Introduction to Complex Analysis

Qitian Liao

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1 Algebra of the Complex Plane

1.1 Introduction to Complex Numbers

Let $z = a + ib \in \mathbb{C}$ where $a, b \in \mathbb{R}$ and $i^2 = -1$.
This number can be thought of as a point in 2-space, $\mathbb{R}^2$, (a, b) or as a position in $\mathbb{C}$.
$\mathbb{R}^2$: $\oplus$ addition; $\odot$ scalar multiplication.
$\mathbb{C}$: $\oplus$ addition; $\odot$ scalar multiplication; a vector space; have multiplication of elements, $\mathbb{C}$ is a field.

$$\text{If } z = a + ib,\ w = c + id, \text{ then } zw = (ac - bd) + i(ad + cb)$$

$$zw = wz$$

$$z(w + \alpha) = zw + z\alpha$$

$$(zw)\alpha = z(w\alpha)$$

1.2 Conjugate of Complex Numbers

Definition of Conjugate.
The complex conjugate of z, $\overline{z}$, is defined by

$$\overline{z} = a - ib$$

Geometric representation: The image of $\bar{z}$ is the reflection of z about the Real axis.

Properties of Conjugate.

$$\overline{\overline{z}} = z$$

$$\overline{zw} = \overline{zw}$$

$$\overline{z + w} = \overline{z} + \overline{w}$$

$$\overline{z} = z \text{ if and only if } z \in \mathbb{R}$$

Real and Imaginary Parts.

We can project z onto the Real or Imaginary axis and measure its distance from 0:

$$\Re(z) = a$$

$$\Im(z) = b, \text{ not } ib$$

Each function is a map $\mathbb{C} \to \mathbb{R}$. Then

$$\Re(z) = \frac{z + \overline{z}}{2}$$

$$\Im(z) = \frac{z - \overline{z}}{2i}$$

This is similar to the pattern with even/odd functions.

1.3 Modulus of Complex Numbers.

Note: $z\overline{z} = (a+ib)(a-ib) = a^2 + b^2 \in \mathbb{R}$

Definition of Modulus.
$|z|$ length/modulus of z is defined by:

$$|z| = (a^2 + b^2)^{\frac{1}{2}} = (z\overline{z})^{\frac{1}{2}} \in \mathbb{R}$$

Properties of Modulus.

$$|zw| = |z||w|$$

$$|z| = |\overline{z}|$$

$$|z| \geqslant 0$$

$$|z| = 0 \text{ if and only if } z = 0$$

Triangle Inequality and Reverse Triangle Inequality.

$$\begin{cases} |z+w| \leqslant |z| + |w| \\ |z| - |w| \leqslant |z-w| \end{cases}$$

$$z = z - w + w \Rightarrow |z| = |z-w+w| \Rightarrow |z| \leqslant |z-w| + |w| \Rightarrow |z| - |w| \leqslant |z-w|$$

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Complex Division.
With $z\overline{z} \in \mathbb{R}$, we can define complex division by reducing it to a multiplication problem.

$$\frac{z}{w} = \frac{z\overline{w}}{w\overline{w}} = \frac{1}{w\overline{w}}(z\overline{w})$$

We also have

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|} \text{ for } w \neq 0$$

Distance in the plane.
A disk in the complex plane centered at c of radius $r \in \mathbb{R}$ is of the form

$$\{z \in \mathbb{C} \mid |z - c| \leqslant r\}$$

1.4 Complex Polynomial

A complex polynomial $p(z)$ of degree n is of the form:

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

where $a_n \neq 0$ and $a_i \in \mathbb{C}$ for $i = 0, \cdots, n$

Fundamental Theorem of Algebra.

The factorization of $p(z)$ factors over $\mathbb{C}$ is unique,

$$p(z) = c(z - z_1)^{m_1} ... (z - z_k)^{m_k}$$

We have roots $z_i \in \mathbb{C}$ of $p(z)$ with order $m_i \in \mathbb{N}$.
For example, if $p(z) = z^2 + 4 = (z + 2i)(z - 2i)$, then it factors over $\mathbb{C}$ but not $\mathbb{R}$.
Note: $\mathbb{C}$ is an algebraically closed field, there are no irreducible polynomials in $\mathbb{C}$.
Note: $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$ are not algebraically closed.

2 Geometry of the Complex Plane

2.1 Properties of Polar Forms

Complex numbers can be represented in polar forms:

$$z = |z|(\cos\theta + i\sin\theta)$$

with modulus $|z|$ and argument θ. To change between the coordinate systems it follows:

$$|z| = (a^2 + b^2)^{\frac{1}{2}}$$

$$\tan\theta = \frac{b}{a}$$

$$a = |z|\cos\theta = \Re(z)$$

$$b = |z|\sin\theta = \Im(z)$$

Note: $\theta_R = \arctan(\frac{b}{a})$ is a reference angle of z. To find θ from θ_R, you need to consider the signs of a and b.
Example:
$z = -3 + 3i = 3\sqrt{2}(\cos\frac{3\pi}{4} + \sin\frac{3\pi}{4})$
$\theta_R = \arctan(\frac{3}{-3}) = -\frac{\pi}{4}$
$\theta = \pi + \theta_R = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$, since θ is in II.

2.2 Definition of Argument and argument

Arg(z) is z's principle polar angle θ, $z \neq 0$, where $\theta \in (-\pi, \pi]$.
arg(z) is all of z's polar angles, $\theta + 2k\pi$, $k \in \mathbb{Z}$.

2.3 Euler's Formula

Euler's Formula is defined as a linear combination of $\cos\theta$ and $\sin\theta$, $\mathbb{R}$-valued functions.

$$e^{i\theta} = \cos\theta + i\sin\theta$$

It allows us to express z in polar form by

$$z = |z|e^{i\theta}$$

-1 has polar angle π and modulus 1,

$$-1 = e^{i\pi} \text{ or } e^{i\pi} + 1 = 0$$

By the angle addition formulas from trigonometry we find:

$$e^{i\theta}e^{i\varphi} = e^{i(\theta+\varphi)}$$

$$(e^{i\theta})^k = e^{i\theta k}$$

2.4 Geometric Understanding of Multiplication

The polar angle of zw is the sum of the polar angles of z and w. The modulus is the product of the moduli.

$$\operatorname{Arg}(zw) = \operatorname{Arg}(z) + \operatorname{Arg}(w)$$

$$\operatorname{Arg}(\overline{z}) = -\operatorname{Arg}(z)$$

Question: How about $\frac{z}{w}$ and z^4?
It follows from trigonometry that $|e^{i\theta}| = 1$, if $\theta \in (-\pi, \pi]$ we get a parametrization of the unit circle.

Example. Discover all solutions to $w^3 = i = z$
Let $p(z) = w^3 - i$. By Fundamental Theorem of Algebra, there are 3 roots of $p(z)$.
Therefore, $3\theta = \frac{\pi}{2} + 2\pi k$, $k \in \mathbb{Z}$
This gives us infinitely many solutions, but the solutions form 3 equivalence classes.
All we need is $k = 0, 1, 2$, which gives $\theta_1 = \frac{\pi}{6}$, $\theta_2 = \frac{5\pi}{6}$, $\theta_3 = \frac{3\pi}{2}$
Our solutions partitioned the unit circle into 3 equally spaced wedges.
The solutions to $w^3 = i$ are $w_1 = \frac{\sqrt{3}}{2} + \frac{1}{2}i$, $w_2 = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$ and $w_3 = -i$.
This problem of unity can be extended to solving $w^k = z$ for $k \in \mathbb{N}$, $z \in \mathbb{C}$ for unknown k-solutions w.

3 Stereographic Projections, Exponentials and Logs

3.1 Stereographic Projections

We can express the complex plane on the unit sphere in $\mathbb{R}^3$. To perform this we project points on the surface of the sphere along the line from the North Pole $(0,0,1)$ through the point and onto the plane $z = 0, \mathbb{C}$
$p_1 = (x_1, x_2, x_3) \to z = a + ib = \frac{x_1 + ix_2}{1 - x_3}$
$x_1 = \frac{2a}{|z|^2+1}$, $x_2 = \frac{2b}{|z|^2+1}$, $x_3 = \frac{|z|^2-1}{|z|^2+1}$
Points in the northern hemisphere P_1, have $|z_1| > 1$; while points in the southern hemisphere P_2, have $|z_2| < 1$.

Mapping from Stereographic Space to the Complex Plane.

$$\begin{aligned} \mathbb{S}^2 &\to \mathbb{C} \\ N = (0,0,1) &\to \infty \\ S = (0,0,-1) &\to 0 \\ \text{lines of latitude } &\to |z| = r, \text{ circles} \\ \text{lines of longitude } &\to \operatorname{Arg}(z) = \pm\theta, \text{ lines through } (0,0) \end{aligned}$$

Note: In general, circles on $\mathbb{S}^2$ map to circles and lines in $\mathbb{C}$, orientation is not always preserved.

3.2 Complex Logarithm

Logarithm of Real Numbers.
Anytime we are dealing with power, the log function is very useful.

$$\log x = \int_1^x \frac{1}{t} dt \text{ for } x \in \mathbb{R}$$

$$\frac{d}{dx} x^x = \frac{d}{dx} e^{\ln x^x} = \frac{d}{dx} e^{x \ln x} = e^{x \ln x}(x \cdot \frac{1}{x} + \ln x) = x^x(1 + \ln x)$$

Logarithm of Complex Numbers.
Remember from Euler's Formula, $e^{i\theta} = \cos\theta + i \sin\theta$.

$$e^z = e^{a+ib} = e^a e^{ib}$$

$$\operatorname{Arg}(e^z) = b$$

$$|e^z| = e^a > 0$$

Therefore, if a is held fixed, e^z maps to a circle as b changes.
On the other hand, if b is held fixed, e^z maps to a line through $(0,0)$.

Derivation of Complex Logarithm.
We want $e^{\log(z)} = z$ for all $z \neq 0$, and thus

$$e^{\Re(\log(z)) + i\Im(\log(z))} = e^{\Re(\log(z))} e^{i\Im(\log(z))} = |z| e^{i\theta} = z$$

$$\Rightarrow |z| = e^{\Re(\log(z))}$$

$$\Rightarrow \Re(\log(z)) = \log|z|$$

From the imaginary part we find

$$e^{i\theta} = e^{i\Im(\log(z))}$$

$$\Rightarrow \arg(z) = \theta = \Im(\log(z))$$

$$\Rightarrow \Im(\log(z)) = \mathrm{Arg}(z)$$

because $arg(z)$ is not well defined.
Our constructed inverse of e^z is a multi-valued function

$$\log(z) = \log|z| + i\arg(z)$$

Conclusion from Derivation.

$$\log(z) = \log|z| + i\arg(z)$$

$$\mathrm{Log}(z) = \log|z| + i\,\mathrm{Arg}(z)$$

Note: $\mathrm{Log}(z)$ does not have all the nice behavior as $\mathbb{R}$-valued $\log(x)$: $\mathrm{Log}\,(z^k)$.
Sometimes they are co-terminal angles, but they are not equal. See the following example:

$$\begin{cases} \mathrm{Log}(i^3) = \mathrm{Log}(-i) = -i\frac{\pi}{2} \\ 3\,\mathrm{Log}(i) = 3 \cdot (i\frac{\pi}{2}) = i\frac{3\pi}{2} \end{cases}$$

Example. Compute 3^i:

$$3^i = e^{\mathrm{Log}\,3^i} = e^{i\,\mathrm{Log}\,3} = \cos(\mathrm{Log}\,3) + i\sin(\mathrm{Log}\,3)$$

How Logarithm acts on curves.

$$\begin{cases} \text{Maps a circle with radius } r \text{ to a vertical line passing through } (\ln(r), 0) \\ \text{Maps a line with angle } \theta \text{ passing through the origin to a horizontal line passing through } (0, i\theta) \end{cases}$$

4 Topology in $\mathbb{C}$

4.1 Sequence

Let $\{Z_n\}$ be a sequence in $\mathbb{C}$.

Cauchy Sequence.
The sequence is Cauchy if for all $\epsilon > 0$, there is a $N \in \mathbb{N}$ such that for all $n, m > N$, $|z_n - z_m| < \epsilon$.

Convergence of Sequence.
The sequence converges if $|z_n - z| \to 0$ as $n \to \infty$. The distance between z_n and z vanishes.

Completeness of $\mathbb{C}$.
$\{z_n\}$ converges if and only if $\{z_n\}$ is Cauchy.
Proof.
We show this by treating $\mathbb{C}$ as $\mathbb{R}^2$ and exploiting $\{X_n\}$ converges if and only if $\{X_n\}$ is Cauchy.
($\Longrightarrow$) (If $z_n \to z$, then $\Re(z_n) \to \Re(z)$ and $\Im(z_n) \to \Im(z)$. Since the sequences of $\mathbb{R}^2$ converge, they are Cauchy.
$|Z_n - Z_m| \leqslant |\Re(Z_n - Z_m)| + |\Im(Z_n - Z_m)| = |\Re(Z_n) - \Re(Z_m)| + |\Im(Z_n) - \Im(Z_m)|$
Upper bounds can be picked to be less than $\frac{\epsilon}{2}$ for some N. Therefore, $|Z_n - Z_m| \to 0$.

($\Longleftarrow$) If $\{Z_n\}$ is Cauchy, so are $\{\Re(Z_n)\}$ and $\{\Im(Z_n)\}$. But these are $\mathbb{R}$-sequences that converge. Therefore, $\{Z_n\}$ converges.

4.2 Complex Set

Let $\Omega \subset \mathbb{C}$. Sets can be open, closed, both, or neither.

Open Set.
If for any $z_0 \in \mathbb{C}$, there exist some $\epsilon > 0$, such that the set $B_\epsilon(z_0) = \{z||z - z_0| < \epsilon\}$ is contained in Ω, then Ω is open.
Ω is open if and only if Ω^c is closed.
Ω is open if and only if Ω is equal to its own interior, which means it does not contain its boundary points $\partial\Omega$, i.e. it does not contain its closure.

Closed Set.
If Ω contains its limit point, then Ω is closed.
Ω is closed if and only if Ω^c is open.
Ω is closed if and only if Ω contains its boundary points.

Compact Set.
If Ω can be contained in a disk of finite radius, then Ω is bounded.

Compact Set.
If Ω is closed and bounded, then Ω is compact. This resembles $[a, b]$ in $\mathbb{R}$.

Connected Set.
If any two points in Ω can be connected by a path, then Ω is connected.
Simply Connected Set: A simply connected set has no "holes" in it. For example, $\Omega = \{z||z - c| < 4\}$.

A connected but not simply connected set is an annulus, $\Omega = \{z|2 < |z-c| < 4\}$

Boundary of Set.
The boundary of Ω, $\partial\Omega$ is all points with ϵ-balls intersecting Ω and Ω^c for all $\epsilon > 0$.

Interior of Set.
The interior of Ω, $\mathrm{Int}(\Omega)$, is all points in Ω with a ϵ-ball contained in Ω for some $\epsilon > 0$. "Largest open set in Ω".

Closure of Set.
The closure of Ω is the union of Ω and its boundary $\partial\Omega$.

Domain.
If a set is open and connected in $\mathbb{C}$, it is a domain.
A domain can be traversed by a path of horizontal and vertical line segments.

Example.
Determine whether the following sets are open or closed.

1. $\Omega = \mathbb{C}\backslash\{0\}$
   Ω is open since it does not contain its closure, the point 0.
   Ω is not closed since it does not contain its limit points. Let $z_n = \frac{1}{n}$. Then $z_n = \frac{1}{n} \to 0 \notin \Omega$.
   Therefore, Ω is open.

2. $\Omega = \{z||z| \geqslant 1\}$
   Ω is not open since any ϵ-ball at 1 intersects Ω^c.
   Ω is closed since Ω^c is open.
   Therefore, Ω is closed.

3. $\Omega = \{z||z| > 1\}$
   Ω is open since Ω^c is closed.
   Ω is not closed since it does not contain its limit points. Let $z_n = \frac{1}{n} + 1$. Then $z_n = \frac{1}{n} + 1 \to 1 \notin \Omega$.

4. $\Omega = \mathbb{C}\backslash(0,1)$
   Ω is not open. Its complement is $[0,1]$. Even though it is closed in $\mathbb{R}$, it is not closed in $\mathbb{C}$, because any 2D ϵ-ball will always extend outside of the set $z \in (0,i)$. Hence, Ω^c is not open and not closed.
   Ω is not closed since it does not contain its limit points. Let $z_n = \frac{1}{3} + i\frac{1}{n}$. Then $z_n = \frac{1}{3} + i\frac{1}{n} \to \frac{1}{3} \notin \Omega$.
   Therefore, Ω is neither open nor closed.

5. $\Omega = \mathbb{C}\backslash[0,1]$ Ω is open since $\Omega^c = [0,1]$ is closed in $\mathbb{C}$.
   Ω is not closed since it does not contain its limit points. Let $z_n = \frac{1}{3} + i\frac{1}{n}$. Then $z_n = \frac{1}{3} + i\frac{1}{n} \to \frac{1}{3} \notin \Omega$.
   Therefore, Ω is open.
   Note: Ω^c is not open in $\mathbb{C}$.

5 Continuity and Branch Cuts

5.1 Complex Continuity

Let $f : \Omega \to \mathbb{C}$, Ω is open and connected. If $z_n \to z_0$ implies $f(z_n) \to f(z_0)$, then f is continuous at z_0. Also, f is bounded near z_0.
f is continuous if for every $\epsilon > 0$, there is $\delta > 0$ such that $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon$.

In either case, $\Re(f(z))$ and $\Im(f(z))$ are each continuous if and only if $f(z)$ is continuous. This follows the pattern as $\mathbb{C}$ being complete.

If f and g are continuous, then so are $f + g$m $f \times g$ and $\frac{f}{g}$ (provided $g(z) \nrightarrow 0$)

5.2 Complex Limits

Just like in $\mathbb{R}^2$, limits are direction independent. Do not restrict limits to just $\Re \to 0$ or $\Im \to 0$. See the following example.

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4 + y^2} \text{ does not exist}$$

as $x \to 0$, $y = 0$, then $f \to 0$, while $y = x^2, x \to 0$, then $f \to 1$.

5.3 Branch Cuts

Log, $z^{\frac{1}{2}}$ and $\arctan(z)$ are constructed by restricting the range of e^z, z^2 and $\tan(z)$.
For example, in creating $\operatorname{Log}(z) = \ln|z| + i \operatorname{Arg}(z)$, we made a choice that $\operatorname{Arg}(z) \in (-\pi, \pi]$, $\operatorname{Arg}(0)$ does not exist.

Example. Consider a path around $z_0 \neq 0$, $\gamma(t) = z_0 + re^{it}$. $\theta(t) = \arg(\gamma(t))$.
As we traverse the circle, $t \in (-\pi, \pi]$,

$$\theta(t) = \arg(\gamma(t)) = \operatorname{Arg}(z_0 + re^{it}) + 2\pi k = \operatorname{Arg}(z_0 + re^{i(t+2\pi)}) + 2\pi k = \arg(\gamma(t + 2\pi)) = \theta(t + 2\pi)$$

Therefore, the angle $\theta(t)$ changes smoothly for all t and we stay on the same branch of $\operatorname{Arg}(\gamma(t))$. That is to say, the $k \in \mathbb{Z}$ is the same for all t.

Compare this with any circular path about $z = 0$, γ_0. Let $\gamma_0(t) = re^{it}$, $t \in (-\pi, \pi]$. As we traverse the circle once, we have a discontinuity in the principal angle of $\gamma_0(t)$. In particular, $\theta(\gamma_0(t)) \neq \theta(\gamma_0(t+2\pi))$

$$\theta(t) = \arg(\gamma(t)) = \operatorname{Arg}(re^{it}) + 2\pi k \neq \operatorname{Arg}(re^{i(t+2\pi)}) + 2\pi(k + 1) = \arg(\gamma(t + 2\pi)) = \theta(t + 2\pi)$$

We jump from the kth to the $(k + 1)$th branch of Arg. Therefore, $\operatorname{Arg}(z)$ has a branch point at $z = 0$.

Definition of Branch Cuts and Branch Points.
If every neighborhood of z_0 contains a path $\gamma(t)$ around z_0 that leads to a jump discontinuity in f, then z_0 is a branch point of $f(z)$.
In order to find branches, at this point, it suffices to study paths of the form $\gamma(t) = z_0 + re^{it}$ for $t \in (-\pi, \pi)$, and see if $f(\gamma(t)) = f(\gamma(t + 2\pi))$ holds for all t.

Example. Arg is discontinuous for all x on the negative $\mathbb{R}$-axis, $\mathbb{R}^-$.

We call this the principal branch cut of the multi-valued function arg. Specifically,

$$\mathrm{Arg}(\gamma_0(t)) \to \pi \text{ as } t \to \pi^-$$

$$\mathrm{Arg}(\gamma_0(t)) \to -\pi \text{ as } t \to -\pi^+$$

but $\gamma_0(\pi) = \gamma_0(-\pi)$ since π and $-\pi$ are coterminal.
$\mathbb{R}^-$ is the principal branch of Log, Arg, and $z^{\frac{1}{2}}$.
The endpoints of a branch cut are branch points, Arg has 0 and ∞ as its branch points.

6 Differentiability in $\mathbb{C}$

Let $f : \Omega \to \mathbb{C}$ for some domain Ω. Then f is differentiable at z_0 if the following exists.

$$\frac{d}{dz} f(z)|_{z=z_0} = f'(z_0) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

This limit must exist on all paths to z_0, since $h \in \mathbb{C}$. We could also take $z_n \to z_0$ and use $\frac{f(z_0)-f(z_n)}{z_0-z_n} \to f'(z_0)$. Remember limits are computed by looking at the difference in the modulus, $|\frac{f(z_0)-f(z_n)}{z_0-z_n} - f'(z_0)| \to 0$ as $n \to \infty$.

If $f'(z_0)$ exists on all points $z_0 \in \Omega$, open and connected in $\mathbb{C}$, then f is holomorphic/$\mathbb{C}$-differentiable/analytic on Ω. The connection between $\mathbb{R}$ and $\mathbb{C}$ analytic will be clear when we cover $\mathbb{C}$-power series.

If $f'(z)$ exists everywhere in $\mathbb{C}$, then f is an entire/meromorphic function.

6.1 Difference between $\mathbb{R}$ and $\mathbb{C}$ differentiability

• $f : \mathbb{R} \to \mathbb{R}$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

has only two paths to x, namely $h \to 0^+$ and $h \to 0^-$.

Tangent plane or linear approximation:

$$f(x) \approx f(a) + f'(a)(x-a)$$

• $f : \mathbb{R}^2 \to \mathbb{R}$

$$\lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} = f_x$$

is also a 1D limit and a partial derivative.

Tangent plane or linear approximation:

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

• $f : \mathbb{R}^2 \to \mathbb{R}^2$ $f(x,y) = (u(x,y), v(x,y))$

Then f is differentiable if the Jacobian Matrix

$$J(f) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

can approximate the local change in f.

In each of the cases above, we are only measuring change in a few directions. However, $h \to 0$ in $\mathbb{C}$ can be from any direction in 2-space. Therefore, $f'(z)$ existing is a much stronger condition for f on $\mathbb{C}$ than on $\mathbb{R}$.

Complications with $\overline{z}$.

Consider the following example:

Let $g(z) = \Re(z) = \frac{z+\overline{z}}{2}$, which is a linear combination of continuous functions.

Assume $h \in \mathbb{R}$,

$$\frac{g(z+ih) - g(z)}{h} = \frac{\Re(z) - \Re(z)}{h} \to 0 \text{ as } h \to 0$$

Compare this with

$$\frac{g(z+ih) - g(z)}{h} = \frac{\Re(z) + h - \Re(z)}{h} = 1 \to 1 \text{ as } h \to 0$$

Therefore, the function is nowhere differentiable in $\mathbb{C}$. The problem with $g'(z)$ had to do with $\overline{z}$, despite reflection in $\mathbb{R}^2$ about $y = 0$ is differentiable. We will discover conditions on u_x, u_y, v_x, and v_y that ensure f' exists for $f(z) = u(x,y) + iv(x,y)$ in the next chapter.

6.2 Properties of $f'(z)$

Proposition: If f is differentiable on Ω, it is continuous on Ω.
Proposition: The power rule holds too: $\frac{d}{dz}z^n = nz^{z-1}$
As does the product, quotient, L'Hospital's and chain rule. In fact, most old results hold as well. In each of the following cases, there are branch points where $f'(z)$ does not exist.
1. If $f(z) = \text{Log}(z)$, then $f'(z) = \frac{1}{z}$
2. If $f(z) = \tan^{-1}(z)$, then $f'(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$, which does not exist for $z = \pm i$
3. If $f(z) = z^{\frac{1}{2}}$, then $f'(z) = \frac{1}{2}z^{-\frac{1}{2}}$

Proposition: Suppose $f(z)$ is holomorphic on Ω. then $g(z) = \overline{f(\overline{z})}$ is holomorphic on $\Omega^* = \{z|\overline{z} \in \Omega\}$
Proof:
Suppose f is holomorphic on Ω. Let $z_0 \in \Omega^*$ and $z_n \in \Omega^*$ for all n and $z_n \to z$. Then $\overline{z_n} \to \overline{z_0}$ in Ω and for $\epsilon > 0$, there is a $N \in \mathbb{N}$ such that for $n > N$.

$$\left|\frac{f(\overline{z_0}) - f(\overline{z_n})}{\overline{z_0} - \overline{z_n}} - f'(\overline{z_0})\right| < \epsilon$$

$$\left|\frac{f(\overline{z_0}) - f(\overline{z_n})}{\overline{z_0} - \overline{z_n}} - f'(\overline{z_0})\right| = \left|\overline{\frac{f(\overline{z_0}) - f(\overline{z_n})}{\overline{z_0} - \overline{z_n}} - f'(\overline{z_0})}\right| = \left|\frac{\overline{f(\overline{z_0}) - f(\overline{z_n})}}{\overline{\overline{z_0} - \overline{z_n}}} - \overline{f'(\overline{z_0})}\right|$$

$$= \left|\frac{\overline{f(\overline{z_0}) - f(\overline{z_n})}}{z_0 - z_n} - \overline{f'(\overline{z_0})}\right| = \left|\frac{g(z_0) - g(z_n)}{z_0 - z_n} - \overline{f'(\overline{z_0})}\right| < \epsilon$$

$$\implies g'(z_0) = \overline{f'(\overline{z_0})}$$

Therefore, g is holomorphic on Ω^*.
This proof is different from when we showed $\frac{d}{dz}\overline{z}$ does not exist. Conjugation must be handled with care.

6.3 Geometric behavior of $f'(z)$

Dilation.
$w = f(z) \approx f'(a)(z-a) + f(a)$. Small changes in z should give small changes in w.
The functions $|z|$ and $\text{Arg}(z)$ are continuous on their domains.
If $f'(z_0) \neq 0$ and f' exists on Ω, then

$$|f'(z_0)| = \left|\lim_{h\to 0}\frac{f(z_0+h) - f(z_0)}{h}\right| = \lim_{h\to 0}\left|\frac{f(z_0+h) - f(z_0)}{h}\right| = \lim_{h\to 0}\frac{|f(z_0+h) - f(z_0)|}{|h|}$$

$$\implies |f'(z_0)||h| \approx |f(z+h) - f(z)|$$

The size of $|f'(z_0)|$ tells us how much f is contracting/dilating near z_0.

Rotation.

$$\text{Arg}(f'(z_0)) = \text{Arg}\left(\lim_{h\to 0}\frac{f(z_0+h) - f(z_0)}{h}\right) = \lim_{h\to 0}\text{Arg}\left(\frac{f(z_0+h) - f(z)}{h}\right)$$

$$
\begin{aligned}
&= \lim_{h\to 0} \operatorname{Arg}(f(z_0+h) - f(z_0)) - \operatorname{Arg}(h) \\
&\Longrightarrow \operatorname{Arg}(f'(z_0)) \approx \operatorname{Arg}(f(z_0+h) - f(z_0)) \\
&\Longrightarrow \operatorname{Arg}(f'(z_0)) + \operatorname{Arg}(h) \approx \operatorname{Arg}(f(z_0+h) - f(z_0))
\end{aligned}
$$

Therefore, f rotates vectors from z_0 to $z_0 + h$ by the angle $\operatorname{Arg}(f'(z_0))$.

Conclusion.
In conclusion, $w = f(z) \approx f(z_0) + f'(z_0)(z - z_0) = c + \rho e^{i\theta}(z - z_0)$

$$
\begin{cases}
c\text{: Translation} \\
\rho\text{: Dilation} \\
e^{i\theta}\text{: rotation about } z_0 \text{ or complex multiplication.}
\end{cases}
$$

7 The Cauchy Riemann Equations

7.1 The Cauchy Riemann Equations

Let $f(z) = f(x+iy) = u(x,y)+iv(x,y)$, then $f(z)$ is holomorphic implies the Cauchy Riemann Equations:

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

Proof. For f, which is $\mathbb{C}$-differentiable, the following representation of $f'(z)$ holds for any path to z.

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

Along the path $x + h + iy \to x + iy$, we get

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{f(x+h+iy) - f(x+iy)}{h} = f_x = u_x(x,y) + iv_x(x,y)$$

Along the path $x + iy + ih \to x + iy$, we get

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{f(x+i(y+h)) - f(x+iy)}{ih} = \frac{f_y}{i} = -if_y = -i(u_y(x,y)+iv_y(x,y)) = v_y(x,y)-iu_y(x,y)$$

Equate components in $f_x = -if_y$, and it is proven that $u_x = v_y$ and $u_y = -v_x$.

Proposition. If the Cauchy Riemann Equations do not hold at z_0, then $f'(z_0)$ does not exist.
Proposition. If f is holomorphic on a domain Ω, an open and connected set in $\mathbb{C}$, then the Cauchy Riemann Equations hold at all points in Ω.

Example.
If $f(x + iy) = x^2 + iy^2$, $u_x = 2x$, $v_x = 0$, $u_y = 0$, $v_y = 2y$. Then $2x = 2y \Rightarrow x = y$, which is a line.
The set of points on the line is not open in $\mathbb{C}$. Therefore, f is nowhere holomorphic in $\mathbb{C}$. However, we will see that $f'(z)$ does exist on the line $y = x$.

Sufficiency of the Cauchy Riemann Equations to f'.
The Cauchy Riemann Equations do a great job showing f' does not exist. But what about it being sufficient for f'? We claim that satisfying the Cauchy Riemann Equations at z_0 implies that f' exists at z_0.
Proof.
f is $\mathbb{C}$-differentiable at z_0 if and only if $u(x,y)$ and $v(x,y)$ have continuous partial derivatives that satisfy the Cauchy Riemann Equations at z_0. This requires us to treat $f(x + iy)$ as a function on $\mathbb{R}^2$, or $f(z)$ induces a map on $\mathbb{R}^2$.
Let $h = \Delta x + i\Delta y$,

$$\frac{f(z+h) - f(z)}{h} = \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)}{\Delta x + i\Delta y} - \frac{u(x,y) + iv(x,y)}{\Delta x + i\Delta y}$$

$$u(x+\Delta x, y+\Delta y) - u(x,y) = u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y) + u(x, y+\Delta y) - u(x,y)$$

The function $u(\cdot,\cdot)$ is differentiable in x and y, we can use the M.V.T (Mean Value Theorem) from $\mathbb{R}$ to rewrite our difference in u by

$$u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y) = \Delta x U_x(\underline{x}, y+\Delta y)$$

where $\underline{x} \in (x, x+\Delta x)$.

If u_x is continuous, $u_x(\underline{x}, y+\Delta y) \approx u_x(x,y) + \epsilon_1$, and as $\Delta y \to 0$ and $\underline{x} \to x$, by Taylor approximation and linear approximation on u_x, we have the error function $\epsilon_1 \to 0$.
Next $u(x, y+\Delta y) - u(x,y) = \Delta y u_y(x, \overline{y})$ and $u_y(x,\overline{y}) \approx u_y(x,y) + \epsilon_2$.
Likewise, for the function $v(x,y)$, we get a v_x and v_y with error terms ϵ_3 and ϵ_4. s

$$\frac{f(z+h)-f(z)}{h} = \frac{\Delta x(u_x + \epsilon_1 + iv_x + i\epsilon_3) + \Delta y(u_y + \epsilon_2 + iv_y + i\epsilon_4)}{\Delta x + i\Delta y}$$

From the Cauchy Riemann Equations, we get $f_x = \frac{f_y}{i} \Rightarrow if_x = f_y \Rightarrow i(u_x + iv_x) = u_y + iv_y$. Substituting the terms, we have

$$f'(z) = \frac{\Delta x(u_x + iv_x) + i\Delta y(u_x + iv_x)}{\Delta x + i\Delta y} + \frac{\lambda}{\Delta x + i\Delta y}$$

where $\lambda = \Delta x(\epsilon_1 + i\epsilon_2) + \Delta y(\epsilon_3 + i\epsilon_4)$. However,

$$\left|\frac{\lambda}{\Delta x + i\Delta y}\right| \leqslant \left|\frac{\Delta x(\epsilon_1 + i\epsilon_2)}{\Delta x + i\Delta y}\right| + \left|\frac{\Delta y(\epsilon_3 + i\epsilon_4)}{\Delta x + i\Delta y}\right| \leqslant |\epsilon_1 + i\epsilon_2| + |\epsilon_3 + i\epsilon_4|$$

because $\left|\frac{\Delta x}{\Delta x + i\Delta y}\right| \leqslant 1$.
As $\Delta z \to 0$, $\left|\frac{\lambda}{\Delta x + i\Delta y}\right| \to 0$, and thus $f'(z) = u_x + iv_x = f_x = \frac{f_y}{i}$.
Therefore, the Cauchy Riemann Equations are an easy way to show $f'(z)$ exists and they provide a set of partial differential equations that f must satisfy.

Example. Let $f(z) = e^z = e^x(\cos(y) + i\sin(y))$

$$u = e^x\cos(y), v = e^x\sin(y)$$

$$u_x = e^x\cos(y), v_x = e^x\sin(y)$$
$$u_y = e^x\sin(y), v_x = e^x\cos(y)$$

Therefore, $f(z)$ is $\mathbb{C}$-differentiable on $\mathbb{C}$, f is entire/meromorphic. $f'(z) = f_x = u_x + iv_x = f(z)$.

7.2 Cauchy Riemann with Logarithm

$$e^{\mathrm{Log(z)}} = z \Rightarrow \frac{d}{dz}e^{\mathrm{Log(z)}} = 1 \Rightarrow z\frac{d}{dz}\mathrm{Log}(z) = 1 \Rightarrow \frac{d}{dz}\mathrm{Log}(z) = \frac{1}{z}$$

We have a branch point in $\mathrm{Log}(z)$ where its derivative is undefined. Then $\mathrm{Log}(z)$ is $\mathbb{C}$-differentiable on $\mathbb{C}\backslash\{0\}$. This is true regardless of the branch cut on $\mathrm{Log}(z)$.

7.3 Lack of Complex Mean Value Theorem

Claim: $\frac{f(z)-f(w)}{z-w} \neq f'(c)$ for some c between z and w.
Proof: Let $z = 1$, $w = 0$ and $f(t) = e^{i\pi t}$, then $f(1) - f(0) = e^{i\pi} - 1 = -2$. However, $|f'(t)| = \pi$ for all $t \in [0, 1]$.
Follow-Up Question: Does the lack of a Mean Value Theorem for $f'(z)$ suggest $f'(z) = 0$ not imply f is constant?
Answer: Suppose f is $\mathbb{C}$-differentiable on Ω and one of the following holds, then f is constant on Ω.

$$\begin{cases} f'(z) = 0 \\ |f(z)| \text{ is constant} \\ Re(f(z)) \text{ is constant} \\ \text{f's conjugate is } \mathbb{C}\text{-differentiable on } \Omega \end{cases}$$

7.4 Wirtinger Equations

There is another way to study the Cauchy Riemann Equations by introducing two operators:

$$\frac{\partial f}{\partial z} = f_z \text{ and } \frac{\partial f}{\partial \overline{z}} = f_{\overline{z}}$$

$$f(x, y) \equiv f(x + iy) = u(x, y) + iv(x, y)$$

$$f(x, y) = f(\Re(z), \Im(z)) = f(\frac{z + \overline{z}}{2}, \frac{z - \overline{z}}{2i})$$

From the chain rule, we get

$$f_z = f_x x_z + f_y y_z = \frac{1}{2} f_x + \frac{1}{2i} f_y = \frac{1}{2} f_x - \frac{i}{2} f_y$$

$$f_{\overline{z}} = f_x x_{\overline{z}} + f_y y_{\overline{z}} = \frac{1}{2} f_x - \frac{1}{2i} f_y = \frac{1}{2} f_x + \frac{i}{2} f_y$$

where $f_x = u_x + iv_x$ and $f_y = u_y + iv_y$
These are the Wirtinger Equations.

$$\begin{cases} \frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}) \\ \frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}) \end{cases}$$

Relationship with the Cauchy Riemann Equations.
From the Cauchy Riemann Equations $if_x = f_y$ we get:

$$f_{\overline{z}} = \frac{1}{2} f_x + \frac{i}{2} f_y = \frac{1}{2} f_x - \frac{1}{2} f_x = 0$$

$$f_z = \frac{1}{2} f_x - \frac{i}{2} f_y = \frac{1}{2} f_x - \frac{i^2}{2} f_x = f_x = f'(z)$$

f is $\mathbb{C}$-differentiable at z_0 if and only if $f(x, y) = u(x, y) + iv(x, y)$ is $\mathbb{R}$-differentiable at z_0 and $f_{\overline{z}}(z_0) = 0$. Then $f'(z_0) = f_z(z_0)$. In other words, $f(z)$ does not depend on $\overline{z}$.

8 Harmonic Functions

8.1 Laplacian

Let $u : \mathbb{R}^2 \to \mathbb{R}$, then the Laplacian of u is

$$\Delta u = u_{xx} + u_{yy} = \nabla \cdot \nabla u$$

where $\nabla = [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}]^T$ is the divergence operator and $\nabla u = [u_x, u_y]^T$ is the gradient of u.

8.2 Harmonic Functions

If $\Delta u = 0$, then $u(x, y)$ satisfies Laplace's (partial differential) equation or u is a harmonic function.
This means:

$$\begin{cases} u \text{ is continuous} \\ u\text{'s } 1^{\text{st}} \text{ and } 2^{\text{nd}} \text{ order partial derivatives exist and are smooth.} \end{cases}$$

Proposition. Suppose $f = u + iv$ is holomorphic on Ω where $u(x, y)$ and $v(x, y)$ have continuous 2^{nd} order partial derivatives, then u and v are harmonic and v is the harmonic conjugate of u.
Proof.
By the Cauchy Riemann Equations, $u_x = v_y$ and $v_x = -u_y$, then $u_{xx} = v_{yx}$ and $v_{xy} = -u_{yy}$. By continuity of v_{yx} and v_{xy}, $v_{yx} = v_{xy}$. This implies $u_{xx} = -u_{yy} \Rightarrow u_{xx} + u_{yy} = 0$
Later on, we will find that the conditions on 2^{nd} order partial derivatives is implied by f being holomorphic on Ω, or f'' exists.

Definition of Harmonic Conjugate.
The harmonic conjugate to $u(x, y)$ is a function $v(x, y)$, such that $f(x, y) = u(x, y) + iv(x, y)$ is holomorphic.

Example. Show that $u(x, y) = x^3 - 3xy^2 + y$ is a harmonic function.
$u_x = 3x^2 - 3y^2$, $u_y = -6xy + 1$
$u_{xx} = 6x$, $u_{yy} = -6x$. Therefore, $u_{xx} + u_{yy} = 0$

Example. Find the harmonic conjugate of $u(x, y) = x^3 - 3xy^2 + y$.
$u_x = 3x^2 - 3y^2 = v_y$
$u_y = -6xy + 1 = -v_x$
$\Rightarrow v = 3x^2y - y^3 + C(x)$ or $v = 3x^2y - x + C(y)$
Therefore, $v = 3x^2y - y^3 - x + C$ is u's harmonic conjugate.

Proposition.
If u is harmonic on a domain Ω, then u_x is the real part of a holomorphic function on Ω. If Ω is simply connected, unlike $\mathbb{C}\backslash\{0\}$, then u is the real part of a holomorphic function on Ω.
Proof.
Assume u is harmonic and Ω is connected. If $f = u_x - iu_y$, then $f_y = if_x$. Hence, f is differentiable on Ω. The simply connected statement requires future theorems to show $F'(z) = f(z)$ for some holomorphic antiderivative $F(z)$.

9 Conformal Maps

Example. Let $f(z) - (x+iy)^2 + 2(x+iy) = (x^2+2x-y^2) + i2(xy+y)$. When are the component functions, $u(x,y)$ and $v(x,y)$ constant?
When are the component functions , $u(x,y)$ and $v(x,y)$, constant?
The function $f(z) = e^z = e^x(\cos(y) + i\sin(y))$ maps the set $\Omega = \{z : |\Im(z)| < \pi\}$ to circles of radius $r \in (-\infty, \infty)$, or all points in $\mathbb{C}\backslash\mathbb{R}^-$. This coincides with the branch cut of $\mathrm{Log}(z)$, or how we made e^z invertible.

9.1 Preservation of Angles

We will now show e^z preserves the angles between curves in Ω. Let us first look at the following example.
Let $\gamma_1(t) = 2i\pi t - i\pi$, $\gamma_2(t) = t + i\frac{\pi}{4}$. $\gamma_1(0) = -i\pi$, $\gamma_1(1) = i\pi$.
The curves γ_1 and γ_2 intersect at an angle $\frac{\pi}{2}$. Also $f(\gamma_1)$ is a circle centered at 0 while $f(\gamma_2)$ is a line through $z = 0$. Their intersection in the w-plane is $\frac{\pi}{2}$ as well.
We will show why $f(z) = e^z$ does this by studying the angles between curves γ_1 and γ_2 and curves $\tau_1 = f(\gamma_1)$ and $\tau_2 = f(\gamma_2)$. If $\gamma(t)$ parameterizes a smooth curve in $\mathbb{C}$, then its tangent vector is $\gamma'(t)$. The angle between any two curves at z_0 is the angle between their tangent vectors at z_0.
Assume the curves intersect at $\gamma(r_0) = \gamma(s_0) = z_0$.
Let the angle of intersection, θ, measured from γ_1' to γ_2' in the counter-clockwise direction.
Let the angle of intersection after transformation of f, φ, measured from τ_1' to τ_2' in the counter-clockwise direction. From past chapters, we know $\theta \approx \varphi$ if f is holomorphic. Now, let us assume f is only $\mathbb{R}$-differentiable and see how f acts on the angle θ.
Curve: $\gamma(t) = (x(t), y(t))$
New curve, f on γ: $\tau(t) = f(\gamma(t)) = u(\gamma(t)) + iv(\gamma(t)) = (\underline{X}(t), \underline{Y}(t))$
New tangent vector: $\tau'(t) = \frac{t}{dt}\tau(t) = (\underline{X}'(t), \underline{Y}'(t))$, where we can invoke the chain rule:

$$\underline{X}'(t) = u_x(\gamma(t))x'(t) + u_y(\gamma(t))y'(t)$$
$$\underline{Y}'(t) = v_x(\gamma(t))x'(t) + v_y(\gamma(t))y'(t)$$

If we have $f = u(x,y) + iv(x,y)$ is $\mathbb{R}$-differentiable, then

$$J(f) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

is the Jacobian Matrix of f and $\tau'(t) = \gamma'(t) \cdot J(f)^T$
If f is $\mathbb{C}$-differentiable and $\gamma(r_0) = z_0$ where $\gamma(t) = x(t) + iy(t)$, then $f'(z) = u_x + iv_x$ and $\gamma'(t) = x' + iy'$.

$$f'(z_0)\gamma'(r_0) = f'(\gamma(r_0))\gamma'(r_0) = (u_x + iv_x)(x' + iy') = (u_x x' - v_x y') + i(u_x y' + v_x x')$$

Applying the Cauchy Riemann Equations,

$$(u_x x' - v_x y') + i(u_x y' + v_x x') = (u_x x' + u_y y') + i(v_x x' + v_y y') = (u_x x' + u_y y', v_x x' + v_y y') \text{ in } \mathbb{R}^2$$
$$= \gamma'(r_0)\mathrm{Jf}(z_0)^T = \tau'$$

By now, we have an understanding of how f acts on tangent vectors when $f' \neq 0$, namely $\theta = \varphi$.

9.2 Conformal Function

Conditions of Conformal Functions.
We say f is a conformal map at z_0 if the following hold:
1 f is $\mathbb{R}^2$-differentiable at z_0
2 $|\text{Jf}| \neq 0$
3 f preserves the oriented angle θ, between γ_1 and γ_2 and τ_1 and τ_2 at z_0 and $f(z_0)$.

e^z is conformal.
Now let us take a closer look at e^z and figure out why it is conformal.
1 holds apparently.
2 $f(z) = e^z = e^x \cos(y) + ie^x \sin(y)$, then

$$J(f(x,y)) = \begin{bmatrix} e^x \cos(y) & -e^x \sin(y) \\ e^x \sin(y) & e^x \cos(y) \end{bmatrix} = e^x \begin{bmatrix} \cos(y) & -\sin(y) \\ \sin(y) & \cos(y) \end{bmatrix}$$

$\Rightarrow |J(f(x,y))| \neq 0$
3 Now we have shown in the previous part that Jf is the product of a dilation matrix, $e^x I$, and a rotation matrix, which means f preserves the angles between γ_1' and γ_2'. Their image under f:

$$\tau_1'(t) = \gamma_1'(t) \cdot J(f)^T$$
$$\tau_2'(t) = \gamma_2'(t) \cdot J(f)^T$$

In fact, f preserving oriented angles implies Jf is a rotation $\otimes$ dilation matrix.
Hence, it is proven that e^z is conformal.

9.3 Conformal Map

Definition. If f is conformal, infinitely differentiable, and one-to-one on a domain Ω to V, then f is a conformal map from Ω to V.
For example, e^z is conformal map from $\Omega = \{z : |\Im(z)| < \pi\}$ to $V = \mathbb{C} \backslash \mathbb{R}^-$.
Proposition. If f is complex differentiable and $f'(z_0) \neq 0$, it is a linear transform of a dilation by $|f'(z_0)|$ and a rotation by $\text{Arg}(f'(z_0))$. Hence, f is conformal because 3 is satisfied.
Example. $f(z) = z^2$ on $\Omega = \{z | 1 < |z| < 3 \text{ and } \Im(z) > 0\}$ is conformal.

Inverse Function Theorem.
If f is a continuously differentiable function with nonzero derivative at the point a, then f is invertible in a neighborhood of a, the inverse is continuously differentiable, and the derivative of the inverse function at $b = f(a)$ is the reciprocal of the derivative of f at a:

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}$$

Proposition. If f is invertible at z_0 and conformal, then f^{-1} is conformal at $f(z_0)$ by the inverse function theorem provided f is continuously differentiable.

From the proposition above, we know that $\text{Log}(z)$ is conformal on $\mathbb{C} \backslash \mathbb{R}^-$.

10 Bilinear Transformations

In complex analysis, the term linear transformation is used to describe affine transformations, $f(z) = az + b$.

10.1 Definition of a Möbius transformation

A bilinear/Möbius transformation is of the form

$$\frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$.
Now, $f(\infty) = \frac{a}{c}$ by L'Hospital argument and we say $f(-\frac{d}{c}) = \infty$.
If $ad - bc \neq 0$, then $f' \neq 0$ by the quotient rule and f is not constant. The constants are not unique, as $f(z) = \frac{az+b}{cz+d} = \frac{(az+b)k}{(cz+d)k}$. Therefore, we only have 3 degrees of freedom.

10.2 Brief Review of other Transformations

We have already seen these functions of this form before f is:
1 Composition of a finite number of

$$\begin{cases} \text{Translations, } f(z) = z + k \\ \text{Rotations, } f(z) = e^{i\theta} z \\ \text{Dilations, } f(z) = kz, k \in \mathbb{R} \\ \text{Inversions, } f(z) = \frac{1}{z} \end{cases}$$

2 Conformal, f is holomorphic away from $z = -\frac{d}{c}$, $f' \neq 0$, and f is one-to-one.
3 Maps circles/lines to either lines or circles, "lines are circles of ∞-radius in $\mathbb{C}$ or $\mathbb{C} \cup \{\infty\}$.
If the line or circle passes through $z = -\frac{d}{c}$, where f is undefined, then it will be mapped to a line. Otherwise, it is mapped to a circle.
4 f can be identified by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then,

$$\begin{cases} f \circ f \equiv A \circ A = A^2 \\ f^{-1} \equiv A^{-1} \\ f \circ g \equiv AB \end{cases}$$

and so there is a group homomorphism with Möbius transforms and invertible matrices in $\mathbb{C}^{2\times 2}$.
$f(z) = (3+2i)z - i^3 = \frac{(3+2i)z - i^3}{\delta z + 1}$
5 We can conformally map 3 points in $\mathbb{C} \cup \{\infty\}$ to any 3 points in $\mathbb{C} \cup \{\infty\}$.
This type of argument is similar to showing norms are equivalent in $\mathbb{R}^n$ or uniqueness of power series expansions.

10.3 Möbius transforming of a function

Given any 3 points z_0, z_1, $z_2 \in \mathbb{C}$, we can create a Möbius transformation T such that

$$\begin{cases} T(z_0) = 0 \\ T(z_1) = 1 \\ T(z_2) = \infty \end{cases} \qquad \begin{cases} T^{-1}(0) = z_0 \\ T^{-1}(1) = z_1 \\ T^{-1}(\infty) = z_2 \end{cases}$$

Then

$$T(z) = (z, z_0, z_1, z_2) = \frac{(z - z_0)(z_1 - z_2)}{(z - z_2)(z_1 - z_0)}$$

is called the cross-ratio of z, z_0, z_1, and z_2.
There are some special cases:

$$\begin{cases} (z, \infty, z_1, z_2) = \frac{z_1 - z_2}{z - z_2} \\ (z, z_0, \infty, z_2) = \frac{z - z_0}{z - z_2} \\ (z, z_0, z_1, \infty) = \frac{z - z_0}{z_1 - z_0} \end{cases}$$

Given 3 more points $w_0, w_1, w_2 \in \mathbb{C}$, we get $S(w) = (w, w_0, w_1, w_2)$. Then, we can construct the function map:

$$z_0 \xrightarrow{T} 0 \xleftarrow{S} w_0$$

$$z_1 \longrightarrow 0 \longleftarrow w_1$$

$$z_2 \longrightarrow 0 \longleftarrow w_2$$

Then $w = f(z) = S^{-1} \circ T(z)$ maps $z_0 \to w_0$, $z_1 \to w_1$, and $z_2 \to w_2$.
The points must be distinct, because f is one-to-one.
To find f from above, we solve $(z, z_0, z_1, z_2) = (w, w_0, w_1, w_2)$ for w.

Example.
Suppose

$$z_0 = 1 \to i = w_0$$

$$z_1 = -1 \to 1 = w_1$$

$$z_2 = 1 \to -1 = w_2$$

then

$$\frac{(w - i)(1 - (-1))}{(w - (-1))(1 - i)} = \frac{2(w - i)}{(w + 1)(1 - i)} = \frac{(z - i)(-1 - 1)}{(z - 1)(-1 - i)} = \frac{-2(z - i)}{(z - 1)(-1 - i)} = \frac{2(z - i)}{(z - 1)(1 + i)}$$

$$\Rightarrow \frac{w - i}{(w + 1)(1 - i)} = \frac{z - i}{(z - 1)(1 + i)}$$

$$\Rightarrow (w - i)(z - 1)(1 + i) = (w + 1)(1 - i)(z - i)$$

$$\Rightarrow w = -\frac{1}{z}$$

It is much work to find a simple function. Mapping a set of three points to another set of three points is time-consuming. If we study Möbius transforms as conformal mappings, then we can introduce another way to move the three points around.

Example. Find a Möbius transform to map the unit disk $|z| < 1$ to $\Im(z) > 0$.
If we pick where 3 points go, say on the boundary of Ω, to the boundary of $\Im(z) > 0$, which is the $\mathbb{R}$-axis, then we can construct $f(z)$. We also need one point to get mapped to ∞, but we have ∞-many choices for the three points.
We want $f(-1) = 0$, $f(-1) = 1$, and $f(1) = \infty$, that will be

$$f(z) = (z, -1, -i, 1) = \frac{(z - z_0)(z_1 - z_2)}{(z - z_2)(z_1 - z_0)} = \frac{(z + 1)(-i - 1)}{(z - 1)(-i + 1)} = -i\frac{z + 1}{z - 1}$$

Our path's direction around the circle has to be preserved because f is conformal.
This means points on the interior of the circle will get mapped to the upper half of $\mathbb{C}$.
To check, we see that $f(0) = i, \Im(f(0)) > 0$. Therefore, the function we found out is correct.

Example. Find a Möbius transform to map the unit disk $|z| < 1$ to $\Im(z) > 0$ and $\Re(z) > 0$.
Using the conclusion from the example above, the new desired function is simply $g(z) = (f(z))^{\frac{1}{2}}$.

11 Contour Integral in $\mathbb{C}$

Let $\gamma(t) = x(t) + iy(t)$ be a curve in $\mathbb{C}$ where $\gamma(a) = z_0$ and $\gamma(b) = z_1$.
Let C be the graph of $\gamma(t)$, $C = \{z | z = \gamma(t) \text{ for some } t \in [a, b]\}$.

11.1 Piecewise Differentiable, Smooth, Simple, Closed curves

Piecewise Differentiable curves.
The curve determined by γ, its graph C, is considered piecewise differentiable if

1. x and y are continuous on [a, b]
2. x' and y' are continuous on a partition of $[a, b]$, $[x_0, x_1] \cup [x_1, x_2] \cup [x_2, x_3] \cup \cdots \cup [x_{n-1}, x_n]$

Smooth curves.
If $\gamma' \neq 0$ for only finitely many points, then the curve is considered smooth.

Simple curves.
A curve is simple if it does not intersect itself, i.e. $\gamma(t) = \gamma(s)$ if and only if $s = t$.

Closed curves.
C is a closed curve if it starts and stops at the same point, i.e. $\gamma(a) = \gamma(b), t \in [a, b]$

11.2 Interior and Exterior of curves

A closed and simple curve keeps the interior of the set on its left side and its exterior to the right.
This means we traverse circles counter-clockwise to describe their interior correctly.

Jordan Curve Theorem.
A closed and simple curve partitions $\mathbb{C}$ into two regions, one of them bounded, defined as the interior of the curve.

11.3 Smoothly Equivalent

The parameter t provides an orientation or direction to C.
Let

$$\begin{cases} C_1 : \gamma_1(t), t \in [a, b] \\ C_2 : \gamma_2(t), t \in [c, d] \end{cases}$$

We say C_1 and C_2 are smoothly equivalent if there exists a one-to-one, continuous derivative mapping $\lambda(t)$,

$$\lambda(t) : [c, d] \to [a, b]$$

$$\lambda(c) = a$$

$$\lambda(d) = b$$

$$\lambda'(t) > 0$$

$$\text{where } \gamma_1(\lambda(t)) = \gamma_2(t)$$

Example.

$$\begin{cases} C_1 : \gamma_1(t) = \cos(t) + i \sin(t), t \in [0, 2\pi] \\ C_2 : \gamma_2(t) = \cos(2t) + i \sin(2t), t \in [0, \pi] \end{cases}$$

Here C_1 and C_2 are smoothly equivalent, since we can let $\lambda(t) = 2t$.
Both parametrize the unit circle, preserve the orientation, and pass through each point the same number of times.
Let us look at other two curves:

$$\begin{cases} \gamma_3(t) = \cos(4t) + i\sin(4t), t \in [0, \pi] \\ \gamma_4(t) = \cos(t) + i\sin(-t), t \in [0, 2\pi] \end{cases}$$

γ_3 traverses the circle multiple times and γ_4 has the opposite orientation of γ_1 and γ_2.

Let $-C$ be the curve C but with a reversed orientation, $\gamma_R(t) = \gamma(b + a - t)$.

$$\begin{cases} C_1 : \gamma_1(t) = \cos(t) + i\sin(t), t \in [0, 2\pi] \\ -C_1 : \gamma_4(t) = \cos(t) + i\sin(-t), t \in [0, 2\pi] \end{cases}$$

We want to integrate $f(z)$ over curves C in $\mathbb{C}$. These will factor into the computation:

$$\begin{cases} \text{Orientation of } C \\ \text{Number of times } C \text{ traverses itself} \\ C \text{ is closed} \\ f \text{ is holomorphic on the interior of } C \text{ and } C \end{cases}$$

11.4 Line Integral

The line integral of f over C is given by

$$\int_C f(z)\,dz = \int_C u(z) + iv(z)\,dz = \int_C u(z) + iv(z)\,(dx + idy) = \int_a^b f(\gamma(t))\gamma'(t)\,dt$$

If C is a closed curve, $\gamma(a) = \gamma(b)$, then we can use a closed loop in our $\int$ symbol, $\oint_C f(z)\,dz$
The term $\gamma'(t)dt$ controls for how fast we traverse the curve. The integral is independent of our choice of smoothly equivalent curve, C_1 or C_2.

Proposition. Let C_1 and C_2 be smoothly equivalent,

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

Proof. Let $\gamma_1(\lambda(t)) = \gamma_2(t)$ and apply change of variables.
Because $u = \lambda(t), u(c) = a, u(d) = b$ and $\gamma_1'(\lambda(t))\lambda'(t) = \gamma_2'(t)$,

$$\int_c^d f(\gamma_2(t))\gamma_2'(t)\,dt = \int_c^d f(\gamma_1(\lambda(t)))\gamma_1'(\lambda(t))\lambda'(t)\,dt = \int_a^b f(\gamma_1(u))\gamma_1'(u)\,du$$

Proposition.

$$-\int_C f(z)\,dz = \int_{-C} f(z)\,dz$$

Proof.

$$\int_{-C} f(z)\,dz = \int_b^a f(\gamma_R(t))\gamma_R'(t)\,dt = \int_b^a f(\gamma(t))\gamma'(t)\,dt$$

Proposition. Linearity holds:

$$\int_C \alpha f(z) + g(z)\, dz = \alpha \int_C f(z)\, dz + \int_C g(z)\, dz$$

Example. Find $\oint_{|z|=1} \frac{1}{z}\, dz = \int_0^{2\pi} f(\gamma(t))\gamma'(t)\, dt$
We parametrize the curve as follows:
C: $\gamma(t) = \cos(t) + i\sin(t)$, $t \in [0, 2\pi]$, $\gamma'(t) = -\sin(t) + i\cos(t)$

$$\frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

$$\int_0^{2\pi} f(\gamma(t))\gamma'(t)\, dt = \int_0^{2\pi} (\cos(t) - i\sin(t))(-\sin(t) + i\cos(t))\, dt = \int_0^{2\pi} i\, dt = 2\pi i$$

There is another way to parametrize the curve:
$\gamma(t) = re^{it}$, $\gamma'(t) = ire^{it}$

$$\oint_{|z|=r} \frac{1}{z}\, dz = \int_0^{2\pi} f(re^{it})ire^{it}\, dt = i\int_0^{2\pi} e^{-it}e^{it}\, dt = \int_0^{2\pi} i\, dt = 2\pi i$$

From this example, we have the following famous result.

Proposition. The following holds with proof shown above.

$$\oint_{|z|=r} z^k\, dz = 0 \text{ for all } k \in \mathbb{Z}, k \neq -1, r > 0.$$

$$\oint_{|z|=r} \frac{1}{z}\, dz = \int_0^{2\pi} i\, dt = 2\pi i \text{ for r} > 0$$

ML Estimate.
Let $|f(z)| < M$ on C, a curve of length L, then

$$|\int_C f(z)\, dz| \leqslant M \cdot L$$

Proof.

$$\left|\int_a^b f(\gamma(t))\gamma'(t)\, dt\right| \leqslant \int_a^b |f(\gamma(t))\gamma'(t)|\, dt \leqslant M\int_a^b |\gamma_1'(t)|\, dt = M \cdot L$$

where $\int_a^b |\gamma_1'(t)|\, dt$ is the arclength of γ.
We do not have to worry about the orientation of C,

$$\left|\int_{-C} f(z)\, dz\right| \leqslant M \cdot L$$

12 Cauchy's Closed Curve Theorem and the Fundamental Theorem of Calculus

13 Cauchy's Integral Formula

14 Growth Conditions of Holomorphic Functions

15 Convergence of Infinite Series in $\mathbb{C}$

16 Power Series in $\mathbb{C}$

17 Series Expansion of Holomorphic Functions

18 Open Mapping Theorem and Reflection Principle

19 Laurent Series

20 Residue Theorem

21 Improper Integrals

22 Argument Principle and Rouche's Theorem

$$f(x) = x^2$$

this formula is an example $f(x) = x$

$$\begin{aligned} 1 + 2 &= 3 \\ 1 &= 3 - 2 \end{aligned}$$

$$\begin{aligned} f(x) &= x^2 \\ g(x) &= \frac{1}{x} \\ h(x) &= \int_b^a \frac{1}{x} x^3 \\ F(x) &= \frac{1}{\sqrt{x}} \end{aligned}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\left(\frac{1}{\sqrt{x}} \right)$$

Core Material: 1. Finding patterns in data; using them to make predictions. 2. Models and statistics help us understand patterns. 3. Optimization algorithms ”learn” the patterns.
Classification: 1.