Глава 1. Интегральное исчисление функций одной переменной. Неопределенный интеграл

1.1 Простейшие интегралы

Определение

Функция $F(x)$ называется первообразной функции $f(x)$, заданной на некотором множестве X, если $F'(x) = f(x)$ для всех $x \in X$.

Первообразные одной и той же функции $f(x)$ отличаются друг от друга на $const$: $\quad \Phi(x) = F(x) + C$.

Множество всех первообразных функции $f(x)$ называется неопределенным интегралом от функции $f(x)$ и обозначается знаком $\int f(x)dx$:

$$\int f(x)dx = \underbrace{F(x) + C.}_{\text{мн-во всех первообразных}} \tag{1.1}$$

<u>Свойства неопределенного интеграла</u>

1. $\left(\int f(x)dx\right)' = f(x). \qquad (1.2)$

2. $\int f'(x)dx = f(x) + C. \qquad (1.3)$

3. $\int af(x)dx = a\int f(x)dx, \quad a \neq 0. \qquad (1.4)$

4. $\int (f_1(x) + f_2(x))dx = \int f_1(x)dx + \int f_2(x)dx. \qquad (1.5)$

<u>Таблица основных неопределенных интегралов</u>

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1). \text{ В частности: } \int dx = x + C. \tag{1.6}$$

$$\int \frac{dx}{x} = \ln|x| + C. \tag{1.7}$$

$$\int a^x dx = \frac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1). \text{ В частности: } \int e^x dx = e^x + C. \tag{1.8}$$

$$\int \sin x dx = -\cos x + C. \tag{1.9}$$

$$\int \cos x dx = \sin x + C. \tag{1.10}$$

$$\int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C. \tag{1.11}$$

$$\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C. \tag{1.12}$$

$$\int \frac{dx}{\sin x} = \ln \left| \operatorname{tg} \frac{x}{2} \right| + C. \tag{1.13}$$

$$\int \frac{dx}{\cos x} = \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right| + C. \tag{1.14}$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + C \quad (a \neq 0). \tag{1.15}$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{x + a}{x - a} \right| + C. \tag{1.16}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C, \quad |x| < |a|. \tag{1.17}$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left| x + \sqrt{x^2 - a^2} \right| + C, \quad |x| > |a|. \tag{1.18}$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln(x + \sqrt{x^2 + a^2}) + C. \tag{1.19}$$

$$\int \operatorname{sh} x dx = \operatorname{ch} x + C. \tag{1.20}$$

$$\int \operatorname{ch} x dx = \operatorname{sh} x + C. \tag{1.21}$$

$$\int \frac{dx}{\operatorname{ch}^2 x} = \operatorname{th} x + C. \tag{1.22}$$

$$\int \frac{dx}{\operatorname{sh}^2 x} = -\operatorname{cth} x + C. \tag{1.23}$$

Таблица производных основных элементарных функций

$$(x^a)' = ax^{a-1}, \quad a \neq 0. \tag{1.24}$$

$$(a^x)' = a^x \ln a, \quad a > 0. \text{ В частности: } (e^x)' = e^x. \tag{1.25}$$

$$(\log_a x)' = \log_a e \cdot \frac{1}{x}, \quad a > 0,\ a \neq 1. \text{ В частности: } (\ln x)' = \frac{1}{x}. \tag{1.26}$$

$$(\sin x)' = \cos x. \tag{1.27}$$

$$(\cos x)' = -\sin x. \tag{1.28}$$

$$(\operatorname{tg} x)' = \frac{1}{\cos^2 x}. \tag{1.29}$$

$$(\operatorname{ctg} x)' = -\frac{1}{\sin^2 x}. \tag{1.30}$$

$$(\arcsin x)' = -(\arccos x)' = \frac{1}{\sqrt{1-x^2}}. \tag{1.31}$$

$$(\operatorname{arctg} x)' = -(\operatorname{arcctg} x)' = \frac{1}{1+x^2}. \tag{1.32}$$

1. $$\int \left(\frac{3}{x} + \frac{5}{x^2}\right) dx = 3\ln|x| - \frac{5}{x} + C.$$

2. $$\int \left(1 - \sin^2\frac{x}{2}\right) dx = \Big/ \sin^2\frac{x}{2} = \frac{1-\cos x}{2} \Big/ = \int \left(1 - \frac{1}{2} + \frac{1}{2}\cos x\right) dx =$$
$$= \frac{1}{2}\int (1+\cos x)dx = \frac{1}{2}x + \frac{1}{2}\sin x + C.$$

3. $$\int \frac{(\sqrt{a}+\sqrt{x})^2}{\sqrt{ax}} dx = \int \frac{a + 2\sqrt{ax} + x}{\sqrt{ax}} dx = \int (a^{\frac{1}{2}}x^{-\frac{1}{2}} + 2 + a^{-\frac{1}{2}}x^{\frac{1}{2}})dx =$$
$$= a^{\frac{1}{2}} \cdot \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + 2x + a^{-\frac{1}{2}} \cdot \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = 2\sqrt{ax} + 2x + \frac{2x\sqrt{x}}{3\sqrt{a}} + C.$$

4. $$\int 2^x e^x dx = \int (2e)^x dx = \frac{(2e)^x}{\ln 2e} + C = \frac{2^x e^x}{\ln 2 + 1} + C.$$

5. $$\int \frac{\sqrt{x^2-3} - \sqrt{x^2+3}}{\sqrt{x^4-9}} dx = \int \frac{\sqrt{x^2+3} - \sqrt{x^2+3}}{\sqrt{(x^2-3)(x^2+3)}} dx =$$

$$= \int \frac{1}{\sqrt{x^2+3}}dx - \int \frac{1}{\sqrt{x^2-3}}dx = \ln\left|x+\sqrt{x^2+3}\right| - \ln\left|x+\sqrt{x^2-3}\right| + C.$$

6. $$\int \frac{\cos^2 x + 3\cos x - 2}{\cos^2 x}dx = \int dx + \int \frac{3}{\cos x}dx - \int \frac{2}{\cos^2 x}dx =$$
$$= x + 3\ln\left|\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| - 2\operatorname{tg} x + C.$$

1.2 Подведение под знак дифференциала

Рассмотрим некоторую функцию: $y = f(x)$

$dy = f'(x)dx$ $\left[f'(x) = \frac{dy}{dx}\right]$

Вычислим дифференциалы для различных функций y :

$y = x^2$ $dy = d(x^2) = 2xdx$

$y = \ln x$ $dy = d(\ln x) = \frac{1}{x}dx$

$y = \cos x$ $dy = d(\cos x) = -\sin x dx$

Пример

$$\int \sqrt{4x+3}dx = \frac{1}{4}\int \sqrt{4x+3}\cdot \underbrace{4dx}_{d(4x+3)} = \frac{1}{4}\int \sqrt{4x+3}\; d(4x+3) =$$

/Мы получили интеграл вида: $\int \sqrt{u}du$/

$$= \frac{1}{4}\cdot\frac{(4x+3)^{\frac{3}{2}}}{\frac{3}{2}} + C.$$

7. $$\int (3-4\sin x)^{\frac{1}{3}}\underbrace{\cos x dx}_{d(\sin x)} = \int (3-4\sin x)^{\frac{1}{3}} \underbrace{d(\sin x)}_{=-\frac{1}{4}d(3-4\sin x)} =$$

$$= -\frac{1}{4}\int (3-4\sin x)^{\frac{1}{3}}d(3-4\sin x) = \Big/ \int u^{\frac{1}{3}}du \Big/ = -\frac{1}{4}\frac{(3-4\sin x)^{\frac{4}{3}}}{\frac{4}{3}} + C =$$
$$= -\frac{3}{16}(3-4\sin x)^{\frac{4}{3}} + C.$$

8. $$\int \frac{dx}{x\ln^2 x} = \Big/ d(\ln x) = \frac{1}{x} \Big/ = \int \frac{d(\ln x)}{\ln^2 x} =$$

/Сделаем замену переменной в интеграле: $u = \ln x$/

$$= \int \frac{du}{u^2} = \int u^{-2} du = \frac{u^{-1}}{-1} + C = -\frac{1}{u} + C = /u = \ln x/ = -\frac{1}{\ln x} + C.$$

9. $$\int \frac{dx}{a+bx} = \frac{1}{b} \int \frac{d(bx)}{a+bx} = \frac{1}{b} \int \frac{d(a+bx)}{a+bx} = \frac{1}{b} \ln \Big| a+bx \Big| + C$$

10. $$\int \frac{1}{\sqrt{a+bx}} dx = \frac{1}{b} \int (a+bx)^{-\frac{1}{2}} d(a+bx) = \frac{2}{b} (a+bx)^{\frac{1}{2}} + C.$$

11. $$\int \frac{1}{\sqrt[3]{5^x}} = \int 5^{-\frac{1}{3}x} dx = -3 \int 5^{-\frac{1}{3}x} d\left(-\frac{1}{3}x\right) = -3 \frac{5^{-\frac{1}{3}x}}{\ln 5} + C.$$

12. $$\int \frac{\sec^2 x}{a - b \operatorname{tg} x} dx = \Big/ \sec x = \frac{1}{\cos x} \Big/ = \int \frac{\frac{1}{\cos^2 x}}{a - b \operatorname{tg} x} dx = *$$

$$d(\operatorname{tg} x) = d\left(\frac{\sin x}{\cos x}\right) = \frac{1}{\cos^2 x} dx$$

$$\left(\frac{\sin x}{\cos x}\right)' = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$* = \int \frac{d(\operatorname{tg} x)}{a - b \operatorname{tg} x} = -\frac{1}{b} \int \frac{d(a - b \operatorname{tg} x)}{a - b \operatorname{tg} x} = -\frac{1}{b} \ln \Big| a - b \operatorname{tg} x \Big| + C.$$

13. $$\int \frac{\cos \frac{x}{\sqrt{2}}}{2 - 3 \sin \frac{x}{\sqrt{2}}} dx = \sqrt{2} \int \frac{\cos \frac{x}{\sqrt{2}}}{2 - 3 \sin \frac{x}{\sqrt{2}}} d\left(\frac{x}{\sqrt{2}}\right) = \sqrt{2} \int \frac{d\left(\sin \frac{x}{\sqrt{2}}\right)}{2 - 3 \sin \frac{x}{\sqrt{2}}} =$$

$$= -\frac{\sqrt{2}}{3} \int \frac{d\left(2 - 3 \sin \frac{x}{\sqrt{2}}\right)}{2 - 3 \sin \frac{x}{\sqrt{2}}} = -\frac{\sqrt{2}}{3} \ln \Big| 2 - 3 \sin \frac{x}{\sqrt{2}} \Big| + C.$$

14. $$\int \operatorname{ctg} x dx = \int \frac{\cos x}{\sin x} dx = \Big/ \cos x dx = d(\sin x) \Big/ = \int \frac{d(\sin x)}{\sin x} =$$

$$= \ln \Big| \sin x \Big| + C.$$

15. $$\int \cos(ax+b) dx = \frac{1}{a} \int \cos(ax+b) d(ax+b) = \frac{1}{a} \sin(ax+b) + C.$$

16. $$\int \sin \sqrt{x} \cdot \frac{dx}{\sqrt{x}} = 2 \int \sin \sqrt{x} d(\sqrt{x}) = \Big/ d(\sqrt{x}) = \frac{1}{2\sqrt{x}} \Big/ =$$

$= -2\cos\left(\sqrt{x}\right) + C.$

17. $\displaystyle\int x\cdot 5^{-x^2}dx = \frac{1}{2}\int 5^{-x^2}d(x^2) = /d(x^2) = 2xdx/ = -\frac{1}{2}\int 5^{-x^2}d(-x^2) =$
$\displaystyle= -\frac{1}{2}\cdot\frac{5^{-x^2}}{\ln 5} + C.$

18. $\displaystyle\int \frac{e^{-ax}}{1+e^{-2ax}}dx =$
$\displaystyle\Big/ u = e^{-ax};\ du = -de^{-ax}dx \ \Rightarrow\ dx = -\frac{1}{a}e^{ax}du = -\frac{1}{a}\cdot\frac{1}{u}du\Big/$
$\displaystyle= -\int\frac{u}{1+u^2}\cdot\frac{1}{au}du = -\frac{1}{a}\int\frac{1}{1+u^2}du = -\frac{1}{a}\operatorname{arctg} u + C =$
$\displaystyle= -\frac{1}{a}\operatorname{arctg}\left(e^{-ax}\right) + C.$

19. $\displaystyle\int\frac{x^2dx}{\sqrt{x^6+1}} = \frac{1}{3}\int\frac{d(x^3)}{\sqrt{(x^3)^2+1}} = /d(x^3) = 3x^2dx/ =$
$\displaystyle= \frac{1}{3}\ln\left|x^3+\sqrt{x^6+1}\right| + C.$

20. $\displaystyle\int\frac{x^2-9}{x^2-8}dx = \int\frac{x^2-8-1}{x^2-8}dx = \int dx - \int\frac{1}{x^2-8}dx = x+$
$\displaystyle+\int\frac{dx}{8-x^2} = x + \frac{1}{2\sqrt{8}}\ln\left|\frac{x+\sqrt{8}}{x-\sqrt{8}}\right| + C.$

21. $\displaystyle\int\frac{dx}{\sqrt{5-3x^2}} = \int\frac{dx}{\sqrt{3}\sqrt{\frac{5}{3}-x^2}} = \frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{\left(\sqrt{\frac{5}{3}}\right)^2-x^2}} =$
$\displaystyle= \frac{1}{\sqrt{3}}\arcsin\frac{x}{\sqrt{\frac{5}{3}}} + C.$

22. $\displaystyle\int\operatorname{tg} x dx = \int\frac{\sin x}{\cos x}dx = -\int\frac{d(\cos x)}{\cos x} = -\ln\left|\cos x\right| + C.$

23. $\displaystyle\int\frac{xdx}{4x^2+7} = \frac{1}{2}\int\frac{d(x^2)}{4x^2+7} = /d(x^2) = 2xdx/ = \frac{1}{8}\int\frac{d(4x^2+7)}{4x^2+7} =$