The Forgotten Mathematical Masterpiece: How Euler Explained Music with Mathematics? Introduction to Tentamen Novae Theoriae Musicae

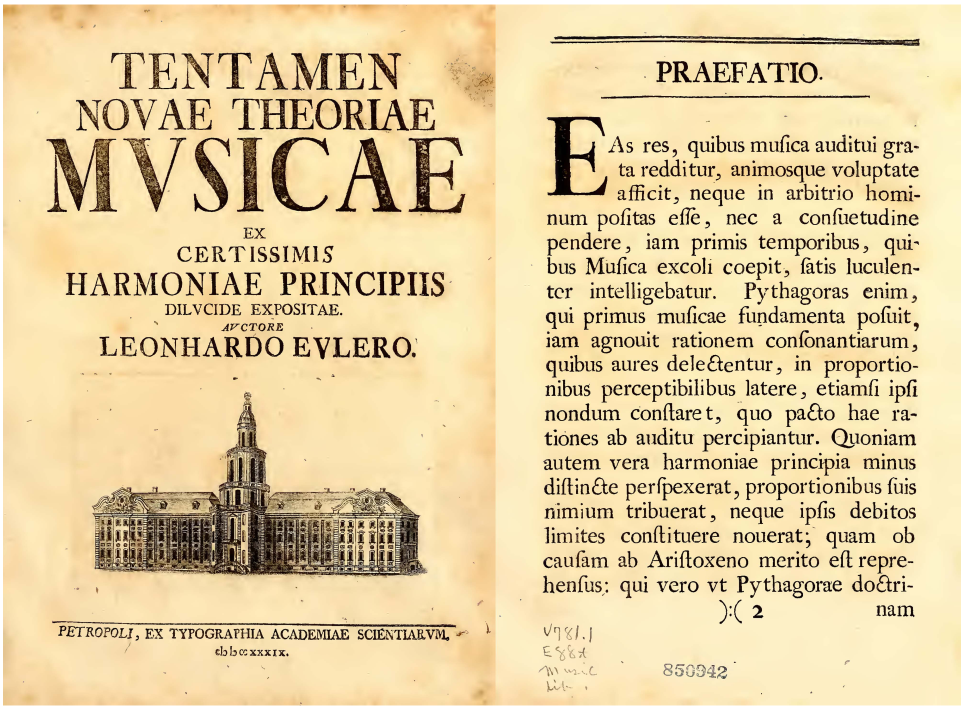
> **Attention**: This article was originally written in Chinese and published on this website. The English version was translated using GPT-4o. Apologies, as I am not a professional columnist and did not have enough time to personally translate or thoroughly proofread the content. I hope this does not affect your reading experience. If you notice any language or content errors, or have any suggestions for improvement, feel free to contact me at liaoziqiang7@gmail.com.

1. Introduction

Anyone with an interest in mathematics is undoubtedly familiar with the legendary name Leonhard Euler. Known as the mathematician who derived the famous Euler's formula $e^{i\pi} + 1 = 0$, Euler is widely regarded as one of the greatest mathematicians of all time. His contributions spanned almost every field of mathematics and extended to physics, engineering, economics, and more. It's hard to find a subject without a formula named after Euler.

> **Image**: Not Leonhard Euler conducting music theory research

However, few people are aware that Euler also ventured into the realm of music theory. In 1739, at the age of 32, Euler authored *Tentamen Novae Theoriae Musicae: Ex Certissimis Harmoniae Principiis Dilucide Expositae* ("An Attempt at a New Theory of Music: Clearly Explained Based on the Most Certain Principles of Harmony"), in which he sought to incorporate music theory into the framework of mathematics. Unfortunately, this book did not garner much attention, and little material about it exists today. Fortunately, some academic papers have studied this work, and an independent researcher, the host of [17th Century Math], has even translated the book from Latin into English, making it accessible to general readers. The Latin original can also be downloaded from [this website].



TENTAMEN
NOVAE THEORIAE
MVSICAE
EX
CERTISSIMIS
HARMONIAE PRINCIPIIS
DILVCIDE EXPOSITAE.
AVCTORE
LEONHARDO EVLERO.

PETROPOLI, EX TYPOGRAPHIA ACADEMIAE SCIENTIARVM.
cIↃIↃCCXXXIX.

PRAEFATIO.

EAs res, quibus muſica auditui grata redditur, animosque voluptate afficit, neque in arbitrio hominum poſitas eſſe, nec a conſuetudine pendere, iam primis temporibus, quibus Muſica excoli coepit, ſatis luculenter intelligebatur. Pythagoras enim, qui primus muſicae fundamenta poſuit, iam agnouit rationem conſonantiarum, quibus aures delectentur, in proportionibus perceptibilibus latere, etiamſi ipſi nondum conſtaret, quo pacto hae rationes ab auditu percipiantur. Quoniam autem vera harmoniae principia minus diſtincte perſpexerat, proportionibus ſuis nimium tribuerat, neque ipſis debitos limites conſtituere nouerat; quam ob cauſam ab Ariſtoxeno merito eſt reprehenſus: qui vero vt Pythagorae doctri-

):(2 nam

The connection between music and mathematics has a long history. From Pythagoras's discovery of the relationship between intervals and string lengths, to the ancient Chinese *San Fen Sun Yi* method, to the calculation of the twelve-tone equal temperament, mathematics has always provided a theoretical foundation for music. Following the Renaissance, the rise of rationalism inspired both mathematicians and musicians to describe music using mathematical language. Euler was one of them. Drawing inspiration from ancient Greek philosophy and Baroque music, he used mathematics to study scales, intervals, and harmony. He proposed a new system that balances various tuning methods, mathematically describing harmony and modes, and even offering guidance for composition.

As a mere enthusiast with limited knowledge of both music and mathematics, I can only provide a superficial exploration of some parts of *Tentamen Novae Theoriae Musicae*. This article is more of a curiosity-driven endeavor, aimed at sparking interest in this fascinating book. I hope to inspire more readers in the Chinese-speaking internet community to delve into this forgotten masterpiece and uncover its wonders.

> For a better reading experience, visit this webpage. Don't forget to leave a like and bookmark before you go!

2. Overview of the Book

Although this article focuses on just a few simple and intriguing aspects of the book, given the near-total absence of introductions to it, I will first provide a general overview of the book's structure. The table of contents of the Latin original is as follows (page numbers correspond to the Latin edition):

I think the book can be roughly divided into the following parts:

1. The first part consists of Chapters 1 and 3, which primarily explore the concepts of sound and music, providing the necessary background knowledge.
2. The second part includes Chapters 2, 4, and 7, focusing on the issue of harmonic intervals and their aesthetic qualities.
3. The third part spans Chapters 5 and 6, which expand the scope from individual chords to entire musical compositions.
4. The fourth part, comprising Chapters 8, 9, and 10, sees Euler applying mathematical methods to generate musical scales.
5. The fifth part consists of Chapters 11, 12, 13, and 14, where Euler builds a new theoretical framework for music, offering guidance on tuning, composition, and more.

Although this book is only about 300 pages long—a mere drop in the ocean compared to Euler's total body of work—it contains enough brilliant ideas to keep an ordinary person busy researching for a year or two. This article will guide readers through the fascinating ways in which Euler used mathematics to explain music theory during the Baroque period.

3. The Mathematical Principles of Harmony and Pleasure

This section corresponds primarily to Chapters 2 and 4 of the original book. To begin, let us clarify: Euler's theory of music focuses on only two dimensions—**pitch** and **duration (rhythm)**. While factors such as volume may also influence musical effect and exhibit some patterns, Euler considered volume to be highly subjective and arbitrary, and thus excluded it from his system.

This reminds me of the Bilibili content creator Leyou Wang, who, in one of his videos, defined the concept of "texture." He explained that texture refers to how one organizes sound to create rich sonorities when tone color cannot be controlled. Considering the significant limitations of the harpsichord in terms of volume control during Bach's era, one could argue that pitch and rhythm form the **skeleton** of music, while tone color and volume serve as its **flesh and blood**. Euler's goal was to strip away the "flesh and blood" and focus solely on the "skeleton" of music.

3.1 Harmony: The Hidden Order Within Notes

> All pleasure in music stems from the perception of ratios, which exist among numbers, as the duration of time can also be expressed numerically.
> — Euler

In tonal music, whether a piece "sounds good" involves at least two key factors from a harmonic perspective: the **consonance** of the harmony itself and the **smoothness of harmonic transitions**. For now, we focus exclusively on the first factor. It is well known that the degree of consonance varies significantly between intervals. For example, a perfect fifth is highly consonant, whereas a minor second or major seventh sounds much harsher. Similarly, different chords evoke different feelings: a major triad (C-E-G) is stable and pleasing, while a diminished seventh chord (B-D-F-A♭) is full of tension. Euler believed that the essence of harmony lies in the sense of "**order**" hidden within sounds, and this sense of order brings pleasure. This order, according to Euler, is neither purely subjective nor a phenomenon specific to certain cultures but a universal principle that can be mathematically described.

This theory was not invented by Euler but has ancient origins. Readers with some knowledge of music theory may already be familiar with it. For example, when the frequencies of two notes are in a 2:1 ratio (i.e., one frequency is double the other), they form an octave. This proportional relationship is simple, and humans perceive this sense of order most clearly. Similarly, the 3:2 ratio (i.e., a perfect fifth) is also considered consonant because it represents another simple mathematical relationship. In general, harmonious sounds are perceived as pleasing because their frequencies exhibit **simple, clear integer ratios**. The simpler the ratio, the more consonant the sound; the more complex the ratio (approaching irrational numbers), the harsher it sounds.

Euler further emphasized that harmony and pleasure are related but not identical. In his book, Euler wrote:

> It is thus evident that what brings us pleasure is not the same as what makes us laugh, and what makes us feel sadness is not the same as what displeases us. We have, to some extent, explained the reasoning behind this: anything that allows us to perceive order will bring pleasure. Among such things, those with simpler and more easily comprehensible order will evoke joy, while those with more complex and less easily discernible order will often evoke sadness.

Here, it is clear that Euler's concept of "pleasure" does not exclusively refer to joy but also encompasses feelings such as being "moved," "captivated," or experiencing "beauty." In summary, harmonious sounds evoke joy, while dissonant sounds evoke sadness—provided they adhere to some form of order. The difference lies in how easily this order can be perceived. Sounds that lack any order altogether, however, elicit aversion. As Euler stated:

> If we cannot perceive order in something, our sense of pleasure diminishes; if we perceive no order at all, we will no longer find the object appealing. Moreover, if we find that something not only lacks order but actively disrupts any potential order, violating reason itself, we will experience aversion, even to the point of discomfort.

Of course, after tonal music reached its peak in the 20th century, Schoenberg proposed the system of atonal music, which broke away from these long-held principles. However, that is beyond the scope of our discussion here.

Naturally, as a mathematician, Euler did not stop at conceptual exploration. In his book, he proposed a **formula to measure dissonance**, which can be used to quantify the degree of consonance for any chord. The formula is as follows:

$$E(n) = 1 + \sum_{k=1}^{r} a_k (p_k - 1)$$

Where:

- $E(n)$ represents the degree of dissonance, and n is the least common multiple (LCM) of the frequency ratios.
- n is expressed as a product of prime factors: $n = p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r}$, where p_k are distinct prime numbers, and $a_k \geq 1$.

Let us consider two examples:

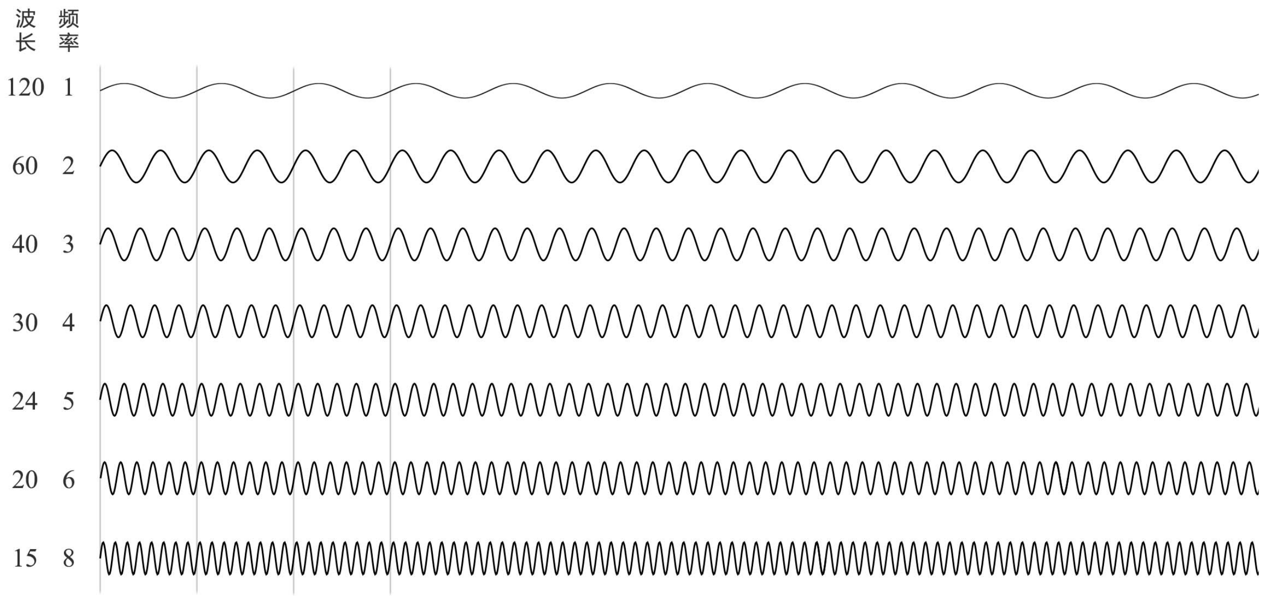
1. **Perfect Fifth**: Ratio is $3 : 2$.
   - LCM: $\mathrm{LCM}(3, 2) = 6$.
   - Prime factorization: $6 = 2^1 \cdot 3^1$.
   - Calculation: $E(6) = 1 + 1 \cdot (2 - 1) + 1 \cdot (3 - 1) = 1 + 1 + 2 = 4$.
2. **Major Third**: Ratio is $5 : 4$.
   - LCM: $\mathrm{LCM}(5, 4) = 20$.
   - Prime factorization: $20 = 2^2 \cdot 5^1$.
   - Calculation: $E(20) = 1 + 2 \cdot (2 - 1) + 1 \cdot (5 - 1) = 1 + 2 + 4 = 7$.

This algorithm can be implemented in Python, but we won't demonstrate it here. Interested readers can download the code from the GitHub repository and run it themselves for verification.

3.1.1 Least Common Multiple and Dissonance

We already know that simpler ratios tend to sound more harmonious. But how can we quantify this "simplicity"? Assume the ratio is expressed in its simplest, irreducible form. Intuitively, we might think the sum of the numerator and denominator reflects its complexity. For example, for the ratio $1/2$, the sum of the numerator and denominator is $1 + 2 = 3$; for $1/3$, it is $1 + 3 = 4$; and for $9/11$, it is $9 + 11 = 20$. Clearly, the more complex the ratio, the higher the sum of the numerator and denominator. Alternatively, one could use the product of the numerator and denominator instead of their sum.

However, Euler introduced a different metric with greater extensibility and physical significance: the **least common multiple (LCM)**. The first advantage of this method is that it not only applies to the frequency ratio of two notes but can also be easily extended to multiple notes. Moreover, it is closely related to the physical nature of sound. Here's a possibly imperfect analogy: sound is essentially the vibration of waves. Suppose the frequency ratio of two notes is $2 : 3$. This can be likened to two balls bouncing: the first ball bounces every 3 seconds, and the second bounces every 2 seconds (note that the period is inversely related to the frequency). When will they land together? The answer is after 6 seconds, which is the least common multiple of their frequencies. Thus, the **LCM quantitatively describes the length of time it takes for the waveforms of multiple notes to "align"**. To make this concept more intuitive, we can refine the image Euler provided in his book:



A more complex example is $4:5:6$, where the least common multiple is 60, meaning the waveforms of these three notes align only once every 60 time units. The principle might be the following: during this period, the phases of the sound waves are mostly misaligned, with varying degrees of misalignment in different intervals. As a result, within these 60 time units, the waveform of the combined sound is complex and irregular, producing a composite waveform that appears intricate. In fact, for the composite waveform function, this 60 corresponds to its period (which is proportional to the period; the exact value requires frequency calculations).

3.1.2 Prime Factorization and Dissonance

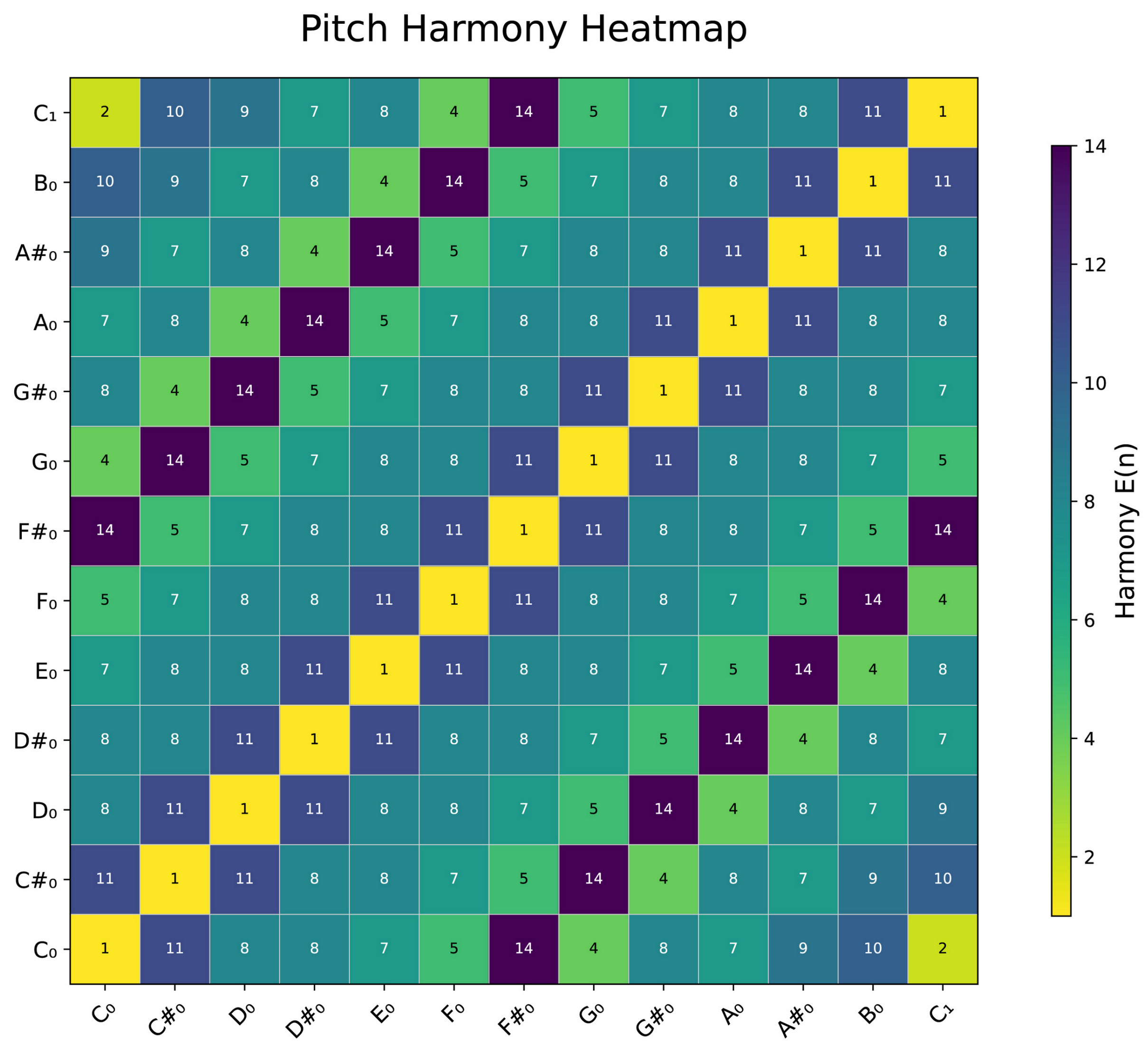
So, does the least common multiple allow us to measure the dissonance of a chord? The answer is no. Euler argued that prime factorization is also required (perhaps a reader more familiar with acoustics can provide a physical explanation for this). The prime factorization of a number reveals its fundamental building blocks. Primes can be seen as the "basic units" of integers, and their unique combinations construct all integers. Therefore, prime factorization provides a clear depiction of a number's complexity. Furthermore, Euler believed that smaller prime factors are simpler, which is why the summation term $\sum_{k=1}^{r} a_k(p_k - 1)$ in the formula is weighted by the prime factor itself. This ensures that larger prime factors significantly increase the score. Essentially, this approach applies a weighted treatment to the prime factorization of the least common multiple. Overall, dissonance depends on two aspects: the number of prime factors and the magnitude of those prime factors.

3.2 Visualization of Interval Dissonance

By substituting approximate ratios for various intervals, we can obtain the following table:

Name	Approx. Ratio	Least Common Multiple (n)	Prime Factorization	Count E(n)
Unison (Perfect 1st)	1:1	1	-	1
Minor Second (m2nd)	16:15	240	$2^4 \cdot 3 \cdot 5$	11
Major Second (M2nd)	9:8	72	$2^3 \cdot 3^2$	8
Minor Third (m3rd)	6:5	30	$2 \cdot 3 \cdot 5$	8
Major Third (M3rd)	5:4	20	$2^2 \cdot 5$	7
Perfect Fourth (P4th)	4:3	12	$2^2 \cdot 3$	5
Tritone (TT)	45:32	1440	$2^5 \cdot 3^2 \cdot 5$	14
Diminished Fifth (d5th)	64:45	2880	$2^6 \cdot 3^2 \cdot 5$	15
Perfect Fifth (P5th)	3:2	6	$2 \cdot 3$	4
Minor Sixth (m6th)	8:5	40	$2^3 \cdot 5$	8
Major Sixth (M6th)	5:3	15	$3 \cdot 5$	7
Minor Seventh (m7th)	9:5	45	$3^2 \cdot 5$	9
Major Seventh (M7th)	15:8	120	$2^3 \cdot 3 \cdot 5$	10
Octave (Perfect 8th)	2:1	2	2	2

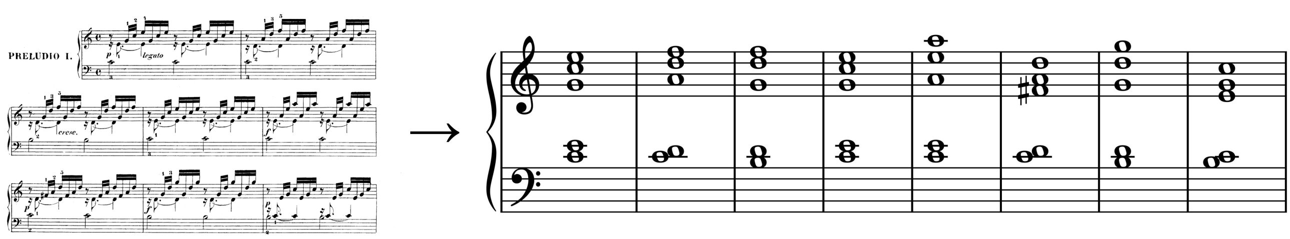
We can visualize this as a heatmap:



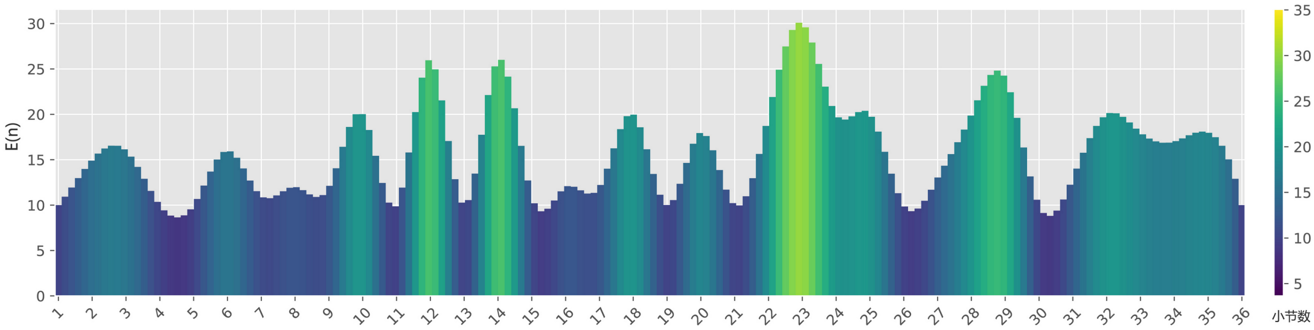
As shown, the results in the chart generally align with our intuition. A single note has a dissonance score of 1, making it the most harmonious. The dominant and subdominant notes are relatively harmonious, while intervals such as the minor second and seventh are more dissonant. The tritone's value is higher than expected, but this is reasonable. In ancient Chinese music, this interval was called the "变徵" (altered zhi) tone, which could disrupt the pentatonic scale, expressing sadness, lamentation, melancholy, or other complex emotions. Note that this table is based on approximate ratios and is for reference only.

3.3 Visualization of Dissonance in Harmony

As mentioned earlier, Euler's algorithm can be extended to analyze chords consisting of multiple notes. Here, we take Bach's classic *Prelude in C Major* as an example. First, we reconstruct the chords into block chords:



Next, we analyze the chords using Python code, resulting in the following graph (interpolation was applied to ensure smoothness):



From the graph, we can clearly observe the variations in dissonance. The music rises and falls like breathing, constantly developing in a rhythmic ebb and flow. This provides a new perspective for understanding the underlying logic of traditional music theory.

> The code used to generate the above images is available in the [Github repository](#).

At this point, readers might wonder: in the harmonic progression of T-S-D-T, if all chords are limited to root-position triads, won't the dissonance levels remain the same? How, then, does the music achieve its sense of development? This brings us to the next topic: the dissonance of harmonic connections.

Euler further explored harmonic theory in Chapter 4, which will not be covered here. Interested readers can refer to the original text. Additionally, Euler extended his method of quantifying individual chords to series of chords in Chapters 5 and 6, allowing for an analysis of the overall harmony of a musical piece. This will also not be discussed here.

4. Scales That Defy Harmony – Euler's New Musical System

This section primarily corresponds to Chapter 8 of Euler's work. Anyone with a basic understanding of music theory is likely familiar with common scales such as the major scale, minor scale, Dorian mode, chromatic scale, pentatonic scale, etc., which are generated by various methods, such as the Chinese "three-fifths method," the ancient Greek system of fifths, just intonation, or the more modern twelve-tone equal temperament.

The earliest methods of generating scales were based on simple ratios, such as $2:1$ for the octave, $3:2$ for the perfect fifth, and so on. However, as these systems were developed, an annoying problem became apparent: if the purity of intervals was maintained, the scale could never "close"; if the scale was forced to close, slight compromises in interval purity were unavoidable. For example, in the three-fifths method, the principle involves reducing or lengthening the string by one-third to derive adjacent notes: reducing the string length by one-third (diminishing by a third) gives the perfect fifth, while lengthening it by one-third (augmenting by a third) gives the perfect fourth. Repeating this process generates twelve tones, but after many iterations, these twelve tones fail to close perfectly, resulting in a small discrepancy (commonly known as the "Pythagorean comma") compared to modern twelve-tone equal temperament. Conversely, in twelve-tone equal temperament, the perfect fifth is no longer exactly $3:2$, but rather an approximation.

This creates an almost unsolvable trade-off: music theorists have had to continually tweak tuning systems to strike a balance that satisfies the ear's sense of beauty. Let's now examine how Euler approached the problem of generating scales mathematically.

4.1 Dividing the Octave

Euler began by establishing a fundamental principle: the octave $(1:2)$ is fixed, and all other notes must fall within this range. This serves as the basic premise. Euler represented all the notes within the octave using mathematical "exponents" (denoted as *exponens*), expressed by the following formula:

$$2^m A$$

Where:

- A is an odd number, composed of the products of prime factors 3 and 5, i.e., $A = 3^a 5^b$. Additional primes like 7 can also be included for experimentation. Each unique (a, b) combination corresponds to a different scale.
- 2^m represents a binary exponent, which extends the generated notes across different octaves (e.g., generating a G and then transposing it to other octaves by multiplying by 2^m). Here, m must ensure the resulting frequency remains within the audible range of human hearing.

The complete calculation process is as follows:

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1. Input: a, b
2. Calculate $A = 3^a \times 5^b$
3. Find all positive divisors of A, denoted as $\{d_1, d_2, \ldots, d_n\}$, including 1 and A itself (a total of n divisors). The first divisor is always 1.
4. Scale all divisors of A by 2^m so that they fall within the range $[1, 2)$.
5. Finally, the n resulting frequency ratios, combined with the octave (2), yield a scale with $n + 1$ notes.
6. Sort and normalize to ensure the ratios are expressed in mutually prime integers.

> **Note 1**: To avoid errors caused by floating-point calculations, the actual implementation may differ slightly from the outlined steps.
> **Note 2**: In theory, the method can be extended to $A = 3^a 5^b 7^c 11^d \ldots$ for scale generation (ensuring the bases are prime numbers), though this is rarely done in practice.

Let's illustrate this with an example where $a = 1$ and $b = 1$. First, compute $A = 3^1 \times 5^1 = 3 \times 5 = 15$. The divisors of 15 are $\{1, 3, 5, 15\}$. After scaling, we find $m = 3$ such that:

Divisor d_i	Scaled to $[1, 2)$	Integer Scaling Factor	Final Frequency
1	$2^0 \times 1 = 1$	8	8
3	$2^{-1} \times 3 = 3/2$	8	12
5	$2^{-2} \times 5 = 5/4$	8	10
15	$2^{-3} \times 15 = 15/8$	8	15

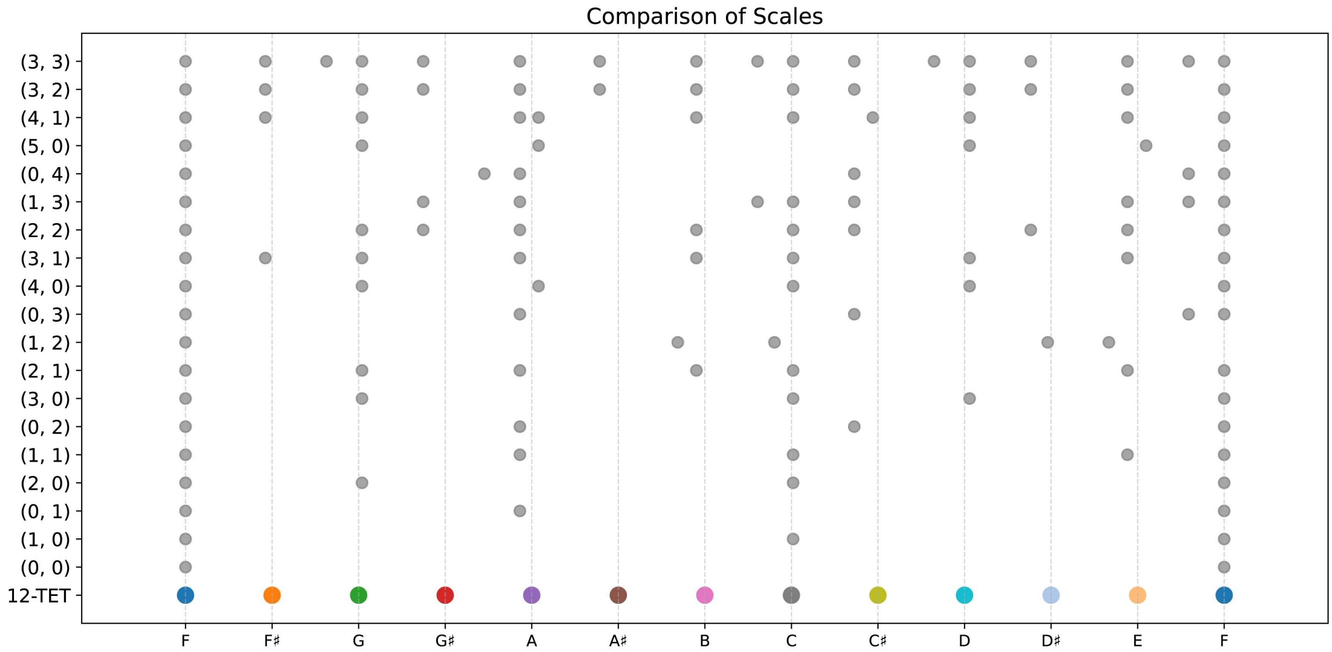
Adding the octave (16), and after sorting and simplifying, the final scale is $8 : 10 : 12 : 15 : 16$, corresponding to the notes C-E-G-B-C'. Using powers of 2, these notes can be extended across multiple octaves. In this way, a scale is constructed. A Python implementation of this algorithm is available in the [Github repository](), which readers can download and verify.

In his book, Euler presented a table where he constructed 17 musical scales using 17 pairs of values, a and b, as follows:

Mode	a	b	Scale Ratios	Scale
I	0	0	1 : 2	F-F
II	1	0	2 : 3 : 4	F-C-F
III	0	1	4 : 5 : 8	F-A-F
IV	2	0	8 : 9 : 12 : 16	F-G-C-F
V	1	1	8 : 10 : 12 : 15 : 16	F-A-C-E-F
VI	0	2	16 : 20 : 25 : 32	F-A-C♯-F
VII	3	0	16 : 18 : 24 : 27 : 32	F-G-C-D-F
VIII	2	1	32 : 36 : 40 : 45 : 48 : 60 : 64	F-G-A-B-C-E-F
IX	1	2	54 : 75 : 80 : 96 : 100 : 120 : 128	F-G♯-A-C-C♯-E-F
X	0	3	64 : 80 : 100 : 125 : 128	F-A-C♯-F*-F
XI	4	0	64 : 72 : 81 : 96 : 108 : 128	F-G-A*-C-D-F
XII	3	1	128 : 135 : 144 : 160 : 180 : 192 : 216 : 240 : 256	F-F♯-G-A-B-C-D-E-F
XIII	2	2	128 : 144 : 150 : 160 : 180 : 192 : 200 : 225 : 240 : 256	F-G-A-B-C-C♯-D♯-E-F
XIV	1	3	256 : 300 : 320 : 375 : 384 : 400 : 480 : 500 : 512	F-G♯-A-B*-C-C♯-E-F*-F
XV	0	4	512 : 625 : 640 : 800 : 1000 : 1024	F-A*-A-C♯-F*-F
XVI	5	0	128 : 144 : 162 : 1922 : 216 : 243 : 256	F-G-A*-C-D-E*-F
XVII	4	1	256 : 270 : 288 : 320 : 324 : 360 : 384 : 405 : 432 : 480 : 512	F-F♯-G-A-A*-B-C-C♯*-D-E-F

> **Note 1:** In Euler's original text, the letter H likely represents B, while B represents B♭.
>
> **Note 2:** F* refers to a microtone that is very close to F.

All these scales can be visualized in the chart below, with the vertical axis representing (a, b). Readers can easily observe the deviations in frequency:



4.2 Euler's New Musical System

In Chapter 9, Euler introduced the 18th scale; in Chapter 10, he explored even more complex scales. He also proposed a practical method for tuning. Overall, some of the generated scales align reasonably well with existing scales, though there are slight deviations in the frequencies of certain notes—these deviations are essentially adjustments to the scales. But which of these adjustments are desirable? At least two criteria must be considered: firstly, aesthetics—the scale should align with human physiological and psychological principles to produce a pleasing sound; and secondly, practicality—music ultimately needs to be played on instruments. If a scale is difficult to execute, hard to transpose, or incompatible with harmony, it will remain purely theoretical.

> Many musicians believe that true music should be based on the equality of intervals rather than the simplicity of interval ratios. Thus, they unhesitatingly divided the octave into 12 equal parts and established the commonly used 12-tone system based on this division. With this system, they became increasingly convinced that all intervals had been equalized, allowing any musical piece to be played in all so-called keys without modification and making it easy to transpose from the original key to any other. In this regard, they were not mistaken. However, they failed to realize that this approach actually eliminated the harmonic characteristics of the keys.
> *— Euler*

Euler's ambition, then, was to devise a method for generating a series of scales: scales that were broadly compatible with traditional ones, ensuring a pleasing auditory experience, while introducing specific adjustments—through the addition of microtones—to make certain chords more pure and harmonious. Euler sought to investigate these scales' intervals, harmonics, tuning, and transposition using mathematical methods, thereby forming a complete theoretical system of music entirely grounded in mathematics. This is precisely the subject of the content starting from Chapter 8 of his book. It is important to note that this theoretical system is not intended to prescribe musical composition but instead to explain the principles behind pleasing music, offering guidance for composition while refraining from stifling creativity. Moreover, the system does not rigidly define the boundaries of consonance and dissonance, leaving it open-ended.

> **Note**: The goal is not to "generate a single scale" but to find a "method for generating a series of scales" so that the system encompasses multiple keys and allows transposition.

4.2.1 The Diatonic-Chromatic Scale

> We call the 18th scale the "diatonic-chromatic scale." This name is evidently derived from its exponential form $2^m \cdot 3^3 \cdot 5^2$, as it is the least common multiple of the exponents for the "diatonic scale" $2^m \cdot 3^3 \cdot 5$ and the "chromatic scale" $2^m \cdot 3^2 \cdot 5^2$. From this, we can infer that this scale might align with the scales widely accepted by contemporary musicians, as they too viewed this scale as a combination of the ancient diatonic and chromatic scales.
> — *Euler*

Simply put, the "diatonic-chromatic scale" is generated using the formula $2^m \cdot 3^3 \cdot 5^2$. Using the Python code provided earlier, we can calculate the following results:

```
Calculation: A = 3^3 * 5^2 * 7^0 * 11^0 = 675
Factors of A: [1, 3, 5, 9, 15, 25, 27, 45, 75, 135, 225, 675]
Generated scale: 512:540:576:600:640:675:720:768:800:864:900:960:1024
Number of notes (excluding octave duplicates): 12
```

Assuming $A = 440\text{Hz}$ as the reference pitch, the following table can be derived:

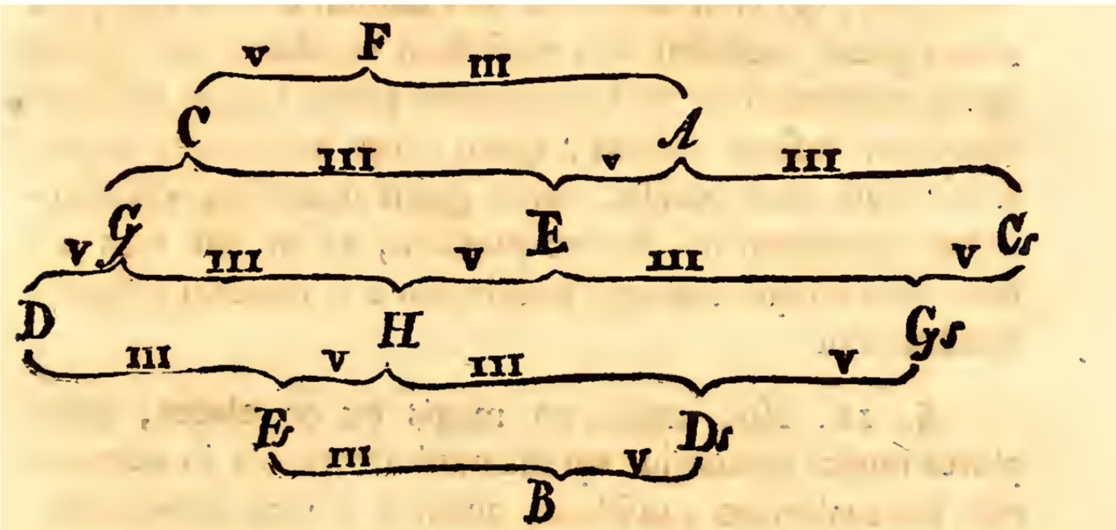
Note	Frequency of the Diatonic-Chromatic Scale (Hz)	Frequency in 12-TET (Hz)	Difference (Hz)	Difference (cents)
A	440.0000	440.0000	0.0000	0.0000
A♯	464.0625	466.1638	2.1013	7.8213
B	495.0000	493.8833	-1.1167	-3.9100
C	515.6250	523.2511	7.6261	25.4176
C♯	550.0000	554.3653	4.3653	13.6863
D	580.0781	587.3295	7.2514	21.5076
D♯	618.7500	622.2540	3.5040	9.7763
E	660.0000	659.2551	-0.7449	-1.9550
F	687.5000	698.4565	10.9565	27.3726
F♯	742.5000	739.9888	-2.5112	-5.8650
G	773.4375	783.9909	10.5534	23.4626
G♯	825.0000	830.6094	5.6094	11.7313
A	880.0000	880.0000	0.0000	0.0000

The graph (Comparison of Scales) for the case of $a = 3, b = 2$ offers a visualization that readers can refer back to. Generally, differences exceeding 20 cents are perceptible to the human ear. Euler believed that this scale could be made usable with some fine-tuning of microtones.

> Therefore, the current method of dividing the octave has been perfected to an extreme level through practice. To make it even more perfect, only one adjustment is needed: slightly lowering the pitch of the note marked with the letter B (i.e., B♭) by a microtone (the difference between a major semitone and a minor semitone). With this adjustment, one can obtain the most perfect scale, which is most suitable for forming harmonies. As for the number of notes included in the scale, this scale will contain exactly the number of notes necessary for harmony—neither more nor fewer. Furthermore, the relationships between these notes fully comply with the proportions dictated by the laws of harmony.
> —Leonhard Euler

> **Note**: Here, B refers to B♭.

4.2.2 Tuning of the Natural Semitone-Half-Tone Scale



> Thus, a person with such keen hearing can tune an instrument in the following sequence. First, determine the note F according to the specific circumstances, and use this note as the basis to obtain all other notes marked with the same letter. Next, find the perfect fifth C and the major third A relative to F, from which the rest of the notes marked with the same letters can be determined according to the first rule mentioned earlier. Then, from the note C, derive its perfect fifth G and major third E. The note E is also the perfect fifth of the note A. From A, derive its major third C♯. Then, from G, obtain its perfect fifth D and major third B. From E, derive its major third G♯, which is also the perfect fifth of C♯. Next, from B, derive its perfect fifth F♯ and major third D♯, or alternatively, D♯ can be derived from G♯. Finally, find the perfect fifth of D♯, which is B. Using this method and repeating the octave, the entire instrument can be correctly tuned.

> This entire tuning process can be better understood with the help of the attached diagram. Since the notes E, B, G♯, F♯, and D♯ can be determined through both perfect fifths and major thirds, this provides significant assistance during the tuning process. If an error occurs, it can be immediately detected and corrected.

The explanation above is taken directly from Euler's writings and should already be quite clear, so no further elaboration is provided here. In the diagram, the snake-like symbol at the bottom right of C, G, and D represents a sharp (♯).

4.2.3 A New Musical System

In the remaining chapters, Euler attempts to construct his new musical system. For example, in Chapter 11, Euler summarizes the concordant tones and chords within this system, classifying them by their degree of dissonance. He also provides practical explanations for the use of different types of chords. In Chapter 12, Euler describes modulation, integrating the scales before and after modulation into the form $2^n \cdot 3^a \cdot 5^b$. He notes that modulation within $2^n \cdot 3^3 \cdot 5^2$ is pure, that modulation beyond this but within $2^n \cdot 3^7 \cdot 5^2$ is impure, and that modulation beyond this range is invalid. In Chapter 13, Euler devotes an enormous amount of text to exploring methods of composition. These topics are too vast and complex for me to study in the short term, so they will not be addressed here.

> Even if the variations within a single system are extremely rich, adhering to the same system for too long will inevitably lead to boredom rather than pleasure. This is because music requires not only the harmony of sounds and chords but also diversity. Thus, the object of hearing must constantly change. —Leonhard Euler

5. Conclusion

Euler's *Tentamen novae theoriae musicae* (*Attempt at a New Theory of Music*) is an extraordinary book that bridges the realms of mathematics and music. It not only showcases Euler's profound mathematical expertise but also reflects his keen sensitivity to art. Using simple elementary calculations, Euler unified music under the principles of order and beauty—truly breathtaking.

Euler was a prolific author throughout his life. From what I know, a comprehensive edition of his collected works—finalized after over a century of effort—was published a few years ago. This edition should include this book. I believe it holds many more treasures of wisdom waiting to be discovered by future generations. Of course, we shouldn't expect this "ancient scroll" to completely overturn or revolutionize the current system. The integration of music theory and mathematics has long been a mature field, and both music and mathematics have made significant progress since Euler's time. Today, for example, we can describe musical phenomena with much greater precision, such as using group theory to analyze scales. Nevertheless, discovering fragments of ancient thought and experiencing the wisdom of our predecessors is both exciting and meaningful, isn't it?