全矢量BPM广角推导

基本公式推导

$$\nabla \times \vec{H} = \mathrm{j}\omega\varepsilon_0\vec{\varepsilon}\vec{\mathrm{E}}$$
$$\nabla \times \vec{E} = -j\omega\mu_0\vec{H}$$
$$\nabla \cdot \left(\vec{\varepsilon}\vec{E}\right) = 0$$

$$\vec{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix}$$

$$\overrightarrow{\varepsilon_{tt}} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix}$$

$$\nabla \times \left(\nabla \times \vec{E}\right) = \nabla \times \left(-j\omega\mu_0\vec{H}\right) = k_0^2\vec{\varepsilon}\vec{\mathrm{E}}$$

$$\nabla \times \nabla \times = \nabla(\nabla \cdot) - \nabla^2$$

$$\nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2\vec{E} = k_0^2\vec{\varepsilon}\vec{E}$$

$$\nabla^2\vec{E} + k_0^2\vec{\varepsilon}\vec{E} = \nabla\left(\nabla \cdot \vec{E}\right)$$

取横向分量

$$\nabla^2\overrightarrow{E_t} + k_0^2\overrightarrow{\varepsilon_{tt}}\overrightarrow{E_t} = \nabla_t\left[\nabla_t \cdot \overrightarrow{E_t} + \frac{\partial}{\partial_z}E_z\right]$$

$$\nabla\cdot\left(\vec{\varepsilon}\vec{E}\right)=0$$

$$\nabla_t\cdot\left(\overrightarrow{\varepsilon_{tt}}\overrightarrow{E_t}\right)+\nabla_z\varepsilon_{zz}E_z=0$$

$$\nabla_t\cdot\left(\overrightarrow{\varepsilon_{tt}}\overrightarrow{E_t}\right)+\varepsilon_{zz}\frac{\partial}{\partial_z}E_z+E_z\frac{\partial}{\partial_z}\varepsilon_{zz}=0$$

假设介电常数沿着z向传播方向缓慢变化

$$\frac{\partial}{\partial_z}\varepsilon_{zz}\approx 0$$

$$\frac{\partial}{\partial_z}E_z\approx-\frac{1}{\varepsilon_{zz}}\nabla_t\cdot\left(\overrightarrow{\varepsilon_{tt}}\overrightarrow{E_t}\right)$$

$$\nabla^2\overrightarrow{E_t}+k_0^2\overrightarrow{\varepsilon_{tt}}\overrightarrow{E_t}=\nabla_t\left[\nabla_t\cdot\overrightarrow{E_t}-\frac{1}{\varepsilon_{zz}}\nabla_t\cdot\left(\overrightarrow{\varepsilon_{tt}}\overrightarrow{E_t}\right)\right]$$

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

$$\nabla_t^2\overrightarrow{E_t}+\frac{\partial^2}{\partial z^2}\overrightarrow{E_t}+k_0^2\overrightarrow{\varepsilon_{tt}}\overrightarrow{E_t}=\nabla_t\left[\nabla_t\cdot\overrightarrow{E_t}-\frac{1}{\varepsilon_{zz}}\nabla_t\cdot\left(\overrightarrow{\varepsilon_{tt}}\overrightarrow{E_t}\right)\right]$$

将场分解为缓变包络和振荡项

$$\overrightarrow{E_t}=\overrightarrow{\Psi_t}e^{-iKz}$$

$K=n_0k_0$是参考传播常数

与全矢量模式求解不同的是，模式求解的Ψ_t是常量，但是BPM的是沿着z轴缓慢变化的函数

$$\frac{\partial^2}{\partial z^2}\overrightarrow{E_t}=\left(\frac{\partial^2}{\partial z^2}\overrightarrow{\Psi_t}-2iK\frac{\partial}{\partial z}\overrightarrow{\Psi_t}-K^2\overrightarrow{\Psi_t}\right)e^{-iKz}$$

代入消除振荡项e^{-iKz}

$$\nabla_t^2\overrightarrow{\Psi_t}+\frac{\partial^2}{\partial z^2}\overrightarrow{\Psi_t}-2iK\frac{\partial}{\partial z}\overrightarrow{\Psi_t}-K^2\overrightarrow{\Psi_t}+k_0^2\overrightarrow{\varepsilon_{tt}}\overrightarrow{\Psi_t}$$
$$=\nabla_t\left[\nabla_t\cdot\overrightarrow{\Psi_t}-\frac{1}{\varepsilon_{zz}}\nabla_t\cdot\left(\overrightarrow{\varepsilon_{tt}}\overrightarrow{\Psi_t}\right)\right]$$

$$\frac{\partial^2}{\partial z^2}\overrightarrow{\Psi_t} - 2iK\frac{\partial}{\partial z}\overrightarrow{\Psi_t} + \nabla_t^2\overrightarrow{\Psi_t} + \left(k_0^2\overrightarrow{\varepsilon_{tt}} - K^2\right)\overrightarrow{\Psi_t} - \nabla_t\left[\nabla_t\cdot\overrightarrow{\Psi_t} - \frac{1}{\varepsilon_{zz}}\nabla_t\cdot\left(\overrightarrow{\varepsilon_{tt}}\overrightarrow{\Psi_t}\right)\right] = 0$$

假设（上式对z的偏导为0变为BPM解模方程）

$$\mathcal{P}\overrightarrow{\Psi_t} = \frac{1}{K^2}\left\{\nabla_t^2\overrightarrow{\Psi_t} + \left(k_0^2\overrightarrow{\varepsilon_{tt}} - K^2\right)\overrightarrow{\Psi_t} - \nabla_t\left[\nabla_t\cdot\overrightarrow{\Psi_t} - \frac{1}{\varepsilon_{zz}}\nabla_t\cdot\left(\overrightarrow{\varepsilon_{tt}}\overrightarrow{\Psi_t}\right)\right]\right\}$$

两边同时除以K^2变成

$$\frac{1}{K^2}\frac{\partial^2}{\partial z^2}\overrightarrow{\Psi_t} - \frac{2i}{K}\frac{\partial}{\partial z}\overrightarrow{\Psi_t} + \mathcal{P}\overrightarrow{\Psi_t} = 0$$

微分算子$D=\frac{\partial}{\partial z}$

$$\frac{1}{K^2}D^2-\frac{2i}{K}D+\mathcal{P}=0$$

$$D=\text{-}iK\left(\pm\sqrt{1+\mathcal{P}}\text{-}1\right)$$

向前传波之前的符号为负

则括号里面要大于0

$$D=\frac{\partial}{\partial z}=-iK\left(\sqrt{1+\mathcal{P}}-1\right)$$

依据Pade近似

$$\left(\sqrt{1+\mathcal{P}}-1\right) \approx \frac{M_m(\mathcal{P})}{N_n(\mathcal{P})}$$

$$\frac{\partial}{\partial z} \approx -iK\frac{M_m(\mathcal{P})}{N_n(\mathcal{P})}$$

如何求Pade展开?

$$\left.\frac{\partial}{\partial z}\right|_n = -iK\frac{\frac{\mathcal{P}}{2}}{1+\frac{i}{2K}\frac{\partial}{\partial z}\Big|_{n-1}}$$

$$\left.\frac{\partial}{\partial z}\right|_0 = 0$$

$$\frac{M_m(\mathcal{P})}{N_n(\mathcal{P})} \approx \frac{m_0(0) + m_1\mathcal{P} + m_2\mathcal{P}^2 + m_3\mathcal{P}^3 + \cdots}{n_0(1) + n_1\mathcal{P} + n_2\mathcal{P}^2 + \cdots}$$

多项式系数高次到低次排列，方便求根

阶数	M_m	N_n
(1,0)	[1, 0]/2	[0, 2]/2
(1,1)	[2, 0]/4	[1, 4]/4
(2,2)	[4, 8, 0]/16	[1, 12, 16]/16
(3,3)	[6, 32, 32, 0]/64	[1, 24, 80, 64]/64
(4,4)	[8, 80, 192, 128, 0]/256	[1, 40, 240, 448, 256]/256
(5,5)	[10, 160, 672, 1024, 512, 0]/1024	[1, 60, 560, 1792, 2304, 1024]/1024

采用CN差分格式

$$\frac{\left(\vec{\Psi}^{m+1}-\vec{\Psi}^{m}\right)}{\Delta z}= -iK\frac{M}{N}\left[\,(1-\alpha)\,\vec{\Psi}^{m}+\alpha\vec{\Psi}^{m+1}\right]$$

$$\vec{\Psi}^{m+1}=\frac{N-iK\Delta z(1-\alpha)M}{N+iK\Delta z\alpha M}\vec{\Psi}^{m}$$

假设最高次数为n，分子分母合并$\mathcal{P}$的多项式然后因式分解，得到

$$\vec{\Psi}^{m+1}=\frac{\sum_i(1+a_i\mathcal{P})}{\sum_i(1+b_i\mathcal{P})}\vec{\Psi}^{m}$$

其中a_i ,b_i计算方法（最高次为n，则有n个根）

$$-1/root(a_n,a_{n-1},\ldots a_0)$$

分n步求解该式子$i = 1, \ldots, n$

$$\overrightarrow{\Psi}^{m+\frac{i}{n}} = \frac{1+\alpha_i \mathcal{P}}{1+\beta_i \mathcal{P}} \overrightarrow{\Psi}^{m+\frac{i-1}{n}}$$

接下来求解

$$\mathcal{P}\overrightarrow{\Psi_t} = \frac{1}{K^2}\left\{\nabla_t^2 \overrightarrow{\Psi_t} + \left(k_0^2 \overrightarrow{\varepsilon_{tt}} - K^2\right)\overrightarrow{\Psi_t} - \nabla_t \left[\nabla_t \cdot \overrightarrow{\Psi_t} - \frac{1}{\varepsilon_{zz}} \nabla_t \cdot \left(\overrightarrow{\varepsilon_{tt}}\overrightarrow{\Psi_t}\right)\right]\right\}$$

$$\mathcal{P}\begin{bmatrix} \mathrm{E}_x \\ \mathrm{E}_y \end{bmatrix} = \frac{1}{K^2}\begin{bmatrix} P_{xx} & P_{xy} \\ P_{yx} & P_{yy} \end{bmatrix}\begin{bmatrix} \mathrm{E}_x \\ \mathrm{E}_y \end{bmatrix}$$

$$P_{xx}=\frac{\partial}{\partial x}\frac{1}{\epsilon_{zz}}\frac{\partial(\epsilon_{xx}E_x)}{\partial x}+\frac{\partial^2 E_x}{\partial y^2}+\left(k_0^2\epsilon_{xx}-K^2\right)E_x+\frac{\partial}{\partial x}\frac{1}{\epsilon_{zz}}\frac{\partial(\epsilon_{yx}E_x)}{\partial y}$$

$$P_{yy}=\frac{\partial}{\partial y}\frac{1}{\epsilon_{zz}}\frac{\partial(\epsilon_{yy}E_y)}{\partial y}+\frac{\partial^2 E_y}{\partial x^2}+(k_0^2\epsilon_{yy}-K^2)E_y+\frac{\partial}{\partial y}\frac{1}{\epsilon_{zz}}\frac{\partial(\epsilon_{xy}E_y)}{\partial x}$$

$$P_{xy}=\frac{\partial}{\partial x}\frac{1}{\epsilon_{zz}}\frac{\partial(\epsilon_{yy}E_y)}{\partial y}-\frac{\partial}{\partial x}\left(\frac{\partial E_y}{\partial y}\right)+k_0^2(\epsilon_{xy}E_y)+\frac{\partial}{\partial x}\frac{1}{\epsilon_{zz}}\frac{\partial(\epsilon_{xy}E_y)}{\partial x}$$

$$P_{yx}=\frac{\partial}{\partial y}\frac{1}{\epsilon_{zz}}\frac{\partial(\epsilon_{xx}E_x)}{\partial x}-\frac{\partial}{\partial y}\left(\frac{\partial E_x}{\partial x}\right)+k_0^2\left(\epsilon_{yx}E_x\right)+\frac{\partial}{\partial y}\frac{1}{\epsilon_{zz}}\frac{\partial\left(\epsilon_{yx}E_x\right)}{\partial y}$$

文献里面把四个公式的最后一项给去掉了，否则不能使用显示交替法利用托马斯求解三对角矩阵。

分解P_{xx}

$$P_{xx} = A_x + A_y$$

$$A_x = \frac{\partial}{\partial x}\frac{1}{\epsilon_{zz}}\frac{\partial(\epsilon_{xx}E_x)}{\partial x} + \frac{1}{2}\left(k_0^2\epsilon_{xx} - K^2\right)E_x$$

$$A_y = \frac{\partial^2 E_x}{\partial y^2} + \frac{1}{2}\left(k_0^2\epsilon_{xx} - K^2\right)E_x$$

分解P_{yy}

$$P_{yy} = B_x + B_y$$

$$B_x = \frac{\partial^2 E_y}{\partial x^2} + \frac{1}{2}(k_0^2\epsilon_{yy} - K^2)E_y$$

$$B_y = \frac{\partial}{\partial y}\frac{1}{\epsilon_{zz}}\frac{\partial(\epsilon_{yy}E_y)}{\partial y} + \frac{1}{2}(k_0^2\epsilon_{yy} - K^2)E_y$$

$$C = \frac{\partial}{\partial x}\frac{1}{\epsilon_{zz}}\frac{\partial(\epsilon_{yy}E_y)}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial E_y}{\partial y}\right) + k_0^2(\epsilon_{xy}E_y)$$

$$D = \frac{\partial}{\partial y}\frac{1}{\epsilon_{zz}}\frac{\partial(\epsilon_{xx}E_x)}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial E_x}{\partial x}\right) + k_0^2(\epsilon_{yx}E_x)$$

CN差分形如以下形式

$$\vec{\Psi}^{m+1} = \frac{1 + a\mathcal{P}}{1 + b\mathcal{P}} \vec{\Psi}^{m}$$

一小步，表示方变$u = E_x, v = E_y$

$$\begin{bmatrix} u \\ v \end{bmatrix}^{m+1} = \frac{\left(1 + \frac{a}{K^2}\begin{bmatrix} A_x & C \\ 0 & B_x \end{bmatrix}\right)\left(1 + \frac{a}{K^2}\begin{bmatrix} A_y & 0 \\ D & B_y \end{bmatrix}\right)}{\left(1 + \frac{b}{K^2}\begin{bmatrix} A_x & C \\ 0 & B_x \end{bmatrix}\right)\left(1 + \frac{b}{K^2}\begin{bmatrix} A_y & 0 \\ D & B_y \end{bmatrix}\right)} \begin{bmatrix} u \\ v \end{bmatrix}^{m}$$

第一步

$$\begin{bmatrix} u \\ v \end{bmatrix}^{m+\frac{1}{2}} = \frac{\left(1+\frac{a}{K^2}\begin{bmatrix} A_y & 0 \\ D & B_y \end{bmatrix}\right)}{\left(1+\frac{b}{K^2}\begin{bmatrix} A_y & 0 \\ D & B_y \end{bmatrix}\right)} \begin{bmatrix} u \\ v \end{bmatrix}^{m}$$

$$\left(1+\frac{b}{K^2}A_y\right)u^{m+\frac{1}{2}} = \left(1+\frac{a}{K^2}A_y\right)u^m$$

$$\left(1+\frac{b}{K^2}B_y\right)v^{m+\frac{1}{2}} + \frac{b}{K^2}Du^{m+\frac{1}{2}} = \left(1+\frac{a}{K^2}B_y\right)v^m + \frac{a}{K^2}Du^m$$

第二步

$$\begin{bmatrix} u \\ v \end{bmatrix}^{m+1} = \frac{\left(1 + \frac{a}{K^2}\begin{bmatrix} A_x & C \\ 0 & B_x \end{bmatrix}\right)}{\left(1 + \frac{b}{K^2}\begin{bmatrix} A_x & C \\ 0 & B_x \end{bmatrix}\right)} \begin{bmatrix} u \\ v \end{bmatrix}^{m+\frac{1}{2}}$$

$$\left(1 + \frac{b}{K^2} A_x\right) u^{m+1} + \frac{b}{K^2} C v^{m+1} = \left(1 + \frac{a}{K^2} A_x\right) u^{m+\frac{1}{2}} + \frac{a}{K^2} C v^{m+\frac{1}{2}}$$

$$\left(1 + \frac{b}{K^2} B_x\right) v^{m+1} = \left(1 + \frac{a}{K^2} B_x\right) v^{m+\frac{1}{2}}$$