

Introduction to Complex Analysis

Qitian Liao

Aug 3, 2020

Contents

1	Algebra of the Complex Plane	2
1.1	Introduction to Complex Numbers	2
1.2	Conjugate of Complex Numbers	2
1.3	Modulus of Complex Numbers.	3
1.4	Complex Polynomial	4
2	Geometry of the Complex Plane	5
2.1	Properties of Polar Forms	5
2.2	Definition of Argument and argument	5
2.3	Euler's Formula	5
2.4	Geometric Understanding of Multiplication	6
3	Stereographic Projections, Exponentials and Logs	7
3.1	Stereographic Projections	7
3.2	Complex Logarithm	7
4	Topology in \mathbb{C}	9
4.1	Sequence	9
4.2	Complex Set	9
5	Continuity and Branch Cuts	11
5.1	Complex Continuity	11
5.2	Complex Limits	11
5.3	Branch Cuts	11
6	Differentiability in \mathbb{C}	13
6.1	Difference between \mathbb{R} and \mathbb{C} differentiability	13
6.2	Properties of $f'(z)$	14
6.3	Geometric behavior of $f'(z)$	14
7	The Cauchy Riemann Equations	16
7.1	The Cauchy Riemann Equations	16
7.2	Cauchy Riemann with Logarithm	17
7.3	Lack of Complex Mean Value Theorem	18
7.4	Wirtinger Equations	18
8	Harmonic Functions	19
8.1	Laplacian	19
8.2	Harmonic Functions	19
9	Conformal Maps	20
9.1	Preservation of Angles	20
9.2	Conformal Function	21
9.3	Conformal Map	21

10 Bilinear Transformations	22
10.1 Definition of a Möbius transformation	22
10.2 Brief Review of other Transformations	22
10.3 Möbius transforming of a function	23
11 Contour Integral in \mathbb{C}	25
11.1 Piecewise Differentiable, Smooth, Simple, Closed curves	25
11.2 Interior and Exterior of curves	25
11.3 Smoothly Equivalent	25
11.4 Line Integral	26
12 Cauchy's Closed Curve Theorem and the Fundamental Theorem of Calculus	28
12.1 Cauchy's Closed Curve Theorem	28
12.2 Fundamental Theorem of Calculus (F.T.C)	29
13 Cauchy's Integral Formula	32
14 Growth Conditions of Holomorphic Functions	35
14.1 Maximum/Minimum Modulus Principles	35
14.2 Mapping and \mathbb{C} -differentiability	36
15 Convergence of Infinite Series in \mathbb{C}	40
16 Power Series in \mathbb{C}	41
17 Series Expansion of Holomorphic Functions	41
18 Open Mapping Theorem and Reflection Principle	41
19 Laurent Series	41
20 Residue Theorem	41
21 Improper Integrals	41
22 Argument Principle and Rouché's Theorem	41

1 Algebra of the Complex Plane

1.1 Introduction to Complex Numbers

Let $z = a + ib \in \mathbb{C}$ where $a, b \in \mathbb{R}$ and $i^2 = -1$.

This number can be thought of as a point in 2-space, \mathbb{R}^2 , (a, b) or as a position in \mathbb{C} .

\mathbb{R}^2 : \oplus addition; \odot scalar multiplication.

\mathbb{C} : \oplus addition; \odot scalar multiplication; a vector space; have multiplication of elements, \mathbb{C} is a field.

If $z = a + ib$, $w = c + id$, then $zw = (ac - bd) + i(ad + cb)$

$$zw = wz$$

$$z(w + \alpha) = zw + z\alpha$$

$$(zw)\alpha = z(w\alpha)$$

1.2 Conjugate of Complex Numbers

Definition of Conjugate.

The complex conjugate of z , \bar{z} , is defined by

$$\bar{z} = a - ib$$

Geometric representation: The image of \bar{z} is the reflection of z about the Real axis.

Properties of Conjugate.

$$\overline{\bar{z}} = z$$

$$\overline{zw} = \bar{z}\bar{w}$$

$$\overline{z + w} = \bar{z} + \bar{w}$$

$$\bar{z} = z \text{ if and only if } z \in \mathbb{R}$$

Real and Imaginary Parts.

We can project z onto the Real or Imaginary axis and measure its distance from 0:

$$\Re(z) = a$$

$$\Im(z) = b, \text{ not } ib$$

Each function is a map $\mathbb{C} \rightarrow \mathbb{R}$. Then

$$\Re(z) = \frac{z + \bar{z}}{2}$$

$$\Im(z) = \frac{z - \bar{z}}{2i}$$

This is similar to the pattern with even/odd functions.

1.3 Modulus of Complex Numbers.

$$z\bar{z} = (a + ib)(a - ib) = a^2 + b^2 \in \mathbb{R}$$

Definition of Modulus.

$|z|$ length/modulus of z is defined by:

$$|z| = (a^2 + b^2)^{\frac{1}{2}} = (z\bar{z})^{\frac{1}{2}} \in \mathbb{R}$$

Properties of Modulus.

$$|zw| = |z||w|$$

$$|z| = |\bar{z}|$$

$$|z| \geq 0$$

$$|z| = 0 \text{ if and only if } z = 0$$

Triangle Inequality and Reverse Triangle Inequality.

$$\begin{cases} |z + w| \leq |z| + |w| \\ |z| - |w| \leq |z - w| \end{cases}$$

$$z = z - w + w \Rightarrow |z| = |z - w + w| \Rightarrow |z| \leq |z - w| + |w| \Rightarrow |z| - |w| \leq |z - w|$$

Complex Division.

With $z\bar{z} \in \mathbb{R}$, we can define complex division by reducing it to a multiplication problem.

$$\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} = \frac{1}{w\bar{w}}(z\bar{w})$$

We also have

$$\left| \frac{z}{w} \right| = \frac{|z|}{|w|} \text{ for } w \neq 0$$

Distance in the plane.

A disk in the complex plane centered at c of radius $r \in \mathbb{R}$ is of the form

$$\{z \in \mathbb{C} \mid |z - c| \leq r\}$$

1.4 Complex Polynomial

A complex polynomial $p(z)$ of degree n is of the form:

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

where $a_n \neq 0$ and $a_i \in \mathbb{C}$ for $i = 0, \dots, n$

Fundamental Theorem of Algebra.

The factorization of $p(z)$ factors over \mathbb{C} is unique,

$$p(z) = c(z - z_1)^{m_1} \cdots (z - z_k)^{m_k}$$

We have roots $z_i \in \mathbb{C}$ of $p(z)$ with order $m_i \in \mathbb{N}$.

For example, if $p(z) = z^2 + 4 = (z + 2i)(z - 2i)$, then it factors over \mathbb{C} but not \mathbb{R} .

Note: \mathbb{C} is an algebraically closed field, there are no irreducible polynomials in \mathbb{C} .

Note: $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$ are not algebraically closed.

2 Geometry of the Complex Plane

2.1 Properties of Polar Forms

Complex numbers can be represented in polar forms:

$$z = |z|(\cos \theta + i \sin \theta)$$

with modulus $|z|$ and argument θ . To change between the coordinate systems it follows:

$$|z| = (a^2 + b^2)^{\frac{1}{2}}$$

$$\tan \theta = \frac{b}{a}$$

$$a = |z| \cos \theta = \Re(z)$$

$$b = |z| \sin \theta = \Im(z)$$

Note that $\theta_R = \arctan(\frac{b}{a})$ is a reference angle of z . To find θ from θ_R , you need to consider the signs of a and b .

Example.

$$z = -3 + 3i = 3\sqrt{2}(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4})$$

$$\theta_R = \arctan(\frac{3}{-3}) = -\frac{\pi}{4}$$

$$\theta = \pi + \theta_R = \pi - \frac{\pi}{4} = \frac{3\pi}{4}, \text{ since } \theta \text{ is in } II.$$

2.2 Definition of Argument and argument

$\text{Arg}(z)$ is z 's principle polar angle θ , $z \neq 0$, where $\theta \in (-\pi, \pi]$.

$\arg(z)$ is all of z 's polar angles, $\theta + 2k\pi$, $k \in \mathbb{Z}$.

2.3 Euler's Formula

Euler's Formula is defined as a linear combination of $\cos \theta$ and $\sin \theta$, \mathbb{R} -valued functions.

$$e^{i\theta} = \cos \theta + i \sin \theta$$

It allows us to express z in polar form by

$$z = |z|e^{i\theta}$$

-1 has polar angle π and modulus 1,

$$-1 = e^{i\pi} \text{ or } e^{i\pi} + 1 = 0$$

By the angle addition formulas from trigonometry we find:

$$e^{i\theta}e^{i\varphi} = e^{i(\theta+\varphi)}$$

$$(e^{i\theta})^k = e^{i\theta k}$$

2.4 Geometric Understanding of Multiplication

The polar angle of zw is the sum of the polar angles of z and w . The modulus is the product of the moduli.

$$\text{Arg}(zw) = \text{Arg}(z) + \text{Arg}(w)$$

$$\text{Arg}(\bar{z}) = -\text{Arg}(z)$$

Question: How about $\frac{z}{w}$ and z^4 ?

It follows from trigonometry that $|e^{i\theta}| = 1$, if $\theta \in (-\pi, \pi]$ we get a parametrization of the unit circle.

Example. Discover all solutions to $w^3 = i = z$

Let $p(z) = w^3 - i$. By Fundamental Theorem of Algebra, there are 3 roots of $p(z)$.

Therefore, $3\theta = \frac{\pi}{2} + 2\pi k$, $k \in \mathbb{Z}$

This gives us infinitely many solutions, but the solutions form 3 equivalence classes.

All we need is $k = 0, 1, 2$, which gives $\theta_1 = \frac{\pi}{6}$, $\theta_2 = \frac{5\pi}{6}$, $\theta_3 = \frac{3\pi}{2}$

Our solutions partitioned the unit circle into 3 equally spaced wedges.

The solutions to $w^3 = i$ are $w_1 = \frac{\sqrt{3}}{2} + \frac{1}{2}i$, $w_2 = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$ and $w_3 = -i$.

This problem of unity can be extended to solving $w^k = z$ for $k \in \mathbb{N}$, $z \in \mathbb{C}$ for unknown k-solutions w .

3 Stereographic Projections, Exponentials and Logs

3.1 Stereographic Projections

We can express the complex plane on the unit sphere in \mathbb{R}^3 . To perform this we project points on the surface of the sphere along the line from the North Pole $(0, 0, 1)$ through the point and onto the plane $z = 0, \mathbb{C}$

$$p_1 = (x_1, x_2, x_3) \rightarrow z = a + ib = \frac{x_1 + ix_2}{1 - x_3}$$

$$x_1 = \frac{2a}{|z|^2 + 1}$$

$$x_2 = \frac{2b}{|z|^2 + 1}$$

$$x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}$$

Points in the northern hemisphere P_1 , have $|z_1| > 1$.

Points in the southern hemisphere P_2 , have $|z_2| < 1$.

Mapping from Stereographic Space to the Complex Plane.

$$\mathbb{S}^2 \rightarrow \mathbb{C}$$

$$N = (0, 0, 1) \rightarrow \infty$$

$$S = (0, 0, -1) \rightarrow 0$$

$$\text{lines of latitude} \rightarrow |z| = r, \text{ circles}$$

$$\text{lines of longitude} \rightarrow \text{Arg}(z) = \pm\theta, \text{ lines through } (0, 0)$$

Note that in general, circles on \mathbb{S}^2 map to circles and lines in \mathbb{C} , orientation is not always preserved.

3.2 Complex Logarithm

Logarithm of Real Numbers.

Anytime we are dealing with power, the log function is very useful.

$$\log x = \int_1^x \frac{1}{t} dt \text{ for } x \in \mathbb{R}$$

$$\frac{d}{dx} x^x = \frac{d}{dx} e^{\ln x^x} = \frac{d}{dx} e^{x \ln x} = e^{x \ln x} (x \cdot \frac{1}{x} + \ln x) = x^x (1 + \ln x)$$

Logarithm of Complex Numbers.

Remember from Euler's Formula, $e^{i\theta} = \cos \theta + i \sin \theta$.

$$e^z = e^{a+ib} = e^a e^{ib}$$

$$\text{Arg}(e^z) = b$$

$$|e^z| = e^a > 0$$

Therefore, if a is held fixed, e^z maps to a circle as b changes.

On the other hand, if b is held fixed, e^z maps to a line through $(0, 0)$.

Derivation of Complex Logarithm.

We want $e^{\log(z)} = z$ for all $z \neq 0$, and thus

$$e^{\Re(\log(z)) + i\Im(\log(z))} = e^{\Re(\log(z))} e^{i\Im(\log(z))} = |z| e^{i\theta} = z$$

$$\Rightarrow |z| = e^{\Re(\log(z))}$$

$$\Rightarrow \Re(\log(z)) = \log |z|$$

From the imaginary part we find

$$e^{i\theta} = e^{i\Im(\log(z))}$$

$$\Rightarrow \arg(z) = \theta = \Im(\log(z))$$

$$\Rightarrow \Im(\log(z)) = \text{Arg}(z)$$

because $\arg(z)$ is not well defined.

Our constructed inverse of e^z is a multi-valued function

$$\log(z) = \log |z| + i \arg(z)$$

Conclusion from Derivation.

$$\log(z) = \log |z| + i \arg(z)$$

$$\text{Log}(z) = \log |z| + i \text{Arg}(z)$$

Note: $\text{Log}(z)$ does not have all the nice behavior as \mathbb{R} -valued $\log(x)$: $\text{Log}(z^k)$.

Sometimes they are co-terminal angles, but they are not equal. See the following example:

$$\begin{cases} \text{Log}(i^3) = \text{Log}(-i) = -i\frac{\pi}{2} \\ 3\text{Log}(i) = 3 \cdot (i\frac{\pi}{2}) = i\frac{3\pi}{2} \end{cases}$$

Example. Compute 3^i :

$$3^i = e^{\text{Log } 3^i} = e^{i \text{Log } 3} = \cos(\text{Log } 3) + i \sin(\text{Log } 3)$$

How Logarithm acts on curves.

$$\begin{cases} \text{Maps a circle with radius } r \text{ to a vertical line passing through } (\ln(r), 0) \\ \text{Maps a line with angle } \theta \text{ passing through the origin to a horizontal line passing through } (0, i\theta) \end{cases}$$

4 Topology in \mathbb{C}

4.1 Sequence

Let $\{Z_n\}$ be a sequence in \mathbb{C} .

Cauchy Sequence.

The sequence is Cauchy if for all $\epsilon > 0$, there is a $N \in \mathbb{N}$ such that for all $n, m > N$, $|z_n - z_m| < \epsilon$.

Convergence of Sequence.

The sequence converges if $|z_n - z| \rightarrow 0$ as $n \rightarrow \infty$. The distance between z_n and z vanishes.

Completeness of \mathbb{C} .

$\{z_n\}$ converges if and only if $\{z_n\}$ is Cauchy.

Proof. We show this by treating \mathbb{C} as \mathbb{R}^2 and exploiting $\{X_n\}$ converges if and only if $\{X_n\}$ is Cauchy. (\implies) If $z_n \rightarrow z$, then $\Re(z_n) \rightarrow \Re(z)$ and $\Im(z_n) \rightarrow \Im(z)$. Since the sequences of \mathbb{R}^2 converge, they are Cauchy.

$$|Z_n - Z_m| \leq |\Re(Z_n - Z_m)| + |\Im(Z_n - Z_m)| = |\Re(Z_n) - \Re(Z_m)| + |\Im(Z_n) - \Im(Z_m)|$$

Upper bounds can be picked to be less than $\frac{\epsilon}{2}$ for some N . Therefore, $|Z_n - Z_m| \rightarrow 0$.

(\impliedby) If $\{Z_n\}$ is Cauchy, so are $\{\Re(Z_n)\}$ and $\{\Im(Z_n)\}$. But these are \mathbb{R} -sequences that converge. Therefore, $\{Z_n\}$ converges.

4.2 Complex Set

Let $\Omega \subset \mathbb{C}$. Sets can be open, closed, both, or neither.

Open Set.

If for any $z_0 \in \Omega$, there exist some $\epsilon > 0$, such that the set $B_\epsilon(z_0) = \{z \mid |z - z_0| < \epsilon\}$ is contained in Ω , then Ω is open.

Ω is open if and only if Ω^c is closed.

Ω is open if and only if Ω is equal to its own interior, which means it does not contain its boundary points $\partial\Omega$, i.e. it does not contain its closure.

Closed Set.

If Ω contains its limit point, then Ω is closed.

Ω is closed if and only if Ω^c is open.

Ω is closed if and only if Ω contains its boundary points.

Compact Set.

If Ω can be contained in a disk of finite radius, then Ω is bounded.

Compact Set.

If Ω is closed and bounded, then Ω is compact. This resembles $[a, b]$ in \mathbb{R} .

Connected Set.

If any two points in Ω can be connected by a path, then Ω is connected.

Simply Connected Set: A simply connected set has no "holes" in it. For example, $\Omega = \{z \mid |z - c| < 4\}$.

A connected but not simply connected set is an annulus, $\Omega = \{z \mid 2 < |z - c| < 4\}$

Boundary of Set.

The boundary of Ω , $\partial\Omega$ is all points with ϵ -balls intersecting Ω and Ω^c for all $\epsilon > 0$.

Interior of Set.

The interior of Ω , $\text{Int}(\Omega)$, is all points in Ω with a ϵ -ball contained in Ω for some $\epsilon > 0$. "Largest open set in Ω ".

Closure of Set.

The closure of Ω is the union of Ω and its boundary $\partial\Omega$.

Domain.

If a set is open and connected in \mathbb{C} , it is a domain.

A domain can be traversed by a path of horizontal and vertical line segments.

Example.

Determine whether the following sets are open or closed.

1. $\Omega = \mathbb{C} \setminus \{0\}$

Ω is open since it does not contain its closure, the point 0.

Ω is not closed since it does not contain its limit points. Let $z_n = \frac{1}{n}$. Then $z_n = \frac{1}{n} \rightarrow 0 \notin \Omega$.

Therefore, Ω is open.

2. $\Omega = \{z \mid |z| \geq 1\}$

Ω is not open since any ϵ -ball at 1 intersects Ω^c .

Ω is closed since Ω^c is open.

Therefore, Ω is closed.

3. $\Omega = \{z \mid |z| > 1\}$

Ω is open since Ω^c is closed.

Ω is not closed since it does not contain its limit points. Let $z_n = \frac{1}{n} + 1$. Then $z_n = \frac{1}{n} + 1 \rightarrow 1 \notin \Omega$.

4. $\Omega = \mathbb{C} \setminus (0, 1)$

Ω is not open. Its complement is $[0, 1]$. Even though it is closed in \mathbb{R} , it is not closed in \mathbb{C} , because any 2D ϵ -ball will always extend outside of the set $z \in (0, i)$. Hence, Ω^c is not open and not closed.

Ω is not closed since it does not contain its limit points. Let $z_n = \frac{1}{3} + i\frac{1}{n}$. Then $z_n = \frac{1}{3} + i\frac{1}{n} \rightarrow \frac{1}{3} \notin \Omega$.

Therefore, Ω is neither open nor closed.

5. $\Omega = \mathbb{C} \setminus [0, 1]$ Ω is open since $\Omega^c = [0, 1]$ is closed in \mathbb{C} .

Ω is not closed since it does not contain its limit points. Let $z_n = \frac{1}{3} + i\frac{1}{n}$. Then $z_n = \frac{1}{3} + i\frac{1}{n} \rightarrow \frac{1}{3} \notin \Omega$.

Therefore, Ω is open.

Note: Ω^c is not open in \mathbb{C} .

5 Continuity and Branch Cuts

5.1 Complex Continuity

Let $f : \Omega \rightarrow \mathbb{C}$, Ω is open and connected. If $z_n \rightarrow z_0$ implies $f(z_n) \rightarrow f(z_0)$, then f is continuous at z_0 . Also, f is bounded near z_0 .

f is continuous if for every $\epsilon > 0$, there is $\delta > 0$ such that $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon$.

In either case, $\Re(f(z))$ and $\Im(f(z))$ are each continuous if and only if $f(z)$ is continuous. This follows the pattern as \mathbb{C} being complete.

If f and g are continuous, then so are $f + g$, $f \times g$ and $\frac{f}{g}$ (provided $g(z) \neq 0$)

5.2 Complex Limits

Just like in \mathbb{R}^2 , limits are direction independent. Do not restrict limits to just $\Re \rightarrow 0$ or $\Im \rightarrow 0$. See the following example.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2y}{x^4 + y^2} \text{ does not exist}$$

as $x \rightarrow 0, y = 0$, then $f \rightarrow 0$, while $y = x^2, x \rightarrow 0$, then $f \rightarrow 1$.

5.3 Branch Cuts

$\text{Log}, z^{\frac{1}{2}}$ and $\arctan(z)$ are constructed by restricting the range of e^z, z^2 and $\tan(z)$.

For example, in creating $\text{Log}(z) = \ln|z| + i \text{Arg}(z)$, we made a choice that $\text{Arg}(z) \in (-\pi, \pi]$, $\text{Arg}(0)$ does not exist.

Example. Consider a path around $z_0 \neq 0$, $\gamma(t) = z_0 + re^{it}$. $\theta(t) = \arg(\gamma(t))$.

As we traverse the circle, $t \in (-\pi, \pi]$,

$$\theta(t) = \arg(\gamma(t)) = \text{Arg}(z_0 + re^{it}) + 2\pi k = \text{Arg}(z_0 + re^{i(t+2\pi)}) + 2\pi k = \arg(\gamma(t+2\pi)) = \theta(t+2\pi)$$

Therefore, the angle $\theta(t)$ changes smoothly for all t and we stay on the same branch of $\text{Arg}(\gamma(t))$. That is to say, the $k \in \mathbb{Z}$ is the same for all t .

Compare this with any circular path about $z = 0$, γ_0 . Let $\gamma_0(t) = re^{it}$, $t \in (-\pi, \pi]$. As we traverse the circle once, we have a discontinuity in the principal angle of $\gamma_0(t)$. In particular, $\theta(\gamma_0(t)) \neq \theta(\gamma_0(t+2\pi))$

$$\theta(t) = \arg(\gamma(t)) = \text{Arg}(re^{it}) + 2\pi k \neq \text{Arg}(re^{i(t+2\pi)}) + 2\pi(k+1) = \arg(\gamma(t+2\pi)) = \theta(t+2\pi)$$

We jump from the k th to the $(k+1)$ th branch of Arg . Therefore, $\text{Arg}(z)$ has a branch point at $z = 0$.

Definition of Branch Cuts and Branch Points.

If every neighborhood of z_0 contains a path $\gamma(t)$ around z_0 that leads to a jump discontinuity in f , then z_0 is a branch point of $f(z)$.

In order to find branches, at this point, it suffices to study paths of the form $\gamma(t) = z_0 + re^{it}$ for $t \in (-\pi, \pi)$, and see if $f(\gamma(t)) = f(\gamma(t+2\pi))$ holds for all t .

Example. Arg is discontinuous for all x on the negative \mathbb{R} -axis, \mathbb{R}^- .

We call this the principal branch cut of the multi-valued function \arg . Specifically,

$$\begin{aligned}\operatorname{Arg}(\gamma_0(t)) &\rightarrow \pi \text{ as } t \rightarrow \pi^- \\ \operatorname{Arg}(\gamma_0(t)) &\rightarrow -\pi \text{ as } t \rightarrow -\pi^+\end{aligned}$$

but $\gamma_0(\pi) = \gamma_0(-\pi)$ since π and $-\pi$ are coterminal.

\mathbb{R}^- is the principal branch of Log , Arg , and $z^{\frac{1}{2}}$.

The endpoints of a branch cut are branch points, Arg has 0 and ∞ as its branch points.

6 Differentiability in \mathbb{C}

Let $f : \Omega \rightarrow \mathbb{C}$ for some domain Ω . Then f is differentiable at z_0 if the following exists.

$$\frac{d}{dz}f(z)|_{z=z_0} = f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$$

This limit must exist on all paths to z_0 , since $h \in \mathbb{C}$. We could also take $z_n \rightarrow z_0$ and use $\frac{f(z_0) - f(z_n)}{z_0 - z_n} \rightarrow f'(z_0)$. Remember limits are computed by looking at the difference in the modulus, $|\frac{f(z_0) - f(z_n)}{z_0 - z_n} - f'(z_0)| \rightarrow 0$ as $n \rightarrow \infty$.

If $f'(z_0)$ exists on all points $z_0 \in \Omega$, open and connected in \mathbb{C} , then f is holomorphic/ \mathbb{C} -differentiable/analytic on Ω . The connection between \mathbb{R} and \mathbb{C} analytic will be clear when we cover \mathbb{C} -power series.

If $f'(z)$ exists everywhere in \mathbb{C} , then f is an entire/meromorphic function.

6.1 Difference between \mathbb{R} and \mathbb{C} differentiability

- $f : \mathbb{R} \rightarrow \mathbb{R}$

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

has only two paths to x , namely $h \rightarrow 0^+$ and $h \rightarrow 0^-$.

Tangent plane or linear approximation:

$$f(x) \approx f(a) + f'(a)(x - a)$$

- $f : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$\lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h} = f_x$$

is also a 1D limit and a partial derivative.

Tangent plane or linear approximation:

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ $f(x, y) = (u(x, y), v(x, y))$

Then f is differentiable if the Jacobian Matrix

$$J(f) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

can approximate the local change in f .

In each of the cases above, we are only measuring change in a few directions. However, $h \rightarrow 0$ in \mathbb{C} can be from any direction in 2-space. Therefore, $f'(z)$ existing is a much stronger condition for f on \mathbb{C} than on \mathbb{R} .

Complications with \bar{z} .

Consider the following example:

Let $g(z) = \Re(z) = \frac{z+\bar{z}}{2}$, which is a linear combination of continuous functions.

Assume $h \in \mathbb{R}$,

$$\frac{g(z+ih) - g(z)}{h} = \frac{\Re(z) - \Re(z)}{h} \rightarrow 0 \text{ as } h \rightarrow 0$$

Compare this with

$$\frac{g(z+ih) - g(z)}{h} = \frac{\Re(z) + h - \Re(z)}{h} = 1 \rightarrow 1 \text{ as } h \rightarrow 0$$

Therefore, the function is nowhere differentiable in \mathbb{C} . The problem with $g'(z)$ had to do with \bar{z} , despite reflection in \mathbb{R}^2 about $y = 0$ is differentiable. We will discover conditions on u_x, u_y, v_x , and v_y that ensure f' exists for $f(z) = u(x, y) + iv(x, y)$ in the next chapter.

6.2 Properties of $f'(z)$

Proposition. If f is differentiable on Ω , it is continuous on Ω .

Proposition. The power rule holds too: $\frac{d}{dz} z^n = n z^{n-1}$

As does the product, quotient, L'Hospital's and chain rule. In fact, most old results hold as well. In each of the following cases, there are branch points where $f'(z)$ does not exist.

1. If $f(z) = \text{Log}(z)$, then $f'(z) = \frac{1}{z}$
2. If $f(z) = \tan^{-1}(z)$, then $f'(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$, which does not exist for $z = \pm i$
3. If $f(z) = z^{\frac{1}{2}}$, then $f'(z) = \frac{1}{2} z^{-\frac{1}{2}}$

Proposition. Suppose $f(z)$ is holomorphic on Ω . then $g(z) = \overline{f(\bar{z})}$ is holomorphic on $\Omega^* = \{z | \bar{z} \in \Omega\}$

Proof. Suppose f is holomorphic on Ω . Let $z_0 \in \Omega^*$ and $z_n \in \Omega^*$ for all n and $z_n \rightarrow z_0$. Then $\bar{z}_n \rightarrow \bar{z}_0$ in Ω and for $\epsilon > 0$, there is a $N \in \mathbb{N}$ such that for $n > N$.

$$\begin{aligned} \left| \frac{f(\bar{z}_0) - f(\bar{z}_n)}{\bar{z}_0 - \bar{z}_n} - f'(\bar{z}_0) \right| &< \epsilon \\ \left| \frac{f(\bar{z}_0) - f(\bar{z}_n)}{\bar{z}_0 - \bar{z}_n} - f'(\bar{z}_0) \right| &= \left| \overline{\frac{f(\bar{z}_0) - f(\bar{z}_n)}{\bar{z}_0 - \bar{z}_n} - f'(\bar{z}_0)} \right| = \left| \frac{\overline{f(\bar{z}_0) - f(\bar{z}_n)}}{\overline{\bar{z}_0 - \bar{z}_n}} - \overline{f'(\bar{z}_0)} \right| \\ &= \left| \frac{\overline{f(\bar{z}_0) - f(\bar{z}_n)}}{z_0 - z_n} - \overline{f'(\bar{z}_0)} \right| = \left| \frac{g(z_0) - g(z_n)}{z_0 - z_n} - \overline{f'(\bar{z}_0)} \right| < \epsilon \\ &\implies g'(z_0) = \overline{f'(\bar{z}_0)} \end{aligned}$$

Therefore, g is holomorphic on Ω^* .

This proof is different from when we showed $\frac{d}{dz} \bar{z}$ does not exist. Conjugation must be handled with care.

6.3 Geometric behavior of $f'(z)$

Dilation.

$w = f(z) \approx f'(a)(z - a) + f(a)$. Small changes in z should give small changes in w .

The functions $|z|$ and $\text{Arg}(z)$ are continuous on their domains.

If $f'(z_0) \neq 0$ and f' exists on Ω , then

$$\begin{aligned} |f'(z_0)| &= \left| \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} \right| = \lim_{h \rightarrow 0} \left| \frac{f(z_0 + h) - f(z_0)}{h} \right| = \lim_{h \rightarrow 0} \frac{|f(z_0 + h) - f(z_0)|}{|h|} \\ &\implies |f'(z_0)||h| \approx |f(z_0 + h) - f(z_0)| \end{aligned}$$

The size of $|f'(z_0)|$ tells us how much f is contracting/dilating near z_0 .

Rotation.

$$\text{Arg}(f'(z_0)) = \text{Arg} \left(\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} \right) = \lim_{h \rightarrow 0} \text{Arg} \left(\frac{f(z_0 + h) - f(z_0)}{h} \right)$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \text{Arg}(f(z_0 + h) - f(z_0)) - \text{Arg}(h) \\
&\implies \text{Arg}(f'(z_0)) \approx \text{Arg}(f(z_0 + h) - f(z_0)) \\
&\implies \text{Arg}(f'(z_0)) + \text{Arg}(h) \approx \text{Arg}(f(z_0 + h) - f(z_0))
\end{aligned}$$

Therefore, f rotates vectors from z_0 to $z_0 + h$ by the angle $\text{Arg}(f'(z_0))$.

Conclusion.

In conclusion, $w = f(z) \approx f(z_0) + f'(z_0)(z - z_0) = c + \rho e^{i\theta}(z - z_0)$

$$\begin{cases} c: \text{Translation} \\ \rho: \text{Dilation} \\ e^{i\theta}: \text{rotation about } z_0 \text{ or complex multiplication.} \end{cases}$$

7 The Cauchy Riemann Equations

7.1 The Cauchy Riemann Equations

Let $f(z) = f(x+iy) = u(x, y) + iv(x, y)$, then $f(z)$ is holomorphic implies the Cauchy Riemann Equations:

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

Proof. For f , which is \mathbb{C} -differentiable, the following representation of $f'(z)$ holds for any path to z .

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$$

Along the path $x+h+iy \rightarrow x+iy$, we get

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \rightarrow 0} \frac{f(x+h+iy) - f(x+iy)}{h} = f_x = u_x(x, y) + iv_x(x, y)$$

Along the path $x+iy+ih \rightarrow x+iy$, we get

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \rightarrow 0} \frac{f(x+i(y+h)) - f(x+iy)}{ih} = \frac{f_y}{i} = -if_y = -i(u_y(x, y) + iv_y(x, y)) = v_y(x, y) - iu_y(x, y)$$

Equate components in $f_x = -if_y$, and it is proven that $u_x = v_y$ and $u_y = -v_x$.

Proposition. If the Cauchy Riemann Equations do not hold at z_0 , then $f'(z_0)$ does not exist.

Proposition. If f is holomorphic on a domain Ω , an open and connected set in \mathbb{C} , then the Cauchy Riemann Equations hold at all points in Ω .

Example.

If $f(x+iy) = x^2 + iy^2$, $u_x = 2x$, $v_x = 0$, $u_y = 0$, $v_y = 2y$. Then $2x = 2y \Rightarrow x = y$, which is a line.

The set of points on the line is not open in \mathbb{C} . Therefore, f is nowhere holomorphic in \mathbb{C} . However, we will see that $f'(z)$ does exist on the line $y = x$.

Sufficiency of the Cauchy Riemann Equations to f' .

The Cauchy Riemann Equations do a great job showing f' does not exist. But what about it being sufficient for f' ? We claim that satisfying the Cauchy Riemann Equations at z_0 implies that f' exists at z_0 .

Proof.

f is \mathbb{C} -differentiable at z_0 if and only if $u(x, y)$ and $v(x, y)$ have continuous partial derivatives that satisfy the Cauchy Riemann Equations at z_0 . This requires us to treat $f(x+iy)$ as a function on \mathbb{R}^2 , or $f(z)$ induces a map on \mathbb{R}^2 .

Let $h = \Delta x + i\Delta y$,

$$\frac{f(z+h) - f(z)}{h} = \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)}{\Delta x + i\Delta y} - \frac{u(x, y) + iv(x, y)}{\Delta x + i\Delta y}$$

$$u(x+\Delta x, y+\Delta y) - u(x, y) = u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y) + u(x, y+\Delta y) - u(x, y)$$

The function $u(\cdot, \cdot)$ is differentiable in x and y , we can use the M.V.T (Mean Value Theorem) from \mathbb{R} to rewrite our difference in u by

$$u(x + \Delta x, y + \Delta y) - u(x, y + \Delta y) = \Delta x U_x(\underline{x}, y + \Delta y)$$

where $\underline{x} \in (x, x + \Delta x)$.

If u_x is continuous, $u_x(\underline{x}, y + \Delta y) \approx u_x(x, y) + \epsilon_1$, and as $\Delta y \rightarrow 0$ and $\underline{x} \rightarrow x$, by Taylor approximation and linear approximation on u_x , we have the error function $\epsilon_1 \rightarrow 0$.

Next $u(x, y + \Delta y) - u(x, y) = \Delta y u_y(x, \bar{y})$ and $u_y(x, \bar{y}) \approx u_y(x, y) + \epsilon_2$.

Likewise, for the function $v(x, y)$, we get a v_x and v_y with error terms ϵ_3 and ϵ_4 . s

$$\frac{f(z + h) - f(z)}{h} = \frac{\Delta x(u_x + \epsilon_1 + iv_x + i\epsilon_3) + \Delta y(u_y + \epsilon_2 + iv_y + i\epsilon_4)}{\Delta x + i\Delta y}$$

From the Cauchy Riemann Equations, we get $f_x = \frac{f_y}{i} \Rightarrow if_x = f_y \Rightarrow i(u_x + iv_x) = u_y + iv_y$. Substituting the terms, we have

$$f'(z) = \frac{\Delta x(u_x + iv_x) + i\Delta y(u_x + iv_x)}{\Delta x + i\Delta y} + \frac{\lambda}{\Delta x + i\Delta y}$$

where $\lambda = \Delta x(\epsilon_1 + i\epsilon_2) + \Delta y(\epsilon_3 + i\epsilon_4)$. However,

$$\left| \frac{\lambda}{\Delta x + i\Delta y} \right| \leq \left| \frac{\Delta x(\epsilon_1 + i\epsilon_2)}{\Delta x + i\Delta y} \right| + \left| \frac{\Delta y(\epsilon_3 + i\epsilon_4)}{\Delta x + i\Delta y} \right| \leq |\epsilon_1 + i\epsilon_2| + |\epsilon_3 + i\epsilon_4|$$

because $\left| \frac{\Delta x}{\Delta x + i\Delta y} \right| \leq 1$.

As $\Delta z \rightarrow 0$, $\left| \frac{\lambda}{\Delta x + i\Delta y} \right| \rightarrow 0$, and thus $f'(z) = u_x + iv_x = f_x = \frac{f_y}{i}$.

Therefore, the Cauchy Riemann Equations are an easy way to show $f'(z)$ exists and they provide a set of partial differential equations that f must satisfy.

Example. Let $f(z) = e^z = e^x(\cos(y) + i \sin(y))$

$$u = e^x \cos(y), v = e^x \sin(y)$$

$$u_x = e^x \cos(y), v_x = e^x \sin(y)$$

$$u_y = -e^x \sin(y), v_y = e^x \cos(y)$$

Therefore, $f(z)$ is \mathbb{C} -differentiable on \mathbb{C} , f is entire/meromorphic. $f'(z) = f_x = u_x + iv_x = f(z)$.

7.2 Cauchy Riemann with Logarithm

$$e^{\text{Log}(z)} = z \Rightarrow \frac{d}{dz} e^{\text{Log}(z)} = 1 \Rightarrow z \frac{d}{dz} \text{Log}(z) = 1 \Rightarrow \frac{d}{dz} \text{Log}(z) = \frac{1}{z}$$

We have a branch point in $\text{Log}(z)$ where its derivative is undefined. Then $\text{Log}(z)$ is \mathbb{C} -differentiable on $\mathbb{C} \setminus \{0\}$. This is true regardless of the branch cut on $\text{Log}(z)$.

7.3 Lack of Complex Mean Value Theorem

Claim: $\frac{f(z)-f(w)}{z-w} \neq f'(c)$ for some c between z and w .

Proof: Let $z = 1$, $w = 0$ and $f(t) = e^{i\pi t}$, then $f(1) - f(0) = e^{i\pi} - 1 = -2$. However, $|f'(t)| = \pi$ for all $t \in [0, 1]$.

Follow-Up Question: Does the lack of a Mean Value Theorem for $f'(z)$ suggest $f'(z) = 0$ not imply f is constant?

Answer: Suppose f is \mathbb{C} -differentiable on Ω and one of the following holds, then f is constant on Ω .

$$\begin{cases} f'(z) = 0 \\ |f(z)| \text{ is constant} \\ \operatorname{Re}(f(z)) \text{ is constant} \\ f\text{'s conjugate is } \mathbb{C}\text{-differentiable on } \Omega \end{cases}$$

7.4 Wirtinger Equations

There is another way to study the Cauchy Riemann Equations by introducing two operators:

$$\begin{aligned} \frac{\partial f}{\partial z} &= f_z \text{ and } \frac{\partial f}{\partial \bar{z}} = f_{\bar{z}} \\ f(x, y) &\equiv f(x + iy) = u(x, y) + iv(x, y) \\ f(x, y) &= f(\Re(z), \Im(z)) = f\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) \end{aligned}$$

From the chain rule, we get

$$\begin{aligned} f_z &= f_x x_z + f_y y_z = \frac{1}{2}f_x + \frac{1}{2i}f_y = \frac{1}{2}f_x - \frac{i}{2}f_y \\ f_{\bar{z}} &= f_x x_{\bar{z}} + f_y y_{\bar{z}} = \frac{1}{2}f_x - \frac{1}{2i}f_y = \frac{1}{2}f_x + \frac{i}{2}f_y \end{aligned}$$

where $f_x = u_x + iv_x$ and $f_y = u_y + iv_y$

These are the Wirtinger Equations.

$$\begin{cases} \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \end{cases}$$

Relationship with the Cauchy Riemann Equations.

From the Cauchy Riemann Equations $if_x = f_y$ we get:

$$\begin{aligned} f_{\bar{z}} &= \frac{1}{2}f_x + \frac{i}{2}f_y = \frac{1}{2}f_x - \frac{1}{2}f_x = 0 \\ f_z &= \frac{1}{2}f_x - \frac{i}{2}f_y = \frac{1}{2}f_x - \frac{i^2}{2}f_x = f_x = f'(z) \end{aligned}$$

f is \mathbb{C} -differentiable at z_0 if and only if $f(x, y) = u(x, y) + iv(x, y)$ is \mathbb{R} -differentiable at z_0 and $f_{\bar{z}}(z_0) = 0$. Then $f'(z_0) = f_z(z_0)$. In other words, $f'(z)$ does not depend on \bar{z} .

8 Harmonic Functions

8.1 Laplacian

Let $u : \mathbb{R}^2 \rightarrow \mathbb{R}$, then the Laplacian of u is

$$\Delta u = u_{xx} + u_{yy} = \nabla \cdot \nabla u$$

where $\nabla = [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}]^T$ is the divergence operator and $\nabla u = [u_x, u_y]^T$ is the gradient of u .

8.2 Harmonic Functions

If $\Delta u = 0$, then $u(x, y)$ satisfies Laplace's (partial differential) equation or u is a harmonic function.

This means:

$$\begin{cases} u \text{ is continuous} \\ u\text{'s 1}^{\text{st}} \text{ and 2}^{\text{nd}} \text{ order partial derivatives exist and are smooth.} \end{cases}$$

Proposition. Suppose $f = u + iv$ is holomorphic on Ω where $u(x, y)$ and $v(x, y)$ have continuous 2nd order partial derivatives, then u and v are harmonic and v is the harmonic conjugate of u .

Proof.

By the Cauchy Riemann Equations, $u_x = v_y$ and $v_x = -u_y$, then $u_{xx} = v_{yx}$ and $v_{xy} = -u_{yy}$. By continuity of v_{yx} and v_{xy} , $v_{yx} = v_{xy}$. This implies $u_{xx} = -u_{yy} \Rightarrow u_{xx} + u_{yy} = 0$

Later on, we will find that the conditions on 2nd order partial derivatives is implied by f being holomorphic on Ω , or f'' exists.

Definition of Harmonic Conjugate.

The harmonic conjugate to $u(x, y)$ is a function $v(x, y)$, such that $f(x, y) = u(x, y) + iv(x, y)$ is holomorphic.

Example. Show that $u(x, y) = x^3 - 3xy^2 + y$ is a harmonic function.

$$u_x = 3x^2 - 3y^2, u_y = -6xy + 1$$

$$u_{xx} = 6x, u_{yy} = -6x. \text{ Therefore, } u_{xx} + u_{yy} = 0$$

Example. Find the harmonic conjugate of $u(x, y) = x^3 - 3xy^2 + y$.

$$u_x = 3x^2 - 3y^2 = v_y$$

$$u_y = -6xy + 1 = -v_x$$

$$\Rightarrow v = 3x^2y - y^3 + C(x) \text{ or } v = 3x^2y - x + C(y)$$

Therefore, $v = 3x^2y - y^3 - x + C$ is u 's harmonic conjugate.

Proposition.

If u is harmonic on a domain Ω , then u_x is the real part of a holomorphic function on Ω . If Ω is simply connected, unlike $\mathbb{C} \setminus \{0\}$, then u is the real part of a holomorphic function on Ω .

Proof.

Assume u is harmonic and Ω is connected. If $f = u_x - iu_y$, then $f_y = if_x$. Hence, f is differentiable on Ω . The simply connected statement requires future theorems to show $F'(z) = f(z)$ for some holomorphic antiderivative $F(z)$.

9 Conformal Maps

Example. Let $f(z) = (x + iy)^2 + 2(x + iy) = (x^2 + 2x - y^2) + i2(xy + y)$. When are the component functions, $u(x, y)$ and $v(x, y)$ constant?

When are the component functions, $u(x, y)$ and $v(x, y)$, constant?

The function $f(z) = e^z = e^x(\cos(y) + i\sin(y))$ maps the set $\Omega = \{z : |\Im(z)| < \pi\}$ to circles of radius $r \in (-\infty, \infty)$, or all points in $\mathbb{C} \setminus \mathbb{R}^-$. This coincides with the branch cut of $\text{Log}(z)$, or how we made e^z invertible.

9.1 Preservation of Angles

We will now show e^z preserves the angles between curves in Ω . Let us first look at the following example.

Let $\gamma_1(t) = 2i\pi t - i\pi$, $\gamma_2(t) = t + i\frac{\pi}{4}$. $\gamma_1(0) = -i\pi$, $\gamma_1(1) = i\pi$.

The curves γ_1 and γ_2 intersect at an angle $\frac{\pi}{2}$. Also $f(\gamma_1)$ is a circle centered at 0 while $f(\gamma_2)$ is a line through $z = 0$. Their intersection in the w -plane is $\frac{\pi}{2}$ as well.

We will show why $f(z) = e^z$ does this by studying the angles between curves γ_1 and γ_2 and curves $\tau_1 = f(\gamma_1)$ and $\tau_2 = f(\gamma_2)$. If $\gamma(t)$ parameterizes a smooth curve in \mathbb{C} , then its tangent vector is $\gamma'(t)$. The angle between any two curves at z_0 is the angle between their tangent vectors at z_0 .

Assume the curves intersect at $\gamma(r_0) = \gamma(s_0) = z_0$.

Let the angle of intersection, θ , measured from γ'_1 to γ'_2 in the counter-clockwise direction.

Let the angle of intersection after transformation of f , φ , measured from τ'_1 to τ'_2 in the counter-clockwise direction. From past chapters, we know $\theta \approx \varphi$ if f is holomorphic. Now, let us assume f is only \mathbb{R} -differentiable and see how f acts on the angle θ .

Curve: $\gamma(t) = (x(t), y(t))$

New curve, f on γ : $\tau(t) = f(\gamma(t)) = u(\gamma(t)) + iv(\gamma(t)) = (\underline{X}(t), \underline{Y}(t))$

New tangent vector: $\tau'(t) = \frac{d}{dt}\tau(t) = (\underline{X}'(t), \underline{Y}'(t))$, where we can invoke the chain rule:

$$\underline{X}'(t) = u_x(\gamma(t))x'(t) + u_y(\gamma(t))y'(t)$$

$$\underline{Y}'(t) = v_x(\gamma(t))x'(t) + v_y(\gamma(t))y'(t)$$

If we have $f = u(x, y) + iv(x, y)$ is \mathbb{R} -differentiable, then

$$J(f) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

is the Jacobian Matrix of f and $\tau'(t) = \gamma'(t) \cdot J(f)^T$

If f is \mathbb{C} -differentiable and $\gamma(r_0) = z_0$ where $\gamma(t) = x(t) + iy(t)$, then $f'(z) = u_x + iv_x$ and $\gamma'(t) = x' + iy'$.

$$f'(z_0)\gamma'(r_0) = f'(\gamma(r_0))\gamma'(r_0) = (u_x + iv_x)(x' + iy') = (u_x x' - v_x y') + i(u_x y' + v_x x')$$

Applying the Cauchy Riemann Equations,

$$\begin{aligned} (u_x x' - v_x y') + i(u_x y' + v_x x') &= (u_x x' + u_y y') + i(v_x x' + v_y y') = (u_x x' + u_y y', v_x x' + v_y y') \text{ in } \mathbb{R}^2 \\ &= \gamma'(r_0) J(f(z_0))^T = \tau' \end{aligned}$$

By now, we have an understanding of how f acts on tangent vectors when $f' \neq 0$, namely $\theta = \varphi$.

9.2 Conformal Function

Conditions of Conformal Functions.

We say f is a conformal map at z_0 if the following hold:

- ① f is \mathbb{R}^2 -differentiable at z_0
- ② $|Jf| \neq 0$
- ③ f preserves the oriented angle θ , between γ_1 and γ_2 and τ_1 and τ_2 at z_0 and $f(z_0)$.

e^z is conformal.

Now let us take a closer look at e^z and figure out why it is conformal.

- ① holds apparently.
- ② $f(z) = e^z = e^x \cos(y) + ie^x \sin(y)$, then

$$J(f(x, y)) = \begin{bmatrix} e^x \cos(y) & -e^x \sin(y) \\ e^x \sin(y) & e^x \cos(y) \end{bmatrix} = e^x \begin{bmatrix} \cos(y) & -\sin(y) \\ \sin(y) & \cos(y) \end{bmatrix}$$

$$\Rightarrow |J(f(x, y))| \neq 0$$

- ③ Now we have shown in the previous part that Jf is the product of a dilation matrix, $e^x I$, and a rotation matrix, which means f preserves the angles between γ'_1 and γ'_2 . Their image under f :

$$\begin{aligned} \tau'_1(t) &= \gamma'_1(t) \cdot J(f)^T \\ \tau'_2(t) &= \gamma'_2(t) \cdot J(f)^T \end{aligned}$$

In fact, f preserving oriented angles implies Jf is a rotation \otimes dilation matrix.

Hence, it is proven that e^z is conformal.

9.3 Conformal Map

Definition. If f is conformal, infinitely differentiable, and one-to-one on a domain Ω to V , then f is a conformal map from Ω to V .

For example, e^z is conformal map from $\Omega = \{z : |\Im(z)| < \pi\}$ to $V = \mathbb{C} \setminus \mathbb{R}^-$.

Proposition. If f is complex differentiable and $f'(z_0) \neq 0$, it is a linear transform of a dilation by $|f'(z_0)|$ and a rotation by $\text{Arg}(f'(z_0))$. Hence, f is conformal because ③ is satisfied.

Example. $f(z) = z^2$ on $\Omega = \{z | 1 < |z| < 3 \text{ and } \Im(z) > 0\}$ is conformal.

Inverse Function Theorem.

If f is a continuously differentiable function with nonzero derivative at the point a , then f is invertible in a neighborhood of a , the inverse is continuously differentiable, and the derivative of the inverse function at $b = f(a)$ is the reciprocal of the derivative of f at a :

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}$$

Proposition. If f is invertible at z_0 and conformal, then f^{-1} is conformal at $f(z_0)$ by the inverse function theorem provided f is continuously differentiable.

From the proposition above, we know that $\text{Log}(z)$ is conformal on $\mathbb{C} \setminus \mathbb{R}^-$.

10 Bilinear Transformations

In complex analysis, the term linear transformation is used to describe affine transformations, $f(z) = az + b$.

10.1 Definition of a Möbius transformation

A bilinear/Möbius transformation is of the form

$$\frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$.

Now, $f(\infty) = \frac{a}{c}$ by L'Hospital argument and we say $f(-\frac{d}{c}) = \infty$.

If $ad - bc \neq 0$, then $f' \neq 0$ by the quotient rule and f is not constant. The constants are not unique, as $f(z) = \frac{az+b}{cz+d} = \frac{(az+b)k}{(cz+d)k}$. Therefore, we only have 3 degrees of freedom.

10.2 Brief Review of other Transformations

We have already seen these functions of this form before f is:

① Composition of a finite number of

$$\left\{ \begin{array}{l} \text{Translations, } f(z) = z + k \\ \text{Rotations, } f(z) = e^{i\theta} z \\ \text{Dilations, } f(z) = kz, k \in \mathbb{R} \\ \text{Inversions, } f(z) = \frac{1}{z} \end{array} \right.$$

② Conformal, f is holomorphic away from $z = -\frac{d}{c}$, $f' \neq 0$, and f is one-to-one.

③ Maps circles/lines to either lines or circles, "lines are circles of ∞ -radius in \mathbb{C} or $\mathbb{C} \cup \{\infty\}$.

If the line or circle passes through $z = -\frac{d}{c}$, where f is undefined, then it will be mapped to a line. Otherwise, it is mapped to a circle.

④ f can be identified by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then,

$$\left\{ \begin{array}{l} f \circ f \equiv A \circ A = A^2 \\ f^{-1} \equiv A^{-1} \\ f \circ g \equiv AB \end{array} \right.$$

and so there is a group homomorphism with Möbius transforms and invertible matrices in $\mathbb{C}^{2 \times 2}$.

$$f(z) = (3 + 2i)z - i^3 = \frac{(3+2i)z - i^3}{\delta z + 1}$$

⑤ We can conformally map 3 points in $\mathbb{C} \cup \{\infty\}$ to any 3 points in $\mathbb{C} \cup \{\infty\}$.

This type of argument is similar to showing norms are equivalent in \mathbb{R}^n or uniqueness of power series expansions.

10.3 Möbius transforming of a function

Given any 3 points $z_0, z_1, z_2 \in \mathbb{C}$, we can create a Möbius transformation T such that

$$\begin{cases} T(z_0) = 0 \\ T(z_1) = 1 \\ T(z_2) = \infty \end{cases} \quad \begin{cases} T^{-1}(0) = z_0 \\ T^{-1}(1) = z_1 \\ T^{-1}(\infty) = z_2 \end{cases}$$

Then

$$T(z) = (z, z_0, z_1, z_2) = \frac{(z - z_0)(z_1 - z_2)}{(z - z_2)(z_1 - z_0)}$$

is called the cross-ratio of z, z_0, z_1 , and z_2 .

There are some special cases:

$$\begin{cases} (z, \infty, z_1, z_2) = \frac{z_1 - z_2}{z - z_2} \\ (z, z_0, \infty, z_2) = \frac{z - z_0}{z - z_2} \\ (z, z_0, z_1, \infty) = \frac{z - z_0}{z_1 - z_0} \end{cases}$$

Given 3 more points $w_0, w_1, w_2 \in \mathbb{C}$, we get $S(w) = (w, w_0, w_1, w_2)$. Then, we can construct the function map:

$$\begin{aligned} z_0 &\xrightarrow{T} 0 \xleftarrow{S} w_0 \\ z_1 &\longrightarrow 0 \longleftarrow w_1 \\ z_2 &\longrightarrow 0 \longleftarrow w_2 \end{aligned}$$

Then $w = f(z) = S^{-1} \circ T(z)$ maps $z_0 \rightarrow w_0, z_1 \rightarrow w_1$, and $z_2 \rightarrow w_2$.

The points must be distinct, because f is one-to-one.

To find f from above, we solve $(z, z_0, z_1, z_2) = (w, w_0, w_1, w_2)$ for w .

Example.

Suppose

$$\begin{aligned} z_0 &= 1 \rightarrow i = w_0 \\ z_1 &= -1 \rightarrow 1 = w_1 \\ z_2 &= 1 \rightarrow -1 = w_2 \end{aligned}$$

then

$$\begin{aligned} \frac{(w - i)(1 - (-1))}{(w - (-1))(1 - i)} &= \frac{2(w - i)}{(w + 1)(1 - i)} = \frac{(z - i)(-1 - 1)}{(z - 1)(-1 - i)} = \frac{-2(z - i)}{(z - 1)(-1 - i)} = \frac{2(z - i)}{(z - 1)(1 + i)} \\ &\Rightarrow \frac{w - i}{(w + 1)(1 - i)} = \frac{z - i}{(z - 1)(1 + i)} \\ &\Rightarrow (w - i)(z - 1)(1 + i) = (w + 1)(1 - i)(z - i) \\ &\Rightarrow w = -\frac{1}{z} \end{aligned}$$

It is much work to find a simple function. Mapping a set of three points to another set of three points is time-consuming. If we study Möbius transforms as conformal mappings, then we can introduce another

way to move the three points around.

Example. Find a Möbius transform to map the unit disk $|z| < 1$ to $\Im(z) > 0$.

If we pick where 3 points go, say on the boundary of Ω , to the boundary of $\Im(z) > 0$, which is the \mathbb{R} -axis, then we can construct $f(z)$. We also need one point to get mapped to ∞ , but we have ∞ -many choices for the three points.

We want $f(-1) = 0$, $f(-i) = 1$, and $f(1) = \infty$, that will be

$$f(z) = (z, -1, -i, 1) = \frac{(z - z_0)(z_1 - z_2)}{(z - z_2)(z_1 - z_0)} = \frac{(z + 1)(-i - 1)}{(z - 1)(-i + 1)} = -i \frac{z + 1}{z - 1}$$

Our path's direction around the circle has to be preserved because f is conformal.

This means points on the interior of the circle will get mapped to the upper half of \mathbb{C} .

To check, we see that $f(0) = i$, $\Im(f(0)) > 0$. Therefore, the function we found out is correct.

Example. Find a Möbius transform to map the unit disk $|z| < 1$ to $\Im(z) > 0$ and $\Re(z) > 0$.

Using the conclusion from the example above, the new desired function is simply $g(z) = (f(z))^{\frac{1}{2}}$.

11 Contour Integral in \mathbb{C}

Let $\gamma(t) = x(t) + iy(t)$ be a curve in \mathbb{C} where $\gamma(a) = z_0$ and $\gamma(b) = z_1$.
Let C be the graph of $\gamma(t)$, $C = \{z | z = \gamma(t) \text{ for some } t \in [a, b]\}$.

11.1 Piecewise Differentiable, Smooth, Simple, Closed curves

Piecewise Differentiable curves.

The curve determined by γ , its graph C , is considered piecewise differentiable if

1. x and y are continuous on $[a, b]$
2. x' and y' are continuous on a partition of $[a, b]$, $[x_0, x_1] \cup [x_1, x_2] \cup [x_2, x_3] \cup \cdots \cup [x_{n-1}, x_n]$

Smooth curves.

If $\gamma' \neq 0$ for only finitely many points, then the curve is considered smooth.

Simple curves.

A curve is simple if it does not intersect itself, i.e. $\gamma(t) = \gamma(s)$ if and only if $s = t$.

Closed curves.

C is a closed curve if it starts and stops at the same point, i.e. $\gamma(a) = \gamma(b)$, $t \in [a, b]$

11.2 Interior and Exterior of curves

A closed and simple curve keeps the interior of the set on its left side and its exterior to the right.
This means we traverse circles counter-clockwise to describe their interior correctly.

Jordan Curve Theorem.

A closed and simple curve partitions \mathbb{C} into two regions, one of them bounded, defined as the interior of the curve.

11.3 Smoothly Equivalent

The parameter t provides an orientation or direction to C .

Let

$$\begin{cases} C_1 : \gamma_1(t), t \in [a, b] \\ C_2 : \gamma_2(t), t \in [c, d] \end{cases}$$

We say C_1 and C_2 are smoothly equivalent if there exists a one-to-one, continuous derivative mapping $\lambda(t)$,

$$\lambda(t) : [c, d] \rightarrow [a, b]$$

$$\lambda(c) = a$$

$$\lambda(d) = b$$

$$\lambda'(t) > 0$$

$$\text{where } \gamma_1(\lambda(t)) = \gamma_2(t)$$

Example.

$$\begin{cases} C_1 : \gamma_1(t) = \cos(t) + i \sin(t), t \in [0, 2\pi] \\ C_2 : \gamma_2(t) = \cos(2t) + i \sin(2t), t \in [0, \pi] \end{cases}$$

Here C_1 and C_2 are smoothly equivalent, since we can let $\lambda(t) = 2t$.

Both parametrize the unit circle, preserve the orientation, and pass through each point the same number of times.

Let us look at other two curves:

$$\begin{cases} \gamma_3(t) = \cos(4t) + i \sin(4t), t \in [0, \pi] \\ \gamma_4(t) = \cos(t) + i \sin(-t), t \in [0, 2\pi] \end{cases}$$

γ_3 traverses the circle multiple times and γ_4 has the opposite orientation of γ_1 and γ_2 .

Let $-C$ be the curve C but with a reversed orientation, $\gamma_R(t) = \gamma(b + a - t)$.

$$\begin{cases} C_1 : \gamma_1(t) = \cos(t) + i \sin(t), t \in [0, 2\pi] \\ -C_1 : \gamma_4(t) = \cos(t) + i \sin(-t), t \in [0, 2\pi] \end{cases}$$

We want to integrate $f(z)$ over curves C in \mathbb{C} . These will factor into the computation:

$$\begin{cases} \text{Orientation of } C \\ \text{Number of times } C \text{ traverses itself} \\ C \text{ is closed} \\ f \text{ is holomorphic on the interior of } C \text{ and } C \end{cases}$$

11.4 Line Integral

The line integral of f over C is given by

$$\int_C f(z) dz = \int_C u(z) + iv(z) dz = \int_C u(z) + iv(z) (dx + idy) = \int_a^b f(\gamma(t)) \gamma'(t) dt$$

If C is a closed curve, $\gamma(a) = \gamma(b)$, then we can use a closed loop in our \int symbol, $\oint_C f(z) dz$

The term $\gamma'(t)dt$ controls for how fast we traverse the curve. The integral is independent of our choice of smoothly equivalent curve, C_1 or C_2 .

Proposition. Let C_1 and C_2 be smoothly equivalent,

$$\int_{C_1} f(z) dz = \int_{C_2} f(z) dz$$

Proof. Let $\gamma_1(\lambda(t)) = \gamma_2(t)$ and apply change of variables.

Because $u = \lambda(t)$, $u(c) = a$, $u(d) = b$ and $\gamma'_1(\lambda(t))\lambda'(t) = \gamma'_2(t)$,

$$\int_c^d f(\gamma_2(t))\gamma'_2(t) dt = \int_c^d f(\gamma_1(\lambda(t)))\gamma'_1(\lambda(t))\lambda'(t) dt = \int_a^b f(\gamma_1(u))\gamma'_1(u) du$$

Proposition.

$$-\int_C f(z) dz = \int_{-C} f(z) dz$$

Proof.

$$\int_{-C} f(z) dz = \int_b^a f(\gamma_R(t))\gamma'_R(t) dt = \int_b^a f(\gamma(t))\gamma'(t) dt$$

Proposition. Linearity holds:

$$\int_C \alpha f(z) + g(z) dz = \alpha \int_C f(z) dz + \int_C g(z) dz$$

Example. Find $\oint_{|z|=1} \frac{1}{z} dz = \int_0^{2\pi} f(\gamma(t))\gamma'(t) dt$

We parametrize the curve as follows:

$C: \gamma(t) = \cos(t) + i \sin(t), t \in [0, 2\pi], \gamma'(t) = -\sin(t) + i \cos(t)$

$$\frac{1}{z} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2}$$

$$\int_0^{2\pi} f(\gamma(t))\gamma'(t) dt = \int_0^{2\pi} (\cos(t) - i \sin(t))(-\sin(t) + i \cos(t)) dt = \int_0^{2\pi} i dt = 2\pi i$$

There is another way to parametrize the curve:

$\gamma(t) = re^{it}, \gamma'(t) = ire^{it}$

$$\oint_{|z|=r} \frac{1}{z} dz = \int_0^{2\pi} f(re^{it})ire^{it} dt = i \int_0^{2\pi} e^{-it}e^{it} dt = \int_0^{2\pi} i dt = 2\pi i$$

From this example, we have the following famous result.

Proposition. The following holds with proof shown above.

$$\oint_{|z|=r} z^k dz = 0 \text{ for all } k \in \mathbb{Z}, k \neq -1, r > 0.$$

$$\oint_{|z|=r} \frac{1}{z} dz = \int_0^{2\pi} i dt = 2\pi i \text{ for } r > 0$$

ML Estimate.

Let f be a complex-valued, continuous function and $|f(z)| < M$ on C , a curve of length L , then

$$\left| \int_C f(z) dz \right| \leq M \cdot L$$

Proof.

$$\left| \int_a^b f(\gamma(t))\gamma'(t) dt \right| \leq \int_a^b |f(\gamma(t))\gamma'(t)| dt \leq M \int_a^b |\gamma_1'(t)| dt = M \cdot L$$

where $\int_a^b |\gamma_1'(t)| dt$ is the arclength of γ .

We do not have to worry about the orientation of C ,

$$\left| \int_{-C} f(z) dz \right| \leq M \cdot L$$

12 Cauchy's Closed Curve Theorem and the Fundamental Theorem of Calculus

We now study contour integrals of holomorphic functions over closed curves.

12.1 Cauchy's Closed Curve Theorem

Green's Theorem.

If $P(x, y)$ and $Q(x, y)$ have continuous first order derivatives on Ω and its boundary $\partial\Omega$, Ω is a domain,

$$\oint_{\partial\Omega} P dx + Q dy = \int_{\Omega} Q_x - P_y dx dy$$

Cauchy-Goursat Theorem.

If $f(z)$ is holomorphic on Ω and C is any closed curve in Ω , where Ω is simply connected. Then

$$\oint_C f(z) dz = 0$$

Proof.

$$\oint_{\partial\Omega} f(z) dz = \oint_{\partial\Omega} u + iv (dx + idy) = \int_{\Omega} if_x - f_y dA$$

and the Cauchy Riemann Equations imply

$$if_x = f_y$$

Therefore,

$$\oint_{\partial\Omega} f(z) dz = 0$$

Corollary of Cauchy's Theorem.

Let C_1 and C_2 be paths from z_0 to z_1 . Then $C_1 \cup -C_2$ is a closed loop in \mathbb{C} . C_1 and C_2 are in a simply connected subset of where f is holomorphic. Then by Cauchy-Goursat Theorem,

$$0 = \oint_{C_1 \cup -C_2} f(z) dz = \int_{C_1} f(z) dz + \int_{-C_2} f(z) dz = \int_{C_1} f(z) dz - \int_{C_2} f(z) dz$$

Hence,

$$\int_{C_1} f(z) dz = \int_{C_2} f(z) dz$$

This means it does not matter how you go from z_0 to z_1 as long as f is \mathbb{C} -differentiable along the way. This means we can use

$$\int_{z_0}^{z_1} f(z) dz$$

to denote the integral from z_0 to z_1 . But we still have to watch out for branch cuts.

Even non-simple curves are allowed, because Cauchy-Goursat Theorem tells us the integral over the loops in the non-simple curves are 0.

Path Independence. We can characterize Cauchy-Goursat Theorem with path independence. We have path independence of $\int_{z_0}^{z_1} f(z) dz$ in Ω if and only if

$$\oint_C f(z) dz = 0 \text{ for all closed curves } C \text{ in } \Omega$$

These conditions tell us holomorphic functions must have antiderivatives.

12.2 Fundamental Theorem of Calculus (F.T.C)

Path independence leads to introduction of Fundamental Theorem of Calculus.

Fundamental Theorem of Calculus (Part One).

Assume f is holomorphic on Ω , open and simply connected in \mathbb{C} . The function

$$F(z) = \int_a^z f(u) du$$

is holomorphic on Ω and $F'(z) = f(z)$.

Proof. This is completely analogous to the proof on \mathbb{R} , which has two steps.

1. Compute difference quotient
2. Bound integrand by invoking continuity of f

①

$$\begin{aligned} \frac{F(z+h) - F(z)}{h} &= \frac{1}{h} \int_z^{z+h} f(u) du \\ \frac{1}{h} \int_z^{z+h} du &= \frac{h}{h} = 1 \end{aligned}$$

implies

$$f(z) = f(z) \frac{1}{h} \int_z^{z+h} du = \frac{1}{h} \int_z^{z+h} f(z) du$$

Then

$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h} \int_z^{z+h} f(u) - f(z) du$$

② Since integrating over any closed curve C gives

$$\oint_C f(z) dz = 0$$

We can pick how we go from z to $z+h$, say along a line from z to $z+h$. We can do this for h sufficiently small so that our path is in Ω . This path is selected to make our upcoming estimate on $f(u) - f(z)$ easier to work with.

Along the path we can have $|u - z| < \delta$, which implies $|f(z) - f(u)| < \epsilon$ by continuity of f where u is on the path. From ML-Estimate, we get

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| \leq \frac{1}{|h|} \int_z^{z+h} |f(u) - f(z)| du < \frac{1}{|h|} |h| \epsilon = \epsilon$$

Therefore, it is proven that $F'(z) = f(z)$.

Let us take a closer look at our proof above. We needed f to be continuous so that we could invoke ML-Estimate. We also needed to pick a nice path from z to $z+h$. But we did not need f to be holomorphic. This leads to the next Theorem.

Morera's Theorem. Let f be a continuous, complex-valued function defined on an open set D in the complex plane. If for all closed paths C in D , $\oint_C f(z) dz = 0$, then $F(z)$ is holomorphic on D , $F'(z) = f(z)$ and f is holomorphic.

Later we will show that $f'(z)$ is holomorphic when f is holomorphic.

Let $f_n \rightarrow f$ uniformly on every compact subset of D . We say f_n converges on compacta to f . If f_n is holomorphic on D for all n and C is any closed curve in D ,

$$\oint_C f_n(z) dz = 0 \text{ for all } n$$

$$\Rightarrow \lim_{n \rightarrow \infty} \oint_C f_n(z) dz = 0$$

Since the convergence is uniform, f is continuous and

$$\lim_{n \rightarrow \infty} \oint_C f_n(z) dz = \oint_C f(z) dz = 0$$

Hence f is holomorphic on D by Morera's Theorem.

Example. Let $f(z) = \int_0^\infty \frac{e^{zt}}{1+t} dt$, $\Re(z) < 0$.

Because $\Re(z) < 0$,

$$\int_0^\infty \frac{|e^{zt}|}{1+t} dt \leq \int_0^\infty e^{\Re(z)t} dt = \frac{e^{\Re(z)t}}{\Re(z)} \Big|_0^\infty = \frac{-1}{\Re(z)}$$

And so we have an absolute convergent integral that is bounded.

$$|f(z)| \leq \frac{1}{|\Re(z)|}$$

Then, since the integral converges absolutely,

$$\oint_C f(z) dz = \oint_C \int_0^\infty \frac{e^{zt}}{1+t} dt dz = \int_0^\infty \frac{1}{1+t} \oint_C e^{zt} dz dt = \int_0^\infty \frac{1}{1+t} 0 dt = 0$$

by Cauchy-Goursat Theorem since e^{zt} is entire for t fixed. Therefore, $f(z)$ is holomorphic on $\Re(z) < 0$.

We can generalize our result to handle other types of integral transforms $F(z) = \int_0^\infty K(t, z, f(t)) dt$.

We also have an easy way to evaluate the line integrals.

Fundamental Theorem of Calculus (Part Two).

Let $F(z)$ be \mathbb{C} -differentiable on a smooth curve C where $F'(z) = f(z)$, then

$$\int_a^b f(z) dz = F(z) \Big|_{z=a}^b$$

Proof. Let $a, b, c \in D$, where c is in the middle of a and b . Then,

$$F(b) - F(a) = \int_a^b f(u) du = \int_a^c f(u) du + \int_c^b f(u) du = \int_a^b f(z) dz$$

The result can also be shown with variable substitution.

$$\int_b^a f(z) dz = \int_0^1 f(\gamma(t))\gamma'(t) dt = F(\gamma(t)) \Big|_{t=0}^1 = F(b) - F(a)$$

Proposition. If $f'(z) = 0$ on Ω , then f is constant on Ω . i.e. $0 = u_x = v_y$ and $u_y = -v_x = 0$.

Proof. Integrate f' along a staircase path in Ω ,

$$f(z) = f(a) + \int_a^z 0 du = f(a)$$

Or we can use the Cauchy Riemann Equations.

Let $F'(z) = G'(z) = f(z)$, then $H(z) = F(z) - G(z)$ satisfies $H'(z) = F'(z) - G'(z) = 0$. This means H is constant. This leads to the theorem below.

Theorem. Any antiderivative of holomorphic $f(z)$ is of the form

$$F(z) = \int_a^z f(u) du + C, C \in \mathbb{C}$$

Proposition. Let Ω be simply-connected, $0 \notin \Omega$. Fix $z_0 \in \Omega$ and pick the value of $\log z_0$. Then

$$F(z) = \int_{z_0}^z \frac{1}{w} dw + \log z_0$$

is a branch of $\text{Log } z$ in Ω .

Proof. F is well-defined since $\frac{1}{w}$ is holomorphic in Ω , we have path independence from z_0 to z , $f' = \frac{1}{z}$. We still need $e^{F(z)} = z$. Define

$$g(z) = ze^{-F(z)}$$

$$g'(z) = e^{-F(z)}(1 - zF'(z)) = e^{-F(z)}(1 - \frac{z}{z}) = 0$$

Therefore, $g(z)$ is constant.

$$g(z) = g(z_0) = z_0 e^{-F(z_0)} = z_0 e^{-\log(z_0)} = 1$$

Therefore,

$$e^{F(z)} = z$$

13 Cauchy's Integral Formula

First Extension of Cauchy-Goursat Theorem.

Let Ω be the domain between two simple closed curves C_1 and C_2 , each oriented counter-clockwise. If f is holomorphic on Ω , then

$$\oint_{C_1} f(z) dz = \oint_{C_2} f(z) dz$$

or

$$\oint_{C_1 \cup -C_2} f(z) dz$$

Second Extension of Cauchy-Goursat Theorem.

Let $g(z)$ be holomorphic on $\Omega \setminus \{z_0\}$, $g(z)$ is continuous on Ω , $g(z_0)$ is finite. Then

$$\oint_C g(z) dz = 0$$

for any closed curve C in Ω and $g(z)$ is holomorphic in Ω .

Proof. By First Extension of Cauchy-Goursat Theorem

$$\oint_C g(z) dz = \oint_{|z-z_0|=\epsilon} g(z) dz$$

and

$$\oint_{|z-z_0|=\epsilon} |g(z)| dz \leq M_\epsilon \cdot 2\pi \cdot \epsilon$$

Since g is continuous, M_ϵ is bounded and as $\epsilon \rightarrow 0$, we have

$$\oint_C g(z) dz = 0$$

Therefore, g is holomorphic on Ω , $g'(z_0)$ exists by Morera's Theorem.

The second extension of Cauchy-Goursat Theorem gives us Cauchy Integral Formula.

Assume f is holomorphic on Ω , $0 \in \Omega$. Let

$$g(z) = \frac{f(z) - f(0)}{z - 0} = \frac{f(z) - f(0)}{z}$$

as $z \rightarrow 0$, $g(0) = f'(0)$. Our function $g(z)$ is holomorphic on $\Omega \setminus \{z_0\}$ and continuous on Ω , so the second extension of Cauchy-Goursat Theorem holds.

Let C be a curve in Ω , by our last result

$$\begin{aligned} \oint_C g(z) dz &= \oint_C \frac{f(z) - f(0)}{z} dz = 0 \\ \Rightarrow \oint_C \frac{f(z)}{z} dz &= \oint_C \frac{f(0)}{z} dz = f(0) \oint_C \frac{dz}{z} = 2\pi i f(0) \end{aligned}$$

We showed before that

$$\oint_C \frac{dz}{z} = 2\pi i$$

Therefore,

$$f(0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z} dz$$

This means that the value of f at $z = 0$ only depends on the value of $\frac{f(z)}{z}$ over circles centered at 0. This result can be further generalized to Cauchy Integral Formula.

Cauchy Integral Formula. (C.I.F)

Let C be a smooth closed curve with interior domain Ω . If f is holomorphic on Ω and continuous on Ω 's closure,

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w - z} dw, z \in \Omega$$

Example.

$$\oint_{|z|=6} \frac{e^{z^2}}{z - 4} = 2\pi i f(4) = 2\pi i e^{16}, f(z) = e^{z^2}$$

However,

$$\oint_{|z|=3} \frac{e^{z^2}}{z - 4} = 0 \text{ by Cauchy-Goursat Theorem}$$

Extension of Cauchy Integral Formula.

Cauchy Integral Formula can be naturally expanded to $f'(z)$,

$$f'(z) = \frac{1}{2\pi i} \oint_C \frac{f'(w)}{w - z} dw = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w - z)^2} dw, z \in \Omega$$

Therefore, the result can be generalized to

$$f^{(k)}(z) = \frac{k!}{2\pi i} \oint_C \frac{f(w)}{(w - z)^{k+1}} dw, z \in \Omega$$

provided we can justify

$$\frac{d}{dz} \oint_C dw = \oint_C \frac{d}{dz} dw$$

Proof. The proof of this depends on $\frac{1}{w-z}$ being differentiable in w along any curve that does not pass through z .

Let f be holomorphic on Ω , then for any circle C in Ω about z_0 , we have a formula for $f^{(k)}(z_0)$ in terms of integrating over C . Hence $f^{(k)}$ exists on Ω , $f^{(k+1)}$ does

- $\Rightarrow f^{(k)}(z)$ is holomorphic on Ω for all k
- $\Rightarrow f$ is infinitely differentiable on Ω
- $\Rightarrow f = u + iv$ holomorphic
- \Rightarrow All partials of u and v exist and are continuous, $u_{xy} = u_{yx}$

Example. Given an integral in the form of C.I.F, we can take multiple approaches. Evaluate

$$\oint_{|z|=9} \frac{\sin(z)}{z^2} dz$$

The first approach is to apply C.I.F directly. Here, $f(z) = \sin(z)$, $f'(z) = \cos(z)$.

$$\oint_{|z|=9} \frac{\sin(z)}{z^2} dz = 2\pi i f'(0) = 2\pi i \cos(0) = 2\pi i$$

The second approach is to apply C.I.F indirectly. The function

$$g(z) = \begin{cases} \frac{\sin(z)-\sin(0)}{z-0}, & z \neq 0 \\ 1, & z = 0 \end{cases}$$

is entire by the second extension of Cauchy-Goursat Theorem.

$$\oint_{|z|=9} \frac{\frac{\sin(z)}{z}}{z} dz = 2\pi i g(0) = 2\pi i$$

Gauss's Mean Value Theorem.

Let $f(z)$ be an analytic function in $|z - z_0| < r$. Then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta$$

Proof. Let $\Omega = \{z ||z - z_0| < r\}$

Let

$$\gamma(t) = z_0 + re^{2\pi it}$$

Then,

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{f(z)}{z - z_0} dz \\ &= \frac{1}{2\pi i} \int_0^1 \frac{f(\gamma(t))}{\gamma(t) - z_0} \gamma'(t) dt \\ &= \int_0^1 f(z_0 + re^{2\pi it}) dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) dt \end{aligned}$$

14 Growth Conditions of Holomorphic Functions

14.1 Maximum/Minimum Modulus Principles

If f has an antiderivative on a domain Ω , then f is holomorphic on Ω . However, its converse is not true. If f is holomorphic on Ω , f does not necessarily have an antiderivative. Here is a counter-example.

Example. Let $f(z) = \frac{1}{z}$, $\Omega = \mathbb{C} \setminus \{0\}$.
 $f(z)$ does not have an antiderivative on Ω because

$$\oint_{|z|=r} \frac{1}{z} dz = 2\pi i \neq 0$$

The problem above is with Ω , which is not simply-connected.

Proposition. If Ω is simply-connected and f is holomorphic on Ω , then f has an antiderivative by F.T.C, a corollary of Cauchy-Goursat Theorem.

Maximum Modulus Principle.

Let f be holomorphic on a bounded domain Ω . If $|f|$ has a local maximum at z_0 , or in other words,

$$|f(z_0)| > |f(z)|$$

for all z in the neighborhood of z_0 , then f is constant near z_0 .

In fact this also implies f is constant in Ω with some additional theorems about uniqueness of holomorphic functions.

If f is continuous on $\partial\Omega$, then either f is constant or the absolute maximum of $|f|$ occurs only on the boundary of Ω , $\partial\Omega$.

Proof. Let f be holomorphic on a domain Ω and suppose the function $|f|$ takes on a local maximum value at z_0 . Near z_0 ,

$$|z - z_0| < r \Rightarrow |f(z)| \leq |f(z_0)|$$

Then by Gauss's Mean Value Theorem,

$$\begin{aligned} |f(z_0)| &= \left| \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) dt \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{it})| dt \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)| dt = |f(z_0)| \\ \Rightarrow |f(z_0)| &= \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{it})| dt = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)| dt \end{aligned}$$

Therefore,

$$\int_0^{2\pi} |f(z_0)| - |f(z_0 + re^{it})| dt = 0$$

But

$$|f(z_0)| \geq |f(z_0 + re^{it})|$$

by our hypothesis and so our integrand is nonnegative. Therefore, the integral is zero.

$$\Rightarrow |f(z_0)| - |f(z_0 + re^{it})| = 0 \Rightarrow |f(z_0)| = |f(z_0 + re^{it})|$$

Hence, the Maximum Modulus Principle is proven.

Claim. If f and \bar{f} are holomorphic on Ω , then f is constant on Ω .

Proof. By Cauchy Riemann Equations,

f is holomorphic $\Rightarrow u_x = v_y$ and $u_y = -v_x$.

\bar{f} is holomorphic $\Rightarrow u_x = -v_y$ and $u_y = v_x$.

This means $u_x = -u_x$, $u_y = -u_y$, $v_x = -v_x$, $v_y = -v_y$ in Ω .

Therefore, u and v are constant, and so is $f(z)$.

Claim. If $|f| = C$ on Ω , then f is constant on Ω . If $f(z_0) = 0$, then $f(z) = 0$ everywhere in Ω because $|f(z_0)| = 0$.

Proof. Let $f(z) = C$, then $g(z) = \frac{1}{\bar{f}(z)}$ is holomorphic, since it is a ratio of holomorphic functions.

Since $|f|^2 = C^2$, then

$$\overline{f(z)} = \frac{f(z)\overline{f(z)}}{f(z)} = \frac{|f(z)|^2}{f(z)} = C^2 g(z)$$

is holomorphic too. Hence, f is constant by the first claim.

Therefore, f is constant along all points of distance r from z_0 .

Minimum Modulus Principle.

Let f be analytic on a domain $\Omega \subseteq \mathbb{C}$, and assume that f never vanishes. Then if there is a point $z_0 \in \Omega$ such that $|f(z_0)| \leq |f(z)|$ for all $z \in \Omega$, then f is constant.

Let $\Omega \subseteq \mathbb{C}$ be a bounded domain, let f be a continuous function on the closed set $\bar{\Omega}$ that is analytic on Ω , and assume that f never vanishes on $\bar{\Omega}$. Then the minimum value of $|f|$ on $\bar{\Omega}$ (which always exists) must occur on $\partial\Omega$.

14.2 Mapping and \mathbb{C} -differentiability

Next we want to study functions that map the open disk to itself where $f(0) = 0$.

Schwarz's Lemma.

If f is holomorphic, $|f(z)| \leq 1$ on $|z| < 1$, and $f(0) = 0$, then

$$|f'(0)| \leq 1 \text{ and } |f(z)| \leq |z| \text{ on } |z| < 1$$

Moreover, if $|f(z_0)| = |z_0|$ for some non-zero z_0 or $|f'(0)| = 1$, then

$$f(z) = e^{i\theta}z, \theta \in [0, 2\pi)$$

Proof. Consider

$$g(z) = \begin{cases} \frac{f(z)-f(0)}{z-0} = \frac{f(z)}{z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases}$$

which we know is holomorphic on $|z| < 1$.

By Maximum Modulus for all $|z| < r$,

$$|g(z)| \leq |g(z_r)|$$

for some fixed $|z_r| = r$. Therefore,

$$\frac{|f(z_r)|}{|z_r|} \leq \frac{1}{|z_r|} = \frac{1}{r}$$

as $r \rightarrow 1$ we have $|g(z)| \leq 1$ or $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$ since $g(0) = f'(0)$.
 If $|g(z_0)| = 1$, then g has a local maximum for $|z| < 1$,

$$\begin{aligned} &\Rightarrow g \text{ is constant on } |z| < 1, |g(z)| = 1 \\ &\Rightarrow |f(z)| = |z| \\ &\Rightarrow f(z) = e^{i\theta} z \end{aligned}$$

Unit Circle Mapping (Schwarz's Lemma). Consider $B_a(z) : \{z \mid |z| < 1\} \rightarrow \{z \mid |z| < 1\}$
 It is a invertible, bilinear transform from the unit disk to itself, or an automorphism on the unit disk.

$$B_a(z) = \frac{z - a}{1 - \bar{a}z}, |a| < 1$$

The function has several interesting qualities.

$$\begin{aligned} |B_a(z)| &< 1 \text{ for } |z| < 1 \\ |B_a(z)| &= 1 \text{ for } |z| = 1 \\ B_a(a) &= 0 \\ B_a(0) &= -a \Rightarrow B_a^{-1}(z) = B_{-a}(z) \\ B'_a(0) &= 1 - |a|^2 \\ B'_a(a) &= \frac{1}{1 - |a|^2} \end{aligned}$$

To use Schwarz's Lemma if $|f(z)| < 1$, but $f(z_0) = 0$, for $z \neq 0$, we try to study

$$g(z) = f \circ B_{z_0}(z), \frac{f(z)}{B_{z_0}(z)} \text{ or } f(z)B_{z_0}(z).$$

If $|f(z)| < 1$ on $\Re(z) > 0$, then consider

$$g(z) = \frac{z + 1}{z - 1}$$

which maps $|z| < 1$ to $\Re(z) > 0$. Then we have $f \circ g^{-1} : \{z \mid |z| < 1\} \rightarrow \{z \mid |z| < 1\}$.

Example. Assume f is holomorphic and bounded by 1 for $|z| < 1$, $f(\frac{1}{2}) = 0$. Estimate $|f(\frac{3}{5})|$. Let

$$g(z) = \begin{cases} \frac{f(z)}{B_{1/2}(z)}, & z \neq \frac{1}{2} \\ \frac{f'(\frac{1}{2})}{B'_{1/2}(z)} = \frac{3}{4}f'(\frac{1}{2}), & z = \frac{1}{2} \end{cases}$$

is holomorphic on $|z| < 1$. (We used L'Hospital's Rule to define $g(\frac{1}{2})$).

As $|z| \rightarrow 1$, $|g| \leq 1$ since $|f| \leq 1$ and $|B_{\frac{1}{2}}(e^{i\theta})| = 1$.

This means $|f(z)| \leq |B_{\frac{1}{2}}(z)|$ for all $|z| < 1$.

Hence,

$$\left| f\left(\frac{3}{5}\right) \right| \leq \left| B_{\frac{1}{2}}\left(\frac{3}{5}\right) \right| = \left| \frac{\frac{3}{5} - \frac{1}{2}}{1 - \frac{1}{2} \cdot \frac{3}{5}} \right| = \frac{1}{7}$$

Note $B_{\frac{1}{2}}(z)$ is a holomorphic function with $B_{\frac{1}{2}}(\frac{1}{2}) = 0$ that hits the upper bound. Therefore, we cannot estimate better than that.

Now we want to use bounds on f to bound $f^{(k)}$.

Cauchy's Inequality.

If $|f(z)| \leq M$ on $|z - z_0| < R$, then for $r < R$ and $C_r = \{z \mid |z - z_0| = r\}$,

$$\left| f^{(k)}(z_0) \right| \leq \frac{k!M}{R^k}$$

Proof.

$$\begin{aligned} \left| f^{(k)}(z_0) \right| &\leq \frac{k!}{2\pi} \oint_{C_r} \frac{|f(w)|}{|w - z_0|^{k+1}} dw \\ &\leq \frac{k!}{2\pi} \frac{M_R}{r^{k+1}} \oint_{C_r} dw \\ &= \frac{k!}{2\pi} \frac{M}{r^{k+1}} 2\pi \\ &= \frac{k!M}{r^k} \end{aligned}$$

Let $r \rightarrow R^-$, then

$$\left| f^{(k)}(z_0) \right| \leq \frac{k!M}{R^k}$$

Liouville's Theorem.

A bounded entire function is constant.

Proof. From Cauchy's Inequality,

$$|f'(z_0)| \leq \frac{M}{R}$$

then as $R \rightarrow \infty$, $f'(z_0) \rightarrow 0$. Therefore, f is constant.

Lemma. Let $p(z)$ be a non-constant polynomial, then

$$\lim_{z \rightarrow \infty} |p(z)| = \infty$$

Proof.

$$\begin{aligned} p(z) &= a_0 + a_1z + \cdots + a_nz^n \\ &= z^n(a_0z^{-n} + a_1z^{-n+1} + \cdots + a_n) \end{aligned}$$

Then

$$|p(z)| = |z^n| |a_0z^{-n} + a_1z^{-n+1} + \cdots + a_n| \rightarrow \infty \text{ as } n \rightarrow \infty$$

Proof of the Fundamental Theorem of Algebra (F.T.A).

The last two results have major consequences. Let $p(z)$ be a non-constant polynomial and $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then

$$f(z) = \frac{1}{p(z)}$$

is entire and non-constant and thus unbounded. As $z \rightarrow \infty$, $p(z) \rightarrow \infty$ and hence

$$\lim_{z \rightarrow \infty} f(z) = 0$$

or $|f(z)| < 1$ for $|z| > R$.

Since f is continuous on $|z| \leq R$, it has a bound $|f(z)| \leq M$, for $|z| < R$. Hence, $f(z)$ is constant and thus, $p(z)$ is constant. This is a contradiction. Therefore, $f(z)$ must not be entire.

Hence, the F.T.A is proven. All non-constant polynomials have a root in \mathbb{C} .

Let $p(z)$ be of degree n . For each root z_i of $p(z)$, we can compute

$$p(z) - p(z_i) = (z - z_i)q(z)$$

Repeat the process of $q(z)$ and show that we have exactly n roots.

Example. Suppose f is entire,

$$|f'(z)| < |f(z)|$$

Then

$$\frac{f'(z)}{f(z)}$$

is entire and bounded. By Liouville's Theorem, $f'(z) = kf(z)$ and $|k| < 1$.

Therefore,

$$f(z) = e^{kz}, |k| > 1$$

Proposition. Suppose f is entire and $|f(z)| > 1$, then $\frac{1}{f(z)}$ is entire and bounded and thus constant.

Example. Non-constant entire functions are unbounded. Therefore, $\sin(z)$ and $\cos(z)$ are unbounded since $\sin(0) \neq \sin(\frac{\pi}{2})$ and $\cos(0) \neq \cos(\frac{\pi}{2})$

Example. Let f and g be entire, $\Re(h(z)) \leq k\Re(g(z))$ for some fixed k . Consider

$$h(z) = f(z) - kg(z)$$

entire where $\Re(h(z)) < 0$. Then $e^{h(z)}$ is bounded by 1.

$$|e^{h(z)}| \leq e^{\Re(h(z))} \leq 1$$

Therefore, $h(z)$ is constant. This means $f(z) - kg(z) = C$ for some constant C .

$$f(z) = kg(z) + C$$

15 Convergence of Infinite Series in \mathbb{C}

We want to study series in \mathbb{C} .

$$\sum_{k=0}^{\infty} a_k$$

- 16 Power Series in \mathbb{C}
- 17 Series Expansion of Holomorphic Functions
- 18 Open Mapping Theorem and Reflection Principle
- 19 Laurent Series
- 20 Residue Theorem
- 21 Improper Integrals
- 22 Argument Principle and Rouché's Theorem

$$f(x) = x^2$$

this formula is an example $f(x) = x$

$$1 + 2 = 3$$

$$1 = 3 - 2$$

$$f(x) = x^2$$

$$g(x) = \frac{1}{x}$$

$$h(x) = \int_b^a \frac{1}{x} x^3$$

$$F(x) = \frac{1}{\sqrt{x}}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left(\frac{1}{\sqrt{x}} \right)$$

Core Material: 1. Finding patterns in data; using them to make predictions. 2. Models and statistics help us understand patterns. 3. Optimization algorithms "learn" the patterns.
Classification: 1.