

# Introduction to Complex Analysis

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Aug 3, 2020

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# 1 Algebra of the Complex Plane

## 1.1 Introduction to Complex Numbers

Let  $z = a + ib \in \mathbb{C}$  where  $a, b \in \mathbb{R}$  and  $i^2 = -1$ .

This number can be thought of as a point in 2-space,  $\mathbb{R}^2$ ,  $(a, b)$  or as a position in  $\mathbb{C}$ .

$\mathbb{R}^2$ :  $\oplus$  addition;  $\odot$  scalar multiplication.

$\mathbb{C}$ :  $\oplus$  addition;  $\odot$  scalar multiplication; a vector space; have multiplication of elements,  $\mathbb{C}$  is a field.

$$\text{If } z = a + ib, w = c + id, \text{ then } zw = (ac - bd) + i(ad + cb)$$

$$zw = wz$$

$$z(w + \alpha) = zw + z\alpha$$

$$(zw)\alpha = z(w\alpha)$$

## 1.2 Conjugate of Complex Numbers

### 1.2.1 Definition of Conjugate

The complex conjugate of  $z$ ,  $\bar{z}$ , is defined by

$$\bar{z} = a - ib$$

Geometric representation: The image of  $\bar{z}$  is the reflection of  $z$  about the Real axis.

### 1.2.2 Properties of Conjugate

$$\overline{\bar{z}} = z$$

$$\overline{zw} = \bar{z}\bar{w}$$

$$\overline{z + w} = \bar{z} + \bar{w}$$

$$\bar{z} = z \text{ if and only if } z \in \mathbb{R}$$

### 1.2.3 Real and Imaginary Parts

We can project  $z$  onto the Real or Imaginary axis and measure its distance from 0:

$$\Re(z) = a$$

$$\Im(z) = b, \text{ not } ib$$

Each function is a map  $\mathbb{C} \rightarrow \mathbb{R}$ . Then

$$\Re(z) = \frac{z + \bar{z}}{2}$$

$$\Im(z) = \frac{z - \bar{z}}{2i}$$

This is similar to the pattern with even/odd functions.

## 1.3 Modulus of Complex Numbers

Note:  $z\bar{z} = (a + ib)(a - ib) = a^2 + b^2 \in \mathbb{R}$

### 1.3.1 Definition of Modulus

$|z|$  length/modulus of  $z$  is defined by:

$$|z| = (a^2 + b^2)^{\frac{1}{2}} = (z\bar{z})^{\frac{1}{2}} \in \mathbb{R}$$

### 1.3.2 Properties of Modulus

$$|zw| = |z||w|$$

$$|z| = |\bar{z}|$$

$$|z| \geq 0$$

$$|z| = 0 \text{ if and only if } z = 0$$

### 1.3.3 Triangle Inequality

Triangle Inequality:

$$|z + w| \leq |z| + |w|$$

Reverse Triangle Inequality:

$$|z| - |w| \leq |z - w|$$

$$z = z - w + w \Rightarrow |z| = |z - w + w| \Rightarrow |z| \leq |z - w| + |w| \Rightarrow |z| - |w| \leq |z - w|.$$

### 1.3.4 Complex Division

With  $z\bar{z} \in \mathbb{R}$ , we can define complex division by reducing it to a multiplication problem.

$$\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} = \frac{1}{w\bar{w}}(z\bar{w})$$

We also have

$$\left| \frac{z}{w} \right| = \frac{|z|}{|w|} \text{ for } w \neq 0$$

### 1.3.5 Distance in the plane

A disk in the complex plane centered at  $c$  of radius  $r \in \mathbb{R}$  is of the form

$$\{z \in \mathbb{C} \mid |z - c| \leq r\}$$

## 1.4 Complex Polynomial

A complex polynomial  $p(z)$  of degree  $n$  is of the form:

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

where  $a_n \neq 0$  and  $a_i \in \mathbb{C}$  for  $i = 0, \dots, n$

### 1.4.1 Fundamental Theorem of Algebra

The factorization of  $p(z)$  factors over  $\mathbb{C}$  is unique,

$$p(z) = c(z - z_1)^{m_1} \cdots (z - z_k)^{m_k}$$

We have roots  $z_i \in \mathbb{C}$  of  $p(z)$  with order  $m_i \in \mathbb{N}$ .

For example, if  $p(z) = z^2 + 4 = (z + 2i)(z - 2i)$ , then it factors over  $\mathbb{C}$  but not  $\mathbb{R}$ .

**Note:**  $\mathbb{C}$  is an algebraically closed field, there are no irreducible polynomials in  $\mathbb{C}$ .

**Note:**  $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$  are not algebraically closed.

## 2 Geometry of the Complex Plane

### 2.1 Properties of Polar Forms

Complex numbers can be represented in polar forms:

$$z = |z|(\cos \theta + i \sin \theta)$$

with modulus  $|z|$  and argument  $\theta$ . To change between the coordinate systems it follows:

$$|z| = (a^2 + b^2)^{\frac{1}{2}}$$

$$\tan \theta = \frac{b}{a}$$

$$a = |z| \cos \theta = \Re(z)$$

$$b = |z| \sin \theta = \Im(z)$$

**Note:**  $\theta_R = \arctan(\frac{b}{a})$  is a reference angle of  $z$ . To find  $\theta$  from  $\theta_R$ , you need to consider the signs of  $a$  and  $b$ .

Example:

$$z = -3 + 3i = 3\sqrt{2}(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4})$$

$$\theta_R = \arctan(\frac{3}{-3}) = -\frac{\pi}{4}$$

$$\theta = \pi + \theta_R = \pi - \frac{\pi}{4} = \frac{3\pi}{4}, \text{ since } \theta \text{ is in } II.$$

### 2.2 Definition of Argument and argument

$\text{Arg}(z)$  is  $z$ 's principle polar angle  $\theta$ ,  $z \neq 0$ , where  $\theta \in (-\pi, \pi]$ .  
 $\arg(z)$  is all of  $z$ 's polar angles,  $\theta + 2k\pi$ ,  $k \in \mathbb{Z}$ .

### 2.3 Euler's Formula

Euler's Formula is defined as a linear combination of  $\cos \theta$  and  $\sin \theta$ ,  $\mathbb{R}$ -valued functions.

$$e^{i\theta} = \cos \theta + i \sin \theta$$

It allows us to express  $z$  in polar form by

$$z = |z|e^{i\theta}$$

-1 has polar angle  $\pi$  and modulus 1,

$$-1 = e^{i\pi} \text{ or } e^{i\pi} + 1 = 0$$

By the angle addition formulas from trigonometry we find:

$$e^{i\theta}e^{i\varphi} = e^{i(\theta+\varphi)}$$

$$(e^{i\theta})^k = e^{i\theta k}$$

## 2.4 Geometric Understanding of Multiplication

The polar angle of  $zw$  is the sum of the polar angles of  $z$  and  $w$ . The modulus is the product of the moduli.

$$\text{Arg}(zw) = \text{Arg}(z) + \text{Arg}(w)$$

$$\text{Arg}(\bar{z}) = -\text{Arg}(z)$$

Question: How about  $\frac{z}{w}$  and  $z^4$ ?

It follows from trigonometry that  $|e^{i\theta}| = 1$ , if  $\theta \in (-\pi, \pi]$  we get a parametrization of the unit circle.

**Example:** Discover all solutions to  $w^3 = i = z$

Let  $p(z) = w^3 - i$ . By Fundamental Theorem of Algebra, there are 3 roots of  $p(z)$ .

Therefore,  $3\theta = \frac{\pi}{2} + 2\pi k$ ,  $k \in \mathbb{Z}$

This gives us infinitely many solutions, but the solutions form 3 equivalence classes.

All we need is  $k = 0, 1, 2$ , which gives  $\theta_1 = \frac{\pi}{6}$ ,  $\theta_2 = \frac{5\pi}{6}$ ,  $\theta_3 = \frac{3\pi}{2}$

Our solutions partitioned the unit circle into 3 equally spaced wedges.

The solutions to  $w^3 = i$  are  $w_1 = \frac{\sqrt{3}}{2} + \frac{1}{2}i$ ,  $w_2 = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$  and  $w_3 = -i$ .

This problem of unity can be extended to solving  $w^k = z$  for  $k \in \mathbb{N}$ ,  $z \in \mathbb{C}$  for unknown  $k$ -solutions  $w$ .



### 3 Stereographic Projections, Exponentials and Logs

#### 3.1 Stereographic Projections

We can express the complex plane on the unit sphere in  $\mathbb{R}^3$ . To perform this we project points on the surface of the sphere along the line from the North Pole  $(0, 0, 1)$  through the point and onto the plane  $z = 0, \mathbb{C}$

$$p_1 = (x_1, x_2, x_3) \rightarrow z = a + ib = \frac{x_1 + ix_2}{1 - x_3}$$

$$x_1 = \frac{2a}{|z|^2 + 1}, x_2 = \frac{2b}{|z|^2 + 1}, x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}$$

Points in the northern hemisphere  $P_1$ , have  $|z_1| > 1$ ; while points in the southern hemisphere  $P_2$ , have  $|z_2| < 1$ .

##### 3.1.1 Mapping

$$\mathbb{S}^2 \rightarrow \mathbb{C}$$

$$N = (0, 0, 1) \rightarrow \infty$$

$$S = (0, 0, -1) \rightarrow 0$$

lines of latitude  $\rightarrow |z| = r$ , circles

lines of longitude  $\rightarrow \text{Arg}(z) = \pm\theta$ , lines through  $(0, 0)$

**Note:** In general, circles on  $\mathbb{S}^2$  map to circles and lines in  $\mathbb{C}$ , orientation is not always preserved.

#### 3.2 Complex Logarithm

##### 3.2.1 Logarithm of Real Numbers

Anytime we are dealing with power, the log function is very useful.

$$\log x = \int_1^x \frac{1}{t} dt \text{ for } x \in \mathbb{R} \quad (1)$$

$$\frac{d}{dx} x^x = \frac{d}{dx} e^{\ln x^x} = \frac{d}{dx} e^{x \ln x} = e^{x \ln x} (x \cdot \frac{1}{x} + \ln x) = x^x (1 + \ln x)$$

##### 3.2.2 Logarithm of Complex Numbers

Remember from Euler's Formula,  $e^{i\theta} = \cos \theta + i \sin \theta$ .

$$e^z = e^{a+ib} = e^a e^{ib}$$

$$\text{Arg}(e^z) = b, |e^z| = e^a > 0$$

Therefore, if  $a$  is held fixed,  $e^z$  maps to a circle as  $b$  changes.

On the other hand, if  $b$  is held fixed,  $e^z$  maps to a line through  $(0, 0)$ .

##### 3.2.3 Derivation of Complex Logarithm

We want  $e^{\log(z)} = z$  for all  $z \neq 0$ , and thus

$$e^{\Re(\log(z)) + i\Im(\log(z))} = e^{\Re(\log(z))} e^{i\Im(\log(z))} = |z| e^{i\theta} = z$$

$$\Rightarrow |z| = e^{\Re(\log(z))}$$

$$\Rightarrow \Re(\log(z)) = \log |z|$$

From the imaginary part we find

$$e^{i\theta} = e^{i\Im(\log(z))}$$

$$\Rightarrow \arg(z) = \theta = \Im(\log(z))$$

$$\Rightarrow \Im(\log(z)) = \text{Arg}(z)$$

because  $\arg(z)$  is not well defined.

Our constructed inverse of  $e^z$  is a multi-valued function

$$\log(z) = \log|z| + i \arg(z)$$

### 3.2.4 Conclusion from Derivation

$$\log(z) = \log|z| + i \arg(z)$$

$$\text{Log}(z) = \log|z| + i \text{Arg}(z)$$

**Note:**  $\text{Log}(z)$  does not have all the nice behavior as  $\mathbb{R}$ -valued  $\log(x)$ :  $\text{Log}(z^k)$ .

Sometimes they are co-terminal angles, but they are not equal. See the following example:

$$\begin{cases} \text{Log}(i^3) = \text{Log}(-i) = -i\frac{\pi}{2} \\ 3\text{Log}(i) = 3 \cdot (i\frac{\pi}{2}) = i\frac{3\pi}{2} \end{cases}$$

**Example:** Compute  $3^i$ :

$$3^i = e^{\text{Log } 3^i} = e^{i \text{Log } 3} = \cos(\text{Log } 3) + i \sin(\text{Log } 3)$$

### 3.2.5 How Logarithm acts on curves

$$\begin{cases} \text{Maps a circle with radius } r \text{ to a vertical line passing through } (\ln(r), 0) \\ \text{Maps a line with angle } \theta \text{ passing through the origin to a horizontal line passing through } (0, i\theta) \end{cases}$$

## 4 Topology in $\mathbb{C}$

### 4.1 Complex Sequence

Let  $\{Z_n\}$  be a sequence in  $\mathbb{C}$ .

#### 4.1.1 Cauchy Sequence

The sequence is Cauchy if for all  $\epsilon > 0$ , there is a  $N \in \mathbb{N}$  such that for all  $n, m > N$ ,  $|z_n - z_m| < \epsilon$ .

#### 4.1.2 Sequence Convergence

The sequence converges if  $|z_n - z| \rightarrow 0$  as  $n \rightarrow \infty$ . The distance between  $z_n$  and  $z$  vanishes.

#### 4.1.3 Completeness of $\mathbb{C}$

$\{z_n\}$  converges if and only if  $\{z_n\}$  is Cauchy.

**Proof:**

We show this by treating  $\mathbb{C}$  as  $\mathbb{R}^2$  and exploiting  $\{X_n\}$  converges if and only if  $\{X_n\}$  is Cauchy.

( $\implies$ ) (If  $z_n \rightarrow z$ , then  $\Re(z_n) \rightarrow \Re(z)$  and  $\Im(z_n) \rightarrow \Im(z)$ . Since the sequences of  $\mathbb{R}^2$  converge, they are Cauchy.

$$|Z_n - Z_m| \leq |\Re(Z_n - Z_m)| + |\Im(Z_n - Z_m)| = |\Re(Z_n) - \Re(Z_m)| + |\Im(Z_n) - \Im(Z_m)|$$

Upper bounds can be picked to be less than  $\frac{\epsilon}{2}$  for some  $N$ . Therefore,  $|Z_n - Z_m| \rightarrow 0$ .

( $\impliedby$ ) If  $\{Z_n\}$  is Cauchy, so are  $\{\Re(Z_n)\}$  and  $\{\Im(Z_n)\}$ . But these are  $\mathbb{R}$ -sequences that converge. Therefore,  $\{Z_n\}$  converges.

### 4.2 Complex Set

Let  $\Omega \subset \mathbb{C}$ . Sets can be open, closed, both, or neither.

#### 4.2.1 Open Set

If for any  $z_0 \in \mathbb{C}$ , there exist some  $\epsilon > 0$ , such that the set  $B_\epsilon(z_0) = \{z \mid |z - z_0| < \epsilon\}$  is contained in  $\Omega$ , then  $\Omega$  is open.

$\Omega$  is open if and only if  $\Omega^c$  is closed.

$\Omega$  is open if and only if  $\Omega$  is equal to its own interior, which means it does not contain its boundary points  $\partial\Omega$ , i.e. it does not contain its closure.

#### 4.2.2 Closed Set

If  $\Omega$  contains its limit point, then  $\Omega$  is closed.

$\Omega$  is closed if and only if  $\Omega^c$  is open.

$\Omega$  is closed if and only if  $\Omega$  contains its boundary points.

#### 4.2.3 Compact Set

If  $\Omega$  can be contained in a disk of finite radius, then  $\Omega$  is bounded.

#### 4.2.4 Compact Set

If  $\Omega$  is closed and bounded, then  $\Omega$  is compact. This resembles  $[a, b]$  in  $\mathbb{R}$ .

#### 4.2.5 Connected Set

If any two points in  $\Omega$  can be connected by a path, then  $\Omega$  is connected.

Simply Connected Set: A simply connected set has no "holes" in it. For example,  $\Omega = \{z \mid |z - c| < 4\}$ .

A connected but not simply connected set is an annulus,  $\Omega = \{z \mid 2 < |z - c| < 4\}$

#### 4.2.6 Boundary of Set

The boundary of  $\Omega$ ,  $\partial\Omega$  is all points with  $\epsilon$ -balls intersecting  $\Omega$  and  $\Omega^c$  for all  $\epsilon > 0$ .

#### 4.2.7 Interior of Set

The interior of  $\Omega$ ,  $\text{Int}(\Omega)$ , is all points in  $\Omega$  with a  $\epsilon$ -ball contained in  $\Omega$  for some  $\epsilon > 0$ . "Largest open set in  $\Omega$ ".

#### 4.2.8 Closure of Set

The closure of  $\Omega$  is the union of  $\Omega$  and its boundary  $\partial\Omega$ .

#### 4.2.9 Domain

If a set is open and connected in  $\mathbb{C}$ , it is a domain.

A domain can be traversed by a path of horizontal and vertical line segments.

#### 4.2.10 Practice Examples

Determine whether the following sets are open or closed.

1.  $\Omega = \mathbb{C} \setminus \{0\}$

$\Omega$  is open since it does not contain its closure, the point 0.

$\Omega$  is not closed since it does not contain its limit points. Let  $z_n = \frac{1}{n}$ . Then  $z_n = \frac{1}{n} \rightarrow 0 \notin \Omega$ .

Therefore,  $\Omega$  is open.

2.  $\Omega = \{z \mid |z| \geq 1\}$

$\Omega$  is not open since any  $\epsilon$ -ball at 1 intersects  $\Omega^c$ .

$\Omega$  is closed since  $\Omega^c$  is open.

Therefore,  $\Omega$  is closed.

3.  $\Omega = \{z \mid |z| > 1\}$

$\Omega$  is open since  $\Omega^c$  is closed.

$\Omega$  is not closed since it does not contain its limit points. Let  $z_n = \frac{1}{n} + 1$ . Then  $z_n = \frac{1}{n} + 1 \rightarrow 1 \notin \Omega$ .

4.  $\Omega = \mathbb{C} \setminus (0, 1)$

$\Omega$  is not open. Its complement is  $[0, 1]$ . Even though it is closed in  $\mathbb{R}$ , it is not closed in  $\mathbb{C}$ , because any 2D  $\epsilon$ -ball will always extend outside of the set  $z \in (0, i)$ . Hence,  $\Omega^c$  is not open and not closed.

$\Omega$  is not closed since it does not contain its limit points. Let  $z_n = \frac{1}{3} + i\frac{1}{n}$ . Then  $z_n = \frac{1}{3} + i\frac{1}{n} \rightarrow \frac{1}{3} \notin \Omega$ .

Therefore,  $\Omega$  is neither open nor closed.

5.  $\Omega = \mathbb{C} \setminus [0, 1]$   $\Omega$  is open since  $\Omega^c = [0, 1]$  is closed in  $\mathbb{C}$ .

$\Omega$  is not closed since it does not contain its limit points. Let  $z_n = \frac{1}{3} + i\frac{1}{n}$ . Then  $z_n = \frac{1}{3} + i\frac{1}{n} \rightarrow \frac{1}{3} \notin \Omega$ .

Therefore,  $\Omega$  is open.

Note:  $\Omega^c$  is not open in  $\mathbb{C}$ .

## 5 Continuity and Branch Cuts

### 5.1 Complex Continuity

Let  $f : \Omega \rightarrow \mathbb{C}$ ,  $\Omega$  is open and connected. If  $z_n \rightarrow z_0$  implies  $f(z_n) \rightarrow f(z_0)$ , then  $f$  is continuous at  $z_0$ . Also,  $f$  is bounded near  $z_0$ .

$f$  is continuous if for every  $\epsilon > 0$ , there is  $\delta > 0$  such that  $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon$ .

- In either case,  $\Re(f(z))$  and  $\Im(f(z))$  are each continuous if and only if  $f(z)$  is continuous. This follows the pattern as  $\mathbb{C}$  being complete.

- If  $f$  and  $g$  are continuous, then so are  $f + g$ ,  $f \times g$  and  $\frac{f}{g}$  (provided  $g(z) \neq 0$ )

### 5.2 Complex Limits

Just like in  $\mathbb{R}^2$ , limits are direction independent. Do not restrict limits to just  $\Re \rightarrow 0$  or  $\Im \rightarrow 0$ . See the following example.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2y}{x^4 + y^2} \text{ does not exist}$$

as  $x \rightarrow 0$ ,  $y = 0$ , then  $f \rightarrow 0$ , while  $y = x^2$ ,  $x \rightarrow 0$ , then  $f \rightarrow 1$ .

### 5.3 Branch Cuts

Log,  $z^{\frac{1}{2}}$  and  $\arctan(z)$  are constructed by restricting the range of  $e^z$ ,  $z^2$  and  $\tan(z)$ .

For example, in creating  $\text{Log}(z) = \ln|z| + i \text{Arg}(z)$ , we made a choice that  $\text{Arg}(z) \in (-\pi, \pi]$ ,  $\text{Arg}(0)$  does not exist.

#### 5.3.1 Example of a Branch Cut

Consider a path around  $z_0 \neq 0$ ,  $\gamma(t) = z_0 + re^{it}$ .  $\theta(t) = \arg(\gamma(t))$

As we traverse the circle,  $t \in (-\pi, \pi]$ ,

$$\theta(t) = \arg(\gamma(t)) = \text{Arg}(z_0 + re^{it}) + 2\pi k = \text{Arg}(z_0 + re^{i(t+2\pi)}) + 2\pi k = \arg(\gamma(t + 2\pi)) = \theta(t + 2\pi)$$

Therefore, the angle  $\theta(t)$  changes smoothly for all  $t$  and we stay on the same branch of  $\text{Arg}(\gamma(t))$ . That is to say, the  $k \in \mathbb{Z}$  is the same for all  $t$ .

Compare this with any circular path about  $z = 0$ ,  $\gamma_0$ . Let  $\gamma_0(t) = re^{it}$ ,  $t \in (-\pi, \pi]$ . As we traverse the circle once, we have a discontinuity in the principal angle of  $\gamma_0(t)$ . In particular,  $\theta(\gamma_0(t)) \neq \theta(\gamma_0(t+2\pi))$

$$\theta(t) = \arg(\gamma(t)) = \text{Arg}(re^{it}) + 2\pi k \neq \text{Arg}(re^{i(t+2\pi)}) + 2\pi(k+1) = \arg(\gamma(t + 2\pi)) = \theta(t + 2\pi)$$

We jump from the  $k$ th to the  $(k+1)$ th branch of  $\text{Arg}$ . Therefore,  $\text{Arg}(z)$  has a branch point at  $z = 0$ .

#### 5.3.2 Definition of Branch Cuts and Branch Points

If every neighborhood of  $z_0$  contains a path  $\gamma(t)$  around  $z_0$  that leads to a jump discontinuity in  $f$ , then  $z_0$  is a branch point of  $f(z)$ .

**Finding branches:** At this point, it suffices to study paths of the form  $\gamma(t) = z_0 + re^{it}$  for  $t \in (-\pi, \pi)$ , and see if  $f(\gamma(t)) = f(\gamma(t + 2\pi))$  holds for all  $t$ .

- $\text{Arg}$  is discontinuous for all  $x$  on the negative  $\mathbb{R}$ -axis,  $\mathbb{R}^-$ . We call this the principal branch cut of the

multi-valued function  $\arg$ . Specifically,

$$\operatorname{Arg}(\gamma_0(t)) \rightarrow \pi \text{ as } t \rightarrow \pi^-$$

$$\operatorname{Arg}(\gamma_0(t)) \rightarrow -\pi \text{ as } t \rightarrow -\pi^+$$

but  $\gamma_0(\pi) = \gamma_0(-\pi)$  since  $\pi$  and  $-\pi$  are coterminal.

$\mathbb{R}^-$  is the principal branch of  $\operatorname{Log}$ ,  $\operatorname{Arg}$ , and  $z^{\frac{1}{2}}$ .

- The endpoints of a branch cut are branch points,  $\operatorname{Arg}$  has 0 and  $\infty$  as its branch points.

## 6 Differentiability in $\mathbb{C}$

Let  $f : \Omega \rightarrow \mathbb{C}$  for some domain  $\Omega$ . Then  $f$  is differentiable at  $z_0$  if the following exists.

$$\frac{d}{dz}f(z)|_{z=z_0} = f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$$

This limit must exist on all paths to  $z_0$ , since  $h \in \mathbb{C}$ . We could also take  $z_n \rightarrow z_0$  and use  $\frac{f(z_0) - f(z_n)}{z_0 - z_n} \rightarrow f'(z_0)$ . Remember limits are computed by looking at the difference in the modulus,  $|\frac{f(z_0) - f(z_n)}{z_0 - z_n} - f'(z_0)| \rightarrow 0$  as  $n \rightarrow \infty$ .

If  $f'(z_0)$  exists on all points  $z_0 \in \Omega$ , open and connected in  $\mathbb{C}$ , then  $f$  is holomorphic/ $\mathbb{C}$ -differentiable/analytic on  $\Omega$ . The connection between  $\mathbb{R}$  and  $\mathbb{C}$  analytic will be clear when we cover  $\mathbb{C}$ -power series.

If  $f'(z)$  exists everywhere in  $\mathbb{C}$ , then  $f$  is an entire/meromorphic function.

### 6.1 Difference between $\mathbb{R}$ and $\mathbb{C}$ differentiability

- $f : \mathbb{R} \rightarrow \mathbb{R}$

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

has only two paths to  $x$ , namely  $h \rightarrow 0^+$  and  $h \rightarrow 0^-$ .

Tangent plane or linear approximation:

$$f(x) \approx f(a) + f'(a)(x - a)$$

- $f : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$\lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h} = f_x$$

is also a 1D limit and a partial derivative.

Tangent plane or linear approximation:

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$   $f(x, y) = (u(x, y), v(x, y))$

Then  $f$  is differentiable if the Jacobian Matrix

$$J(f) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

can approximate the local change in  $f$ .

In each of the cases above, we are only measuring change in a few directions. However,  $h \rightarrow 0$  in  $\mathbb{C}$  can be from any direction in 2-space. Therefore,  $f'(z)$  existing is a much stronger condition for  $f$  on  $\mathbb{C}$  than on  $\mathbb{R}$ .

#### 6.1.1 Complications with $\bar{z}$

Consider the following example:

Let  $g(z) = \Re(z) = \frac{z+\bar{z}}{2}$ , which is a linear combination of continuous functions.

Assume  $h \in \mathbb{R}$ ,  $\frac{g(z+ih) - g(z)}{h} = \frac{\Re(z) - \Re(z)}{h} \rightarrow 0$  as  $h \rightarrow 0$

Compare this with  $\frac{g(z+ih) - g(z)}{h} = \frac{\Re(z) + h - \Re(z)}{h} = 1 \rightarrow 1$  as  $h \rightarrow 0$ .

Therefore, the function is nowhere differentiable in  $\mathbb{C}$ . The problem with  $g'(z)$  had to do with  $\bar{z}$ , despite reflection in  $\mathbb{R}^2$  about  $y = 0$  is differentiable. We will discover conditions on  $u_x$ ,  $u_y$ ,  $v_x$ , and  $v_y$  that ensure  $f'$  exists for  $f(z) = u(x, y) + iv(x, y)$  in the next chapter.

## 6.2 Properties of $f'(z)$

**Proposition:** If  $f$  is differentiable on  $\Omega$ , it is continuous on  $\Omega$ .

**Proposition:** The power rule holds too:  $\frac{d}{dz}z^n = nz^{n-1}$

As does the product, quotient, L'Hospital's and chain rule. In fact, most old results hold as well. In each of the following cases, there are branch points where  $f'(z)$  does not exist.

1. If  $f(z) = \text{Log}(z)$ , then  $f'(z) = \frac{1}{z}$
2. If  $f(z) = \tan^{-1}(z)$ , then  $f'(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$ , which does not exist for  $z = \pm i$
3. If  $f(z) = z^{\frac{1}{2}}$ , then  $f'(z) = \frac{1}{2}z^{-\frac{1}{2}}$

**Proposition:** Suppose  $f(z)$  is holomorphic on  $\Omega$ . then  $g(z) = \overline{f(\bar{z})}$  is holomorphic on  $\Omega^* = \{z | \bar{z} \in \Omega\}$

**Proof:**

Suppose  $f$  is holomorphic on  $\Omega$ . Let  $z_0 \in \Omega^*$  and  $z_n \in \Omega^*$  for all  $n$  and  $z_n \rightarrow z_0$ . Then  $\bar{z}_n \rightarrow \bar{z}_0$  in  $\Omega$  and for  $\epsilon > 0$ , there is a  $N \in \mathbb{N}$  such that for  $n > N$ .

$$\begin{aligned} \left| \frac{f(\bar{z}_0) - f(\bar{z}_n)}{\bar{z}_0 - \bar{z}_n} - f'(\bar{z}_0) \right| &< \epsilon \\ \left| \frac{f(\bar{z}_0) - f(\bar{z}_n)}{\bar{z}_0 - \bar{z}_n} - f'(\bar{z}_0) \right| &= \left| \overline{\frac{f(\bar{z}_0) - f(\bar{z}_n)}{\bar{z}_0 - \bar{z}_n} - f'(\bar{z}_0)} \right| = \left| \frac{\overline{f(\bar{z}_0) - f(\bar{z}_n)}}{\overline{\bar{z}_0 - \bar{z}_n}} - \overline{f'(\bar{z}_0)} \right| \\ &= \left| \frac{\overline{f(\bar{z}_0) - f(\bar{z}_n)}}{z_0 - z_n} - \overline{f'(\bar{z}_0)} \right| = \left| \frac{g(z_0) - g(z_n)}{z_0 - z_n} - \overline{f'(\bar{z}_0)} \right| < \epsilon \\ &\implies g'(z_0) = \overline{f'(\bar{z}_0)} \end{aligned}$$

Therefore,  $g$  is holomorphic on  $\Omega^*$ .

This proof is different from when we showed  $\frac{d}{dz}\bar{z}$  does not exist. Conjugation must be handled with care.

## 6.3 Geometric behavior of $f'(z)$

### 6.3.1 Dilation

$w = f(z) \approx f'(a)(z - a) + f(a)$ . Small changes in  $z$  should give small changes in  $w$ .

The functions  $|z|$  and  $\text{Arg}(z)$  are continuous on their domains.

If  $f'(z_0) \neq 0$  and  $f'$  exists on  $\Omega$ , then

$$\begin{aligned} |f'(z_0)| &= \left| \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} \right| = \lim_{h \rightarrow 0} \left| \frac{f(z_0 + h) - f(z_0)}{h} \right| = \lim_{h \rightarrow 0} \frac{|f(z_0 + h) - f(z_0)|}{|h|} \\ &\implies |f'(z_0)||h| \approx |f(z_0 + h) - f(z_0)| \end{aligned}$$

The size of  $|f'(z_0)|$  tells us how much  $f$  is contracting/dilating near  $z_0$ .

### 6.3.2 Rotation

$$\begin{aligned} \text{Arg}(f'(z_0)) &= \text{Arg} \left( \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} \right) = \lim_{h \rightarrow 0} \text{Arg} \left( \frac{f(z_0 + h) - f(z_0)}{h} \right) \\ &= \lim_{h \rightarrow 0} \text{Arg}(f(z_0 + h) - f(z_0)) - \text{Arg}(h) \\ &\implies \text{Arg}(f'(z_0)) \approx \text{Arg}(f(z_0 + h) - f(z_0)) \end{aligned}$$



$$\implies \text{Arg}(f'(z_0)) + \text{Arg}(h) \approx \text{Arg}(f(z_0 + h) - f(z_0))$$

Therefore,  $f$  rotates vectors from  $z_0$  to  $z_0 + h$  by the angle  $\text{Arg}(f'(z_0))$ .

### 6.3.3 Conclusion

In conclusion,  $w = f(z) \approx f(z_0) + f'(z_0)(z - z_0) = c + \rho e^{i\theta}(z - z_0)$

$$\begin{cases} c: \text{ Translation} \\ \rho: \text{ Dilation} \\ e^{i\theta}: \text{ rotation about } z_0 \text{ or complex multiplication.} \end{cases}$$

## 7 The Cauchy Riemann Equations

### 7.1 The Cauchy Riemann Equations

Let  $f(z) = f(x+iy) = u(x, y) + iv(x, y)$ , then  $f(z)$  is holomorphic implies the Cauchy Riemann Equations:

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

**Proof:** For  $f$ , which is  $\mathbb{C}$ -differentiable, the following representation of  $f'(z)$  holds for any path to  $z$ .

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$$

Along the path  $x+h+iy \rightarrow x+iy$ , we get

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \rightarrow 0} \frac{f(x+h+iy) - f(x+iy)}{h} = f_x = u_x(x, y) + iv_x(x, y)$$

Along the path  $x+iy+ih \rightarrow x+iy$ , we get

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \rightarrow 0} \frac{f(x+i(y+h)) - f(x+iy)}{ih} = \frac{f_y}{i} = -if_y = -i(u_y(x, y) + iv_y(x, y)) = v_y(x, y) - iu_y(x, y)$$

Equate components in  $f_x = -if_y$ , and it is proven that  $u_x = v_y$  and  $u_y = -v_x$ .

#### 7.1.1 Properties of the Cauchy Riemann Equations

**Proposition:** If the Cauchy Riemann Equations do not hold at  $z_0$ , then  $f'(z_0)$  does not exist.

**Proposition:** If  $f$  is holomorphic on a domain  $\Omega$ , an open and connected set in  $\mathbb{C}$ , then the Cauchy Riemann Equations hold at all points in  $\Omega$ .

**Example:**

If  $f(x+iy) = x^2 + iy^2$ ,  $u_x = 2x$ ,  $v_x = 0$ ,  $u_y = 0$ ,  $v_y = 2y$ . Then  $2x = 2y \Rightarrow x = y$ , which is a line.

The set of points on the line is not open in  $\mathbb{C}$ . Therefore,  $f$  is nowhere holomorphic in  $\mathbb{C}$ . However, we will see that  $f'(z)$  does exist on the line  $y = x$ .

#### 7.1.2 Sufficiency of the Cauchy Riemann Equations to $f'$

**Follow-Up Question:** The Cauchy Riemann Equations do a great job showing  $f'$  does not exist. But what about it being sufficient for  $f'$ ?

**Claim:** Satisfying the Cauchy Riemann Equations at  $z_0$  implies that  $f'$  exists at  $z_0$ .

**Proof:**

$f$  is  $\mathbb{C}$ -differentiable at  $z_0$  if and only if  $u(x, y)$  and  $v(x, y)$  have continuous partial derivatives that satisfy the Cauchy Riemann Equations at  $z_0$ . This requires us to treat  $f(x+iy)$  as a function on  $\mathbb{R}^2$ , or  $f(z)$  induces a map on  $\mathbb{R}^2$ .

Let  $h = \Delta x + i\Delta y$ ,

$$\frac{f(z+h) - f(z)}{h} = \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)}{\Delta x + i\Delta y} - \frac{u(x, y) + iv(x, y)}{\Delta x + i\Delta y}$$

$$u(x+\Delta x, y+\Delta y) - u(x, y) = u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y) + u(x, y+\Delta y) - u(x, y)$$

The function  $u(\cdot, \cdot)$  is differentiable in  $x$  and  $y$ , we can use the M.V.T (Mean Value Theorem) from  $\mathbb{R}$  to rewrite our difference in  $u$  by

$$u(x + \Delta x, y + \Delta y) - u(x, y + \Delta y) = \Delta x U_x(\underline{x}, y + \Delta y)$$

where  $\underline{x} \in (x, x + \Delta x)$ .

If  $u_x$  is continuous,  $u_x(\underline{x}, y + \Delta y) \approx u_x(x, y) + \epsilon_1$ , and as  $\Delta y \rightarrow 0$  and  $\underline{x} \rightarrow x$ , by Taylor approximation and linear approximation on  $u_x$ , we have the error function  $\epsilon_1 \rightarrow 0$ .

Next  $u(x, y + \Delta y) - u(x, y) = \Delta y u_y(x, \bar{y})$  and  $u_y(x, \bar{y}) \approx u_y(x, y) + \epsilon_2$ .

Likewise, for the function  $v(x, y)$ , we get a  $v_x$  and  $v_y$  with error terms  $\epsilon_3$  and  $\epsilon_4$ .

$$\frac{f(z + h) - f(z)}{h} = \frac{\Delta x(u_x + \epsilon_1 + iv_x + i\epsilon_3) + \Delta y(u_y + \epsilon_2 + iv_y + i\epsilon_4)}{\Delta x + i\Delta y}$$

From the Cauchy Riemann Equations, we get  $f_x = \frac{f_y}{i} \Rightarrow if_x = f_y \Rightarrow i(u_x + iv_x) = u_y + iv_y$ . Substituting the terms, we have

$$f'(z) = \frac{\Delta x(u_x + iv_x) + i\Delta y(u_x + iv_x)}{\Delta x + i\Delta y} + \frac{\lambda}{\Delta x + i\Delta y}$$

where  $\lambda = \Delta x(\epsilon_1 + i\epsilon_2) + \Delta y(\epsilon_3 + i\epsilon_4)$ . However,

$$\left| \frac{\lambda}{\Delta x + i\Delta y} \right| \leq \left| \frac{\Delta x(\epsilon_1 + i\epsilon_2)}{\Delta x + i\Delta y} \right| + \left| \frac{\Delta y(\epsilon_3 + i\epsilon_4)}{\Delta x + i\Delta y} \right| \leq |\epsilon_1 + i\epsilon_2| + |\epsilon_3 + i\epsilon_4|$$

because  $\left| \frac{\Delta x}{\Delta x + i\Delta y} \right| \leq 1$ .

As  $\Delta z \rightarrow 0$ ,  $\left| \frac{\lambda}{\Delta x + i\Delta y} \right| \rightarrow 0$ , and thus  $f'(z) = u_x + iv_x = f_x = \frac{f_y}{i}$ .

Therefore, the Cauchy Riemann Equations are an easy way to show  $f'(z)$  exists and they provide a set of partial differential equations that  $f$  must satisfy.

**Example:** Let  $f(z) = e^z = e^x(\cos(y) + i\sin(y))$

$$u = e^x \cos(y), v = e^x \sin(y)$$

$$u_x = e^x \cos(y), v_x = e^x \sin(y)$$

$$u_y = -e^x \sin(y), v_y = e^x \cos(y)$$

Therefore,  $f(z)$  is  $\mathbb{C}$ -differentiable on  $\mathbb{C}$ ,  $f$  is entire/meromorphic.  $f'(z) = f_x = u_x + iv_x = f(z)$

## 7.2 Cauchy Riemann with Logarithm

$$e^{\text{Log}(z)} = z \Rightarrow \frac{d}{dz} e^{\text{Log}(z)} = 1 \Rightarrow z \frac{d}{dz} \text{Log}(z) = 1 \Rightarrow \frac{d}{dz} \text{Log}(z) = \frac{1}{z}$$

We have a branch point in  $\text{Log}(z)$  where its derivative is undefined. Then  $\text{Log}(z)$  is  $\mathbb{C}$ -differentiable on  $\mathbb{C} \setminus \{0\}$ . This is true regardless of the branch cut on  $\text{Log}(z)$ .

## 7.3 Lack of Complex Mean Value Theorem

**Claim:**  $\frac{f(z) - f(w)}{z - w} \neq f'(c)$  for some  $c$  between  $z$  and  $w$ .

**Proof:** Let  $z = 1$ ,  $w = 0$  and  $f(t) = e^{i\pi t}$ , then  $f(1) - f(0) = e^{i\pi} - 1 = -2$ . However,  $|f'(t)| = \pi$  for all  $t \in [0, 1]$ .

**Follow-Up Question:** Does the lack of a Mean Value Theorem for  $f'(z)$  suggest  $f'(z) = 0$  not imply  $f$

is constant?

**Answer:** Suppose  $f$  is  $\mathbb{C}$ -differentiable on  $\Omega$  and one of the following holds, then  $f$  is constant on  $\Omega$ .

$$\begin{cases} f'(z) = 0 \\ |f(z)| \text{ is constant} \\ \operatorname{Re}(f(z)) \text{ is constant} \\ f\text{'s conjugate is } \mathbb{C}\text{-differentiable on } \Omega \end{cases}$$

## 7.4 Wirtinger Equations

There is another way to study the Cauchy Riemann Equations by introducing two operators:

$$\frac{\partial f}{\partial z} = f_z \text{ and } \frac{\partial f}{\partial \bar{z}} = f_{\bar{z}}$$

$$f(x, y) \equiv f(x + iy) = u(x, y) + iv(x, y)$$

$$f(x, y) = f(\Re(z), \Im(z)) = f\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right)$$

From the chain rule, we get

$$f_z = f_x x_z + f_y y_z = \frac{1}{2}f_x + \frac{1}{2i}f_y = \frac{1}{2}f_x - \frac{i}{2}f_y$$

$$f_{\bar{z}} = f_x x_{\bar{z}} + f_y y_{\bar{z}} = \frac{1}{2}f_x - \frac{1}{2i}f_y = \frac{1}{2}f_x + \frac{i}{2}f_y$$

where  $f_x = u_x + iv_x$  and  $f_y = u_y + iv_y$

These are the Wirtinger Equations.

$$\begin{cases} \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \end{cases}$$

### 7.4.1 Relationship with the Cauchy Riemann Equations

From the Cauchy Riemann Equations  $if_x = f_y$  we get:

$$f_{\bar{z}} = \frac{1}{2}f_x + \frac{i}{2}f_y = \frac{1}{2}f_x - \frac{1}{2}f_x = 0$$

$$f_z = \frac{1}{2}f_x - \frac{i}{2}f_y = \frac{1}{2}f_x - \frac{i^2}{2}f_x = f_x = f'(z)$$

$f$  is  $\mathbb{C}$ -differentiable at  $z_0$  if and only if  $f(x, y) = u(x, y) + iv(x, y)$  is  $\mathbb{R}$ -differentiable at  $z_0$  and  $f_{\bar{z}}(z_0) = 0$ . Then  $f'(z_0) = f_z(z_0)$ . In other words,  $f'(z)$  does not depend on  $\bar{z}$ .

## 8 Harmonic Functions

### 8.1 Laplacian

Let  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ , then the Laplacian of  $u$  is

$$\Delta u = u_{xx} + u_{yy} = \nabla \cdot \nabla u$$

where  $\nabla = [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}]^T$  is the divergence operator and  $\nabla u = [u_x, u_y]^T$  is the gradient of  $u$ .

### 8.2 Harmonic Functions

If  $\Delta u = 0$ , then  $u(x, y)$  satisfies Laplace's (partial differential) equation or  $u$  is a harmonic function.

This means:

$$\begin{cases} u \text{ is continuous} \\ u\text{'s 1st and 2nd order partial derivatives exist and are smooth.} \end{cases}$$

**Proposition:** Suppose  $f = u + iv$  is holomorphic on  $\Omega$  where  $u(x, y)$  and  $v(x, y)$  have continuous 2<sup>nd</sup> order partial derivatives, then  $u$  and  $v$  are harmonic and  $v$  is the harmonic conjugate of  $u$ .

**Proof:**

By the Cauchy Riemann Equations,  $u_x = v_y$  and  $v_x = -u_y$ , then  $u_{xx} = v_{yx}$  and  $v_{xy} = -u_{yy}$ . By continuity of  $v_{yx}$  and  $v_{xy}$ ,  $v_{yx} = v_{xy}$ . This implies  $u_{xx} = -u_{yy} \Rightarrow u_{xx} + u_{yy} = 0$

Later on, we will find that the conditions on 2<sup>nd</sup> order partial derivatives is implied by  $f$  being holomorphic on  $\Omega$ , or  $f''$  exists.

**Definition: Harmonic Conjugate**

The harmonic conjugate to  $u(x, y)$  is a function  $v(x, y)$ , such that  $f(x, y) = u(x, y) + iv(x, y)$  is holomorphic.

**Example.** Show that  $u(x, y) = x^3 - 3xy^2 + y$  is a harmonic function.

$$u_x = 3x^2 - 3y^2, u_y = -6xy + 1$$

$$u_{xx} = 6x, u_{yy} = -6x. \text{ Therefore, } u_{xx} + u_{yy} = 0$$

**Example.** Find the harmonic conjugate of  $u(x, y) = x^3 - 3xy^2 + y$ .

$$u_x = 3x^2 - 3y^2 = v_y$$

$$u_y = -6xy + 1 = -v_x$$

$$\Rightarrow v = 3x^2y - y^3 + C(x) \text{ or } v = 3x^2y - x + C(y)$$

Therefore,  $v = 3x^2y - y^3 - x + C$  is  $u$ 's harmonic conjugate.

**Proposition:**

If  $u$  is harmonic on a domain  $\Omega$ , then  $u_x$  is the real part of a holomorphic function on  $\Omega$ . If  $\Omega$  is simply connected, unlike  $\mathbb{C} \setminus \{0\}$ , then  $u$  is the real part of a holomorphic function on  $\Omega$ .

**Proof:**

Assume  $u$  is harmonic and  $\Omega$  is connected. If  $f = u_x - iu_y$ , then  $f_y = if_x$ . Hence,  $f$  is differentiable on  $\Omega$ . The simply connected statement requires future theorems to show  $F'(z) = f(z)$  for some holomorphic antiderivative  $F(z)$ .

## 9 Conformal Maps

**Example.** Let  $f(z) = (x + iy)^2 + 2(x + iy) = (x^2 + 2x - y^2) + i2(xy + y)$ . When are the component functions,  $u(x, y)$  and  $v(x, y)$  constant?

When are the component functions,  $u(x, y)$  and  $v(x, y)$ , constant?

The function  $f(z) = e^z = e^x(\cos(y) + i\sin(y))$  maps the set  $\Omega = \{z : |\Im(z)| < \pi\}$  to circles of radius  $r \in (-\infty, \infty)$ , or all points in  $\mathbb{C} \setminus \mathbb{R}^-$ . This coincides with the branch cut of  $\text{Log}(z)$ , or how we made  $e^z$  invertible.

### 9.1 Preservation of Angles

We will now show  $e^z$  preserves the angles between curves in  $\Omega$ . Let us first look at the following example.

Let  $\gamma_1(t) = 2i\pi t - i\pi$ ,  $\gamma_2(t) = t + i\frac{\pi}{4}$ .  $\gamma_1(0) = -i\pi$ ,  $\gamma_1(1) = i\pi$ .

The curves  $\gamma_1$  and  $\gamma_2$  intersect at an angle  $\frac{\pi}{2}$ . Also  $f(\gamma_1)$  is a circle centered at 0 while  $f(\gamma_2)$  is a line through  $z = 0$ . Their intersection in the  $w$ -plane is  $\frac{\pi}{2}$  as well.

We will show why  $f(z) = e^z$  does this by studying the angles between curves  $\gamma_1$  and  $\gamma_2$  and curves  $\tau_1 = f(\gamma_1)$  and  $\tau_2 = f(\gamma_2)$ . If  $\gamma(t)$  parameterizes a smooth curve in  $\mathbb{C}$ , then its tangent vector is  $\gamma'(t)$ . The angle between any two curves at  $z_0$  is the angle between their tangent vectors at  $z_0$ .

Assume the curves intersect at  $\gamma(r_0) = \gamma(s_0) = z_0$ .

Let the angle of intersection,  $\theta$ , measured from  $\gamma'_1$  to  $\gamma'_2$  in the counter-clockwise direction.

Let the angle of intersection after transformation of  $f$ ,  $\varphi$ , measured from  $\tau'_1$  to  $\tau'_2$  in the counter-clockwise direction. From past chapters, we know  $\theta \approx \varphi$  if  $f$  is holomorphic. Now, let us assume  $f$  is only  $\mathbb{R}$ -differentiable and see how  $f$  acts on the angle  $\theta$ .

Curve:  $\gamma(t) = (x(t), y(t))$

New curve,  $f$  on  $\gamma$ :  $\tau(t) = f(\gamma(t)) = u(\gamma(t)) + iv(\gamma(t)) = (\underline{X}(t), \underline{Y}(t))$

New tangent vector:  $\tau'(t) = \frac{d}{dt}\tau(t) = (\underline{X}'(t), \underline{Y}'(t))$ , where we can invoke the chain rule:

$$\underline{X}'(t) = u_x(\gamma(t))x'(t) + u_y(\gamma(t))y'(t)$$

$$\underline{Y}'(t) = v_x(\gamma(t))x'(t) + v_y(\gamma(t))y'(t)$$

If we have  $f = u(x, y) + iv(x, y)$  is  $\mathbb{R}$ -differentiable, then

$$J(f) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

is the Jacobian Matrix of  $f$  and  $\tau'(t) = \gamma'(t) \cdot J(f)^T$

If  $f$  is  $\mathbb{C}$ -differentiable and  $\gamma(r_0) = z_0$  where  $\gamma(t) = x(t) + iy(t)$ , then  $f'(z) = u_x + iv_x$  and  $\gamma'(t) = x' + iy'$ .

$$f'(z_0)\gamma'(r_0) = f'(\gamma(r_0))\gamma'(r_0) = (u_x + iv_x)(x' + iy') = (u_x x' - v_x y') + i(u_x y' + v_x x')$$

Applying the Cauchy Riemann Equations,

$$\begin{aligned} (u_x x' - v_x y') + i(u_x y' + v_x x') &= (u_x x' + u_y y') + i(v_x x' + v_y y') = (u_x x' + u_y y', v_x x' + v_y y') \text{ in } \mathbb{R}^2 \\ &= \gamma'(r_0) J(f(z_0))^T = \tau' \end{aligned}$$

By now, we have an understanding of how  $f$  acts on tangent vectors when  $f' \neq 0$ , namely  $\theta = \varphi$ .

## 9.2 Conformal Function

### 9.2.1 Conditions of Conformal Functions

We say  $f$  is a conformal map at  $z_0$  if the following hold:

- ①  $f$  is  $\mathbb{R}^2$ -differentiable at  $z_0$
- ②  $|Jf| \neq 0$
- ③  $f$  preserves the oriented angle  $\theta$ , between  $\gamma_1$  and  $\gamma_2$  and  $\tau_1$  and  $\tau_2$  at  $z_0$  and  $f(z_0)$ .

### 9.2.2 $e^z$ is conformal

Now let us take a closer look at  $e^z$  and figure out why it is conformal.

- ① holds apparently.
- ②  $f(z) = e^z = e^x \cos(y) + ie^x \sin(y)$ , then

$$J(f(x, y)) = \begin{bmatrix} e^x \cos(y) & -e^x \sin(y) \\ e^x \sin(y) & e^x \cos(y) \end{bmatrix} = e^x \begin{bmatrix} \cos(y) & -\sin(y) \\ \sin(y) & \cos(y) \end{bmatrix}$$

$$\Rightarrow |J(f(x, y))| \neq 0$$

- ③ Now we have shown in the previous part that  $Jf$  is the product of a dilation matrix,  $e^x I$ , and a rotation matrix, which means  $f$  preserves the angles between  $\gamma'_1$  and  $\gamma'_2$ . Their image under  $f$ :

$$\begin{aligned} \tau'_1(t) &= \gamma'_1(t) \cdot J(f)^T \\ \tau'_2(t) &= \gamma'_2(t) \cdot J(f)^T \end{aligned}$$

In fact,  $f$  preserving oriented angles implies  $Jf$  is a rotation  $\otimes$  dilation matrix. Hence, it is proven that  $e^z$  is conformal.

## 9.3 Conformal Map

**Definition:** If  $f$  is conformal, infinitely differentiable, and one-to-one on a domain  $\Omega$  to  $V$ , then  $f$  is a conformal map from  $\Omega$  to  $V$ .

For example,  $e^z$  is conformal map from  $\Omega = \{z : |\Im(z)| < \pi\}$  to  $V = \mathbb{C} \setminus \mathbb{R}^-$ .

**Proposition:** If  $f$  is complex differentiable and  $f'(z_0) \neq 0$ , it is a linear transform of a dilation by  $|f'(z_0)|$  and a rotation by  $\text{Arg}(f'(z_0))$ . Hence,  $f$  is conformal because ③ is satisfied.

**Example.**  $f(z) = z^2$  on  $\Omega = \{z | 1 < |z| < 3 \text{ and } \Im(z) > 0\}$  is conformal.

### Inverse Function Theorem:

If  $f$  is a continuously differentiable function with nonzero derivative at the point  $a$ , then  $f$  is invertible in a neighborhood of  $a$ , the inverse is continuously differentiable, and the derivative of the inverse function at  $b = f(a)$  is the reciprocal of the derivative of  $f$  at  $a$ :

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}$$

**Proposition:** If  $f$  is invertible at  $z_0$  and conformal, then  $f^{-1}$  is conformal at  $f(z_0)$  by the inverse function theorem provided  $f$  is continuously differentiable.

From the proposition above, we know that  $\text{Log}(z)$  is conformal on  $\mathbb{C} \setminus \mathbb{R}^-$ .

## 10 Bilinear Transformations

In complex analysis, the term linear transformation is used to describe affine transformations,  $f(z) = az + b$ .

A bilinear/Möbius transformation is of the form

$$\frac{az + b}{cz + d}$$

where  $a, b, c, d \in \mathbb{C}$ .

Now,  $f(\infty) = \frac{a}{c}$  by L'Hospital argument and we say  $f(-\frac{d}{c}) = \infty$ .

If  $ad - bc \neq 0$ , then  $f' \neq 0$  by the quotient rule and  $f$  is not constant. The constants are not unique, as  $f(z) = \frac{az+b}{cz+d} = \frac{(az+b)k}{(cz+d)k}$ . Therefore, we only have 3 degrees of freedom.

We have already seen these functions of this form before  $f$  is:

① Composition of a finite number of

$$\begin{cases} \text{Translations, } f(z) = z + k \\ \text{Rotations, } f(z) = e^{i\theta} z \\ \text{Dilations, } f(z) = kz, k \in \mathbb{R} \\ \text{Inversions, } f(z) = \frac{1}{z} \end{cases}$$

② Conformal,  $f$  is holomorphic away from  $z = -\frac{d}{c}$ ,  $f' \neq 0$ , and  $f$  is one-to-one.

③ Maps circles/lines to either lines or circles, "lines are circles of  $\infty$ -radius in  $\mathbb{C}$  or  $\mathbb{C} \cup \{\infty\}$ .

If the line or circle passes through  $z = -\frac{d}{c}$ , where  $f$  is undefined, then it will be mapped to a line. Otherwise, it is mapped to a circle.

④  $f$  can be identified by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then,

$$\begin{cases} f \circ f \equiv A \circ A = A^2 \\ f^{-1} \equiv A^{-1} \\ f \circ g \equiv AB \end{cases}$$

and so there is a group homomorphism with Möbius transforms and invertible matrices in  $\mathbb{C}^{2 \times 2}$ .

$$f(z) = (3 + 2i)z - i^3 = \frac{(3+2i)z - i^3}{\delta z + 1}$$

⑤ We can conformally map 3 points in  $\mathbb{C} \cup \{\infty\}$  to any 3 points in  $\mathbb{C} \cup \{\infty\}$ .

This type of argument is similar to showing norms are equivalent in  $\mathbb{R}^n$  or uniqueness of power series expansions.

Given any 3 points  $z_0, z_1, z_2 \in \mathbb{C}$ , we can create a Möbius transformation  $T$  such that

$$\begin{cases} T(z_0) = 0 \\ T(z_1) = 1 \\ T(z_2) = \infty \end{cases}$$

then  $T(z) = (z, z_0, z_1, z_2) = \frac{(z-z_0)(z_1-z_2)}{(z-z_2)(z_1-z_0)}$  is called the cross-ratio of  $z, z_0, z_1$ , and  $z_2$ .

Special cases:

$$\begin{cases} (z, \infty, z_1, z_2) = \frac{z_1 - z_2}{z - z_2} \\ (z, z_0, \infty, z_2) = \frac{z - z_0}{z - z_2} \\ (z, z_0, z_1, \infty) = \frac{z - z_0}{z_1 - z_0} \end{cases}$$



Then

$$\begin{cases} T^{-1}(0) = z_0 \\ T^{-1}(1) = z_1 \\ T^{-1}(\infty) = z_2 \end{cases}$$

- 11 Contour Integral in  $\mathbb{C}$
- 12 Cauchy's Closed Curve Theorem and the Fundamental Theorem of Calculus
- 13 Cauchy's Integral Formula
- 14 Growth Conditions of Holomorphic Functions
- 15 Convergence of Infinite Series in  $\mathbb{C}$
- 16 Power Series in  $\mathbb{C}$
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- 19 Laurent Series
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## Chapter 1: Algebra in $\mathbb{C}$

$$f(x) = x^2$$

this formula is an example  $f(x) = x$

$$1 + 2 = 3$$

$$1 = 3 - 2$$

$$f(x) = x^2$$

$$g(x) = \frac{1}{x}$$

$$h(x) = \int_b^a \frac{1}{x} x^3$$

$$F(x) = \frac{1}{\sqrt{x}}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left( \frac{1}{\sqrt{x}} \right)$$

Core Material: 1. Finding patterns in data; using them to make predictions. 2. Models and statistics help us understand patterns. 3. Optimization algorithms "learn" the patterns.  
Classification: 1.