FIT2086 Lecture 2
Note on Taylor Series Expansion

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Let X be a random variable. In Lecture 2 we covered a technique for approximating the expectation and variance of a function $f(x)$ of X using Taylor series. The result is below.

Fact 1. Let $f(x)$ be a twice differentiable function in x, and let X be a random variable satisfying $\mathbb{E}[X] = \mu < \infty$ and $\mathbb{V}[X] = \sigma^2 < \infty$; then we have

$$\begin{aligned}
\mathbb{E}[f(X)] &\approx f(\mu) + \left[\left. \frac{d^2 f(x)}{dx^2} \right|_{x=\mu} \right] \frac{\sigma^2}{2} \\
\mathbb{V}[f(X)] &\approx \left[\left. \frac{df(x)}{dx} \right|_{x=\mu} \right]^2 \sigma^2
\end{aligned}$$

To apply this result all we need to do is find the first and second derivatives of $f(x)$, and know the mean $\mathbb{E}[X]$ and variance $\mathbb{V}[X]$ of the random variable X.

Notation: There have been some questions about the origin of this approximation, as well as some possible confusion arising from the notation. The first thing to note is that many of you may be familiar with Lagrange's notation for derivatives, i.e., $f'(x)$ denoting the first derivative, $f''(x)$ denoting the second etc. This notation can be written in terms of the above notation (Leibniz') by noting that

$$f'(x) = \left[\left. \frac{df(\bar{x})}{d\bar{x}} \right|_{\bar{x}=x} \right]^2$$

with a similar relation for $f''(x)$. Basically the Leibniz notation tells us to differentiate the function $f(\bar{x})$ with respect to (a dummy variable) $\bar{x}$, and evaluate the resulting derivative (which is itself a function) at $\bar{x} = x$, with a similar interpretation for $f''(x)$. The Leibniz notation is often used (and you should have encountered this in mathematics) because it extends to multiple variables in a much more convenient fashion. We can rewrite the above fact in Lagrange's notation as

$$\begin{aligned}
\mathbb{E}[f(X)] &\approx f(\mu) + f''(\mu) \frac{\sigma^2}{2} \\
\mathbb{V}[f(X)] &\approx (f'(\mu))^2 \sigma^2
\end{aligned}$$

Derivation: To help understand the method more clearly I will give a simple derivation of the two results in Fact 1. To find $\mathbb{E}\left[f(X)\right]$ we first perform a *second-order* Taylor series expansion on $f(x)$ around the point $x = \mathbb{E}\left[X\right] = \mu$ so that

$$f(x) \approx f(\mu) + f'(\mu)(x-\mu) + \frac{f''(\mu)}{2}(x-\mu)^2$$

and then we take expectations of the right-hand-side

$$\begin{aligned}
\mathbb{E}\left[f(X)\right] &\approx \mathbb{E}\left[f(\mu) + f'(\mu)(X-\mu) + \frac{f''(\mu)}{2}(X-\mu)^2\right] \\
&= f(\mu) + f'(\mu)\,\mathbb{E}\left[(X-\mu)\right] + \frac{f''(\mu)}{2}\,\mathbb{E}\left[(X-\mu)^2\right] \\
&= f(\mu) + \frac{f''(\mu)}{2}\,\mathbb{V}\left[X\right] \\
&= f(\mu) + \frac{f''(\mu)}{2}\sigma^2
\end{aligned}$$

where $\mathbb{E}\left[X-\mu\right] = 0$ as $\mathbb{E}\left[X\right] = \mu$ by definition, and $\mathbb{E}\left[(X-\mu)^2\right] = \mathbb{V}\left[X\right] = \sigma^2$ again by definition of variance. So the contribution of the first derivative disappears. To approximate $\mathbb{V}\left[f(X)\right]$ we perform a *first-order* Taylor series expansion around $x = \mu$

$$f(x) \approx f(\mu) + f'(\mu)(x-\mu)$$

and take the variance of the right-hand-side

$$\begin{aligned}
\mathbb{V}\left[f(X)\right] &\approx \mathbb{V}\left[f(\mu) + f'(\mu)(X-\mu)\right] \\
&= \mathbb{V}\left[f'(\mu)(X-\mu)\right] \\
&= \left(f'(\mu)\right)^2\mathbb{V}\left[(X-\mu)\right] \\
&= \left(f'(\mu)\right)^2\sigma^2
\end{aligned}$$

where this follows from $\mathbb{V}\left[c\right] = 0$ if c is a constant and $\mathbb{V}\left[c\,X\right] = c^2\mathbb{V}\left[X\right]$. The reason we only use a first-order Taylor series expansion for the variance is that if we used a second order expansion we would need to find the variance of X^2, which we do not know in our assumptions.