

# A SHORT PROOF OF NONHOMOGENEITY OF THE PSEUDO-CIRCLE

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*Dedicated to James T. Rogers, Jr. on the occasion of his 65th birthday*

**ABSTRACT.** The pseudo-circle is known to be nonhomogeneous. The original proofs of this fact were discovered independently by L. Fearnley [6] and J.T. Rogers, Jr. [17]. The purpose of this paper is to provide an alternative, very short proof based on a result of D. Bellamy and W. Lewis [4].

## 1. INTRODUCTION

A *pseudo-arc* is a hereditarily indecomposable, chainable continuum. In 1948, E.E. Moise [16] constructed a pseudo-arc as an indecomposable continuum homeomorphic to each of its subcontinua. Moise correctly conjectured that the hereditarily indecomposable continuum given by B. Knaster [11] in 1922 is a pseudo-arc. Also in 1948, R.H. Bing [1] proved that Moise's example is homogeneous. In 1951, Bing [2] proved that every hereditarily indecomposable chainable continuum is a pseudo-arc and that all pseudo-arcs are homeomorphic. In 1959, Bing [3] gave another characterization of the pseudo-arc: a homogeneous chainable continuum.

The history of many other aspects of the pseudo-arc can be found in survey papers by W. Lewis [14] and [15].

In 1951, Bing [2] described a *pseudo-circle*, a planar hereditarily indecomposable circularly chainable continuum which separates the plane and whose every proper subcontinuum is a pseudo-arc. It has been shown by L. Fearnley in [6] and J. T. Rogers, Jr. in [17] that the pseudo-circle is not homogeneous. Fearnley also proved that the pseudo-circle is unique [5] and [7]. The fact that the pseudo-circle is not homogeneous also follows from more general theorems proved in [8], [10], [13], and [18].

This paper offers yet another, very short proof, a consequence of a result of D. Bellamy and W. Lewis [4]. Similarly as in [18], an infinite covering space of a plane separating continuum is used.

## 2. PRELIMINARIES

Throughout the paper, a *continuum* will refer to a nondegenerate compact and connected metric space. A continuum is *indecomposable* if it is not the union of two proper subcontinua. A continuum is *hereditarily indecomposable* if every subcontinuum is also indecomposable. For a point  $a$  in  $X$ , the *composant*  $K(a)$  of  $a$  in  $X$  is the union of all proper subcontinua of  $X$  containing  $a$ . An indecomposable continuum contains uncountably many pairwise disjoint composants, see [12] Theorem 7, page 212.

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A topological space  $X$  is *homogeneous* if for any two points in  $X$  there is a homeomorphism of  $X$  onto itself mapping one point onto the other.

Let  $C$  denote the pseudo-circle. We may assume that  $C$  is contained in a planar annulus  $A$  in such a way that the winding number of each circular chain in the sequence of crooked circular chains defining  $C$  is one. Any homeomorphism  $h : C \rightarrow C$  extends to a continuous map  $f : A \rightarrow A$  of degree  $\pm 1$ . (First extend  $h$  to a map  $\overline{U} \rightarrow A$  for some closed annular neighborhood  $\overline{U}$  of  $C$ , then compose a retraction of  $A$  onto  $\overline{U}$  with this extension.)

Let  $\tilde{A}$  be the universal covering space of  $A$  with projection  $p$ . For any  $\tilde{x} \in \tilde{A}$  and  $\tilde{y} \in p^{-1}(f(p(\tilde{x})))$  there is a map  $\tilde{f}$  such that the diagram

$$\begin{array}{ccc} & \tilde{f} & \\ \tilde{A} & \longrightarrow & \tilde{A} \\ p \downarrow & & \downarrow p \\ A & \longrightarrow & A \\ & f & \end{array}$$

commutes and  $\tilde{f}(\tilde{x}) = \tilde{y}$ ; see for example [9], Theorem 16.3. Let  $\hat{A}$  be the disc that is a two-point compactification of  $\tilde{A}$ . Denote the two added points of the compactification by  $a$  and  $b$ . The map  $\tilde{f}$  extends uniquely to a map  $F : \hat{A} \rightarrow \hat{A}$ .

Let  $\tilde{C} = p^{-1}(C)$ , and let  $P = \tilde{C} \cup \{a, b\}$ , a two-point compactification of  $p^{-1}(C)$ . D. Bellamy and W. Lewis considered this set in [4] and proved that  $P$  is a pseudo-arc.

Denote by  $H$  the restriction of  $F$  to  $P$  and note that

- (1) either  $H(a) = a$  and  $H(b) = b$ , or  $H(a) = b$  and  $H(b) = a$ ,
- (2)  $\tilde{f}(\tilde{C}) = \tilde{C}$  and hence  $H(P) = P$ ,
- (3)  $\tilde{f}|_{\tilde{C}}$  is one-to-one.

Thus

**Lemma.**  *$H$  is a homeomorphism from  $P$  to  $P$ .*

### 3. PROOF OF NONHOMOGENEITY OF THE PSEUDO-CIRCLE

**Theorem 1.** *The pseudo-circle is not homogeneous.*

*Proof.* Let  $K(a)$  and  $K(b)$  be the composants of  $a$  and  $b$ , respectively, in the pseudo-arc  $P$ . Let  $\tilde{x}$  and  $\tilde{y}$  be two points in  $P$  such that  $\tilde{x} \in (K(a) \cup K(b)) - \{a, b\}$  and  $\tilde{y} \in P - (K(a) \cup K(b))$ . If  $C$  were homogeneous, then there would be a homeomorphism  $h : C \rightarrow C$  taking  $x = p(\tilde{x})$  onto  $y = p(\tilde{y})$ . Then there would be a homeomorphism  $H : P \rightarrow P$  as described in section 2 taking  $\tilde{x}$  onto  $\tilde{y}$ . This is not possible since under every such homeomorphism, the set  $K(a) \cup K(b)$  is invariant; the image of a componant is a componant.  $\square$

**Remark.** It is not important for this proof that  $K(a)$  and  $K(b)$  are not the same set, but the authors are grateful to D. Bellamy and W. Lewis for showing that  $K(a)$  and  $K(b)$  were indeed different componants.

**Theorem 2.** *If for some  $x$ , the componant  $K(a)$  intersects the fiber  $p^{-1}(x)$ , then it contains  $p^{-1}(x)$ .*

*Proof.* If  $y \in p^{-1}(x) \cap K(a)$ , then by the definition of a composant, there is a proper subcontinuum  $W$  of  $P$  that contains both  $a$  and  $y$ . Let  $g : \tilde{C} \rightarrow \tilde{C}$  be a deck transformation such that  $p^{-1}(x) = \{g^n(y)\}_{n \in \mathbb{Z}}$ ,  $\mathbb{Z}$  being the set of integers. Denote by  $G$  the extension of  $g$  to  $P$ . The set  $W_n = G^n(W)$  is a continuum containing  $a$  and  $g^n(y)$ . Thus  $p^{-1}(x) \subset K(a)$ .  $\square$

Note that Theorem 2 and Remark above imply that  $p(K(a) - \{a\}) \cap p(K(b) - \{b\}) = \emptyset$ .

**Question.** Can the sets  $p(K(a) - \{a\})$  and  $p(K(b) - \{b\})$  be used to classify the composants of the pseudo-circle  $C$ ?

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#### REFERENCES

- [1] R.H. Bing, *A homogeneous indecomposable plane continuum*, Duke Math. J. 15 (1948), 729-742.
- [2] R.H. Bing, *Concerning hereditarily indecomposable continua*, Pacific J. Math. 1 (1951), 43-51.
- [3] R.H. Bing, *Each homogeneous nondegenerate chainable continuum is a pseudo-arc*, Proc. Amer. Math. Soc. 10 (1959), 345-346.
- [4] D.P. Bellamy and W. Lewis, *An orientation reversing homeomorphism of the plane with invariant pseudo-arc*, Proc. Amer. Math. Soc. 114 (1992), 1145-1149.
- [5] L. Fearnley, *The pseudo-circle is unique*, Bull. Amer. Math. Soc. 75 (1969), 398-401.
- [6] L. Fearnley, *The pseudo-circle is not homogeneous*, Bull. Amer. Math. Soc. 75 (1969), 554-558.
- [7] L. Fearnley, *The pseudo-circle is unique*, Trans. Amer. Math. Soc. 149 (1970), 45-64.
- [8] C.L. Hagopian, *The fixed-point property for almost chainable homogeneous continua*, Illinois J. Math. 20 (1976), 650-652.
- [9] S.T. Hu, *Homotopy Theory*, Elsevier Science and Technology Books, 1959.
- [10] J. Kennedy, and J.T. Rogers, Jr., *Orbits of the pseudocircle*, Trans. Amer. Math. Soc. 296 (1986), 327-340.
- [11] B. Knaster, *Un continu dont tout sous-continu est indécomposable*, Fund. Math. 3 (1922), 247-286.
- [12] K. Kuratowski, *Topology*, Vol. II, Academic Press, 1968.
- [13] W. Lewis, *Almost chainable homogeneous continua are chainable*, Houston J. Math. 7 (1981), 373-377.
- [14] W. Lewis, *The pseudo-arc*, Contemp. Math. 117 (1991), 103-123.
- [15] W. Lewis, *The pseudo-arc*, Bol. Soc. Mat. Mexicana 5 (1999), 25-77.
- [16] E.E. Moise, *An indecomposable plane continuum which is homeomorphic to each of its nondegenerate subcontinua*, Trans. Amer. Math. Soc. 63 (1948), 581-594.
- [17] J.T. Rogers, Jr., *The pseudo-circle is not homogeneous*, Trans. Amer. Math. Soc. 148 (1970), 417-428.
- [18] J.T. Rogers, Jr., *Homogeneous, separating plane continua are decomposable*, Michigan Math. J. 28 (1981), 317-322.

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