3 Quantization of the K-G Field

3.1 The K-G Field as Harmonic Oscillators

For real K-G field, we intend to promote ϕ and π to operators.

Recall that for a discrete system, the commutation relations:

$$[q_i, p_j] = i\delta_{ij} \tag{3.1.1}$$

$$[q_i, q_j] = [p_i, p_j] = 0 \tag{3.1.2}$$

For a continuous system:

$$[\phi(\boldsymbol{x}), \pi(\boldsymbol{y})] = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}) \tag{3.1.3}$$

$$[\phi(\boldsymbol{x}), \phi(\boldsymbol{y})] = [\pi(\boldsymbol{x}), \pi(\boldsymbol{y})] = 0 \tag{3.1.4}$$

To find the spectrum from the Hamiltonian, first expand the classical K-G field in Fourier space:

$$\phi(\boldsymbol{x}, t) = \int \frac{d^3p}{(2\pi)^3} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \phi(\boldsymbol{p}, t) \tag{3.1.5}$$

where $\phi^*(\boldsymbol{p}) = \phi(-\boldsymbol{p})$ so that $\phi(\boldsymbol{x})$ is real.

Substituting the ϕ into the K-G equation:

$$\begin{aligned}
\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right) \int \frac{d^3p}{(2\pi)^3} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \phi(\boldsymbol{p}, t) = 0 \\
\Longrightarrow \int \frac{d^3p}{(2\pi)^3} \left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right) e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \phi(\boldsymbol{p}, t) = 0 \\
\Longrightarrow \int \frac{d^3p}{(2\pi)^3} \left(\frac{\partial^2}{\partial t^2} + |\boldsymbol{p}|^2 + m^2\right) e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \phi(\boldsymbol{p}, t) = 0 \\
\Longrightarrow \left(\frac{\partial^2}{\partial t^2} + |\boldsymbol{p}|^2 + m^2\right) e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \phi(\boldsymbol{p}, t) = 0 \\
\Longrightarrow \left(\frac{\partial^2}{\partial t^2} + |\boldsymbol{p}|^2 + m^2\right) \phi(\boldsymbol{p}, t) = 0
\end{aligned} \tag{3.1.6}$$

where

$$\begin{aligned}
\nabla^2 \left[e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \phi(\boldsymbol{p}, t)\right] &= \nabla \cdot \left[\nabla e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \phi(\boldsymbol{p}, t)\right] \\
&= \nabla \cdot \left[i\boldsymbol{p} \cdot e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \phi(\boldsymbol{p}, t)\right] \\
&= -|\boldsymbol{p}|^2 \cdot e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \phi(\boldsymbol{p}, t)
\end{aligned} \tag{3.1.7}$$

Therefore, the classical K-G equation is the same as a harmonic oscillator in p-space with frequency:

$$\omega_{\boldsymbol{p}} = \sqrt{|\boldsymbol{p}|^2 + m^2} \tag{3.1.8}$$

Recall for the simple harmonic oscillator:

$$\phi = \frac{1}{\sqrt{2\omega}} \left(a + a^\dagger\right); \qquad p = -i\sqrt{\frac{\omega}{2}} \left(a - a^\dagger\right); \qquad \left[a, a^\dagger\right] = 1 \tag{3.1.9}$$

$$H_{SHO} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2\phi^2 = \omega\left(a^\dagger a + \frac{1}{2}\right) \tag{3.1.10}$$

The zero state $|0\rangle$ with the zero-point energy $\frac{1}{2}\omega$:

$$a\,|0\rangle = 0 \tag{3.1.11}$$

The commutators:

$$\left[H_{SHO}, a^\dagger\right] = \omega a^\dagger, \qquad [H_{SHO}, a] = -\omega a \tag{3.1.12}$$

$$|n\rangle \equiv (a^\dagger)^n |0\rangle \quad \text{(with eigenvalues: } (n + \frac{1}{2})\omega) \tag{3.1.13}$$

In analogy for K-G Hamiltonian:

$$\phi(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + a^\dagger_{\boldsymbol{p}} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p}} + a^\dagger_{-\boldsymbol{p}}\right) e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \tag{3.1.14}$$

$$\pi(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\boldsymbol{p}}}{2}} \left(a_{\boldsymbol{p}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} - a^\dagger_{\boldsymbol{p}} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\boldsymbol{p}}}{2}} \left(a_{\boldsymbol{p}} - a^\dagger_{-\boldsymbol{p}}\right) e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \tag{3.1.15}$$

where

$$\phi^\dagger(\boldsymbol{x}) = \phi(\boldsymbol{x}); \quad \pi^\dagger(\boldsymbol{x}) = \pi(\boldsymbol{x}) \tag{3.1.16}$$

The commutation relation:

$$\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}'}\right] = (2\pi)^3 \delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}') \tag{3.1.17}$$

which leads the commutator of ϕ and π:

$$\begin{aligned}
\left[\phi(\boldsymbol{x}), \pi(\boldsymbol{x}')\right] &= \left[\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p}} + a^\dagger_{-\boldsymbol{p}}\right) e^{i\boldsymbol{p}\cdot\boldsymbol{x}}, \int \frac{d^3p'}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\boldsymbol{p}'}}{2}} \left(a_{\boldsymbol{p}'} - a^\dagger_{-\boldsymbol{p}'}\right) e^{i\boldsymbol{p}'\cdot\boldsymbol{x}'}\right] \\
&= \iint \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \frac{(-i)}{2} \sqrt{\frac{\omega_{\boldsymbol{p}'}}{\omega_{\boldsymbol{p}}}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} e^{i\boldsymbol{p}'\cdot\boldsymbol{x}'} \left[\left(a_{\boldsymbol{p}} + a^\dagger_{-\boldsymbol{p}}\right), \left(a_{\boldsymbol{p}'} - a^\dagger_{-\boldsymbol{p}'}\right)\right] \\
&= \iint \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \frac{(-i)}{2} \sqrt{\frac{\omega_{\boldsymbol{p}'}}{\omega_{\boldsymbol{p}}}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} e^{i\boldsymbol{p}'\cdot\boldsymbol{x}'} F \\
&= \iint \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \frac{(-i)}{2} \sqrt{\frac{\omega_{\boldsymbol{p}'}}{\omega_{\boldsymbol{p}}}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} e^{i\boldsymbol{p}'\cdot\boldsymbol{x}'} \left[(-2)(2\pi)^3 \delta^{(3)}(\boldsymbol{p} + \boldsymbol{p}')\right] \\
&= i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_{-\boldsymbol{p}}}{\omega_{\boldsymbol{p}}}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}'} \quad \left(\omega_{\boldsymbol{p}} = \sqrt{|\boldsymbol{p}|^2 + m^2} \longrightarrow \omega_{-\boldsymbol{p}} = \omega_{\boldsymbol{p}}\right) \\
&= i \int \frac{d^3p}{(2\pi)^3} e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')} = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}')
\end{aligned} \tag{3.1.18}$$

with

$$\begin{aligned}
F &= \left[\left(a_{\boldsymbol{p}} + a^\dagger_{-\boldsymbol{p}}\right), \left(a_{\boldsymbol{p}'} - a^\dagger_{-\boldsymbol{p}'}\right)\right] \\
&= \left[\left(a_{\boldsymbol{p}} + a^\dagger_{-\boldsymbol{p}}\right), a_{\boldsymbol{p}'}\right] - \left[\left(a_{\boldsymbol{p}} + a^\dagger_{-\boldsymbol{p}}\right), a^\dagger_{-\boldsymbol{p}'}\right] \\
&= \left[a_{\boldsymbol{p}}, a_{\boldsymbol{p}'}\right] + \left[a^\dagger_{-\boldsymbol{p}}, a_{\boldsymbol{p}'}\right] - \left[a_{\boldsymbol{p}}, a^\dagger_{-\boldsymbol{p}'}\right] - \left[a^\dagger_{-\boldsymbol{p}}, a^\dagger_{-\boldsymbol{p}'}\right] \\
&= 0 + \left[a^\dagger_{-\boldsymbol{p}}, a_{\boldsymbol{p}'}\right] - \left[a_{\boldsymbol{p}}, a^\dagger_{-\boldsymbol{p}'}\right] - 0 \\
&= (-2)(2\pi)^3 \delta^{(3)}(\boldsymbol{p} + \boldsymbol{p}')
\end{aligned} \tag{3.1.19}$$

Express the Hamiltonian in terms of ladder operators:

$$
\begin{aligned}
\hat{H} &= \int d^3x \mathscr{H} = \int d^3x \left[\frac{1}{2}\hat{\pi}^2 + \frac{1}{2}\left(\nabla\hat{\phi}\right)^2 + \frac{1}{2}m^2\phi^2\right] \\
&= \iint \frac{d^3p d^3p'}{(2\pi)^6} \int d^3x e^{i(\boldsymbol{p}+\boldsymbol{p}')\cdot\boldsymbol{x}} \left\{-\frac{\sqrt{\omega_{\boldsymbol{p}}\omega_{\boldsymbol{p}'}}}{4}\left(a_{\boldsymbol{p}} - a^\dagger_{-\boldsymbol{p}}\right)\left(a_{\boldsymbol{p}'} - a^\dagger_{-\boldsymbol{p}'}\right)\right. \\
&\left. +\frac{1}{2}\frac{1}{\sqrt{4\omega_{\boldsymbol{p}}\omega_{\boldsymbol{p}'}}}(-\boldsymbol{p}\cdot\boldsymbol{p}')\left(a_{\boldsymbol{p}} + a^\dagger_{-\boldsymbol{p}}\right)\left(a_{\boldsymbol{p}'} + a^\dagger_{-\boldsymbol{p}'}\right) + \frac{1}{2}m^2\frac{1}{\sqrt{4\omega_{\boldsymbol{p}}\omega_{\boldsymbol{p}'}}}\left(a_{\boldsymbol{p}} + a^\dagger_{-\boldsymbol{p}}\right)\left(a_{\boldsymbol{p}'} + a^\dagger_{-\boldsymbol{p}'}\right)\right\} \\
&= (2\pi)^3\delta^{(3)}(\boldsymbol{p}+\boldsymbol{p}') \iint \frac{d^3p d^3p'}{(2\pi)^6}\left\{-\frac{\sqrt{\omega_{\boldsymbol{p}}\omega_{\boldsymbol{p}'}}}{4}\left(a_{\boldsymbol{p}} - a^\dagger_{-\boldsymbol{p}}\right)\left(a_{\boldsymbol{p}'} - a^\dagger_{-\boldsymbol{p}'}\right) + \frac{-\boldsymbol{p}\cdot\boldsymbol{p}' + m^2}{4\sqrt{\omega_{\boldsymbol{p}}\cdot\omega_{\boldsymbol{p}'}}}\left(a_{\boldsymbol{p}} + a^\dagger_{-\boldsymbol{p}}\right)\left(a_{\boldsymbol{p}'} + a^\dagger_{-\boldsymbol{p}'}\right)\right\} \\
&= \int \frac{d^3p}{(2\pi)^3}\left\{-\frac{\sqrt{\omega_{\boldsymbol{p}}\omega_{-\boldsymbol{p}}}}{4}\left(a_{\boldsymbol{p}} - a^\dagger_{-\boldsymbol{p}}\right)\left(a_{-\boldsymbol{p}} - a^\dagger_{\boldsymbol{p}}\right) + \frac{\boldsymbol{p}\cdot\boldsymbol{p} + m^2}{4\sqrt{\omega_{\boldsymbol{p}}\cdot\omega_{-\boldsymbol{p}}}}\left(a_{\boldsymbol{p}} + a^\dagger_{-\boldsymbol{p}}\right)\left(a_{-\boldsymbol{p}} + a^\dagger_{\boldsymbol{p}}\right)\right\} \\
&= \int \frac{d^3p}{(2\pi)^3}\left\{-\frac{\omega_{\boldsymbol{p}}}{4}\left(a_{\boldsymbol{p}} - a^\dagger_{-\boldsymbol{p}}\right)\left(a_{-\boldsymbol{p}} - a^\dagger_{\boldsymbol{p}}\right) + \frac{\omega^2_{\boldsymbol{p}}}{4\omega_{\boldsymbol{p}}}\left(a_{\boldsymbol{p}} + a^\dagger_{-\boldsymbol{p}}\right)\left(a_{-\boldsymbol{p}} + a^\dagger_{\boldsymbol{p}}\right)\right\} \\
&= \frac{\omega_{\boldsymbol{p}}}{4}\int \frac{d^3p}{(2\pi)^3}\left\{-\left(a_{\boldsymbol{p}} - a^\dagger_{-\boldsymbol{p}}\right)\left(a_{-\boldsymbol{p}} - a^\dagger_{\boldsymbol{p}}\right) + \left(a_{\boldsymbol{p}} + a^\dagger_{-\boldsymbol{p}}\right)\left(a_{-\boldsymbol{p}} + a^\dagger_{\boldsymbol{p}}\right)\right\} \\
&= \frac{\omega_{\boldsymbol{p}}}{2}\int \frac{d^3p}{(2\pi)^3}\left(a_{\boldsymbol{p}}a^\dagger_{\boldsymbol{p}} + a^\dagger_{-\boldsymbol{p}}a_{-\boldsymbol{p}}\right) \\
&= \frac{\omega_{\boldsymbol{p}}}{2}\int \frac{d^3p}{(2\pi)^3}\left(a_{\boldsymbol{p}}a^\dagger_{\boldsymbol{p}} + a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}}\right) \\
&= \frac{\omega_{\boldsymbol{p}}}{2}\int \frac{d^3p}{(2\pi)^3}\left(\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}}\right] + 2a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}}\right) \\
&= \int \frac{d^3p}{(2\pi)^3}\omega_{\boldsymbol{p}}\left(a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}} + \frac{1}{2}\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}}\right]\right)
\end{aligned}
\tag{3.1.20}
$$

The vacuum energy (we will ignore this infinite constant term):

$$
\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}'}\right] = (2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}') \Longrightarrow \frac{1}{2}\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}}\right] \propto \delta^{(3)}(0) \Longrightarrow \int \frac{d^3p}{(2\pi)^3}\omega_{\boldsymbol{p}}\left(\frac{1}{2}\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}}\right]\right) \to \infty \tag{3.1.21}
$$

The vacuum state with $E = 0$ (for all $\boldsymbol{p}$):

$$
a_{\boldsymbol{p}}\left|0\right\rangle = 0 \tag{3.1.22}
$$

The commutators:

$$
\begin{aligned}
\left[H, a^\dagger_{\boldsymbol{p}}\right] &= \left[\int \frac{d^3p}{(2\pi)^3}\omega_{\boldsymbol{p}}a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}}\right] \\
&= \int \frac{d^3p}{(2\pi)^3}\omega_{\boldsymbol{p}}\left[a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}}\right] \\
&= \int \frac{d^3p}{(2\pi)^3}\omega_{\boldsymbol{p}}\left\{a^\dagger_{\boldsymbol{p}}\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}}\right] + \left[a^\dagger_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}}\right]a_{\boldsymbol{p}}\right\} \\
&= \int \frac{d^3p}{(2\pi)^3}\omega_{\boldsymbol{p}}\left\{a^\dagger_{\boldsymbol{p}}(2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p})\right\} \\
&= \omega_{\boldsymbol{p}}a^\dagger_{\boldsymbol{p}}
\end{aligned}
\tag{3.1.23}
$$

$$
[H, a_{\boldsymbol{p}}] = -\omega_{\boldsymbol{p}}a_{\boldsymbol{p}} \tag{3.1.24}
$$

where the eigenstate of H with energy $\omega_{\boldsymbol{p}} + \omega_{\boldsymbol{q}} + \cdots$:

$$
a^\dagger_{\boldsymbol{p}}a^\dagger_{\boldsymbol{q}}\cdots\left|0\right\rangle \tag{3.1.25}
$$

Example:

$$H\left(a_{\boldsymbol{p}}^\dagger|0\rangle\right)=\left\{\left[H,a_{\boldsymbol{p}}^\dagger\right]+a_{\boldsymbol{p}}^\dagger H\right\}|0\rangle=\omega_{\boldsymbol{p}}a_{\boldsymbol{p}}^\dagger|0\rangle+0=\omega_{\boldsymbol{p}}\left(a_{\boldsymbol{p}}^\dagger|0\rangle\right) \tag{3.1.26}$$

$$\begin{aligned}H\left(a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{q}}^\dagger|0\rangle\right)&=\left\{\left[Ha_{\boldsymbol{p}}^\dagger,a_{\boldsymbol{q}}^\dagger\right]+a_{\boldsymbol{q}}^\dagger Ha_{\boldsymbol{p}}^\dagger\right\}|0\rangle\\&=\left[Ha_{\boldsymbol{p}}^\dagger,a_{\boldsymbol{q}}^\dagger\right]|0\rangle+a_{\boldsymbol{q}}^\dagger\left(\omega_{\boldsymbol{p}}\left(a_{\boldsymbol{p}}^\dagger|0\rangle\right)\right)\\&=H\left[a_{\boldsymbol{p}}^\dagger,a_{\boldsymbol{q}}^\dagger\right]|0\rangle+\left[H,a_{\boldsymbol{q}}^\dagger\right]a_{\boldsymbol{p}}^\dagger|0\rangle+\omega_{\boldsymbol{p}}a_{\boldsymbol{q}}^\dagger a_{\boldsymbol{p}}^\dagger|0\rangle\\&=0+\omega_{\boldsymbol{q}}a_{\boldsymbol{q}}^\dagger a_{\boldsymbol{p}}^\dagger|0\rangle+\omega_{\boldsymbol{p}}a_{\boldsymbol{q}}^\dagger a_{\boldsymbol{p}}^\dagger|0\rangle\\&=\left(\omega_{\boldsymbol{p}}+\omega_{\boldsymbol{q}}\right)a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{q}}^\dagger|0\rangle\end{aligned} \tag{3.1.27}$$

Similarly, from the equation:

$$P^i=\int T^{0i}d^3x=-\int\pi\partial_i\phi d^3x \tag{3.1.28}$$

we can get the total momentum operator:

$$P=-\int d^3x\pi(\boldsymbol{x})\nabla\phi(\boldsymbol{x})=\int\frac{d^3p}{(2\pi)^3}\boldsymbol{p}a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}} \tag{3.1.29}$$

Bose-Einstein statistics:

$$\left[a_{\boldsymbol{p}}^\dagger,a_{\boldsymbol{q}}^\dagger\right]=0\Longrightarrow|\boldsymbol{p},\boldsymbol{q}\rangle\equiv a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{q}}^\dagger|0\rangle=|\boldsymbol{q},\boldsymbol{p}\rangle \tag{3.1.30}$$

3.2 Normalization of the single state

We naturally choose the vacuum state:

$$\langle0|0\rangle=1 \tag{3.2.1}$$

The simplest normalization is as follows but it is not Lorentz invariant:

$$\langle\boldsymbol{p}|\boldsymbol{q}\rangle=(2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}) \tag{3.2.2}$$

Consider a boost in the 3-direction:

$$\left\{\begin{array}{l}p_3'=\gamma(p_3+\beta E)\\E'=\gamma(E+\beta p_3)\end{array}\right.,\quad\text{where }\gamma=\frac{1}{\sqrt{1-\beta^2}} \tag{3.2.3}$$

Using the delta function identity:

$$\delta(f(x)-f(x_0))=\frac{1}{|f'(x_0)|}\delta(x-x_0) \tag{3.2.4}$$

$$\begin{aligned}\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})&=\delta^{(3)}(\boldsymbol{p}'-\boldsymbol{q}')\frac{dp_3'}{dp_3}\\&=\delta^{(3)}(\boldsymbol{p}'-\boldsymbol{q}')\gamma\left(1+\beta\frac{dE}{dp_3}\right)\\&=\delta^{(3)}(\boldsymbol{p}'-\boldsymbol{q}')\gamma\left(1+\beta\frac{d(p_1^2+p_2^2+p_3^2)^{\frac{1}{2}}}{dp_3}\right)\\&=\delta^{(3)}(\boldsymbol{p}'-\boldsymbol{q}')\gamma\left(1+\beta\frac{p_3}{E}\right)\\&=\delta^{(3)}(\boldsymbol{p}'-\boldsymbol{q}')\frac{\gamma}{E}(E+\beta p_3)\\&=\delta^{(3)}(\boldsymbol{p}'-\boldsymbol{q}')\frac{E'}{E}\end{aligned} \tag{3.2.5}$$

It is not Lorentz invariant, but it is easy to find:

$$E\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}) = E'\delta^{(3)}(\boldsymbol{p}'-\boldsymbol{q}') \tag{3.2.6}$$

Therefore, we define:

$$|\boldsymbol{p}\rangle = \sqrt{2E_{\boldsymbol{p}}}a^\dagger_{\boldsymbol{p}}|0\rangle \tag{3.2.7}$$

$$\begin{aligned}\langle\boldsymbol{p}|\boldsymbol{q}\rangle &= \left(\sqrt{2E_{\boldsymbol{p}}}\langle 0|a_{\boldsymbol{p}}\right)\left(\sqrt{2E_{\boldsymbol{q}}}a^\dagger_{\boldsymbol{q}}|0\rangle\right)\\ &= 2\sqrt{E_{\boldsymbol{p}}E_{\boldsymbol{q}}}\langle 0|a_{\boldsymbol{p}}a^\dagger_{\boldsymbol{q}}|0\rangle\\ &= 2\sqrt{E_{\boldsymbol{p}}E_{\boldsymbol{q}}}\langle 0|\left\{\left[a_{\boldsymbol{p}},a^\dagger_{\boldsymbol{q}}\right]+a^\dagger_{\boldsymbol{q}}a_{\boldsymbol{p}}\right\}|0\rangle\\ &= 2E_{\boldsymbol{p}}(2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})\end{aligned} \tag{3.2.8}$$

On the Hilbert space of quantum states, Lorentz transformation will be like some unitary operator:

$$U^\dagger(\Lambda) = U^{-1}(\Lambda) \tag{3.2.9}$$

For a single particle state:

$$U(\Lambda)|\boldsymbol{p}\rangle = |\Lambda\boldsymbol{p}\rangle \tag{3.2.10}$$

Acting on the operator:

$$U(\Lambda)a^\dagger_{\boldsymbol{p}}U^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda\boldsymbol{p}}}{E_{\boldsymbol{p}}}}a^\dagger_{\Lambda\boldsymbol{p}} \tag{3.2.11}$$

which is because:

$$U(\Lambda)|\boldsymbol{p},\boldsymbol{p}_1,\boldsymbol{p}_2,\cdots\rangle = \sqrt{2E_{\boldsymbol{p}}}U(\Lambda)a^\dagger_{\boldsymbol{p}}|\boldsymbol{0},\boldsymbol{p}_1,\boldsymbol{p}_2,\cdots\rangle \tag{3.2.12}$$

$$U(\Lambda)|\boldsymbol{p},\boldsymbol{p}_1,\boldsymbol{p}_2,\cdots\rangle = |\Lambda\boldsymbol{p},\Lambda\boldsymbol{p}_1,\Lambda\boldsymbol{p}_2,\cdots\rangle = \sqrt{2E_{\Lambda\boldsymbol{p}}}a^\dagger_{\Lambda\boldsymbol{p}}|\boldsymbol{0},\Lambda\boldsymbol{p}_1,\Lambda\boldsymbol{p}_2,\cdots\rangle = \sqrt{2E_{\Lambda\boldsymbol{p}}}a^\dagger_{\Lambda\boldsymbol{p}}U(\Lambda)|\boldsymbol{0},\boldsymbol{p}_1,\boldsymbol{p}_2,\cdots\rangle \tag{3.2.13}$$

$$\Longrightarrow \sqrt{2E_{\boldsymbol{p}}}U(\Lambda)a^\dagger_{\boldsymbol{p}} = \sqrt{2E_{\Lambda\boldsymbol{p}}}a^\dagger_{\Lambda\boldsymbol{p}}U(\Lambda) \Longrightarrow U(\Lambda)a^\dagger_{\boldsymbol{p}}U^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda\boldsymbol{p}}}{E_{\boldsymbol{p}}}}a^\dagger_{\Lambda\boldsymbol{p}} \tag{3.2.14}$$

With this normalization, the completeness relation for the one-particle states:

$$(\boldsymbol{1})_{\text{1-particle}} = \int\frac{d^3p}{(2\pi)^3}|\boldsymbol{p}\rangle\frac{1}{2E_{\boldsymbol{p}}}\langle\boldsymbol{p}| \tag{3.2.15}$$

The integral of this form is a Lorentz-invariant 3-momentum integral:

$$\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}} = \int\frac{d^4p}{(2\pi)^4}(2\pi)\delta(p^2-m^2)\Big|_{p^0>0} \tag{3.2.16}$$

Next, consider the interpretation of the $\phi(\boldsymbol{x})$.

$$\begin{aligned}\phi(\boldsymbol{x})|0\rangle &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_{\boldsymbol{p}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}}+a^\dagger_{\boldsymbol{p}}e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right)|0\rangle\\ &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}a^\dagger_{\boldsymbol{p}}|0\rangle e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\\ &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\frac{1}{\sqrt{2E_p}}|\boldsymbol{p}\rangle e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\\ &= \frac{1}{2E_p}\int\frac{d^3p}{(2\pi)^3}e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}|\boldsymbol{p}\rangle\end{aligned} \tag{3.2.17}$$

We will claim that the operator $\phi(\boldsymbol{x})$, acting on the vacuum, creates a particle at position $\boldsymbol{x}$.
We can further confirm that:

$$
\begin{aligned}
\langle 0|\,\phi(\boldsymbol{x})\,|\boldsymbol{p}\rangle &= \langle 0| \int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{p'}}}\left(a_{\boldsymbol{p}'}e^{i\boldsymbol{p}'\cdot\boldsymbol{x}} + a^\dagger_{\boldsymbol{p}'}e^{-i\boldsymbol{p}'\cdot\boldsymbol{x}}\right)\sqrt{2E_{\boldsymbol{p}}}a^\dagger_{\boldsymbol{p}}\,|0\rangle \\
&= e^{i\boldsymbol{p}\cdot\boldsymbol{x}}
\end{aligned} \tag{3.2.18}
$$

which is the position-space representation of the single-particle wavefunction. (like $\langle \boldsymbol{x}|\boldsymbol{p}\rangle \propto e^{i\boldsymbol{p}\cdot\boldsymbol{x}}$ in NRQM)

3.3 The K-G Field in Space-Time

In the Heisenberg picture,

$$\phi(x) = \phi(\boldsymbol{x},t) = e^{iHt}\phi(\boldsymbol{x})e^{-iHt} \tag{3.3.1}$$

$$\pi(x) = \pi(\boldsymbol{x},t) = e^{iHt}\pi(\boldsymbol{x})e^{-iHt} \tag{3.3.2}$$

The Heisenberg equation of motion:

$$i\frac{\partial}{\partial t}\mathscr{O} = [\mathscr{O}, H] \tag{3.3.3}$$

Compute the time dependence of ϕ and π:

$$
\begin{aligned}
i\frac{\partial}{\partial t}\phi(\boldsymbol{x},t) &= \left[\phi(\boldsymbol{x},t), \int d^3x' \left\{\frac{1}{2}\pi^2(\boldsymbol{x}',t) + \frac{1}{2}(\nabla\phi(\boldsymbol{x}',t))^2 + \frac{1}{2}m^2\phi^2(\boldsymbol{x}',t)\right\}\right] \\
&= \left[\phi(\boldsymbol{x},t), \int d^3x' \left\{\frac{1}{2}\pi^2(\boldsymbol{x}',t)\right\}\right] \\
&= \int d^3x' \left(i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')\pi(\boldsymbol{x}',t)\right) \\
&= i\pi(\boldsymbol{x},t)
\end{aligned} \tag{3.3.4}
$$

$$
\begin{aligned}
i\frac{\partial}{\partial t}\pi(\boldsymbol{x},t) &= \left[\pi(\boldsymbol{x},t), \int d^3x' \left\{\frac{1}{2}\pi^2(\boldsymbol{x}',t) + \frac{1}{2}(\nabla\phi(\boldsymbol{x}',t))^2 + \frac{1}{2}m^2\phi^2(\boldsymbol{x}',t)\right\}\right] \\
&= \left[\pi(\boldsymbol{x},t), \int d^3x' \left\{\frac{1}{2}(\nabla\phi(\boldsymbol{x}',t))^2 + \frac{1}{2}m^2\phi^2(\boldsymbol{x}',t)\right\}\right] \\
&= \int d^3x' \left(-i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')\left(-\nabla^2+m^2\right)\phi(\boldsymbol{x}',t)\right) \\
&= -i\left(-\nabla^2+m^2\right)\phi(\boldsymbol{x},t)
\end{aligned} \tag{3.3.5}
$$

Combining the two results gives the K-G equation:

$$\frac{\partial^2}{\partial t^2}\phi = (\nabla^2 - m^2)\phi \tag{3.3.6}$$

where

$$
\begin{aligned}
\phi(\boldsymbol{x},t) &= e^{iHt}\phi(\boldsymbol{x})e^{-iHt} \\
&= e^{iHt}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + a^\dagger_{\boldsymbol{p}}e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right)e^{-iHt} \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\left(e^{iHt}a_{\boldsymbol{p}}e^{-iHt}e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + e^{iHt}a^\dagger_{\boldsymbol{p}}e^{-iHt}e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}}(t)e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + a^\dagger_{\boldsymbol{p}}(t)e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right)
\end{aligned} \tag{3.3.7}
$$

Note:

$$a_{\boldsymbol{p}}(t) = e^{iHt}a_{\boldsymbol{p}}e^{-iHt} \tag{3.3.8}$$

$$a_{\boldsymbol{p}}^{\dagger}(t) = e^{iHt} a_{\boldsymbol{p}}^{\dagger} e^{-iHt} \tag{3.3.9}$$

We can perform the same manipulations with P instead of H.
The motion-equation for $a_{\boldsymbol{p}}(t)$:

$$\begin{aligned}
\frac{d}{dt} a_{\boldsymbol{p}}(t) &= -i\left[a_{\boldsymbol{p}}(t), H\right] \\
&= i\left[H, a_{\boldsymbol{p}}(t)\right] \\
&= e^{iHt} i\left[H, a_{\boldsymbol{p}}\right] e^{-iHt} \\
&= e^{iHt} i(-E_{\boldsymbol{p}} a_{\boldsymbol{p}}) e^{-iHt} \\
&= -iE_{\boldsymbol{p}} a_{\boldsymbol{p}}(t)
\end{aligned} \tag{3.3.10}$$

We can solve that:

$$a_{\boldsymbol{p}}(t) = a_{\boldsymbol{p}} e^{-iE_{\boldsymbol{p}}t}; \quad a_{\boldsymbol{p}}^{\dagger}(t) = a_{\boldsymbol{p}}^{\dagger} e^{iE_{\boldsymbol{p}}t} \tag{3.3.11}$$

Set $p^{\mu} = (E_{\boldsymbol{p}}, \boldsymbol{p}),\ x^{\mu} = (x^0, \boldsymbol{x})$:

$$p \cdot x = p^{\mu} x_{\mu} = \eta_{\mu\nu} p^{\mu} x^{\nu} = E_{\boldsymbol{p}} t - \boldsymbol{p} \cdot \boldsymbol{x} \tag{3.3.12}$$

Then we can get:

$$\begin{aligned}
\phi(\boldsymbol{x}, t) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p}}(t) e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + a_{\boldsymbol{p}}^{\dagger}(t) e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \right) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p}} e^{-iE_{\boldsymbol{p}}t} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + a_{\boldsymbol{p}}^{\dagger} e^{iE_{\boldsymbol{p}}t} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \right) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p}} e^{-ip\cdot x} + a_{\boldsymbol{p}}^{\dagger} e^{ip\cdot x} \right) \Big|_{p^0 = E_{\boldsymbol{p}}}
\end{aligned} \tag{3.3.13}$$

$$\pi(\boldsymbol{x}, t) = \frac{\partial}{\partial t} \phi(\boldsymbol{x}, t) \tag{3.3.14}$$

In analogy:

$$e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} a_{\boldsymbol{p}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} = a_{\boldsymbol{p}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}}; \quad e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} a_{\boldsymbol{p}}^{\dagger} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} = a_{\boldsymbol{p}}^{\dagger} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \tag{3.3.15}$$

Therefore,

$$\phi(x) = e^{i(Ht - \boldsymbol{p}\cdot\boldsymbol{x})} \phi(0) e^{-i(Ht - \boldsymbol{p}\cdot\boldsymbol{x})} = e^{ip\cdot x} \phi(0) e^{-ip\cdot x} \tag{3.3.16}$$

3.4 Causality

In the Heisenberg picture, the amplitude for a particle to propagate from y to x:

$$\begin{aligned}
D(x - y) &= \langle x| e^{-iH(x^0 - y^0)} |y\rangle \\
&= \langle x| e^{-iHx^0} e^{iHy^0} |y\rangle \\
&= \langle 0| e^{iHx^0} \phi(\boldsymbol{x}) e^{-iHx^0} e^{iHy^0} \phi(\boldsymbol{y}) e^{-iHy^0} |0\rangle \\
&= \langle 0| \phi(x)\phi(y) |0\rangle
\end{aligned} \tag{3.4.1}$$

where

$$|x\rangle = \phi(\boldsymbol{x}) e^{-iHx^0} |0\rangle; \quad |y\rangle = \phi(\boldsymbol{y}) e^{-iHy^0} |0\rangle \tag{3.4.2}$$

Substituting the following equations:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p}} e^{-ip\cdot x} + a_{\boldsymbol{p}}^{\dagger} e^{ip\cdot x} \right) \Big|_{p^0 = E_{\boldsymbol{p}}} \tag{3.4.3}$$

$$\phi(y) = \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{q}}}} \left(a_{\boldsymbol{q}} e^{-iq\cdot y} + a_{\boldsymbol{q}}^\dagger e^{iq\cdot y} \right)\Big|_{q^0=E_{\boldsymbol{q}}} \tag{3.4.4}$$

Then

$$\begin{aligned} D(x-y) &= \langle 0|\, \phi(x)\phi(y)\, |0\rangle \\ &= \langle 0| \iint \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_{\boldsymbol{p}}E_{\boldsymbol{q}}}} \left(a_{\boldsymbol{p}} e^{-ip\cdot x} + a_{\boldsymbol{p}}^\dagger e^{ip\cdot x} \right) \left(a_{\boldsymbol{q}} e^{-iq\cdot y} + a_{\boldsymbol{q}}^\dagger e^{iq\cdot y} \right) |0\rangle \\ &= \iint \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_{\boldsymbol{p}}E_{\boldsymbol{q}}}} \langle 0|\, a_{\boldsymbol{p}} a_{\boldsymbol{q}}^\dagger \,|0\rangle\, e^{-ip\cdot x} e^{iq\cdot y} \\ &= \iint \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_{\boldsymbol{p}}E_{\boldsymbol{q}}}} \langle 0| \left[a_{\boldsymbol{p}}, a_{\boldsymbol{q}}^\dagger \right] + a_{\boldsymbol{q}}^\dagger a_{\boldsymbol{p}} \,|0\rangle\, e^{-ip\cdot x} e^{iq\cdot y} \\ &= \iint \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_{\boldsymbol{p}}E_{\boldsymbol{q}}}} \langle 0| \left[a_{\boldsymbol{p}}, a_{\boldsymbol{q}}^\dagger \right] |0\rangle\, e^{-ip\cdot x} e^{iq\cdot y} \\ &= \iint \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_{\boldsymbol{p}}E_{\boldsymbol{q}}}} \left((2\pi)^3 \delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}) \right) e^{-ip\cdot x} e^{iq\cdot y} \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} e^{-ip\cdot(x-y)} \end{aligned} \tag{3.4.5}$$

which we have proved this integral is Lorentz invariant.

① $x^0 - y^0 = t$, $\boldsymbol{x} - \boldsymbol{y} = 0$:

$$\begin{aligned} D(x-y) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} e^{-ip^0\cdot(x^0-y^0)} \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} e^{-iE_{\boldsymbol{p}}\cdot t} \\ &= \int \frac{4\pi p^2 dp}{(2\pi)^3} \frac{1}{2\sqrt{|\boldsymbol{p}|^2+m^2}} e^{-it\sqrt{|\boldsymbol{p}|^2+m^2}} \end{aligned} \tag{3.4.6}$$

$$E_{\boldsymbol{p}} = \sqrt{|\boldsymbol{p}|^2+m^2} \Longrightarrow \frac{dE}{d|\boldsymbol{p}|} = \frac{p}{\sqrt{|\boldsymbol{p}|^2+m^2}} \Longrightarrow dp = \frac{\sqrt{|\boldsymbol{p}|^2+m^2}}{p} dE \tag{3.4.7}$$

$$\begin{aligned} D(x-y) &= \frac{4\pi}{(2\pi)^3} \int_0^{+\infty} \frac{\sqrt{|\boldsymbol{p}|^2+m^2}}{p} dE \frac{p^2}{2\sqrt{|\boldsymbol{p}|^2+m^2}} e^{-iEt} \\ &= \frac{1}{4\pi^2} \int_0^{+\infty} dE p e^{-iEt} \\ &= \frac{1}{4\pi^2} \int_0^{+\infty} dE \sqrt{E^2-m^2} e^{-iEt} \\ &\sim (t\to\infty)\ e^{-imt} \end{aligned} \tag{3.4.8}$$

② $x^0 - y^0 = 0$, $\boldsymbol{x} - \boldsymbol{y} = \boldsymbol{r}$:

$$\begin{aligned} D(x-y) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} e^{-i\boldsymbol{p}\cdot\boldsymbol{r}} \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} e^{-iprcos\theta} \\ &= \int_0^{+\infty} \int_0^{\pi} \int_0^{2\pi} \frac{p^2 sin\theta dp d\theta d\phi}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} e^{-iprcos\theta} \\ &\sim (r \to \infty)\ e^{-mr} \end{aligned} \tag{3.4.9}$$

which shows that outside the light-cone, the propagation amplitude is exponentially vanishing but nonzero.

To really discuss causality, however, we should ask whether a measurement performed at one point can affect a measurement at another point whose separation from the first is spacelike.

If the commutator $[\phi(x), \phi(y)]$ vanishes for spacelike $(x-y)^2 < 0$, causality is preserved quite generally.

Do the more general computation:

$$\begin{aligned} [\phi(x), \phi(y)] &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{q}}}} \left[\left(a_{\boldsymbol{p}} e^{-ip\cdot x} + a_{\boldsymbol{p}}^\dagger e^{ip\cdot x} \right), \left(a_{\boldsymbol{q}} e^{-iq\cdot y} + a_{\boldsymbol{q}}^\dagger e^{iq\cdot y} \right) \right] \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right) \\ &= D(x-y) - D(y-x) \end{aligned} \tag{3.4.10}$$

where

$$\begin{aligned} \left[\left(a_{\boldsymbol{p}} e^{-ip\cdot x} + a_{\boldsymbol{p}}^\dagger e^{ip\cdot x} \right), \left(a_{\boldsymbol{q}} e^{-iq\cdot y} + a_{\boldsymbol{q}}^\dagger e^{iq\cdot y} \right) \right] &= \left[a_{\boldsymbol{p}} e^{-ip\cdot x}, \left(a_{\boldsymbol{q}} e^{-iq\cdot y} + a_{\boldsymbol{q}}^\dagger e^{iq\cdot y} \right) \right] + \left[a_{\boldsymbol{p}}^\dagger e^{ip\cdot x}, \left(a_{\boldsymbol{q}} e^{-iq\cdot y} + a_{\boldsymbol{q}}^\dagger e^{iq\cdot y} \right) \right] \\ &= 0 + \left[a_{\boldsymbol{p}} e^{-ip\cdot x}, a_{\boldsymbol{q}}^\dagger e^{iq\cdot y} \right] + \left[a_{\boldsymbol{p}}^\dagger e^{ip\cdot x}, a_{\boldsymbol{q}} e^{-iq\cdot y} \right] + 0 \\ &= (2\pi)^3 \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}) e^{-ip\cdot x + iq\cdot y} - (2\pi)^3 \delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}) e^{ip\cdot x - iq\cdot y} \end{aligned} \tag{3.4.11}$$

When $(x-y)^2 < 0$, we can perform a Lorentz transformation on $(x-y)$ to $-(x-y)$.

Since $D(x-y)$ is Lorentz-invarient:

$$D(x-y) = D(-(x-y)) = D(y-x) \Longrightarrow [\phi(x), \phi(y)] = 0 \tag{3.4.12}$$

Thus no measurement in the K-G theory can affect another measurement outside the light-cone, causality is preserved.

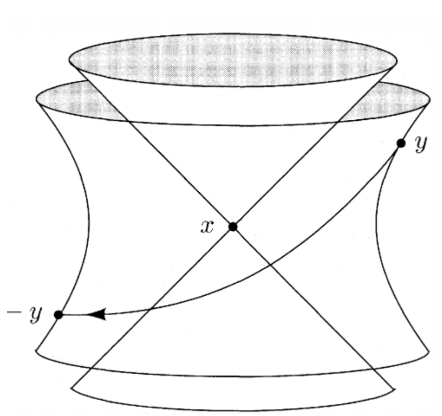

Figure 1: When $x - y$ is spacelike, a continuous Lorentz transformation can take $(x-y)$ to $-(x-y)$.

3.5 The relation between causality and antiparticle

Now discuss a complex Klein-Gordon field:

$$\phi(x) = \psi(x) + i\tilde{\psi}(x) \tag{3.5.1}$$

After the quantization, since now $\phi^\dagger(x) \neq \phi(x)$, we will need different operator $a_{\boldsymbol{p}}$ and $b_{\boldsymbol{p}}$:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p}} e^{-ip\cdot x} + b^\dagger_{\boldsymbol{p}} e^{ip\cdot x} \right) \tag{3.5.2}$$

where $b^\dagger_{\boldsymbol{p}}$ stands for the creation of an antiparticle.
Now we study:

$$\begin{aligned}
\langle 0| \left[\phi(x), \phi^\dagger(y)\right] |0\rangle &= \langle 0| \phi(x)\phi^\dagger(y) |0\rangle - \langle 0| \phi^\dagger(y)\phi(x) |0\rangle \\
&= \left[\phi^\dagger(x) |0\rangle\right]^* \left[\phi^\dagger(y) |0\rangle\right] - [\phi(y) |0\rangle]^* [\phi(x) |0\rangle] \\
&= \iint \frac{d^3p d^3p}{(2\pi)^6} \frac{1}{2E_{\boldsymbol{p}}} \left\{ \langle 0| a_{\boldsymbol{p}} a^\dagger_{\boldsymbol{p}} |0\rangle - \langle 0| b_{\boldsymbol{p}} b^\dagger_{\boldsymbol{p}} |0\rangle \right\}
\end{aligned} \tag{3.5.3}$$

Because of causality, we need:

$$\left[\phi(x), \phi^\dagger(y)\right] = 0 \Longrightarrow \langle 0| \left[\phi(x), \phi^\dagger(y)\right] |0\rangle = 0 \Longrightarrow \langle 0| a_{\boldsymbol{p}} a^\dagger_{\boldsymbol{p}} |0\rangle - \langle 0| b_{\boldsymbol{p}} b^\dagger_{\boldsymbol{p}} |0\rangle = 0 \tag{3.5.4}$$

which shows the necessity to introduce the antiparticle, which has the same m with the particle.

3.6 The Klein-Gordon Propagator

Since $[\phi(x), \phi(y)]$ is a c-number, we can write:

$$[\phi(x), \phi(y)] = \langle 0| [\phi(x), \phi(y)] |0\rangle \tag{3.6.1}$$

Suppose $x^0 > y^0$, rewrite the equation as a 4-dimensional integral:

$$\begin{aligned}
\langle 0| [\phi(x), \phi(y)] |0\rangle &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right) \\
&= \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{1}{2E_{\boldsymbol{p}}} e^{-ip\cdot(x-y)}\Big|_{p^0=E_{\boldsymbol{p}}} + \frac{1}{-2E_{\boldsymbol{p}}} e^{-ip\cdot(x-y)}\Big|_{p^0=-E_{\boldsymbol{p}}} \right\} \\
&= \int \frac{d^3p}{(2\pi)^3} \int \frac{dp^0}{2\pi i} \frac{-1}{p^2 - m^2} e^{-ip\cdot(x-y)} \\
&= \int \frac{d^4p}{(2\pi)^3} \frac{i}{p^2 - m^2} e^{-ip\cdot(x-y)}
\end{aligned} \tag{3.6.2}$$

① Retarded Green's Function:

$$D_R(x-y) \equiv \theta(x^0 - y^0) \langle 0| [\phi(x), \phi(y)] |0\rangle \tag{3.6.3}$$

Do the computation:

$$\begin{aligned}
\left(\partial^2 + m^2\right) D_R(x-y) &= \left(\partial^2 \theta(x^0 - y^0)\right) \langle 0| [\phi(x), \phi(y)] |0\rangle + 2 \left(\partial_\mu \theta(x^0 - y^0)\right) \left(\partial^\mu \langle 0| [\phi(x), \phi(y)] |0\rangle\right) \\
&\quad + \theta(x^0 - y^0) \left(\partial^2 + m^2\right) \langle 0| [\phi(x), \phi(y)] |0\rangle \\
&= -\delta(x^0 - y^0) \langle 0| [\pi(x), \phi(y)] |0\rangle + 2\delta(x^0 - y^0) \langle 0| [\pi(x), \phi(y)] |0\rangle + 0 \\
&= -\delta^{(4)}(x-y)
\end{aligned} \tag{3.6.4}$$

which shows that $D_R(x-y)$ is a Green's function of the K-G operator.
We could also find it by Fourier transformation:

$$D_R(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \tilde{D}_R(p) \tag{3.6.5}$$

where

$$(-p^2+m^2)\tilde{D}_R(p) = -i \tag{3.6.6}$$

② Feynman propagator:

$$D_F \equiv \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon} e^{-ip\cdot(x-y)} \tag{3.6.7}$$

We have:

$$\begin{aligned} D_F(x-y) &= \begin{cases} D(x-y) & \text{for } x^0 > y^0 \\ D(y-x) & \text{for } x^0 < y^0 \end{cases} \\ &= \theta(x^0-y^0) \langle 0| \phi(x)\phi(y) |0\rangle + \theta(y^0-x^0) \langle 0| \phi(y)\phi(x) |0\rangle \\ &\equiv \langle 0| T\phi(x)\phi(y) |0\rangle \end{aligned} \tag{3.6.8}$$

where the symbol T is the time-ording operator.
$D_F(x-y)$ is a Green's function of the K-G operator, it will represent the propagation of virtual particles.

3.7 Particle Creation by a Classical Source

Consider a K-G field coupled to an external, classical source field $j(x)$:

$$\left(\partial^2+m^2\right)\phi(x) = j(x) \tag{3.7.1}$$

The field equation follows from the Lagrangian:

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}m^2\phi^2 + j(x)\phi(x) \tag{3.7.2}$$

where $j(x)$ is nonzero only for a finite time interval.
$\phi_0(x)$ is the initial condition of the field:

$$\left(\partial^2+m^2\right)\phi_0(x) = 0 \tag{3.7.3}$$

and $\phi_0(x)$ has the form:

$$\phi_0(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p}} e^{-ip\cdot x} + a^\dagger_{\boldsymbol{p}} e^{ip\cdot x}\right) \tag{3.7.4}$$

Now the solution of the equation of motion can be constructed:

$$\begin{aligned} \phi(x) &= \phi_0(x) + i\int d^4y D_R(x-y) j(y) \\ &= \phi_0(x) + i\int d^4y \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}} \theta(x^0-y^0)\left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right) j(y) \\ &= \phi_0(x) + i\int d^4y \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right) j(y) \\ &= \phi_0(x) + i\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}} \left[e^{-ip\cdot x}\int d^4y e^{ip\cdot y} j(y) + e^{ip\cdot x}\int d^4y e^{-ip\cdot y} j(y)\right] \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}} e^{-ip\cdot x} + a^\dagger_{\boldsymbol{p}} e^{ip\cdot x}\right) + i\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}\left[e^{-ip\cdot x}\tilde{j}(p) + e^{ip\cdot x}\tilde{j}^\dagger(p)\right] \end{aligned} \tag{3.7.5}$$

Group the positive-frequency terms together and negative-frequency terms together:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left\{ \left(a_{\boldsymbol{p}} + \frac{i}{\sqrt{2E_{\boldsymbol{p}}}} \tilde{j}(p) \right) e^{-ip\cdot x} + h.c. \right\} \tag{3.7.6}$$

We can also get the form of the Hamiltonian after $j(x)$ has acted:

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\boldsymbol{p}} \left(a_{\boldsymbol{p}}^\dagger - \frac{i}{\sqrt{2E_{\boldsymbol{p}}}} \tilde{j}^*(p) \right) \left(a_{\boldsymbol{p}} + \frac{i}{\sqrt{2E_{\boldsymbol{p}}}} \tilde{j}(p) \right) \tag{3.7.7}$$

The energy of the system after the source has been turned off is:

$$\langle 0| H |0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \left|\tilde{j}(p)\right|^2 = \int \frac{d^3p}{(2\pi)^3} E_{\boldsymbol{p}} \frac{1}{2E_{\boldsymbol{p}}} \left|\tilde{j}(p)\right|^2 \tag{3.7.8}$$

which shows that the probability density for creating a particle in the mode $p = (E_{\boldsymbol{p}}, \boldsymbol{p})$ is:

$$\frac{1}{2E_{\boldsymbol{p}}} \left|\tilde{j}(p)\right|^2 \tag{3.7.9}$$

Then the total number of particles produced is:

$$\int dN = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} \left|\tilde{j}(p)\right|^2 = \int \frac{d^4p}{(2\pi)^4} \delta(p^2 - m^2) \left|\tilde{j}(p)\right|^2 \tag{3.7.10}$$