Review Chap 2: Loss Distributions

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TA: Yuan Zhuang (Instructor: Prof. Lianzeng Zhang)

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目录

Distribution in Actuarial Modelling

- Fit a probability distribution with reasonably tractable mathematical properties
- Concern about the chances and sizes of large claims

What's each section about?

- **Section 2.2**: What are the basic properties of some commonly used and classic loss distributions? How to fit data to a loss distribution? **(Classical loss distributions)**
- **Section 2.3**: How well does our selected distribution fit the data? **(Analyzing fit)**
- **Section 2.4**: Well, what's going on if the parameter is not a constant, but follows a certain distribution? **(Mixture distribution)**
- **Section 2.5**: What will change under reinsurance contract? **(Loss distributions and reinsurance)**

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Exponential Distribution (Page 36 - 39)

- PDF, CDF, MGF, Mean, Variance (Have a look at the form of CDF!)
- Memoryless property
- Thiner tail than the dataset `Theft`
- Failure (or hazard) rate function

$$r_X(x) = \lim_{h\to 0} \frac{F_X(x+h) - F_X(x)}{h} \frac{1}{\bar{F}_X(x)} = \frac{f_X(x)}{\bar{F}_X(x)} = \frac{\lambda e^{-\lambda x}}{e^{-\lambda x}} = \lambda.$$

- Method of moments and maximum likelihood
- $\bar{F}_X(x) = e^{-\lambda x}$ converges to 0 really fast

Pareto Distribution (Page 39 - 42)

- PDF, CDF, Mean, Variance

$$\bar{F}_X(x) = \left(\frac{\lambda}{\lambda + x}\right)^{\alpha}$$

- Dealing with inflation in claims and deductibles
- Method of moments and maximum likelihood (numerical methods needed)
- Much closer to observed relative frequencies of these events for the `Theft` claim data than the ML fitted exponential distribution

Gamma Distribution (Page 43 - 44)

- Shape and scale parameter
- Method of moments and maximum likelihood (Reparametrization needed)

Weibull Distribution (Page 45 - 46)

- Shape and scale parameter c and γ
  - $\gamma = 1$, exponential
  - $\gamma < 1$, tail is fatter (heavier) than that of any exponential distribution, but not as heavy as that of a Pareto
- $$r_X(x) = \frac{f_X(x)}{\bar{F}_X(x)} = \frac{c\gamma x^{\gamma-1} e^{-cx^\gamma}}{e^{-cx^\gamma}} = c\gamma x^{\gamma-1}$$
- Moments (Gamma function needed)
- Maximum Likelihood and method of percentiles

Lognormal Distribution (Page 47 - 51)

- Normal after Logged
- PDF, Mean, Variance
- Method of moments or Maximum likelihood

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Tests on Goodness of Fit

- Kolmogorov–Smirnoff test
- Anderson–Darling test
- Chi-square goodness-of-fit tests (*Important)

Chi-square goodness-of-fit tests (Page 54 - 56)

- Maximum likelihood on the grouped data
- $$\chi^2_{GF} = \sum_1^k \left(O_i - E_i\right)^2 / E_i$$
- Degrees of Freedom: $d = k - 1 - r$
  - k: Number of groups
  - r: Number of parameters to be estimated

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Mixture Distributions (Page 58 - 60)

Simple Case

$$F(x) = pF_1(x) + qF_2(x)$$

General Case

$$F(x) = \int_\Theta F_\theta(x)dG(\theta) = \int_\Theta F_\theta(x)g(\theta)d\theta$$

Focus on Example 2.5

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Assumptions (Page 61)

- Claim-by-claim Based
- $$X = Y + Z = h_I(X) + h_R(X)$$

Proportional reinsurance (Page 62)

- $$h_I(X) = Y = \alpha X,\ h_R(X) = Z = (1-\alpha)X$$
- Closed under multiplication by a positive scalar (exponential, Pareto, gamma, Weibull and lognormal)

Excess of Loss Reinsurance (Page 62 - 63)

- $$Y = \begin{cases} X & \text{if } X \leq M \\ M & \text{if } X > M \end{cases}$$

$$E(Y) = \int_0^M x f_X(x) dx + M \bar{F}_X(M) = E(X) - \int_0^\infty y f_X(y+M) dy$$

- Maximum Likelihood:

$$\mathbf{x} = x_1, x_2, M, x_4, M, x_6, x_7, M, \ldots$$

$$L(\theta) = \prod_1^n f_X\left(x_i, \theta\right) \prod_1^m \bar{F}_X(M, \theta)$$

The Reinsurer's View of Excess of Loss Reinsurance (Page 63 - 64)

- $$E(Z) = 0 \cdot F_X(M) + E\left(Z_R\right) \bar{F}_X(M) = E\left(Z_R\right) P(X > M)$$
- $$\bar{F}_{Z_R}(z) = P(X > M + z \mid X > M) = \frac{\bar{F}_X(M+z)}{\bar{F}_X(M)}$$

Dealing with Claims Inflation (Page 64 - 65)

- $$E\left(Y^*\right) = \int_0^{M/k} kxf_X(x)dx + \int_{M/k}^{\infty} Mf_X(x)dx$$
- $$E\left(Y^*\right) = k\left[E(X) - \int_0^{\infty} yf_X(y + M/k)dy\right]$$
- The increase overall in payment by the (ceding) insurer is less than k

Policy Excess and Deductibles (Page 66 - 68)

- $$Y = \begin{cases} 0 & \text{if } X \leq D \\ X - D & \text{if } X > D \end{cases}$$
- $$E(Y) = \int_D^\infty (x - D) f_X(x) dx = \int_0^\infty y f_X(y + D) dy$$

Focus on Example 2.7