

Introduction to Complex Analysis

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Aug 3, 2020

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1 Algebra of the Complex Plane

1.1 Introduction to Complex Numbers

Let $z = a + ib \in \mathbb{C}$ where $a, b \in \mathbb{R}$ and $i^2 = -1$.

This number can be thought of as a point in 2-space, \mathbb{R}^2 , (a, b) or as a position in \mathbb{C} .

\mathbb{R}^2 : \oplus addition; \odot scalar multiplication.

\mathbb{C} : \oplus addition; \odot scalar multiplication; a vector space; have multiplication of elements, \mathbb{C} is a field.

$$\text{If } z = a + ib, w = c + id, \text{ then } zw = (ac - bd) + i(ad + cb)$$

$$zw = wz$$

$$z(w + \alpha) = zw + z\alpha$$

$$(zw)\alpha = z(w\alpha)$$

1.2 Conjugate of Complex Numbers

1.2.1 Definition of Conjugate

The complex conjugate of z , \bar{z} , is defined by

$$\bar{z} = a - ib$$

Geometric representation: The image of \bar{z} is the reflection of z about the Real axis.

1.2.2 Properties of Conjugate

$$\overline{\bar{z}} = z$$

$$\overline{zw} = \bar{z}\bar{w}$$

$$\overline{z + w} = \bar{z} + \bar{w}$$

$$\bar{z} = z \text{ if and only if } z \in \mathbb{R}$$

1.2.3 Real and Imaginary Parts

We can project z onto the Real or Imaginary axis and measure its distance from 0:

$$\Re(z) = a$$

$$\Im(z) = b, \text{ not } ib$$

Each function is a map $\mathbb{C} \rightarrow \mathbb{R}$. Then

$$\Re(z) = \frac{z + \bar{z}}{2}$$

$$\Im(z) = \frac{z - \bar{z}}{2i}$$

This is similar to the pattern with even/odd functions.

1.3 Modulus of Complex Numbers

Note: $z\bar{z} = (a + ib)(a - ib) = a^2 + b^2 \in \mathbb{R}$

1.3.1 Definition of Modulus

$|z|$ length/modulus of z is defined by:

$$|z| = (a^2 + b^2)^{\frac{1}{2}} = (z\bar{z})^{\frac{1}{2}} \in \mathbb{R}$$

1.3.2 Properties of Modulus

$$|zw| = |z||w|$$

$$|z| = |\bar{z}|$$

$$|z| \geq 0$$

$$|z| = 0 \text{ if and only if } z = 0$$

1.3.3 Triangle Inequality

Triangle Inequality:

$$|z + w| \leq |z| + |w|$$

Reverse Triangle Inequality:

$$|z| - |w| \leq |z - w|$$

$$z = z - w + w \Rightarrow |z| = |z - w + w| \Rightarrow |z| \leq |z - w| + |w| \Rightarrow |z| - |w| \leq |z - w|.$$

1.3.4 Complex Division

With $z\bar{z} \in \mathbb{R}$, we can define complex division by reducing it to a multiplication problem.

$$\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} = \frac{1}{w\bar{w}}(z\bar{w})$$

We also have

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|} \text{ for } w \neq 0$$

1.3.5 Distance in the plane

A disk in the complex plane centered at c of radius $r \in \mathbb{R}$ is of the form

$$\{z \in \mathbb{C} \mid |z - c| \leq r\}$$

1.4 Complex Polynomial

A complex polynomial $p(z)$ of degree n is of the form:

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

where $a_n \neq 0$ and $a_i \in \mathbb{C}$ for $i = 0, \dots, n$

1.4.1 Fundamental Theorem of Algebra

The factorization of $p(z)$ factors over \mathbb{C} is unique,

$$p(z) = c(z - z_1)^{m_1} \dots (z - z_k)^{m_k}$$

We have roots $z_i \in \mathbb{C}$ of $p(z)$ with order $m_i \in \mathbb{N}$.

For example, if $p(z) = z^2 + 4 = (z + 2i)(z - 2i)$, then it factors over \mathbb{C} but not \mathbb{R} .

Note: \mathbb{C} is an algebraically closed field, there are no irreducible polynomials in \mathbb{C} .

Note: $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$ are not algebraically closed.

2 Geometry of the Complex Plane

2.1 Properties of Polar Forms

Complex numbers can be represented in polar forms:

$$z = |z|(\cos \theta + i \sin \theta)$$

with modulus $|z|$ and argument θ . To change between the coordinate systems it follows:

$$|z| = (a^2 + b^2)^{\frac{1}{2}}$$

$$\tan \theta = \frac{b}{a}$$

$$a = |z| \cos \theta = \Re(z)$$

$$b = |z| \sin \theta = \Im(z)$$

Note: $\theta_R = \arctan(\frac{b}{a})$ is a reference angle of z . To find θ from θ_R , you need to consider the signs of a and b .

Example:

$$z = -3 + 3i = 3\sqrt{2}(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4})$$

$$\theta_R = \arctan(\frac{3}{-3}) = -\frac{\pi}{4}$$

$$\theta = \pi + \theta_R = \pi - \frac{\pi}{4} = \frac{3\pi}{4}, \text{ since } \theta \text{ is in } II.$$

2.2 Definition of Argument and argument

$\text{Arg}(z)$ is z 's principle polar angle θ , $z \neq 0$, where $\theta \in (-\pi, \pi]$.

$\arg(z)$ is all of z 's polar angles, $\theta + 2k\pi$, $k \in \mathbb{Z}$.

2.3 Euler's Formula

Euler's Formula is defined as a linear combination of $\cos \theta$ and $\sin \theta$, \mathbb{R} -valued functions.

$$e^{i\theta} = \cos \theta + i \sin \theta$$

It allows us to express z in polar form by

$$z = |z|e^{i\theta}$$

-1 has polar angle π and modulus 1,

$$-1 = e^{i\pi} \text{ or } e^{i\pi} + 1 = 0$$

By the angle addition formulas from trigonometry we find:

$$e^{i\theta} e^{i\varphi} = e^{i(\theta+\varphi)}$$

$$(e^{i\theta})^k = e^{i\theta k}$$

2.4 Geometric Understanding of Multiplication

The polar angle of zw is the sum of the polar angles of z and w . The modulus is the product of the moduli.

$$\text{Arg}(zw) = \text{Arg}(z) + \text{Arg}(w)$$

$$\text{Arg}(\bar{z}) = -\text{Arg}(z)$$

Question: How about $\frac{z}{w}$ and z^4 ?

It follows from trigonometry that $|e^{i\theta}| = 1$, if $\theta \in (-\pi, \pi]$ we get a parametrization of the unit circle.

Example: Discover all solutions to $w^3 = i = z$

Let $p(z) = w^3 - i$. By Fundamental Theorem of Algebra, there are 3 roots of $p(z)$.

Therefore, $3\theta = \frac{\pi}{2} + 2\pi k$, $k \in \mathbb{Z}$

This gives us infinitely many solutions, but the solutions form 3 equivalence classes.

All we need is $k = 0, 1, 2$, which gives $\theta_1 = \frac{\pi}{6}$, $\theta_2 = \frac{5\pi}{6}$, $\theta_3 = \frac{3\pi}{2}$

Our solutions partitioned the unit circle into 3 equally spaced wedges.

The solutions to $w^3 = i$ are $w_1 = \frac{\sqrt{3}}{2} + \frac{1}{2}i$, $w_2 = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$ and $w_3 = -i$.

This problem of unity can be extended to solving $w^k = z$ for $k \in \mathbb{N}$, $z \in \mathbb{C}$ for unknown k -solutions w .

3 Stereographic Projections, Exponentials and Logs

3.1 Stereographic Projections

We can express the complex plane on the unit sphere in \mathbb{R}^3 . To perform this we project points on the surface of the sphere along the line from the North Pole $(0, 0, 1)$ through the point and onto the plane $z = 0, \mathbb{C}$

$$p_1 = (x_1, x_2, x_3) \rightarrow z = a + ib = \frac{x_1 + ix_2}{1 - x_3}$$

$$x_1 = \frac{2a}{|z|^2 + 1}, x_2 = \frac{2b}{|z|^2 + 1}, x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}$$

Points in the northern hemisphere P_1 , have $|z_1| > 1$; while points in the southern hemisphere P_2 , have $|z_2| < 1$.

3.1.1 Mapping

$$\mathbb{S}^2 \rightarrow \mathbb{C}$$

$$N = (0, 0, 1) \rightarrow \infty$$

$$S = (0, 0, -1) \rightarrow 0$$

$$\text{lines of latitude} \rightarrow |z| = r, \text{ circles}$$

$$\text{lines of longitude} \rightarrow \text{Arg}(z) = \pm\theta, \text{ lines through } (0, 0)$$

Note: In general, circles on \mathbb{S}^2 map to circles and lines in \mathbb{C} , orientation is not always preserved.

3.2 Complex Logarithm

3.2.1 Logarithm of Real Numbers

Anytime we are dealing with power, the log function is very useful.

$$\log x = \int_1^x \frac{1}{t} dt \text{ for } x \in \mathbb{R} \quad (1)$$

$$\frac{d}{dx} x^x = \frac{d}{dx} e^{\ln x^x} = \frac{d}{dx} e^{x \ln x} = e^{x \ln x} (x \cdot \frac{1}{x} + \ln x) = x^x (1 + \ln x)$$

3.2.2 Logarithm of Complex Numbers

Remember from Euler's Formula, $e^{i\theta} = \cos \theta + i \sin \theta$.

$$e^z = e^{a+ib} = e^a e^{ib}$$

$$\text{Arg}(e^z) = b, |e^z| = e^a > 0$$

Therefore, if a is held fixed, e^z maps to a circle as b changes.

On the other hand, if b is held fixed, e^z maps to a line through $(0, 0)$.

3.2.3 Derivation of Complex Logarithm

We want $e^{\log(z)} = z$ for all $z \neq 0$, and thus

$$e^{\Re(\log(z)) + i\Im(\log(z))} = e^{\Re(\log(z))} e^{i\Im(\log(z))} = |z| e^{i\theta} = z$$

$$\Rightarrow |z| = e^{\Re(\log(z))}$$

$$\Rightarrow \Re(\log(z)) = \log |z|$$

From the imaginary part we find

$$e^{i\theta} = e^{i\Im(\log(z))}$$

$$\Rightarrow \arg(z) = \theta = \Im(\log(z))$$

$$\Rightarrow \Im(\log(z)) = \text{Arg}(z)$$

because $\arg(z)$ is not well defined.

Our constructed inverse of e^z is a multi-valued function

$$\log(z) = \log |z| + i \arg(z)$$

3.2.4 Conclusion from Derivation

$$\log(z) = \log |z| + i \arg(z)$$

$$\operatorname{Log}(z) = \log |z| + i \operatorname{Arg}(z)$$

Note: $\operatorname{Log}(z)$ does not have all the nice behavior as \mathbb{R} -valued $\log(x)$: $\operatorname{Log}(z^k)$.

Sometimes they are co-terminal angles, but they are not equal. See the following example:

$$\begin{cases} \operatorname{Log}(i^3) = \operatorname{Log}(-i) = -i\frac{\pi}{2} \\ 3\operatorname{Log}(i) = 3 \cdot (i\frac{\pi}{2}) = i\frac{3\pi}{2} \end{cases}$$

Example: Compute 3^i :

$$3^i = e^{\operatorname{Log} 3^i} = e^{i \operatorname{Log} 3} = \cos(\operatorname{Log} 3) + i \sin(\operatorname{Log} 3)$$

3.2.5 How Logarithm acts on curves

$$\begin{cases} \text{Maps a circle with radius } r \text{ to a vertical line passing through } (\ln(r), 0) \\ \text{Maps a line with angle } \theta \text{ passing through the origin to a horizontal line passing through } (0, i\theta) \end{cases}$$

4 Topology in \mathbb{C}

4.1 Complex Sequence

Let $\{Z_n\}$ be a sequence in \mathbb{C} .

4.1.1 Cauchy Sequence

The sequence is Cauchy if for all $\epsilon > 0$, there is a $N \in \mathbb{N}$ such that for all $n, m > N$, $|z_n - z_m| < \epsilon$.

4.1.2 Sequence Convergence

The sequence converges if $|z_n - z| \rightarrow 0$ as $n \rightarrow \infty$. The distance between z_n and z vanishes.

4.1.3 Completeness of \mathbb{C}

$\{z_n\}$ converges if and only if $\{z_n\}$ is Cauchy.

Proof:

We show this by treating \mathbb{C} as \mathbb{R}^2 and exploiting $\{X_n\}$ converges if and only if $\{X_n\}$ is Cauchy.

(\implies) (If $z_n \rightarrow z$, then $\Re(z_n) \rightarrow \Re(z)$ and $\Im(z_n) \rightarrow \Im(z)$. Since the sequences of \mathbb{R}^2 converge, they are Cauchy.

$$|Z_n - Z_m| \leq |\Re(Z_n - Z_m)| + |\Im(Z_n - Z_m)| = |\Re(Z_n) - \Re(Z_m)| + |\Im(Z_n) - \Im(Z_m)|$$

Upper bounds can be picked to be less than $\frac{\epsilon}{2}$ for some N . Therefore, $|Z_n - Z_m| \rightarrow 0$.

(\impliedby) If $\{Z_n\}$ is Cauchy, so are $\{\Re(Z_n)\}$ and $\{\Im(Z_n)\}$. But these are \mathbb{R} -sequences that converge. Therefore, $\{Z_n\}$ converges.

4.2 Complex Set

Let $\Omega \subset \mathbb{C}$. Sets can be open, closed, both, or neither.

4.2.1 Open Set

If for any $z_0 \in \mathbb{C}$, there exist some $\epsilon > 0$, such that the set $B_\epsilon(z_0) = \{z \mid |z - z_0| < \epsilon\}$ is contained in Ω , then Ω is open.

Ω is open if and only if Ω^c is closed.

Ω is open if and only if Ω is equal to its own interior, which means it does not contain its boundary points $\partial\Omega$, i.e. it does not contain its closure.

4.2.2 Closed Set

If Ω contains its limit point, then Ω is closed.

Ω is closed if and only if Ω^c is open.

Ω is closed if and only if Ω contains its boundary points.

4.2.3 Compact Set

If Ω can be contained in a disk of finite radius, then Ω is bounded.

4.2.4 Compact Set

If Ω is closed and bounded, then Ω is compact. This resembles $[a, b]$ in \mathbb{R} .

4.2.5 Connected Set

If any two points in Ω can be connected by a path, then Ω is connected.

Simply Connected Set: A simply connected set has no "holes" in it. For example, $\Omega = \{z \mid |z - c| < 4\}$.

A connected but not simply connected set is an annulus, $\Omega = \{z \mid 2 < |z - c| < 4\}$

4.2.6 Boundary of Set

The boundary of Ω , $\partial\Omega$ is all points with ϵ -balls intersecting Ω and Ω^c for all $\epsilon > 0$.

4.2.7 Interior of Set

The interior of Ω , $\text{Int}(\Omega)$, is all points in Ω with a ϵ -ball contained in Ω for some $\epsilon > 0$. "Largest open set in Ω ".

4.2.8 Closure of Set

The closure of Ω is the union of Ω and its boundary $\partial\Omega$.

4.2.9 Domain

If a set is open and connected in \mathbb{C} , it is a domain.

A domain can be traversed by a path of horizontal and vertical line segments.

4.2.10 Practice Examples

Determine whether the following sets are open or closed.

1. $\Omega = \mathbb{C} \setminus \{0\}$

Ω is open since it does not contain its closure, the point 0.

Ω is not closed since it does not contain its limit points. Let $z_n = \frac{1}{n}$. Then $z_n = \frac{1}{n} \rightarrow 0 \notin \Omega$.

Therefore, Ω is open.

2. $\Omega = \{z \mid |z| \geq 1\}$

Ω is not open since any ϵ -ball at 1 intersects Ω^c .

Ω is closed since Ω^c is open.

Therefore, Ω is closed.

3. $\Omega = \{z \mid |z| > 1\}$

Ω is open since Ω^c is closed.

Ω is not closed since it does not contain its limit points. Let $z_n = \frac{1}{n} + 1$. Then $z_n = \frac{1}{n} + 1 \rightarrow 1 \notin \Omega$.

4. $\Omega = \mathbb{C} \setminus (0, 1)$

Ω is not open. Its complement is $[0, 1]$. Even though it is closed in \mathbb{R} , it is not closed in \mathbb{C} , because any 2D ϵ -ball will always extend outside of the set $z \in (0, i)$. Hence, Ω^c is not open and not closed.

Ω is not closed since it does not contain its limit points. Let $z_n = \frac{1}{3} + i\frac{1}{n}$. Then $z_n = \frac{1}{3} + i\frac{1}{n} \rightarrow \frac{1}{3} \notin \Omega$.

Therefore, Ω is neither open nor closed.

5. $\Omega = \mathbb{C} \setminus [0, 1]$ Ω is open since $\Omega^c = [0, 1]$ is closed in \mathbb{C} .

Ω is not closed since it does not contain its limit points. Let $z_n = \frac{1}{3} + i\frac{1}{n}$. Then $z_n = \frac{1}{3} + i\frac{1}{n} \rightarrow \frac{1}{3} \notin \Omega$.

Therefore, Ω is open.

Note: Ω^c is not open in \mathbb{C} .

5 Continuity and Branch Cuts

5.1 Complex Continuity

Let $f : \Omega \rightarrow \mathbb{C}$, Ω is open and connected. If $z_n \rightarrow z_0$ implies $f(z_n) \rightarrow f(z_0)$, then f is continuous at z_0 . Also, f is bounded near z_0 .

f is continuous if for every $\epsilon > 0$, there is $\delta > 0$ such that $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon$.

- In either case, $\Re(f(z))$ and $\Im(f(z))$ are each continuous if and only if $f(z)$ is continuous. This follows the pattern as \mathbb{C} being complete.

- If f and g are continuous, then so are $f + g$, $f \times g$ and $\frac{f}{g}$ (provided $g(z) \neq 0$)

5.2 Complex Limits

Just like in \mathbb{R}^2 , limits are direction independent. Do not restrict limits to just $\Re \rightarrow 0$ or $\Im \rightarrow 0$. See the following example.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2y}{x^4 + y^2} \text{ does not exist}$$

as $x \rightarrow 0$, $y = 0$, then $f \rightarrow 0$, while $y = x^2$, $x \rightarrow 0$, then $f \rightarrow 1$.

5.3 Branch Cuts

Log , $z^{\frac{1}{2}}$ and $\arctan(z)$ are constructed by restricting the range of e^z , z^2 and $\tan(z)$.

For example, in creating $\text{Log}(z) = \ln|z| + i \text{Arg}(z)$, we made a choice that $\text{Arg}(z) \in (-\pi, \pi]$, $\text{Arg}(0)$ does not exist. Consider a path around $z_0 \neq 0$, $\gamma(t) = z_0 + re^{it}$. $\theta(t) = \arg(\gamma(t))$

As we traverse the circle, $t \in (-\pi, \pi]$,

$$\theta(t) = \arg(\gamma(t)) = \text{Arg}(z_0 + re^{it}) + 2\pi k = \text{Arg}(z_0 + re^{i(t+2\pi)}) + 2\pi k = \arg(\gamma(t+2\pi)) = \theta(t+2\pi)$$

Therefore, the angle $\theta(t)$ changes smoothly for all t and we stay on the same branch of $\text{Arg}(\gamma(t))$. That is to say, the $k \in \mathbb{Z}$ is the same for all t .

Compare this with any circular path about $z = 0$, γ_0 . Let $\gamma_0(t) = re^{it}$, $t \in (-\pi, \pi]$. As we traverse the circle once, we have a discontinuity in the principal angle of $\gamma_0(t)$. In particular, $\theta(\gamma_0(t)) \neq \theta(\gamma_0(t+2\pi))$

$$\theta(t) = \arg(\gamma(t)) = \text{Arg}(re^{it}) + 2\pi k \neq \text{Arg}(re^{i(t+2\pi)}) + 2\pi(k+1) = \arg(\gamma(t+2\pi)) = \theta(t+2\pi)$$

We jump from the k th to the $(k+1)$ th branch of Arg . Therefore, $\text{Arg}(z)$ has a branch point at $z = 0$.

If every neighborhood of z_0 contains a path $\gamma(t)$ around z_0 that leads to a jump discontinuity in f , then z_0 is a branch point of $f(z)$.

At this point, it suffices to study paths of the form $\gamma(t) = z_0 + re^{it}$ for $t \in (-\pi, \pi)$, and see if $f(\gamma(t)) = f(\gamma(t+2\pi))$ holds for all t .

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Chapter 1: Algebra in C

$$f(x) = x^2$$

this formula is an example $f(x) = x$

$$1 + 2 = 3$$

$$1 = 3 - 2$$

$$f(x) = x^2$$

$$g(x) = \frac{1}{x}$$

$$h(x) = \int_b^a \frac{1}{x} x^3$$

$$F(x) = \frac{1}{\sqrt{x}}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left(\frac{1}{\sqrt{x}} \right)$$

Core Material: 1. Finding patterns in data; using them to make predictions. 2. Models and statistics help us understand patterns. 3. Optimization algorithms "learn" the patterns.
Classification: 1.