

A SHORT PROOF OF NONHOMOGENEITY OF THE PSEUDO-CIRCLE

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Dedicated to James T. Rogers, Jr. on the occasion of his 65th birthday

ABSTRACT. The pseudo-circle is known to be nonhomogeneous. The original proofs of this fact were discovered independently by L. Fearnley [6] and J.T. Rogers, Jr. [17]. The purpose of this paper is to provide an alternative, very short proof based on a result of D. Bellamy and W. Lewis [4].

1. INTRODUCTION

A *pseudo-arc* is a hereditarily indecomposable, chainable continuum. In 1948, E.E. Moise [16] constructed a pseudo-arc as an indecomposable continuum homeomorphic to each of its subcontinua. Moise correctly conjectured that the hereditarily indecomposable continuum given by B. Knaster [11] in 1922 is a pseudo-arc. Also in 1948, R.H. Bing [1] proved that Moise's example is homogeneous. In 1951, Bing [2] proved that every hereditarily indecomposable chainable continuum is a pseudo-arc and that all pseudo-arcs are homeomorphic. In 1959, Bing [3] gave another characterization of the pseudo-arc: a homogeneous chainable continuum.

The history of many other aspects of the pseudo-arc can be found in survey papers by W. Lewis [14] and [15].

In 1951, Bing [2] described a *pseudo-circle*, a planar hereditarily indecomposable circularly chainable continuum which separates the plane and whose every proper subcontinuum is a pseudo-arc. It has been shown by L. Fearnley in [6] and J. T. Rogers, Jr. in [17] that the pseudo-circle is not homogeneous. Fearnley also proved that the pseudo-circle is unique [5] and [7]. The fact that the pseudo-circle is not homogeneous also follows from more general theorems proved in [8], [10], [13], and [18].

This paper offers yet another, very short proof, a consequence of a result of D. Bellamy and W. Lewis [4]. Similarly as in [18], an infinite covering space of a plane separating continuum is used.

2. PRELIMINARIES

Throughout the paper, a *continuum* will refer to a nondegenerate compact and connected metric space. A continuum is *indecomposable* if it is not the union of two proper subcontinua. A continuum is *hereditarily indecomposable* if every subcontinuum is also indecomposable. For a point a in X , the *composant* $K(a)$ of a in X is the union of all proper subcontinua of X containing a . An indecomposable continuum contains uncountably many pairwise disjoint composants, see [12] Theorem 7, page 212.

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A topological space X is *homogeneous* if for any two points in X there is a homeomorphism of X onto itself mapping one point onto the other.

Let C denote the pseudo-circle. We may assume that C is contained in a planar annulus A in such a way that the winding number of each circular chain in the sequence of crooked circular chains defining C is one. Any homeomorphism $h : C \rightarrow C$ extends to a continuous map $f : A \rightarrow A$ of degree ± 1 . (First extend h to a map $\overline{U} \rightarrow A$ for some closed annular neighborhood \overline{U} of C , then compose a retraction of A onto \overline{U} with this extension.)

Let \tilde{A} be the universal covering space of A with projection p . For any $\tilde{x} \in \tilde{A}$ and $\tilde{y} \in p^{-1}(f(p(\tilde{x})))$ there is a map \tilde{f} such that the diagram

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{\tilde{f}} & \tilde{A} \\ p \downarrow & & \downarrow p \\ A & \xrightarrow{f} & A \end{array}$$

commutes and $\tilde{f}(\tilde{x}) = \tilde{y}$; see for example [9], Theorem 16.3. Let \hat{A} be the disc that is a two-point compactification of \tilde{A} . Denote the two added points of the compactification by a and b . The map \tilde{f} extends uniquely to a map $F : \hat{A} \rightarrow \hat{A}$.

Let $\tilde{C} = p^{-1}(C)$, and let $P = \tilde{C} \cup \{a, b\}$, a two-point compactification of $p^{-1}(C)$. D. Bellamy and W. Lewis considered this set in [4] and proved that P is a pseudo-arc.

Denote by H the restriction of F to P and note that

- (1) either $H(a) = a$ and $H(b) = b$, or $H(a) = b$ and $H(b) = a$,
- (2) $\tilde{f}(\tilde{C}) = \tilde{C}$ and hence $H(P) = P$,
- (3) $\tilde{f}|_{\tilde{C}}$ is one-to-one.

Thus

Lemma. H is a homeomorphism from P to P .

3. PROOF OF NONHOMOGENEITY OF THE PSEUDO-CIRCLE

Theorem 1. *The pseudo-circle is not homogeneous.*

Proof. Let $K(a)$ and $K(b)$ be the composants of a and b , respectively, in the pseudo-arc P . Let \tilde{x} and \tilde{y} be two points in P such that $\tilde{x} \in (K(a) \cup K(b)) - \{a, b\}$ and $\tilde{y} \in P - (K(a) \cup K(b))$. If C were homogeneous, then there would be a homeomorphism $h : C \rightarrow C$ taking $x = p(\tilde{x})$ onto $y = p(\tilde{y})$. Then there would be a homeomorphism $H : P \rightarrow P$ as described in section 2 taking \tilde{x} onto \tilde{y} . This is not possible since under every such homeomorphism, the set $K(a) \cup K(b)$ is invariant; the image of a composant is a composant. \square

Remark. It is not important for this proof that $K(a)$ and $K(b)$ are not the same set, but the authors are grateful to D. Bellamy and W. Lewis for showing that $K(a)$ and $K(b)$ were indeed different composants.

Theorem 2. *If for some x , the composant $K(a)$ intersects the fiber $p^{-1}(x)$, then it contains $p^{-1}(x)$.*

Proof. If $y \in p^{-1}(x) \cap K(a)$, then by the definition of a composant, there is a proper subcontinuum W of P that contains both a and y . Let $g : \tilde{C} \rightarrow \tilde{C}$ be a deck transformation such that $p^{-1}(x) = \{g^n(y)\}_{n \in \mathbb{Z}}$, \mathbb{Z} being the set of integers. Denote by G the extension of g to P . The set $W_n = G^n(W)$ is a continuum containing a and $g^n(y)$. Thus $p^{-1}(x) \subset K(a)$. \square

Note that Theorem 2 and Remark above imply that $p(K(a) - \{a\}) \cap p(K(b) - \{b\}) = \emptyset$.

Question. *Can the sets $p(K(a) - \{a\})$ and $p(K(b) - \{b\})$ be used to classify the composants of the pseudo-circle C ?*

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