

# Introduction to Complex Analysis

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# 1 Algebra of the Complex Plane

## 1.1 Introduction to Complex Numbers

Let  $z = a + ib \in \mathbb{C}$  where  $a, b \in \mathbb{R}$  and  $i^2 = -1$ .

This number can be thought of as a point in 2-space,  $\mathbb{R}^2$ ,  $(a, b)$  or as a position in  $\mathbb{C}$ .

$\mathbb{R}^2$ :  $\oplus$  addition;  $\odot$  scalar multiplication.

$\mathbb{C}$ :  $\oplus$  addition;  $\odot$  scalar multiplication; a vector space; have multiplication of elements,  $\mathbb{C}$  is a field.

If  $z = a + ib$ ,  $w = c + id$ , then  $zw = (ac - bd) + i(ad + cb)$

$$zw = wz$$

$$z(w + \alpha) = zw + z\alpha$$

$$(zw)\alpha = z(w\alpha)$$

## 1.2 Conjugate of Complex Numbers

### Definition of Conjugate.

The complex conjugate of  $z$ ,  $\bar{z}$ , is defined by

$$\bar{z} = a - ib$$

Geometric representation: The image of  $\bar{z}$  is the reflection of  $z$  about the Real axis.

### Properties of Conjugate.

$$\overline{\bar{z}} = z$$

$$\overline{zw} = \bar{z}\bar{w}$$

$$\overline{z + w} = \bar{z} + \bar{w}$$

$$\bar{z} = z \text{ if and only if } z \in \mathbb{R}$$

### Real and Imaginary Parts.

We can project  $z$  onto the Real or Imaginary axis and measure its distance from 0:

$$\Re(z) = a$$

$$\Im(z) = b, \text{ not } ib$$

Each function is a map  $\mathbb{C} \rightarrow \mathbb{R}$ . Then

$$\Re(z) = \frac{z + \bar{z}}{2}$$

$$\Im(z) = \frac{z - \bar{z}}{2i}$$

This is similar to the pattern with even/odd functions.

### 1.3 Modulus of Complex Numbers.

$$z\bar{z} = (a + ib)(a - ib) = a^2 + b^2 \in \mathbb{R}$$

#### Definition of Modulus.

$|z|$  length/modulus of  $z$  is defined by:

$$|z| = (a^2 + b^2)^{\frac{1}{2}} = (z\bar{z})^{\frac{1}{2}} \in \mathbb{R}$$

#### Properties of Modulus.

$$|zw| = |z||w|$$

$$|z| = |\bar{z}|$$

$$|z| \geq 0$$

$$|z| = 0 \text{ if and only if } z = 0$$

#### Triangle Inequality and Reverse Triangle Inequality.

$$\begin{cases} |z + w| \leq |z| + |w| \\ |z| - |w| \leq |z - w| \end{cases}$$

$$z = z - w + w \Rightarrow |z| = |z - w + w| \Rightarrow |z| \leq |z - w| + |w| \Rightarrow |z| - |w| \leq |z - w|$$

#### Complex Division.

With  $z\bar{z} \in \mathbb{R}$ , we can define complex division by reducing it to a multiplication problem.

$$\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} = \frac{1}{w\bar{w}}(z\bar{w})$$

We also have

$$\left| \frac{z}{w} \right| = \frac{|z|}{|w|} \text{ for } w \neq 0$$

#### Distance in the plane.

A disk in the complex plane centered at  $c$  of radius  $r \in \mathbb{R}$  is of the form

$$\{z \in \mathbb{C} \mid |z - c| \leq r\}$$

## 1.4 Complex Polynomial

A complex polynomial  $p(z)$  of degree  $n$  is of the form:

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

where  $a_n \neq 0$  and  $a_i \in \mathbb{C}$  for  $i = 0, \dots, n$

### Fundamental Theorem of Algebra.

The factorization of  $p(z)$  factors over  $\mathbb{C}$  is unique,

$$p(z) = c(z - z_1)^{m_1} \cdots (z - z_k)^{m_k}$$

We have roots  $z_i \in \mathbb{C}$  of  $p(z)$  with order  $m_i \in \mathbb{N}$ .

For example, if  $p(z) = z^2 + 4 = (z + 2i)(z - 2i)$ , then it factors over  $\mathbb{C}$  but not  $\mathbb{R}$ .

**Note:**  $\mathbb{C}$  is an algebraically closed field, there are no irreducible polynomials in  $\mathbb{C}$ .

**Note:**  $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$  are not algebraically closed.

## 2 Geometry of the Complex Plane

### 2.1 Properties of Polar Forms

Complex numbers can be represented in polar forms:

$$z = |z|(\cos \theta + i \sin \theta)$$

with modulus  $|z|$  and argument  $\theta$ . To change between the coordinate systems it follows:

$$|z| = (a^2 + b^2)^{\frac{1}{2}}$$

$$\tan \theta = \frac{b}{a}$$

$$a = |z| \cos \theta = \Re(z)$$

$$b = |z| \sin \theta = \Im(z)$$

Note that  $\theta_R = \arctan(\frac{b}{a})$  is a reference angle of  $z$ . To find  $\theta$  from  $\theta_R$ , you need to consider the signs of  $a$  and  $b$ .

**Example.**

$$z = -3 + 3i = 3\sqrt{2}(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4})$$

$$\theta_R = \arctan(\frac{3}{-3}) = -\frac{\pi}{4}$$

$$\theta = \pi + \theta_R = \pi - \frac{\pi}{4} = \frac{3\pi}{4}, \text{ since } \theta \text{ is in } II.$$

### 2.2 Definition of Argument and argument

$\text{Arg}(z)$  is  $z$ 's principle polar angle  $\theta$ ,  $z \neq 0$ , where  $\theta \in (-\pi, \pi]$ .

$\arg(z)$  is all of  $z$ 's polar angles,  $\theta + 2k\pi$ ,  $k \in \mathbb{Z}$ .

### 2.3 Euler's Formula

Euler's Formula is defined as a linear combination of  $\cos \theta$  and  $\sin \theta$ ,  $\mathbb{R}$ -valued functions.

$$e^{i\theta} = \cos \theta + i \sin \theta$$

It allows us to express  $z$  in polar form by

$$z = |z|e^{i\theta}$$

-1 has polar angle  $\pi$  and modulus 1,

$$-1 = e^{i\pi} \text{ or } e^{i\pi} + 1 = 0$$

By the angle addition formulas from trigonometry we find:

$$e^{i\theta}e^{i\varphi} = e^{i(\theta+\varphi)}$$

$$(e^{i\theta})^k = e^{i\theta k}$$

## 2.4 Geometric Understanding of Multiplication

The polar angle of  $zw$  is the sum of the polar angles of  $z$  and  $w$ . The modulus is the product of the moduli.

$$\text{Arg}(zw) = \text{Arg}(z) + \text{Arg}(w)$$

$$\text{Arg}(\bar{z}) = -\text{Arg}(z)$$

Question: How about  $\frac{z}{w}$  and  $z^4$ ?

It follows from trigonometry that  $|e^{i\theta}| = 1$ , if  $\theta \in (-\pi, \pi]$  we get a parametrization of the unit circle.

**Example.** Discover all solutions to  $w^3 = i = z$

Let  $p(z) = w^3 - i$ . By Fundamental Theorem of Algebra, there are 3 roots of  $p(z)$ .

Therefore,  $3\theta = \frac{\pi}{2} + 2\pi k$ ,  $k \in \mathbb{Z}$

This gives us infinitely many solutions, but the solutions form 3 equivalence classes.

All we need is  $k = 0, 1, 2$ , which gives  $\theta_1 = \frac{\pi}{6}$ ,  $\theta_2 = \frac{5\pi}{6}$ ,  $\theta_3 = \frac{3\pi}{2}$

Our solutions partitioned the unit circle into 3 equally spaced wedges.

The solutions to  $w^3 = i$  are  $w_1 = \frac{\sqrt{3}}{2} + \frac{1}{2}i$ ,  $w_2 = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$  and  $w_3 = -i$ .

This problem of unity can be extended to solving  $w^k = z$  for  $k \in \mathbb{N}$ ,  $z \in \mathbb{C}$  for unknown  $k$ -solutions  $w$ .



### 3 Stereographic Projections, Exponentials and Logs

#### 3.1 Stereographic Projections

We can express the complex plane on the unit sphere in  $\mathbb{R}^3$ . To perform this we project points on the surface of the sphere along the line from the North Pole  $(0, 0, 1)$  through the point and onto the plane  $z = 0, \mathbb{C}$

$$p_1 = (x_1, x_2, x_3) \rightarrow z = a + ib = \frac{x_1 + ix_2}{1 - x_3}$$

$$x_1 = \frac{2a}{|z|^2 + 1}$$

$$x_2 = \frac{2b}{|z|^2 + 1}$$

$$x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}$$

Points in the northern hemisphere  $P_1$ , have  $|z_1| > 1$ .

Points in the southern hemisphere  $P_2$ , have  $|z_2| < 1$ .

#### Mapping from Stereographic Space to the Complex Plane.

$$\mathbb{S}^2 \rightarrow \mathbb{C}$$

$$N = (0, 0, 1) \rightarrow \infty$$

$$S = (0, 0, -1) \rightarrow 0$$

$$\text{lines of latitude} \rightarrow |z| = r, \text{ circles}$$

$$\text{lines of longitude} \rightarrow \text{Arg}(z) = \pm\theta, \text{ lines through } (0, 0)$$

Note that in general, circles on  $\mathbb{S}^2$  map to circles and lines in  $\mathbb{C}$ , orientation is not always preserved.

#### 3.2 Complex Logarithm

##### Logarithm of Real Numbers.

Anytime we are dealing with power, the log function is very useful.

$$\log x = \int_1^x \frac{1}{t} dt \text{ for } x \in \mathbb{R}$$

$$\frac{d}{dx} x^x = \frac{d}{dx} e^{\ln x^x} = \frac{d}{dx} e^{x \ln x} = e^{x \ln x} (x \cdot \frac{1}{x} + \ln x) = x^x (1 + \ln x)$$

##### Logarithm of Complex Numbers.

Remember from Euler's Formula,  $e^{i\theta} = \cos \theta + i \sin \theta$ .

$$e^z = e^{a+ib} = e^a e^{ib}$$

$$\text{Arg}(e^z) = b$$

$$|e^z| = e^a > 0$$

Therefore, if  $a$  is held fixed,  $e^z$  maps to a circle as  $b$  changes.

On the other hand, if  $b$  is held fixed,  $e^z$  maps to a line through  $(0, 0)$ .

**Derivation of Complex Logarithm.**

We want  $e^{\log(z)} = z$  for all  $z \neq 0$ , and thus

$$e^{\Re(\log(z)) + i\Im(\log(z))} = e^{\Re(\log(z))} e^{i\Im(\log(z))} = |z| e^{i\theta} = z$$

$$\Rightarrow |z| = e^{\Re(\log(z))}$$

$$\Rightarrow \Re(\log(z)) = \log|z|$$

From the imaginary part we find

$$e^{i\theta} = e^{i\Im(\log(z))}$$

$$\Rightarrow \arg(z) = \theta = \Im(\log(z))$$

$$\Rightarrow \Im(\log(z)) = \text{Arg}(z)$$

because  $\arg(z)$  is not well defined.

Our constructed inverse of  $e^z$  is a multi-valued function

$$\log(z) = \log|z| + i \arg(z)$$

**Conclusion from Derivation.**

$$\log(z) = \log|z| + i \arg(z)$$

$$\text{Log}(z) = \log|z| + i \text{Arg}(z)$$

**Note:**  $\text{Log}(z)$  does not have all the nice behavior as  $\mathbb{R}$ -valued  $\log(x)$ :  $\text{Log}(z^k)$ .

Sometimes they are co-terminal angles, but they are not equal. See the following example:

$$\begin{cases} \text{Log}(i^3) = \text{Log}(-i) = -i\frac{\pi}{2} \\ 3\text{Log}(i) = 3 \cdot (i\frac{\pi}{2}) = i\frac{3\pi}{2} \end{cases}$$

**Example.** Compute  $3^i$ :

$$3^i = e^{\text{Log } 3^i} = e^{i \text{Log } 3} = \cos(\text{Log } 3) + i \sin(\text{Log } 3)$$

**How Logarithm acts on curves.**

$$\begin{cases} \text{Maps a circle with radius } r \text{ to a vertical line passing through } (\ln(r), 0) \\ \text{Maps a line with angle } \theta \text{ passing through the origin to a horizontal line passing through } (0, i\theta) \end{cases}$$

## 4 Topology in $\mathbb{C}$

### 4.1 Sequence

Let  $\{Z_n\}$  be a sequence in  $\mathbb{C}$ .

#### Cauchy Sequence.

The sequence is Cauchy if for all  $\varepsilon > 0$ , there is a  $N \in \mathbb{N}$  such that for all  $n, m > N$ ,  $|z_n - z_m| < \varepsilon$ .

#### Convergence of Sequence.

The sequence converges if  $|z_n - z| \rightarrow 0$  as  $n \rightarrow \infty$ . The distance between  $z_n$  and  $z$  vanishes.

#### Completeness of $\mathbb{C}$ .

$\{z_n\}$  converges if and only if  $\{z_n\}$  is Cauchy.

**Proof.** We show this by treating  $\mathbb{C}$  as  $\mathbb{R}^2$  and exploiting  $\{X_n\}$  converges if and only if  $\{X_n\}$  is Cauchy.

( $\implies$ ) If  $z_n \rightarrow z$ , then  $\Re(z_n) \rightarrow \Re(z)$  and  $\Im(z_n) \rightarrow \Im(z)$ . Since the sequences of  $\mathbb{R}^2$  converge, they are Cauchy.

$$|Z_n - Z_m| \leq |\Re(Z_n - Z_m)| + |\Im(Z_n - Z_m)| = |\Re(Z_n) - \Re(Z_m)| + |\Im(Z_n) - \Im(Z_m)|$$

Upper bounds can be picked to be less than  $\frac{\varepsilon}{2}$  for some  $N$ . Therefore,  $|Z_n - Z_m| \rightarrow 0$ .

( $\impliedby$ ) If  $\{Z_n\}$  is Cauchy, so are  $\{\Re(Z_n)\}$  and  $\{\Im(Z_n)\}$ . But these are  $\mathbb{R}$ -sequences that converge. Therefore,  $\{Z_n\}$  converges.

### 4.2 Complex Set

Let  $\Omega \subset \mathbb{C}$ . Sets can be open, closed, both, or neither.

#### Open Set.

If for any  $z_0 \in \Omega$ , there exist some  $\varepsilon > 0$ , such that the set  $B_\varepsilon(z_0) = \{z \mid |z - z_0| < \varepsilon\}$  is contained in  $\Omega$ , then  $\Omega$  is open.

$\Omega$  is open if and only if  $\Omega^c$  is closed.

$\Omega$  is open if and only if  $\Omega$  is equal to its own interior, which means it does not contain its boundary points  $\partial\Omega$ , i.e. it does not contain its closure.

#### Closed Set.

If  $\Omega$  contains its limit point, then  $\Omega$  is closed.

$\Omega$  is closed if and only if  $\Omega^c$  is open.

$\Omega$  is closed if and only if  $\Omega$  contains its boundary points.

#### Compact Set.

If  $\Omega$  can be contained in a disk of finite radius, then  $\Omega$  is bounded.

#### Compact Set.

If  $\Omega$  is closed and bounded, then  $\Omega$  is compact. This resembles  $[a, b]$  in  $\mathbb{R}$ .

#### Connected Set.

If any two points in  $\Omega$  can be connected by a path, then  $\Omega$  is connected.

Simply Connected Set: A simply connected set has no "holes" in it. For example,  $\Omega = \{z \mid |z - c| < 4\}$ .

A connected but not simply connected set is an annulus,  $\Omega = \{z \mid 2 < |z - c| < 4\}$

**Boundary of Set.**

The boundary of  $\Omega$ ,  $\partial\Omega$  is all points with  $\varepsilon$ -balls intersecting  $\Omega$  and  $\Omega^c$  for all  $\varepsilon > 0$ .

**Interior of Set.**

The interior of  $\Omega$ ,  $\text{Int}(\Omega)$ , is all points in  $\Omega$  with a  $\varepsilon$ -ball contained in  $\Omega$  for some  $\varepsilon > 0$ . "Largest open set in  $\Omega$ ".

**Closure of Set.**

The closure of  $\Omega$  is the union of  $\Omega$  and its boundary  $\partial\Omega$ .

**Domain.**

If a set is open and connected in  $\mathbb{C}$ , it is a domain.

A domain can be traversed by a path of horizontal and vertical line segments.

**Example.**

Determine whether the following sets are open or closed.

1.  $\Omega = \mathbb{C} \setminus \{0\}$

$\Omega$  is open since it does not contain its closure, the point 0.

$\Omega$  is not closed since it does not contain its limit points. Let  $z_n = \frac{1}{n}$ . Then  $z_n = \frac{1}{n} \rightarrow 0 \notin \Omega$ .

Therefore,  $\Omega$  is open.

2.  $\Omega = \{z \mid |z| \geq 1\}$

$\Omega$  is not open since any  $\varepsilon$ -ball at 1 intersects  $\Omega^c$ .

$\Omega$  is closed since  $\Omega^c$  is open.

Therefore,  $\Omega$  is closed.

3.  $\Omega = \{z \mid |z| > 1\}$

$\Omega$  is open since  $\Omega^c$  is closed.

$\Omega$  is not closed since it does not contain its limit points. Let  $z_n = \frac{1}{n} + 1$ . Then  $z_n = \frac{1}{n} + 1 \rightarrow 1 \notin \Omega$ .

4.  $\Omega = \mathbb{C} \setminus (0, 1)$

$\Omega$  is not open. Its complement is  $[0, 1]$ . Even though it is closed in  $\mathbb{R}$ , it is not closed in  $\mathbb{C}$ , because any 2D  $\varepsilon$ -ball will always extend outside of the set  $z \in (0, i)$ . Hence,  $\Omega^c$  is not open and not closed.

$\Omega$  is not closed since it does not contain its limit points. Let  $z_n = \frac{1}{3} + i\frac{1}{n}$ . Then  $z_n = \frac{1}{3} + i\frac{1}{n} \rightarrow \frac{1}{3} \notin \Omega$ .

Therefore,  $\Omega$  is neither open nor closed.

5.  $\Omega = \mathbb{C} \setminus [0, 1]$   $\Omega$  is open since  $\Omega^c = [0, 1]$  is closed in  $\mathbb{C}$ .

$\Omega$  is not closed since it does not contain its limit points. Let  $z_n = \frac{1}{3} + i\frac{1}{n}$ . Then  $z_n = \frac{1}{3} + i\frac{1}{n} \rightarrow \frac{1}{3} \notin \Omega$ .

Therefore,  $\Omega$  is open.

Note:  $\Omega^c$  is not open in  $\mathbb{C}$ .

## 5 Continuity and Branch Cuts

### 5.1 Complex Continuity

Let  $f : \Omega \rightarrow \mathbb{C}$ ,  $\Omega$  is open and connected. If  $z_n \rightarrow z_0$  implies  $f(z_n) \rightarrow f(z_0)$ , then  $f$  is continuous at  $z_0$ . Also,  $f$  is bounded near  $z_0$ .

$f$  is continuous if for every  $\varepsilon > 0$ , there is  $\delta > 0$  such that  $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$ .

In either case,  $\Re(f(z))$  and  $\Im(f(z))$  are each continuous if and only if  $f(z)$  is continuous. This follows the pattern as  $\mathbb{C}$  being complete.

If  $f$  and  $g$  are continuous, then so are  $f + g$ ,  $f \times g$  and  $\frac{f}{g}$  (provided  $g(z) \neq 0$ )

### 5.2 Complex Limits

Just like in  $\mathbb{R}^2$ , limits are direction independent. Do not restrict limits to just  $\Re \rightarrow 0$  or  $\Im \rightarrow 0$ . See the following example.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2y}{x^4 + y^2} \text{ does not exist}$$

as  $x \rightarrow 0, y = 0$ , then  $f \rightarrow 0$ , while  $y = x^2, x \rightarrow 0$ , then  $f \rightarrow 1$ .

### 5.3 Branch Cuts

$\text{Log}, z^{\frac{1}{2}}$  and  $\arctan(z)$  are constructed by restricting the range of  $e^z, z^2$  and  $\tan(z)$ .

For example, in creating  $\text{Log}(z) = \ln|z| + i \text{Arg}(z)$ , we made a choice that  $\text{Arg}(z) \in (-\pi, \pi]$ ,  $\text{Arg}(0)$  does not exist.

**Example.** Consider a path around  $z_0 \neq 0$ ,  $\gamma(t) = z_0 + re^{it}$ .  $\theta(t) = \arg(\gamma(t))$ .

As we traverse the circle,  $t \in (-\pi, \pi]$ ,

$$\theta(t) = \arg(\gamma(t)) = \text{Arg}(z_0 + re^{it}) + 2\pi k = \text{Arg}(z_0 + re^{i(t+2\pi)}) + 2\pi k = \arg(\gamma(t+2\pi)) = \theta(t+2\pi)$$

Therefore, the angle  $\theta(t)$  changes smoothly for all  $t$  and we stay on the same branch of  $\text{Arg}(\gamma(t))$ . That is to say, the  $k \in \mathbb{Z}$  is the same for all  $t$ .

Compare this with any circular path about  $z = 0$ ,  $\gamma_0$ . Let  $\gamma_0(t) = re^{it}$ ,  $t \in (-\pi, \pi]$ . As we traverse the circle once, we have a discontinuity in the principal angle of  $\gamma_0(t)$ . In particular,  $\theta(\gamma_0(t)) \neq \theta(\gamma_0(t+2\pi))$

$$\theta(t) = \arg(\gamma(t)) = \text{Arg}(re^{it}) + 2\pi k \neq \text{Arg}(re^{i(t+2\pi)}) + 2\pi(k+1) = \arg(\gamma(t+2\pi)) = \theta(t+2\pi)$$

We jump from the  $k$ th to the  $(k+1)$ th branch of  $\text{Arg}$ . Therefore,  $\text{Arg}(z)$  has a branch point at  $z = 0$ .

#### Definition of Branch Cuts and Branch Points.

If every neighborhood of  $z_0$  contains a path  $\gamma(t)$  around  $z_0$  that leads to a jump discontinuity in  $f$ , then  $z_0$  is a branch point of  $f(z)$ .

In order to find branches, at this point, it suffices to study paths of the form  $\gamma(t) = z_0 + re^{it}$  for  $t \in (-\pi, \pi)$ , and see if  $f(\gamma(t)) = f(\gamma(t+2\pi))$  holds for all  $t$ .

**Example.**  $\text{Arg}$  is discontinuous for all  $x$  on the negative  $\mathbb{R}$ -axis,  $\mathbb{R}^-$ .

We call this the principal branch cut of the multi-valued function  $\arg$ . Specifically,

$$\begin{aligned}\operatorname{Arg}(\gamma_0(t)) &\rightarrow \pi \text{ as } t \rightarrow \pi^- \\ \operatorname{Arg}(\gamma_0(t)) &\rightarrow -\pi \text{ as } t \rightarrow -\pi^+\end{aligned}$$

but  $\gamma_0(\pi) = \gamma_0(-\pi)$  since  $\pi$  and  $-\pi$  are coterminal.

$\mathbb{R}^-$  is the principal branch of  $\operatorname{Log}$ ,  $\operatorname{Arg}$ , and  $z^{\frac{1}{2}}$ .

The endpoints of a branch cut are branch points,  $\operatorname{Arg}$  has 0 and  $\infty$  as its branch points.

## 6 Differentiability in $\mathbb{C}$

Let  $f : \Omega \rightarrow \mathbb{C}$  for some domain  $\Omega$ . Then  $f$  is differentiable at  $z_0$  if the following exists.

$$\frac{d}{dz}f(z)|_{z=z_0} = f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$$

This limit must exist on all paths to  $z_0$ , since  $h \in \mathbb{C}$ . We could also take  $z_n \rightarrow z_0$  and use  $\frac{f(z_0) - f(z_n)}{z_0 - z_n} \rightarrow f'(z_0)$ . Remember limits are computed by looking at the difference in the modulus,  $|\frac{f(z_0) - f(z_n)}{z_0 - z_n} - f'(z_0)| \rightarrow 0$  as  $n \rightarrow \infty$ .

If  $f'(z_0)$  exists on all points  $z_0 \in \Omega$ , open and connected in  $\mathbb{C}$ , then  $f$  is holomorphic/ $\mathbb{C}$ -differentiable/analytic on  $\Omega$ . The connection between  $\mathbb{R}$  and  $\mathbb{C}$  analytic will be clear when we cover  $\mathbb{C}$ -power series.

If  $f'(z)$  exists everywhere in  $\mathbb{C}$ , then  $f$  is an entire/meromorphic function.

### 6.1 Difference between $\mathbb{R}$ and $\mathbb{C}$ differentiability

- $f : \mathbb{R} \rightarrow \mathbb{R}$

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

has only two paths to  $x$ , namely  $h \rightarrow 0^+$  and  $h \rightarrow 0^-$ .

Tangent plane or linear approximation:

$$f(x) \approx f(a) + f'(a)(x - a)$$

- $f : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$\lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h} = f_x$$

is also a 1D limit and a partial derivative.

Tangent plane or linear approximation:

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$   $f(x, y) = (u(x, y), v(x, y))$

Then  $f$  is differentiable if the Jacobian Matrix

$$J(f) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

can approximate the local change in  $f$ .

In each of the cases above, we are only measuring change in a few directions. However,  $h \rightarrow 0$  in  $\mathbb{C}$  can be from any direction in 2-space. Therefore,  $f'(z)$  existing is a much stronger condition for  $f$  on  $\mathbb{C}$  than on  $\mathbb{R}$ .

### Complications with $\bar{z}$ .

Consider the following example:

Let  $g(z) = \Re(z) = \frac{z+\bar{z}}{2}$ , which is a linear combination of continuous functions.

Assume  $h \in \mathbb{R}$ ,

$$\frac{g(z+ih) - g(z)}{h} = \frac{\Re(z) - \Re(z)}{h} \rightarrow 0 \text{ as } h \rightarrow 0$$

Compare this with

$$\frac{g(z+ih) - g(z)}{h} = \frac{\Re(z) + h - \Re(z)}{h} = 1 \rightarrow 1 \text{ as } h \rightarrow 0$$

Therefore, the function is nowhere differentiable in  $\mathbb{C}$ . The problem with  $g'(z)$  had to do with  $\bar{z}$ , despite reflection in  $\mathbb{R}^2$  about  $y = 0$  is differentiable. We will discover conditions on  $u_x, u_y, v_x$ , and  $v_y$  that ensure  $f'$  exists for  $f(z) = u(x, y) + iv(x, y)$  in the next chapter.

## 6.2 Properties of $f'(z)$

**Proposition.** If  $f$  is differentiable on  $\Omega$ , it is continuous on  $\Omega$ .

**Proposition.** The power rule holds too:  $\frac{d}{dz} z^n = n z^{n-1}$

As does the product, quotient, L'Hospital's and chain rule. In fact, most old results hold as well. In each of the following cases, there are branch points where  $f'(z)$  does not exist.

1. If  $f(z) = \text{Log}(z)$ , then  $f'(z) = \frac{1}{z}$
2. If  $f(z) = \tan^{-1}(z)$ , then  $f'(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$ , which does not exist for  $z = \pm i$
3. If  $f(z) = z^{\frac{1}{2}}$ , then  $f'(z) = \frac{1}{2} z^{-\frac{1}{2}}$

**Proposition.** Suppose  $f(z)$  is holomorphic on  $\Omega$ . then  $g(z) = \overline{f(\bar{z})}$  is holomorphic on  $\Omega^* = \{z | \bar{z} \in \Omega\}$

**Proof.** Suppose  $f$  is holomorphic on  $\Omega$ . Let  $z_0 \in \Omega^*$  and  $z_n \in \Omega^*$  for all  $n$  and  $z_n \rightarrow z_0$ . Then  $\bar{z}_n \rightarrow \bar{z}_0$  in  $\Omega$  and for  $\varepsilon > 0$ , there is a  $N \in \mathbb{N}$  such that for  $n > N$ .

$$\begin{aligned} \left| \frac{f(\bar{z}_0) - f(\bar{z}_n)}{\bar{z}_0 - \bar{z}_n} - f'(\bar{z}_0) \right| &< \varepsilon \\ \left| \frac{f(\bar{z}_0) - f(\bar{z}_n)}{\bar{z}_0 - \bar{z}_n} - f'(\bar{z}_0) \right| &= \left| \overline{\frac{f(\bar{z}_0) - f(\bar{z}_n)}{\bar{z}_0 - \bar{z}_n} - f'(\bar{z}_0)} \right| = \left| \frac{\overline{f(\bar{z}_0) - f(\bar{z}_n)}}{\overline{\bar{z}_0 - \bar{z}_n}} - \overline{f'(\bar{z}_0)} \right| \\ &= \left| \frac{\overline{f(\bar{z}_0) - f(\bar{z}_n)}}{z_0 - z_n} - \overline{f'(\bar{z}_0)} \right| = \left| \frac{g(z_0) - g(z_n)}{z_0 - z_n} - \overline{f'(\bar{z}_0)} \right| < \varepsilon \\ &\implies g'(z_0) = \overline{f'(\bar{z}_0)} \end{aligned}$$

Therefore,  $g$  is holomorphic on  $\Omega^*$ .

This proof is different from when we showed  $\frac{d}{dz} \bar{z}$  does not exist. Conjugation must be handled with care.

## 6.3 Geometric behavior of $f'(z)$

**Dilation.**

$w = f(z) \approx f'(a)(z - a) + f(a)$ . Small changes in  $z$  should give small changes in  $w$ .

The functions  $|z|$  and  $\text{Arg}(z)$  are continuous on their domains.

If  $f'(z_0) \neq 0$  and  $f'$  exists on  $\Omega$ , then

$$\begin{aligned} |f'(z_0)| &= \left| \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} \right| = \lim_{h \rightarrow 0} \left| \frac{f(z_0 + h) - f(z_0)}{h} \right| = \lim_{h \rightarrow 0} \frac{|f(z_0 + h) - f(z_0)|}{|h|} \\ &\implies |f'(z_0)||h| \approx |f(z_0 + h) - f(z_0)| \end{aligned}$$

The size of  $|f'(z_0)|$  tells us how much  $f$  is contracting/dilating near  $z_0$ .

**Rotation.**

$$\text{Arg}(f'(z_0)) = \text{Arg} \left( \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} \right) = \lim_{h \rightarrow 0} \text{Arg} \left( \frac{f(z_0 + h) - f(z_0)}{h} \right)$$



$$\begin{aligned}
&= \lim_{h \rightarrow 0} \text{Arg}(f(z_0 + h) - f(z_0)) - \text{Arg}(h) \\
&\implies \text{Arg}(f'(z_0)) \approx \text{Arg}(f(z_0 + h) - f(z_0)) \\
&\implies \text{Arg}(f'(z_0)) + \text{Arg}(h) \approx \text{Arg}(f(z_0 + h) - f(z_0))
\end{aligned}$$

Therefore,  $f$  rotates vectors from  $z_0$  to  $z_0 + h$  by the angle  $\text{Arg}(f'(z_0))$ .

**Conclusion.**

In conclusion,  $w = f(z) \approx f(z_0) + f'(z_0)(z - z_0) = c + \rho e^{i\theta}(z - z_0)$

$$\begin{cases} c: \text{Translation} \\ \rho: \text{Dilation} \\ e^{i\theta}: \text{rotation about } z_0 \text{ or complex multiplication.} \end{cases}$$

## 7 The Cauchy Riemann Equations

### 7.1 The Cauchy Riemann Equations

Let  $f(z) = f(x+iy) = u(x, y) + iv(x, y)$ , then  $f(z)$  is holomorphic implies the Cauchy Riemann Equations:

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

**Proof.** For  $f$ , which is  $\mathbb{C}$ -differentiable, the following representation of  $f'(z)$  holds for any path to  $z$ .

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$$

Along the path  $x+h+iy \rightarrow x+iy$ , we get

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \rightarrow 0} \frac{f(x+h+iy) - f(x+iy)}{h} = f_x = u_x(x, y) + iv_x(x, y)$$

Along the path  $x+iy+ih \rightarrow x+iy$ , we get

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \rightarrow 0} \frac{f(x+i(y+h)) - f(x+iy)}{ih} = \frac{f_y}{i} = -if_y = -i(u_y(x, y) + iv_y(x, y)) = v_y(x, y) - iu_y(x, y)$$

Equate components in  $f_x = -if_y$ , and it is proven that  $u_x = v_y$  and  $u_y = -v_x$ .

**Proposition.** If the Cauchy Riemann Equations do not hold at  $z_0$ , then  $f'(z_0)$  does not exist.

**Proposition.** If  $f$  is holomorphic on a domain  $\Omega$ , an open and connected set in  $\mathbb{C}$ , then the Cauchy Riemann Equations hold at all points in  $\Omega$ .

**Example.**

If  $f(x+iy) = x^2 + iy^2$ ,  $u_x = 2x$ ,  $v_x = 0$ ,  $u_y = 0$ ,  $v_y = 2y$ . Then  $2x = 2y \Rightarrow x = y$ , which is a line.

The set of points on the line is not open in  $\mathbb{C}$ . Therefore,  $f$  is nowhere holomorphic in  $\mathbb{C}$ . However, we will see that  $f'(z)$  does exist on the line  $y = x$ .

**Sufficiency of the Cauchy Riemann Equations to  $f'$ .**

The Cauchy Riemann Equations do a great job showing  $f'$  does not exist. But what about it being sufficient for  $f'$ ? We claim that satisfying the Cauchy Riemann Equations at  $z_0$  implies that  $f'$  exists at  $z_0$ .

**Proof.**

$f$  is  $\mathbb{C}$ -differentiable at  $z_0$  if and only if  $u(x, y)$  and  $v(x, y)$  have continuous partial derivatives that satisfy the Cauchy Riemann Equations at  $z_0$ . This requires us to treat  $f(x+iy)$  as a function on  $\mathbb{R}^2$ , or  $f(z)$  induces a map on  $\mathbb{R}^2$ .

Let  $h = \Delta x + i\Delta y$ ,

$$\frac{f(z+h) - f(z)}{h} = \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)}{\Delta x + i\Delta y} - \frac{u(x, y) + iv(x, y)}{\Delta x + i\Delta y}$$

$$u(x+\Delta x, y+\Delta y) - u(x, y) = u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y) + u(x, y+\Delta y) - u(x, y)$$

The function  $u(\cdot, \cdot)$  is differentiable in  $x$  and  $y$ , we can use the M.V.T (Mean Value Theorem) from  $\mathbb{R}$  to rewrite our difference in  $u$  by

$$u(x + \Delta x, y + \Delta y) - u(x, y + \Delta y) = \Delta x U_x(\underline{x}, y + \Delta y)$$

where  $\underline{x} \in (x, x + \Delta x)$ .

If  $u_x$  is continuous,  $u_x(\underline{x}, y + \Delta y) \approx u_x(x, y) + \varepsilon_1$ , and as  $\Delta y \rightarrow 0$  and  $\underline{x} \rightarrow x$ , by Taylor approximation and linear approximation on  $u_x$ , we have the error function  $\varepsilon_1 \rightarrow 0$ .

Next  $u(x, y + \Delta y) - u(x, y) = \Delta y u_y(x, \bar{y})$  and  $u_y(x, \bar{y}) \approx u_y(x, y) + \varepsilon_2$ .

Likewise, for the function  $v(x, y)$ , we get a  $v_x$  and  $v_y$  with error terms  $\varepsilon_3$  and  $\varepsilon_4$ .

$$\frac{f(z + h) - f(z)}{h} = \frac{\Delta x(u_x + \varepsilon_1 + iv_x + i\varepsilon_3) + \Delta y(u_y + \varepsilon_2 + iv_y + i\varepsilon_4)}{\Delta x + i\Delta y}$$

From the Cauchy Riemann Equations, we get  $f_x = \frac{f_y}{i} \Rightarrow if_x = f_y \Rightarrow i(u_x + iv_x) = u_y + iv_y$ . Substituting the terms, we have

$$f'(z) = \frac{\Delta x(u_x + iv_x) + i\Delta y(u_x + iv_x)}{\Delta x + i\Delta y} + \frac{\lambda}{\Delta x + i\Delta y}$$

where  $\lambda = \Delta x(\varepsilon_1 + i\varepsilon_2) + \Delta y(\varepsilon_3 + i\varepsilon_4)$ . However,

$$\left| \frac{\lambda}{\Delta x + i\Delta y} \right| \leq \left| \frac{\Delta x(\varepsilon_1 + i\varepsilon_2)}{\Delta x + i\Delta y} \right| + \left| \frac{\Delta y(\varepsilon_3 + i\varepsilon_4)}{\Delta x + i\Delta y} \right| \leq |\varepsilon_1 + i\varepsilon_2| + |\varepsilon_3 + i\varepsilon_4|$$

because  $\left| \frac{\Delta x}{\Delta x + i\Delta y} \right| \leq 1$ .

As  $\Delta z \rightarrow 0$ ,  $\left| \frac{\lambda}{\Delta x + i\Delta y} \right| \rightarrow 0$ , and thus  $f'(z) = u_x + iv_x = f_x = \frac{f_y}{i}$ .

Therefore, the Cauchy Riemann Equations are an easy way to show  $f'(z)$  exists and they provide a set of partial differential equations that  $f$  must satisfy.

**Example.** Let  $f(z) = e^z = e^x(\cos(y) + i \sin(y))$

$$u = e^x \cos(y), v = e^x \sin(y)$$

$$u_x = e^x \cos(y), v_x = e^x \sin(y)$$

$$u_y = -e^x \sin(y), v_y = e^x \cos(y)$$

Therefore,  $f(z)$  is  $\mathbb{C}$ -differentiable on  $\mathbb{C}$ ,  $f$  is entire/meromorphic.  $f'(z) = f_x = u_x + iv_x = f(z)$ .

## 7.2 Cauchy Riemann with Logarithm

$$e^{\text{Log}(z)} = z \Rightarrow \frac{d}{dz} e^{\text{Log}(z)} = 1 \Rightarrow z \frac{d}{dz} \text{Log}(z) = 1 \Rightarrow \frac{d}{dz} \text{Log}(z) = \frac{1}{z}$$

We have a branch point in  $\text{Log}(z)$  where its derivative is undefined. Then  $\text{Log}(z)$  is  $\mathbb{C}$ -differentiable on  $\mathbb{C} \setminus \{0\}$ . This is true regardless of the branch cut on  $\text{Log}(z)$ .

### 7.3 Lack of Complex Mean Value Theorem

**Claim:**  $\frac{f(z)-f(w)}{z-w} \neq f'(c)$  for some  $c$  between  $z$  and  $w$ .

**Proof:** Let  $z = 1$ ,  $w = 0$  and  $f(t) = e^{i\pi t}$ , then  $f(1) - f(0) = e^{i\pi} - 1 = -2$ . However,  $|f'(t)| = \pi$  for all  $t \in [0, 1]$ .

**Follow-Up Question:** Does the lack of a Mean Value Theorem for  $f'(z)$  suggest  $f'(z) = 0$  not imply  $f$  is constant?

**Answer:** Suppose  $f$  is  $\mathbb{C}$ -differentiable on  $\Omega$  and one of the following holds, then  $f$  is constant on  $\Omega$ .

$$\begin{cases} f'(z) = 0 \\ |f(z)| \text{ is constant} \\ \operatorname{Re}(f(z)) \text{ is constant} \\ f\text{'s conjugate is } \mathbb{C}\text{-differentiable on } \Omega \end{cases}$$

### 7.4 Wirtinger Equations

There is another way to study the Cauchy Riemann Equations by introducing two operators:

$$\begin{aligned} \frac{\partial f}{\partial z} &= f_z \text{ and } \frac{\partial f}{\partial \bar{z}} = f_{\bar{z}} \\ f(x, y) &\equiv f(x + iy) = u(x, y) + iv(x, y) \\ f(x, y) &= f(\Re(z), \Im(z)) = f\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) \end{aligned}$$

From the chain rule, we get

$$\begin{aligned} f_z &= f_x x_z + f_y y_z = \frac{1}{2}f_x + \frac{1}{2i}f_y = \frac{1}{2}f_x - \frac{i}{2}f_y \\ f_{\bar{z}} &= f_x x_{\bar{z}} + f_y y_{\bar{z}} = \frac{1}{2}f_x - \frac{1}{2i}f_y = \frac{1}{2}f_x + \frac{i}{2}f_y \end{aligned}$$

where  $f_x = u_x + iv_x$  and  $f_y = u_y + iv_y$

These are the Wirtinger Equations.

$$\begin{cases} \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \end{cases}$$

#### Relationship with the Cauchy Riemann Equations.

From the Cauchy Riemann Equations  $if_x = f_y$  we get:

$$\begin{aligned} f_{\bar{z}} &= \frac{1}{2}f_x + \frac{i}{2}f_y = \frac{1}{2}f_x - \frac{1}{2}f_x = 0 \\ f_z &= \frac{1}{2}f_x - \frac{i}{2}f_y = \frac{1}{2}f_x - \frac{i^2}{2}f_x = f_x = f'(z) \end{aligned}$$

$f$  is  $\mathbb{C}$ -differentiable at  $z_0$  if and only if  $f(x, y) = u(x, y) + iv(x, y)$  is  $\mathbb{R}$ -differentiable at  $z_0$  and  $f_{\bar{z}}(z_0) = 0$ . Then  $f'(z_0) = f_z(z_0)$ . In other words,  $f'(z)$  does not depend on  $\bar{z}$ .

## 8 Harmonic Functions

### 8.1 Laplacian

Let  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ , then the Laplacian of  $u$  is

$$\Delta u = u_{xx} + u_{yy} = \nabla \cdot \nabla u$$

where  $\nabla = [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}]^T$  is the divergence operator and  $\nabla u = [u_x, u_y]^T$  is the gradient of  $u$ .

### 8.2 Harmonic Functions

If  $\Delta u = 0$ , then  $u(x, y)$  satisfies Laplace's (partial differential) equation or  $u$  is a harmonic function.

This means:

$$\begin{cases} u \text{ is continuous} \\ u\text{'s 1st and 2nd order partial derivatives exist and are smooth.} \end{cases}$$

**Proposition.** Suppose  $f = u + iv$  is holomorphic on  $\Omega$  where  $u(x, y)$  and  $v(x, y)$  have continuous 2<sup>nd</sup> order partial derivatives, then  $u$  and  $v$  are harmonic and  $v$  is the harmonic conjugate of  $u$ .

**Proof.**

By the Cauchy Riemann Equations,  $u_x = v_y$  and  $v_x = -u_y$ , then  $u_{xx} = v_{yx}$  and  $v_{xy} = -u_{yy}$ . By continuity of  $v_{yx}$  and  $v_{xy}$ ,  $v_{yx} = v_{xy}$ . This implies  $u_{xx} = -u_{yy} \Rightarrow u_{xx} + u_{yy} = 0$

Later on, we will find that the conditions on 2<sup>nd</sup> order partial derivatives is implied by  $f$  being holomorphic on  $\Omega$ , or  $f''$  exists.

**Definition of Harmonic Conjugate.**

The harmonic conjugate to  $u(x, y)$  is a function  $v(x, y)$ , such that  $f(x, y) = u(x, y) + iv(x, y)$  is holomorphic.

**Example.** Show that  $u(x, y) = x^3 - 3xy^2 + y$  is a harmonic function.

$$u_x = 3x^2 - 3y^2, u_y = -6xy + 1$$

$$u_{xx} = 6x, u_{yy} = -6x. \text{ Therefore, } u_{xx} + u_{yy} = 0$$

**Example.** Find the harmonic conjugate of  $u(x, y) = x^3 - 3xy^2 + y$ .

$$u_x = 3x^2 - 3y^2 = v_y$$

$$u_y = -6xy + 1 = -v_x$$

$$\Rightarrow v = 3x^2y - y^3 + C(x) \text{ or } v = 3x^2y - x + C(y)$$

Therefore,  $v = 3x^2y - y^3 - x + C$  is  $u$ 's harmonic conjugate.

**Proposition.**

If  $u$  is harmonic on a domain  $\Omega$ , then  $u_x$  is the real part of a holomorphic function on  $\Omega$ . If  $\Omega$  is simply connected, unlike  $\mathbb{C} \setminus \{0\}$ , then  $u$  is the real part of a holomorphic function on  $\Omega$ .

**Proof.**

Assume  $u$  is harmonic and  $\Omega$  is connected. If  $f = u_x - iu_y$ , then  $f_y = if_x$ . Hence,  $f$  is differentiable on  $\Omega$ . The simply connected statement requires future theorems to show  $F'(z) = f(z)$  for some holomorphic antiderivative  $F(z)$ .

## 9 Conformal Maps

**Example.** Let  $f(z) = (x + iy)^2 + 2(x + iy) = (x^2 + 2x - y^2) + i2(xy + y)$ . When are the component functions,  $u(x, y)$  and  $v(x, y)$  constant?

When are the component functions,  $u(x, y)$  and  $v(x, y)$ , constant?

The function  $f(z) = e^z = e^x(\cos(y) + i\sin(y))$  maps the set  $\Omega = \{z : |\Im(z)| < \pi\}$  to circles of radius  $r \in (-\infty, \infty)$ , or all points in  $\mathbb{C} \setminus \mathbb{R}^-$ . This coincides with the branch cut of  $\text{Log}(z)$ , or how we made  $e^z$  invertible.

### 9.1 Preservation of Angles

We will now show  $e^z$  preserves the angles between curves in  $\Omega$ . Let us first look at the following example.

Let  $\gamma_1(t) = 2i\pi t - i\pi$ ,  $\gamma_2(t) = t + i\frac{\pi}{4}$ .  $\gamma_1(0) = -i\pi$ ,  $\gamma_1(1) = i\pi$ .

The curves  $\gamma_1$  and  $\gamma_2$  intersect at an angle  $\frac{\pi}{2}$ . Also  $f(\gamma_1)$  is a circle centered at 0 while  $f(\gamma_2)$  is a line through  $z = 0$ . Their intersection in the  $w$ -plane is  $\frac{\pi}{2}$  as well.

We will show why  $f(z) = e^z$  does this by studying the angles between curves  $\gamma_1$  and  $\gamma_2$  and curves  $\tau_1 = f(\gamma_1)$  and  $\tau_2 = f(\gamma_2)$ . If  $\gamma(t)$  parameterizes a smooth curve in  $\mathbb{C}$ , then its tangent vector is  $\gamma'(t)$ . The angle between any two curves at  $z_0$  is the angle between their tangent vectors at  $z_0$ .

Assume the curves intersect at  $\gamma(r_0) = \gamma(s_0) = z_0$ .

Let the angle of intersection,  $\theta$ , measured from  $\gamma'_1$  to  $\gamma'_2$  in the counter-clockwise direction.

Let the angle of intersection after transformation of  $f$ ,  $\varphi$ , measured from  $\tau'_1$  to  $\tau'_2$  in the counter-clockwise direction. From past chapters, we know  $\theta \approx \varphi$  if  $f$  is holomorphic. Now, let us assume  $f$  is only  $\mathbb{R}$ -differentiable and see how  $f$  acts on the angle  $\theta$ .

Curve:  $\gamma(t) = (x(t), y(t))$

New curve,  $f$  on  $\gamma$ :  $\tau(t) = f(\gamma(t)) = u(\gamma(t)) + iv(\gamma(t)) = (\underline{X}(t), \underline{Y}(t))$

New tangent vector:  $\tau'(t) = \frac{d}{dt}\tau(t) = (\underline{X}'(t), \underline{Y}'(t))$ , where we can invoke the chain rule:

$$\underline{X}'(t) = u_x(\gamma(t))x'(t) + u_y(\gamma(t))y'(t)$$

$$\underline{Y}'(t) = v_x(\gamma(t))x'(t) + v_y(\gamma(t))y'(t)$$

If we have  $f = u(x, y) + iv(x, y)$  is  $\mathbb{R}$ -differentiable, then

$$J(f) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

is the Jacobian Matrix of  $f$  and  $\tau'(t) = \gamma'(t) \cdot J(f)^T$

If  $f$  is  $\mathbb{C}$ -differentiable and  $\gamma(r_0) = z_0$  where  $\gamma(t) = x(t) + iy(t)$ , then  $f'(z) = u_x + iv_x$  and  $\gamma'(t) = x' + iy'$ .

$$f'(z_0)\gamma'(r_0) = f'(\gamma(r_0))\gamma'(r_0) = (u_x + iv_x)(x' + iy') = (u_x x' - v_x y') + i(u_x y' + v_x x')$$

Applying the Cauchy Riemann Equations,

$$\begin{aligned} (u_x x' - v_x y') + i(u_x y' + v_x x') &= (u_x x' + u_y y') + i(v_x x' + v_y y') = (u_x x' + u_y y', v_x x' + v_y y') \text{ in } \mathbb{R}^2 \\ &= \gamma'(r_0) J(f(z_0))^T = \tau' \end{aligned}$$

By now, we have an understanding of how  $f$  acts on tangent vectors when  $f' \neq 0$ , namely  $\theta = \varphi$ .

## 9.2 Conformal Function

### Conditions of Conformal Functions.

We say  $f$  is a conformal map at  $z_0$  if the following hold:

- ①  $f$  is  $\mathbb{R}^2$ -differentiable at  $z_0$
- ②  $|Jf| \neq 0$
- ③  $f$  preserves the oriented angle  $\theta$ , between  $\gamma_1$  and  $\gamma_2$  and  $\tau_1$  and  $\tau_2$  at  $z_0$  and  $f(z_0)$ .

$e^z$  is conformal.

Now let us take a closer look at  $e^z$  and figure out why it is conformal.

- ① holds apparently.
- ②  $f(z) = e^z = e^x \cos(y) + ie^x \sin(y)$ , then

$$J(f(x, y)) = \begin{bmatrix} e^x \cos(y) & -e^x \sin(y) \\ e^x \sin(y) & e^x \cos(y) \end{bmatrix} = e^x \begin{bmatrix} \cos(y) & -\sin(y) \\ \sin(y) & \cos(y) \end{bmatrix}$$

$$\Rightarrow |J(f(x, y))| \neq 0$$

- ③ Now we have shown in the previous part that  $Jf$  is the product of a dilation matrix,  $e^x I$ , and a rotation matrix, which means  $f$  preserves the angles between  $\gamma'_1$  and  $\gamma'_2$ . Their image under  $f$ :

$$\begin{aligned} \tau'_1(t) &= \gamma'_1(t) \cdot J(f)^T \\ \tau'_2(t) &= \gamma'_2(t) \cdot J(f)^T \end{aligned}$$

In fact,  $f$  preserving oriented angles implies  $Jf$  is a rotation  $\otimes$  dilation matrix. Hence, it is proven that  $e^z$  is conformal.

## 9.3 Conformal Map

**Definition.** If  $f$  is conformal, infinitely differentiable, and one-to-one on a domain  $\Omega$  to  $V$ , then  $f$  is a conformal map from  $\Omega$  to  $V$ .

For example,  $e^z$  is conformal map from  $\Omega = \{z : |\Im(z)| < \pi\}$  to  $V = \mathbb{C} \setminus \mathbb{R}^-$ .

**Proposition.** If  $f$  is complex differentiable and  $f'(z_0) \neq 0$ , it is a linear transform of a dilation by  $|f'(z_0)|$  and a rotation by  $\text{Arg}(f'(z_0))$ . Hence,  $f$  is conformal because ③ is satisfied.

**Example.**  $f(z) = z^2$  on  $\Omega = \{z | 1 < |z| < 3 \text{ and } \Im(z) > 0\}$  is conformal.

### Inverse Function Theorem.

If  $f$  is a continuously differentiable function with nonzero derivative at the point  $a$ , then  $f$  is invertible in a neighborhood of  $a$ , the inverse is continuously differentiable, and the derivative of the inverse function at  $b = f(a)$  is the reciprocal of the derivative of  $f$  at  $a$ :

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}$$

**Proposition.** If  $f$  is invertible at  $z_0$  and conformal, then  $f^{-1}$  is conformal at  $f(z_0)$  by the inverse function theorem provided  $f$  is continuously differentiable.

From the proposition above, we know that  $\text{Log}(z)$  is conformal on  $\mathbb{C} \setminus \mathbb{R}^-$ .

## 10 Bilinear Transformations

In complex analysis, the term linear transformation is used to describe affine transformations,  $f(z) = az + b$ .

### 10.1 Definition of a Möbius transformation

A bilinear/Möbius transformation is of the form

$$\frac{az + b}{cz + d}$$

where  $a, b, c, d \in \mathbb{C}$ .

Now,  $f(\infty) = \frac{a}{c}$  by L'Hospital argument and we say  $f(-\frac{d}{c}) = \infty$ .

If  $ad - bc \neq 0$ , then  $f' \neq 0$  by the quotient rule and  $f$  is not constant. The constants are not unique, as  $f(z) = \frac{az+b}{cz+d} = \frac{(az+b)k}{(cz+d)k}$ . Therefore, we only have 3 degrees of freedom.

### 10.2 Brief Review of other Transformations

We have already seen these functions of this form before  $f$  is:

① Composition of a finite number of

$$\left\{ \begin{array}{l} \text{Translations, } f(z) = z + k \\ \text{Rotations, } f(z) = e^{i\theta} z \\ \text{Dilations, } f(z) = kz, k \in \mathbb{R} \\ \text{Inversions, } f(z) = \frac{1}{z} \end{array} \right.$$

② Conformal,  $f$  is holomorphic away from  $z = -\frac{d}{c}$ ,  $f' \neq 0$ , and  $f$  is one-to-one.

③ Maps circles/lines to either lines or circles, "lines are circles of  $\infty$ -radius in  $\mathbb{C}$  or  $\mathbb{C} \cup \{\infty\}$ .

If the line or circle passes through  $z = -\frac{d}{c}$ , where  $f$  is undefined, then it will be mapped to a line. Otherwise, it is mapped to a circle.

④  $f$  can be identified by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then,

$$\left\{ \begin{array}{l} f \circ f \equiv A \circ A = A^2 \\ f^{-1} \equiv A^{-1} \\ f \circ g \equiv AB \end{array} \right.$$

and so there is a group homomorphism with Möbius transforms and invertible matrices in  $\mathbb{C}^{2 \times 2}$ .

$$f(z) = (3 + 2i)z - i^3 = \frac{(3+2i)z - i^3}{\delta z + 1}$$

⑤ We can conformally map 3 points in  $\mathbb{C} \cup \{\infty\}$  to any 3 points in  $\mathbb{C} \cup \{\infty\}$ .

This type of argument is similar to showing norms are equivalent in  $\mathbb{R}^n$  or uniqueness of power series expansions.



### 10.3 Möbius transforming of a function

Given any 3 points  $z_0, z_1, z_2 \in \mathbb{C}$ , we can create a Möbius transformation  $T$  such that

$$\begin{cases} T(z_0) = 0 \\ T(z_1) = 1 \\ T(z_2) = \infty \end{cases} \quad \begin{cases} T^{-1}(0) = z_0 \\ T^{-1}(1) = z_1 \\ T^{-1}(\infty) = z_2 \end{cases}$$

Then

$$T(z) = (z, z_0, z_1, z_2) = \frac{(z - z_0)(z_1 - z_2)}{(z - z_2)(z_1 - z_0)}$$

is called the cross-ratio of  $z, z_0, z_1$ , and  $z_2$ .

There are some special cases:

$$\begin{cases} (z, \infty, z_1, z_2) = \frac{z_1 - z_2}{z - z_2} \\ (z, z_0, \infty, z_2) = \frac{z - z_0}{z - z_2} \\ (z, z_0, z_1, \infty) = \frac{z - z_0}{z_1 - z_0} \end{cases}$$

Given 3 more points  $w_0, w_1, w_2 \in \mathbb{C}$ , we get  $S(w) = (w, w_0, w_1, w_2)$ . Then, we can construct the function map:

$$\begin{aligned} z_0 &\xrightarrow{T} 0 \xleftarrow{S} w_0 \\ z_1 &\longrightarrow 0 \longleftarrow w_1 \\ z_2 &\longrightarrow 0 \longleftarrow w_2 \end{aligned}$$

Then  $w = f(z) = S^{-1} \circ T(z)$  maps  $z_0 \rightarrow w_0, z_1 \rightarrow w_1$ , and  $z_2 \rightarrow w_2$ .

The points must be distinct, because  $f$  is one-to-one.

To find  $f$  from above, we solve  $(z, z_0, z_1, z_2) = (w, w_0, w_1, w_2)$  for  $w$ .

#### Example.

Suppose

$$\begin{aligned} z_0 &= 1 \rightarrow i = w_0 \\ z_1 &= -1 \rightarrow 1 = w_1 \\ z_2 &= 1 \rightarrow -1 = w_2 \end{aligned}$$

then

$$\begin{aligned} \frac{(w - i)(1 - (-1))}{(w - (-1))(1 - i)} &= \frac{2(w - i)}{(w + 1)(1 - i)} = \frac{(z - i)(-1 - 1)}{(z - 1)(-1 - i)} = \frac{-2(z - i)}{(z - 1)(-1 - i)} = \frac{2(z - i)}{(z - 1)(1 + i)} \\ &\Rightarrow \frac{w - i}{(w + 1)(1 - i)} = \frac{z - i}{(z - 1)(1 + i)} \\ &\Rightarrow (w - i)(z - 1)(1 + i) = (w + 1)(1 - i)(z - i) \\ &\Rightarrow w = -\frac{1}{z} \end{aligned}$$

It is much work to find a simple function. Mapping a set of three points to another set of three points is time-consuming. If we study Möbius transforms as conformal mappings, then we can introduce another

way to move the three points around.

**Example.** Find a Möbius transform to map the unit disk  $|z| < 1$  to  $\Im(z) > 0$ .

If we pick where 3 points go, say on the boundary of  $\Omega$ , to the boundary of  $\Im(z) > 0$ , which is the  $\mathbb{R}$ -axis, then we can construct  $f(z)$ . We also need one point to get mapped to  $\infty$ , but we have  $\infty$ -many choices for the three points.

We want  $f(-1) = 0$ ,  $f(i) = 1$ , and  $f(1) = \infty$ , that will be

$$f(z) = (z, -1, -i, 1) = \frac{(z - z_0)(z_1 - z_2)}{(z - z_2)(z_1 - z_0)} = \frac{(z + 1)(-i - 1)}{(z - 1)(-i + 1)} = -i \frac{z + 1}{z - 1}$$

Our path's direction around the circle has to be preserved because  $f$  is conformal.

This means points on the interior of the circle will get mapped to the upper half of  $\mathbb{C}$ .

To check, we see that  $f(0) = i$ ,  $\Im(f(0)) > 0$ . Therefore, the function we found out is correct.

**Example.** Find a Möbius transform to map the unit disk  $|z| < 1$  to  $\Im(z) > 0$  and  $\Re(z) > 0$ .

Using the conclusion from the example above, the new desired function is simply  $g(z) = (f(z))^{\frac{1}{2}}$ .

## 11 Contour Integral in $\mathbb{C}$

Let  $\gamma(t) = x(t) + iy(t)$  be a curve in  $\mathbb{C}$  where  $\gamma(a) = z_0$  and  $\gamma(b) = z_1$ .  
Let  $C$  be the graph of  $\gamma(t)$ ,  $C = \{z | z = \gamma(t) \text{ for some } t \in [a, b]\}$ .

### 11.1 Piecewise Differentiable, Smooth, Simple, Closed curves

#### Piecewise Differentiable curves.

The curve determined by  $\gamma$ , its graph  $C$ , is considered piecewise differentiable if

1.  $x$  and  $y$  are continuous on  $[a, b]$
2.  $x'$  and  $y'$  are continuous on a partition of  $[a, b]$ ,  $[x_0, x_1] \cup [x_1, x_2] \cup [x_2, x_3] \cup \cdots \cup [x_{n-1}, x_n]$

#### Smooth curves.

If  $\gamma' \neq 0$  for only finitely many points, then the curve is considered smooth.

#### Simple curves.

A curve is simple if it does not intersect itself, i.e.  $\gamma(t) = \gamma(s)$  if and only if  $s = t$ .

#### Closed curves.

$C$  is a closed curve if it starts and stops at the same point, i.e.  $\gamma(a) = \gamma(b)$ ,  $t \in [a, b]$

### 11.2 Interior and Exterior of curves

A closed and simple curve keeps the interior of the set on its left side and its exterior to the right.  
This means we traverse circles counter-clockwise to describe their interior correctly.

#### Jordan Curve Theorem.

A closed and simple curve partitions  $\mathbb{C}$  into two regions, one of them bounded, defined as the interior of the curve.

### 11.3 Smoothly Equivalent

The parameter  $t$  provides an orientation or direction to  $C$ .

Let

$$\begin{cases} C_1 : \gamma_1(t), t \in [a, b] \\ C_2 : \gamma_2(t), t \in [c, d] \end{cases}$$

We say  $C_1$  and  $C_2$  are smoothly equivalent if there exists a one-to-one, continuous derivative mapping  $\lambda(t)$ ,

$$\lambda(t) : [c, d] \rightarrow [a, b]$$

$$\lambda(c) = a$$

$$\lambda(d) = b$$

$$\lambda'(t) > 0$$

$$\text{where } \gamma_1(\lambda(t)) = \gamma_2(t)$$

#### Example.

$$\begin{cases} C_1 : \gamma_1(t) = \cos(t) + i \sin(t), t \in [0, 2\pi] \\ C_2 : \gamma_2(t) = \cos(2t) + i \sin(2t), t \in [0, \pi] \end{cases}$$

Here  $C_1$  and  $C_2$  are smoothly equivalent, since we can let  $\lambda(t) = 2t$ .

Both parametrize the unit circle, preserve the orientation, and pass through each point the same number of times.

Let us look at other two curves:

$$\begin{cases} \gamma_3(t) = \cos(4t) + i \sin(4t), t \in [0, \pi] \\ \gamma_4(t) = \cos(t) + i \sin(-t), t \in [0, 2\pi] \end{cases}$$

$\gamma_3$  traverses the circle multiple times and  $\gamma_4$  has the opposite orientation of  $\gamma_1$  and  $\gamma_2$ .

Let  $-C$  be the curve  $C$  but with a reversed orientation,  $\gamma_R(t) = \gamma(b + a - t)$ .

$$\begin{cases} C_1 : \gamma_1(t) = \cos(t) + i \sin(t), t \in [0, 2\pi] \\ -C_1 : \gamma_4(t) = \cos(t) + i \sin(-t), t \in [0, 2\pi] \end{cases}$$

We want to integrate  $f(z)$  over curves  $C$  in  $\mathbb{C}$ . These will factor into the computation:

$$\begin{cases} \text{Orientation of } C \\ \text{Number of times } C \text{ traverses itself} \\ C \text{ is closed} \\ f \text{ is holomorphic on the interior of } C \text{ and } C \end{cases}$$

## 11.4 Line Integral

The line integral of  $f$  over  $C$  is given by

$$\int_C f(z) dz = \int_C u(z) + iv(z) dz = \int_C u(z) + iv(z) (dx + idy) = \int_a^b f(\gamma(t)) \gamma'(t) dt$$

If  $C$  is a closed curve,  $\gamma(a) = \gamma(b)$ , then we can use a closed loop in our  $\int$  symbol,  $\oint_C f(z) dz$

The term  $\gamma'(t)dt$  controls for how fast we traverse the curve. The integral is independent of our choice of smoothly equivalent curve,  $C_1$  or  $C_2$ .

**Proposition.** Let  $C_1$  and  $C_2$  be smoothly equivalent,

$$\int_{C_1} f(z) dz = \int_{C_2} f(z) dz$$

**Proof.** Let  $\gamma_1(\lambda(t)) = \gamma_2(t)$  and apply change of variables.

Because  $u = \lambda(t)$ ,  $u(c) = a$ ,  $u(d) = b$  and  $\gamma'_1(\lambda(t))\lambda'(t) = \gamma'_2(t)$ ,

$$\int_c^d f(\gamma_2(t))\gamma'_2(t) dt = \int_c^d f(\gamma_1(\lambda(t)))\gamma'_1(\lambda(t))\lambda'(t) dt = \int_a^b f(\gamma_1(u))\gamma'_1(u) du$$

**Proposition.**

$$-\int_C f(z) dz = \int_{-C} f(z) dz$$

**Proof.**

$$\int_{-C} f(z) dz = \int_b^a f(\gamma_R(t))\gamma'_R(t) dt = \int_b^a f(\gamma(t))\gamma'(t) dt$$

**Proposition.** Linearity holds:

$$\int_C \alpha f(z) + g(z) dz = \alpha \int_C f(z) dz + \int_C g(z) dz$$

**Example.** Find  $\oint_{|z|=1} \frac{1}{z} dz = \int_0^{2\pi} f(\gamma(t))\gamma'(t) dt$

We parametrize the curve as follows:

$C: \gamma(t) = \cos(t) + i \sin(t), t \in [0, 2\pi], \gamma'(t) = -\sin(t) + i \cos(t)$

$$\frac{1}{z} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2}$$

$$\int_0^{2\pi} f(\gamma(t))\gamma'(t) dt = \int_0^{2\pi} (\cos(t) - i \sin(t))(-\sin(t) + i \cos(t)) dt = \int_0^{2\pi} i dt = 2\pi i$$

There is another way to parametrize the curve:

$\gamma(t) = re^{it}, \gamma'(t) = ire^{it}$

$$\oint_{|z|=r} \frac{1}{z} dz = \int_0^{2\pi} f(re^{it})ire^{it} dt = i \int_0^{2\pi} e^{-it}e^{it} dt = \int_0^{2\pi} i dt = 2\pi i$$

From this example, we have the following famous result.

**Proposition.** The following holds with proof shown above.

$$\oint_{|z|=r} z^k dz = 0 \text{ for all } k \in \mathbb{Z}, k \neq -1, r > 0.$$

$$\oint_{|z|=r} \frac{1}{z} dz = \int_0^{2\pi} i dt = 2\pi i \text{ for } r > 0$$

**ML Estimate.**

Let  $f$  be a complex-valued, continuous function and  $|f(z)| < M$  on  $C$ , a curve of length  $L$ , then

$$\left| \int_C f(z) dz \right| \leq M \cdot L$$

**Proof.**

$$\left| \int_a^b f(\gamma(t))\gamma'(t) dt \right| \leq \int_a^b |f(\gamma(t))\gamma'(t)| dt \leq M \int_a^b |\gamma_1'(t)| dt = M \cdot L$$

where  $\int_a^b |\gamma_1'(t)| dt$  is the arclength of  $\gamma$ .

We do not have to worry about the orientation of  $C$ ,

$$\left| \int_{-C} f(z) dz \right| \leq M \cdot L$$

## 12 Cauchy's Closed Curve Theorem and the Fundamental Theorem of Calculus

We now study contour integrals of holomorphic functions over closed curves.

### 12.1 Cauchy's Closed Curve Theorem

#### Green's Theorem.

If  $P(x, y)$  and  $Q(x, y)$  have continuous first order derivatives on  $\Omega$  and its boundary  $\partial\Omega$ ,  $\Omega$  is a domain,

$$\oint_{\partial\Omega} P dx + Q dy = \int_{\Omega} Q_x - P_y dx dy$$

#### Cauchy-Goursat Theorem.

If  $f(z)$  is holomorphic on  $\Omega$  and  $C$  is any closed curve in  $\Omega$ , where  $\Omega$  is simply connected. Then

$$\oint_C f(z) dz = 0$$

**Proof.**

$$\oint_{\partial\Omega} f(z) dz = \oint_{\partial\Omega} u + iv (dx + idy) = \int_{\Omega} if_x - f_y dA$$

and the Cauchy Riemann Equations imply

$$if_x = f_y$$

Therefore,

$$\oint_{\partial\Omega} f(z) dz = 0$$

#### Corollary of Cauchy's Theorem.

Let  $C_1$  and  $C_2$  be paths from  $z_0$  to  $z_1$ . Then  $C_1 \cup -C_2$  is a closed loop in  $\mathbb{C}$ .  $C_1$  and  $C_2$  are in a simply connected subset of where  $f$  is holomorphic. Then by Cauchy-Goursat Theorem,

$$0 = \oint_{C_1 \cup -C_2} f(z) dz = \int_{C_1} f(z) dz + \int_{-C_2} f(z) dz = \int_{C_1} f(z) dz - \int_{C_2} f(z) dz$$

Hence,

$$\int_{C_1} f(z) dz = \int_{C_2} f(z) dz$$

This means it does not matter how you go from  $z_0$  to  $z_1$  as long as  $f$  is  $\mathbb{C}$ -differentiable along the way. This means we can use

$$\int_{z_0}^{z_1} f(z) dz$$

to denote the integral from  $z_0$  to  $z_1$ . But we still have to watch out for branch cuts.

Even non-simple curves are allowed, because Cauchy-Goursat Theorem tells us the integral over the loops in the non-simple curves are 0.

**Path Independence.** We can characterize Cauchy-Goursat Theorem with path independence. We have path independence of  $\int_{z_0}^{z_1} f(z) dz$  in  $\Omega$  if and only if

$$\oint_C f(z) dz = 0 \text{ for all closed curves } C \text{ in } \Omega$$

These conditions tell us holomorphic functions must have antiderivatives.

## 12.2 Fundamental Theorem of Calculus (F.T.C)

Path independence leads to introduction of Fundamental Theorem of Calculus.

### Fundamental Theorem of Calculus (Part One).

Assume  $f$  is holomorphic on  $\Omega$ , open and simply connected in  $\mathbb{C}$ . The function

$$F(z) = \int_a^z f(u) du$$

is holomorphic on  $\Omega$  and  $F'(z) = f(z)$ .

**Proof.** This is completely analogous to the proof on  $\mathbb{R}$ , which has two steps.

1. Compute difference quotient
2. Bound integrand by invoking continuity of  $f$

①

$$\begin{aligned} \frac{F(z+h) - F(z)}{h} &= \frac{1}{h} \int_z^{z+h} f(u) du \\ \frac{1}{h} \int_z^{z+h} du &= \frac{h}{h} = 1 \end{aligned}$$

implies

$$f(z) = f(z) \frac{1}{h} \int_z^{z+h} du = \frac{1}{h} \int_z^{z+h} f(z) du$$

Then

$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h} \int_z^{z+h} f(u) - f(z) du$$

② Since integrating over any closed curve  $C$  gives

$$\oint_C f(z) dz = 0$$

We can pick how we go from  $z$  to  $z+h$ , say along a line from  $z$  to  $z+h$ . We can do this for  $h$  sufficiently small so that our path is in  $\Omega$ . This path is selected to make our upcoming estimate on  $f(u) - f(z)$  easier to work with.

Along the path we can have  $|u - z| < \delta$ , which implies  $|f(z) - f(u)| < \varepsilon$  by continuity of  $f$  where  $u$  is on the path. From ML-Estimate, we get

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| \leq \frac{1}{|h|} \int_z^{z+h} |f(u) - f(z)| du < \frac{1}{|h|} |h| \varepsilon = \varepsilon$$

Therefore, it is proven that  $F'(z) = f(z)$ .

Let us take a closer look at our proof above. We needed  $f$  to be continuous so that we could invoke ML-Estimate. We also needed to pick a nice path from  $z$  to  $z+h$ . But we did not need  $f$  to be holomorphic. This leads to the next Theorem.

**Morera's Theorem.** Let  $f$  be a continuous, complex-valued function defined on an open set  $D$  in the complex plane. If for all closed paths  $C$  in  $D$ ,  $\oint_C f(z) dz = 0$ , then  $F(z)$  is holomorphic on  $D$ ,  $F'(z) = f(z)$  and  $f$  is holomorphic.

Later we will show that  $f'(z)$  is holomorphic when  $f$  is holomorphic.

Let  $f_n \rightarrow f$  uniformly on every compact subset of  $D$ . We say  $f_n$  converges on compacta to  $f$ . If  $f_n$  is holomorphic on  $D$  for all  $n$  and  $C$  is any closed curve in  $D$ ,

$$\oint_C f_n(z) dz = 0 \text{ for all } n$$

$$\Rightarrow \lim_{n \rightarrow \infty} \oint_C f_n(z) dz = 0$$

Since the convergence is uniform,  $f$  is continuous and

$$\lim_{n \rightarrow \infty} \oint_C f_n(z) dz = \oint_C f(z) dz = 0$$

Hence  $f$  is holomorphic on  $D$  by Morera's Theorem.

**Example.** Let  $f(z) = \int_0^\infty \frac{e^{zt}}{1+t} dt$ ,  $\Re(z) < 0$ .

Because  $\Re(z) < 0$ ,

$$\int_0^\infty \frac{|e^{zt}|}{1+t} dt \leq \int_0^\infty e^{\Re(z)t} dt = \frac{e^{\Re(z)t}}{\Re(z)} \Big|_0^\infty = \frac{-1}{\Re(z)}$$

And so we have an absolute convergent integral that is bounded.

$$|f(z)| \leq \frac{1}{|\Re(z)|}$$

Then, since the integral converges absolutely,

$$\oint_C f(z) dz = \oint_C \int_0^\infty \frac{e^{zt}}{1+t} dt dz = \int_0^\infty \frac{1}{1+t} \oint_C e^{zt} dz dt = \int_0^\infty \frac{1}{1+t} 0 dt = 0$$

by Cauchy-Goursat Theorem since  $e^{zt}$  is entire for  $t$  fixed. Therefore,  $f(z)$  is holomorphic on  $\Re(z) < 0$ .

We can generalize our result to handle other types of integral transforms  $F(z) = \int_0^\infty K(t, z, f(t)) dt$ .

We also have an easy way to evaluate the line integrals.

### Fundamental Theorem of Calculus (Part Two).

Let  $F(z)$  be  $\mathbb{C}$ -differentiable on a smooth curve  $C$  where  $F'(z) = f(z)$ , then

$$\int_a^b f(z) dz = F(z) \Big|_{z=a}^b$$

**Proof.** Let  $a, b, c \in D$ , where  $c$  is in the middle of  $a$  and  $b$ . Then,

$$F(b) - F(a) = \int_a^b f(u) du = \int_a^c f(u) du + \int_c^b f(u) du = \int_a^b f(z) dz$$



The result can also be shown with variable substitution.

$$\int_b^a f(z) dz = \int_0^1 f(\gamma(t))\gamma'(t) dt = F(\gamma(t)) \Big|_{t=0}^1 = F(b) - F(a)$$

**Proposition.** If  $f'(z) = 0$  on  $\Omega$ , then  $f$  is constant on  $\Omega$ . i.e.  $0 = u_x = v_y$  and  $u_y = -v_x = 0$ .

**Proof.** Integrate  $f'$  along a staircase path in  $\Omega$ ,

$$f(z) = f(a) + \int_a^z 0 du = f(a)$$

Or we can use the Cauchy Riemann Equations.

Let  $F'(z) = G'(z) = f(z)$ , then  $H(z) = F(z) - G(z)$  satisfies  $H'(z) = F'(z) - G'(z) = 0$ . This means  $H$  is constant. This leads to the theorem below.

**Theorem.** Any antiderivative of holomorphic  $f(z)$  is of the form

$$F(z) = \int_a^z f(u) du + C, C \in \mathbb{C}$$

**Proposition.** Let  $\Omega$  be simply-connected,  $0 \notin \Omega$ . Fix  $z_0 \in \Omega$  and pick the value of  $\log z_0$ . Then

$$F(z) = \int_{z_0}^z \frac{1}{w} dw + \log z_0$$

is a branch of  $\text{Log } z$  in  $\Omega$ .

**Proof.**  $F$  is well-defined since  $\frac{1}{w}$  is holomorphic in  $\Omega$ , we have path independence from  $z_0$  to  $z$ ,  $f' = \frac{1}{z}$ . We still need  $e^{F(z)} = z$ . Define

$$g(z) = ze^{-F(z)}$$

$$g'(z) = e^{-F(z)}(1 - zF'(z)) = e^{-F(z)}(1 - \frac{z}{z}) = 0$$

Therefore,  $g(z)$  is constant.

$$g(z) = g(z_0) = z_0 e^{-F(z_0)} = z_0 e^{-\log(z_0)} = 1$$

Therefore,

$$e^{F(z)} = z$$

## 13 Cauchy's Integral Formula

### First Extension of Cauchy-Goursat Theorem.

Let  $\Omega$  be the domain between two simple closed curves  $C_1$  and  $C_2$ , each oriented counter-clockwise. If  $f$  is holomorphic on  $\Omega$ , then

$$\oint_{C_1} f(z) dz = \oint_{C_2} f(z) dz$$

or

$$\oint_{C_1 \cup -C_2} f(z) dz$$

### Second Extension of Cauchy-Goursat Theorem.

Let  $g(z)$  be holomorphic on  $\Omega \setminus \{z_0\}$ ,  $g(z)$  is continuous on  $\Omega$ ,  $g(z_0)$  is finite. Then

$$\oint_C g(z) dz = 0$$

for any closed curve  $C$  in  $\Omega$  and  $g(z)$  is holomorphic in  $\Omega$ .

**Proof.** By First Extension of Cauchy-Goursat Theorem

$$\oint_C g(z) dz = \oint_{|z-z_0|=\varepsilon} g(z) dz$$

and

$$\oint_{|z-z_0|=\varepsilon} |g(z)| dz \leq M_\varepsilon \cdot 2\pi \cdot \varepsilon$$

Since  $g$  is continuous,  $M_\varepsilon$  is bounded and as  $\varepsilon \rightarrow 0$ , we have

$$\oint_C g(z) dz = 0$$

Therefore,  $g$  is holomorphic on  $\Omega$ ,  $g'(z_0)$  exists by Morera's Theorem.

The second extension of Cauchy-Goursat Theorem gives us Cauchy Integral Formula.

Assume  $f$  is holomorphic on  $\Omega$ ,  $0 \in \Omega$ . Let

$$g(z) = \frac{f(z) - f(0)}{z - 0} = \frac{f(z) - f(0)}{z}$$

as  $z \rightarrow 0$ ,  $g(0) = f'(0)$ . Our function  $g(z)$  is holomorphic on  $\Omega \setminus \{z_0\}$  and continuous on  $\Omega$ , so the second extension of Cauchy-Goursat Theorem holds.

Let  $C$  be a curve in  $\Omega$ , by our last result

$$\begin{aligned} \oint_C g(z) dz &= \oint_C \frac{f(z) - f(0)}{z} dz = 0 \\ \Rightarrow \oint_C \frac{f(z)}{z} dz &= \oint_C \frac{f(0)}{z} dz = f(0) \oint_C \frac{dz}{z} = 2\pi i f(0) \end{aligned}$$

We showed before that

$$\oint_C \frac{dz}{z} = 2\pi i$$

Therefore,

$$f(0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z} dz$$

This means that the value of  $f$  at  $z = 0$  only depends on the value of  $\frac{f(z)}{z}$  over circles centered at 0. This result can be further generalized to Cauchy Integral Formula.

### Cauchy's Integral Formula. (C.I.F)

Let  $C$  be a smooth closed curve with interior domain  $\Omega$ . If  $f$  is holomorphic on  $\Omega$  and continuous on  $\Omega$ 's closure,

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w - z} dw, z \in \Omega$$

**Example.**

$$\oint_{|z|=6} \frac{e^{z^2}}{z - 4} = 2\pi i f(4) = 2\pi i e^{16}, f(z) = e^{z^2}$$

However,

$$\oint_{|z|=3} \frac{e^{z^2}}{z - 4} = 0 \text{ by Cauchy-Goursat Theorem}$$

### Extension of Cauchy's Integral Formula.

Cauchy's Integral Formula can be naturally expanded to  $f'(z)$ ,

$$f'(z) = \frac{1}{2\pi i} \oint_C \frac{f'(w)}{w - z} dw = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w - z)^2} dw, z \in \Omega$$

Therefore, the result can be generalized to

$$f^{(k)}(z) = \frac{k!}{2\pi i} \oint_C \frac{f(w)}{(w - z)^{k+1}} dw, z \in \Omega$$

provided we can justify

$$\frac{d}{dz} \oint_C dw = \oint_C \frac{d}{dz} dw$$

**Proof.** The proof of this depends on  $\frac{1}{w-z}$  being differentiable in  $w$  along any curve that does not pass through  $z$ .

Let  $f$  be holomorphic on  $\Omega$ , then for any circle  $C$  in  $\Omega$  about  $z_0$ , we have a formula for  $f^{(k)}(z_0)$  in terms of integrating over  $C$ . Hence  $f^{(k)}$  exists on  $\Omega$ ,  $f^{(k+1)}$  does

- $\Rightarrow f^{(k)}(z)$  is holomorphic on  $\Omega$  for all  $k$
- $\Rightarrow f$  is infinitely differentiable on  $\Omega$
- $\Rightarrow f = u + iv$  holomorphic
- $\Rightarrow$  All partials of  $u$  and  $v$  exist and are continuous,  $u_{xy} = u_{yx}$

**Example.** Given an integral in the form of C.I.F, we can take multiple approaches. Evaluate

$$\oint_{|z|=9} \frac{\sin(z)}{z^2} dz$$

The first approach is to apply C.I.F directly. Here,  $f(z) = \sin(z)$ ,  $f'(z) = \cos(z)$ .

$$\oint_{|z|=9} \frac{\sin(z)}{z^2} dz = 2\pi i f'(0) = 2\pi i \cos(0) = 2\pi i$$

The second approach is to apply C.I.F indirectly. The function

$$g(z) = \begin{cases} \frac{\sin(z)-\sin(0)}{z-0}, & z \neq 0 \\ 1, & z = 0 \end{cases}$$

is entire by the second extension of Cauchy-Goursat Theorem.

$$\oint_{|z|=9} \frac{\frac{\sin(z)}{z}}{z} dz = 2\pi i g(0) = 2\pi i$$

### Gauss's Mean Value Theorem.

Let  $f(z)$  be an analytic function in  $|z - z_0| < r$ . Then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta$$

**Proof.** Let  $\Omega = \{z ||z - z_0| < r\}$

Let

$$\gamma(t) = z_0 + re^{2\pi it}$$

Then,

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{f(z)}{z - z_0} dz \\ &= \frac{1}{2\pi i} \int_0^1 \frac{f(\gamma(t))}{\gamma(t) - z_0} \gamma'(t) dt \\ &= \int_0^1 f(z_0 + re^{2\pi it}) dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) dt \end{aligned}$$

## 14 Growth Conditions of Holomorphic Functions

### 14.1 Maximum/Minimum Modulus Principles

If  $f$  has an antiderivative on a domain  $\Omega$ , then  $f$  is holomorphic on  $\Omega$ . However, its converse is not true. If  $f$  is holomorphic on  $\Omega$ ,  $f$  does not necessarily have an antiderivative. Here is a counter-example.

**Example.** Let  $f(z) = \frac{1}{z}$ ,  $\Omega = \mathbb{C} \setminus \{0\}$ .  
 $f(z)$  does not have an antiderivative on  $\Omega$  because

$$\oint_{|z|=r} \frac{1}{z} dz = 2\pi i \neq 0$$

The problem above is with  $\Omega$ , which is not simply-connected.

**Proposition.** If  $\Omega$  is simply-connected and  $f$  is holomorphic on  $\Omega$ , then  $f$  has an antiderivative by F.T.C, a corollary of Cauchy-Goursat Theorem.

#### Maximum Modulus Principle.

Let  $f$  be holomorphic on a bounded domain  $\Omega$ . If  $|f|$  has a local maximum at  $z_0$ , or in other words,

$$|f(z_0)| > |f(z)|$$

for all  $z$  in the neighborhood of  $z_0$ , then  $f$  is constant near  $z_0$ .

In fact this also implies  $f$  is constant in  $\Omega$  with some additional theorems about uniqueness of holomorphic functions.

If  $f$  is continuous on  $\partial\Omega$ , then either  $f$  is constant or the absolute maximum of  $|f|$  occurs only on the boundary of  $\Omega$ ,  $\partial\Omega$ .

**Proof.** Let  $f$  be holomorphic on a domain  $\Omega$  and suppose the function  $|f|$  takes on a local maximum value at  $z_0$ . Near  $z_0$ ,

$$|z - z_0| < r \Rightarrow |f(z)| \leq |f(z_0)|$$

Then by Gauss's Mean Value Theorem,

$$\begin{aligned} |f(z_0)| &= \left| \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) dt \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{it})| dt \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)| dt = |f(z_0)| \\ \Rightarrow |f(z_0)| &= \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{it})| dt = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)| dt \end{aligned}$$

Therefore,

$$\int_0^{2\pi} |f(z_0)| - |f(z_0 + re^{it})| dt = 0$$

But

$$|f(z_0)| \geq |f(z_0 + re^{it})|$$

by our hypothesis and so our integrand is nonnegative. Therefore, the integral is zero.

$$\Rightarrow |f(z_0)| - |f(z_0 + re^{it})| = 0 \Rightarrow |f(z_0)| = |f(z_0 + re^{it})|$$

Hence, the Maximum Modulus Principle is proven.

**Claim.** If  $f$  and  $\bar{f}$  are holomorphic on  $\Omega$ , then  $f$  is constant on  $\Omega$ .

**Proof.** By Cauchy Riemann Equations,

$f$  is holomorphic  $\Rightarrow u_x = v_y$  and  $u_y = -v_x$ .

$\bar{f}$  is holomorphic  $\Rightarrow u_x = -v_y$  and  $u_y = v_x$ .

This means  $u_x = -u_x$ ,  $u_y = -u_y$ ,  $v_x = -v_x$ ,  $v_y = -v_y$  in  $\Omega$ .

Therefore,  $u$  and  $v$  are constant, and so is  $f(z)$ .

**Claim.** If  $|f| = C$  on  $\Omega$ , then  $f$  is constant on  $\Omega$ . If  $f(z_0) = 0$ , then  $f(z) = 0$  everywhere in  $\Omega$  because  $|f(z_0)| = 0$ .

**Proof.** Let  $f(z) = C$ , then  $g(z) = \frac{1}{\bar{f}(z)}$  is holomorphic, since it is a ratio of holomorphic functions.

Since  $|f|^2 = C^2$ , then

$$\overline{f(z)} = \frac{f(z)\overline{f(z)}}{f(z)} = \frac{|f(z)|^2}{f(z)} = C^2 g(z)$$

is holomorphic too. Hence,  $f$  is constant by the first claim.

Therefore,  $f$  is constant along all points of distance  $r$  from  $z_0$ .

### Minimum Modulus Principle.

Let  $f$  be analytic on a domain  $\Omega \subseteq \mathbb{C}$ , and assume that  $f$  never vanishes. Then if there is a point  $z_0 \in \Omega$  such that  $|f(z_0)| \leq |f(z)|$  for all  $z \in \Omega$ , then  $f$  is constant.

Let  $\Omega \subseteq \mathbb{C}$  be a bounded domain, let  $f$  be a continuous function on the closed set  $\bar{\Omega}$  that is analytic on  $\Omega$ , and assume that  $f$  never vanishes on  $\bar{\Omega}$ . Then the minimum value of  $|f|$  on  $\bar{\Omega}$  (which always exists) must occur on  $\partial\Omega$ .

## 14.2 Mapping and $\mathbb{C}$ -differentiability

Next we want to study functions that map the open disk to itself where  $f(0) = 0$ .

### Schwarz's Lemma.

If  $f$  is holomorphic,  $|f(z)| \leq 1$  on  $|z| < 1$ , and  $f(0) = 0$ , then

$$|f'(0)| \leq 1 \text{ and } |f(z)| \leq |z| \text{ on } |z| < 1$$

Moreover, if  $|f(z_0)| = |z_0|$  for some non-zero  $z_0$  or  $|f'(0)| = 1$ , then

$$f(z) = e^{i\theta}z, \theta \in [0, 2\pi)$$

**Proof.** Consider

$$g(z) = \begin{cases} \frac{f(z)-f(0)}{z-0} = \frac{f(z)}{z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases}$$

which we know is holomorphic on  $|z| < 1$ .

By Maximum Modulus for all  $|z| < r$ ,

$$|g(z)| \leq |g(z_r)|$$

for some fixed  $|z_r| = r$ . Therefore,

$$\frac{|f(z_r)|}{|z_r|} \leq \frac{1}{|z_r|} = \frac{1}{r}$$

as  $r \rightarrow 1$  we have  $|g(z)| \leq 1$  or  $|f(z)| \leq |z|$  and  $|f'(0)| \leq 1$  since  $g(0) = f'(0)$ .  
 If  $|g(z_0)| = 1$ , then  $g$  has a local maximum for  $|z| < 1$ ,

$$\begin{aligned} &\Rightarrow g \text{ is constant on } |z| < 1, |g(z)| = 1 \\ &\Rightarrow |f(z)| = |z| \\ &\Rightarrow f(z) = e^{i\theta} z \end{aligned}$$

**Unit Circle Mapping (Schwarz's Lemma).** Consider  $B_a(z) : \{z \mid |z| < 1\} \rightarrow \{z \mid |z| < 1\}$   
 It is a invertible, bilinear transform from the unit disk to itself, or an automorphism on the unit disk.

$$B_a(z) = \frac{z - a}{1 - \bar{a}z}, |a| < 1$$

The function has several interesting qualities.

$$\begin{aligned} |B_a(z)| &< 1 \text{ for } |z| < 1 \\ |B_a(z)| &= 1 \text{ for } |z| = 1 \\ B_a(a) &= 0 \\ B_a(0) &= -a \Rightarrow B_a^{-1}(z) = B_{-a}(z) \\ B'_a(0) &= 1 - |a|^2 \\ B'_a(a) &= \frac{1}{1 - |a|^2} \end{aligned}$$

To use Schwarz's Lemma if  $|f(z)| < 1$ , but  $f(z_0) = 0$ , for  $z \neq 0$ , we try to study

$$g(z) = f \circ B_{z_0}(z), \frac{f(z)}{B_{z_0}(z)} \text{ or } f(z)B_{z_0}(z).$$

If  $|f(z)| < 1$  on  $\Re(z) > 0$ , then consider

$$g(z) = \frac{z + 1}{z - 1}$$

which maps  $|z| < 1$  to  $\Re(z) > 0$ . Then we have  $f \circ g^{-1} : \{z \mid |z| < 1\} \rightarrow \{z \mid |z| < 1\}$ .

**Example.** Assume  $f$  is holomorphic and bounded by 1 for  $|z| < 1$ ,  $f(\frac{1}{2}) = 0$ . Estimate  $|f(\frac{3}{5})|$ . Let

$$g(z) = \begin{cases} \frac{f(z)}{B_{1/2}(z)}, & z \neq \frac{1}{2} \\ \frac{f'(\frac{1}{2})}{B'_{1/2}(z)} = \frac{3}{4}f'(\frac{1}{2}), & z = \frac{1}{2} \end{cases}$$

is holomorphic on  $|z| < 1$ . (We used L'Hospital's Rule to define  $g(\frac{1}{2})$ ).

As  $|z| \rightarrow 1$ ,  $|g| \leq 1$  since  $|f| \leq 1$  and  $|B_{\frac{1}{2}}(e^{i\theta})| = 1$ .

This means  $|f(z)| \leq |B_{\frac{1}{2}}(z)|$  for all  $|z| < 1$ .

Hence,

$$\left| f\left(\frac{3}{5}\right) \right| \leq \left| B_{\frac{1}{2}}\left(\frac{3}{5}\right) \right| = \left| \frac{\frac{3}{5} - \frac{1}{2}}{1 - \frac{1}{2} \cdot \frac{3}{5}} \right| = \frac{1}{7}$$

Note  $B_{\frac{1}{2}}(z)$  is a holomorphic function with  $B_{\frac{1}{2}}(\frac{1}{2}) = 0$  that hits the upper bound. Therefore, we cannot estimate better than that.

Now we want to use bounds on  $f$  to bound  $f^{(k)}$ .

**Cauchy's Inequality.**

If  $|f(z)| \leq M$  on  $|z - z_0| < R$ , then for  $r < R$  and  $C_r = \{z \mid |z - z_0| = r\}$ ,

$$\left| f^{(k)}(z_0) \right| \leq \frac{k!M}{R^k}$$

**Proof.**

$$\begin{aligned} \left| f^{(k)}(z_0) \right| &\leq \frac{k!}{2\pi} \oint_{C_r} \frac{|f(w)|}{|w - z_0|^{k+1}} dw \\ &\leq \frac{k!}{2\pi} \frac{M_R}{r^{k+1}} \oint_{C_r} dw \\ &= \frac{k!}{2\pi} \frac{M}{r^{k+1}} 2\pi \\ &= \frac{k!M}{r^k} \end{aligned}$$

Let  $r \rightarrow R^-$ , then

$$\left| f^{(k)}(z_0) \right| \leq \frac{k!M}{R^k}$$

**Liouville's Theorem.**

A bounded entire function is constant.

**Proof.** From Cauchy's Inequality,

$$|f'(z_0)| \leq \frac{M}{R}$$

then as  $R \rightarrow \infty$ ,  $f'(z_0) \rightarrow 0$ . Therefore,  $f$  is constant.

**Lemma.** Let  $p(z)$  be a non-constant polynomial, then

$$\lim_{z \rightarrow \infty} |p(z)| = \infty$$

**Proof.**

$$\begin{aligned} p(z) &= a_0 + a_1z + \cdots + a_nz^n \\ &= z^n(a_0z^{-n} + a_1z^{-n+1} + \cdots + a_n) \end{aligned}$$

Then

$$|p(z)| = |z^n| |a_0z^{-n} + a_1z^{-n+1} + \cdots + a_n| \rightarrow \infty \text{ as } n \rightarrow \infty$$

**Proof of the Fundamental Theorem of Algebra (F.T.A).**

The last two results have major consequences. Let  $p(z)$  be a non-constant polynomial and  $p(z) \neq 0$  for all  $z \in \mathbb{C}$ . Then

$$f(z) = \frac{1}{p(z)}$$

is entire and non-constant and thus unbounded. As  $z \rightarrow \infty$ ,  $p(z) \rightarrow \infty$  and hence

$$\lim_{z \rightarrow \infty} f(z) = 0$$



or  $|f(z)| < 1$  for  $|z| > R$ .

Since  $f$  is continuous on  $|z| \leq R$ , it has a bound  $|f(z)| \leq M$ , for  $|z| < R$ . Hence,  $f(z)$  is constant and thus,  $p(z)$  is constant. This is a contradiction. Therefore,  $f(z)$  must not be entire.

Hence, the F.T.A is proven. All non-constant polynomials have a root in  $\mathbb{C}$ .

Let  $p(z)$  be of degree  $n$ . For each root  $z_i$  of  $p(z)$ , we can compute

$$p(z) - p(z_i) = (z - z_i)q(z)$$

Repeat the process of  $q(z)$  and show that we have exactly  $n$  roots.

**Example.** Suppose  $f$  is entire,

$$|f'(z)| < |f(z)|$$

Then

$$\frac{f'(z)}{f(z)}$$

is entire and bounded. By Liouville's Theorem,  $f'(z) = kf(z)$  and  $|k| < 1$ .

Therefore,

$$f(z) = e^{kz}, |k| > 1$$

**Proposition.** Suppose  $f$  is entire and  $|f(z)| > 1$ , then  $\frac{1}{f(z)}$  is entire and bounded and thus constant.

**Example.** Non-constant entire functions are unbounded. Therefore,  $\sin(z)$  and  $\cos(z)$  are unbounded since  $\sin(0) \neq \sin(\frac{\pi}{2})$  and  $\cos(0) \neq \cos(\frac{\pi}{2})$

**Example.** Let  $f$  and  $g$  be entire,  $\Re(h(z)) \leq k\Re(g(z))$  for some fixed  $k$ . Consider

$$h(z) = f(z) - kg(z)$$

entire where  $\Re(h(z)) < 0$ . Then  $e^{h(z)}$  is bounded by 1.

$$|e^{h(z)}| \leq e^{\Re(h(z))} \leq 1$$

Therefore,  $h(z)$  is constant. This means  $f(z) - kg(z) = C$  for some constant  $C$ .

$$f(z) = kg(z) + C$$

## 15 Convergence of Infinite Series in $\mathbb{C}$

We want to study series in  $\mathbb{C}$ .

$$\sum_{k=0}^{\infty} a_k$$

We start with the most famous series.

### Geometric Series.

$$S_n = a + ar + \cdots + ar^n = \sum_{k=0}^n ar^k$$

Observe

$$\begin{aligned}(1-r)S_n &= S_n - rS_n \\ &= a + ar + \cdots + ar^n - (ar + \cdots + ar^{n+1}) \\ &= a(1 - r^{n+1})\end{aligned}$$

Thus

$$S_n = \sum_{k=0}^n ar^k = \frac{a(1 - r^{n+1})}{1 - r}$$

$r$  is the ratio of the geometric sum.

The geometric series is

$$S = \sum_{k=0}^{\infty} ar^k = \frac{a}{1 - r} \text{ provided } |r| < 1$$

If  $|r| \geq 1$ , then the sum diverges and does not converge.

### Absolute Convergence.

If  $\sum |a_n|$  converges, then  $\sum a_n$  converges absolutely.

**Example.** The following series converges absolutely.

$$\sum \frac{(-1)^n}{n^2}$$

### Conditional Convergence.

If  $\sum |a_n|$  diverges and  $\sum a_n$  converges, then the series converges conditionally.

**Example.** The following series converges conditionally.

$$\sum \frac{(-1)^n}{n}$$

## 15.1 Convergence Tests

### Alternating Series Test (A.S.T).

Suppose that we have a series  $\sum a_n$  and either  $a_n = (-1)^n b_n$  or  $a_n = (-1)^{n+1} b_n$  where  $b_n > 0$  for all  $n$ . Then if,

1.  $\lim_{n \rightarrow \infty} b_n = 0$  and,
2.  $\{b_n\}$  is a decreasing sequence

the series  $\sum a_n$  is convergent.

### Root Test.

Suppose that we have a series  $\sum a_n$ . Define

$$L = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}$$

Then,

1. if  $L < 1$ , the series is absolutely convergent (and hence convergent).
2. if  $L > 1$ , the series is divergent.
3. if  $L = 1$ , the series may be divergent, conditionally convergent, or absolutely convergent.

$$\lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1$$

### Ratio Test.

Suppose that we have a series  $\sum a_n$ . Define

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Then,

1. if  $L < 1$  the series is absolutely convergent (and hence convergent).
2. if  $L > 1$ , the series is divergent.
3. if  $L = 1$ , the series may be divergent, conditionally convergent, or absolutely convergent.

### Basic Comparison Test (B.S.T).

Suppose that we have two series  $\sum a_n$  and  $\sum b_n$  with  $a_n, b_n \geq 0$  for all  $n$  and  $a_n < b_n$  for all  $n$ . Then,

1. If  $\sum b_n$  is convergent, then so is  $\sum a_n$ .
2. If  $\sum a_n$  is divergent, then so is  $\sum b_n$ .

### Limit Comparison Test (L.S.T).

Suppose that we have two series  $\sum a_n$  and  $\sum b_n$  with  $a_n \geq 0$ ,  $b_n > 0$  for all  $n$ . Define,

$$c = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}$$

If  $c$  is positive and is finite, then either both series converge or both series diverge.

### Integral Test.

Suppose that  $f(x)$  is a continuous, positive and decreasing function on the interval  $[k, \infty)$  and that  $f(n) = a_n$  then,

1. If  $\int_k^\infty f(x) dx$  is convergent so is  $\sum_{n=k}^\infty a_n$ .
2. If  $\int_k^\infty f(x) dx$  is divergent so is  $\sum_{n=k}^\infty a_n$ .

**Example.** p-series.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if and only if } p > 1.$$

**Monotone Convergence Theorem.**

If  $(a_n)$  is a monotone sequence of real numbers, then  $(a_n)$  is convergent if and only if  $(a_n)$  is bounded. Let  $a_n > 0$ , then  $S_n = \sum_{k=1}^n a_k$  is increasing and if it is bounded,  $S_n \rightarrow S$ .

**Weierstrass M-Test.**

If  $\{f_n(z)\}$  be a sequence of functions with  $|f_n(z)| \leq M_n, z \in \Omega$  and  $\sum M_n < \infty$ , then  $\sum f_n(z)$  converges uniformly on  $\Omega$ .

**Proposition.**

In general,  $f_n \rightarrow f$  uniformly does imply  $\int f_n \rightarrow \int f$  uniformly, but  $f'_n \not\rightarrow f'$ , even pointwise. However, for holomorphic functions  $f_n \rightarrow f$  uniformly on compacta implies  $f'_n \rightarrow f'$  uniformly.

**Example.**

Let

$$f_n(z) = \frac{\sin(nz)}{\sqrt{n}} \rightarrow 0 \text{ uniformly}$$

and

$$f'_n(z) = \sqrt{n} \cos(nz), f'_n(0) \rightarrow \infty \neq f'(0)$$

**Cauchy Condensation Test.**

Let  $\{a_n\}$  be a series of positive terms with  $a_{n+1} \leq a_n$ . Then  $\sum_{n=1}^{\infty} a_n$  converges if and only if

$$\sum_{k=0}^{\infty} 2^k a_{2^k}$$

converges.

**Proof.**

$$\begin{aligned} & \sum 2^n f(2^n) < \infty \\ \iff & \int_1^{\infty} 2^x f(2^x) dx \\ \iff & \log(2) \int_1^{\infty} 2^x f(2^x) dx < \infty \\ \iff & \int_2^{\infty} f(u) du < \infty \\ \iff & \sum f(n) < \infty \end{aligned}$$

**Example.** Use Cauchy Condensation Test to prove  $f(n) = \frac{1}{n}$  diverges.

$$\begin{aligned} f(n) = \frac{1}{n} & \Rightarrow \frac{2^n}{2^n} = 1 \\ \sum_{n=1}^{\infty} 1 & = \infty \text{ if and only if } \sum_{n=1}^{\infty} \frac{1}{n} = \infty \end{aligned}$$

Despite  $\frac{1}{n} \rightarrow 0$ , the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

**Dirichlet Test.**

Let  $a_1, a_2, \dots$  be a monotonically decreasing infinite real sequence.

Let  $\sum b_n$  be an infinite complex series such that its partial sums are bounded. i.e.

$$\left| \sum_{n=1}^N b_n \right| \leq M$$

for all  $N \in \mathbb{N}$ .

Then

$$\sum_{n=1}^{\infty} a_n b_n$$

converges.

**Example.** Determine whether the following series converge.

$$\sum_{n=1}^{\infty} \frac{(e^{2\pi i/3})}{\sqrt{n}}$$

When viewed as vectors,  $b_1 + b_2 + b_3 = 0$  and  $\sum_{n=1}^{3k} b_n = 0$

We have  $|\sum_{n=1}^{\infty} b_n|$  is bounded but does not converge, just like  $|\sum_{n=1}^{\infty} (-1)^n|$ .  
Therefore,  $\frac{b_n}{\sqrt{n}}$  converges.

Using the same technique, one can also show  $\sum_{n=1}^{\infty} \frac{\sin(n)}{n}$  converges.

## 16 Power Series in $\mathbb{C}$

We want to express holomorphic functions in  $\mathbb{C}$  by series in  $z$  instead of integrals and the Cauchy Riemann Equations. Cauchy-Goursat Theorem will provide the connection between the two forms,

$$f(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Let

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

be a power series centered at  $z_0$ .

The coefficients are unique:

$$a = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w - z_0)^{n+1}} dw$$

where  $C$  is a path about  $z_0$ .

### Derivative and Ant-derivative of a Power Series.

$f'(z)$  and  $F(z)$  are given by

$$f'(z) = \sum_{n=0}^{\infty} a_n \cdot n (z - z_0)^{n-1}$$

$$F(z) = \int f(z) dz = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1} + C_0$$

and are holomorphic and converge for  $|z - z_0| < R$ .  $R$  denotes the radius of convergence.

We have no clue about what happens with  $|z - z_0| = R$ , then we have to check for each  $f(z)$  and point.

If  $R = \infty$ , then  $f(z)$  is entire.

If  $R = 0$ , then  $f$  is not holomorphic on disks centered at  $z_0$ .

Before we move on to more specifics about the radius of convergence, a brief review of terminology is in order.

A reminder about upper and lower limits. Let

$$\text{Largest lower bound: } \lim_{n \rightarrow \infty} \inf |a_n| = L$$

$$\text{Smallest upper bound: } \lim_{n \rightarrow \infty} \sup |a_n| = U$$

then

$$L = \sup_{n \geq 0} (\inf_{k \geq n} |a_k|) = \lim_{n \rightarrow \infty} (\inf \{|a_n|, |a_{n+1}|, \dots\})$$

$$U = \inf_{n \geq 0} (\sup_{k \geq n} |a_k|) = \lim_{n \rightarrow \infty} (\sup \{|a_n|, |a_{n+1}|, \dots\})$$

and  $L \leq U$ . For  $n$  large and  $\varepsilon_0, \varepsilon_1 > 0$ ,

$$L - \varepsilon_0 \leq |a_n| \leq U + \varepsilon_1$$

If we show that  $U \leq L$ , then  $|a_n|$  converges.

$$\lim_{n \rightarrow \infty} |a_n| = L = U$$

## 16.1 Radius of Convergence

**Hadamard's Formula for  $R$ .**

$$\lim_{n \rightarrow \infty} \sup |a_n|^{\frac{1}{n}} = \frac{1}{R}$$

Just like with the geometric series,

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

converges uniformly for  $|z - z_0| \leq r < R$  and pointwise  $|z - z_0| < R$ .

**Proof.** In this proof about convergence of power series, we could assume

$$\lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} = \frac{1}{R}$$

Assume  $z_0 = 0$  and

$$L = \frac{1}{R} = \lim_{n \rightarrow \infty} \sup |a_n|^{\frac{1}{n}}$$

$L \neq 0$  or  $\infty$ . Let  $|z| \leq r < R$ , and from lim sup, we have

$$|a_n|^{\frac{1}{n}} \leq L + \varepsilon$$

for  $n$  large and  $\varepsilon > 0$ . Therefore,

$$|a_n| \leq (L + \varepsilon)^n$$

and

$$|a_n| |z|^n \leq ((L + \varepsilon)|z|)^n \leq ((L + \varepsilon)r)^n$$

Since  $(L + \varepsilon)r = \frac{r}{R} + \varepsilon r$  and  $\frac{r}{R} < 1$ , we can pick  $\varepsilon$  small enough such that the geometric series

$$\sum_{n=0}^{\infty} ((L + \varepsilon)r)^n$$

converges for  $|(L + \varepsilon)r| < 1$ . Therefore, the partial sums

$$|S_n(z)| = \sum_{k=0}^n |a_k z^k|$$

are increasing sequence with respect to  $n$ ,  $z$ -fixed, and bounded by a convergent sum. By the monotone convergence theorem, the series converges for  $|z| < R$ . By the Weierstrass M-Test, the convergence is uniform on  $|z| \leq r < R$  but not on  $|z| < R$ , since we need a fixed radius.

**Example.** If

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!}$$

Then  $a_n = \frac{1}{n!}$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} = \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0 = \frac{1}{R}, R = \infty$$

Therefore,

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!} = e^z - 1$$

converges for all  $z \in \mathbb{C}$ .

**Example.** If

$$f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{3^{2n}}$$

$$|a_n|^{\frac{1}{n}} = \frac{|z^2|}{|9|}$$

If  $\frac{|z|^2}{|9|} < 1$ , the sum converges absolutely on  $|z| < 3$ . Here,

$$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{3^{2n}} = \frac{1}{1 - (-\frac{z}{3})^2} = \frac{1}{1 + \frac{z^2}{9}} = \frac{9}{9 + z^2} = \frac{d}{dz} 3 \arctan\left(\frac{z}{3}\right)$$

Most of the power series with reals still work here in complex.

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, z \in \mathbb{C}$$

However,

$$\sqrt{z} \neq \sum_{n=0}^{\infty} a_n z^n$$

because  $\sqrt{z}$  is not differentiable at  $z = 0$ .

## 16.2 Derivative of Power Series

Now let us prove the claim we made before.

**Claim.** The derivative of a power series is given as below.

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}, |z| < R$$

**Proof.** Define

$$g(z) = \sum_{n=0}^{\infty} a_n \cdot n z^{n-1}$$

and if

$$L = \frac{1}{R} = \lim_{n \rightarrow \infty} \sup |a_n|^{\frac{1}{n}}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \sup |n a_n|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \sup |n|^{\frac{1}{n}} |a_n|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \sup |n|^{\frac{1}{n}} \lim_{n \rightarrow \infty} \sup |a_n|^{\frac{1}{n}}$$

because  $|n^k|^{\frac{1}{n}} \rightarrow 1$  as  $n \rightarrow \infty$ ,  $k \in \mathbb{R}$ ,  $k \neq 0$ . Hence  $g(z)$  and  $f(z)$  have the same radius of convergence. Now we must show  $g(z)$  is the derivative of  $f(z)$ ,

$$f(z) = \sum_{n=0}^N a_n z^n + \sum_{n=N+1}^{\infty} a_n z^n = S_N(z) + E_N(z)$$

where  $|z| < r < R$ ,

$$\frac{f(z+h) - f(z)}{h} - g(z) = \frac{S_N(z+h) - S_N(z)}{h} - S'_N(z) + (S'_N(z) - g(z)) + \frac{E_N(z+h) - E_N(z)}{h}$$



We have three terms to study, and we can set each of them to be  $< \frac{\varepsilon}{3}$ .

$$\frac{S_N(z+h) - S_N(z)}{h} - S'_N(z) \text{ derivative of a polynomial,}$$

$$S'_N(z) - g(z), S'_N(z) \rightarrow g(z) \text{ for } N > M \text{ for some } M,$$

$$\frac{E_N(z+h) - E_N(z)}{h}, \text{ this is more technical and we will investigate more}$$

$$\left| \frac{E_N(z+h) - E_N(z)}{h} \right| \leq \sum_{n=N+1}^{\infty} |a_n| \left| \frac{(z+h)^n - z^n}{h} \right|$$

This means we need a bound on  $(z+h)^n - z^n$ . Remember

$$1 + x + x^2 + \dots + x^{n-1} = \frac{1 - x^n}{1 - x}$$

$$\Rightarrow (1-x)(1+x+x^2+\dots+x^{n-1}) = 1 - x^n$$

If we let  $x = \frac{b}{a}$ , then we have a more generalized formula,

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \dots + ab^{n-2} + b^{n-1})$$

Apply the formula above to  $a = z+h$  and  $b = z$ ,

$$|(z+h)^n - z^n| = |h| |(z+h)^{n-1} + (z+h)^{n-2}z + \dots + z^{n-1}|$$

We have  $n$  terms of the form

$$(z+h)^{n-1-k} \cdot z^k$$

and if  $h$  is small,  $|z+h| \leq r < R$ , then

$$|(z+h)^{n-1-k} \cdot z^k| \leq r^{n-1} \leq R^{n-1}$$

Therefore,

$$|a_n| \frac{|(z+h)^n - z^n|}{|h|} \leq |a_n| \frac{|h| n R^{n-1}}{|h|} = |a_n| n R^{n-1}$$

Hence, we have a bound on

$$\left| \frac{E_N(z+h) - E_N(z)}{h} \right| \leq \sum_{n=N+1}^{\infty} |a_n| n R^{n-1}$$

which we know to be the tail of a convergent sum,  $g(z)$ . Thus, it can be picked such that it is less than  $\frac{\varepsilon}{3}$  because the tail tends to 0 as  $n \rightarrow \infty$ . Hence,

$$\left| \frac{f(z+h) - f(z)}{h} - g(z) \right| < \varepsilon$$

or  $f'(z) = g(z)$ .

### Proposition.

Power series functions are holomorphic in  $|z - z_0| < R$  and are naturally  $\infty$ -differentiable, i.e.  $\mathbb{C}^\infty$ .

## 17 Series Expansion of Holomorphic Functions

Now we want to show if  $f(z)$  is holomorphic on  $\Omega$ , it has a power series expansion on any disk in  $\Omega$ .

### Connection of the Geometric Series to Cauchy's Integral Formula.

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w-z} dw$$

$$\frac{1}{w-z} = \frac{1}{w} \cdot \frac{1}{1-\frac{z}{w}} = \frac{1}{w} \left( 1 + \frac{z}{w} + \left(\frac{z}{w}\right)^2 + \cdots \right), r = \frac{z}{w}$$

If  $|\frac{z}{w}| < 1$ ,  $|z| < |w|$ , the sum converges uniformly,

$$\begin{aligned} \frac{1}{w-z} &= \frac{1}{w} \sum_{n=0}^{\infty} \left(\frac{z}{w}\right)^n \\ \Rightarrow f(z) &= \frac{1}{2\pi i} \oint_C \frac{f(w)}{w-z} dw = \frac{1}{2\pi i} \oint_C \frac{1}{w} \sum_{n=0}^{\infty} f(w) \left(\frac{z}{w}\right)^n dw \\ &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \oint_C f(w) \frac{1}{w} \left(\frac{z}{w}\right)^n dw \text{ because of uniform convergence} \\ &= \sum_{n=0}^{\infty} \frac{1}{2\pi i} \oint_C \frac{f(w)}{w^{n+1}} dw z^n \\ &= \sum_{n=0}^{\infty} a_n z^n \end{aligned}$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w^{n+1}} dw = \frac{f^{(n)}(0)}{n!}$$

### Theorem.

If  $f$  is holomorphic on  $\Omega$ , then for every disk in  $\Omega$  centered at  $z_0$ ,  $f$  has a unique power series expansion at  $z_0$  that converges to  $f(z)$ .

If the disk is closed and in  $\Omega$ , then the power series converges uniformly to  $f(z)$ .

### Isolated and Non-isolated Root.

Let  $f(z_0) = 0$ . If  $f$  does not have a root on  $0 < |z - z_0| < \delta$  for some  $\delta > 0$ , then the zero is *isolated*. This means  $f$  does not have a root on punctured disk centered at  $z_0$ .

If  $f$  has a root on  $0 < |z - z_0| < \delta$  for all  $\delta > 0$ , then the zero is *non-isolated*.  $f$  has a root on every punctured disk centered at  $z_0$ . We can say that  $f$  has a limit point of zeros at  $z_0$ .

**Example.**  $f(z) = z(z-3)$  has isolated roots at  $z = 0, 3$ .

### Theorem.

Suppose  $f(z)$  is holomorphic on  $\Omega$  and there exists a sequence  $\{z_n\}$  in  $\Omega$  where  $z_n \rightarrow z_0 \in \Omega$  and  $f(z_n) = 0$  for all  $n$ . Then  $f(z) = 0$  on  $\Omega$ .

**Proof.** Let

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Suppose  $f$  is not the zero function in the disk of convergence. That means  $a_n \neq 0$  for some  $n \in \mathbb{N}$ . Let  $m \in \mathbb{N}$  be the smallest number such that  $a_m \neq 0$ . In other words,  $f(z)$  has a root of order  $m$  at  $z_0$ . Then,

$$f(z) = a_m(z-z_0)^m + \sum_{n=m+1}^{\infty} a_n(z-z_0)^n = a_m(z-z_0)^m \left( 1 + \sum_{n=m+1}^{\infty} \frac{a_n}{a_m} \frac{(z-z_0)^n}{(z-z_0)^m} \right) = a_m(z-z_0)^m(1+g(z))$$

by letting

$$g(z) = \sum_{n=m+1}^{\infty} \frac{a_n}{a_m} \frac{(z-z_0)^n}{(z-z_0)^m}$$

so  $g(z) \rightarrow 0$  as  $z \rightarrow z_0$ . The function  $g(z)$  exists and is holomorphic and continuous since  $m < n$  in the sum and  $a_m \neq 0$ .

Consider our sequence  $z_k$  that converges to  $z_0$ , then

$$a_m(z_k - z_0)^m \neq 0$$

since  $z_k \neq z_0$  is clearly not a root of  $(z - z_0)^k$  and

$$1 + g(z_k) \rightarrow 1 + 0 = 1$$

because  $g(z)$  is continuous and is tending to zero. In particular, we can find a neighborhood of  $z_0$  containing  $z_k$  for  $k > N$  such that

$$|g(z_k)| < \frac{1}{2} \Rightarrow |1 + g(z_k)| > \frac{1}{2} \Rightarrow 1 + g(z_k) \neq 0$$

Therefore,  $f(z_k) \neq 0$  and  $f(z_k) = 0$  for all  $k > N$ . This is a contradiction.

Hence, it is proven that  $a_n = 0$  for all  $n \in \mathbb{N}$ .

### Uniqueness Theorem.

Let  $f$  and  $g$  be holomorphic on  $\Omega$ . The following are equivalent:

1.  $f(z) = g(z)$  for all  $z \in \Omega$ .
2.  $f(z) = g(z)$  for all  $z \in U$ , where  $U$  is a non-empty, open subset of  $\Omega$ .
3. For some  $a \in \Omega$ ,  $f^n(a) = g^n(a)$  for all  $n \in \mathbb{N}$ .
4.  $f(z_n) = g(z_n)$  for some  $z_n \rightarrow z_0$  in  $\Omega$ , with  $z_n$  distinct.

So  $f(z)$  is defined on its domain by its behavior near  $z_0$ .

### Example.

$$f(z) = \begin{cases} z & , |z| < 1 \\ \frac{z}{|z|} & , |z| \geq 1 \end{cases}$$

$$f(z) = \begin{cases} e^{-\left(\frac{1}{|z|-1}\right)^2} & , |z| > 1 \\ 0 & , |z| \leq 1 \end{cases}$$

The functions above do not have convergent power series at  $z = 1$ .

**Note.**  $\infty$ -many 0's does not imply  $f(z) = 0$  for all  $z$ . We provide a counter-example below.

$$f(z) = \sin(z) = 0 \text{ if and only if } z = k\pi, k \in \mathbb{Z}$$

**Note.** A convergent sequence of 0's does mean  $f = 0$  provided the limit point is in  $\Omega$ ,  $f$ 's domain. Limit points outside of  $\Omega$  do not imply  $f(z) = 0$ . We provide a counter-example below.

$$\begin{aligned} f(z) &= \sin\left(\frac{1}{z}\right) \\ z_n &= \frac{1}{n\pi} \\ z_n &\rightarrow 0 \end{aligned}$$

But  $\sin\left(\frac{1}{0}\right)$  does not exist and  $\sin\left(\frac{1}{z}\right) \neq 0$  for all  $z$ .

## 18 Open Mapping Theorem and Reflection Principle

### 18.1 Open Mapping Theorem

**Question.** Let  $f$  be continuous on  $\mathbb{R}$ , does  $f$  map open sets to open sets?

**Answer.** No. See the following counter-example.

Let  $f(x) = |x|$ , then  $f(-3) = f(3)$  and  $f((-3, 3)) = [0, 3)$

The continuous function does not map open sets to open sets.

**Follow-Up (Alternative Definition of Continuity).**

The pre-image of open sets under a continuous function  $f$  is open.

Let  $f$  be non-constant. Then Cauchy-Riemann Equations imply  $|f(z)|$  is not constant. Uniqueness Theorem implies  $f(\Omega) \neq 0$  on an open subset of  $\Omega$ . Therefore, non-constant holomorphic functions cannot map open sets to points or arcs.

#### Open Mapping Theorem.

Let  $f(z)$  be non-constant and holomorphic on  $\Omega$ . Let  $U$  be open in  $\Omega$  then,

$$f(U) = \{w | f(z) = w \text{ for some } z \in U\}$$

is open.

**Proof.** Let  $a \in \Omega$ , then a neighborhood of  $a$  under  $f$  will contain a neighborhood of  $f(a)$ . Assume  $f(a) = 0$ . If this is not the case, we can study  $g(z) = f(z) - f(a)$ . Since  $f$  is not constant, by the Uniqueness Theorem there is  $r > 0$  such that  $f(z) \neq 0$  for all  $z$  where  $|z - a| = r$ . Consider

$$0 < 2\varepsilon = \min_{|z-a|=r} |f(z)|$$

We will show the image of the disk  $B_r(a) = U$  contains  $B_\varepsilon(f(a)) = B_\varepsilon(0)$  in the  $w$ -plane. Or in other words, interior points are mapped to interior points. Let  $w \in B_\varepsilon(0)$  and look at  $f(z) - w$ . If  $|z - a| = r$ ,

$$|f(z) - w| \geq |f(z)| - |w| \geq \varepsilon$$

because  $|f(z)| \geq 2\varepsilon$  and  $|w| < \varepsilon$ . Now at  $z = a$ , we have  $|f(a) - w| = |-w| = \varepsilon$ .

On the boundary of  $U$ , we have  $|f(z) - w| \geq \varepsilon$ .

But at the center of  $U$ , we have  $|f(a) - w| < \varepsilon$ . To sum up what we have so far:

1.  $f(z) - w$  is not constant.
2.  $|f(z) - w| < \varepsilon$  on  $w$ .
3.  $|f(z) - w| \geq \varepsilon$  on  $\partial w$ .
4.  $|f(z) - w|$  smaller on  $U$  than on  $\partial U$ .

Therefore,  $|f(z) - w|$  has a global minimum in  $U$ .

By Minimum Modulus Principle,  $f(z) - w = 0$  or  $f(z) = w$  for some  $z \in U$ . Hence,  $f(U)$  contains  $B_\varepsilon(0)$ .

This concludes the proof of Open Mapping Theorem.

We can also prove the Open Mapping Theorem without using Minimum Modulus Principle, and instead prove it by counting zeros.

#### Corollary.

Non-constant holomorphic functions map domains to domains.

#### Proof.

(Like the Intermediate Value Theorem) The continuous image of a connected set is connected.

**Corollary.**

Non-constant, invertible holomorphic functions map boundary points to boundary points.

**Proof.**

If  $f(z)$  is a boundary point, then any neighborhood of  $f(z)$  intersects points not in  $f(\Omega)$ . Thus  $f(z)$  has no open neighborhood in  $f(\Omega)$ . Therefore,  $z$  is not an interior in  $\Omega$ .

**18.2 Reflection Principle.****Symmetry Principle.**

Let  $f$  be continuous on an open set  $\Omega$  and holomorphic on  $\Omega$  except on a line segment  $L$ . Then  $f$  is holomorphic on  $\Omega$ .

**Proof.** Assume  $L$  is on  $\mathbb{R}$ -axis, otherwise apply affine transform to move  $L$ ,  $f(z) = f(az + b)$ .

Let  $z \in L$  and consider a ball about  $z$ . Let  $C$  be a closed loop in the ball. If  $C$  does not cross  $L$ , then

$$\oint_C f(z) dz = 0$$

by Cauchy's Theorem. If  $C$  does pass through  $L$ , we can break up the curve as follows:

$$\oint_C f(z) dz = \lim_{\varepsilon \rightarrow 0} \oint_{U_\varepsilon} f(z) dz + \oint_{L_\varepsilon} f(z) dz = 0$$

because  $f$  is continuous. Then by Morera's Theorem,  $f$  is holomorphic in the ball.

**Schwarz's Reflection Principle.**

Let  $\Omega$  be in the upper half plane and  $\Omega$ 's boundary contains a line segment  $L$  that intersects the  $\mathbb{R}$ -axis. If  $f$  is  $\mathbb{R}$  for  $z \in L$  and is holomorphic on  $\Omega$ , then

$$f(z) = \begin{cases} f(z), & z \in \Omega \cup L \\ \overline{f(\bar{z})}, & \bar{z} \in \Omega \end{cases}$$

is holomorphic on  $\Omega$  and  $\Omega^* = \{z | \bar{z} \in \Omega\}$ .

**Note.** To do this extension in a continuous manner is trivial. We are extending  $f$  to a holomorphic function below  $\mathbb{R}$ -axis by reflection. The set  $\Omega \cup L \cup \Omega^*$  is symmetric about the  $\mathbb{R}$ -axis.

**Proof.** Let  $z_0, z_1 \in \Omega^*$

$$\begin{aligned} f(\bar{z}) &= \sum_{n=0}^{\infty} a_n (\bar{z} - \bar{z}_0)^n \\ \overline{f(\bar{z})} &= \sum_{n=0}^{\infty} \overline{a_n} (z - z_0)^n \end{aligned}$$

Therefore,  $\overline{f(\bar{z})}$  is holomorphic on  $\Omega^*$  and we assumed

$$g(x) = \overline{g(x)}$$

for  $x \in L$ . Thus,  $g$  extends continuously up to  $L$ . By Symmetry Principle  $g$  is holomorphic.

**Corollary.** If  $f$  is holomorphic in a region symmetric with respect to  $\mathbb{R}$ -axis and  $f(z)$  is  $\mathbb{R}$  for  $z \in \mathbb{R}$ , then

$$f(z) = \overline{f(\bar{z})}$$

by Uniqueness Theorem.

**Example.** By the corollary above,

$$\cos(z) = \overline{\cos(\bar{z})}$$

for all  $z$ .

**Example.** Let  $f$  be bounded and holomorphic on  $\Im(z) \geq 0$  and  $f(z) \in \mathbb{R}$  for  $z \in \mathbb{R}$ . Then

$$g(z) = \begin{cases} f(z), & \Im(z) \geq 0 \\ \overline{f(\bar{z})}, & \Im(z) < 0 \end{cases}$$

is entire and bounded. Then  $g(z)$  is constant by Liouville's Theorem.

**Note.** Since we can map  $|z| < 1$  to  $\Im(z) > 0$ , the Reflection Principle can be extended to boundaries of circles and curves.

**Example.** If  $f$  is holomorphic,  $|f(z)| < 1$  for  $|z| < 1$ , continuous to the boundary and maps the boundary to the boundary, then

$$g(z) = \begin{cases} f(z), & |z| \leq 1 \\ \frac{1}{\overline{f(\frac{1}{\bar{z}})}}, & |z| > 1 \end{cases}$$

is holomorphic except at the reflected zeros in  $|z| < 1$ . Reflection is about the boundary of  $|z| = 1$ .

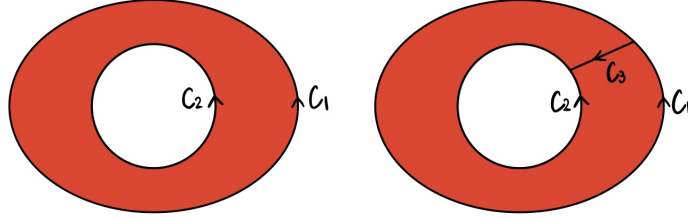
## 19 Laurent Series

### First Extension of Cauchy's Integral Formula.

Let  $\Omega$  be the domain between two simple closed curves  $C_1$  and  $C_2$ , each oriented counter-clockwise. If  $f$  is holomorphic on  $\Omega$ , then

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{w-z} dw, z \in \Omega$$

**Proof.**



Let  $z \in \Omega$  and  $C : C_1 \cup C_3 \cup -C_2 \cup -C_3$  be a closed curve about  $z$  in  $\Omega$ .

Now Cauchy's Integral Formula applies

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w-z} dw = \frac{1}{2\pi i} \oint_{C_1 \cup -C_2} \frac{f(w)}{w-z} dw = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{w-z} dw$$

### Laurent's Theorem.

For  $z \in A$  and  $f$  holomorphic in

$$A = \{z | r_1 < |z - z_0| < r_2\}$$

we have

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w - z_0)^{n+1}} dw$$

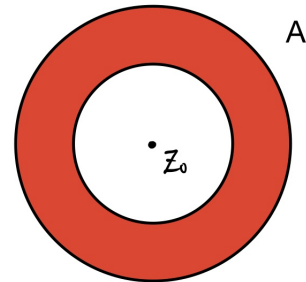
where  $C$  is any closed simple path in  $A$  about  $z_0$ .

We may also use different notation for the coefficients with negative indices,  $b_n = -a_n$ .

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w - z_0)^{n+1}} dw$$

$$b_n = \frac{1}{2\pi i} \oint_C f(w) (w - z_0)^{n-1} dw$$





**Proof.** Let  $A = \{z | r_1 < |z - z_0| < r_2\}$  be an annulus centered at  $z_0$ . Choose  $C_1$  and  $C_2$ , counter-clockwise circles in  $A$  such that  $z$  is in the region bounded by between them. We will assume  $z_0 = 0$ . By First Extension of Cauchy's Integral Formula,

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w - z} dw - \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{w - z} dw$$

As shown earlier with Taylor Series:

$$\begin{aligned} \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w - z} dw &= \sum_{n=0}^{\infty} a_n z^n \\ a_n &= \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{w^{n+1}} dw \end{aligned}$$

provided  $|z| < r_2$ . Then we transform the integral over  $C_2$  as follows.

$$\frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{w - z} dw = \frac{-1}{2\pi i} \oint_{C_2} \frac{f(w)}{z} \cdot \frac{1}{1 - \frac{w}{z}} dw = \frac{-1}{2\pi i} \oint_{C_2} \frac{f(w)}{z} \sum_{n=0}^{\infty} \left(\frac{w}{z}\right)^n dw = \frac{-1}{2\pi i} \oint_{C_2} \sum_{n=0}^{\infty} f(w) \frac{w^n}{z^{n+1}} dw$$

Because the function  $f(w)w^n$  is holomorphic in  $A$  for all  $n \in \mathbb{N}$ , we can change the curve we are integrating over to  $C_1$  or any other curve in  $A$ . Hence

$$\frac{-1}{2\pi i} \oint_{C_2} \sum_{n=0}^{\infty} f(w) \frac{w^n}{z^{n+1}} dw = \frac{-1}{2\pi i} \oint_{C_1} \sum_{n=0}^{\infty} f(w) \frac{w^n}{z^{n+1}} dw$$

And since the Geometric Series converges uniformly for  $\left|\frac{w}{z}\right| \leq r < 1$ , we can swap the order of integration and summation.

$$\frac{-1}{2\pi i} \oint_{C_1} \sum_{n=0}^{\infty} f(w) \frac{w^n}{z^{n+1}} dw = \frac{-1}{2\pi i} \sum_{n=0}^{\infty} \oint_{C_1} f(w) w^n dw \frac{1}{z^{n+1}}$$

We rewrite  $\frac{1}{z^{n+1}}$  with  $z^{-n-1}$  and reindex our sum to start at  $n = 1$ . Therefore,

$$\begin{aligned} \frac{1}{2\pi i} \oint_{-C_2} \frac{f(w)}{w - z} dw &= \sum_{n=1}^{\infty} b_n z^{-n}, |z| > r \\ b_n &= \frac{1}{2\pi i} \oint_{C_1} f(w) w^{n-1} dw \end{aligned}$$

**Techniques.** We have two tricks to transform Power Series into forms we can manipulate.

1. Treat  $r = \frac{z}{w}$ . If  $\left|\frac{z}{w}\right| < 1$ ,  $|z| < |w|$ , the sum converges.

$$\frac{1}{w - z} = \frac{1}{w} \cdot \frac{1}{1 - \frac{z}{w}} = \frac{1}{w} \left(1 + \frac{z}{w} + \left(\frac{z}{w}\right)^2 + \dots\right)$$

2. Treat  $r = \frac{w}{z}$ . If  $\left|\frac{w}{z}\right| < 1$ ,  $|w| < |z|$ , the sum converges.

$$\frac{1}{w - z} = \frac{-1}{z} \cdot \frac{1}{1 - \frac{w}{z}} = \frac{-1}{z} \left(1 + \frac{w}{z} + \left(\frac{w}{z}\right)^2 + \dots\right)$$

Laurent Series can be differentiated and integrated term by term to get  $f'(z)$  or  $F(z)$ , since we have uniform convergence on closed annuli in  $A$ . We can understand Laurent Series as a "double-ended" Taylor Series.

$$\begin{aligned} f(z) &= \cdots + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1z + a_2z^2 + \cdots \\ &= \underbrace{\cdots + \frac{b_2}{z^2} + \frac{b_1}{z}}_{\text{Singular/Principal Part}} + \underbrace{a_0 + a_1z + a_2z^2 + \cdots}_{\text{Regular/Analytic Part}} \end{aligned}$$

The structure of the principal part will tell us about  $f$ 's singularities at  $z_0$ . We do not often find  $a_n$ 's from the formula, but follow the three rules.

1. Break up  $f(z)$  by Prime Factor Decomposition.
2. Expand each term in  $|z - z_0| < r$  or  $|z - z_0| > R$ .  
The radius is often the distance to the next closest singularity of  $f(z)$ .
3. Collect the needed terms from the step above for each element of the Prime Factor Decomposition.

**Example.** Expand

$$\frac{z}{z^2 + 1}, z_0 = i$$

There is a root of  $z^2 + 1$  at  $i$ .

$$\frac{z}{z^2 + 1} = \frac{1}{2} \cdot \frac{1}{z - i} + \frac{1}{2} \cdot \frac{1}{z + i}$$

The second term has a Taylor expansion at  $z_0 = i$ .

$$\frac{1}{2} \cdot \frac{1}{z + i} = \frac{1}{2} \cdot \frac{1}{2i} \cdot \frac{1}{1 - \left(-\frac{z-i}{2i}\right)} = \frac{1}{4i} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-i}{2i}\right)^n$$

Therefore, we have the Laurent series of  $\frac{z}{z^2+1}$  around  $z_0 = i$  for  $\left|\frac{z-i}{2i}\right| < 1$ .

$$\Rightarrow \frac{z}{z^2 + 1} = \underbrace{\frac{1}{2} \cdot \frac{1}{z - i}}_{\text{Principal Part}} + \underbrace{\frac{1}{4i} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-i}{2i}\right)^n}_{\text{Regular Part}}$$

**Example.** Expand

$$f(z) = \frac{1}{z+1} + \frac{1}{z+3}, z_0 = 0$$

In this case, we have three annuli. And we have two expressions for each simple partial fraction.

$$\text{For } |z| < 1 : \frac{1}{z+1} = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty} (-1)^n z^n$$

$$\text{For } |z| > 1 : \frac{1}{z+1} = \frac{1}{z} \cdot \frac{1}{1 - \left(\frac{-1}{z}\right)} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{(-1)^n}{z^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}}$$

$$\text{For } |z| < 3 : \frac{1}{z+3} = \frac{1}{3} \cdot \frac{1}{1 - \left(\frac{-z}{3}\right)} = \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{3}\right)^n$$

$$\text{For } |z| > 3 : \frac{1}{z+3} = \frac{1}{z} \cdot \frac{1}{1 - \left(\frac{-3}{z}\right)} = \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n \left(\frac{3}{z}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{z^{n+1}}$$

Therefore,

$$\text{If } |z| < 1 : f(z) = \sum_{n=0}^{\infty} (-1)^n z^n + \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{3}\right)^n$$

It only has regular terms since  $f$  is holomorphic on  $|z| < 1$ .

$$\text{If } 1 < |z| < 3 : f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}} + \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{3}\right)^n$$

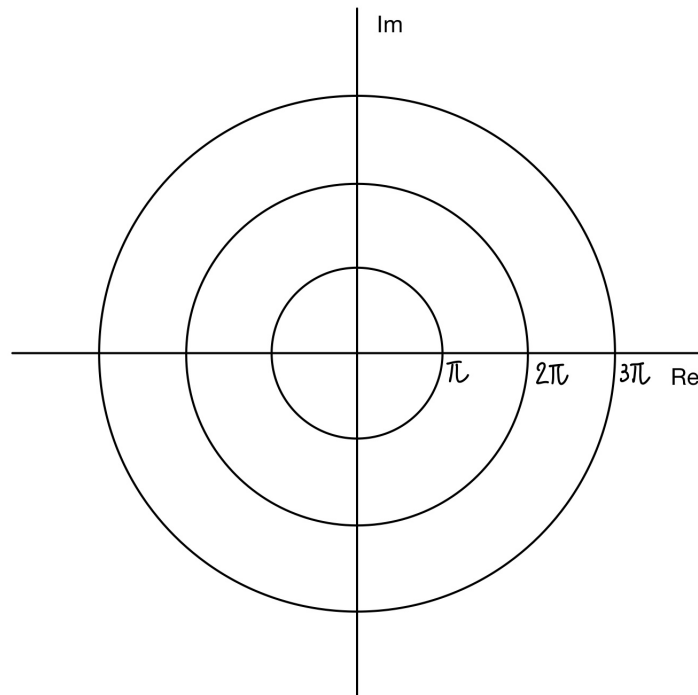
$$\text{If } |z| > 3 : f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}} + \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{z^{n+1}}$$

It only has singular terms.

**Example.** If

$$f(z) = \frac{1}{\sin(z)}$$

has  $\infty$ -many singularities in  $\mathbb{C}$ , we will need expansions for  $k\pi < |z| < (k+1)\pi, k \in \mathbb{N}$ .



## 20 Residue Theorem

### 20.1 Singularity and Root

#### Singularity.

Let  $f : \Omega \rightarrow \mathbb{C}$ . Then  $f$  has a singularity at  $z_0$  if  $f$  is not  $\mathbb{C}$ -differentiable at  $z_0$ .

#### Isolated Singularity.

If  $f$  is holomorphic on the deleted or punctured disk  $0 < |z - z_0| < \delta$  for some  $\delta > 0$ , then the singularity is isolated.

#### Example.

$\text{Log}(z)$  has a branch point at  $z = 0$ , every neighborhood of 0 intersects the branch cut of  $\text{Log}(z)$ . Therefore,  $\text{Log}(z)$  has a non-isolated singularity at 0.

On the other hand,  $\frac{1}{z}$  has an isolated singularity at 0.

There is a natural connection between an isolated singularity and a root of a holomorphic function. If  $f$  is  $\mathbb{C}$ -differentiable at  $z_0$  and its Taylor expansion is of the form

$$f(z) = \sum_{k=n}^{\infty} a_k(z - z_0)^k = a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + \dots$$

If  $n = 1$ , then  $f$  has a simple root at  $z_0$ .

#### Pole.

We want to do the same thing for the principal/singular expansion for  $f(z)$ . For simplicity, let  $b_n = a_{-n}$  to identify the singular coefficients. If

$$f(z) = \sum_{k=-n}^{\infty} a_k(z - z_0)^k = \frac{b_n}{(z - z_0)^n} + \frac{b_{n-1}}{(z - z_0)^{n-1}} + \dots$$

for  $n > 0$  and  $b_n \neq 0$ , then  $f$  has a pole of order  $n$  at  $z_0$ .

If  $b_n = 0$  for  $n > 1$  and  $|b_1| \neq 0$ , then  $f$  has a simple pole at  $z_0$ .

#### Essential Singularity.

If the principal part of  $f(z)$  has  $\infty$ -many terms in it at  $z_0$ , then  $f$  has an essential singularity or a non-removable pole at  $z_0$ . On the other hand, order- $n$  poles are removable.

**Example.**  $\sin(\frac{1}{x})$  has a non-removable pole at 0.

#### Claim.

If  $f$  has an order  $n$  root at  $z_0$ ,  $f(z) \approx a_n(z - z_0)^n$  for  $z \approx z_0$ .

If  $f$  has an order  $n$  pole at  $z_0$ ,  $f(z) \approx \frac{b_n}{(z - z_0)^n}$  for  $z \approx z_0$ .

**Proof.** Suppose  $f$  has an order  $n$  root at  $z_0$ . Since  $a_n \neq 0$ ,

$$f(z) = \sum_{k=n}^{\infty} a_k(z - z_0)^k = a_n(z - z_0)^n \sum_{k=0}^{\infty} \frac{a_{k+n}}{a_n}(z - z_0)^k$$

The proof with poles follows similarly.

We cannot correct for an essential singularity, but we can predict how the function will behave.

**Picard's Theorem.**

If  $f$  has an essential singularity at  $z_0$ , then  $f$  takes on all but possibly one value in  $\mathbb{C}$   $\infty$ -many times in any neighborhood near  $z_0$ .

**Meromorphic Function.**

A function is meromorphic on a domain  $\Omega$  if  $f$  is holomorphic on  $\Omega$  except at isolated poles. Informally, these functions are the ratio of two holomorphic functions on  $\Omega$ .

On a bounded domain, a meromorphic function can only have finitely many isolated poles. Similar to the Uniqueness Theorem for holomorphic functions, zeros have to be isolated.

**Example.**

$\frac{1}{z}$  is meromorphic on  $\mathbb{C}$ .

$\csc(z)$  is meromorphic on  $\mathbb{C}$ , but it has  $\infty$ -many poles, namely  $z = k\pi$  for  $k \in \mathbb{Z}$ .

**Example.** The meromorphic function

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

has no singularity at  $z = 0$ , but does have a simple pole at  $z = 1$ .

$$\frac{1}{1-z} = \frac{-1}{z-1} + 0 \cdot (z-1)^0 + 0 \cdot (z-1)^1 + \dots$$

Namely, the function is its singular part at  $z = 1$ . However,

$$\frac{1}{1-z} = \frac{-1}{z} \cdot \frac{1}{1-\frac{1}{z}} = \sum_{n=1}^{\infty} \frac{-1}{z^n}, |z| > 1$$

does have an essential singularity under a change of variables  $w = \frac{1}{z}$ . Therefore,  $\frac{1}{1-z}$  has an essential singularity at  $\infty$ .

For polynomial denominators, poles are easy to classify.

**Example.**

$$f(z) = \frac{z+1}{z^3(z^2+1)}$$

has 3<sup>rd</sup>-order pole at  $z = 0$  and 1<sup>st</sup>-order poles at  $z = \pm i$

**Example.**

$$f(z) = \frac{\sin(z)}{z} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n)!} = 1 - \frac{z^2}{6} + \frac{z^4}{120} + \dots$$

has no pole at  $z = 0$ , since  $\sin(z)$  and  $z$  have simple roots at 0. Therefore,  $f(z)$  has an isolated, removable singularity at 0.

## 20.2 Residue and Laurent Series

### Residue.

Suppose  $f$  has an isolated singularity at  $z_0$  and a Laurent Series on the annulus

$$A = \{z \mid 0 < |z - z_0| < r\}$$

(The function has no other singularities in  $A$ .)

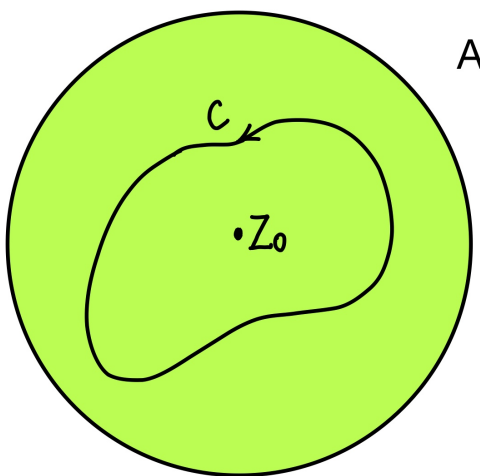
The residue of  $f(z)$  at  $z_0$  is given by

$$\text{Res}(f, z_0) = b_1 = a_{-1}$$

Remember

$$\oint_C z^k dz = \begin{cases} 0, & \text{if } k \neq -1 \\ 2\pi i & \text{if } k = -1 \end{cases}$$

for any closed simple curve about 0.



For any closed simple counter-clockwise curve  $C$  in  $A$  about  $z_0$ , we have

$$\oint_C (z - z_0)^k dz = \begin{cases} 0, & \text{if } k \neq -1 \\ 2\pi i & \text{if } k = -1 \end{cases}$$

Therefore,

$$\oint_C f(z) dz = \oint_C \sum_{k=-n}^{\infty} a_k (z - z_0)^k dz = 2\pi i a_{-1} = 2\pi i b_1 = 2\pi i \text{Res}(f, z_0)$$

**Example.** If

$$f(z) = \frac{1}{z^3} + \frac{2}{z^2} + \frac{4}{z} + 5$$

then we have a pole of order 3 at 0 and  $\text{Res}(f, 0) = 4$ .

Not every Laurent expansion about a pole may be used to compute residues.

**Example.** Let

$$f(z) = \frac{-1}{z} \cdot \frac{1}{1-z} = -\frac{1}{z} \sum_{n=0}^{\infty} z^n = -\frac{1}{z} - 1 - z - z^2 - \dots$$

for  $0 < |z| < 1$ . Therefore,  $f(z)$  has a simple pole and  $\text{Res}(f, 0) = -1$ . However,

$$f(z) = \frac{1}{z^2} \cdot \frac{1}{1-\frac{1}{z}} = \frac{1}{z^2} \sum_{n=0}^{\infty} \frac{1}{z^n} = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \dots$$

for  $|z| > 1$ . But  $\text{Res}(f, 0) \neq 0$  because this Laurent Series is not valid in a punctured disk containing 0. Hence, we must use the first result.

$$\text{Res}\left(\frac{1}{z(z-1)}, 0\right) = -1$$

**Example.**  $\text{Log}(z)$  has a non-isolated singularity at 0. Therefore,  $\text{Res}(\text{Log}(z), 0)$  does not exist.

### Empirical Rules.

1. If

$$f(z) = \sum_{n=-1}^{\infty} a_n(z-z_0)^n$$

then

$$\text{Res}(f(z), z_0) = a_{-1} = b_1$$

and  $f$  has a simple pole at  $z_0$ .

2. If  $g(z) = (z-z_0)f(z)$  is  $\mathbb{C}$ -differentiable at  $z_0$ , then  $\text{Res}(f(z), z_0) = g(z_0)$ .

3. If  $f$  has a simple pole at  $z_0$ , then

$$\lim_{z \rightarrow z_0} (z-z_0)f(z) = \text{Res}(f(z), z_0)$$

(The limit exists and we have equality.)

Conversely, if

$$\lim_{z \rightarrow z_0} (z-z_0)f(z)$$

exists, then  $f$  has a simple pole at  $z_0$  or  $f$  is  $\mathbb{C}$ -differentiable at  $z_0$  and

$$\lim_{z \rightarrow z_0} (z-z_0)f(z) = \text{Res}(f(z), z_0)$$

4. If  $f$  has a simple pole at  $z_0$  and  $g(z)$  is  $\mathbb{C}$ -differentiable at  $z_0$ ,

$$\text{Res}(fg, z_0) = g(z_0) \text{Res}(f, z_0)$$

If  $g(z_0) \neq 0$ ,

$$\text{Res}\left(\frac{f}{g}, z_0\right) = \frac{\text{Res}(f, z_0)}{g(z_0)}$$

5. If  $g$  has a simple root at  $z_0$ , then  $\frac{1}{g}$  has a simple pole at  $z_0$  and

$$\text{Res}\left(\frac{1}{g}, z_0\right) = \frac{1}{g'(z_0)}$$

6. If  $f$  has a pole of order  $k$  at  $z_0$ , then  $g(z) = (z - z_0)^k f(z)$  is holomorphic and for

$$g(z) = a_0 + a_1(z - z_0) + \cdots$$

we have

$$\text{Res}(f, z_0) = a_{k-1} = \frac{g^{(k-1)}(z_0)}{(k-1)!}$$

Each of these can be shown by studying  $f$ 's Laurent expansion in  $0 < |z - z_0| < \delta$ .

**Example.**

$$f(z) = \frac{z^2 + z + 2}{(z - 2)(z - 3)(z - 4)(z - 5)}$$

at  $z = 2$  we have a simple pole and  $g(z) = (z - 2)f(z)$  is holomorphic.

Therefore by (3)  $\text{Res}(f, 2) = g(2) = -\frac{4}{3}$ .

We do not need to compute the Laurent Series at  $z_0 = 2$ .

**Example.**

$$f(z) = \frac{1}{\sin(z)}$$

Then by (5)

$$\text{Res}(f(z), \pi k) = \frac{1}{\cos(\pi k)} = (-1)^k$$

**Example.** Since

$$\sinh(z) = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

Then

$$f(z) = \frac{\sinh(z)}{z^5} = \frac{1}{z^4} + \frac{1}{6z^2} + \frac{1}{120} + \cdots$$

Therefore,  $\text{Res}(f, 0) = 0$ .

**Example.**

$$f(z) = \frac{1}{z(z-2)^2}, g(z) = (z-2)^2 f(z) = \frac{1}{z}$$

Then by (6)

$$\text{Res}(f, 2) = g'(2) = \frac{-1}{z^2} \Big|_{z=2} = -\frac{1}{4}$$

**Example.**

$$f(z) = \frac{1}{z(z-2)^3}, g(z) = (z-2)^3 f(z) = \frac{1}{z}$$

Then by (6)

$$\text{Res}(f, 2) = \frac{g''(2)}{2} = \frac{1}{z^3} \Big|_{z=2} = \frac{1}{8}$$



**Example.**

$$\cot(z) = \frac{\cos(t)}{\sin(t)} = \frac{b_1}{z} + a_0 + a_1z + a_2z^2 + \dots = \frac{1 - \frac{z^2}{2} + \frac{z^4}{4!} - \dots}{z - \frac{z^3}{6} + \frac{z^5}{5!} - \dots}$$

We have a system of equations

$$\left(\frac{b_1}{z} + a_0 + a_1z + a_2z^2 + \dots\right)\left(z - \frac{z^3}{6} + \frac{z^5}{5!} - \dots\right) = 1 - \frac{z^2}{2} + \frac{z^4}{4!} - \dots$$

After we equate the powers of  $z$ , we have

$$b_1 + a_0z + \left(-\frac{b_1}{6} + a_1\right)z^2 + \left(-\frac{a_0}{6} + a_2\right)z^3 + \left(\frac{b_1}{5} - \frac{a_1}{6} + a_3\right)z^4 = 1 - \frac{z^2}{2} + \frac{z^4}{4!}$$

Therefore,  $a_0 = 0, a_1 = -\frac{1}{3}, a_2 = 0, a_3 = -\frac{1}{45}, b_1 = 1$ . This means

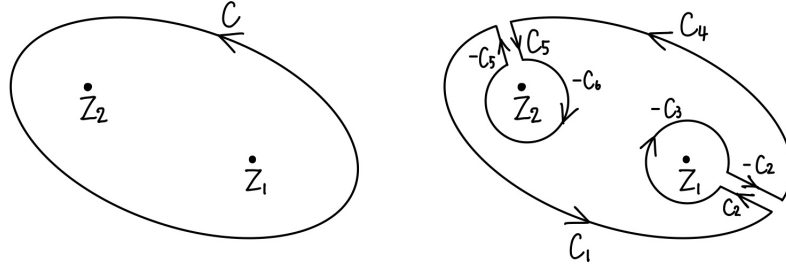
$$\cot(z) = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} + \dots$$

### Cauchy's Residue Theorem.

Let  $\Omega$  be a domain in  $\mathbb{C}$  and  $f$  is holomorphic on  $\Omega$  except at finitely many isolated singularities. Then for any simple-closed curve  $C$  in  $\Omega$  that does not pass through any singularity of  $f$ , oriented counter-clockwise, we have

$$\oint_C f(z) dz = 2\pi i \sum \text{residues of } f \text{ in } C$$

**Proof.** Assume  $f$  has 2 singularities. This proof extends easily to finitely many. Consider a path around  $f$ 's singularities,  $z_1$  and  $z_2$ .



$$C_0 = C_1 \cup C_2 \cup -C_3 \cup -C_4 \cup C_5 \cup -C_6 \cup -C_5$$

$$C = C_1 \cup C_4$$

By Cauchy-Goursat Theorem,

$$\oint_{C_0} f(z) dz = 0$$

And the integrals over  $\pm C_2$  and  $\pm C_5$  sum to zero. Therefore,

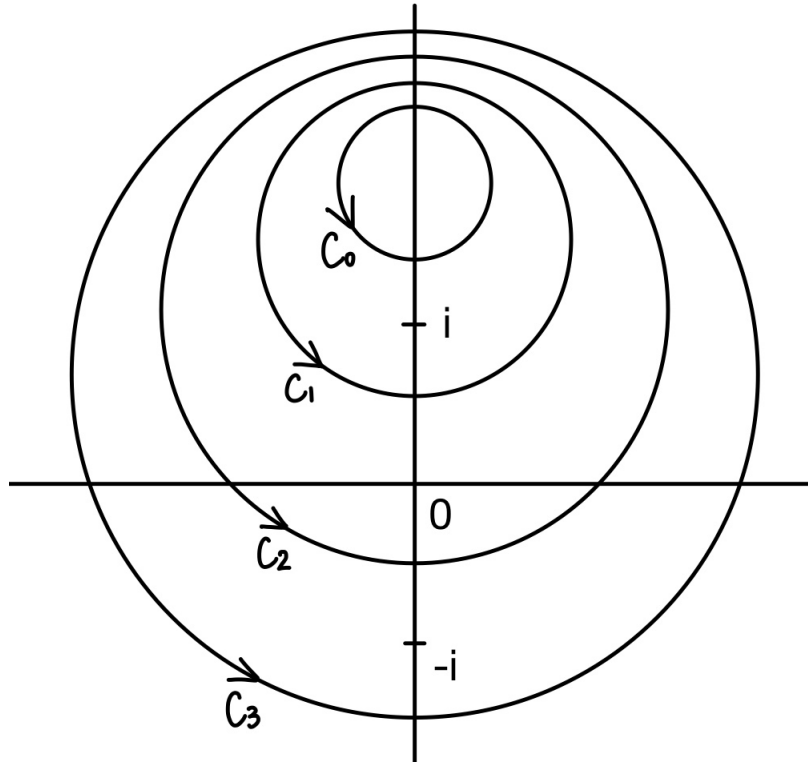
$$\begin{aligned} 0 &= \oint_{C_0} f(z) dz = \oint_{C_1 \cup -C_3 \cup C_4 \cup -C_6} f(z) dz = \oint_{C_1 \cup C_4} f(z) dz + \oint_{-C_3 \cup -C_6} f(z) dz \\ &\Rightarrow \oint_C f(z) dz = \oint_{C_3 \cup C_6} f(z) dz = \oint_{C_3} f(z) dz + \oint_{C_6} f(z) dz = 2\pi i (\text{Res}(f, z_1) + \text{Res}(f, z_2)) \end{aligned}$$

Hence, this concludes the proof.

**Example.**

$$f(z) = \frac{1}{z(z^2 + 1)}$$

Then we can derive the following residues.



$$Res(f, i) = (z - i)f(z) \Big|_{z=i} = \frac{1}{i(i + 1)} = -\frac{1}{2}$$

$$Res(f, 0) = zf(z) \Big|_{z=0} = \frac{1}{0 + 1} = 1$$

$$Res(f, -i) = (z + i)f(z) \Big|_{z=-i} = \frac{1}{-i(-i - 1)} = -\frac{1}{2}$$

When we traverse through the contours illustrated in the graph, we have

$$\oint_{C_0} f(z) dz = 0 \text{ by Cauchy-Goursat Theorem}$$

$$\oint_{C_1} f(z) dz = 2\pi i Res(f, i) = -\pi i$$

$$\oint_{C_2} f(z) dz = 2\pi i (Res(f, i) + Res(f, 0)) = -\pi i + 2\pi i = \pi i$$

$$\oint_{C_3} f(z) dz = 2\pi i (Res(f, i) + Res(f, 0) + Res(f, -i)) = -\pi i + 2\pi i - \pi i = 0$$

But  $f$  is not holomorphic on the interior of  $C_3$ .

**Example.** Integrate

$$f(z) = \frac{1}{z(z-2)^4}$$

over  $|z-2| = \frac{1}{3}$ . Therefore, we need  $\text{Res}(f, 2)$ .

$$\begin{aligned}\frac{1}{z} &= \frac{1}{2 + (z-2)} = \frac{1}{2} \cdot \frac{1}{1 - (-\frac{z-2}{2})} = \frac{1}{2} \left( 1 - \frac{z-2}{2} + \frac{(z-2)^2}{4} - \frac{(z-2)^3}{8} + \dots \right) \\ f(z) &= \frac{1}{2(z-2)^4} - \frac{1}{4(z-2)^3} + \frac{1}{8(z-2)^2} - \frac{1}{16(z-2)} + \dots\end{aligned}$$

Therefore,  $\text{Res}(f, 2) = -\frac{1}{16}$ . Hence,

$$\oint_{|z-2|=\frac{1}{3}} f(z) dz = 2\pi i \text{Res}(f, 2) = -\frac{\pi i}{8}$$

## 21 Improper Integrals

### 21.1 Principal Value

Suppose

$$I = \int_{\mathbb{R}} f(x) dx$$

converges absolutely. Then

$$I = \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx = P.V. \int_{\mathbb{R}} f(x) dx$$

Here

$$P.V. \int_{\mathbb{R}} f(x) dx = \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx$$

denotes the principal value of

$$\int_{\mathbb{R}} f(x) dx$$

However, without absolute convergence of  $I$ , this does not hold.

**Example.**

$$\lim_{R \rightarrow \infty} \int_{-R}^R \sin(x) dx = 0$$

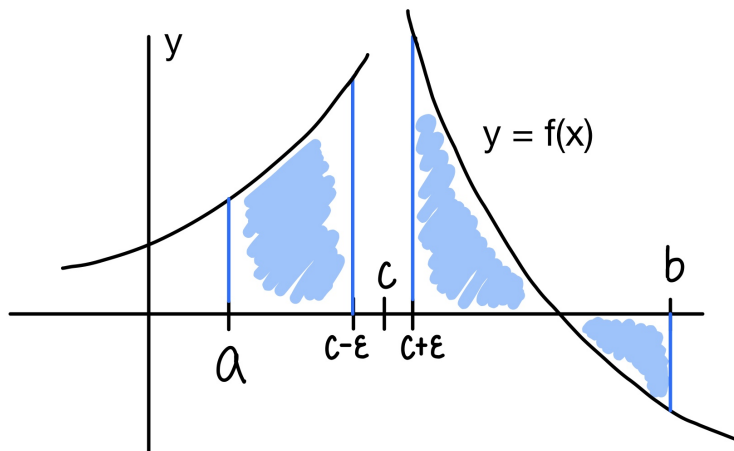
but

$$\int_{\mathbb{R}} \sin(x) dx \text{ does not exist}$$

$$\int_{\mathbb{R}} |\sin(x)| dx = \infty$$

If  $f(x)$  is undefined at  $c \in (a, b)$ , then

$$P.V. \int_a^b f(x) dx = \lim_{\varepsilon \rightarrow 0^+} \left( \int_a^{c-\varepsilon} f(x) dx + \int_{c+\varepsilon}^b f(x) dx \right)$$



**Example.** Let

$$f(x) = \frac{1}{x^3}$$

then

$$\begin{aligned} I &= P.V. \int_{-2}^3 f(x) dx \\ &= \lim_{\varepsilon \rightarrow 0^+} \int_{-2}^{-\varepsilon} f(x) dx + \int_{\varepsilon}^3 f(x) dx \\ &= \lim_{\varepsilon \rightarrow 0^+} F(3) - F(\varepsilon) + F(-\varepsilon) - F(-2) \\ &= \lim_{\varepsilon \rightarrow 0^+} F(3) - F(-2) \\ &= \left. \frac{-1}{2x^2} \right|_{x=-2}^3 \end{aligned}$$

since  $F$  is even. We are "ignoring" the singularity by exploiting

$$\int_{-a}^a f(x) dx = 0$$

for  $f$  odd and piecewise-smooth.

**Lemma.**

If  $|f(x)| \leq g(x)$  for all  $x \in \mathbb{R}$  and  $\int_{\mathbb{R}} g(x) dx$  converges, then

$$I = \int_{\mathbb{R}} f(x) dx$$

converges absolutely.

## 21.2 Clever Choice of Contours.

**Example\*.** The improper integral

$$\int_{\mathbb{R}} \frac{1}{(x^2 + 1)^2} dx$$

converges absolutely because

$$\frac{1}{(x^2 + 1)^2} \leq \frac{1}{(x^2 + 1)} \text{ and } \int_{\mathbb{R}} \frac{1}{(x^2 + 1)} dx < \infty$$

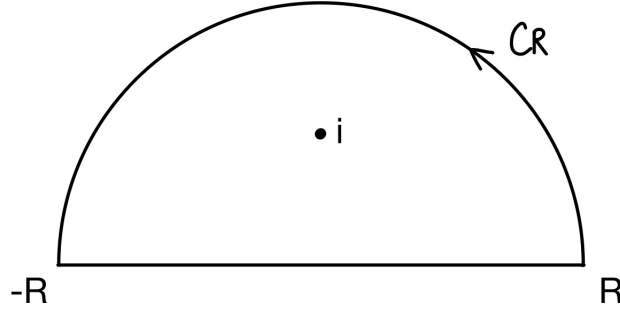
Let

$$f(z) = \frac{1}{(z^2 + 1)^2} = \frac{1}{(z + i)^2(z - i)^2}$$

and by Residue Theorem,

$$\oint_C f(z) dz = 2\pi i \operatorname{Res}(f, i)$$

for any closed curve  $C$  about  $i$ , but not  $-i$ .



We pick a path along  $[-R, R]$  and the upper half of the disk  $|z| = R$  traversed counter-clockwise. Therefore,

$$2\pi i \operatorname{Res}(f, i) = \int_{C_R} f(z) dz + \oint_{-R}^R f(z) dz$$

then

$$\int_{\mathbb{R}} f(x) dx = \lim_{R \rightarrow \infty} \oint_C f(z) dz = \lim_{R \rightarrow \infty} \int_{-R}^R f(z) dz$$

provided

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0$$

That is to say, the integral over the upper curve vanishes. We can establish this with an ML-Estimate,

$$\begin{aligned} |f(z)| &\leq \frac{A}{|z|^4} \text{ for } z\text{-large, } A > 0 \\ \Rightarrow \left| \int_{C_R} f(z) dz \right| &\leq \int_{C_R} |f(z)| |dz| \leq \underbrace{\frac{A}{R^4}}_M \underbrace{\pi R}_L = \frac{A\pi}{R^3} \end{aligned}$$

Therefore, its upper bound  $\rightarrow 0$  as  $R \rightarrow \infty$ . Therefore,

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0$$

Next we solve  $\operatorname{Res}(f, i)$ .

$$\begin{aligned} g(z) &= (z - i)^2 f(z) = \frac{1}{(z + 2)^2} \\ \operatorname{Res}(f, i) &= g'(i) = \frac{d}{dz} \frac{1}{(z + i)^2} \Big|_{z=i} = \frac{-2}{(z + i)^3} \Big|_{z=i} = \frac{1}{4i} \end{aligned}$$

Hence,

$$\int_{\mathbb{R}} f(x) dx = 2\pi i \operatorname{Res}(f, i) = 2\pi i \cdot \frac{1}{4i} = \frac{\pi}{2}$$

This technique works whenever  $|f(z)| \leq \frac{A}{|z|^a}$ ,  $a > 1$  and  $z$ -large.

**Example (Trigonometry).** Find

$$I = \int_{\mathbb{R}} \frac{\cos(x)}{(x-1)^2 + 1} dx$$

$$|I| = \int_{\mathbb{R}} \left| \frac{\cos(x)}{(x-1)^2 + 1} \right| dx \leq \int_{\mathbb{R}} \frac{A}{x^2 + 1} dx < \infty$$

Therefore,  $I$  converges absolutely. Here  $\cos(z) \rightarrow \infty$  as  $z \rightarrow \infty$ .

Replace  $\cos(x)$  with  $e^{ix} = \cos(x) + i \sin(x)$ .

$$I^* = \int_{\mathbb{R}} \frac{e^{ix}}{(x-1)^2 + 1} dx, \quad I = \Re(I^*)$$

$$f(z) = \frac{e^{iz}}{(z-1)^2 + 1}, \quad |f(z)| = \frac{|e^{i(x+iy)}|}{|z^2 - 2z + 2|} = \frac{e^{-y}}{|z^2 - 2z + 2|}$$

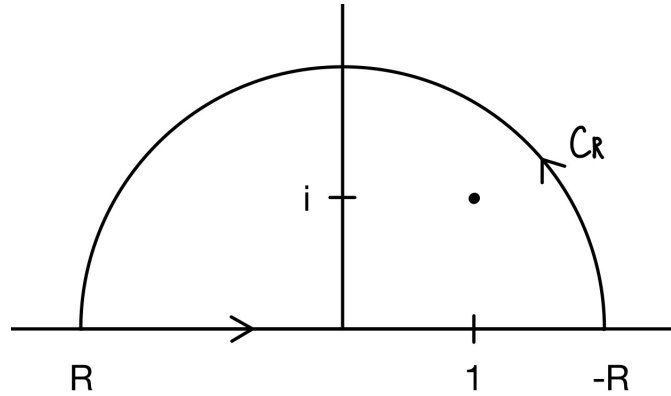
We evaluate the integral over the same contour,  $C = C_R \cup [-R, R]$ . With  $y > 0$ , for  $z$ -large we have

$$\frac{e^{-y}}{|z^2 - 2z + 2|} \leq \frac{A}{|z|^2}$$

Repeating the argument from the previous example, we have

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0$$

We have two poles,  $z = 1 \pm i$  and only one is inside the curve  $C$ ,  $z = 1 + i$ .



$$I^* = 2\pi i \operatorname{Res}(f, 1+i) = 2\pi i \left( -\frac{i}{2} e^{i-1} \right) = \frac{\pi e^i}{e}$$

$$\operatorname{Res}(f, 1+i) = \left. \frac{e^{iz}}{z - (1-i)} \right|_{z=1+i} = -\frac{i}{2} e^{i-1}$$

$$I = \Re(I^*) = \Re \left( \pi \frac{e^i}{e} \right) = \frac{\pi}{e} \cos(1)$$

**Note.** The approach we used above  $I = \Re(I^*)$  fails for

$$I = \int_{\mathbb{R}} \frac{\cos(x)}{(x-1)^3} dx$$

because  $I \neq \bar{I}$  or  $I \notin \mathbb{C} \setminus \mathbb{R}$ .

**Proof.**

$$I = \int_{\mathbb{R}} \frac{\cos(x)}{(x-i)^3} dx = \int_{\mathbb{R}} \frac{\cos(x)}{(x^2+i)^3} (x+i)^3 dx = \int_{\mathbb{R}} \frac{\cos(x)}{(x^2+i)^3} (x^3 + 3ix^2 - 3x - i) dx$$

Let  $f(x) = \frac{\cos(x)}{(x^2+1)^3}$ , it is an even function. Thus,

$$\begin{aligned} I &= \int_{\mathbb{R}} f(x)(x^3 + 3ix^2 - 3x - i) dx \\ &= \int_{\mathbb{R}} f(x)x^3 dx + \int_{\mathbb{R}} f(x)3ix^2 dx - \int_{\mathbb{R}} f(x)3x dx - \int_{\mathbb{R}} if(x) dx \\ &= \int_{\mathbb{R}} f(x)3ix^2 dx - \int_{\mathbb{R}} if(x) dx \end{aligned}$$

The equality holds because

$$\int_{\mathbb{R}} f(x)x^3 dx = \int_{\mathbb{R}} f(x)3x dx = 0$$

as  $f(x)x^3$  and  $f(x)3x$  are both odd functions. Similarly,

$$\begin{aligned} \bar{I} &= \int_{\mathbb{R}} \frac{\cos(x)}{(x+i)^3} dx \\ &= \int_{\mathbb{R}} \frac{\cos(x)}{(x^2+i)^3} (x-i)^3 dx \\ &= \int_{\mathbb{R}} \frac{\cos(x)}{(x^2+i)^3} (x^3 - 3ix^2 - 3x + i) dx \\ &= - \int_{\mathbb{R}} f(x)3ix^2 dx + \int_{\mathbb{R}} if(x) dx \end{aligned}$$

Therefore,  $I = \bar{I}$ , which obviously does not satisfy  $I = \bar{I}$ .

**Example (Change of Variable).**

For

$$\int_0^{2\pi} g(\theta) d\theta$$

Let  $z = e^{i\theta} \Rightarrow dz = ie^{i\theta} d\theta \Rightarrow d\theta = \frac{dz}{iz}$ . Then

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{5 + 2 \cos \theta} &= \oint_{|z|=1} \frac{1}{iz} \cdot \frac{dz}{5 + z + \frac{1}{z}} \\ &\text{for } \cos \theta = \frac{z + z^{-1}}{2} \end{aligned}$$



Here the poles of the integrand are

$$z = \frac{-5 \pm \sqrt{21}}{2}, \left| \frac{-5 + \sqrt{21}}{2} \right| < 1 \text{ and } \left| \frac{-5 - \sqrt{21}}{2} \right| > 1$$

$$\Rightarrow \frac{1}{i} \oint_{|z|=1} \frac{dz}{z^2 + 5z + 1} = \frac{2\pi i}{i} \frac{1}{\left( z - \frac{1}{2}(-5 - \sqrt{21}) \right)} \Big|_{z = \frac{-5 + \sqrt{21}}{2}} = \frac{2\pi}{\sqrt{21}}$$

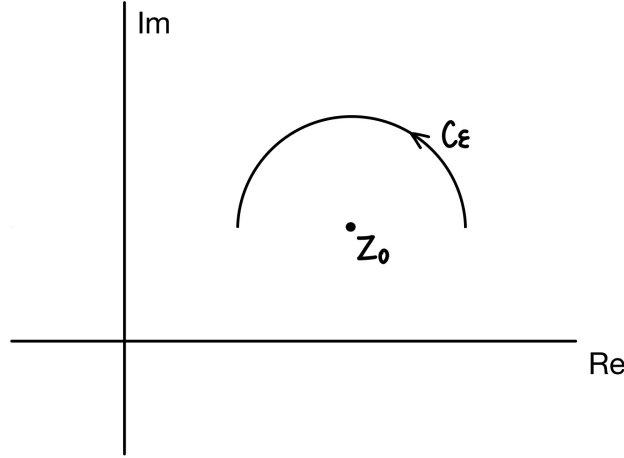
**Semicircle Lemma.**

Let  $f(z)$  have a simple pole at  $z_0$  and let  $C_\varepsilon$  be the curve parametrized by

$$\gamma(t) = z_0 + \varepsilon e^{it}, t \in [0, \pi]$$

Then

$$\lim_{\varepsilon \rightarrow 0^+} \int_{C_\varepsilon} f(z) dz = \pi i \operatorname{Res}(f, z_0)$$



**Proof.** Assume  $\varepsilon > 0$  but small enough for  $f$  to have a Laurent expansion for  $0 < |z - z_0| < \varepsilon$ . Then

$$f(z) = \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + \dots$$

$$\int_C f(z) dz = \int_0^\pi f(z_0 + \varepsilon e^{it}) \varepsilon i e^{it} dt$$

$$= \int_0^\pi b_1 i + a_0 i \varepsilon e^{it} + a_1 i \varepsilon^2 e^{2it} + \dots dt$$

and

$$\lim_{\varepsilon \rightarrow 0^+} \int_C f(z) dz = \pi i b_1 = \pi i \operatorname{Res}(f, z_0)$$

For non-simple poles, the limit does not exist and we can extend this argument to any angular range.

**Corollary of Semicircle Lemma.**

$$\lim_{\varepsilon \rightarrow 0^+} \int_{C_\varepsilon} f(z) dz = \theta i \operatorname{Res}(f, z_0)$$

where  $C_\varepsilon$  is the curve parametrized by

$$\gamma(t) = z_0 + \varepsilon e^{it}, t \in [\alpha, \alpha + \theta]$$

**Jordan's Lemma.**

If for  $z$ -large,  $|f(z)| \leq \frac{A}{|z|}$  and  $C_R$  is the curve parametrized by

$$\gamma(t) = R e^{it}, t \in [0, \pi]$$

then for  $\alpha > 0$ ,

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) e^{i\alpha z} dz = 0$$

**Example.** Find

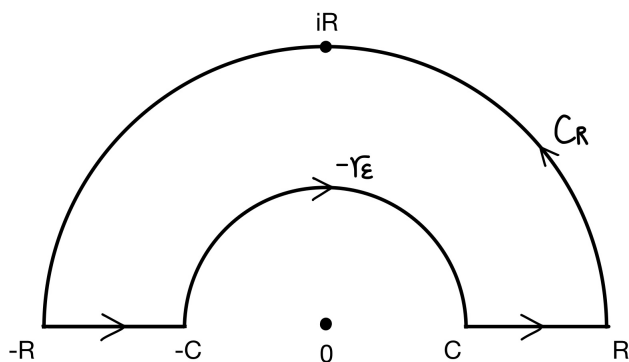
$$I = P.V. \int_{\mathbb{R}} \frac{e^{ix}}{x} dx$$

This integrand is not continuous at  $x = 0$ ,

$$I = \lim_{R \rightarrow \infty, \varepsilon \rightarrow 0^+} \int_{-R}^{-\varepsilon} f(x) dx + \int_{\varepsilon}^R f(x) dx$$

We approach  $x = 0$  from the left and right at the same rate and distance.

Let  $f(z) = \frac{e^{iz}}{z}$  and consider the contour  $C$ . Then  $f$  is holomorphic on  $C$ 's interior and by Cauchy-Goursat



Theorem,

$$0 = \oint_C f(z) dz = \int_{-R}^{-\varepsilon} f(z) dz + \int_{-\varepsilon}^{\varepsilon} f(z) dz + \int_{\varepsilon}^R f(z) dz + \int_{C_R} f(z) dz$$

By Jordan's Lemma,

$$\int_{C_R} f(z) dz \rightarrow 0 \text{ as } R \rightarrow \infty$$

By Semicircle Lemma,

$$\int_{-\tau_\varepsilon} f(z) dz \rightarrow -\pi i \operatorname{Res}(f, 0)$$

when  $\varepsilon \rightarrow 0^+$  because there is a simple pole at 0. Therefore,

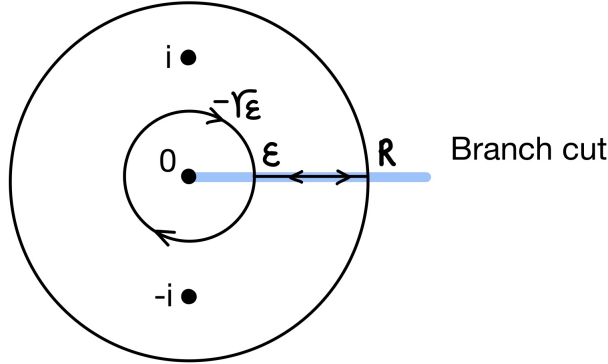
$$\begin{aligned} \int_{-R}^{-\varepsilon} f(z) dz + \int_{\varepsilon}^R f(z) dz &= \int_{\tau_\varepsilon} f(z) dz = \pi i \operatorname{Res}(f, 0) \text{ as } R \rightarrow \infty \text{ and } \varepsilon \rightarrow 0^+ \\ \Rightarrow P.V. \int_{\mathbb{R}} \frac{e^{ix}}{x} dx &= \pi i \operatorname{Res}(f, 0) = \pi i e^{iz} \Big|_{z=0} = \pi i \end{aligned}$$

**Example.** Find  $I = \int_0^\infty \frac{\sqrt{x}}{x^2+1} dx$  where  $\sqrt{x} > 0$ .

$$I = \lim_{R \rightarrow \infty, \varepsilon \rightarrow 0^+} \int_{\varepsilon}^R \frac{\sqrt{x}}{x^2+1} dx$$

In this case, we need a contour in  $\mathbb{C}$  that does the following:

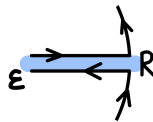
1.  $f$  is holomorphic in its interior.
2. The contour coincides with  $\mathbb{R}^+$  for  $f(z) = \frac{\sqrt{z}}{z^2+1}$ .
3. Avoid 0, the branch point.



This means the branch cut for  $\sqrt{z}$  must be along  $\mathbb{R}^+$ . Else,  $f$  is not meromorphic on the interior of  $C$ .

$$\begin{aligned} \sqrt{z} &= e^{\frac{\operatorname{Log}|z| + i\theta}{2}}, \theta \in (0, 2\pi) \\ \oint_C f(z) dz &= 2\pi i (\operatorname{Res}(f, i) + \operatorname{Res}(f, -1)) \\ &= \int_{\varepsilon}^R f(z) dz + \int_{C_R} f(z) dz + \int_R^{\varepsilon} f(z) dz + \int_{-\tau_\varepsilon} f(z) dz \end{aligned}$$

Now along the branch cut, we need to study the sign of  $\sqrt{x}$ . On  $[\varepsilon, R]$  and  $[R, \varepsilon]$ , we are using different



branches of  $\sqrt{z}$  but the same branch cut.

For  $y > 0$  and  $x \in (\varepsilon, R)$ ,  $\sqrt{x} > 0$ .

For  $y < 0$  and  $x \in (\varepsilon, R)$ ,  $\sqrt{x} < 0$ .

$$\begin{aligned}\int_{\varepsilon}^R f(z) dz &= \int_{\varepsilon}^R \frac{\sqrt{z}}{z^2 + 1} dz \\ - \int_R^{\varepsilon} f(z) dz &= \int_R^{\varepsilon} \frac{-\sqrt{z}}{z^2 + 1} dz = \int_{\varepsilon}^R \frac{\sqrt{z}}{z^2 + 1} dz\end{aligned}$$

because  $\varepsilon < R$ . Therefore,

$$2 \int_{\varepsilon}^R \frac{\sqrt{z}}{z^2 + 1} dz = \int_{\varepsilon}^R f(z) dz - \int_R^{\varepsilon} f(z) dz$$

Now

$$\left| \int_{C_R} f(z) dz \right| \leq 2\pi R \frac{A\sqrt{R}}{R^2} \rightarrow 0 \text{ as } R \rightarrow \infty$$

Observe for  $\varepsilon \in (0, 1)$ ,

$$1 - \varepsilon^2 \leq |\varepsilon^2 + 1| = \varepsilon^2 + 1 \Rightarrow \frac{1}{|\varepsilon^2 + 1|} \leq \frac{1}{1 - \varepsilon^2}$$

And so by an ML-Estimate,

$$\left| \int_{-\tau_{\varepsilon}} f(z) dz \right| \leq \underbrace{\frac{\varepsilon^{\frac{1}{2}}}{1 - \varepsilon^2}}_M \underbrace{2\pi\varepsilon}_L \rightarrow 0 \text{ as } \varepsilon \rightarrow 0^+$$

As  $\varepsilon \rightarrow 0^+$ ,  $R \rightarrow \infty$ ,

$$2I = 2 \int_0^{\infty} \frac{\sqrt{x}}{x^2 + 1} dx = 2\pi i (Res(f, i) + Res(f, -i))$$

The residues require us to use our branch of  $\sqrt{z}$ ,  $\arg(z) \in (0, 2\pi)$ .

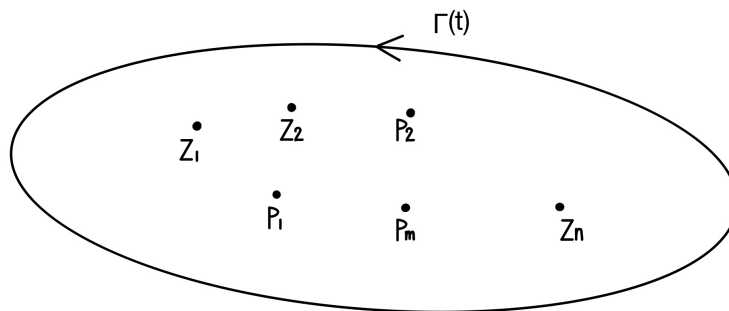
$$\begin{aligned}
 \operatorname{Res}(f, i) &= \left. \frac{e^{\frac{(\operatorname{Log} |z| + i \arg(z))}{2}}}{z + i} \right|_{z=i} \\
 &= \frac{e^{\frac{0+i\frac{\pi}{2}}{2i}}}{2i} \\
 &= \frac{e^{i\frac{\pi}{4}}}{2i} \\
 &= \frac{1}{2i} \left( \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right)
 \end{aligned}$$

$$\begin{aligned}
 \operatorname{Res}(f, -i) &= \left. \frac{e^{\frac{(\operatorname{Log} |z| + i \arg(z))}{2}}}{z - i} \right|_{z=-i} \\
 &= \frac{e^{\frac{0+i\frac{3\pi}{2}}{2i}}}{-2i} \\
 &= \frac{e^{i\frac{3\pi}{4}}}{-2i} \\
 &= -\frac{1}{2i} \left( -\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right)
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow I &= \int_0^\infty \frac{\sqrt{x}}{x^2 + 1} dx \\
 &= \pi i \left( \frac{1}{2i} \left( \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) - \frac{1}{2i} \left( -\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) \right) \\
 &= \frac{\pi}{\sqrt{2}}
 \end{aligned}$$

## 22 Argument Principle and Rouché's Theorem

Let  $\gamma$  be a simple, piecewise smooth, closed curve in  $\mathbb{C}$  oriented counter-clockwise. Let  $f$  be meromorphic inside and on  $\gamma$  but with no poles or zeros on  $\gamma$ .



Let  $p_1, p_2, \dots, p_m$  be  $f$ 's poles in  $\gamma$  and  $z_1, \dots, z_n$  be  $f$ 's zeros in  $\gamma$ . Define  $\text{mult}(z_k)$  and  $\text{mult}(p_k)$  as the orders of the root  $z_k$  and pole  $p_k$ . Define

$$Z_{f,\gamma} = \sum \text{mult}(z_k)$$

$$P_{f,\gamma} = \sum \text{mult}(p_k)$$

as the sum of the orders of the poles and roots of  $f$  in  $\gamma$ .

**Lemma.**

With  $f$  and  $\gamma$  defined as above:

$$\oint_{\gamma} \frac{f'(z)}{f(z)} dz = 2\pi i \left( \sum \text{mult}(z_k) - \sum \text{mult}(p_k) \right) = 2\pi i (Z_{f,\gamma} - P_{f,\gamma})$$

**Proof.** Let  $f$  have a zero of order  $k$  at  $z_0$ .

$$\begin{aligned} f(z) &= (z - z_0)^k g(z), g(z_0) \neq 0 \\ \Rightarrow f'(z) &= (z - z_0)^{k-1} (kg(z) + (z - z_0)g'(z)) \\ \Rightarrow f' &\text{ has a root of order } k - 1 \text{ at } z_0 \\ \Rightarrow \frac{f'}{f} &= \underbrace{\frac{k}{z - z_0}}_{\text{Singular Part}} + \underbrace{\frac{g'(z)}{g(z)}}_{\text{Holomorphic Part}} \end{aligned}$$

Therefore,  $\frac{f'}{f}$  has a simple pole and  $\text{Res}(\frac{f'}{f}, z_0) = k$ , which is also the order of the root at  $z_0$  of  $f$ . Repeat the argument with a pole at  $p_0$  of order  $k$ , we have

$$\frac{f'}{f} = \frac{-k}{z - p_0} + \frac{g'(z)}{g(z)}$$

The result then follows by the Residue Theorem.

**Example.** Consider

$$\oint_{|z|=4} \frac{f'(z)}{f(z)} dz \text{ for } f(z) = \frac{(z^2 + 1)^2}{(z^2 + 2z + 2)^3}$$

$f$  has “4” roots in  $|z| < 2$  and “6” poles in  $|z| < 4$ . Therefore,

$$\oint_{|z|=4} \frac{f'(z)}{f(z)} dz = 2\pi i(4 - 6) = -4\pi i$$

**Example.** Consider

$$f(z) = z^5 - 3iz + 2z - 1 + i$$

$f(z)$  has 5 roots in  $\mathbb{C}$  and

$$\oint_{|z|=R} \frac{f'(z)}{f(z)} dz = 2\pi i(5) = 10\pi i$$

for  $R = \max |z_i|$  where  $f(z_i) = 0$ .

**Winding Number (Index).**

For any closed curve  $\gamma$  in  $\mathbb{C}$ , its winding number (index) about  $z_0$  is

$$Ind(\gamma, z_0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{1}{z - z_0} dz \in \mathbb{Z}$$

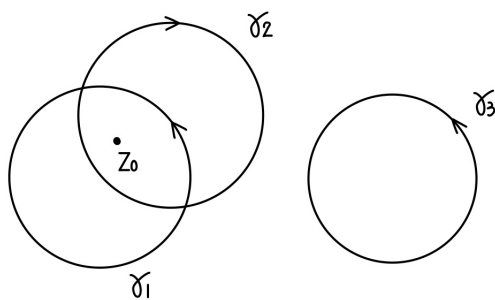
If  $\gamma$  is simple, then  $Ind(\gamma, z_0) \in \{1, 0, -1\}$ .

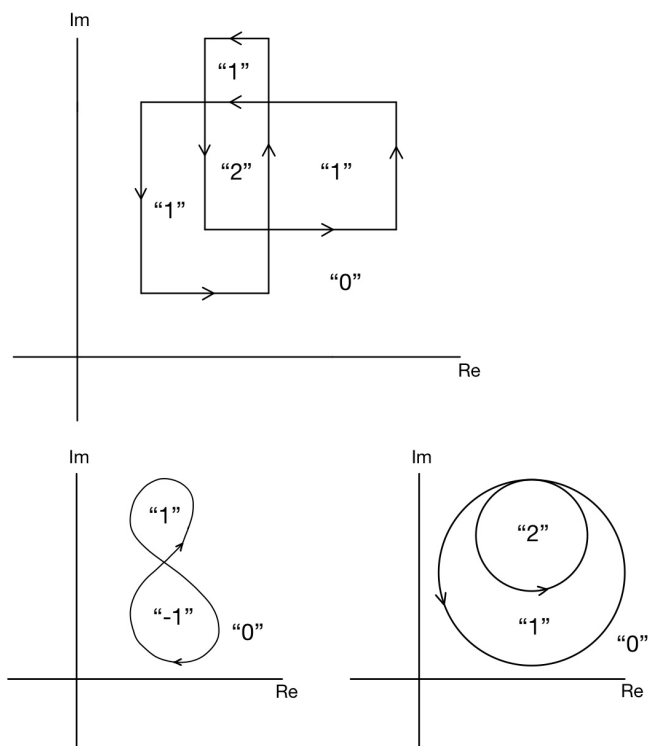
**Example.**

$$Ind(\gamma_1, z_0) = 1$$

$$Ind(\gamma_2, z_0) = -1$$

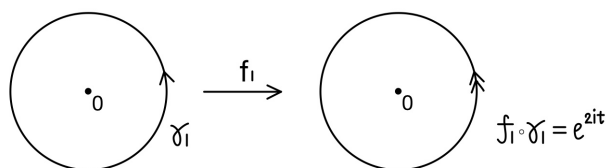
$$Ind(\gamma_3, z_0) = 0$$





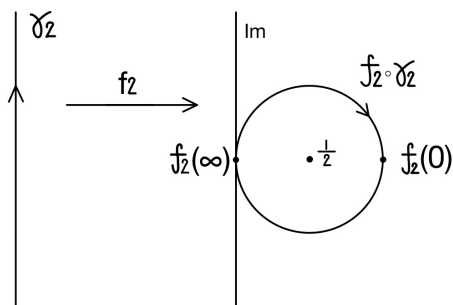
**Example.**

If  $\gamma_1(t) = e^{it}$  and  $f_1(z) = z^2$ , then  $f_1 \circ \gamma_1 = e^{2it}$  traverses the unit circle twice for  $t \in [0, 2\pi]$ . Therefore,  $Ind(f_1 \circ \gamma_1, 0) = 2$ .



If  $\gamma_2(t) = it, t \in \mathbb{R}$  and  $f_2 = \frac{1}{z+1}$ , then  $f_2 \circ \gamma_2$  is a circle about  $\frac{1}{2}$  of radius  $\frac{1}{2}$  by the Möbius transform theory.

Therefore,  $Ind(f_2 \circ \gamma_2, \frac{1}{2}) = -1$ .



The number of times  $f \circ \gamma$  wraps around  $w_0$  implies how many solutions exist to  $f(z) = w_0$  for  $z$  inside  $\gamma$ .



### Argument Principle.

For  $f$  and  $\gamma$  as in our lemma,

$$\oint_{\gamma} \frac{f'(z)}{f(z)} dz = 2\pi i \operatorname{Ind}(f \circ \gamma, 0) = 2\pi i (Z_{f,\gamma} - P_{f,\gamma})$$

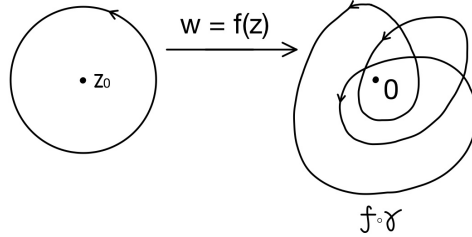
**Proof.** Let  $w = f(z)$ , then  $z = \gamma(t)$ . Then,  $w = f \circ \gamma$  and  $dw = f'(\gamma(t)) \gamma'(t) dt$ .

$$\oint_{\gamma} \frac{f'(z)}{f(z)} dz = \int_{f \circ \gamma} \frac{dw}{w} = 2\pi i \operatorname{Ind}(f \circ \gamma, 0) = 2\pi i (Z_{f,\gamma} - P_{f,\gamma})$$

We can measure how many times a curve in  $\mathbb{C}$  is wrapped around a point in the  $w$ -plane by  $w = f(z)$  or measure the change in the curve's argument.

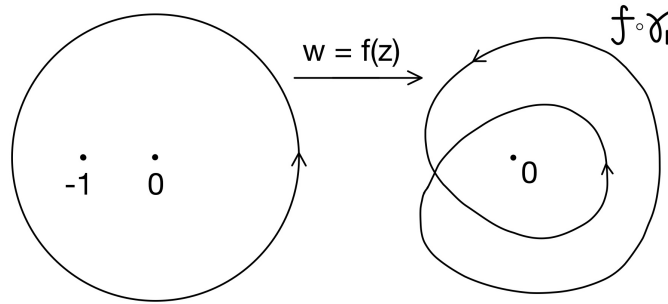
$$\oint_{\gamma} \frac{f'}{f} dz = \frac{1}{2\pi i} \operatorname{Log}(f(z)) \Big|_{z=\gamma(0)}^{\gamma(1)} = \frac{1}{2\pi} \triangle \operatorname{Arg}(f(z))$$

for  $\gamma(0) = \gamma(1)$ . (The  $\mathbb{R}$ -part vanishes since  $|f(\gamma(0))| = |f(\gamma(1))|$ ).

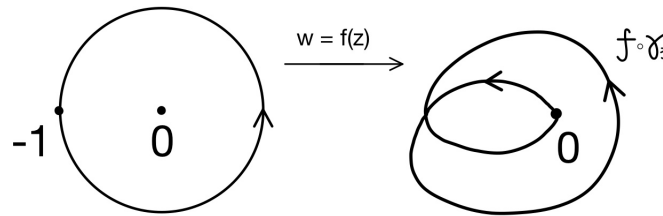


**Example.** Let  $f(z) = z^2 + z$ . It has two roots 0 and -1 and no poles.

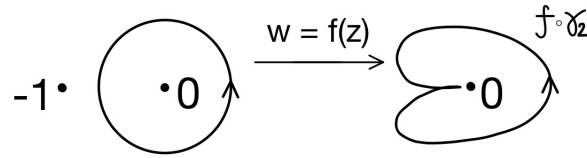
Let  $\gamma_1$  be a circle of radius 2, and both roots are inside  $f \circ \gamma_1$ . Therefore,  $\operatorname{Ind}(f \circ \gamma_1, 0) = 2$ .



Let  $\gamma_2$  be a circle of radius  $\frac{1}{2}$ , and one root is inside  $f \circ \gamma_2$ . Therefore,  $\operatorname{Ind}(f \circ \gamma_2, 0) = 1$ .



Let  $\gamma_3$  be a circle of radius 1, and it passes through the root -1. Therefore, the Argument Principle fails.



Imagine you are always 10 feet or more from a tree and you are walking a dog with an 8 feet leash. Everything you walk around the tree, so must the dog.

Let  $f$  be a function and  $|h| < |f|$ , then the perturbed function  $f + h$  has its roots near the roots of  $f$ . Here  $h$  is a “small” change to  $f$ .

### Rouche's Theorem.

Let  $\gamma$  be simple closed curve in  $\mathbb{C}$ . Suppose  $f$  and  $h$  are meromorphic inside and on  $\gamma$ , but with no poles on  $\gamma$ , and  $|h| < |f|$  on  $\gamma$ . Then

$$Z_{f,\gamma} - P_{f,\gamma} = Z_{f+h,\gamma} - P_{f+h,\gamma}$$

### Proof.

Claim 1:  $f$ ,  $f + h$  and  $\frac{f+h}{f}$  have no roots on  $\gamma$ .

$$\begin{aligned} 0 \leq |h| < |f| &\Rightarrow f \text{ has no roots on } \gamma \\ &\Rightarrow f + h \text{ has no roots on } \gamma, \text{ since } |h| - |f| \neq 0 \text{ on } \gamma \\ &\Rightarrow \frac{f+h}{f} \text{ has no roots on } \gamma \text{ since } f + h \neq 0 \text{ on } \gamma \end{aligned}$$

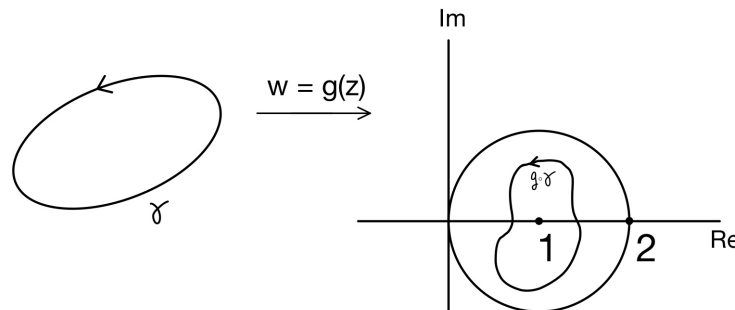
Claim 2:  $f$ ,  $f + h$  and  $\frac{f+h}{f}$  have no poles on  $\gamma$ .

$f$  and  $f + h$  have no poles by our hypothesis and  $f \neq 0$  on  $\gamma$ . Therefore,  $\frac{f+h}{f}$  has no poles on  $\gamma$ . By the Argument Principle,

$$\begin{aligned} \oint_{\gamma} \frac{f'}{f} dz &= 2\pi i \text{Ind}(f \circ \gamma, 0) = 2\pi i(Z_{f,\gamma} - P_{f,\gamma}) \\ \oint_{\gamma} \frac{f' + h'}{f + h} dz &= 2\pi i \text{Ind}((f + h) \circ \gamma, 0) = 2\pi i(Z_{f+h,\gamma} - P_{f+h,\gamma}) \end{aligned}$$

Now  $\left| \frac{h}{f} \right| < 1$  and so  $\frac{h}{f} \circ \gamma$  is in the unit circle.

Then  $g = 1 + \frac{h}{f}$  maps  $\gamma$  to the interior of a circle centered at 1 with radius 1.



Since the curve cannot contain 0,  $Ind(g, 0) = 0$ . Then,

$$\begin{aligned}\frac{g'}{g} &= \frac{f' + h'}{f + h} - \frac{f'}{f} \\ \Rightarrow 0 &= \oint_{\gamma} \frac{g'}{g} dz \\ \Rightarrow &\oint_{\gamma} \frac{f' + h'}{f + h} dz - \oint_{\gamma} \frac{f'}{f} dz \\ \Rightarrow Ind(f + h, 0) &= Ind(f, 0)\end{aligned}$$

**Corollary.**

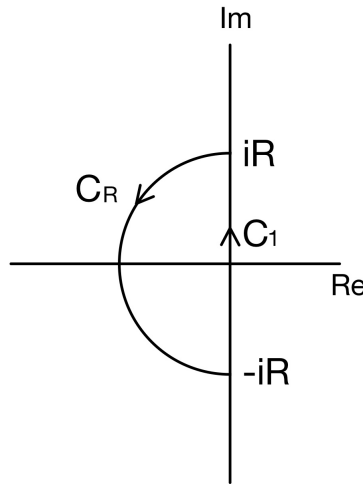
If  $f$  and  $g$  are also holomorphic inside and on  $\gamma$ , then

$$Z_{f,\gamma} = Z_{f+h,\gamma}$$

Rouche's Theorem gives us an easy way to locate roots.

**Example.** Locate the roots of  $\alpha(z) = z + 3 + 2e^z$ .

Let  $f = z + 3$  and  $h = 2e^z$  so that  $\alpha = f + h$ . For  $R$  large, we want  $|f| > |h|$  on  $C_R$  and  $C_1$ , a contour similar to one seen before.



If  $z \in C_1, z = iy$ .

$$\begin{aligned}|f(z)| &= |3 + iy| > 3 \\ |h(z)| &= 2|e^{iy}| = 2\end{aligned}$$

Hence,  $|h| < |f|$  on  $C_1$ . For  $z \in C_R, z = x + iy, |z| = R$  and  $x < 0$ ,

$$\begin{aligned}|f(z)| &> R - 3 \text{ for } R\text{-large} \\ |h(z)| &= 2|e^{x+iy}| = 2e^x < 2 \text{ because } x < 0 \\ \Rightarrow |h| &< |f| \text{ on } C \text{ and } f(z) = 3 + z \text{ has only one root for all } R \text{ large} \\ \Rightarrow \alpha &\text{ has only one root in } \Re(z) < 0.\end{aligned}$$

**Example.** Find how many roots  $2z^{10} + 4z^2 + 1$  has in  $|z| < 1$ .

On  $|z| = 1$ , let  $f = 4z^2$  and  $h = 2z^{10} + 1$ ,  $|f| > |h|$ . Therefore,  $f + h = 2z^{10} + 4z^2 + 1$  have exactly two roots in  $|z| < 1$ .

**Example.** Let  $\frac{1}{3}e^z - z = 0$ . Does it have a solution in  $|z| < 1$ ?

Observe  $|z| > \left|\frac{1}{3}e^z\right|$  for  $|z| = 1$  and  $f(z) = -z$  has one root in  $|z| < 1$ . Therefore,  $f + \left|\frac{1}{3}e^z\right|$  has exactly one root in  $|z| < 1$ .

**Example.** Locate the roots of  $\alpha(z) = z^7 - 5z^3 + 12$ .

On  $|z| = 1$ , let  $f = 12$ ,  $h = z^7 - 5z^3$ .  $|f| = 12$  and  $|h| \leq |1| + |5| = 6 < 12$ . Since  $f$  does not have a root on  $|z| < 1$ ,  $f + h = \alpha$  has no roots in  $|z| < 1$ .

On  $|z| = 2$ , let  $f = z^7$ ,  $h = -5z^3 + 12$ .  $|f| = 2^7 = 128$ ,  $|h| \leq 5|z|^3 + 12 = 52$ . So on  $f$  and  $f + h$  have the same number of roots on  $|z| < 2$ , which is 7.

Therefore, all of  $\alpha$ 's 7 roots are in  $1 < |z| < 2$ .

### Corollary: Fundamental Theorem of Algebra.

Let  $\alpha(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$  and  $f(z) = z^n$ .

Let  $h(z) = \alpha(z) - f(z) \Rightarrow f + h = \alpha$

Let  $R = \max\{1, n|a_{n-1}|, \cdots, n|a_0|\} + 1$ , then on  $|z| = R$

$$\begin{aligned} |h| &\leq |a_{n-1}|R^{n-1} + |a_{n-2}|R^{n-2} + \cdots + |a_0| \\ &\leq \frac{R}{n}R^{n-1} + \frac{R}{n}R^{n-2} + \cdots + |a_0| \\ &\leq \underbrace{\frac{R^n}{n} + \frac{R^n}{n} + \cdots + \frac{R^n}{n}}_{n\text{-terms}} \\ &= R^n \end{aligned}$$

On  $|z| = R$ ,  $|f(z)| = R^n \Rightarrow |h| < |f|$  for  $|z| = R$ .

Then  $f + h$  and  $f$  have the same number of zeros on  $|z| < R$ ,  $n$ , by Rouché's Theorem. As  $R \rightarrow \infty$ , we have exactly  $n$  roots in  $\mathbb{C}$ .