Introduction to Complex Analysis

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Contents

1 Algebra of the Complex Plane

1.1 Introduction to Complex Numbers

Let $z = a + ib \in \mathbb{C}$ where $a, b \in \mathbb{R}$ and $i^2 = -1$.
This number can be thought of as a point in 2-space, $\mathbb{R}^2$, (a, b) or as a position in $\mathbb{C}$.
$\mathbb{R}^2$: $\oplus$ addition; $\odot$ scalar multiplication.
$\mathbb{C}$: $\oplus$ addition; $\odot$ scalar multiplication; a vector space; have multiplication of elements, $\mathbb{C}$ is a field.

$$\text{If } z = a + ib,\ w = c + id, \text{ then } zw = (ac - bd) + i(ad + cb)$$

$$zw = wz$$

$$z(w + \alpha) = zw + z\alpha$$

$$(zw)\alpha = z(w\alpha)$$

1.2 Conjugate of Complex Numbers

1.2.1 Definition of Conjugate

The complex conjugate of z, $\overline{z}$, is defined by

$$\overline{z} = a - ib$$

Geometric representation: The image of $\bar{z}$ is the reflection of z about the Real axis.

1.2.2 Properties of Conjugate

$$\overline{\overline{z}} = z$$

$$\overline{zw} = \overline{z}\overline{w}$$

$$\overline{z + w} = \overline{z} + \overline{w}$$

$$\overline{z} = z \text{ if and only if } z \in \mathbb{R}$$

1.2.3 Real and Imaginary Parts

We can project z onto the Real or Imaginary axis and measure its distance from 0:

$$\Re(z) = a$$

$$\Im(z) = b, \text{ not } ib$$

Each function is a map $\mathbb{C} \to \mathbb{R}$. Then

$$\Re(z) = \frac{z + \overline{z}}{2}$$

$$\Im(z) = \frac{z - \overline{z}}{2i}$$

This is similar to the pattern with even/odd functions.

1.3 Modulus of Complex Numbers

Note: $z\overline{z} = (a + ib)(a - ib) = a^2 + b^2 \in \mathbb{R}$

1.3.1 Definition of Modulus

$|z|$ length/modulus of z is defined by:

$$|z| = (a^2 + b^2)^{\frac{1}{2}} = (z\overline{z})^{\frac{1}{2}} \in \mathbb{R}$$

1.3.2 Properties of Modulus

$$|zw| = |z||w|$$
$$|z| = |\overline{z}|$$
$$|z| \geqslant 0$$
$$|z| = 0 \text{ if and only if } z = 0$$

1.3.3 Triangle Inequality

Triangle Inequality:
$$|z + w| \leqslant |z| + |w|$$
Reverse Triangle Inequality:
$$|z| - |w| \leqslant |z - w|$$
$z = z - w + w \Rightarrow |z| = |z - w + w| \Rightarrow |z| \leqslant |z - w| + |w| \Rightarrow |z| - |w| \leqslant |z - w|$.

1.3.4 Complex Division

With $z\overline{z} \in \mathbb{R}$, we can define complex division by reducing it to a multiplication problem.
$$\frac{z}{w} = \frac{z\overline{w}}{w\overline{w}} = \frac{1}{w\overline{w}}(z\overline{w})$$
We also have
$$|\frac{z}{w}| = \frac{|z|}{|w|} \text{ for } w \neq 0$$

1.3.5 Distance in the plane

A disk in the complex plane centered at c of radius $r \in \mathbb{R}$ is of the form
$$\{z \in \mathbb{C} \mid |z - c| \leqslant r\}$$

1.4 Complex Polynomial

A complex polynomial $p(z)$ of degree n is of the form:
$$p(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$$
where $a_n \neq 0$ and $a_i \in \mathbb{C}$ for $i = 0, ..., n$

1.4.1 Fundamental Theorem of Algebra

The factorization of $p(z)$ factors over $\mathbb{C}$ is unique,
$$p(z) = c(z - z_1)^{m_1} ... (z - z_k)^{m_k}$$
We have roots $z_i \in \mathbb{C}$ of $p(z)$ with order $m_i \in \mathbb{N}$.
For example, if $p(z) = z^2 + 4 = (z + 2i)(z - 2i)$, then it factors over $\mathbb{C}$ but not $\mathbb{R}$.
Note: $\mathbb{C}$ is an algebraically closed field, there are no irreducible polynomials in $\mathbb{C}$.
Note: $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$ are not algebraically closed.

2 Geometry of the Complex Plane

2.1 Properties of Polar Forms

Complex numbers can be represented in polar forms:

$$z = |z|(\cos\theta + i\sin\theta)$$

with modulus $|z|$ and argument θ. To change between the coordinate systems it follows:

$$|z| = (a^2 + b^2)^{\frac{1}{2}}$$
$$\tan\theta = \frac{b}{a}$$
$$a = |z|\cos\theta = \Re(z)$$
$$b = |z|\sin\theta = \Im(z)$$

Note: $\theta_R = \arctan(\frac{b}{a})$ is a reference angle of z. To find θ from θ_R, you need to consider the signs of a and b.
Example:
$z = -3 + 3i = 3\sqrt{2}(\cos\frac{3\pi}{4} + \sin\frac{3\pi}{4})$
$\theta_R = \arctan(\frac{3}{-3}) = -\frac{\pi}{4}$
$\theta = \pi + \theta_R = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$, since θ is in II.

2.2 Definition of Argument and argument

Arg(z) is z's principle polar angle θ, $z \neq 0$, where $\theta \in (-\pi, \pi]$.
arg(z) is all of z's polar angles, $\theta + 2k\pi$, $k \in \mathbb{Z}$.

2.3 Euler's Formula

Euler's Formula is defined as a linear combination of $\cos\theta$ and $\sin\theta$, $\mathbb{R}$-valued functions.

$$e^{i\theta} = \cos\theta + i\sin\theta$$

It allows us to express z in polar form by

$$z = |z|e^{i\theta}$$

-1 has polar angle π and modulus 1,

$$-1 = e^{i\pi} \text{ or } e^{i\pi} + 1 = 0$$

By the angle addition formulas from trigonometry we find:

$$e^{i\theta}e^{i\varphi} = e^{i(\theta+\varphi)}$$
$$(e^{i\theta})^k = e^{i\theta k}$$

2.4 Geometric Understanding of Multiplication

The polar angle of zw is the sum of the polar angles of z and w. The modulus is the product of the moduli.

$$\operatorname{Arg}(zw) = \operatorname{Arg}(z) + \operatorname{Arg}(w)$$
$$\operatorname{Arg}(\overline{z}) = -\operatorname{Arg}(z)$$

Question: How about $\frac{z}{w}$ and z^4?
It follows from trigonometry that $|e^{i\theta}| = 1$, if $\theta \in (-\pi, \pi]$ we get a parametrization of the unit circle.

Example: Discover all solutions to $w^3 = i = z$
Let $p(z) = w^3 - i$. By Fundamental Theorem of Algebra, there are 3 roots of $p(z)$.
Therefore, $3\theta = \frac{\pi}{2} + 2\pi k$, $k \in \mathbb{Z}$
This gives us infinitely many solutions, but the solutions form 3 equivalence classes.
All we need is $k = 0, 1, 2$, which gives $\theta_1 = \frac{\pi}{6}$, $\theta_2 = \frac{5\pi}{6}$, $\theta_3 = \frac{3\pi}{2}$
Our solutions partitioned the unit circle into 3 equally spaced wedges.
The solutions to $w^3 = i$ are $w_1 = \frac{\sqrt{3}}{2} + \frac{1}{2}i$, $w_2 = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$ and $w_3 = -i$.
This problem of unity can be extended to solving $w^k = z$ for $k \in \mathbb{N}$, $z \in \mathbb{C}$ for unknown k-solutions w.

3 Stereographic Projections, Exponentials and Logs

3.1 Stereographic Projections

We can express the complex plane on the unit sphere in $\mathbb{R}^3$. To perform this we project points on the surface of the sphere along the line from the North Pole $(0,0,1)$ through the point and onto the plane $z=0,\mathbb{C}$
$p_1=(x_1,x_2,x_3)\to z=a+ib=\frac{x_1+ix_2}{1-x_3}$
$x_1=\frac{2a}{|z|^2+1}$, $x_2=\frac{2b}{|z|^2+1}$, $x_3=\frac{|z|^2-1}{|z|^2+1}$
Points in the northern hemisphere P_1, have $|z_1|>1$; while points in the southern hemisphere P_2, have $|z_2|<1$.

3.1.1 Mapping

$\mathbb{S}^2\to\mathbb{C}$
$N=(0,0,1)\to\infty$
$S=(0,0,-1)\to 0$
lines of latitude $\to |z|=r$, circles
lines of longitude $\to \mathrm{Arg}(z)=\pm\theta$, lines through $(0,0)$
Note: In general, circles on $\mathbb{S}^2$ map to circles and lines in $\mathbb{C}$, orientation is not always preserved.

3.2 Complex Logarithm

3.2.1 Logarithm of Real Numbers

Anytime we are dealing with power, the log function is very useful.

$$\log x=\int_1^x \frac{1}{t}dt \text{ for } x\in\mathbb{R} \tag{1}$$

$$\frac{d}{dx}x^x=\frac{d}{dx}e^{\ln x^x}=\frac{d}{dx}e^{x\ln x}=e^{x\ln x}(x\cdot\frac{1}{x}+\ln x)=x^x(1+\ln x)$$

3.2.2 Logarithm of Complex Numbers

Remember from Euler's Formula, $e^{i\theta}=\cos\theta+i\sin\theta$.
$e^z=e^{a+ib}=e^ae^{ib}$
$\mathrm{Arg}(e^z)=b$, $|e^z|=e^a>0$
Therefore, if a is held fixed, e^z maps to a circle as b changes.
On the other hand, if b is held fixed, e^z maps to a line through $(0,0)$.

3.2.3 Derivation of Complex Logarithm

We want $e^{\log(z)}=z$ for all $z\neq 0$, and thus

$$e^{\Re(\log(z))+i\Im(\log(z))}=e^{\Re(\log(z))}e^{i\Im(\log(z))}=|z|e^{i\theta}=z$$

$$\Rightarrow |z|=e^{\Re(\log(z))}$$

$$\Rightarrow \Re(\log(z))=\log|z|$$

From the imaginary part we find

$$e^{i\theta}=e^{i\Im(\log(z))}$$

$$\Rightarrow \arg(z)=\theta=\Im(\log(z))$$

$$\Rightarrow \Im(\log(z))=\mathrm{Arg}(z)$$

because $arg(z)$ is not well defined.
Our constructed inverse of e^z is a multi-valued function

$$\log(z)=\log|z|+i\arg(z)$$

3.2.4 Conclusion from Derivation

$$\log(z) = \log|z| + i\arg(z)$$

$$\mathrm{Log}(z) = \log|z| + i\,\mathrm{Arg}(z)$$

Note: $\mathrm{Log}(z)$ does not have all the nice behavior as $\mathbb{R}$-valued $\log(x)$: $\mathrm{Log}\left(z^k\right)$.
Sometimes they are co-terminal angles, but they are not equal. See the following example:

$$\begin{cases} \mathrm{Log}(i^3) = \mathrm{Log}(-i) = -i\frac{\pi}{2} \\ 3\,\mathrm{Log}(i) = 3 \cdot (i\frac{\pi}{2}) = i\frac{3\pi}{2} \end{cases}$$

Example: Compute 3^i:

$$3^i = e^{\mathrm{Log}\,3^i} = e^{i\,\mathrm{Log}\,3} = \cos\left(\mathrm{Log}\,3\right) + i\sin\left(\mathrm{Log}\,3\right)$$

3.2.5 How Logarithm acts on curves

$$\begin{cases} \text{Maps a circle with radius } r \text{ to a vertical line passing through } (\ln(r), 0) \\ \text{Maps a line with angle } \theta \text{ passing through the origin to a horizontal line passing through } (0, i\theta) \end{cases}$$

4 Topology in C

4.1 Complex Sequence

Let $\{Z_n\}$ be a sequence in $\mathbb{C}$.

4.1.1 Cauchy Sequence

The sequence is Cauchy if for all $\epsilon > 0$, there is a $N \in \mathbb{N}$ such that for all $n, m > N$, $|z_n - z_m| < \epsilon$.

4.1.2 Sequence Convergence

The sequence converges if $|z_n - z| \to 0$ as $n \to \infty$. The distance between z_n and z vanishes.

4.1.3 Completeness of $\mathbb{C}$

$\{z_n\}$ converges if and only if $\{z_n\}$ is Cauchy.
Proof:
We show this by treating $\mathbb{C}$ as $\mathbb{R}^2$ and exploiting $\{X_n\}$ converges if and only if $\{X_n\}$ is Cauchy.
($\Longrightarrow$) (If $z_n \to z$, then $\Re(z_n) \to \Re(z)$ and $\Im(z_n) \to \Im(z)$. Since the sequences of $\mathbb{R}^2$ converge, they are Cauchy.
$|Z_n - Z_m| \leqslant |\Re(Z_n - Z_m)| + |\Im(Z_n - Z_m)| = |\Re(Z_n) - \Re(Z_m)| + |\Im(Z_n) - \Im(Z_m)|$
Upper bounds can be picked to be less than $\frac{\epsilon}{2}$ for some N. Therefore, $|Z_n - Z_m| \to 0$.

($\Longleftarrow$) If $\{Z_n\}$ is Cauchy, so are $\{\Re(Z_n)\}$ and $\{\Im(Z_n)\}$. But these are $\mathbb{R}$-sequences that converge. Therefore, $\{Z_n\}$ converges.

4.2 Complex Set

Let $\Omega \subset \mathbb{C}$. Sets can be open, closed, both, or neither.

4.2.1 Open Set

If for any $z_0 \in \mathbb{C}$, there exist some $\epsilon > 0$, such that the set $B_\epsilon(z_0) = \{z||z - z_0| < \epsilon\}$ is contained in Ω, then Ω is open.
Ω is open if and only if Ω^c is closed.
Ω is open if and only if Ω is equal to its own interior, which means it does not contain its boundary points $\partial\Omega$.

4.2.2 Closed Set

If Ω contains its limit point, then Ω is closed.
Ω is closed if and only if Ω^c is open.
Ω is closed if and only if Ω contains its boundary points.

4.2.3 Compact Set

If Ω can be contained in a disk of finite radius, then Ω is bounded.

4.2.4 Compact Set

If Ω is closed and bounded, then Ω is compact. This resembles $[a, b]$ in $\mathbb{R}$.

4.2.5 Connected Set

If any two points in Ω can be connected by a path, then Ω is connected.
Simply Connected Set: A simply connected set has no "holes" in it. For example, $\Omega = \{z||z - c| < 4\}$.
A connected but not simply connected set is an annulus, $\Omega = \{z|2 < |z - c| < 4\}$

4.2.6 Boundary of Set

The boundary of Ω, $\partial\Omega$ is all points with ϵ-balls intersecting Ω and Ω^c for all $\epsilon > 0$.

4.2.7 Interior of Set

The interior of Ω, $\text{Int}(\Omega)$, is all points in Ω with a ϵ-ball contained in Ω for some $\epsilon > 0$. "Largest open set in Ω".

4.2.8 Closure of Set

The closure of Ω is the union of Ω and its boundary $\partial\Omega$.

4.2.9 Domain

If a set is open and connected in $\mathbb{C}$, it is a domain.
A domain can be traversed by a path of horizontal and vertical line segments.

$$f(x) = x^2$$

this formula is an example $f(x) = x$

$$\begin{aligned} 1 + 2 &= 3 \\ 1 &= 3 - 2 \end{aligned}$$

$$\begin{aligned} f(x) &= x^2 \\ g(x) &= \frac{1}{x} \\ h(x) &= \int_b^a \frac{1}{x} x^3 \\ F(x) &= \frac{1}{\sqrt{x}} \end{aligned}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\left(\frac{1}{\sqrt{x}} \right)$$

Core Material: 1. Finding patterns in data; using them to make predictions. 2. Models and statistics help us understand patterns. 3. Optimization algorithms "learn" the patterns.
Classification: 1.