Game Theory Homework 2 Report

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一、Code

定義一個 class 名為 Player，裡面變數有 player 的 id、他個人的 utility matrix、他對於對手的 belief 及他的 payoff。其中，於 *calculate_payoff* 函數計算 payoff。

```
private:
    unsigned int id;
    std::array<std::array<int, 2>, 2> utility;
    std::array<double, 2> belief;
    std::array<double, 2> payoff;

    void calculate_payoff() {
        if (id == 1){
            payoff[0] = belief[0] * utility[0][0] + belief[1] * utility[0][1];
            payoff[1] = belief[0] * utility[1][0] + belief[1] * utility[1][1];
        }
        else if (id == 2){
            payoff[0] = belief[0] * utility[0][0] + belief[1] * utility[1][0];
            payoff[1] = belief[0] * utility[0][1] + belief[1] * utility[1][1];
        }
        return;
    }
```

```
public:
    Player(unsigned int player_id, std::array<std::array<int, 2>, 2> game, std::array<double, 2> init_belief) {
        id = player_id;
        std::copy(std::begin(game), std::end(game), std::begin(utility));
        std::copy(std::begin(init_belief), std::end(init_belief), std::begin(belief));
        calculate_payoff();
    }
```

初始的 belief 以 pseudo random function 產生，player 中兩個 strategies 的 belief 加總為 1000，且型別為 float。

```
struct Belief {
    float strategy_1;
    float strategy_2;
};

Belief random_belief() {
    //return initial belief range in [0, 1000] and the sum of 2 stratgy's believes is 1000
    Belief init_belief;
    init_belief.strategy_1 = (float)(rand() % 1000) + ((float)rand()/(float)(RAND_MAX));
    init_belief.strategy_2 = 1000 - init_belief.strategy_1;
    return init_belief;
}
```

這次作業總共有 9 小題，程式中一開始會先請使用者輸入想解哪一題。為了方便解釋，以這邊以 Q1 為例，其他 questions 只是輸入的 utility matrix 不同。先宣告兩個 Player object player1 及 player2，參數依序為 id、utility matrix 及初始 belief。接著進行 game。

```
case 1: {   //Q1: One pure-strategy Nash Equilibrium
    Player player1(1, {{{-1, 1}, {0, 3}}}, {player1_init_belief.strategy_1, player1_init_belief.strategy_2});
    Player player2(2, {{{-1, 0}, {1, 3}}}, {player2_init_belief.strategy_1, player2_init_belief.strategy_2});
    game_run(player1, player2);
}
    break;
```

game_run 函數中，會先 print 出兩個 players 當前的 belief 及 payoff，再以 payoff 決定這輪的 best response，最後用對手的 best response 更新自己的 belief 及 payoff。不斷的重複做迴圈，收斂條件為兩個 players 的 best response 1000 回合都沒改變或該 game 進行超過 3000 回合。

```cpp
void game_run(Player& player1, Player& player2) {
    BestReponse best_response;
    BestReponse pre_best_response;
    int iter = 1;
    int converge_count = 0;

    while(converge_count < 1000 && iter <= 3000) {
        std::cout << "Iteration: " << iter << std::endl;

        player1.print_belief_payoff();
        player2.print_belief_payoff();

        best_response.p1 = player1.respond();
        best_response.p2 = player2.respond();
        std::cout << "best response: " << best_response.p1 << ' ' << best_response.p2 << std::endl;

        player1.update_belief(best_response.p2);
        player2.update_belief(best_response.p1);

        if(pre_best_response.p1 == best_response.p1 && pre_best_response.p2 == best_response.p2)
            converge_count++;
        else
            converge_count = 0;
        pre_best_response.p1 = best_response.p1;
        pre_best_response.p2 = best_response.p2;
        iter++;

        std::cout << "--------------------------------" << std::endl;
    }

    return;
}
```

update_belief、*respond* 及 *print_belief_payoff* 函數定義如下，值得注意的是在 *respond* 中，若 player 的兩個 payoff 相同，就隨機選擇 best response；以及在 *update_belief* 中，除了更新 belief，還會更新計算 payoff。

```cpp
void update_belief(unsigned int strategy) {
    belief[strategy]++;
    calculate_payoff();
    return;
}

unsigned int respond() {
    if (payoff[0] > payoff[1])
        return 0;
    else if (payoff[1] > payoff[0])
        return 1;
    else
        return rand() % 2;
}

void print_belief_payoff() const {
    std::cout << "player" << id << "'s belief: " << belief[0] << "  " << belief[1] << std::endl;
    std::cout << "player" << id << "'s payoff: " << payoff[0] << "  " << payoff[1] << std::endl << std::endl;
    return;
}
```

二、Questions

1. One Pure-Strategy Nash Equilibrium

這個 game 會收斂到 pure-strategy NE (r_2, c_2)。設 player1's belief (a, b)、player2's belief (c, d)，則 player1's payoff 為 (-a+b, 3b)、player2's payoff 為 (-c+d, 3d)，因 a, b, c, d 皆為非負的 float，兩 players 的 best response 永遠都是(r_2, c_2)。下圖為程式執行範例，左圖為起始點，右圖為結束點。

```
Iteration: 1
player1's belief: 110.56  889.44
player1's payoff: 778.89  2668.33

player2's belief: 640.14  359.86
player2's payoff: -280.28  1079.58

best response: 1 1
--------------------------------
Iteration: 2
player1's belief: 110.56  890.44
player1's payoff: 779.89  2671.33

player2's belief: 640.14  360.86
player2's payoff: -279.28  1082.58

best response: 1 1
--------------------------------
```

```
Iteration: 1000
player1's belief: 110.56  1888.44
player1's payoff: 1777.89  5665.33

player2's belief: 640.14  1358.86
player2's payoff: 718.72  4076.58

best response: 1 1
--------------------------------
Iteration: 1001
player1's belief: 110.56  1889.44
player1's payoff: 1778.89  5668.33

player2's belief: 640.14  1359.86
player2's payoff: 719.72  4079.58

best response: 1 1
--------------------------------
```

2. Two or More Pure-Strategy NE

這個 game 有可能收斂到兩個 pure-strategy NE (r_1, c_1)、(r_2, c_2)，也有可能在(r_1, c_2)、(r_2, c_1)之間一直跳，收斂到 mixed-strategy NE，$P(r_1) = \frac{1}{2}$、$P(r_2) = \frac{1}{2}$、$P(c_1) = \frac{1}{2}$、$P(c_2) = \frac{1}{2}$。

(1) 收斂到 pure-strategy NE (r_1, c_1)，自從 game 進入到(r_1, c_1)後，兩個 players 的 payoff 變動皆為(+2, 0)，這會使得接下來所有回合都選擇(r_1, c_1)。

```
Iteration: 1075
player1's belief: 520.28  1553.72
player1's payoff: 2594.28  4661.16

player2's belief: 537.50  1536.50
player2's payoff: 2611.50  4609.50

best response: 1 1
--------------------------------
Iteration: 1076
player1's belief: 520.28  1554.72
player1's payoff: 2595.28  4664.16

player2's belief: 537.50  1537.50
player2's payoff: 2612.50  4612.50

best response: 1 1
--------------------------------
```

```
Iteration: 1088
player1's belief: 1870.98  216.02
player1's payoff: 3957.98  648.05

player2's belief: 1543.24  543.76
player2's payoff: 3630.24  1631.27

best response: 0 0
--------------------------------
Iteration: 1089
player1's belief: 1871.98  216.02
player1's payoff: 3959.98  648.05

player2's belief: 1544.24  543.76
player2's payoff: 3632.24  1631.27

best response: 0 0
--------------------------------
```

(2) 收斂到 pure-strategy NE (r_2, c_2)，自從 game 進入到(r_2, c_2)後，兩個 players 的 payoff 變動皆為(+1, +3)，這使得接下來所有回合都選擇(r_2, c_2)。

```
Iteration: 1
player1's belief: 445.28  554.72
player1's payoff: 1445.28  1664.16

player2's belief: 537.50  462.50
player2's payoff: 1537.50  1387.50

best response: 1 0
--------------------------------
Iteration: 2
player1's belief: 446.28  554.72
player1's payoff: 1447.28  1664.16

player2's belief: 537.50  463.50
player2's payoff: 1538.50  1390.50

best response: 1 0
--------------------------------
```

```
Iteration: 1075
player1's belief: 520.28  1553.72
player1's payoff: 2594.28  4661.16

player2's belief: 537.50  1536.50
player2's payoff: 2611.50  4609.50

best response: 1 1
--------------------------------
Iteration: 1076
player1's belief: 520.28  1554.72
player1's payoff: 2595.28  4664.16

player2's belief: 537.50  1537.50
player2's payoff: 2612.50  4612.50

best response: 1 1
--------------------------------
```

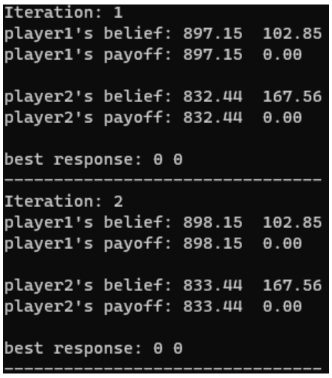
(3) 收斂到 mixed-strategy NE，如果起始 belief 接近$P(r_1)=\frac{1}{2}$、$P(r_2)=\frac{1}{2}$、$P(c_1)=\frac{1}{2}$、$P(c_2)=\frac{1}{2}$，但有些微不同，則會造成結果於(r_1, c_2)、(r_2, c_1)之間一直跳。因為各 player 之內的初始 payoff 差值小於 1，在 best response 為(r_1, c_2)、(r_2, c_1)狀況下，都是造成比較小的 payoff 再加 1，這樣上一輪 payoff 較大的一方在下一輪會變成比較小，如此不斷循環，最後結果為 mixed-strategy NE。

```
Iteration: 1
player1's belief: 500.01  499.99
player1's payoff: 1500.01  1499.97

player2's belief: 499.96  500.04
player2's payoff: 1499.96  1500.12

best response: 0 1
--------------------------------
Iteration: 2
player1's belief: 500.01  500.99
player1's payoff: 1501.01  1502.97

player2's belief: 500.96  500.04
player2's payoff: 1501.96  1500.12

best response: 1 0
--------------------------------
```

```
Iteration: 2999
player1's belief: 1999.01  1998.99
player1's payoff: 5997.01  5996.97

player2's belief: 1998.96  1999.04
player2's payoff: 5996.96  5997.12

best response: 0 1
--------------------------------
Iteration: 3000
player1's belief: 1999.01  1999.99
player1's payoff: 5998.01  5999.97

player2's belief: 1999.96  1999.04
player2's payoff: 5998.96  5997.12

best response: 1 0
--------------------------------
```

3. Two or More Pure-Strategy NE (Conti.)

這個 game 會收斂到 pure-strategy NE (r_1, c_1)。設 player1's belief (a, b)、player2's belief (c, d)，則 player1's payoff 為 (a, 0)、player2's payoff 為 (c, 0)，因 a, c 為非負的 float，players 的 best response 會很快的收斂到(r_1, c_1)。



```
Iteration: 1
player1's belief: 897.15  102.85
player1's payoff: 897.15  0.00

player2's belief: 832.44  167.56
player2's payoff: 832.44  0.00

best response: 0 0
--------------------------------
Iteration: 2
player1's belief: 898.15  102.85
player1's payoff: 898.15  0.00

player2's belief: 833.44  167.56
player2's payoff: 833.44  0.00

best response: 0 0
--------------------------------
```

```
Iteration: 1000
player1's belief: 1896.15  102.85
player1's payoff: 1896.15  0.00

player2's belief: 1831.44  167.56
player2's payoff: 1831.44  0.00

best response: 0 0
--------------------------------
Iteration: 1001
player1's belief: 1897.15  102.85
player1's payoff: 1897.15  0.00

player2's belief: 1832.44  167.56
player2's payoff: 1832.44  0.00

best response: 0 0
--------------------------------
```

4. Mixed-Strategy Nash Equilibrium

這個 game 會收斂到 mixed-strategy NE，$P(r_1)=\frac{4}{5}$、$P(r_2)=\frac{1}{5}$、$P(c_1)=\frac{1}{2}$、$P(c_2)=\frac{1}{2}$。Best response 為(r_1, c_2)時，player1 的 payoff 變動為(+2, 0)、player2 的 payoff 變動為(+1, 0)，直至 player2 best response 變為 c_1。在 best response 為(r_1, c_1)，player1 的 payoff 變動為(0, +2)、player2 的 payoff 變動為(+1, 0)，直至 player1 best response 變

為 r_2。在 best response 為(r_2, c_1)， player1 的 payoff 變動為(0, +2)、player2 的 payoff 變動為(0, +4)，直至 player2 best response 變為 c_2。在 best response 為(r_2, c_2)，player1 的 payoff 變動為(+2, 0)、player2 的 payoff 變動為(0, +4)，直至 player1 best response 變為 r_1。所以又回到了(r_1, c_2)，進入無限輪迴。依據以上四種 best response 下 payoff 的變動程度，可得 best response 機率為$P(r_1)=\frac{4}{5}$、$P(r_2)=\frac{1}{5}$、$P(c_1)=\frac{1}{2}$、$P(c_2)=\frac{1}{2}$。

```
Iteration: 1
player1's belief: 724.14  275.86
player1's payoff: 551.73  1448.27

player2's belief: 483.24  516.76
player2's payoff: 483.24  2067.06

best response: 1 1
-------------------------------
Iteration: 2
player1's belief: 724.14  276.86
player1's payoff: 553.73  1448.27

player2's belief: 483.24  517.76
player2's payoff: 483.24  2071.06

best response: 1 1
-------------------------------
```

```
Iteration: 2999
player1's belief: 724.14  3273.86
player1's payoff: 6547.73  1448.27

player2's belief: 3032.24  965.76
player2's payoff: 3032.24  3863.06

best response: 0 1
-------------------------------
Iteration: 3000
player1's belief: 724.14  3274.86
player1's payoff: 6549.73  1448.27

player2's belief: 3033.24  965.76
player2's payoff: 3033.24  3863.06

best response: 0 1
-------------------------------
```

5. Best-Reply Path

這個 game 會收斂到 mixed-strategy NE，$P(r_1)=\frac{1}{2}$、$P(r_2)=\frac{1}{2}$、$P(c_1)=\frac{1}{2}$、$P(c_2)=\frac{1}{2}$。Best response 為(r_1, c_1)時，player1 的 payoff 變動為(0, +1)、player2 的 payoff 變動為(+1, 0)，直至 player1 best response 變為 r_2。在 best response 為(r_2, c_1)，player1 的 payoff 變動為(0, +1)、player2 的 payoff 變動為(0, +1)，直至 player2 best response 變為 c_2。在 best response 為(r_2, c_2)，player1 的 payoff 變動為(+1, 0)、player2 的 payoff 變動為(0, +1)，直至 player1 best response 變為 r_1。在 best response 為(r_1, c_2)，player1 的 payoff 變動為(+1, 0)、player2 的 payoff 變動為(+1, 0)，直至 player2 best response 變為 c_1。所以又回到了(r_1, c_1)，進入無限輪迴。依據以上四種 best response 下 payoff 的變動程度，可得 best response 機率為$P(r_1)=\frac{1}{2}$、$P(r_2)=\frac{1}{2}$、$P(c_1)=\frac{1}{2}$、$P(c_2)=\frac{1}{2}$。

```
player1's belief: 448.75  551.25
player1's payoff: 551.25  448.75

player2's belief: 537.56  462.46
player2's payoff: 537.56  462.46

best response: 0 0
--------------------------------
Iteration: 2
player1's belief: 449.75  551.25
player1's payoff: 551.25  449.75

player2's belief: 538.56  462.46
player2's payoff: 538.56  462.46

best response: 0 0
```

```
Iteration: 2999
player1's belief: 1954.75  2043.25
player1's payoff: 2043.25  1954.75

player2's belief: 2051.56  1946.46
player2's payoff: 2051.56  1946.46

best response: 0 0
--------------------------------
Iteration: 3000
player1's belief: 1955.75  2043.25
player1's payoff: 2043.25  1955.75

player2's belief: 2052.56  1946.46
player2's payoff: 2052.56  1946.46

best response: 0 0
--------------------------------
```

6. Pure-Coordination Game

這個 game 有可能收斂到兩個 pure-strategy NE (r_1, c_1)、(r_2, c_2)，也有可能在(r_1, c_2)、(r_2, c_1)之間一直跳，收斂到 mixed-strategy NE，$P(r_1)=\frac{1}{2}$、$P(r_2)=\frac{1}{2}$、$P(c_1)=\frac{1}{2}$、$P(c_2)=\frac{1}{2}$。

(1) 收斂到 pure-strategy NE (r_1, c_1)，自從 game 進入到(r_1, c_1)後，兩個 players 的 payoff 變動皆為(+10, 0)，這會使得接下來所有回合都選擇(r_1, c_1)。

```
Iteration: 1
player1's belief: 376.12  623.88
player1's payoff: 3761.23  6238.77

player2's belief: 906.70  93.30
player2's payoff: 9066.98  933.02

best response: 1 0
--------------------------------
Iteration: 2
player1's belief: 377.12  623.88
player1's payoff: 3771.23  6238.77

player2's belief: 906.70  94.30
player2's payoff: 9066.98  943.02

best response: 1 0
```

```
Iteration: 1248
player1's belief: 1623.12  623.88
player1's payoff: 16231.23  6238.77

player2's belief: 1905.70  341.30
player2's payoff: 19056.98  3413.02

best response: 0 0
--------------------------------
Iteration: 1249
player1's belief: 1624.12  623.88
player1's payoff: 16241.23  6238.77

player2's belief: 1906.70  341.30
player2's payoff: 19066.98  3413.02

best response: 0 0
```

(2) 收斂到 pure-strategy NE (r_2, c_2)，自從 game 進入到(r_2, c_2)後，兩個 players 的 payoff 變動皆為(0, +10)，這使得接下來所有回合都選擇(r_2, c_2)。

```
Iteration: 1
player1's belief: 23.70  976.30
player1's payoff: 236.97  9763.03

player2's belief: 888.39  111.61
player2's payoff: 8883.88  1116.12

best response: 1 0
--------------------------------
Iteration: 2
player1's belief: 24.70  976.30
player1's payoff: 246.97  9763.03

player2's belief: 888.39  112.61
player2's payoff: 8883.88  1126.12

best response: 1 0
```

```
Iteration: 1777
player1's belief: 800.70  1975.30
player1's payoff: 8006.97  19753.03

player2's belief: 888.39  1887.61
player2's payoff: 8883.88  18876.12

best response: 1 1
--------------------------------
Iteration: 1778
player1's belief: 800.70  1976.30
player1's payoff: 8006.97  19763.03

player2's belief: 888.39  1888.61
player2's payoff: 8883.88  18886.12

best response: 1 1
```

(3) 收斂到 mixed-strategy NE，如果起始 belief 接近$P(r_1)=\frac{1}{2}$、$P(r_2)=\frac{1}{2}$、$P(c_1)=\frac{1}{2}$、$P(c_2)=\frac{1}{2}$，但有些微不同，則會造成結果於(r_1, c_2)、

(r_2, c_1)之間一直跳。因為各 player 之內的初始 payoff 差值小於 10，在 best response 為(r_1, c_2)、(r_2, c_1)狀況下，都是造成比較小的 payoff 再加 10，這樣上一輪 payoff 較大的一方在下一輪會變成比較小，如此不斷循環，最後結果為 mixed-strategy NE。

```
Iteration: 1
player1's belief: 500.01  499.99
player1's payoff: 5000.10  4999.90

player2's belief: 499.96  500.04
player2's payoff: 4999.60  5000.40

best response: 0 1
--------------------------------
Iteration: 2
player1's belief: 500.01  500.99
player1's payoff: 5000.10  5009.90

player2's belief: 500.96  500.04
player2's payoff: 5009.60  5000.40

best response: 1 0
```

```
Iteration: 2999
player1's belief: 1999.01  1998.99
player1's payoff: 19990.10  19989.90

player2's belief: 1998.96  1999.04
player2's payoff: 19989.60  19990.40

best response: 0 1
--------------------------------
Iteration: 3000
player1's belief: 1999.01  1999.99
player1's payoff: 19990.10  19999.90

player2's belief: 1999.96  1999.04
player2's payoff: 19999.60  19990.40

best response: 1 0
```

7. Anti-Coordination Game

這個 game 有可能收斂到兩個 pure-strategy NE (r_1, c_2)、(r_2, c_1)，也有可能在(r_1, c_1)、(r_2, c_2)之間一直跳，收斂到 mixed-strategy NE，$P(r_1) = \frac{1}{2}$、$P(r_2) = \frac{1}{2}$、$P(c_1) = \frac{1}{2}$、$P(c_2) = \frac{1}{2}$。

(1) 收斂到 pure-strategy NE (r_1, c_2)，自從 game 進入到(r_1, c_2)後，player1 的 payoff 變動為(+1, 0)，player2 的 payoff 變動為(0, +1)，這會使得接下來所有回合都選擇(r_1, c_2)。

```
Iteration: 1
player1's belief: 25.77  974.23
player1's payoff: 974.23  25.77

player2's belief: 345.55  654.45
player2's payoff: 654.45  345.55

best response: 0 0
--------------------------------
Iteration: 2
player1's belief: 26.77  974.23
player1's payoff: 974.23  26.77

player2's belief: 346.55  654.45
player2's payoff: 654.45  346.55

best response: 0 0
```

```
Iteration: 1309
player1's belief: 334.77  1973.23
player1's payoff: 1973.23  334.77

player2's belief: 1653.55  654.45
player2's payoff: 654.45  1653.55

best response: 0 1
--------------------------------
Iteration: 1310
player1's belief: 334.77  1974.23
player1's payoff: 1974.23  334.77

player2's belief: 1654.55  654.45
player2's payoff: 654.45  1654.55

best response: 0 1
```

(2) 收斂到 pure-strategy NE (r_2, c_1)，自從 game 進入到(r_2, c_1)後，player1 的 payoff 變動為(0, +1)，player2 的 payoff 變動為(+1, 0)，這使得接下來所有回合都選擇(r_2, c_1)。

```
Iteration: 1
player1's belief: 619.47  380.53
player1's payoff: 380.53  619.47

player2's belief: 592.20  407.80
player2's payoff: 407.80  592.20

best response: 1 1
--------------------------------
Iteration: 2
player1's belief: 619.47  381.53
player1's payoff: 381.53  619.47

player2's belief: 592.20  408.80
player2's payoff: 408.80  592.20

best response: 1 1
```

```
Iteration: 1185
player1's belief: 1618.47  565.53
player1's payoff: 565.53  1618.47

player2's belief: 592.20  1591.80
player2's payoff: 1591.80  592.20

best response: 1 0
--------------------------------
Iteration: 1186
player1's belief: 1619.47  565.53
player1's payoff: 565.53  1619.47

player2's belief: 592.20  1592.80
player2's payoff: 1592.80  592.20

best response: 1 0
```

(3) 收斂到 mixed-strategy NE，如果起始 belief 接近$P(r_1) = \frac{1}{2}$、$P(r_2) = \frac{1}{2}$、$P(c_1) = \frac{1}{2}$、$P(c_2) = \frac{1}{2}$，但有些微不同，則會造成結果於(r_1, c_1)、(r_2, c_2)之間一直跳。因為各 player 之內的初始 payoff 差值小於 1，在 best response 為(r_1, c_1)、(r_2, c_2)狀況下，都是造成比較小的 payoff 再加 1，這樣上一輪 payoff 較大的一方在下一輪會變成比較小，如此不斷循環，最後結果為 mixed-strategy NE。

```
Iteration: 1
player1's belief: 500.01  499.99
player1's payoff: 499.99  500.01

player2's belief: 500.05  499.95
player2's payoff: 499.95  500.05

best response: 1 1
--------------------------------
Iteration: 2
player1's belief: 500.01  500.99
player1's payoff: 500.99  500.01

player2's belief: 500.05  500.95
player2's payoff: 500.95  500.05

best response: 0 0
```

```
Iteration: 2999
player1's belief: 1999.01  1998.99
player1's payoff: 1998.99  1999.01

player2's belief: 1999.05  1998.95
player2's payoff: 1998.95  1999.05

best response: 1 1
--------------------------------
Iteration: 3000
player1's belief: 1999.01  1999.99
player1's payoff: 1999.99  1999.01

player2's belief: 1999.05  1999.95
player2's payoff: 1999.95  1999.05

best response: 0 0
```

8. Battle of the Sexes

這個 game 有可能收斂到兩個 pure-strategy NE (r_1, c_1)、(r_2, c_2)，也有可能在(r_1, c_2)、(r_2, c_1)之間一直跳，收斂到 mixed-strategy NE，$P(r_1) = \frac{3}{5}$、$P(r_2) = \frac{2}{5}$、$P(c_1) = \frac{2}{5}$、$P(c_2) = \frac{3}{5}$。

(1) 收斂到 pure-strategy NE (r_1, c_1)，自從 game 進入到(r_1, c_1)後，player1 的 payoff 變動為(+3, 0)，player2 的 payoff 變動為(+2, 0)，這會使得接下來所有回合都選擇(r_1, c_1)。

```
Iteration: 1
player1's belief: 380.09  619.91
player1's payoff: 1140.27  1239.82

player2's belief: 943.60  56.40
player2's payoff: 1887.20  169.20

best response: 1 0
--------------------------------
Iteration: 2
player1's belief: 381.09  619.91
player1's payoff: 1143.27  1239.82

player2's belief: 943.60  57.40
player2's payoff: 1887.20  172.20

best response: 1 0
```

```
Iteration: 1034
player1's belief: 1413.09  619.91
player1's payoff: 4239.27  1239.82

player2's belief: 1942.60  90.40
player2's payoff: 3885.20  271.20

best response: 0 0
--------------------------------
Iteration: 1035
player1's belief: 1414.09  619.91
player1's payoff: 4242.27  1239.82

player2's belief: 1943.60  90.40
player2's payoff: 3887.20  271.20

best response: 0 0
```

(2) 收斂到 pure-strategy NE (r_2, c_2)，自從 game 進入到(r_2, c_2)後，player1 的 payoff 變動為(0, +2)，player2 的 payoff 變動為(0, +3)，這使得接下來所有回合都選擇(r_2, c_2)。

```
Iteration: 1
player1's belief: 102.21  897.79
player1's payoff: 306.62  1795.59

player2's belief: 816.18  183.82
player2's payoff: 1632.36  551.46

best response: 1 0
--------------------------------
Iteration: 2
player1's belief: 103.21  897.79
player1's payoff: 309.62  1795.59

player2's belief: 816.18  184.82
player2's payoff: 1632.36  554.46

best response: 1 0
```

```
Iteration: 1361
player1's belief: 463.21  1896.79
player1's payoff: 1389.62  3793.59

player2's belief: 816.18  1543.82
player2's payoff: 1632.36  4631.46

best response: 1 1
--------------------------------
Iteration: 1362
player1's belief: 463.21  1897.79
player1's payoff: 1389.62  3795.59

player2's belief: 816.18  1544.82
player2's payoff: 1632.36  4634.46

best response: 1 1
```

(3) 收斂到 mixed-strategy NE，如果起始 belief 接近$P(r_1)=\frac{3}{5}$、$P(r_2)=\frac{2}{5}$、$P(c_1)=\frac{2}{5}$、$P(c_2)=\frac{3}{5}$，但有些微不同，則會造成結果於(r_1, c_2)、(r_2, c_1)之間一直跳。因為各 player 之內的初始 payoff 差值小於 2，在 best response 為(r_1, c_2)、(r_2, c_1)狀況下，都是造成比較小的 payoff 再加 2 或 3，這樣上一輪 payoff 較大的一方在下一輪會變成比較小，如此不斷循環，最後結果為 mixed-strategy NE。

```
Iteration: 1
player1's belief: 399.92  600.08
player1's payoff: 1199.76  1200.16

player2's belief: 600.03  399.97
player2's payoff: 1200.06  1199.91

best response: 1 0
--------------------------------
Iteration: 2
player1's belief: 400.92  600.08
player1's payoff: 1202.76  1200.16

player2's belief: 600.03  400.97
player2's payoff: 1200.06  1202.91

best response: 0 1
```

```
Iteration: 2999
player1's belief: 1598.92  2399.08
player1's payoff: 4796.76  4798.16

player2's belief: 2399.03  1598.97
player2's payoff: 4798.06  4796.91

best response: 1 0
--------------------------------
Iteration: 3000
player1's belief: 1599.92  2399.08
player1's payoff: 4799.76  4798.16

player2's belief: 2399.03  1599.97
player2's payoff: 4798.06  4799.91

best response: 0 1
```

9. Stag Hunt Game

這個 game 有可能收斂到兩個 pure-strategy NE (r_1, c_1)、(r_2, c_2)，也有可能在(r_1, c_2)、(r_2, c_1)之間一直跳，收斂到 mixed-strategy NE，$P(r_1)=\frac{1}{2}$、$P(r_2)=\frac{1}{2}$、$P(c_1)=\frac{1}{2}$、$P(c_2)=\frac{1}{2}$。

(1) 收斂到 pure-strategy NE (r_1, c_1)，自從 game 進入到(r_1, c_1)後，兩個 players 的 payoff 變動皆為(+3, +2)，這會使得接下來所有回合都選擇(r_1, c_1)。

```
Iteration: 1
player1's belief: 976.86  23.14
player1's payoff: 2930.58  1976.86

player2's belief: 647.41  352.59
player2's payoff: 1942.23  1647.41

best response: 0 0
--------------------------------
Iteration: 2
player1's belief: 977.86  23.14
player1's payoff: 2933.58  1978.86

player2's belief: 648.41  352.59
player2's payoff: 1945.23  1649.41

best response: 0 0
```

```
Iteration: 1000
player1's belief: 1975.86  23.14
player1's payoff: 5927.58  3974.86

player2's belief: 1646.41  352.59
player2's payoff: 4939.23  3645.41

best response: 0 0
--------------------------------
Iteration: 1001
player1's belief: 1976.86  23.14
player1's payoff: 5930.58  3976.86

player2's belief: 1647.41  352.59
player2's payoff: 4942.23  3647.41

best response: 0 0
```

(2) 收斂到 pure-strategy NE (r_2, c_2)，自從 game 進入到(r_2, c_2)後，兩個 players 的 payoff 變動皆為(0, +1)，這使得接下來所有回合都選擇(r_2, c_2)。

```
Iteration: 1
player1's belief: 273.71  726.29
player1's payoff: 821.13  1273.71

player2's belief: 654.24  345.76
player2's payoff: 1962.71  1654.24

best response: 1 0
--------------------------------
Iteration: 2
player1's belief: 274.71  726.29
player1's payoff: 824.13  1275.71

player2's belief: 654.24  346.76
player2's payoff: 1962.71  1655.24

best response: 1 0
```

```
Iteration: 1309
player1's belief: 582.71  1725.29
player1's payoff: 1748.13  2890.71

player2's belief: 654.24  1653.76
player2's payoff: 1962.71  2962.24

best response: 1 1
--------------------------------
Iteration: 1310
player1's belief: 582.71  1726.29
player1's payoff: 1748.13  2891.71

player2's belief: 654.24  1654.76
player2's payoff: 1962.71  2963.24

best response: 1 1
```

(3) 收斂到 mixed-strategy NE，如果起始 belief 接近$P(r_1)=\frac{1}{2}$、$P(r_2)=\frac{1}{2}$、$P(c_1)=\frac{1}{2}$、$P(c_2)=\frac{1}{2}$，但有些微不同，則會造成結果於(r_1, c_2)、(r_2, c_1)之間一直跳。因為各 player 之內的初始 payoff 差值小於 1，在 best response 為(r_1, c_2)時，player1 payoff 變動為(0, +1)，player2 payoff 變動為(+3, +2)；在 best response 為(r_2, c_1)時，player1 payoff 變動為(+3, +2)，player2 payoff 變動為(0, +1)。可以觀察出所有 cases 都是較小的 payoff 比較大 payoff 再多加 1，這樣上一輪 payoff 較大的一方在下一輪會變成比較小，如此不斷循環，最後結果為 mixed-strategy NE。

```
Iteration: 1
player1's belief: 499.94  500.06
player1's payoff: 1499.82  1499.94

player2's belief: 500.05  499.95
player2's payoff: 1500.15  1500.05

best response: 1 0
--------------------------------
Iteration: 2
player1's belief: 500.94  500.06
player1's payoff: 1502.82  1501.94

player2's belief: 500.05  500.95
player2's payoff: 1500.15  1501.05

best response: 0 1
```

```
Iteration: 2999
player1's belief: 1998.94  1999.06
player1's payoff: 5996.82  5996.94

player2's belief: 1999.05  1998.95
player2's payoff: 5997.15  5997.05

best response: 1 0
--------------------------------
Iteration: 3000
player1's belief: 1999.94  1999.06
player1's payoff: 5999.82  5998.94

player2's belief: 1999.05  1999.95
player2's payoff: 5997.15  5998.05

best response: 0 1
```

10. Observation and Conclusion

Fictious play 不是永遠都可以找到 NE，假設 utility matrix 如下圖。

	c_1	c_2	c_3
r_1	0, 1	1, 0	0, 0
r_2	1, 0	0, 1	0, 0
r_3	0, 0	0, 0	0.5, 0.5

pure-strategy NE 為(r_3, c_3)，mixed-strategy NE 為$P(r_1) = \frac{1}{3}$、$P(r_2) = \frac{1}{3}$、$P(r_3) = \frac{1}{3}$、$P(c_1) = \frac{1}{3}$、$P(c_2) = \frac{1}{3}$、$P(c_3) = \frac{1}{3}$。

一旦 fictious play 進入了(r_1, c_1)、(r_2, c_1)、(r_2, c_2)、(r_1, c_2)任何一個 state，他就會一直無限的被困在這個 infinite-loop 中，沒有辦法到達 pure-strategy NE(r_3, c_3)，同時他也不會收斂到$P(r_1) = \frac{1}{3}$、$P(r_2) = \frac{1}{3}$、$P(r_3) = \frac{1}{3}$、$P(c_1) = \frac{1}{3}$、$P(c_2) = \frac{1}{3}$、$P(c_3) = \frac{1}{3}$。