi, 0) = g(r, \theta, \varphi)$

By means of eigenfunction expansion or from seperation of variable:

$$u(r, \theta, \varphi, t) = \sum_{j=1}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} B_{jlm}(t) j_l(\lambda_{lj} r) Y_{lm}(\theta, \varphi)$$

and further:

$$B''_{jlm}(t) = -\lambda_{lj}^2 c^2 B_{jlm}(t) \Longrightarrow B_{jlm} = C_{jlm} cosc\lambda_{lj} t + D_{jlm} sinc\lambda_{lj} t$$

As usual C_{jlm} and D_{jlm} can be determined from the IVCs.

$$C_{jlm} = \frac{2}{a^3 j_{l+1}^2(\alpha_{l+\frac{1}{2},j})} \int_0^a r^2 dr \int_0^{\pi} sin\theta d\theta \int_0^{2\pi} d\varphi f(r, \theta, \varphi) j_l(\lambda_{lj} r) \overline{Y}_{lm}(\theta, \varphi)$$

$$D_{jlm} \cdot c\lambda_{lj} = \frac{2}{a^3 j_{l+1}^2(\alpha_{l+\frac{1}{2},j})} \int_0^a r^2 dr \int_0^{\pi} sin\theta d\theta \int_0^{2\pi} d\varphi g(r, \theta, \varphi) j_l(\lambda_{lj} r) \overline{Y}_{lm}(\theta, \varphi)$$

asymptotic behavior of wave solution wrt. r:

① one dimension: $u(x) \sim sin(\lambda x + \varphi)$

② polar case: $u(r) \sim J_n(\lambda r) \sim \frac{1}{\sqrt{r}} cos(\lambda r + \varphi)$

③ spherical case: $u(r) \sim j_l(\lambda r) \sim \frac{1}{r} cos(\lambda r + \varphi)$

energy conservation:

① one dimension: flow $\propto$ constant

② polar case: $\propto \left(\frac{1}{\sqrt{r}}\right)^2 2\pi r =$ constant

③ spherical case: $\propto \left(\frac{1}{r}\right)^2 4\pi r^2 =$ constant

8 Chapter 8. Fourier and Laplace transforms and their applications to PDE

8.1 The Fourier transform

starting from Fourier series for $x \in [-p, p]$

$$f(x) = \sum_{n=0}^{\infty} \left(a_n cos\frac{n\pi x}{p} + b_n sin\frac{n\pi x}{p} \right) = \frac{p}{\pi} \sum_{n=0}^{\infty} \left(a_n cos\frac{n\pi x}{p} + b_n sin\frac{n\pi x}{p} \right) \frac{\pi}{p}$$

with $a_n = \frac{\delta_n}{p}\int_{-p}^{p} f(x)cos\frac{n\pi x}{p}dx, \qquad b_n = \frac{1}{p}\int_{-p}^{p} f(x)sin\frac{n\pi x}{p}dx$

$\omega_n = \frac{n\pi}{p}, \qquad$ let $p \to \infty \Longrightarrow f(x) = \int_0^{\infty}\left[a(\omega)cos\omega x + b(\omega)sin\omega x\right]d\omega$

Suppose $f(x)$ is piecewise continuous on every finite interval and that $\int_{-\infty}^{\infty}|f(x)|\,dx < \infty$.

Then $f(x)$ has the Fourier integral representation of the form:

$$f(x) = \int_0^{\infty}\left[A(\omega)cos\omega x + B(\omega)sin\omega x\right]d\omega \qquad (-\infty < x < +\infty)$$

where for all $\omega \geq 0$:

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x)cos\omega x dx, \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x)sin\omega x dx$$

Integral representation of $f(x)$ converges to $f(x)$ if $f(x)$ is continuous at x, to $\frac{f(x+)+f(x-)}{2}$ otherwise.

$\left(\text{for delta function:}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-i\omega x}\delta(x)dx = \frac{1}{\sqrt{2\pi}}, \quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i\omega x}\frac{1}{\sqrt{2\pi}}d\omega = \delta(x)\right)$

Or using a move compact form:

for any function $f(x)$, define its Fourier transform:

$$F[f(x)](\omega) = \hat{f}(\omega) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-i\omega x}dx \;\; (-\infty < \omega < +\infty)$$

Inverse Fourier transform:

$$F^{-1}[\hat{f}(\omega)](x) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\hat{f}(\omega)e^{i\omega x}d\omega = f(x) \;\; (-\infty < \omega < +\infty)$$

or in a familiar way: $F^{-1}[F[f(x)]] = f(x)$

Example: $\quad f(x) = \begin{cases} 1, & \text{if } |x| < a \\ 0, & \text{if } |x| > a \end{cases}$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{-i\omega x}dx = \sqrt{\frac{2}{\pi}}\frac{sin\omega a}{\omega}$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{cos\omega x sin\omega a}{\omega}d\omega$$

Operation properties:

① linearity: $F[af+bg] = aF[f] + bF[g]$

② Suppose f and f' both be piecewise continuous and integrable and $\lim_{x\to\pm\infty} f(x) = 0$, then $F[f'] = i\omega F[f]$

if in addition: f'' is piecewise continuous and integrable, and $\lim_{x\to\pm\infty} f'(x) = 0$, then $F[f''] = -\omega^2 F[f]$

③ Suppose $f(x)$ and $x^n f(x)$ are integrable:

$$F[x^n f(x)] = i^n \frac{d^n}{d\omega^n} F[f(x)]$$

④ shifting on the ω-axis: $F[e^{iax} f(x)] = F[f(x)](\omega - a)$

shifting on the x-axis: $F^{-1}[e^{-ihx} \hat{f}(\omega)] = F^{-1}[\hat{f}(\omega)](x - h)$

⑤ define convolution of two functions f and g: $\quad f * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x-t) g(t) dt$

then $F[f * g] = F[f] \cdot F[g]$

Example: Fourier transform of the Gaussian function: $f(x) = e^{-\frac{ax^2}{2}},\ a > 0$

we know $f(x)$ satisfy $f'(x) + axf(x) = 0$, taken F on the both side.

$\Longrightarrow i\omega \hat{f}(\omega) + ai\hat{f}'(\omega) = 0 \Longrightarrow \hat{f}(\omega) = Ae^{-\frac{\omega^2}{2a}}$

$\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{ax^2}{2}} e^{-i0x} dx = \frac{1}{\sqrt{a}} \Longrightarrow A = \frac{1}{\sqrt{a}}$

Fourier Sine/Cosine transform: defined on $0 \leq x < +\infty,\ 0 \leq \omega < +\infty$

$$F_c[f(x)](\omega) = \hat{f}(\omega) \equiv \sqrt{\frac{2}{\pi}} \int_0^{+\infty} dx f(x) cos\omega x$$

$$F_c^{-1}[\hat{f}(\omega)](x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} d\omega \hat{f}(\omega) cos\omega x = f(x)$$

$$F_s[f(x)](\omega) = \hat{f}(\omega) \equiv \sqrt{\frac{2}{\pi}} \int_0^{+\infty} dx f(x) sin\omega x$$

$$F_s^{-1}[\hat{f}(\omega)](x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} d\omega \hat{f}(\omega) sin\omega x = f(x)$$

derivatives:

$$F_c[f'(x)] = \omega F_s[f(x)] - \sqrt{\frac{2}{\pi}} f(0);\ \ F_c[f''(x)] = -\omega^2 F_c[f(x)] - \sqrt{\frac{2}{\pi}} f'(0)$$

$$F_s[f'(x)] = -\omega F_c[f(x)];\ \ F_s[f''(x)] = -\omega^2 F_s[f(x)] + \sqrt{\frac{2}{\pi}} \omega f(0)$$

8.2 Fourier transform method on PDE on infinte region

When discussing infinite region, the series expansion method fails.

E.g.

① heat equation for an infinite rod, $t \geq 0$

$$\frac{\partial}{\partial t} u = c^2 \frac{\partial^2}{\partial x^2} u,\ (-\infty < x < +\infty) \qquad u(x,0) = f(x)$$

take F on both side wrt. x, considering ω as fixed:

$$\hat{u}(\omega, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u(x,t) e^{-i\omega x} dx, \quad u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{u}(\omega, t) e^{i\omega x} d\omega$$

back to PDE:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{d\hat{u}(\omega,t)}{dt}e^{i\omega x}d\omega = c^2\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\hat{u}(\omega,t)(-\omega^2)e^{i\omega x}d\omega$$

which leads to:

$$\frac{d\hat{u}(\omega,t)}{dt} = -c^2\omega^2\hat{u}(\omega,t),\ \ \hat{u}(\omega,0)=\hat{f}(\omega) \Longrightarrow \hat{u}(\omega,t)=\hat{f}(\omega)e^{-c^2\omega^2 t}$$

thus $u(x,t) = \frac{1}{\sqrt{2c^2t}}(f*g)(x)$, with $g(x)\equiv e^{-\frac{x^2}{4c^2t}}$.

So $u(x,t) = \frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^{+\infty} f(s)e^{-\frac{(x-s)^2}{4c^2t}}ds.$

② wave equation on an infinite string, $t \geq 0$

$$\frac{\partial^2}{\partial t^2}u = c^2\frac{\partial^2}{\partial x^2}u,\ (-\infty < x < +\infty)\quad u(x,0)=f(x),\ u_t(x,0)=g(x)$$

take F on both side wrt. x,

$$\frac{d^2\hat{u}(\omega,t)}{dt^2} = -c^2\omega^2\hat{u}(\omega,t),\ \ \hat{u}(\omega,0)=\hat{f}(\omega),\quad \hat{u}'(\omega,0)=\hat{g}(\omega)$$

thus

$$\hat{u}(\omega,t)=\hat{f}(\omega)cos\omega ct + \frac{\hat{g}(\omega)}{\omega c}sin\omega ct$$

$$u(x,t) = F^{-1}[\hat{u}(\omega,t)] = F^{-1}[\frac{\hat{f}(\omega)}{2}(e^{i\omega ct}+e^{-i\omega ct})] + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}d\omega\hat{g}(\omega)e^{i\omega x}\int_0^t dt' cos\omega ct'$$

$$= \frac{1}{2}[f(x+ct)+f(x-ct)] + \int_0^t dt'(\frac{1}{2}g(x+ct')+\frac{1}{2}g(x-ct')) \qquad \left(\int_0^t dt' cos\omega ct' = \frac{1}{\omega c}sin\omega ct'\Big|_0^t = \frac{sin\omega t}{\omega c}\right)$$

$$= \frac{1}{2}[f(x+ct)+f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}g(s)ds$$

③ Laplace's Equation on the upper $x-y$ plane $\quad (f(\pm\infty)=0)$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\quad -\infty<x<+\infty,\ y\geq 0,\ u\big|_{y=0}=f(x)$$

take F on both side wrt. x,

$$\frac{d^2\hat{u}(\omega,y)}{dy^2}-\omega^2\hat{u}=0,\ \ \hat{u}(\omega,0)=\hat{f}(\omega),\ \ \hat{u}(\omega,+\infty)=0$$

thus $\hat{u}(\omega,y)=\hat{f}(\omega)e^{-|\omega|y}$, and

$$u(x,y) = F^{-1}[\hat{u}(\omega,y)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}d\omega\hat{f}(\omega)e^{i\omega x}e^{-|\omega|y}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}dx' f(x')\int_{-\infty}^{+\infty}d\omega e^{-i\omega(x'-x)}e^{-|\omega|y} = \frac{y}{\pi}\int_{-\infty}^{+\infty}dx'\frac{f(x')}{(x-x')^2+y^2}$$

④ wave equation in 3-D without boundaries

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u, \quad u|_{t=0} = f(\vec{r}), \quad u_t|_{t=0} = g(\vec{r})$$

Now introducing 3-D F wrt. $\vec{r} \equiv \{x, y, z\} \longrightarrow \vec{\omega} \equiv \{\omega_x, \omega_y, \omega_z\}$

$$\frac{d^2\hat{u}(\vec{\omega}, t)}{dt^2} = -\omega^2 c^2 \hat{u}(\vec{\omega}, t), \qquad \hat{u}(\vec{\omega}, 0) = \hat{f}(\vec{\omega}), \ \hat{u}'(\vec{\omega}, 0) = \hat{g}(\vec{\omega})$$

thus, $\hat{u}(\vec{\omega}, t) = \hat{f}(\vec{\omega}) cos\omega ct + \dfrac{\hat{g}(\vec{\omega})}{\omega c} sin\omega ct$

and $u(\vec{r}, t) = F^{-1}[\hat{u}(\vec{\omega}, t)] = \dfrac{1}{(2\pi)^{\frac{3}{2}}} \displaystyle\int d\vec{\omega} \cdot e^{i\vec{\omega}\vec{r}} \hat{u}(\vec{\omega}, t)$

$$= \frac{1}{(2\pi)^3}\left\{\int d\vec{r'} f(\vec{r'}) \int d\vec{\omega} \cdot e^{-i\vec{\omega}(\vec{r'}-\vec{r})} cos\omega ct + \int d\vec{r'} g(\vec{r'}) \int d\vec{\omega} \cdot e^{-i\vec{\omega}(\vec{r'}-\vec{r})} \frac{sin\omega ct}{\omega c}\right\}$$

now $\displaystyle\int d\vec{r'} f(\vec{r'}) \int d\vec{\omega} e^{-i\vec{\omega}\cdot(\vec{r'}-\vec{r})} cos\omega ct = \int d\vec{r'} f(\vec{r} + \vec{r'}) \int d\vec{\omega} \cdot e^{-i\vec{\omega}\cdot\vec{r'}} cos\omega ct$

$$= \frac{\partial}{\partial t}\int d\vec{r'} f(\vec{r} + \vec{r'}) \int d\vec{\omega} \frac{sin\omega ct}{\omega c} \cdot e^{-i\omega r' cos\theta} = \frac{\partial}{\partial t}\int d\vec{r'} f(\vec{r} + \vec{r'})$$

$(\displaystyle\int d\vec{\omega} = \int_0^\infty \omega^2 d\omega \int_{-1}^1 dcos\theta \int_0^{2\pi} d\varphi)$

$\displaystyle\int_0^\infty \omega^2 d\omega \frac{2\pi}{-i\omega^2 c r'} sin\omega ct(e^{-i\omega r'} - e^{i\omega r'}) = \frac{(2\pi)^2}{2c}\frac{\partial}{\partial t}\int d\vec{r'} f(\vec{r} + \vec{r'}) \frac{\delta(r' - ct)}{r'} = \frac{(2\pi)^2}{2c}\frac{\partial}{\partial t}\iint_{S_{ct}^{\vec{r}}} \frac{f(\vec{r'})}{ct} ds'$

$S_{ct}^{\vec{r}} \longrightarrow$ surface of sphere centered at $\vec{r}$, with radius ct.

similar for the second term.

finally:

$$u(\vec{r}, t) = \frac{1}{4\pi c}\frac{\partial}{\partial t}\iint_{S_{ct}^{\vec{r}}} \frac{f(\vec{r'})}{ct} ds' + \frac{1}{4\pi c}\iint_{S_{ct}^{\vec{r}}} \frac{g(\vec{r'})}{ct} ds'$$

Fourier transform in 3-D space: $\quad u(x, y, z, t) \longrightarrow \hat{u}(\omega_x, \omega_y, \omega_z, t)$

$F[u(x, y, z)](\omega_x, \omega_y, \omega_z) = \dfrac{1}{(2\pi)^{\frac{3}{2}}} \displaystyle\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz\, u(x, y, z) e^{-i\omega_x x} e^{-i\omega_y y} e^{-i\omega_z z}$

$(x, y, z) \to \vec{r}, \qquad (\omega_x, \omega_y, \omega_z) \to \vec{\omega}, \qquad dxdydz \to d\vec{r}, \qquad d\omega_x d\omega_y d\omega_z \to d\vec{\omega}$

$\delta(x)\delta(y)\delta(z) \equiv \delta(\vec{r}), \qquad \displaystyle\int d\vec{r}\delta(\vec{r}) = 1, \qquad \int d\vec{\omega} e^{i\vec{\omega}\cdot\vec{r}} = (2\pi)^3\delta(\vec{r})$

$$F[u(\vec{r})](\vec{\omega}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d\vec{r} u(\vec{r}) e^{-i\vec{\omega}\cdot\vec{r}}$$
$$F^{-1}[u(\vec{\omega})](\vec{r}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d\vec{\omega} \hat{u}(\vec{\omega}) e^{i\vec{\omega}\cdot\vec{r}}$$

⑤ non-homogeneous Wave equation in 3-D without any boundaries

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u + f(\vec{r}, t), \qquad u|_{t=0} = 0, \quad u_t|_{t=0} = 0$$

using 3-D F wrt. $\vec{r}$:

$$\hat{u}(\vec{\omega},t) = cos\omega ct\int_0^t dt'\hat{f}(\vec{\omega},t')\frac{-sin\omega ct'}{\omega c} + sin\omega ct\int_0^t dt'\hat{f}(\vec{\omega},t')\frac{cos\omega ct'}{\omega c}$$

now apply F^{-1}:

$$u(\vec{r},t) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d\vec{\omega}e^{i\vec{\omega}\cdot\vec{r}}\hat{u}(\vec{\omega},t)$$

$$=\frac{1}{(2\pi)^3}\int d\vec{r'}\int_0^t dt' f(\vec{r'},t')\int d\vec{\omega}\frac{1}{\omega c}e^{i\vec{\omega}(\vec{r}-\vec{r'})}sin\omega c(t-t')=\frac{1}{4\pi c^2}\iiint_{T^{\vec{r}}_{ct}}\frac{f(\vec{r'},t-\frac{|\vec{r}-\vec{r'}|}{c})}{|\vec{r}-\vec{r'}|}d\vec{r'}$$

⑥ semi-infinite region, 1-D heat equation

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2}{\partial x^2}u,\ \ x\geq 0,\ \ t\geq 0,\quad u(x,t)|_{x=0} = u_0,\ \ u(x,t)|_{t=0} = 0$$

using F_s wrt. x:

$$\frac{d\hat{u}(\omega,t)}{dt} = c^2\left(-\omega^2\hat{u}(\omega,t) + \sqrt{\frac{2}{\pi}}\omega u(0,t)\right),\qquad \hat{u}(\omega,0) = 0$$

thus $\hat{u}(\omega,t) = \sqrt{\frac{2}{\pi}}\frac{u_0}{\omega}(1-e^{-\omega^2c^2t})$, and the inverse

$u(x,t) = \frac{2}{\pi}\int_0^\infty d\omega\frac{u_0}{\omega}(1-e^{-\omega^2c^2t})sin\omega x = u_0\frac{2}{\pi}\int_0^\infty\frac{d\omega}{\omega}(1-e^{-\omega^2})sin\frac{\omega x}{c\sqrt{t}}$, let $z = \frac{x}{2c\sqrt{t}}$

$= u_0\frac{4}{\pi}\int_0^z dy\int_0^\infty d\omega(1-e^{-\omega^2})cos(2\omega y) = \frac{u_0}{\pi}\int_0^z dy\int_{-\infty}^{+\infty}d\omega(1-e^{-\omega^2})(e^{i2\omega y}+e^{-i2\omega y})$

$= u_0(1-\frac{2}{\sqrt{\pi}}\int_0^z e^{-y^2}dy)$

$(= u_0\ erfc(\frac{x}{2c\sqrt{t}}) = 1-\frac{2}{\sqrt{\pi}}\int_0^z e^{-y^2}dy,\qquad erfc(z) = 1-erf(z))$

8.3 The Laplace transform

Definition:

Suppose $f(t)$ is defined for all $t\geq 0$, the Laplace transform of f is the function:

$$L[f(t)](s) = \int_0^{+\infty}f(t)e^{-st}dt\qquad \text{(s can be on the full complex-plane)}$$

If f is piecewise continuous on $[0,+\infty)$, and is exponential ordered, namely exist M and S_0,

$$|f(t)|\leq Me^{S_0t}\text{ for all }t\geq 0$$

Then the Laplace transform exist for all $s > S_0$.

E.g.

$$L[t^\beta](s) = \int_0^{+\infty} t^\beta e^{-st}dt = \frac{1}{s^{\beta+1}}\int_0^{+\infty} x^\beta e^{-x}dx = \frac{\Gamma(\beta+1)}{s^{\beta+1}}$$
$$L[e^{\alpha t}](s) = \int_0^{+\infty} e^{\alpha t}e^{-st}dt = \frac{1}{s-\alpha}, \text{ for } s > \alpha$$
$$L[sin\omega t](s) = \int_0^{+\infty} sin\omega t e^{-st}dt = \frac{\omega}{s^2+\omega^2}, \quad L[cos\omega t] = \frac{s}{s^2+\omega^2}$$

Properties of Laplace transform:

① linearity: $L[\alpha f + \beta g] = \alpha L[f] + \beta L[g]$;

② $L[f'(t)] = \int_0^{+\infty} f'(t)e^{-st}dt = f(t)e^{-st}\Big|_0^{+\infty} + s\int_0^{+\infty} f(t)e^{-st}dt = sL[f(t)] - f(0)$

$L[f^{(n)}(t)] = s^nL[f] - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$

③ $L[t^nf(t)] = (-1)^n\dfrac{d^nL[f(t)]}{ds^n}$

④ $L[e^{\alpha t}f(t)] = L[f(t)](s-\alpha)$

⑤ Convolutions: define $f * g(t) = \int_0^t f(x)g(t-x)dx$, then

$$L[f * g(t)] = \int_0^{+\infty} dt\int_0^t dxf(x)g(t-x)e^{-st}, \text{ let } y = x, \;\; z = t-x$$
$$= \int_0^{+\infty} dy\int_0^{+\infty} dzf(y)g(z)e^{-s(y+z)} = L[f(t)]\cdot L[g(t)]$$

Inverse of Laplace transform:

given $\hat{f}(s)$, find $f(t)$, let $L[f(t)] = \hat{f}(s)$ or $f(t) \equiv L^{-1}[\hat{f}(s)]$

e.g.

$L^{-1}[\frac{1}{s}] = 1, \quad L^{-1}[\frac{1}{s^2}] = t, \quad L^{-1}[\frac{1}{s^3}] = \frac{1}{2}t^2, \cdots$

$L^{-1}[\frac{1}{s^3(s+\alpha)}] = L^{-1}[\frac{1}{\alpha}\frac{1}{s^3} - \frac{1}{\alpha^2}\frac{1}{s^2} + \frac{1}{\alpha^3}\frac{1}{s} - \frac{1}{\alpha^3}\frac{1}{s+\alpha}] = \frac{1}{2\alpha}t^2 - \frac{1}{\alpha^2}t + \frac{1}{\alpha^3} - \frac{1}{\alpha^3}e^{-\alpha t}$

Supposing $\hat{f}(s)$ being analytic, and $\lim\limits_{|s|\to\infty}\hat{f}(s) = 0$, on $\mathrm{Re}\, s > S_0$

further $\int_{p-i\infty}^{p+i\infty} |\hat{f}(s)|d\sigma$ exist for $p = Re\, s > S_0$, then

$$f(t) \equiv L^{-1}[\hat{f}(s)] = \frac{1}{2\pi i}\int_{p-i\infty}^{p+i\infty} \hat{f}(s)e^{st}ds, \quad p > S_0$$

Application to PDE on semi-infinite region, e.g. heat equation:

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}, \;\; x \geq 0, \;\; t \geq 0, \quad u|_{x=0} = f(t), \;\; u|_{t=0} = 0, \;\; u|_{x=+\infty} = 0$$

now apply L on both side wrt. t:

$$c^2\frac{d^2\hat{u}(x,s)}{dx^2} = s\hat{u}(x,s)+0, \quad \hat{u}(0,s)=\hat{f}(s), \;\; \hat{u}(+\infty,s)=0$$

thus $\hat{u}(x,s)=\hat{f}(s)e^{-\frac{\sqrt{s}}{c}x}$
$L^{-1}[\frac{1}{\sqrt{s}}e^{-\alpha\sqrt{s}}] = \frac{1}{\sqrt{\pi t}}e^{-\frac{\alpha^2}{4t}} \Longrightarrow L^{-1}[e^{-\frac{\sqrt{s}}{c}x}] = \frac{x}{2c\sqrt{\pi t^3}}e^{-\frac{x^2}{4tc^2}}$
from convolutions:

$$u(x,t)=\int_0^t f(t-z)\frac{x}{2c\sqrt{\pi z^3}}e^{-\frac{x^2}{4zc^2}}dz$$

E.g.
wave equation on 1-D ,$0\le x<+\infty, \quad t>0$

$$\begin{cases} u_{tt}-c^2u_{xx}=0 \\ u\big|_{t=0}=u_t\big|_{t=0}=0 \\ u\big|_{x=0}=f(t), \text{ or } u_x\big|_{x=0}=f(t) \end{cases}$$

Solution:

① In case of 1-st kind BVCs, using F_s wrt. x, $\hat{u}(\omega,t)\equiv F_s[u(x,t)]$, thus:

$$\frac{d^2\hat{u}}{dt^2}+\omega^2c^2\hat{u}-\sqrt{\frac{2}{\pi}}c^2\omega f(t)=0, \quad \hat{u}\big|_{t=0}=\hat{u}_t\big|_{t=0}=0$$

② In case of 2-nd kind BVCs, using F_c wrt. x, $\hat{u}(\omega,t)\equiv F_c[u(x,t)]$, thus:

$$\frac{d^2\hat{u}}{dt^2}+\omega^2c^2\hat{u}+\sqrt{\frac{2}{\pi}}c^2 f(t)=0, \quad \hat{u}\big|_{t=0}=\hat{u}_t\big|_{t=0}=0$$

In both cases: $y_1=cos\omega ct, \quad y_2=sin\omega ct$, and $W[y_1,y_2]=\omega c$
while the non-hom. term is $\sqrt{\frac{2}{\pi}}c^2\omega f(t)$ and $-\sqrt{\frac{2}{\pi}}c^2 f(t)$ respectively.
for case①:
$y_p=\sqrt{\frac{2}{\pi}}c\int_0^t f(\tau)sin\omega c(t-\tau)d\tau$
the solution of IVP: $y=y_p+c_1y_1+c_2y_2$
since $y_p'=\sqrt{\frac{2}{\pi}}c\left(f(t)\cdot 0+\int_0^t f(\tau)\omega c\cdot cos\omega c(t-\tau)d\tau\right), \; y_p'\big|_{t=0}=0.$
Thus from $y\big|_{t=0}=y_t\big|_{t=0}=0$, we know

$$y=y_p=\sqrt{\frac{2}{\pi}}c\int_0^t f(\tau)sin\omega c(t-\tau)d\tau$$

Similarly for case ②, $y=-\sqrt{\frac{2}{\pi}}\frac{c}{\omega}\int_0^t f(\tau)sin\omega c(t-\tau)d\tau$

In the last step we need to do inverse transform,
case ①:

$$
\begin{aligned}
u(x,t) &= F_s^{-1}[\hat{u}] = \frac{2}{\pi}c\int_0^{+\infty} d\omega sin\omega x\int_0^t f(\tau)sin\omega c(t-\tau)d\tau \\
&= \frac{2}{\pi}c\int_0^t d\tau f(\tau)\int_0^{+\infty}\frac{d\omega}{2}[cos\omega(x-c(t-\tau))-cos\omega(x+c(t-\tau))] \\
&= \frac{2}{\pi}c\int_0^t d\tau f(\tau)\frac{1}{4}\int_{-\infty}^{+\infty}d\omega\left[\frac{1}{2}(e^{i\omega(x-c(t-\tau))}+e^{-i\omega(x-c(t-\tau))})-\frac{1}{2}(e^{i\omega(x+c(t-\tau))}+e^{-i\omega(x+c(t-\tau))})\right] \\
&= c\int_0^t d\tau f(\tau)(\delta(x-c(t-\tau))-\delta(x+c(t-\tau))) \\
&= \theta(t-\frac{x}{c})\cdot f(t-\frac{x}{c})
\end{aligned}
$$

case②:

$$
\begin{aligned}
u(x,t) &= F_c^{-1}[\hat{u}] = -\frac{2}{\pi}c\int_0^{+\infty}d\omega cos\omega x\int_0^t\frac{f(\tau)}{\omega}sin\omega c(t-\tau)d\tau \\
&= -\frac{2}{\pi}c\int_0^t d\tau f(\tau)\int_0^{+\infty}\frac{d\omega}{2\omega}\left[sin\omega(x-c(t-\tau))+sin\omega(x+c(t-\tau))\right] \\
&= -\frac{2}{\pi}c\int_0^t d\tau f(\tau)\theta(c(t-\tau)-x)\int_0^{+\infty}d\omega\frac{sin\omega}{\omega} \\
&= -c\theta(t-\frac{x}{c})\int_0^{t-\frac{x}{c}}d\tau f(\tau)
\end{aligned}
$$

One can try Laplace transform for case①,

L wrt. t:

$$
s^2\hat{u}(x,s)-su(x,0)-u_t(x,0)-c^2\frac{d^2\hat{u}(x,s)}{dx^2}=0
$$

$\Longrightarrow s^2\hat{u}(x,s)-c^2\frac{d^2\hat{u}(x,s)}{dx^2}=0$, further $\hat{u}(0,s)=\hat{f}(s)$, with $\hat{f}(s)=L[f(t)]$

considering $u(+\infty,t)$ is finite, only one of the two solutions of ODE servives,

$$
\hat{u}(x,s)=\hat{f}(s)exp[\frac{-sx}{c}]
$$

the inverse:

$$
u(x,t)=L^{-1}[\hat{u}]=L^{-1}[\hat{f}(s)exp[\frac{-sx}{c}]]
$$

we know : $\quad L^{-1}[\hat{f}(s)]=f(t), \quad L^{-1}[exp[\frac{-sx}{c}]]=\delta(t-\frac{x}{c})$

thus using convolution:

$$
u(x,t)=\int_0^t d\tau f(\tau)\delta(t-\tau-\frac{x}{c})=\theta(t-\frac{x}{c})\cdot f(t-\frac{x}{c})
$$

Intuitive picture on wave equations

examples:

① 1D space with $x\in[0,+\infty)$:

$$
\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}, \quad u|_{t=0}=0, u_t|_{t=0}=0
$$

and boundary conditions, either $u|_{x=0} = f(t)$ or $u_x|_{x=0} = g(t)$

(From general solution of PDEs, we know $u(x,t) = \phi_1(x-ct) + \phi_2(x+ct)$, namely two travelling waves, and only $\phi_1(x-ct)$ in this case (v=+c). Thus at any point x, its movement can be think as duplicate of that at $x_0 = 0$ but with time delay $\frac{x}{c}$. The latter starts to propagate at $t = 0$.)

For first case, $u(x,t) = \theta(t - \frac{x}{c}) \cdot f(t - \frac{x}{c})$

For second case, since $u_t(x,t) = \theta(t - \frac{x}{c}) \cdot u_t(0, t - \frac{x}{c})$, the total momentum:

$$P = \int_0^{+\infty} u_t(x,t)\rho dx = \rho \int_0^{ct} u_t(0, t - \frac{x}{c}) dx = \rho c \int_0^t u_t(0,t')dt' = \rho c u(0,t) = \int_0^t (-Tg(t'))dt'$$

(note $c^2 = \frac{T}{\rho}$)

So we derive: $u(0,t) = -\frac{T}{\rho c} \int_0^t g(t')dt' = -c \int_0^t g(t')dt'$

and with the propagation picture $u(x,t) = -\theta(t - \frac{x}{c}) c \int_0^{t-\frac{x}{c}} g(t')dt'$

② 3D space non-homogeneous equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u + f(\vec{r}, t), \quad u|_{t=0} = 0, \quad u_t|_{t=0} = 0$$

Now imagine $f(\vec{r},t) = \int d\vec{r'} \int_0^t dt' f(\vec{r'}, t') \delta(\vec{r'} - \vec{r}) \delta(t' - t)$

using superposition principle, each small part induces an initial speed/thrust at $\vec{r} = \vec{r'}, \;\; t = t'$

initial speed: $u_t^\delta(\vec{r}, t') = \delta(\vec{r'} - \vec{r}); \quad u^\delta(\vec{r}, t') = 0$

thus

$$u^\delta(\vec{r}, t) = \frac{1}{4\pi c} \iint_{S^{\vec{r}}_{c(t-t')}} \frac{u_t^\delta(\vec{R}, t')}{c(t-t')} dS = \frac{1}{4\pi c} \frac{\delta\left(c(t-t') - |\vec{r} - \vec{r'}|\right)}{|\vec{r} - \vec{r'}|}$$

and finally:

$$u(\vec{r}, t) = \int d\vec{r'} \int_0^t dt' u^\delta(\vec{r}, t) \cdot f(\vec{r'}, t') = \frac{1}{4\pi c^2} \int_{T^{\vec{r}}_{ct}} d\vec{r'} \frac{f(\vec{r'}, t - \frac{|\vec{r} - \vec{r'}|}{c})}{|\vec{r} - \vec{r'}|}$$

9 Chapter 9. Method of Green's functions on PDEs

9.1 Green's function for Poisson's equation

9.1.1 Integral formular for poisson's equation (3D case)

Supposing $u(\vec{r})$ and $v(\vec{r})$ have continuous 2nd derivatives in region T and continuous 1st derivatives on its surface $\sum$,

i.e. $u(\vec{r})$ and $v(\vec{r}) \in C^2(T) \cap C^1(\sum)$, thus

$$\iint_{\sum} u\vec{\nabla} v \cdot d\vec{s} = \iiint_T \vec{\nabla} \cdot (u\vec{\nabla} v) dV = \iiint_T (\vec{\nabla} u \cdot \vec{\nabla} v + u\nabla^2 v) dV \quad \text{(first Green's formular)}$$

$$\iint\limits_{\sum} (u\vec{\nabla} v - v\vec{\nabla} u) \cdot d\vec{s} = \iiint_T (u\nabla^2 v - v\nabla^2 u) dV \quad \text{(second Green's formular)}$$

Now to solve for $\nabla^2 u = f(\vec{r}), \vec{r} \in T$ with a general BVC $\left[\alpha \frac{\partial u}{\partial n} + \beta u\right]\Big|_{\sum} = \varphi(\vec{r}),\ \ \vec{r} \in \sum$.

We first introduce an auxiliary/Green's function,satisfying:

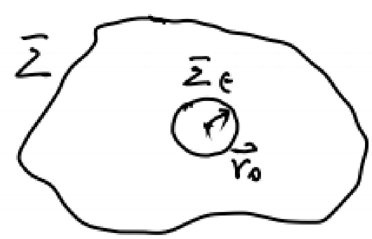
$$\nabla^2 v(\vec{r}, \vec{r}_0) = \delta(\vec{r} - \vec{r}_0) \qquad \text{(no BVCs, thus not unique)}$$

and consider T_ε being $T - T_{\vec{r}_0}^{\varepsilon}$ (subtracting spherical region centered at $\vec{r}_0$ with radius ε)
apply Green's second formular for T_ε,

$$\iint\limits_{\sum} (u\vec{\nabla} v - v\vec{\nabla} u) \cdot d\vec{s} - \iint_{\sum_\varepsilon} (u\vec{\nabla} v - v\vec{\nabla} u) \cdot d\vec{s} = \iiint_{T_\varepsilon} (u\nabla^2 v - v\nabla^2 u) dV$$

with $v = v(\vec{r}, \vec{r}_0)$, in the limit $\varepsilon \to 0$ (note $v \to -\frac{1}{4\pi}\frac{1}{|\vec{r} - \vec{r}_0|}$)
one get

$$u(\vec{r}_0) = \iiint_T v f(\vec{r}) dV + \iint\limits_{\sum} (u\vec{\nabla} v - v\vec{\nabla} u) \cdot d\vec{s} \qquad \text{integral formular for poisson's equation!}$$



9.1.2 Applications to Laplace's equation ($f(\vec{r}) \equiv 0$), $\nabla^2 u = 0$

one can simply let $v(\vec{r}, \vec{r}_0) = -\frac{1}{4\pi}\frac{1}{|\vec{r} - \vec{r}_0|}$, then

$$u(\vec{r}_0) = \iint\limits_{\sum} (u\vec{\nabla} v - v\vec{\nabla} u) \cdot d\vec{s} = \iint\limits_{\sum} \left(\frac{1}{|\vec{r} - \vec{r}_0|}\frac{\partial u}{\partial n} - u\frac{\partial}{\partial n}\left(\frac{1}{|\vec{r} - \vec{r}_0|}\right)\right)\frac{ds}{4\pi}$$

further note for Laplace's equation:

$$\nabla^2 u = 0 \to \iiint_T \nabla^2 u dV = 0 \to \iint_{\sum} \vec{\nabla} u \cdot d\vec{s} = 0 = \iint_{\sum} \frac{\partial u}{\partial n} ds$$

$$\text{(Consistency condition for Neunman's BVC, } \iint_{\sum} \frac{\partial u}{\partial n} ds = 0)$$

Now if T is a spherical region centered at $\vec{r}_0$ with radius a,

$$u(\vec{r}_0) = \frac{1}{4\pi a} \iint_{\sum} \frac{\partial u}{\partial n} ds + \frac{1}{4\pi a^2} \iint_{\sum} u ds =< u(\vec{r}) >_{\sum} \qquad \text{equals mean temparature on sphere!}$$

9.1.3 Green's functions for Poisson's equation

① Dirichlet's BVCs:

$$\begin{cases} \nabla^2 u = f(\vec{r}),\ \vec{r} \in T \\ u(\vec{r})\big|_{\sum} = \varphi(\vec{r}),\ \vec{r} \in \sum \end{cases} \xrightarrow[Green's\ function]{} \begin{cases} \nabla^2 G(\vec{r}, \vec{r}_0) = \delta(\vec{r} - \vec{r}_0),\ \vec{r} \in T \\ G(\vec{r}, \vec{r}_0)\big|_{\sum} = 0,\ \vec{r} \in \sum \end{cases}$$

in integral formular of Poisson's equation, let $v = G(\vec{r}, \vec{r}_0)$,

$$u(\vec{r}_0) = \iiint_T G(\vec{r}, \vec{r}_0) f(\vec{r}) dV + \iint_{\sum} \varphi(\vec{r}) \frac{\partial G(\vec{r}, \vec{r}_0)}{\partial n} ds$$

using symmetric relation of Green's function, $G(\vec{r}_1, \vec{r}_2) = G(\vec{r}_2, \vec{r}_1)$

that comes to:

$$u(\vec{r}_0) = \iiint_T G(\vec{r}_0, \vec{r}) f(\vec{r}) dV + \iint_{\sum} \varphi(\vec{r}) \frac{\partial G(\vec{r}_0, \vec{r})}{\partial n} ds$$

where $\iiint_T G(\vec{r}_0, \vec{r}) f(\vec{r}) dV$ is by charge inside T $\qquad \iint_{\sum} \varphi(\vec{r}) \frac{\partial G(\vec{r}_0, \vec{r})}{\partial n} ds$ is by potential on $\sum$

Proof of the symmetric relation:

$$\iiint_T \left(G(\vec{r}, \vec{r}_1) \nabla^2 G(\vec{r}, \vec{r}_2) - G(\vec{r}, \vec{r}_2) \nabla^2 G(\vec{r}, \vec{r}_1) \right) dV$$

$$= \iint_{\sum} \left(G(\vec{r}, \vec{r}_1) \vec{\nabla} G(\vec{r}, \vec{r}_2) - G(\vec{r}, \vec{r}_2) \vec{\nabla} G(\vec{r}, \vec{r}_1) \right) \cdot d\vec{s}$$

$= 0$, due to BVCs.

further from PDE, above $= \iiint_T \left(G(\vec{r}, \vec{r}_1) \delta(\vec{r} - \vec{r}_2) - G(\vec{r}, \vec{r}_2) \delta(\vec{r} - \vec{r}_1) \right) dV = G(\vec{r}_2, \vec{r}_1) - G(\vec{r}_1, \vec{r}_2)$

thus $G(\vec{r}_2, \vec{r}_1) = G(\vec{r}_1, \vec{r}_2)$, valid also for Robin's homogeneous BVCs.

② Robin's BVCs:

$$\begin{cases} \nabla^2 u = f(\vec{r}),\ \vec{r} \in T \\ \left[\alpha \frac{\partial u}{\partial n} + \beta u \right] \Big|_{\sum} = \varphi(\vec{r}),\ \vec{r} \in \sum \end{cases} \xrightarrow[Green's\ function]{} \begin{cases} \nabla^2 G(\vec{r}, \vec{r}_0) = \delta(\vec{r} - \vec{r}_0),\ \vec{r} \in T \\ \left[\alpha \frac{\partial G}{\partial n} + \beta G \right] \Big|_{\sum} = 0,\ \vec{r} \in \sum \end{cases}$$

using integral formular of Poisson's equation, Similar to Dirichlet's BVCs.

$$u(\vec{r}_0) = \iiint_T G(\vec{r}_0, \vec{r}) f(\vec{r}) dV + \frac{1}{\beta} \iint_{\sum} \varphi(\vec{r}) \frac{\partial G(\vec{r}_0, \vec{r})}{\partial n} ds$$

$$= \iiint_T G(\vec{r}_0,\vec{r})f(\vec{r})dV - \frac{1}{\alpha}\iint_{\sum} \varphi(\vec{r})G(\vec{r}_0,\vec{r})ds$$

when $\alpha = 0$, $\beta = 1$, that goes back to Dirichlet's BVCs.

③ Neuman's BVCs:

$$\begin{cases} \nabla^2 u = f(\vec{r}),\ \vec{r}\in T \\ \left.\dfrac{\partial u}{\partial n}\right|_{\sum} = \varphi(\vec{r}),\ \vec{r}\in \sum \end{cases} \xrightarrow[Green's\ function]{} \begin{cases} \nabla^2 G(\vec{r},\vec{r}_0) = \delta(\vec{r}-\vec{r}_0) - \dfrac{1}{V_T},\ \vec{r}\in T \\ \left.\dfrac{\partial G(\vec{r},\vec{r}_0)}{\partial n}\right|_{\sum} = 0,\ \vec{r}\in \sum \end{cases}$$

here V_T is the total volumn of region T.

The $\dfrac{1}{V_T}$ term is introduced otherwise no solution for G.

(Imagine total charge in $T = 0$, since $\left.\vec{E}\right|_{\sum} = 0$. In this case it is called generalized Green's function)

$$u(\vec{r}_0) = \iiint_T G(\vec{r},\vec{r}_0)f(\vec{r})dV - \iint_{\sum} \varphi(\vec{r})G(\vec{r},\vec{r}_0)ds$$

Note for Neuman's BVCs, either u or G can be only determined up to a constant.

9.2 Green's functions and electric-image method

9.2.1 Green's function in unbounded space

$$\text{Definition:} \qquad \nabla^2 G(\vec{r},\vec{r}_0) = \delta(\vec{r}-\vec{r}_0)$$

in connection with EM of a point electric charge, thus

$$G(\vec{r},\vec{r}_0) = -\frac{1}{4\pi}\frac{1}{|\vec{r}-\vec{r}_0|} \quad \text{for 3-D}$$
$$G(\vec{r},\vec{r}_0) = \frac{1}{2\pi} ln|\vec{r}-\vec{r}_0| \quad \text{for 2-D}$$
$$G(x,x_0) = \frac{1}{2}|x-x_0| \quad \text{for 1-D}$$

note in unbounded case G can be only determined upto a constant!

9.2.2 Green's functions from method of eigenfunction expansion

$$\text{Definition: (1-st kind)} \qquad \nabla^2 G(\vec{r},\vec{r}_0) = \delta(\vec{r}-\vec{r}_0), \quad G|_{\sum} = 0$$

Now suppose we know $u_N(\vec{r})$ being eigenfunctions of the Helmheltz's equation with same BVCs, namely

$$\nabla^2 u_N(\vec{r}) = -\lambda_N u_N(\vec{r}),\ \ u_N|_{\sum} = 0,\ \ (u_N,u_M) = \iiint_T u_N u_M dV \propto \delta_{MN}$$

then we know must exist:

$$G(\vec{r},\vec{r}_0) = \sum_N a_N(\vec{r}_0)u_N(\vec{r})$$
$$\Longrightarrow \nabla^2 G(\vec{r},\vec{r}_0) = \sum_N (-\lambda_N a_N(\vec{r}_0))u_N(\vec{r}) = \delta(\vec{r}-\vec{r}_0)$$

taken inner product with $u_M(\vec{r}) \Longrightarrow a_M(\vec{r}) = -\dfrac{u_M(\vec{r}_0)}{\lambda_M(u_M,u_M)}$

$\left(\dfrac{(u_N,f(\vec{r}))}{(u_N,u_N)} = \dfrac{\iiint_V d\vec{r} u_N(\vec{r})\delta(\vec{r}-\vec{r}_0)}{\|u_N\|^2} = \dfrac{u_N(\vec{r}_0)}{\|u_N\|^2}\right)$

Consider a rectangle region: $\nabla^2 G(\vec{r},\vec{r}_0) = \delta(\vec{r}-\vec{r}_0) = \delta(x-x_0)\delta(y-y_0)$
for $0 \le x \le a, \quad 0 \le y \le b$, and $G|_{\sum} = 0$.
We've already know:

$$u_{mn}(\vec{r}) = sin\frac{m\pi x}{a}sin\frac{n\pi y}{b}, \quad \nabla^2 u_{mn} = -\left[(\frac{m}{a})^2 + (\frac{n}{b})^2\right]\pi^2 u_{mn} \text{ and}$$
$$(u_{mn},u_{mn}) = \int_0^a dx \int_0^b dy u_{mn}^2 = \frac{ab}{4}, \; m,n = 1,2,\cdots$$

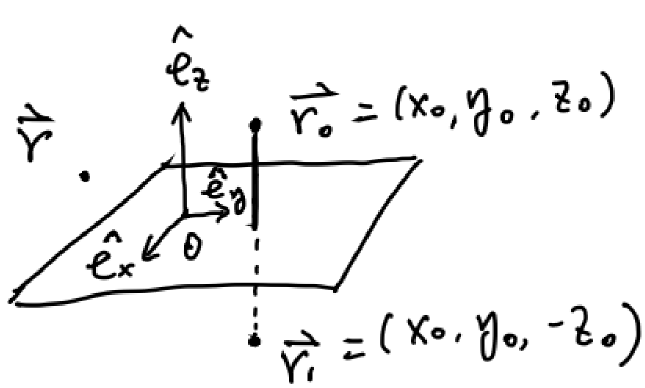
Thus

$$G(\vec{r},\vec{r}_0) = -\sum_{m,n} \frac{4sin\frac{m\pi x}{a}sin\frac{m\pi x_0}{a}sin\frac{n\pi y}{b}sin\frac{n\pi y_0}{b}}{ab\pi^2\left[(\frac{m}{a})^2 + (\frac{n}{b})^2\right]}$$

Now according to Green's solution of Poisson's equation:

$$\begin{cases} \nabla^2 u(\vec{r}) = f(\vec{r}) \\ 0 \le x \le a, \;\; 0 \le x \le b \\ u|_{\sum} = 0 \end{cases} \Longrightarrow u(\vec{r}) = \iint_T d\vec{r}_0 G(\vec{r},\vec{r}_0)f(\vec{r}_0) = -\sum_{m,n} sin\frac{m\pi x}{a}sin\frac{n\pi y}{b}\frac{\int dx_0 dy_0 f(\vec{r}_0)u_{mn}(\vec{r}_0)}{\frac{ab}{4}\pi^2\left[(\frac{m}{a})^2 + (\frac{n}{b})^2\right]}$$

9.2.3 Method of electric image



$$\begin{cases} \nabla^2 G(\vec{r}, \vec{r}_0) = \delta(\vec{r} - \vec{r}_0) \\ z \geq 0, \ -\infty < x < +\infty, \ -\infty < y < +\infty \qquad\qquad \text{in 3-D region with } z \geq 0 \\ G|_{z=0} = 0 \end{cases}$$

The solution of Green's function is the potential sum of a point-like charge at $\vec{r}_0$ and another with same but opposite charge ar $\vec{r}_1$

$$G(\vec{r}, \vec{r}_0) = -\frac{1}{4\pi}\left(\frac{1}{|\vec{r} - \vec{r}_0|} - \frac{1}{|\vec{r} - \vec{r}_1|}\right)$$

and on the boundary $z = 0$,

$$\left.\frac{\partial G(\vec{r}, \vec{r}_0)}{\partial z}\right|_{z=0} = -\frac{1}{4\pi}\frac{2z_0}{((x - x_0)^2 + (y - y_0)^2 + z_0^2)^{\frac{3}{2}}}$$

Now solutions for the poisson's equation can be obtained,

$$\begin{cases} \nabla^2 u(\vec{r}) = f(\vec{r}) \\ z \geq 0, \ -\infty < x < +\infty, \ -\infty < y < +\infty \qquad\qquad \text{with Dirichlet BVCs.} \\ u|_{z=0} = \varphi(\vec{r}) \end{cases}$$

Solution:

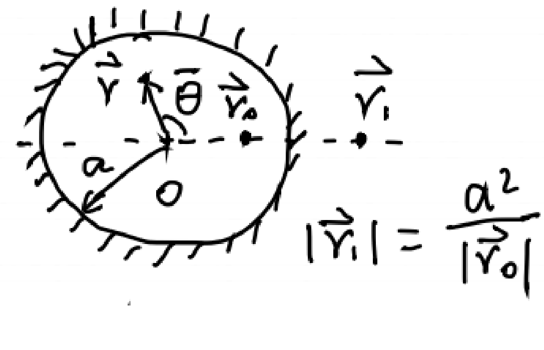
$$u(\vec{r}) = \iiint_T G(\vec{r}, \vec{r}_0) f(\vec{r}_0) dV_0 + \iint_{\sum} \varphi(\vec{r}_0)\frac{\partial G(\vec{r}, \vec{r}_0)}{\partial n_0} ds_0$$

if Laplace's equation, $f(\vec{r}) = 0$,

$$u(\vec{r}) = \int_{-\infty}^{+\infty} dx_0 \int_{-\infty}^{+\infty} dy_0 \varphi(x_0, y_0)\frac{z}{2\pi\left((x - x_0)^2 + (y - y_0)^2 + z_0^2\right)^{\frac{3}{2}}}$$

recall similar formular we derived for 2-D case with the Fourier transform's method.

inside a sphere with radius a:



$$\begin{cases} \nabla^2 G(\vec{r},\vec{r}_0)=\delta(\vec{r}-\vec{r}_0) \\ |\vec{r}|\le a \\ G|_{|\vec{r}|=a}=0 \end{cases}$$

The solution of G is the potential sum of a point-like charge at $\vec{r}_0$ and its image at $\vec{r}_1$ with opposite charge $\frac{a}{|\vec{r}_0|}$, namely

$$G(\vec{r},\vec{r}_0)=-\frac{1}{4\pi}\left(\frac{1}{|\vec{r}-\vec{r}_0|}-\frac{\frac{a}{|\vec{r}_0|}}{|\vec{r}-\vec{r}_1|}\right)$$

and on the boundary

$$\left.\frac{\partial G(\vec{r},\vec{r}_0)}{\partial n}\right|_{|\vec{r}|=a}=-\frac{1}{4\pi}\left(\frac{-(a-|\vec{r}_0|cos\bar{\theta})}{|\vec{r}-\vec{r}_0|^3}+\frac{\frac{a^2}{|\vec{r}_0|}(1-\frac{a}{|\vec{r}_0|}cos\bar{\theta})}{|\vec{r}-\vec{r}_1|^3}\right)=-\frac{1}{4\pi a}\frac{r_0^2-a^2}{|\vec{r}-\vec{r}_0|^3}$$

Now solution for the Poisson's equation can be obtained,

$$\begin{cases} \nabla^2 u(\vec{r})=f(\vec{r}) \\ |\vec{r}|\le a \qquad\qquad \text{with Dirichlet's BVCs.} \\ u|_{\sum}=g(\vec{r}) \end{cases}$$

Solution:

$$u(\vec{r})=\iiint_V G(\vec{r},\vec{r}_0)f(\vec{r}_0)dV_0+\iint_{\sum} g(\vec{r}_0)\frac{\partial G(\vec{r},\vec{r}_0)}{\partial n_0}ds_0$$

if Laplace's equation, $f(\vec{r})=0$: $\qquad$ (g($\vec{r}$) $\equiv 1 \Longrightarrow u(\vec{r})\equiv 1$))

$$u(\vec{r})=\int d\Omega_0 a^2 g(\theta_0,\varphi_0)(-\frac{1}{4\pi a})\frac{r^2-a^2}{|\vec{r}-\vec{r}_0|^3}=-\frac{a}{4\pi}\int_{-1}^{1}dcos\theta_0\int_0^{2\pi}d\varphi_0 g(\theta_0,\varphi_0)\frac{r^2-a^2}{\left(r^2+a^2-2arcos\bar{\Theta}\right)^{\frac{3}{2}}}$$

note $|\vec{r}_0|=a$, $\qquad$ with $cos\bar{\Theta}=cos\theta cos\theta_0+sin\theta sin\theta_0 cos(\varphi-\varphi_0)$

9.3 Green's function method for nonhomogeneous heat/wave equations

Wave equation

$$\begin{cases} u_{tt}-c^2\nabla^2 u=f(\vec{r},t),\ \ \vec{r}\in T \\ \left.\left[\alpha\frac{\partial u}{\partial n}+\beta u\right]\right|_{\sum}=\theta(\vec{r},t),\ \ \vec{r}\in\sum \\ u|_{t=0}=\varphi(\vec{r}),\ \ \vec{r}\in T \\ u_t|_{t=0}=\psi(\vec{r}),\ \ \vec{r}\in T \end{cases}$$

Green's function

$$\begin{cases} G_{tt}-c^2\nabla^2 G=\delta(\vec{r}-\vec{r}_0)\delta(t-t_0) \\ \left.\left[\alpha\frac{\partial G}{\partial n}+\beta G\right]\right|_{\sum}=0 \\ G|_{t=0}=0 \\ G_t|_{t=0}=0 \end{cases}$$

if $G(\vec{r},t;\vec{r}_0,t_0)$ is known, $\quad$ [symmetric relation $G(\vec{r},t;\vec{r}_0,t_0)=G(\vec{r}_0,-t_0;\vec{r},-t)$]

$$u(\vec{r},t)=\iiint_{T_0}\int_0^t Gf(\vec{r}_0,t_0)d\vec{r}_0dt_0+c^2\iint_{\sum_0}\int_0^t\left(G\frac{\partial u}{\partial n_0}-u\frac{\partial G}{\partial n_0}\right)ds_0dt_0+\iiint_{T_0}\left.\left(G\frac{\partial u}{\partial t_0}-u\frac{\partial G}{\partial t_0}\right)\right|_{t_0=0}d\vec{r}_0$$

note in the integral: $G = G(\vec{r}, t; \vec{r}_0, t_0), \quad u = u(\vec{r}_0, t_0)$

Proof:

starting with

$u_{t_0t_0} - c^2\nabla_0^2 u = f(\vec{r}_0, t_0) ------ \otimes G(\vec{r}, t; \vec{r}_0, t_0)$

$G_{t_0t_0} - c^2\nabla_0^2 G = \delta(t_0 - t)\delta(\vec{r}_0 - \vec{r}) ------ \otimes u(\vec{r}_0, t_0)$

and integrating over t_0 from 0 to $t + \epsilon$, $(\epsilon > 0)$, and on T_0,

$$\iiint_{T_0} dV_0 \int_0^{t+\epsilon} (Gu_{t_0t_0} - uG_{t_0t_0})dt_0 - c^2 \int_0^{t+\epsilon} dt_0 \iiint_{T_0} dV_0 (G\nabla_0^2 u - u_0\nabla_0^2 G) = \int_0^{t+\epsilon} dt_0 \iiint_{T_0} dV_0 f(\vec{r}_0, t_0)G(\vec{r}, t; \vec{r}_0, t_0) - u(\vec{r}, t)$$

integrating for the first term and using Green's 2nd formular for second term,

$$u(\vec{r}, t) = \int_0^t dt_0 \iiint_{T_0} dV_0 f(\vec{r}_0, t_0)G + c^2 \int_0^t dt_0 \iint_{\sum_0} \left(G\frac{\partial u}{\partial n_0} - u_0 \frac{\partial G}{\partial n_0} \right) ds_0 + \iiint_{T_0} dV_0 \left. (Gu_{t_0} - uG_{t_0}) \right|_{t_0=0} - \iiint_{T_0} dV_0 \left. (Gu_{t_0} - uG_{t_0}) \right|_{t_0=t+\epsilon}$$

heat equation

$$\begin{cases} u_t - c^2\nabla^2 u = f(\vec{r}, t), & \vec{r} \in T \\ \left. \left[\alpha \dfrac{\partial u}{\partial n} + \beta u \right] \right|_{\sum} = \theta(\vec{r}, t), & \vec{r} \in \sum \\ u|_{t=0} = \varphi(\vec{r}), & \vec{r} \in T \end{cases}$$

Green's function

$$\begin{cases} G_t - c^2\nabla^2 G = \delta(\vec{r} - \vec{r}_0)\delta(t - t_0) \\ \left. \left[\alpha \dfrac{\partial G}{\partial n} + \beta G \right] \right|_{\sum} = 0 \\ G|_{t=0} = 0 \end{cases}$$

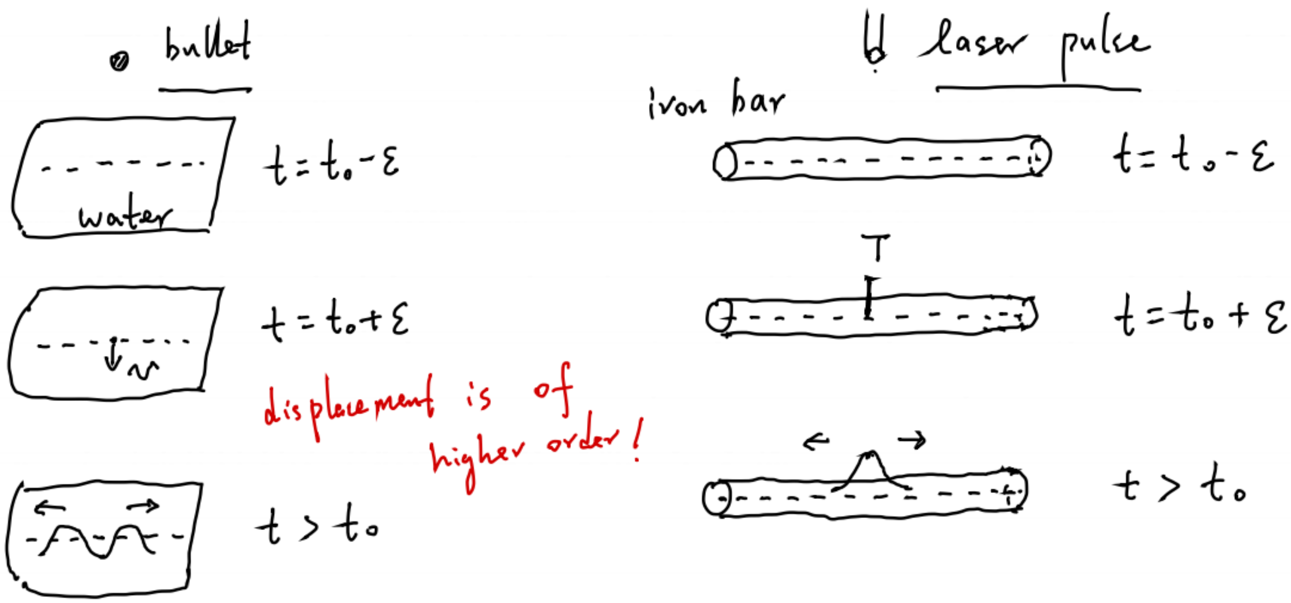
if $G(\vec{r}, t; \vec{r}_0, t_0)$ is known

$$u(\vec{r}, t) = \iiint_{T_0} \int_0^t Gf(\vec{r}_0, t_0)f(\vec{r}_0, t_0)dt_0 d\vec{r}_0 + c^2 \iint_{\sum_0} \int_0^t \left(G\frac{\partial u}{\partial n_0} - u\frac{\partial G}{\partial n_0} \right) dt_0 ds_0 + \iiint_{T_0} G \left. u \right|_{t_0=0} d\vec{r}_0$$

note in the integral $G = G(\vec{r}, t; \vec{r}_0, t_0) \qquad u = u(\vec{r}_0, t_0)$

9.4 Evaluation of Green's function by means of impulse theorem

The problem of Green's function is a non-homogeneous equation with homogeneous BVCs and zero initial conditions, that can be related to homogeneous equation.



Example 1: 1-D wave equation:

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u|_{t=0} = 0, \quad u_t|_{t=0} = 0 \end{cases} \Longrightarrow \begin{cases} G_{tt} - c^2 G_{xx} = \delta(x-x_0)\delta(t-t_0) \\ G|_{t=0} = 0, \quad G_t|_{t=0} = 0 \end{cases}$$

$$-\infty < x < +\infty, \ \ t > 0 \qquad\qquad -\infty < x, x_0 < +\infty, \ \ t, t_0 > 0$$

now using impulse theorem, we know $G \equiv 0$ when $t < t_0$, and

$$\begin{cases} G_{tt} - c^2 G_{xx} = 0 \\ G|_{t=t_0} = 0, \quad G_t|_{t=t_0} = \delta(x-x_0) \end{cases}$$

By D'Alembert's method: $G = \dfrac{1}{2c}\displaystyle\int_{x-c(t-t_0)}^{x+c(t-t_0)} \delta(s-x_0)ds = \Theta(t-t_0)\cdot \begin{cases} \dfrac{1}{2c}, & x_0 - c(t-t_0) < x < x_0 + c(t-t_0) \\ 0, & \text{otherwise} \end{cases}$

thus

$$u(x,t) = \int_{-\infty}^{+\infty} dx_0 \int_0^{+\infty} dt_0 G(x,t;x_0,t_0) f(x_0,t_0) = \frac{1}{2c}\int_0^t dt_0 \int_{x-c(t-t_0)}^{x+c(t-t_0)} dx_0 f(x_0,t_0)$$

Example 2: 1-D heat equation:

$$\begin{cases} u_t - c^2 u_{xx} = f(x,t) \\ u|_{t=0} = 0 \end{cases} \Longrightarrow \begin{cases} G_t - c^2 G_{xx} = \delta(x-x_0)\delta(t-t_0) \\ G|_{t=0} = 0 \end{cases}$$

$$-\infty < x < +\infty, \quad t > 0; \qquad\qquad t, t_0 > 0, -\infty < x < +\infty$$

using impluse theorem, we know $G \equiv 0$ when $t < t_0$, and

$$\begin{cases} G_t - c^2 G_{xx} = 0, \quad t > t_0, -\infty < x < +\infty \\ G|_{t=t_0} = \delta(x-x_0) \end{cases}$$

using F method: $G = \Theta(t-t_0)\dfrac{1}{2c\sqrt{\pi(t-t_0)}} e^{-\dfrac{(x-x_0)^2}{4c^2(t-t_0)}}$

thus,

$$u(x,t)=\int_0^{+\infty}dt_0\int_{-\infty}^{+\infty}dx_0G(x,t;x_0,t_0)f(x_0,t_0)=\int_0^t\int_{-\infty}^{+\infty}dx_0\frac{f(x_0,t_0)}{2c\sqrt{\pi(t-t_0)}}e^{-\dfrac{(x-x_0)^2}{4c^2(t-t_0)}}$$

Example 3: 1-D wave equation (finite inteval) :

$$\begin{cases} u_{tt}-c^2u_{xx}=Acos\dfrac{\pi x}{l}sin\omega t, \quad 0<x<l, t>0 \\ u_x|_{x=0}=0, \quad u_x|_{x=l}=0, \quad t>0 \\ u|_{t=0}=0, \quad u_t|_{t=0}=0, \quad 0<x<l \end{cases} \Longrightarrow \begin{cases} G_{tt}-c^2G_{xx}=\delta(x-x_0)\delta(t-t_0) \\ G_x|_{x=0}=0, \quad G_x|_{x=l}=0 \\ G|_{t=0}=0, \quad G_t|_{t=0}=0 \end{cases}$$

using impulse theorem, we know $G\equiv 0$, if $t<t_0$, and

$$\begin{cases} G_{tt}-c^2G_{xx}=0, \quad 0<x<l, t>t_0 \\ G_x|_{x=0}=0, \quad G_x|_{x=l}=0 \\ G|_{t=t_0}=0, \quad G_t|_{t=t_0}=\delta(x-x_0) \end{cases}$$

solution $G(x,t;x_0,t)=\sum_{n=1}^{\infty}\dfrac{2cos\frac{n\pi x_0}{l}}{n\pi c}cos\dfrac{n\pi x}{l}sin\dfrac{n\pi c(t-t_0)}{l}+\dfrac{t-t_0}{l}$

thus,

$$u(x,t)=\int_0^t dt_0\int_0^l dx_0\cdot G\cdot Acos\frac{\pi x_0}{l}sin\omega t_0=\left(\int_0^t dt_0 sin\frac{\pi c(t-t_0)}{l}sin\omega t_0\right)\frac{lA}{\pi c}cos\frac{\pi x}{l}$$
$$=cos\frac{\pi x}{l}\left(Asin\omega t\frac{1}{(\frac{\pi c}{l})^2-\omega^2}-\frac{\omega Al}{\pi c}sin\frac{\pi ct}{l}\frac{1}{(\frac{\pi c}{l})^2-\omega^2}\right)$$