This proves that f(y)−y is isotropic for any nonzero isotropic vector y. Since by hypothesis f(u)−u is isotropic for every nonisotropic vector u, we proved that f(u)−u is isotropic for every u ∈ E. If we let W = Im(f − id), then every vector in W is isotropic, and thus W is totally isotropic (recall that we assumed that char(K) = 26 , so ϕ is determined by Φ). For any u ∈ E and any v ∈ W ⊥, since W is totally isotropic, we have  
这证明了f（y）−y对任何非零各向同性向量y都是各向同性的。由于假设f（u）−u对每个非各向同性向量u都是各向同性的，我们证明了f（u）−u对每个u∈e都是各向同性的。如果我们让w=im（f−id），那么w中的每个向量都是各向同性的，因此w是完全各向同性的。pic（回想一下，我们假设char（k）=26，因此，a由Φ决定）。对于任何u∈e和任何v∈w，由于w是完全各向同性的，我们有

ϕ(f(u) − u,f(v) − v) = 0,  
⑨（f（u）−u，f（v）−v）=0，

and since f(u) − u ∈ W and v ∈ W ⊥, we have ϕ(f(u) − u,v) = 0, and so  
既然f（u）−u∈w和v∈w，我们就有（f（u）−u，v）=0，所以

0 = ϕ(f(u) − u,f(v) − v)  
0=\_（f（u）−u，f（v）−v）

= ϕ(f(u),f(v)) − ϕ(u,f(v)) − ϕ(f(u) − u,v)  
=\_（f（u），f（v））−\_（u，f（v））−\_（f（u）−u，v）

= ϕ(u,v) − ϕ(u,f(v))  
=\_（u，v）-\_（u，f（v））

= ϕ(u,v − f(v)),  
=⑨（u，v−f（v）），

for all u ∈ E. Since ϕ is nonsingular, this means that f(v) = v, for all v ∈ W ⊥. However, by hypothesis, no nonisotropic vector is left fixed, which implies that W ⊥ is also totally isotropic. In summary, we proved that W ⊆ W ⊥ and W ⊥ ⊆ W ⊥⊥ = W, that is,  
对于所有u e，由于\_是非奇异的，这意味着f（v）=v，对于所有v w。然而，根据假设，没有非各向同性向量是固定的，这意味着w也是完全各向同性的。总之，我们证明了w w和w w=w，即，

W = W ⊥.  
W=W。

Since, dim(W) + dim(W ⊥) = n, we conclude that W is a totally isotropic subspace of E such that dim(W) = n/2.  
由于dim（w）+dim（w）=n，我们得出w是e的完全各向同性子空间，因此dim（w）=n/2。

By Proposition 28.29, the space E is an Artinian space of dimension n = 2m. Since W = W ⊥ and f(W) = W, by Proposition 28.42, the isometry f is a rotation.   
根据命题28.29，空间E是一个尺寸为n=2米的Artian空间。由于w=w和f（w）=w，根据命题28.42，等距线f是一个旋转。

Remarks:  
评论：

1. Another way to finish the proof of Proposition 28.43 is to prove that if f is an isometry, then  
   完成28.43号提案的证明的另一种方法是证明，如果f是等距线，那么

Ker(f − id) = (Im(f − id))⊥.  
ker（f−id）=（im（f−id））。

After having proved that W = Im(f − id) is totally isotropic, we get  
在证明w=im（f-id）是完全各向同性之后，我们得到

Ker(f − id) = Im(f − id),  
ker（f-id）=im（f-id）、

which implies that (f − id)2 = 0. From this, we deduce that det(f) = 1. For details, see Jacobson [96] (Chapter 6, Section 6).  
这意味着（f-id）2=0。由此，我们推导出Det（f）=1。有关详细信息，请参见Jacobson[96]（第6章第6节）。

1. If f = τHk ◦ ··· ◦ τH1, where the Hi are hyperplanes, then it can be shown that  
   如果f=τhk····τh1，其中hi是超平面，则可以证明

dim(H1 ∩ H2 ∩ ··· ∩ Hs) ≥ n − s.  
尺寸（h1 h2··hs）≥N s。

Now, since each Hi is left fixed by τHi, we see that every vector in H1 ∩ ··· ∩ Hs is left fixed by f. In particular, if s < n, then f has some nonzero fixed point. As a consequence, an isometry without fixed points requires n hyperplane reflections.  
现在，由于每个hi都由τhi保持不变，我们看到h1··hs中的每个向量都由f保持不变。特别是，如果s<n，那么f有一些非零固定点。因此，没有固定点的等距测量需要n个超平面反射。

# 28.10 Witt’s Theorem 28.10维特定理

Witt’s theorem was referred to as a “scandal” by Emil Artin. What he meant by this is that one had to wait until 1936 (Witt [184]) to formulate and prove a theorem at once so simple in its statement and underlying concepts, and so useful in various domains (geometry, arithmetic of quadratic forms).  
维特定理被埃米尔·阿丁称为“丑闻”。他所指的是，一个人必须等到1936年（witt[184]）才能立刻制定和证明一个定理，它的表述和基本概念如此简单，在各个领域（几何、二次型算术）也如此有用。

Besides Witt’s original proof (Witt [184]), Chevalley’s proof [37] seems to be the “best” proof that applies to the symmetric as well as the skew-symmetric case. The proof in Bourbaki [24] is based on Chevalley’s proof, and so are a number of other proofs. This is the one we follow (slightly reorganized). In the symmetric case, Serre’s exposition is hard to beat (see Serre [152], Chapter IV).  
除了维特的原始证明（维特[184]），契瓦利的证明[37]似乎是适用于对称和斜对称情况的“最佳”证明。Bourbaki[24]中的证明基于Chevalley的证明，其他一些证明也是如此。这是我们所遵循的（稍微重组）。在对称的情况下，Serre的论述很难打破（见Serre[152]，第四章）。

The following observation is one of the key ingredients in the proof of Theorem 28.45.  
下面的观察是定理28.45证明的关键成分之一。

Proposition 28.44. Given a finite-dimensional space E equipped with an -Hermitan form ϕ, if U1 and U2 are two subspaces of E such that U1 ∩U2 = (0) and if we have metric linear maps f1 : U1 → E and f2 : U2 → E such that  
提案28.44。给定一个有限维空间e，如果u1和u2是e的两个子空间，那么u1 u2=（0），如果我们有公制线性映射f1:u1→e和f2:u2→e，那么

ϕ(f1(u1),f2(u2)) = ϕ(u1,u2) for ui ∈ Ui (i = 1,2), (∗)  
⑨（f1（u1），f2（u2））=⑨（u1，u2）代表ui∈ui（i=1,2），（）

then the linear map f : U1 ⊕ U2 → E given by f(u1 + u2) = f1(u1) + f2(u2) extends f1 and f2 and is metric. Furthermore, if f1 and f2 are injective, then so if f.  
然后，由f（u1+u2）=f1（u1）+f2（u2）给出的线性图f:u1 u2→e延伸f1和f2，为公制。此外，如果f1和f2是内射的，那么如果f。

Proof. Indeed, since f1 and f2 are metric and using (∗), we have  
证据。事实上，因为f1和f2是公制的，并且使用（），我们有

ϕ(f1(u1) + f2(u2),f1(v1) + f2(v2)) = ϕ(f1(u1),f1(v1)) + ϕ(f1(u1),f2(v2))  
Ⅷ（f1（u1）+f2（u2），f1（v1）+f2（v2））=Ⅷ（f1（u1），f1（v1））+Ⅷ（f1（u1），f2（v2））

+ ϕ(f2(u2),f1(v1)) + ϕ(f2(u2),f2(v2)) = ϕ(u1,v1) + ϕ(u1,v2) + ϕ(u2,v1) + ϕ(u2,v2) = ϕ(u1 + u2,v2 + v2).  
+Ⅷ（F2（U2），F1（V1））+Ⅷ（F2（U2），F2（V2））=Ⅷ（U1，V1）+Ⅷ（U1，V2）+Ⅷ（U2，V1）+Ⅷ（U2，V2）=Ⅷ（U1+U2，V2+V2）。

Thus f is a metric map extending f1 and f2.   
因此，F是延伸f1和f2的度量图。

Theorem 28.45. (Witt, 1936) Let E and E0 be two finite-dimensional spaces respectively equipped with two nondegenerate -Hermitan forms ϕ and ϕ0 satisfying condition (T), and assume that there is an isometry between (E,ϕ) and (E0,ϕ0). For any subspace U of E, every injective metric linear map f from U into E0 extends to an isometry from E to E0.  
定理28.45。（witt，1936）让e和e0分别为两个有限维空间，分别配备两个满足条件（t）的非简并-厄米坦形式，并假设（e，\_）和（e0，\_0）之间存在一个等距线。对于e的任何子空间u，从u到e0的每个内射度量线性映射f延伸到从e到e0的等距线。

Proof. Since (E,ϕ) and (E0,ϕ0) are isometric, we may assume that E0 = E and ϕ0 = ϕ (if h: E → E0 is an isometry, then h−1 ◦f is an injective metric map from U into E. The details are left to the reader).  
证据。因为（e，\_）和（e0，\_）是等距的，我们可以假设e0=e和\_0=\_（如果h:e→e0是等距的，那么h-1 f是从u到e的一个注入式公制图。细节留给读者）。

We proceed by induction on the dimension r of U. Since the proof is quite intricate, we spell out the general plan of attack. For the induction step, we first show that we can reduce the situation to what we call Case (H), namely that the subspace of U left fixed by f is a hyperplane H in U. Then, the set D = {f(u) − u | u ∈ U} is a line in U and it turns out that D⊥ is a hyperplane in E. We now introduce Hypothesis (V), which says we can find a nontrivial subspace V of E orthogonal to D and such that V ∩ U = V ∩ f(U) = (0). We show that if Hypothesis (V) holds, then f can be extended to an isometry of U ⊕ V . It is then possible to further extend f to an isometry of E.  
我们对U的维R进行归纳，由于证明是相当复杂的，所以我们阐明了攻击的总体方案。对于诱导步骤，我们首先证明了我们可以将情形简化为（h），也就是说，由f左固定的u的子空间是u中的超平面h，然后，集合d=f（u）−u u∈u是u中的一条线，结果表明d是e中的超平面，现在我们引入假设。（v），也就是说我们可以找到一个与d正交的e的非平凡子空间v，这样v u=v f（u）=0。我们证明，如果假设（v）成立，那么f可以推广到u v的等距测量。然后可以进一步将f扩展到e的等值线。

To prove that Hypothesis (V) holds we consider two cases. In Case (a), we obtain some V such that E = U ⊕ V and we are done. In Case (b), we obtain some V such that D⊥ = U ⊕V . We are then reduced to the situation where U = D⊥ is a hyperplane in E and f is an isometry of U. To finish the proof we pick any v /∈ U, so that E = U ⊕ Kv, and we find some v1 ∈ E such that  
为了证明假设（v）成立，我们考虑了两个案例。在（a）种情况下，我们得到一些v，这样e=u\_v，我们就完成了。在（b）种情况下，我们得到一些v，这样d=u v。然后，我们将其简化为u=d是e中的超平面，f是u的一个等距，为了完成这个证明，我们选取了任意v/∈u，使e=u\_kv，我们发现了一些v1∈e，这样

ϕ(f(u),v1) = ϕ(u,v) for all u ∈ U  
所有u∈u的（f（u），v1）=（u，v）

ϕ(v1,v1) = ϕ(v,v).  
⑨（v1，v1）=⑨（v，v）。

Then, by Proposition 28.44, we can extend f to a metric map g of U + Kv = E such that g(v) = v1. The argument used to find v1 makes use of (†) (see below) and is bit tricky. We also makes use of Property (T) in the form of Lemma 28.28.  
然后，根据命题28.44，我们可以把f扩展到u+kv=e的度量图g，这样g（v）=v1。用于查找v1的参数使用了（†）（见下文）并有点棘手。我们还使用了属性（t）的形式lemma 28.28。

We now go back to the proof. The case r = 0 is trivial. For the induction step, r ≥ 1 so U = (0)6 , and let H be any hyperplane in U. Let f : U → E be an injective metric linear map. By the induction hypothesis, the restriction f0 of f to H extends to an isometry g0 of E. If g0 extends f, we are done. Otherwise, H is the subspace of elements of U left fixed by g0−1 ◦f. If the theorem holds in this situation, namely the subspace of U left fixed by g0−1 ◦f is a hyperplane H in U, then we have an isometry g1 of E extending g0−1 ◦ f, and g0 ◦ g1 is an isometry of E extending f. Therefore, we are reduced to the following situation:  
我们现在回到证据上来。r=0的情况是微不足道的。对于诱导步骤，r≥1，所以u=（0）6，并且h是u中的任何超平面。让f:u→e是一个内射度量线性映射。根据诱导假设，F对H的限制性f0延伸到E的等值线g0，如果g0延伸到F，我们就完成了。否则，h是由g0−1 f固定的u元素的子空间。如果定理在这种情况下成立，即由g0−1 f固定的u元素的子空间是u中的超平面h，则我们有e延伸g0−1 f的等距g1，而g0 g1是e延伸f的等距。关于，我们将减少到以下情况：

Case (H). The subspace of U left fixed by f is a hyperplane H in U.  
案例（h）。F左固定的U的子空间是U中的超平面H。

In this case, the set D = {f(u) − u | u ∈ U} is a line in U (a one-dimensional subspace).  
在这种情况下，集合d=f（u）−u u∈u是u（一维子空间）中的一条线。

For all u,v ∈ U, we have ϕ(f(u),f(v) − v) = ϕ(f(u),f(v)) − ϕ(f(u),v) = ϕ(u,v) − ϕ(f(u),v) = ϕ(u − f(u),v),  
对于所有u，v∈u，我们都有\_（f（u），f（v）−v）=（f（u），f（v））−\_（f（u），v）=（u，v）−（f（u），v）=（u−f（u），v），

that is  
那就是

ϕ(f(u),f(v) − v) = ϕ(u − f(u),v) for all u,v ∈ U, (∗∗)  
所有u，v∈u，（\_）

and if u ∈ H, which means that f(u) = u, we get u ∈ D⊥. Therefore, H ⊆ D⊥. Since ϕ is nondegenerate, we have dim(D) + dim(D⊥) = dim(E), and since dim(D) = 1, the subspace D⊥ is a hyperplane in E.  
如果u∈h，这意味着f（u）=u，我们得到u∈d。因此，H D。由于ω是非退化的，所以我们有dim（d）+dim（d）=dim（e），由于dim（d）=1，子空间d是e中的超平面。

Hypothesis (V). We can find a nontrivial subspace V of E orthogonal to D and such that  
假设（v）。我们可以找到一个与d正交的e的非平凡子空间v，这样

V ∩ U = V ∩ f(U) = (0).  
V U=V F（U）=（0）。

Claim. Hypothesis (V) implies that f can be extended to an isometry of U ⊕ V .  
索赔。假设（v）意味着f可以扩展到u v的等距测量。

Proof of Claim. If Hypothesis (V) holds, then we have  
索赔证明。如果假设（v）成立，那么我们有

ϕ(f(u),v) = ϕ(u,v) for all u ∈ U and all v ∈ V ,  
（f（u），v）=（u，v），对于所有u∈u和所有v∈v，

since ϕ(f(u),v) − ϕ(u,v) = ϕ(f(uf)1−=u,vf ) = 0and f, with2 the inclusion off(u) − u ∈ VD intoand Ev , we can extend∈ V orthogonal to D. By Proposition 28.44 with f{ to an injective metric map on U ⊕ V Dleaving all vectors in. V fixed. In this case, the set f(w) − w | w ∈ U ⊕ V } is still the line   
既然\_（f（u），v）−\_（u，v）=（f（uf）1−u，vf）=0和f，在2不包含（u）−u∈v d intoand ev的情况下，我们可以通过命题28.44，用f扩展到u v上的一个内射度量图，而不保留所有的向量。V固定。在这种情况下，集合f（w）−w w∈u v仍然是线

We show below that the fact that f can be extended to U ⊕ V implies that f can be  
我们在下面表明，f可以扩展到u v这一事实意味着f可以

extended to the whole ofIn case (b), D⊥ = U ⊕ V Ewhere. There are two cases. In Case (a),D⊥fis a hyperplane incan be extended to an isometry ofE and fEis an isometry of= U ⊕VEand we are done.. D⊥. By a subtle argument, we will show that  
扩展到整个案例（b），d=u v e这里。有两种情况。在（a）种情况下，d fi是一个超平面inca，扩展到e和fei的等距，等距等于u vea，我们就完成了..d。通过微妙的争论，我们将展示

We are reduced to proving that a subspace V as above exists. We distinguish between two cases.  
我们可以证明上面的子空间V存在。我们区分两种情况。

Case (a). U ⊆6 D⊥.  
案例（a）。U 6 D。

Proof of Case (a). In this case, formula (∗∗) show that f(U) is not contained in D⊥ (check this!). Consequently,  
案件证明（a）。在这种情况下，公式（）表示f（u）不包含在d中（勾选此项！）因此，

U ∩ D⊥ = f(U) ∩ D⊥ = H.  
U D=F（U）D=H.

We can pickV ∩fU(U⊕) = (0)V 6=VDto be any supplement of. Since⊥ (sinceUU⊕is not contained inV contains the hyperplaneH in DD⊥, and the above formula shows that⊥ andDV⊥⊆(sinceD⊥), we must haveD⊥ = H V and HV ∩UU=), and ⊕ E = U⊆⊕ V , and as we showed as a consequence of hypothesis (V), f can be extended to an isometry of  
我们可以选择v fu（u）=（0）v 6=vd作为任何补充。由于（since u u不包含，in v在dd中包含超平面，并且上述公式表明和dv（sinced），我们必须有d=h v和hv uu=）和e=u v，并且，正如我们在假设（v）的结果中所示，f可以扩展为

U ⊕ V = E.   
u v=e.

Case (b). U ⊆ D⊥.  
案例（b）。U D。

sinceProof of Case (b).D = {f(u) − uIn this case, formula (| u ∈ U}, we have D∗∗⊆) Dshows that⊥; that is, the linef(U) ⊆ DD⊥ sois isotropic.U +f(U) ⊆ D⊥, and  
鉴于情况（b）.d=f（u）−ui，在这种情况下，公式（u∈u，我们有d）表示，即，线f（u）dd sois各向同性.u+f（u）d，和

We show that there exists a subspace V of D⊥, such that  
我们证明存在d的子空间v，这样

D⊥ = U ⊕ V = f(U) ⊕ V.  
D=U V=F（U）V.

Thus, case (b) shows that we are reduced to the situation where U = D⊥ and f is an isometry of U.  
因此，情况（b）表明，我们被简化到u=d和f是u的等距测量。

x /∈IfHU, and let= f(Uy)U∈we pick, we havef(U) withV to be a supplement ofy /∈ H. Since f(H) =UH in(pointwise),D⊥. Otherwise, letf is injective, andx ∈ U withH is a hyperplane in  
x/∈ifhu，且设=f（uy）u∈we pick，我们将f（u）与v作为y/∈h的补充，因为f（h）=uh in（pointwise），d。否则，Letf是内射的，x∈u和h是中的超平面。

U = H ⊕ Kx, f(U) = H ⊕ Ky.  
u=h kx，f（u）=h ky。

We claim that x + y /∈ U. Otherwise, since y = x + y − x, with x + y,x ∈ U and since y ∈ f(U), we would have y ∈ U ∩ f(U) = H, a contradiction. Similarly, x + y /∈ f(U). It follows that  
我们声称x+y/∈u，否则，因为y=x+y−x，有x+y，x∈u，既然y∈f（u），我们就有y∈u f（u）=h，一个矛盾。同样，x+y/∈f（u）。接下来是

* 1. + f(U) = U ⊕ K(x + y) = f(U) ⊕ K(x + y).  
     +f（u）=u k（x+y）=f（u）k（x+y）。

Now, pick W to be any supplement of U + f(U) in D⊥ so that D⊥ = (U + f(U)) ⊕ W, and let  
现在，选择w作为d中u+f（u）的任何补充，使d=（u+f（u））w，并让

* 1. = K(x + y) + W.  
     =K（x+y）+W。

Then, since x ∈ U,y ∈ f(U), W ⊆ D⊥, and U + f(U) ⊆ D⊥, we have V ⊆ D⊥. We also have  
然后，由于x∈u，y∈f（u），w d，和u+f（u）d，我们得到v d。我们也有

U ⊕ V = U ⊕ K(x + y) ⊕ W = (U + f(U)) ⊕ W = D⊥  
u v=u k（x+y）w=（u+f（u））w=d

and  
和

f(U) ⊕ V = f(U) ⊕ K(x + y) ⊕ W = (U + f(U)) ⊕ W = D⊥,  
F（U）V=F（U）K（X+Y）W=（U+F（U））W=D，

so as we showed as a consequence of hypothesis (V), f can be extended to an isometry of the hyperplane D⊥ = U ⊕ V , and D is still the line {f(w) − w | w ∈ U ⊕ V }.   
因此，根据假设（v）的结果，f可以扩展到超平面d=u v的等距线，d仍然是线f（w）−w w∈u v。

The argument in the proof of Case (b) shows that we are reduced to the situation where U = D⊥ is a hyperplane in E and f is an isometry of U. If we pick any v /∈ U, then E = U ⊕ Kv, so suppose we can find some v1 ∈ E such that  
例（b）证明中的论点表明，我们简化到u=d是e中的超平面，f是u的一个等值线的情形，如果我们选取任何v/∈u，那么e=u kv，那么假设我们可以找到一些v1∈e，这样

ϕ(f(u),v1) = ϕ(u,v) for all u ∈ U  
所有u∈u的（f（u），v1）=（u，v）

ϕ(v1,v1) = ϕ(v,v).  
⑨（v1，v1）=⑨（v，v）。

The first condition is condition (∗) of Proposition 28.44, and the second condition asserts that the map λv 7→ λv2 from the line Kv to the line Kv1 is a metric map. Then, by Proposition 28.44, we can extend f to a metric map g of U + Kv = E such that g(v) = v1. To find v1, let us prove that for every v ∈ E, there is some v0 ∈ E such that  
第一个条件是命题28.44的条件（），第二个条件断言从Kv线到Kv1线的映射λv 7→λv2是公制映射。然后，根据命题28.44，我们可以把f扩展到u+kv=e的度量图g，这样g（v）=v1。为了找到v1，让我们证明对于每一个v∈e，都有一些v0∈e，这样

ϕ(f(u),v0) = ϕ(u,v) for all u ∈ U. (†)  
所有u∈u.（†）的（f（u），v0）=（u，v）

This is because the linear form u 7→ ϕ(f−1(u),v) (u ∈ U) is the restriction of a linear form ψ ∈ E∗, and since ϕ is nondegenerate, there is some (unique) v0 ∈ E, such that  
这是因为线性形式u 7→（f−1（u），v）（u∈u）是线性形式ψ∈e的限制，并且由于\_是非退化的，所以有一些（唯一的）v0∈e，这样

ψ(x) = ϕ(x,v0) for all x ∈ E,  
ψ（x）＝（x，v0），对于所有x∈e，

which implies that ϕ(u,v0) = ϕ(f−1(u),v) for all u ∈ U,  
也就是说，对于所有u∈u，（u，v0）=（f−1（u），v）

and since f is an automorphism of U, that (†) holds. Furthermore, observe that formula (†) still holds if we add to v0 any vector y in D, since f(U) = U = D⊥. Therefore, for any v1 = v0 + y with y ∈ D, if we extend f to a linear map of E by setting g(v) = v1, then by (†) we have ϕ(g(u),g(v)) = ϕ(u,v) for all u ∈ U.  
因为f是u的自同构，所以（†）成立。此外，如果我们在v0中加上d中的任何向量y，则可以观察到公式（†）仍然成立，因为f（u）=u=d。因此，对于任何带有y∈d的v1=v0+y，如果我们通过设置g（v）=v1将f扩展到e的线性映射，那么通过（†）我们得到了所有u∈u的（g（u），g（v））=（u，v）。

We still need to pick y ∈ D so that v1 = v0 + y satisfies ϕ(v1,v1) = ϕ(v,v). However, since v /∈ U = D⊥, the vector v is not orthogonal D, and by Lemma 28.28, there is some y0 ∈ D such that ϕ(v0 + y0,v0 + y0) = ϕ(v,v).  
我们仍然需要选择y∈d，以便v1=v0+y满足\_（v1，v1）=（v，v）。然而，由于v/∈u=d，向量v不是正交d，并且根据引理28.28，有一些y0∈d，这样使得饨（v0+y0，v0+y0）=（v，v）。

Then, if we let v1 = v0 + y0, by Proposition 28.44, we can extend f to a metric map g of U + Kv = E by setting g(v) = v1. Since ϕ is nondegenerate, g is an isometry.   
然后，如果我们让v1=v0+y0，根据命题28.44，我们可以通过设置g（v）=v1，将f扩展到u+kv=e的公制图g。由于\_是非简并的，g是等距测量。

The first corollary of Witt’s theorem is sometimes called the Witt’s cancellation theorem.  
维特定理的第一个推论有时被称为维特对消定理。

Theorem 28.46. (Witt Cancellation Theorem) Let (E1,ϕ1) and (E2,ϕ2) be two pairs of finite-dimensional spaces and nondegenerate -Hermitian forms satisfying condition (T), and assume that (E1,ϕ1) and (E2,ϕ2) are isometric. For any subspace U of E1 and any subspace V of E2, if there is an isometry f : U → V , then there is an isometry g: U⊥ → V ⊥.  
定理28.46。（维特相消定理）设（e1，\_）和（e2，\_）为满足条件（t）的两对有限维空间和非退化厄米特形式，并假定（e1，\_）和（e2，\_）是等距的。对于e1的任何子空间u和e2的任何子空间v，如果存在等距f:u→v，则存在等距g:u→v。

Proof. If f : U → V is an isometry between U and V , by Witt’s theorem (Theorem 28.46), the linear map f extends to an isometry g between E1 and E2. We claim that g maps U⊥ into V ⊥. This is because if v ∈ U⊥, we have ϕ1(u,v) = 0 for all u ∈ U, so  
证据。如果f:u→v是u和v之间的等值线，根据维特定理（定理28.46），线性映射f延伸到e1和e2之间的等值线g。我们声称G把u映射成v。这是因为如果v u，我们将所有u u都取\_（u，v）=0，所以

ϕ2(g(u),g(v)) = ϕ1(u,v) = 0 for all u ∈ U,  
\_（g（u），g（v））=\_（u，v）=0，对于所有u∈u，

and since g is a bijection between U and V , we have g(U) = V , so we see that g(v) is orthogonal to V for every v ∈ U⊥; that is, g(U⊥) ⊆ V ⊥. Since g is a metric map and since ϕ1 is nondegenerate, the restriction of g to U⊥ is an isometry from U⊥ to V ⊥.   
因为g是u和v之间的双射，所以我们得到了g（u）=v，所以我们看到g（v）对于每个v∈u是与v正交的，也就是说，g（u）v。由于g是一个公制地图，且由于\_是非退化的，因此g对u的限制是从u到v的等距测量。

A pair (E,ϕ) where E is finite-dimensional and ϕ is a nondegenerate -Hermitian form is often called an -Hermitian space. When = 1 and ϕ is symmetric, we use the term Euclidean space or quadratic space. When is alternating, we use the term  
其中e是有限维的，而a是非简并的-厄米特形式的对（e，a），通常称为-厄米特空间。当=1和\_对称时，我们使用欧几里得空间或二次空间。当是交替的，我们用这个词

symplectic space. When = 1 and the automorphism λ 7→ λ is not the identity we use the term Hermitian space, and when 1, we use the term skew-Hermitian space.  
辛空间。当=1且自同构λ7→λ不是恒等式时，我们使用术语Hermitian空间；当1时，我们使用术语Skew Hermitian空间。

We also have the following result showing that the group of isometries of an -Hermitian space is transitive on totally isotropic subspaces of the same dimension.  
结果还表明，厄米空间的等轴测群在同一维的完全各向同性子空间上是可传递的。

Theorem 28.47. Let E be a finite-dimensional vector space and let ϕ be a nondegenerate -Hermitian form on E satisfying condition (T). Then for any two totally isotropic subspaces U and V of the same dimension, there is an isometry f ∈ Isom(ϕ) such that f(U) = V . Furthermore, every linear automorphism of U is induced by an isometry of E.  
定理28.47。设e为有限维向量空间，当e满足条件（t）时，设\_为非退化厄米形式。那么对于任意两个完全各向同性的同维子空间u和v，有一个等距f∈isom（），这样f（u）=v。此外，u的每一个线性自同构都是由e的一个等距引起的。

Remark: Witt’s cancelation theorem can be used to define an equivalence relation on Hermitian spaces and to define a group structure on these equivalence classes. This way, we obtain the Witt group, but we will not discuss it here.  
注：维特的抵消定理可以用来定义厄米特空间上的等价关系，也可以用来定义这些等价类上的群结构。这样，我们就得到了维特集团，但我们不会在这里讨论它。

Witt’s Theorem can be sharpened to isometries in SO(ϕ), but some condition on U is needed.  
威特定理可以在so中锐化为等轴测，但需要在u上有一些条件。

Theorem 28.48. (Witt–Sharpened Version) Let E be a finite-dimensional space equipped with a nondegenerate symmetric bilinear forms ϕ. For any subspace U of E, every linear injective metric map f from U into E extends to an isometry g of E with a prescribed value  
定理28.48。（witt–锐化版本）让e是一个配备非退化对称双线性形式的有限维空间。对于e的任何子空间u，从u到e的每一个线性内射度量图f都扩展到具有规定值的e的等距g。

±1 of det(g) iff dim(U) + dim(rad(U)) < dim(E) = n.  
±1 of det（g）iff dim（u）+dim（rad（u））<dim（e）=n.

If  
如果

dim(U) + dim(rad(U)) = dim(E) = n,  
dim（u）+dim（rad（u））=dim（e）=n，

and det(f) = −1, then there is no g ∈ SO(ϕ) extending f.  
且Det（f）=-1，则没有g∈so（）延伸f。

Proof. If g1 and g2 are two extensions of f such that det( 1, then h = g1−1 ◦g2 is an isometry such that det(h) = −1, and h leaves every vector of fixed. Conversely, if h is an isometry such that det(h) = −1, and h(u) = u for all u ∈ U, then for any extesnion g1 of f, the map g2 = h◦g1 is another extension of f such that det(g2) = −det(g1). Therefore, we need to show that a map h as above exists.  
证据。如果g1和g2是f的两个扩展，那么det（1，那么h=g1−1 g2是一个等距测量，这样det（h）=−1，h使每个向量保持固定。相反地，如果h是一个等距测量，使得所有u∈u的Det（h）=-1和h（u）=u，那么对于f的任何外轴g1，图g2=h g1是f的另一个延伸，这样Det（g2）=-Det（g1）。因此，我们需要证明上面的地图h是存在的。

If dim(U)+dim(rad(U)) < dim(E), consider the nondegenerate completion U of U given  
如果dim（u）+dim（rad（u））<dim（e），考虑给定u的非退化完成u

by Proposition 28.32. We know that dim(U) = dim(U) + dim(rad(U)) < n, and since U is nondegenerate, we have  
根据第28.32号提案。我们知道dim（u）=dim（u）+dim（rad（u））<n，由于u是非退化的，我们有

,  
，

with U⊥ = (0)6 . Pick any isometry τ of U⊥ such that det(τ) = −1, and extend it to an  
使用u=（0）6。选取u的任何等距τ，使det（τ）=1，并将其扩展到

isometry h of E whose restriction to U is the identity.  
e的等距H，其对u的限制是同一性。

If dim(U) + dim(rad(U)) = dim(E) = n, then U = V ⊕⊥ W with V = rad(U) and since  
如果dim（u）+dim（rad（u））=dim（e）=n，则u=v w，其中v=rad（u），自

dim(U) = dim(U) + dim(rad(U)) = n, we have  
dim（u）=dim（u）+dim（rad（u））=n，我们有

E = U = (V ⊕ V 0) ⊕⊥ W,  
E=U=（V V 0）W，

where V ⊕V 0 = Ar2r = W ⊥ is an Artinian space. Any isometry h of E which is the identity on U and with det(h) = −1 is the identity on W, and thus it must map W ⊥ = Ar2r = V ⊕V 0 into itself, and the restriction h0 of h to Ar2r has det(h0) = −1. However, h0 is the identity on V = rad(U), a totally isotropic subspace of Ar2r of dimension r, and by Proposition 28.42, we have det(h0) = +1, a contradiction.   
其中v v 0=ar2r=w是一个Artian空间。e的任何等距H是u上的同一性，且具有det（h）=-1，则是w上的同一性，因此它必须将w=ar2r=v v 0映射到自身中，而h对ar2r的限制h0具有det（h0）=-1。然而，h0是v=rad（u）上的恒等式，是r维ar2r的一个完全各向同性子空间，根据命题28.42，我们得到了det（h0）=+1，这是一个矛盾。

It can be shown that the center of O(ϕ) is {id,−id}. For further properties of orthogonal groups, see Grove [83], Jacobson [96], Taylor [169], and Artin [6].  
可以看出，O（\_）的中心是ID、−ID。关于正交群的进一步性质，请参见Grove[83]、Jacobson[96]、Taylor[169]和Artin[6]。

Part IV  
第四部分

Algebra: PID’s, UFD’s, Noetherian Rings, Tensors,  
代数：pid's，ufd's，noetherian环，张量，

Modules over a PID, Normal Forms  
PID上的模块，正常形式

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Chapter 29  
第二十九章

# Polynomials, Ideals and PID’s 多项式、理想和PID

## 29.1 Multisets 29.1多集

This chapter contains a review of polynomials and their basic properties. First, multisets are defined. Polynomials in one variable are defined next. The notion of a polynomial function in one argument is defined. Polynomials in several variable are defined, and so is the notion of a polynomial function in several arguments. The Euclidean division algorithm is presented, and the main consequences of its existence are derived. Ideals are defined, and the characterization of greatest common divisors of polynomials in one variables (gcd’s) in terms of ideals is shown. We also prove the Bezout identity. Next, we consider the factorization of polynomials in one variables into irreducible factors. The unique factorization of polynomials in one variable into irreducible factors is shown. Roots of polynomials and their multiplicity are defined. It is shown that a nonnull polynomial in one variable and of degree m over an integral domain has at most m roots. The chapter ends with a brief treatment of polynomial interpolation: Lagrange, Newton, and Hermite interpolants are introduced.  
本章回顾了多项式及其基本性质。首先，定义多集。接下来定义一个变量的多项式。定义了一个参数中多项式函数的概念。定义了几个变量的多项式，以及几个参数中多项式函数的概念。提出了欧几里得分割算法，并推导了其存在的主要后果。定义了理想，并给出了一个变量（GCD）中多项式最大公约数的理想特征。我们也证明了贝佐特的身份。接下来，我们考虑将一个变量中的多项式因式分解为不可约因子。证明了多项式在一个变量中的唯一因式分解为不可约因子。定义了多项式的根及其多重性。结果表明，一个变量的非零多项式和一个积分域上的m次多项式最多有m个根。最后简要介绍了多项式插值的方法：拉格朗日插值法、牛顿插值法和埃尔米特插值法。

In this chapter, it is assumed that all rings considered are commutative. Recall that a (commutative) ring A is an integral domain (or an entire ring) if 1 = 06 , and if ab = 0, then either a = 0 or b = 0, for all a,b ∈ A. This second condition is equivalent to saying that if a = 06 and b = 06 , then ab = 06 . Also, recall that a = 06 is not a zero divisor if ab 6= 0 whenever b = 06 . Observe that a field is an integral domain.  
在本章中，假设所有考虑的环都是交换的。回想一下，（交换）环A是一个积分域（或一个整环），如果1=06，并且如果a b=0，那么a=0或b=0，对于所有a，b∈a。第二个条件等价于说，如果a=06和b=06，那么ab=06。另外，如果a b 6=0，当b=06时，a=06不是零除数。观察一个场是一个积分域。

Our goal is to define polynomials in one or more indeterminates (or variables) X1,...,Xn, with coefficients in a ring A. This can be done in several ways, and we choose a definition that has the advantage of extending immediately from one to several variables. First, we need to review the notion of a (finite) multiset.  
我们的目标是在一个或多个不确定项（或变量）x1，…，xn中定义多项式，其中系数在环A中。这可以通过多种方式实现，并且我们选择的定义具有立即从一个变量扩展到多个变量的优点。首先，我们需要回顾一下（有限）多重集的概念。

Definition 29.1. Given a set I, a (finite) multiset over I is any function M : I → N such that M(i) = 06 for finitely many i ∈ I. The multiset M such that M(i) = 0 for all i ∈ I is the empty multiset, and it is denoted by 0. If M(i) = k = 06 , we say that i is a member of M of multiplicity k. The union M1 + M2 of two multisets M1 and M2 is defined such that  
定义29.1.给定一个集合i，i上的（有限）多集是任意函数m:i→n，这样m（i）=06表示有限多i∈i。多集m使得m（i）=0表示所有i∈i为空多集，并用0表示。如果m（i）=k=06，我们说i是多重性k的m的成员。两个多集m1和m2的联合m1+m2的定义如下：

(M1 +M2)(i) = M1(i)+M2(i), for every i ∈ I. If I is finite, say I = {1,...,n}, the multiset  
（m1+m2）（i）=m1（i）+m2（i），对于每一个i∈i，如果i是有限的，那么说i=1，…，n，多集

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M such that M(i) = ki for every i, 1 ≤ i ≤ n, is denoted by k1 · 1 + ··· + kn · n, or more simply, by (k1,...,kn), and deg(k1 · 1 + ··· + kn · n) = k1 + ··· + kn is the size or degree of M. The set of all multisets over I is denoted by N(I), and when I = {1,...,n}, by N(n).  
m使得m（i）=ki，对于每一个i，1≤i≤n，用k1·1+·················+kn·n表示，或者更简单地说，用（k1，…，kn）表示，deg（k1·1+·················+kn·n）=k1+·············+kn

Intuitively, the order of the elements of a multiset is irrelevant, but the multiplicity of each element is relevant, contrary to sets. Every i ∈ I is identified with the multiset Mi such that Mi(i) = 1 and Mi(j) = 0 for j =6 i. When I = {1}, the set N(1) of multisets k ·1 can be identified with N and {1}∗. We will denote k · 1 simply by k.  
直观地说，多集元素的顺序是不相关的，但是每个元素的多重性是相关的，与集相反。每一个i∈i都用多集mi来标识，使得mi（i）=1，mi（j）=0（j=6i），当i=1时，多集k·1的n（1）可以用n和1来标识。我们将简单地用k来表示k·1。

 However, beware that when n ≥ 2, the set N(n) of multisets cannot be identified with the set of strings in {1,...,n}∗, because multiset union is commutative, but concatenation of strings in {1,...,n}∗ is not commutative when n ≥ 2. This is because in a multiset k1 · 1 + ··· + kn · n, the order is irrelevant, whereas in a string, the order is relevant. For example, 2 · 1 + 3 · 2 = 3 · 2 + 2 · 1, but 11222 = 222116 , as strings over {1,2}.  
但是，要注意，当n≥2时，多集的集合n（n）不能用1，…，n中的字符串集来标识，因为多集并集是交换的，而1，…，n中的字符串的连接在n≥2时不是交换的。这是因为在多集k1·1+·····+kn·n中，顺序是不相关的，而在字符串中，顺序是相关的。例如，2·1+3·2=3·2+2·1，但11222=222116，作为1,2上的字符串。

Nevertherless, N(n) and the set Nn of ordered n-tuples under component-wise addition are isomorphic under the map  
然而，在映射下，按分量加法的有序n-元组的n（n）和集合nn是同构的。

k1 · 1 + ··· + kn · n 7→ (k1,...,kn).  
k1·1+·····+kn·n 7→（k1，…，kn）。

Thus, since the notation (k1,...,kn) is less cumbersome that k1 · 1 + ··· + kn · n, it will be preferred. We just have to remember that the order of the ki is really irrelevant.  
因此，由于符号（k1，…，kn）比k1·1+········+kn·n更为繁琐，因此最好使用它。我们只需要记住，ki的顺序确实是不相关的。

 But when I is infinite, beware that N(I) and the set NI of ordered I-tuples are not isomorphic.  
但当我无穷大时，要注意有序i-元组的n（i）和集ni不是同构的。

We are now ready to define polynomials.  
我们现在准备定义多项式。

## 29.2 Polynomials 29.2多项式

We begin with polynomials in one variable.  
我们从一个变量的多项式开始。

Definition 29.2. Given a ring A, we define the set PA(1) of polynomials over A in one variable as the set of functions P : N → A such that P(k) = 06 for finitely many k ∈ N. The polynomial such that P(k) = 0 for all k ∈ N is the null (or zero) polynomial and it is denoted by 0. We define addition of polynomials, multiplication by a scalar, and multiplication of polynomials, as follows: Given any three polynomials P,Q,R ∈ PA(1), letting ak = P(k), bk = Q(k), and ck = R(k), for every k ∈ N, we define R = P + Q such that  
定义29.2.对于一个环A，我们将一个变量上多项式的集合PA（1）定义为函数集P:N→A，这样对于有限多个k∈n，p（k）=06。对于所有k∈n，p（k）=0的多项式是零（或零）多项式，用0表示。我们定义了多项式的加法、标量乘法和多项式的乘法，如下：给定任意三个多项式p、q、r∈p a（1），让a k=p（k）、bk=q（k）和ck=r（k），对于每一个k∈n，我们定义r=p+q，使

ck = ak + bk,  
ck=ak+bk，

R = λP such that ck = λak,  
r=λp，因此ck=λak，

where λ ∈ A,  
式中，λ∈a，

and R = PQ such that  
R=PQ，因此

ck = X aibj.  
ck=x aibj.

i+j=k  
I+J＝K

We define the polynomial ek such that ek(k) = 1 and ek(i) = 0 for i =6 k. We also denote e0 by 1 when k = 0. Given a polynomial P, the ak = P(k) ∈ A are called the coefficients of P. If P is not the null polynomial, there is a greatest n ≥ 0 such that an = 0 (6 and thus, ak = 0 for all k > n) called the degree of P and denoted by deg(P). Then, P is written uniquely as  
我们定义多项式Ek，使Ek（k）=1，Ek（i）=0代表i=6 k。当k=0时，我们也用1表示e0。对于一个多项式p，a k=p（k）∈a被称为p的系数。如果p不是零多项式，则有一个最大的n≥0，使得a=0（6，因此，k>n的所有k=0）被称为p的度数，并用deg（p）表示。然后，p被唯一地写为

P = a0e0 + a1e1 + ··· + anen.  
P=a0e0+a1e1+····+anen。

When P is the null polynomial, we let deg(P) = −∞.  
当p是零多项式时，我们让deg（p）=-∞。

There is an injection of A into PA(1) given by the map a 7→ a1 (recall that 1 denotes e0). There is also an injection of N into PA(1) given by the map k →7 ek. Observe that  
在图A 7→A1给出的PA（1）中注入了A（记住1表示e0）。图K→7EK给出的PA（1）中也注入了N。注意

(with = 1). In order to alleviate the notation, we often denote e1 by X, and we call X a variable (or indeterminate). Then, is denoted by Xk. Adopting this notation, given a nonnull polynomial P of degree n, if P(k) = ak, P is denoted by  
（与=1）。为了减少符号，我们通常用x表示e1，我们称x为变量（或不确定）。然后，用xk表示。采用这个符号，给出n次的非零多项式p，如果p（k）=ak，p表示为

P = a0 + a1X + ··· + anXn,  
P=a0+a1x+····+anxn，

or by  
或通过

P = anXn + an−1Xn−1 + ··· + a0,  
P=anxn+an−1Xn−1+····+a0，

if this is more convenient (the order of the terms does not matter anyway). Sometimes, it will also be convenient to write a polynomial as  
如果这样更方便（条款的顺序无论如何都不重要）。有时，将多项式写成

P = a0Xn + a1Xn−1 + ··· + an.  
P=a0xn+a1xn−1+····+an。

The set PA(1) is also denoted by A[X] and a polynomial P may be denoted by P(X). In denoting polynomials, we will use both upper-case and lower-case letters, usually, P,Q, R,S, p,q,r,s, but also f,g,h, etc., if needed (as long as no ambiguities arise).  
集合p a（1）也用[x]表示，多项式p可用p（x）表示。在表示多项式时，我们将同时使用大写和小写字母，通常是P、Q、R、S、P、Q、R、S，如果需要，也可以使用F、G、H等（只要不出现歧义）。

Given a nonnull polynomial P of degree n, the nonnull coefficient an is called the leading coefficient of P. The coefficient a0 is called the constant term of P. A polynomial of the form akXk is called a monomial. We say that akXk occurs in P if ak = 06 . A nonzero polynomial P of degree n is called a monic polynomial (or unitary polynomial, or monic) if an = 1, where an is its leading coefficient, and such a polynomial can be written as  
给定n次的非零多项式p，非零系数an称为p的前导系数。系数a0称为p的常数项。形式为akxk的多项式称为单项式。如果ak=06，则akxk出现在p中。n次的非零多项式p称为monic多项式（或一元多项式，或monic），如果an=1，其中an是其导系数，这样的多项式可以写为

P = Xn + an−1Xn−1 + ··· + a0 or P = Xn + a1Xn−1 + ··· + an.  
P=xn+an−1Xn−1+·····+a0或P=xn+a1xn−1+····+an。

 The choice of the variable X to denote e1 is standard practice, but there is nothing special about X. We could have chosen Y , Z, or any other symbol, as long as no ambiguities arise.  
变量x表示e1的选择是标准做法，但x没有什么特别之处，只要没有歧义，我们可以选择y、z或任何其他符号。

Formally, the definition of PA(1) has nothing to do with X. The reason for using X is simply convenience. Indeed, it is more convenient to write a polynomial as P = a0 + a1X + ··· + anXn rather than as P = a0e0 + a1e1 + ··· + anen.  
从形式上讲，pa（1）的定义与x无关，使用x的原因很简单。实际上，将多项式写成p=a0+a1x+·····+anxn比写成p=a0e0+a1e1+····+anen更方便。

We have the following simple but crucial proposition.  
我们有以下简单但关键的命题。

Proposition 29.1. Given two nonnull polynomials P(X) = a0+a1X+···+amXm of degree m and Q(X) = b0 + b1X + ··· + bnXn of degree n, if either am or bn is not a zero divisor, then ambn = 06 , and thus, PQ = 06 and  
提案29.1.给出了两个非零多项式p（x）=a0+a1x+······················································

deg(PQ) = deg(P) + deg(Q).  
度（PQ）=deg（P）+deg（Q）。

In particular, if A is an integral domain, then A[X] is an integral domain.  
特别地，如果a是一个积分域，那么[x]是一个积分域。

Proof. Since the coefficient of Xm+n in PQ is ambn, and since we assumed that either am or an is not a zero divisor, we have ambn = 06 , and thus, PQ = 06 and  
证据。由于pq中xm+n的系数是ambn，并且由于我们假设am或an不是零除数，我们得到ambn=06，因此pq=06和

deg(PQ) = deg(P) + deg(Q).  
度（PQ）=deg（P）+deg（Q）。

Then, it is obvious that A[X] is an integral domain.   
很明显，a[x]是一个积分域。

It is easily verified that A[X] is a commutative ring, with multiplicative identity 1X0 = 1. It is also easily verified that A[X] satisfies all the conditions of Definition 3.1, but A[X] is not a vector space, since A is not necessarily a field.  
很容易证明[X]是一个交换环，其乘法恒等式为1X0=1。也很容易证明[X]满足定义3.1的所有条件，但[X]不是向量空间，因为A不一定是场。

A structure satisfying the axioms of Definition 3.1 when K is a ring (and not necessarily a field) is called a module. Modules fail to have some of the nice properties that vector spaces have, and thus, they are harder to study. For example, there are modules that do not have a basis. We postpone the study of modules until Chapter 34.  
当k是环（不一定是场）时，满足定义3.1公理的结构称为模块。模件没有向量空间所具有的一些好的性质，因此很难研究。例如，有些模块没有基础。我们将模块的研究推迟到第34章。

However, when the ring A is a field, A[X] is a vector space. But even when A is just a be written in a unique way asring, the family of polynomials (PX(Xk)) =k∈Nais a basis of0 + a1X + ···A[X+]a, since every polynomialnXn (with P(X) = 0 whenP(XP)(canX) is the null polynomial). Thus, A[X] is a free module.  
然而，当环A是一个场时，[X]是一个向量空间。但即使a只是以一种独特的方式写成，多项式族（p x（x k））=k∈nai也是0+a1x+·················································因此，一个[X]是一个自由模块。

Next, we want to define the notion of evaluating a polynomial P(X) at some α ∈ A. For this, we need a proposition.  
接下来，我们要定义在某个α∈a处计算多项式p（x）的概念，为此，我们需要一个命题。

Proposition 29.2. Let A,B be two rings and let h: A → B be a ring homomorphism. For any β ∈ B, there is a unique ring homomorphism ϕ: A[X] → B extending h such that ϕ(X) = β, as in the following diagram (where we denote by h+β the map h+β: A∪{X} → B such that (h + β)(a) = h(a) for all a ∈ A and (h + β)(X) = β):  
提案29.2.让a，b是两个环，让h:a→b是一个环同态。对于任何β∈b，都有一个唯一的环同态：a[x]→b延伸h，使得（x）=β，如下图所示（其中我们用h+β表示图h+β：a x→b，这样（h+β）（a）=h（a）表示所有a∈a和（h+β）（x）=β）：

A ∪ {X} ι / A[X]  
A X/A[X]

LLLLLLLLLL% ϕ h+β  
llllllllll%h+β

B  
乙

Proof. Let ϕ(0) = 0, and for every nonull polynomial P(X) = a0 + a1X + ··· + anXn, let  
证据。设\_（0）=0，对于每个非满多项式p（x）=a0+a1x+····+anxn，设

ϕ(P(X)) = h(a0) + h(a1)β + ··· + h(an)βn.  
⑨（P（x））=H（a0）+H（a1）β+·····+H（an）βn.

It is easily verified that ϕ is the unique homomorphism ϕ: A[X] → B extending h such that ϕ(X) = β.   
很容易证实，\_是唯一的同态\_：a[x]→b延伸h，使得\_（x）=β。

Taking A = B in Proposition 29.2 and h: A → A the identity, for every β ∈ A, there is a unique homomorphism ϕβ : A[X] → A such that ϕβ(X) = β, and for every polynomial P(X), we write ϕβ(P(X)) as P(β) and we call P(β) the value of P(X) at X = β. Thus, we can define a function PA : A → A such that PA(β) = P(β), for all β ∈ A. This function is called the polynomial function induced by P.  
以29.2和h:a→a的恒等式中的a=b为例，对于每一个β∈a，都有一个唯一的同态，即，对于每一个多项式p（x），我们把\_β（p（x））写为p（β），我们把p（β）称为p（x）在x=β时的值。因此，我们可以定义一个函数p a:a→a，使得pa（β）=p（β），对于所有的β∈a，这个函数称为p诱导的多项式函数。

More generally, PB can be defined for any (commutative) ring B such that A ⊆ B. In general, it is possible that PA = QA for distinct polynomials P,Q. We will see shortly conditions for which the map P 7→ PA is injective. In particular, this is true for A = R (in general, any infinite integral domain). We now define polynomials in n variables.  
更一般地说，p b可以定义为任何（交换）环b，这样a b。一般来说，对于不同的多项式p，q，pa=qa是可能的。我们将很快看到map p 7→pa是内射的条件。特别是，对于a=r（一般来说，任何无穷大的积分域），这是正确的。我们现在定义n个变量的多项式。

Definition 29.3. Given n ≥ 1 and a ring A, the set PA(n) of polynomials over A in n variables is the set of functions P : N(n) → A such that P(k1,...,kn) = 06 for finitely many (k1,...,kn) ∈ N(n). The polynomial such that P(k1,...,kn) = 0 for all (k1,...,kn) is the null (or zero) polynomial and it is denoted by 0. We define addition of polynomials, multiplication by a scalar, and multiplication of polynomials, as follows: Given any three polynomials P,Q,R ∈ PA(n), letting a(k1,...,kn) = P(k1,...,kn), b(k1,...,kn) = Q(k1,...,kn), c(k1,...,kn) = R(k1,...,kn), for every (k1,...,kn) ∈ N(n), we define R = P + Q such that  
定义29.3.给定n≥1和环a，n个变量上多项式的集合pa（n）是函数p:n（n）→a这样，p（k1，…，kn）=06表示有限多（k1，…，kn）∈n（n）。所有（k1，…，kn）的p（k1，…，kn）=0的多项式是零（或零）多项式，用0表示。我们定义了多项式的加法、标量乘法和多项式的乘法，如下：给定任意三个多项式p，q，r∈p a（n），让a（k1，…，kn）=p（k1，…，kn），b（k1，…，kn）=q（k1，…，kn），c（k1，…，kn）=r（k1，…，kn），对于每（k1，…，kn）∈n（n），我们定义r=p。+问题是这样的

c(k1,...,kn) = a(k1,...,kn) + b(k1,...,kn),  
c（k1，…，kn）=a（k1，…，kn）+b（k1，…，kn）

R = λP, where λ ∈ A, such that  
r=λp，其中λ∈a，这样

c(k1,...,kn) = λa(k1,...,kn),  
c（k1，…，kn）=λa（k1，…，kn）

and R = PQ, such that  
R=PQ，这样

c(k1,...,kn) = X a(i1,...,in)b(j1,...,jn).  
c（k1，…，kn）=x a（i1，…，in）b（j1，…，jn）。

(i1,...,in)+(j1,...,jn)=(k1,...,kn)  
（i1，…，in）+（j1，…，jn）=（k1，…，kn）

For every (k1,...,kn) ∈ N(n), we let e(k1,...,kn) be the polynomial such that  
对于每一（k1，…，kn）∈n（n），我们让e（k1，…，kn）是多项式，这样

e(k1,...,kn)(k1,...,kn) = 1 and e(k1,...,kn)(h1,...,hn) = 0,  
e（k1，…，kn）（k1，…，kn）=1，e（k1，…，kn）（h1，…，hn）=0，

for (h1,...,hn) = (6 k1,...,kn). We also denote e(0,...,0) by 1. Given a polynomial P, the a(k1,...,kn) = P(k1,...,kn) ∈ A, are called the coefficients of P. If P is not the null polynomial, there is a greatest d ≥ 0 such that a(k1,...,kn) = 06 for some (k1,...,kn) ∈ N(n), with d = k1 + ··· + kn, called the total degree of P and denoted by deg(P). Then, P is written uniquely as  
对于（h1，…，hn）=（6 k1，…，kn）。我们也用1表示e（0，…，0）。给定一个多项式p，a（k1，…，kn）=p（k1，…，kn）∈a，称为p的系数，如果p不是零多项式，则有一个最大d≥0，使得a（k1，…，kn）=06对于某些（k1，…，kn）∈n（n），d=k1+········+kn，称为p的总次数，用deg（p）表示。然后，p被唯一地写为

P = X a(k1,...,kn)e(k1,...,kn).  
P=x a（k1，…，kn）e（k1，…，kn）。

(k1,...,kn)∈N(n)  
（k1，…，kn）∈n（n）

When P is the null polynomial, we let deg(P) = −∞.  
当p是零多项式时，我们让deg（p）=-∞。

There is an injection of A into PA(n) given by the map a 7→ a1 (where 1 denotes e(0,...,0)). There is also an injection of N(n) into PA(n) given by the map (h1,...,hn) 7→ e(h1,...,hn). Note that e(h1,...,hn)e(k1,...,kn) = e(h1+k1,...,hn+kn). In order to alleviate the notation, let X1,...,Xn be n distinct variables and denote e(0,...,0,1,0...,0), where 1 occurs in the position i, by Xi (where 1 ≤ i ≤ n). With this convention, in view of e(h1,...,hn)e(k1,...,kn) = e(h1+k1,...,hn+kn), the polynomial e(k1,...,kn) is denoted by (with = 1) and it is called  
图A 7→A1（其中1表示e（0，…，0））给出了a注入到pa（n）中。图（h1，…，hn）7→e（h1，…，hn）给出的PA（n）中也注入了n（n）。注意e（h1，…，hn）e（k1，…，kn）=e（h1+k1，…，hn+kn）。为了减轻符号，让X1，…，Xn是n个不同的变量，并表示E（0，…，0，1,0…，0），其中1出现在位置I，由Xi（其中1个i i小于n）。根据这个惯例，考虑到e（h1，…，hn）e（k1，…，kn）=e（h1+k1，…，hn+kn），多项式e（k1，…，kn）用（with=1）表示，称为

a primitive monomial. Then, P is also written as  
原始的单项式。那么，p也写为

.  
.

We also denote PA(n) by A[X1,...,Xn]. A polynomial P ∈ A[X1,...,Xn] is also denoted by P(X1,...,Xn).  
我们也用[x1，…，xn]表示pa（n）。多项式p∈a[x1，…，xn]也用p（x1，…，xn）表示。

As in the case n = 1, there is nothing special about the choice of X1,...,Xn as variables (or indeterminates). It is just a convenience. After all, the construction of PA(n) has nothing to do with X1,...,Xn.  
在n=1的情况下，选择x1，…，xn作为变量（或不确定）没有什么特别的。这只是一种方便。毕竟，PA（n）的构造与x1，…，xn无关。

Given a nonnull polynomial P of degree d, the nonnull coefficients a(k1,...,kn) = 06 such that d = k1 + ··· + kn are called the leading coefficients of P. A polynomial of the form  
给定d次的非零多项式p，非零系数a（k1，…，kn）=06，使d=k1+·····+kn称为p的前导系数。该形式的多项式

is called a monomial. Note that deg(.  
称为单项式。注意度数（.

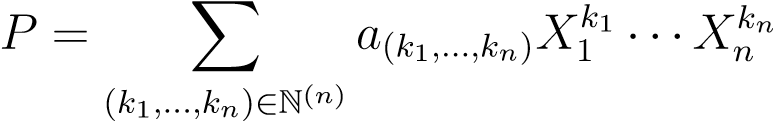
Given a polynomial  
给定多项式

P = X a(k1,...,kn)X1k1 ···Xnkn,  
P=x a（k1，…，kn）x1k1···xnkn，

(k1,...,kn)∈N(n)  
（k1，…，kn）∈n（n）

a monomial occurs in the polynomial P if a(k1,...,kn) = 0.6  
如果a（k1，…，kn）=0.6，多项式p中会出现一个单项式。

A polynomial  
多项式



is homogeneous of degree d if deg(  
D度均匀度（

for every monomialoccurring in P. If P is a polynomial of total degree d, it is clear that P can be written uniquely as  
对于p中的每一个单项式，如果p是总次数d的多项式，很明显p可以唯一地写成

P = P(0) + P(1) + ··· + P(d),  
P=P（0）+P（1）+····+P（d）、

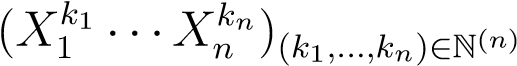
where P(i) is the sum of all monomials of degree i occurring in P, where 0 ≤ i ≤ d.  
式中，p（i）是p中出现的所有i级单项式的总和，其中0≤i≤d。

It is easily verified that A[X1,...,Xn] is a commutative ring, with multiplicative identity  
很容易证明[x1，…，xn]是一个交换环，具有乘法恒等式。

= 1. It is also easily verified that A[X] is a module. When A is a field, A[X] is  
= 1。也很容易验证[X]是一个模块。当a是一个字段时，a[x]是

a vector space.  
向量空间。

Even when A is just a ring, the family of polynomials  
即使a只是一个环，多项式家族



is a basis of A[X1,...,Xn], since every polynomial P(X1,...,Xn) can be written in a unique way as  
是一个[x1，…，xn]的基础，因为每个多项式p（x1，…，xn）都可以用一种独特的方式写为

.  
.

Thus, A[X1,...,Xn] is a free module.  
因此，[x1，…，xn]是一个自由模块。

Remark: The construction of Definition 29.3 can be immediately extended to an arbitrary set I, and not just I = {1,...,n}. It can also be applied to monoids more general that N(I). Proposition 29.2 is generalized as follows.  
注：定义29.3的构造可以立即扩展到任意集i，而不仅仅是i=1，…，n。它也可以应用于比n（i）更一般的单倍体。命题29.2概括如下。

Proposition 29.3. Let A,B be two rings and let h: A → B be a ring homomorphism. For any β = (β1,...,βn) ∈ Bn, there is a unique ring homomorphism ϕ: A[X1,...,Xn] → B extending h such that ϕ(Xi) = βi, 1 ≤ i ≤ n, as in the following diagram (where we denote by h + β the map h + β: A ∪ {X1,...,Xn} → B such that (h + β)(a) = h(a) for all a ∈ A and (h + β)(Xi) = βi, 1 ≤ i ≤ n):  
提案29.3.让a，b是两个环，让h:a→b是一个环同态。对于任何β=（β1，…，βn）BN，存在一个唯一的环同态，即：[x1，…，Xn ] -b扩展h，使得（Xi）＝βi，1±i，n，如下面的图（其中H+β表示图H+β：α{x1，…，xn}）b，使得（a +）（h）＝h（a），对于所有αa和（H+β）（x）（x）i）=βi，1≤i≤n）：

A ∪ {X1,...,Xn} ι / A[X1,...,Xn]  
a x1，…，xn/a[x1，…，xn]

TTTTTTTTTTTTTTTTT) Bϕ h+β  
ttttttttttttttttt）b\_h+β

Proof. Let ϕ(0) = 0, and for every nonull polynomial  
证据。设\_（0）=0，对于每个非满多项式

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，

let  
让

.  
.

It is easily verified that ϕ is the unique homomorphism ϕ: A[X1,...,Xn] → B extending h such that ϕ(Xi) = βi.   
很容易证明，Xn是唯一的同态：A[x1，…，Xn ] -B扩展H，使得（Xi）＝βI。

Taking A = B in Proposition 29.3 and h: A → A the identity, for every β1,...,βn ∈ A, there is a unique homomorphism ϕ: A[X1,...,Xn] → A such that ϕ(Xi) = βi, and for every polynomial P(X1,...,Xn), we write ϕ(P(X1,...,Xn)) as P(β1,...,βn) and we call P(β1,...,βn) the value of P(X1,...,Xn) at X1 = β1,...,Xn = βn. Thus, we can define a function PA : An → A such that PA(β1,...,βn) = P(β1,...,βn), for all β1,...,βn ∈ A. This function is called the polynomial function induced by P.  
取命题29.3中的a= b和h：a，a，β，n，β，n，有一个唯一的同态，即[x1，…，Xn ]：a，（x），βi，对于每个多项式p（x1，…，xn），我们将p（x1，…，Xn）写为p（β1，…，βn），我们称p（β1，…，βn）p（x1，…）的值。，xn）在x1=β1，…，xn=βn。因此，我们可以定义一个函数p a:an→a，使得pa（β1，…，βn）=p（β1，…，βn），对于所有β1，…，βn∈a。这个函数称为p诱导的多项式函数。

More generally, PB can be defined for any (commutative) ring B such that A ⊆ B. As in the case of a single variable, it is possible that PA = QA for distinct polynomials P,Q. We will see shortly that the map P →7 PA is injective when A = R (in general, any infinite integral domain).  
更一般地说，p b可以定义为任何（交换）环b，这样a b。在单变量的情况下，对于不同的多项式p，q，pa=qa是可能的。我们很快就会看到，当a=r时，map p→7pa是内射的（一般来说，任何无限积分域）。

Given any nonnull polynomial in  
给定任意非空多项式

A[X1,...,Xn], where n ≥ 2, P(X1,...,Xn) can be uniquely written as  
a[x1，…，xn]，其中n≥2，p（x1，…，xn）可以唯一地写为

,  
，

where each polynomial Qkn(X1,...,Xn−1) is in A[X1,...,Xn−1]. Even if A is a field, A[X1,...,Xn−1] is not a field, which confirms that it is useful (and necessary!) to consider polynomials over rings that are not necessarily fields.  
其中，每个多项式qkn（x1，…，xn−1）位于[x1，…，xn−1]中。即使a是一个字段，a[x1，…，xn−1]也不是一个字段，这证实了它是有用的（也是必要的！）考虑不一定是场的环上的多项式。

It is not difficult to show that A[X1,...,Xn] and A[X1,...,Xn−1][Xn] are isomorphic rings. This way, it is often possible to prove properties of polynomials in several variables X1,...,Xn, by induction on the number n of variables. For example, given two nonnull polynomials P(X1,...,Xn) of total degree p and Q(X1,...,Xn) of total degree q, since we assumed that A is an integral domain, we can prove that  
不难证明[x1，…，xn]和[x1，…，xn−1][xn]是同构环。这样，通过对变量数n的归纳，通常可以证明多个变量x1，…，xn中多项式的性质。例如，假设总度数p的两个非空多项式p（x1，…，xn）和总度数q的q（x1，…，xn），因为我们假设a是一个积分域，我们可以证明

deg(PQ) = deg(P) + deg(Q),  
度（PQ）=deg（P）+deg（Q）

and that A[X1,...,Xn] is an integral domain.  
而[x1，…，xn]是一个积分域。

Next, we will consider the division of polynomials (in one variable).  
接下来，我们将考虑多项式的除法（在一个变量中）。

## 29.3 Euclidean Division of Polynomials 29.3欧氏多项式划分

We know that every natural number n ≥ 2 can be written uniquely as a product of powers of prime numbers and that prime numbers play a very important role in arithmetic. It would be nice if every polynomial could be expressed (uniquely) as a product of “irreducible” factors. This is indeed the case for polynomials over a field. The fact that there is a division algorithm for the natural numbers is essential for obtaining many of the arithmetical properties of the natural numbers. As we shall see next, there is also a division algorithm for polynomials in A[X], when A is a field.  
我们知道每一个自然数n≥2都可以作为素数幂的乘积唯一地写，素数在算术中起着非常重要的作用。如果每一个多项式都可以（唯一地）表示为“不可约”因子的乘积，那就太好了。这确实是一个领域的多项式的情况。自然数有一个除法，这一事实对于获得自然数的许多算术性质是至关重要的。正如我们接下来将看到的，当a是一个场时，a[x]中还有一个多项式的除法。

Proposition 29.4. Let A be a ring, let f(X),g(X) ∈ A[X] be two polynomials of degree m = deg(f) and n = deg(g) with f(X) 6= 0 , and assume that the leading coefficient am of f(X) is invertible. Then, there exist unique polynomials q(X) and r(X) in A[X] such that  
提案29.4.设a为环，设f（x），g（x）∈a[x]为m=deg（f）和n=deg（g）的两个多项式，f（x）6=0，并假定f（x）的导系数am是可逆的。然后，在a[x]中存在唯一的多项式q（x）和r（x），这样

g = fq + r and deg(r) < deg(f) = m.  
G=FQ+R和deg（R）<deg（F）=m。

Proof. We first prove the existence of q and r. Let  
证据。我们首先证明q和r的存在。

f = amXm + am−1Xm−1 + ··· + a0,  
f=amxm+am−1xm−1+····+a0，

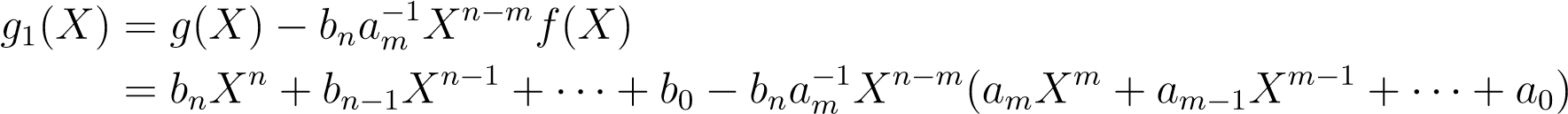
and g = bnXn + bn−1Xn−1 + ··· + b0.  
g=bnxn+bn−1xn−1+····+b0。

If n < m, then let q = 0 and r = g. Since deg(g) < deg(f) and r = g, we have deg(r) < deg(f).  
如果n<m，则q=0，r=g。由于deg（g）<deg（f）和r=g，我们得到deg（r）<deg（f）。

29.3. EUCLIDEAN DIVISION OF POLYNOMIALS  
29.3。欧氏多项式划分

If n ≥ m, we proceed by induction on n. If n = 0, then g = b0, m = 0, f = a0 6= 0 , and we let q = a−0 1b0 and r = 0. Since deg(r) = deg(0) = −∞ and deg(f) = deg(a0) = 0 because a0 = 06 , we have deg(r) < deg(f).  
如果n≥m，我们在n上进行诱导。如果n=0，那么g=b0，m=0，f=a0 6=0，我们让q=a−0 1b0和r=0。因为deg（r）=deg（0）=-∞和deg（f）=deg（a0）=0，因为a0=06，我们得到deg（r）<deg（f）。

If n ≥ 1, since n ≥ m, note that  
如果n≥1，因为n≥m，注意



is a polynomial of degree deg(g1) < n, since the terms bnXn and bnam−1Xn−mamXm of degree n cancel out. Now, since deg(g1) < n, by the induction hypothesis, we can find q1 and r such that g1 = fq1 + r and deg(r) < deg(f) = m,  
是degree deg（g1）<n的多项式，因为degree n的术语bnxn和bnam−1xn−mamxm取消。现在，由于deg（g1）<n，根据诱导假设，我们可以找到q1和r，使得g1=fq1+r和deg（r）<deg（f）=m，

and thus,  
因此，

g1(X) = g(X) − bna−m1Xn−mf(X) = f(X)q1(X) + r(X),  
g1（x）=g（x）−bna−m1xn−mf（x）=f（x）q1（x）+r（x），

from which, letting ), we get  
我们从中得到

g = fq + r and deg(r) < m = deg(f).  
G=FQ+R和deg（R）<M=deg（F）。

We now prove uniqueness. If  
我们现在证明了它的独特性。如果

g = fq1 + r1 = fq2 + r2,  
g=fq1+r1=fq2+r2，

with deg(r1) < deg(f) and deg(r2) < deg(f), we get  
当deg（r1）<deg（f）和deg（r2）<deg（f）时，我们得到

f(q1 − q2) = r2 − r1.  
F（q1−q2）=r2−r1。

If q2 −q1 = 06 , since the leading coefficient am of f is invertible, by Proposition 29.1, we have  
如果q2−q1=06，因为f的超前系数am是可逆的，根据命题29.1，我们得到

deg(r2 − r1) = deg(f(q1 − q2)) = deg(f) + deg(q2 − q1),  
度（r2−r1）=deg（f（q1−q2））=deg（f）+deg（q2−q1），

and so, deg(r2−r1) ≥ deg(f), which contradicts the fact that deg(r1) < deg(f) and deg(r2) < deg(f). Thus, q1 = q2, and then also r1 = r2.   
因此，deg（r2−r1）≥deg（f），这与deg（r1）<deg（f）和deg（r2）<deg（f）的事实相矛盾。因此，q1=q2，然后r1=r2。

It should be noted that the proof of Proposition 29.4 actually provides an algorithm for finding the quotient q and the remainder r of the division of g by f. This algorithm is called the Euclidean algorithm, or division algorithm. Note that the division of g by f is always possible when f is a monic polynomial, since 1 is invertible. Also, when A is a field, am = 06 is always invertible, and thus, the division can always be performed. We say that f divides g when r = 0 in the result of the division g = fq + r. We now draw some important consequences of the existence of the Euclidean algorithm.  
需要指出的是，29.4命题的证明实际上提供了一种求g除以f的商q和余数r的算法。这种算法被称为欧几里得算法或除法。注意，当f是Monic多项式时，g除以f总是可能的，因为1是可逆的。此外，当a是一个字段时，am=06总是可逆的，因此，可以始终执行除法。我们认为，当R=0时，F在G=FQ+R的除法结果中对G进行除法，得出了欧几里得算法存在的一些重要结果。

## 29.4 Ideals, PID’s, and Greatest Common Divisors 29.4理想、PID和最大公约数

First, we introduce the fundamental concept of an ideal.  
首先，我们介绍理想的基本概念。

Definition 29.4. Given a ring A, an ideal of A is any nonempty subset I of A satisfying the following two properties:  
定义29.4.给定环A，A的理想是满足以下两个性质的A的任何非空子集I：

(ID1) If a,b ∈ I, then b − a ∈ I.  
（id1）如果a，b∈i，则b−a∈i。

(ID2) If a ∈ I, then ax ∈ I for every x ∈ A.  
（id2）如果a∈i，则每x∈a的ax∈i。

An ideal I is a principal ideal if there is some a ∈ I, called a generator, such that  
一个理想i是一个主理想，如果有一个∈i，称为发生器，这样

I = {ax | x ∈ A}.  
i=a x x∈a。

The equality I = {ax | x ∈ A} is also written as I = aA or as I = (a). The ideal  
等式i=a x x∈a也写为i=a a或i=（a）。

I = (0) = {0} is called the null ideal (or zero ideal).  
i=（0）=0称为零理想（或零理想）。

An idealEquivalently,In other words,An idealI is aIIis ais a prime ideal ifAprime ideal−maximal idealI is closed under multiplication and 1if I =6 Iif=6AIA=6and ifand ifA and for every idealaba,b∈ ∈I, thenA−I, thena∈∈AJI−ab =6orI.A∈b, ifA∈−III, for all⊆, for allJ, thena,ba,bJ∈∈=AAI...  
一个理想，换句话说，理想是一个基本理想，如果一个理想−最大理想在乘法下闭合，并且1if i=6 iif=6a i a=6和ifand ifa，对于每一个理想，b−i，thena∈aji−ab=6ori，a∈b，ifa−iii，对于所有，对于allj，thena，ba，bj∈∈=AAI…

Note that if I is an ideal, then I = A iff 1 ∈ I. Since by definition, an ideal I is nonempty, there is some a ∈ I, and by (ID1) we get 0 = a − a ∈ I. Then, for every a ∈ I, since 0 ∈ I, by (ID1) we get −a ∈ I. Thus, an ideal is an additive subgroup of A. Because of (ID2), an ideal is also a subring.  
注意，如果我是一个理想，那么i=a iff 1∈i。因为根据定义，一个理想i是非空的，有一些a∈i，通过（id1）我们得到0=a−a∈i。然后，对于每一个a∈i，由于0∈i，通过（id1）我们得到−a∈i。因此，理想是a的加性子群。因为（id2），一个理想也是一个子环。

Observe that if A is a field, then A only has two ideals, namely, the trivial ideal (0) and A itself. Indeed, if I 6= (0), because every nonnull element has an inverse, then 1 ∈ I, and thus, I = A.  
如果A是一个场，那么A只有两个理想，即平凡理想（0）和A本身。实际上，如果i 6=（0），因为每个非空元素都有一个逆元素，那么1∈i，因此i=a。

Definition 29.5.of a and that a dividesGiven a ringb if b = acA, for any two elementsfor some c ∈ A; this is usually denoted bya,b ∈ A we say thatab|is a multipleb.  
a的定义29.5.如果b=aca，则a分格在a环b中，对于某些c∈a的任何两个元素，这通常表示为a，b∈a，我们称ab是一个倍数b。

Note that the principal ideal (a) is the set of all multiples of a, and that a divides b iff b is a multiple of a iff b ∈ (a) iff (b) ⊆ (a).  
注意，主理想（a）是a的所有倍数的集合，并且a除以b iff b是iff b∈（a）iff（b）（a）的倍数。

iff acNote that every= 0 for someac∈= 06 A . With this convention, 0 is a zero divisor unlessdivides 0. However, it is customary to say that a is aA zero divisor= {0} (the trivial ring), and A is an integral domain iff 0 is the only zero divisor in A.  
iff ac注意，someac的每=0∈=06a。按照这个约定，0是零除数，除非除数是0。然而，习惯上说a是aa零除数=0（平凡环），a是积分域，iff 0是a中唯一的零除数。

Given a,b ∈ A with a,b 6= 0, if (a) = (b) then there exist c,d ∈ A such that a = bc and is an integral domain, we getb = ad. From this, we get a =dcadc= 1andandb =cdbcd= 1, that is,, that is,a(1c is invertible with inverse−dc) = 0 and b(1−cd) = 0d. Thus,. If A when A is an integral domain, we have b = ad, with d invertible. The converse is obvious, if b = ad with d invertible, then (a) = (b).  
给定a，b∈a与a，b 6=0，如果（a）=（b），则存在c，d∈a，使a=bc且是一个积分域，我们得到b=ad。由此，我们得到a=dcadc=1 anddb=cdbcd=1，即a（1c是可逆的，逆−dc）=0，b（1−cd）=0d。因此，…如果A是一个积分域，我们有b=a d，d是可逆的。反之很明显，如果b=a d，d可逆，则（a）=（b）。

It is worth recording this fact as the following proposition.  
这一事实值得记录为以下命题。

Proposition 29.5. If A is an integral domain, for any a,b ∈ A with a,b = 06 , we have (a) = (b) iff there exists some invertible d ∈ A such that b = ad.  
提案29.5。如果a是一个积分域，对于任意a，b∈a，b=06，我们得到（a）=（b）如果存在一些可逆d∈a，那么b=ad。

An invertible element u ∈ A is also called a unit. Given two ideals I and J, their sum  
可逆元素u∈a也称为单位。给定两个理想i和j，它们的和

I + J = {a + b | a ∈ I, b ∈ J}  
i+j=a+b a∈i，b∈j

is clearly an ideal. Given any nonempty subset J of A, the set  
显然是个理想。给定a的任何非空子集j，集合

{a1x1 + ··· + anxn | x1,...,xn ∈ A, a1,...,an ∈ J, n ≥ 1}  
a1 x1+······+a n xn x1，…，xn∈a，a1，…，a n∈j，n≥1

is easily seen to be an ideal, and in fact, it is the smallest ideal containing J. It is usually denoted by (J).  
很容易被认为是理想，事实上，它是包含j的最小理想，通常用（j）表示。

Ideals play a very important role in the study of rings. They tend to show up everywhere. For example, they arise naturally from homomorphisms.  
理想在环的研究中起着非常重要的作用。他们经常出现在任何地方。例如，它们自然地产生于同态。

Proposition 29.6. Given any ring homomorphism h: A → B, the kernel Kerh = {a ∈ A | h(a) = 0} of h is an ideal.  
提案29.6.在任意环同态h:a→b下，h的核kerh=a∈a h（a）=0是一个理想。

Proof. Given a,b ∈ A, we have a,b ∈ Kerh iff h(a) = h(b) = 0, and since h is a homomorphism, we get  
证据。给定a，b∈a，我们有a，b∈kerh iff h（a）=h（b）=0，由于h是同态，我们得到

h(b − a) = h(b) − h(a) = 0,  
h（b−a）=h（b）−h（a）=0，

and h(ax) = h(a)h(x) = 0  
h（a x）=h（a）h（x）=0

for all x ∈ A, which shows that Kerh is an ideal.   
对于所有的x∈a，这表明kerh是一个理想。

There is a sort of converse property. Given a ring A and an ideal I ⊆ A, we can define the quotient ring A/I, and there is a surjective homomorphism π: A → A/I whose kernel is precisely I.  
有一种逆向性质。给定一个环A和一个理想的I A，我们可以定义商环A/I，并且有一个同态π：A→A/I，它的核正好是I。

Proposition 29.7. Given any ring A and any ideal I ⊆ A, the equivalence relation ≡I defined by a ≡I b iff b − a ∈ I is a congruence, which means that if a1 ≡I b1 and a2 ≡I b2, then  
提案29.7。对于任何环A和任何理想i a，由a i b iff b−a∈i定义的等价关系i是一个同余，这意味着如果a1 i b1和a2 i b2，那么

1. a1 + a2 ≡I b1 + b2, and  
   A1+A2 I B1+B2，以及
2. a1a2 ≡I b1b2.  
   A1A2 I B1B2。

Then, the set A/I of equivalence classes modulo I is a ring under the operations  
那么，等价类模I的集合A/I就是操作下的一个环

[a] + [b] = [a + b]  
[A]+[B]=[A+B]

[a][b] = [ab].  
[A][B]=[AB]。

The map π: A → A/I such that π(a) = [a] is a surjective homomorphism whose kernel is precisely I.  
图π：a→a/i使得π（a）=[a]是一个主观性同态，它的核正好是i。

Proof. Everything is straightforward. For example, if a1 ≡I b1 and a2 ≡I b2, then b1 −a1 ∈ I and b2 − a2 ∈ I. Since I is an ideal, we get  
证据。一切都很简单。例如，如果a1 i b1和a2 i b2，那么b1 a1∈i和b2−a2∈i，因为i是一个理想，我们得到

(b1 − a1)b2 = b1b2 − a1b2 ∈ I  
（b1−a1）b2=b1b2−a1b2∈i

and  
和

(b2 − a2)a1 = a1b2 − a1a2 ∈ I.  
（b2−a2）a1=a1b2−a1a2∈i。

Since I is an ideal, and thus, an additive group, we get  
因为我是一个理想，因此是一个加性群，我们得到

b1b2 − a1a2 ∈ I,  
b1b2−a1a2∈i，

i.e., a1a2 ≡I b1b2. The equality Kerπ = I holds because I is an ideal.   
即A1A2 I B1B2。等式kπ=我持有，因为我是一个理想。

Example 29.1.  
例29.1。

1. In the ring Z, for every p ∈ Z, the subroup pZ is an ideal, and Z/pZ is a ring, the ring of residues modulo p. This ring is a field iff p is a prime number.  
   在环Z中，对于每一个p∈z，子上的pz是一个理想的，z/pz是一个环，剩余模的环p。这个环是一个域iff p是一个素数。
2. The quotient of the polynomial ring R[X] by a prime ideal I is an integral domain.  
   素理想i对多项式环r[x]的商是一个积分域。
3. The quotient of the polynomial ring R[X] by a maximal ideal I is a field. For example, if I = (X2 + 1), the principal ideal generated by X2 + 1 (which is indeed a maximal ideal since X2 + 1 has no real roots), then R[X]/(X2 + 1) ∼= C.  
   多项式环r[x]被最大理想i的商是一个场。例如，如果i=（x2+1），由x2+1生成的主理想（由于x2+1没有实根，因此实际上是最大理想），则r[x]/（x2+1）=c。

The following proposition yields a characterization of prime ideals and maximal ideals in terms of quotients.  
下面的命题用商来描述素理想和最大理想。

Proposition 29.8. Given a ring A, for any ideal I ⊆ A, the following properties hold.  
提案29.8。对于任何理想i a，给定一个环a，以下属性保持不变。

1. The ideal I is a prime ideal iff A/I is an integral domain.  
   理想i是一个素理想，如果a/i是一个积分域。
2. The ideal I is a maximal ideal iff A/I is a field.  
   理想i是最大理想iff a/i是一个场。

Proof. (1) Assume that I is a prime ideal. Since I is prime, I =6 A, and thus, A/I is not the trivial ring (0). If [a][b] = 0, since [a][b] = [ab], we have ab ∈ I, and since I is prime, then either a ∈ I or b ∈ I, so that either [a] = 0 or [b] = 0. Thus, A/I is an integral domain.  
证据。（1）假设我是一个基本理想。因为i是素数，i=6a，因此a/i不是平凡的环（0）。如果[a][b]=0，因为[a][b]=ab]，我们有ab∈i，并且因为i是素数，那么a∈i或b∈i，所以[a]=0或[b]=0。因此，A/I是一个积分域。

Conversely, assume that A/I is an integral domain. Since A/I is not the trivial ring, I =6 A. Assume that ab ∈ I. Then, we have  
相反，假设a/i是一个积分域。既然a/i不是平凡环，i=6a，假设ab∈i，那么

π(ab) = π(a)π(b) = 0,  
π（a b）=π（a）π（b）=0，

which implies that either π(a) = 0 or π(b) = 0, since A/I is an integral domain (where π: A → A/I is the quotient map). Thus, either a ∈ I or b ∈ I, and I is a prime ideal.  
这意味着π（a）=0或π（b）=0，因为a/i是一个积分域（其中π：a→a/i是商映射）。因此，无论是a∈i还是b∈i，我都是素理想。

(2) Assume that I is a maximal ideal. As in (1), A/I is not the trivial ring (0). Let [a] = 06 in A/I. We need to prove that [a] has a multiplicative inverse. Since [a] = 06 , we have a /∈ I. Let Ia be the ideal generated by I and a. We have  
（2）假设我是最大理想。如（1）所示，A/I不是普通环（0）。假设A/I中[A]=06，我们需要证明[A]有一个乘法逆。从[a]=06开始，我们有一个/∈i。让ia是i和a产生的理想。我们有

I ⊆ Ia and I =6 Ia,  
i ia和i=6 ia，

since a /∈ I, and since I is maximal, this implies that  
既然a/∈i，既然i是极大的，这就意味着

Ia = A.  
IA=A。

However, we know that  
但是，我们知道

Ia = {ax + h | x ∈ A, h ∈ I},  
i a=a x+h x∈a，h∈i，

and thus, there is some x ∈ A so that  
因此，有一些x∈a，所以

ax + h = 1,  
ax+h=1，

which proves that [a][x] = [1], as desired.  
这证明了[a][x]=[1]，如所需。

Conversely, assume that A/I is a field. Again, since A/I is not the trivial ring, I =6 A. Let J be any proper ideal such that I ⊆ J, and assume that I =6 J. Thus, there is some j ∈ J − I, and since Kerπ = I, we have π(j) = 06 . Since A/I is a field and π is surjective, there is some k ∈ A so that π(j)π(k) = 1, which implies that  
相反，假设A/I是一个字段。再者，因为a/i不是平凡环，i=6a。让j是任何合适的理想，这样i j，假设i=6j。因此，有一些j∈j−i，由于kerπ=i，我们有π（j）=06。由于a/i是一个场，π是可射的，所以有一些k∈a，所以π（j）π（k）=1，这意味着

jk − 1 = i  
jk−1=i

for some i ∈ I, and since I ⊂ J and J is an ideal, it follows that 1 = jk − i ∈ J, showing that J = A, a contradiction. Therefore, I = J, and I is a maximal ideal.   
对于某些i i，由于i j和j是一个理想，它遵循1=jk−i j，表明j=a是一个矛盾。因此，i=j，i是最大理想。

As a corollary, we obtain the following useful result. It emphasizes the importance of maximal ideals.  
作为推论，我们得到了以下有用的结果。它强调最大理想的重要性。

Corollary 29.9. Given any ring A, every maximal ideal I in A is a prime ideal.  
推论29.9。对于任何环A，A中的每个最大理想I都是素理想。

Proof. If I is a maximal ideal, then, by Proposition 29.8, the quotient ring A/I is a field. However, a field is an integral domain, and by Proposition 29.8 (again), I is a prime ideal.   
证据。如果我是一个最大理想，那么，根据命题29.8，商环A/I是一个场。然而，场是一个积分域，根据命题29.8（再次），我是一个素理想。

Observe that a ring A is an integral domain iff (0) is a prime ideal. This is an example of a prime ideal which is not a maximal ideal, as immediately seen in A = Z, where (p) is a maximal ideal for every prime number p.  
观察环A是积分域iff（0）是素理想。这是素数理想的一个例子，它不是最大理想，如a=z所示，其中（p）是每个素数p的最大理想。

 A less obvious example of a prime ideal which is not a maximal ideal is the ideal (X) in the ring of polynomials Z[X]. Indeed, (X,2) is also a prime ideal, but (X) is properly contained in (X,2). The ideal (X) is the set of all polynomials of the form XQ(X) for any Q(X) ∈ Z[X], in other words the set of all polynomials in Z[X] with constant term equal to  
不是最大理想的素数理想的一个不太明显的例子是多项式Z[X]环中的理想（X）。实际上，（x，2）也是一个基本理想，但（x）正确地包含在（x，2）中。理想（x）是任何q（x）∈z[x]形式的所有多项式的集合，换句话说，z[x]形式的所有多项式的集合，常数项等于

0, and the ideal (X,2) is the set of all polynomials of the form  
理想（x，2）是形式的所有多项式的集合。

XQ1(X) + 2Q2(X), Q1(X),Q2(X) ∈ Z[X],  
x q1（x）+2q2（x），q1（x），q2（x）∈z[x]，

which is just the set of all polynomials in Z[X] whose constant term is of the form 2c for some c ∈ Z. The ideal (X) is indeed properly contained in the ideal (X,2). If P(X)Q(X) ∈ (X,2), let a be the constant term in P(X) and let b be the constant term in Q(X). Since P(X)Q(X) ∈ (X,2), we must have ab = 2c for some c ∈ Z, and since 2 is prime, either a is divisible by 2 or b is divisible by 2. It follows that either P(X) ∈ (X,2) or Q(X) ∈ (X,2), which shows that (X,2) is a prime ideal.  
它只是z[x]中所有多项式的集合，其常数项对于某些c∈z为2c形式。理想（x）确实正确地包含在理想（x，2）中。如果p（x）q（x）∈（x，2），设a为p（x）中的常数项，设b为q（x）中的常数项。由于p（x）q（x）∈（x，2），对于某些c∈z，我们必须有a b=2c，并且由于2是素数，a可以被2整除，或者b可以被2整除。由此得出p（x）∈（x，2）或q（x）∈（x，2），这表明（x，2）是一个素理想。

Definition 29.6. An integral domain in which every ideal is a principal ideal is called a principal ring or principal ideal domain, for short, a PID.  
定义29.6.每个理想都是主理想的积分域称为主环或主理想域，简称PID。

The ring Z is a PID. This is a consequence of the existence of a (Euclidean) division algorithm. As we shall see next, when K is a field, the ring K[X] is also a principal ring.  
Z环是PID。这是（欧几里得）除法存在的结果。接下来我们将看到，当k是一个场时，环k[x]也是一个主环。

 However, when n ≥ 2, the ring K[X1,...,Xn] is not principal. For example, in the ring  
但是，当n≥2时，环k[x1，…，xn]不是主体。例如，在环中

K[X,Y ], the ideal (X,Y ) generated by X and Y is not principal. First, since (X,Y ) is the set of all polynomials of the form Xq1 + Y q2, where q1,q2 ∈ K[X,Y ], except when Xq1 + Y q2 = 0, we have deg(Xq1 + Y q2) ≥ 1. Thus, 1 ∈/ (X,Y ). Now if there was some p ∈ K[X,Y ] such that (X,Y ) = (p), since 1 ∈/ (X,Y ), we must have deg(p) ≥ 1. But we would also have X = pq1 and Y = pq2, for some q1,q2 ∈ K[X,Y ]. Since deg(X) = deg(Y ) = 1, this is impossible.  
k[x，y]，x和y产生的理想（x，y）不是主体。首先，因为（x，y）是x q1+y q2形式的所有多项式的集合，其中q1，q2∈k[x，y]，除了xq1+y q2=0时，我们有deg（xq1+y q2）≥1。因此，1∈/（x，y）。如果有一些p∈k[x，y]这样（x，y）=（p），既然1∈/（x，y），我们必须有deg（p）≥1。但是我们也可以得到x=pq1和y=pq2，对于一些q1，q2∈k[x，y]。因为deg（x）=deg（y）=1，这是不可能的。

Even though K[X,Y ] is not a principal ring, a suitable version of unique factorization in terms of irreducible factors holds. The ring K[X,Y ] (and more generally K[X1,...,Xn]) is what is called a unique factorization domain, for short, UFD, or a factorial ring.  
即使k[x，y]不是主环，但不可约因子的唯一因式分解的一个合适版本仍然成立。环k[x，y]（通常是k[x1，…，xn]）被称为唯一的因式分解域，简称为ufd或因式分解环。

From this point until Definition 29.11, we consider polynomials in one variable over a field K.  
从这一点到29.11的定义，我们考虑一个变量在一个域k上的多项式。

Remark: Although we already proved part (1) of Proposition 29.10 in a more general situation above, we reprove it in the special case of polynomials. This may offend the purists, but most readers will probably not mind.  
注：虽然在上述更一般的情况下，我们已经证明了29.10命题的第（1）部分，但我们在多项式的特殊情况下对其进行了反驳。这可能会冒犯纯粹主义者，但大多数读者可能不会介意。

Proposition 29.10. Let K be a field. The following properties hold:  
提案29.10。让k成为一个场。以下属性保留：

1. For any two nonzero polynomials f,g ∈ K[X], (f) = (g) iff there is some λ = 06 in K such that g = λf.  
   对于任意两个非零多项式f，g∈k[x]，（f）=（g）iff，k中有一些λ=06，使得g=λf。
2. For every nonnull ideal I in K[X], there is a unique monic polynomial f ∈ K[X] such that I = (f).  
   对于k[x]中的每一个非零理想i，都有一个唯一的Monic多项式f∈k[x]，因此i=（f）。

Proof. (1) If (f) = (g), there are some nonzero polynomials q1,q2 ∈ K[X] such that g = fq1 and f = gq2. Thus, we have f = fq1q2, which implies f(1 − q1q2) = 0. Since K is a field, by Proposition 29.1, K[X] has no zero divisor, and since we assumed f = 06 , we must have q1q2 = 1. However, if either q1 or q2 is not a constant, by Proposition 29.1 again, deg(q1q2) = deg(q1) + deg(q2) ≥ 1, contradicting q1q2 = 1, since deg(1) = 0. Thus, both q1,q2 ∈ K −{0}, and (1) holds with λ = q1. In the other direction, it is obvious that g = λf implies that (f) = (g).  
证据。（1）如果（f）=（g），则存在一些非零多项式q1，q2∈k[x]，使得g=fq1，f=gq2。因此，我们有f=f q1q2，这意味着f（1−q1q2）=0。因为k是一个场，根据29.1，k[x]没有零除数，而且既然我们假设f=06，我们必须得到q1q2=1。然而，如果q1或q2不是一个常数，根据命题29.1，deg（q1q2）=deg（q1）+deg（q2）≥1，与q1q2=1矛盾，因为deg（1）=0。因此，q1，q2∈k−0，和（1）都与λ=q1保持一致。在另一个方向上，显然g=λf意味着（f）=（g）。

(2) Since we are assuming that I is not the null ideal, there is some polynomial of smallest degree in I, and since K is a field, by suitable multiplication by a scalar, we can make sure that this polynomial is monic. Thus, let f be a monic polynomial of smallest degree in I. By (ID2), it is clear that (f) ⊆ I. Now, let g ∈ I. Using the Euclidean algorithm, there exist unique q,r ∈ K[X] such that  
（2）因为我们假设i不是零理想，所以i中有一个最小度数的多项式，因为k是一个域，通过适当的乘一个标量，我们可以确定这个多项式是Monic。因此，设f为i中最小阶的Monic多项式，由（id2），很明显（f）i。现在，设g∈i。使用欧几里得算法，存在唯一的q，r∈k[x]，这样

g = qf + r and deg(r) < deg(f).  
g=qf+r和deg（r）<deg（f）。

If r = 06 , there is some λ = 06 in K such that λr is a monic polynomial, and since λr =  
如果r=06，k中有一些λ=06，因此λr是Monic多项式，因为λr=

, with f,g ∈ I, by (ID1) and (ID2), we have λr ∈ I, where deg(λr) < deg(f) and is a monic polynomial, contradicting the minimality of the degree of f. Thus, r = 0, and  
，对于f，g∈i，通过（id1）和（id2），我们得到了λr∈i，其中deg（λr）<deg（f），是一个Monic多项式，与f阶的极小性相矛盾。因此，r=0，和

g ∈ (f). The uniqueness of the monic polynomial f follows from (1).   
G∈（F）。Monic多项式f的唯一性来自（1）。

Proposition 29.10 shows that K[X] is a principal ring when K is a field.  
命题29.10表明，当k是一个场时，k[x]是一个主环。

We now investigate the existence of a greatest common divisor (gcd) for two nonzero polynomials. Given any two nonzero polynomials f,g ∈ K[X], recall that f divides g if g = fq for some q ∈ K[X].  
我们现在研究两个非零多项式的最大公约数的存在性。对于任意两个非零多项式f，g∈k[x]，回想一下，如果g=f q，f对某些q∈k[x]除以g。

Definition 29.7. Given any two nonzero polynomials f,g ∈ K[X], a polynomial d ∈ K[X] is a greatest common divisor of f and g (for short, a gcd of f and g) if d divides f and g and whenever h ∈ K[X] divides f and g, then h divides d. We say that f and g are relatively prime if 1 is a gcd of f and g.  
定义29.7.对于任意两个非零多项式f，g∈k[x]，一个多项式d∈k[x]是f和g的最大公因数（简而言之，f和g的gcd），如果d除以f和g，当h∈k[x]除以f和g，则h除以d。如果1是f和g的gcd，则f和g是相对素数。

Note that f and g are relatively prime iff all of their gcd’s are constants (scalars in K), or equivalently, if f,g have no divisor q of degree deg(q) ≥ 1.  
注意，f和g是相对素数iff，它们的gcd都是常数（K中的标量），或者等价的，如果f，g没有deg（q）≥1的除数q。

 In particular, note that f and g are relatively prime when f is a nonzero constant polynomial (a scalar λ = 06 in K) and g is any nonzero polynomial.  
特别要注意，当f是非零常数多项式（k中的标量λ=06）且g是任何非零多项式时，f和g是相对素数。

We can characterize gcd’s of polynomials as follows.  
我们可以用下面的方法来描述多项式的gcd。

Proposition 29.11. Let K be a field and let f,g ∈ K[X] be any two nonzero polynomials. For every polynomial d ∈ K[X], the following properties are equivalent:  
提案29.11.设k为场，设f，g∈k[x]为任意两个非零多项式。对于每一个多项式d∈k[x]，下列性质是等价的：

1. The polynomial d is a gcd of f and g.  
   多项式d是f和g的gcd。
2. The polynomial d divides f and g and there exist u,v ∈ K[X] such that  
   多项式d除以f和g，并存在u，v∈k[x]，这样
   1. = uf + vg.  
      =uf+vg。
3. The ideals (f),(g), and (d) satisfy the equation  
   理想（f）、（g）和（d）满足方程
   1. = (f) + (g).  
      =（f）+（g）。

In addition, d = 06 , and d is unique up to multiplication by a nonzero scalar in K.  
此外，d=06，d是唯一的，直到K中的非零标量相乘。

Proof. Given any two nonzero polynomials u,v ∈ K[X], observe that u divides v iff (v) ⊆ (u). Now, (2) can be restated as (f) ⊆ (d), (g) ⊆ (d), and d ∈ (f) + (g), which is equivalent to (d) = (f) + (g), namely (3).  
证据。给定任意两个非零多项式u，v∈k[x]，观察u除以v iff（v）（u）。现在，（2）可以重述为（f）（d）、（g）（d）和d∈（f）+（g），相当于（d）=（f）+（g），即（3）。

If (2) holds, since d = uf + vg, whenever h ∈ K[X] divides f and g, then h divides d, and d is a gcd of f and g.  
如果（2）成立，因为d=uf+vg，当h∈k[x]除以f和g时，h除以d，d是f和g的gcd。

Assume that d is a gcd of f and g. Then, since d divides f and d divides g, we have (f) ⊆ (d) and (g) ⊆ (d), and thus (f) + (g) ⊆ (d), and (f) + (g) is nonempty since f and g are nonzero. By Proposition 29.10, there exists a monic polynomial d1 ∈ K[X] such that (d1) = (f) + (g). Then, d1 divides both f and g, and since d is a gcd of f and g, then d1 divides d, which shows that (d) ⊆ (d1) = (f) + (g). Consequently, (f) + (g) = (d), and (3) holds.  
假设d是f和g的gcd，因为d除以f和d除以g，我们得到（f）（d）和（g）（d），因此（f）+（g）（d）和（f）+（g）是非空的，因为f和g是非零的。根据29.10，存在一个Monic多项式d1∈k[x]，使得（d1）=（f）+（g）。然后，d1将f和g分开，因为d是f和g的gcd，所以d1将d分开，这表明（d）（d1）=（f）+（g）。因此，（f）+（g）=（d）和（3）成立。

Since (d) = (f) + (g) and f and g are nonzero, the last part of the proposition is obvious.   
因为（d）=（f）+（g）和f和g是非零的，所以命题的最后一部分是显而易见的。

As a consequence of Proposition 29.11, two nonzero polynomials f,g ∈ K[X] are relatively prime iff there exist u,v ∈ K[X] such that  
由于29.11命题的结果，两个非零多项式f，g∈k[x]是相对素数，如果存在u，v∈k[x]，那么

uf + vg = 1.  
uf+vg=1.

The identity  
身份

d = uf + vg  
D=uf+vg

of part (2) of Proposition 29.11 is often called the Bezout identity.  
在第29.11号提案的第（2）部分中，常被称为贝佐特身份。

We derive more useful consequences of Proposition 29.11.  
我们得出29.11号提案更有用的结果。

Proposition 29.12. Let K be a field and let f,g ∈ K[X] be any two nonzero polynomials. For every gcd d ∈ K[X] of f and g, the following properties hold:  
提案29.12。设k为场，设f，g∈k[x]为任意两个非零多项式。对于f和g的每一个gcd d∈k[x]，下列性质成立：

1. For every nonzero polynomial q ∈ K[X], the polynomial dq is a gcd of fq and gq.  
   对于每个非零多项式q∈k[x]，多项式dq是fq和gq的gcd。
2. For every nonzero polynomial q ∈ K[X], if q divides f and g, then d/q is a gcd of f/q and g/q.  
   对于每一个非零多项式q∈k[x]，如果q除以f和g，则d/q是f/q和g/q的gcd。

Proof. (1) By Proposition 29.11 (2), d divides f and g, and there exist u,v ∈ K[X], such that  
证据。（1）根据29.11（2）号命题，d将f和g分开，并存在u，v∈k[x]，这样

d = uf + vg.  
d=uf+vg。

Then, dq divides fq and gq, and  
然后，dq将fq和gq分开，并且

dq = ufq + vgq.  
dq=ufq+vgq。

By Proposition 29.11 (2), dq is a gcd of fq and gq. The proof of (2) is similar.   
根据29.11（2）号提案，DQ是FQ和GQ的GCD。（2）的证明类似。

The following proposition is used often.  
通常使用以下命题。

Proposition 29.13. (Euclid’s proposition) Let K be a field and let f,g,h ∈ K[X] be any nonzero polynomials. If f divides gh and f is relatively prime to g, then f divides h.  
提案29.13。（欧几里得命题）设k为场，设f，g，h∈k[x]为任意非零多项式。如果F除以GH，F相对地是G的素数，那么F除以H。

Proof. From Proposition 29.11, f and g are relatively prime iff there exist some polynomials u,v ∈ K[X] such that  
证据。从29.11命题来看，f和g是相对素数iff，存在一些多项式u，v∈k[x]，这样

uf + vg = 1.  
uf+vg=1.

Then, we have  
那么，我们有了

ufh + vgh = h,  
ufh+vgh=h，

and since f divides gh, it divides both ufh and vgh, and so, f divides h.   
因为F划分了GH，所以它同时划分了UFH和VGH，所以，F划分了H。

Proposition 29.14. Let K be a field and let f,g1,...,gm ∈ K[X] be some nonzero polynomials. If f and gi are relatively prime for all i, 1 ≤ i ≤ m, then f and g1 ···gm are relatively prime.  
提案29.14。设k为场，设f，g1，…，gm∈k[x]为非零多项式。如果f和gi对所有i都是相对素数，1≤i≤m，那么f和g1···gm是相对素数。

Proof. We proceed by induction on m. The case m = 1 is trivial. Let h = g2 ···gm. By the induction hypothesis, f and h are relatively prime. Let d be a gcd of f and g1h. We claim that d is relatively prime to g1. Otherwise, d and g1 would have some nonconstant gcd d1 which would divide both f and g1, contradicting the fact that f and g1 are relatively prime. Now, by Proposition 29.13, since d divides g1h and d and g1 are relatively prime, d divides h = g2 ···gm. But then, d is a divisor of f and h, and since f and h are relatively prime, d must be a constant, and f and g1 ···gm are relatively prime.   
证据。我们对m进行归纳，m=1的情况是微不足道的。假设h=g2···gm，根据诱导假设，f和h是相对质数。假设d是f和g1h的gcd，我们认为d是g1的相对素数。否则，d和g1会有一些不稳定的gcd d1，将f和g1分开，这与f和g1相对质数的事实相矛盾。现在，根据命题29.13，由于d除以g1 h和d，g1是相对素数，d除以h=g2····gm，但d是f和h的除数，由于f和h是相对素数，d必须是常数，f和g1···gm是相对素数。

Definition 29.7 is generalized to any finite number of polynomials as follows.  
定义29.7推广到任何有限数量的多项式，如下所示。

Definition 29.8. Given any nonzero polynomials f1,...,fn ∈ K[X], where n ≥ 2, a polynomial d ∈ K[X] is a greatest common divisor of f1,...,fn (for short, a gcd of f1,...,fn) if d divides each fi and whenever h ∈ K[X] divides each fi, then h divides d. We say that f1,...,fn are relatively prime if 1 is a gcd of f1,...,fn.  
定义29.8.给定任意非零多项式f1，…，fn∈k[x]，其中n≥2，多项式d∈k[x]是f1，…，fn的最大公因数（简而言之，f1，…，fn的gcd），如果d对每个fi进行除，当h∈k[x]对每个fi进行除时，h对d进行除。我们说f1，…，fn是相对素数，如果1是gcd，fn是相对素数。一层，…，二层。

It is easily shown that Proposition 29.11 can be generalized to any finite number of polynomials, and similarly for its relevant corollaries. The details are left as an exercise.  
很容易证明29.11命题可以推广到任何有限个多项式，同样也可以推广到它的相关推论。细节留作练习。

Proposition 29.15. Let K be a field and let f1,...,fn ∈ K[X] be any n ≥ 2 nonzero polynomials. For every polynomial d ∈ K[X], the following properties are equivalent:  
提案29.15。设k为场，设f1，…，fn∈k[x]为任意n≥2个非零多项式。对于每一个多项式d∈k[x]，下列性质是等价的：

1. The polynomial d is a gcd of f1,...,fn.  
   多项式d是f1，…，fn的gcd。
2. The polynomial d divides each fi and there exist u1,...,un ∈ K[X] such that  
   多项式d将每一个fi分开，并且存在u1，…，un∈k[x]，这样
   1. = u1f1 + ··· + unfn.  
      =U1F1+·····+UNFN。
3. The ideals (fi), and (d) satisfy the equation  
   理想（fi）和（d）满足方程
   1. = (f1) + ··· + (fn).  
      =（F1）+·····+（FN）。

In addition, d = 06 , and d is unique up to multiplication by a nonzero scalar in K.  
此外，d=06，d是唯一的，直到K中的非零标量相乘。

As a consequence of Proposition 29.15, some polynomials f1,...,fn ∈ K[X] are relatively prime iff there exist u1,...,un ∈ K[X] such that  
由于29.15命题的结果，一些多项式f1，…，fn∈k[x]是相对素数iff存在u1，…，un∈k[x]这样

u1f1 + ··· + unfn = 1.  
U1F1+·····+UNFN=1.

The identity  
身份

u1f1 + ··· + unfn = 1  
U1F1+·····+UNFN=1

of part (2) of Proposition 29.15 is also called the Bezout identity.  
第29.15号提案第（2）部分也被称为贝索特身份。

We now consider the factorization of polynomials of a single variable into irreducible factors.  
我们现在考虑将单个变量的多项式因式分解为不可约因子。

## 29.5 Factorization and Irreducible Factors in K[X] 29.5因子分解和K[X]中的不可约因子

Definition 29.9. Given a field K, a polynomial p ∈ K[X] is irreducible or indecomposable or prime if deg(p) ≥ 1 and if p is not divisible by any polynomial q ∈ K[X] such that 1 ≤ deg(q) < deg(p). Equivalently, p is irreducible if deg(p) ≥ 1 and if p = q1q2, then either q1 ∈ K or q2 ∈ K (and of course, q1 = 06 , q2 = 0).6  
定义29.9.给定一个域k，如果deg（p）≥1，多项式p∈k[x]是不可约或不可分解的或素数，如果p不可被任何多项式q∈k[x]除，则1≤deg（q）<deg（p）。等价地，如果deg（p）≥1，p是不可约的，如果p=q1 q2，则q1∈k或q2∈k（当然，q1=06，q2=0）。6

Example 29.2. Every polynomial aX + b of degree 1 is irreducible. Over the field R, the polynomial X2 + 1 is irreducible (why?), but X3 + 1 is not irreducible, since  
例29.2。1次的每个多项式ax+b都是不可约的。在r域上，多项式x2+1是不可约的（为什么？），但x3+1不是不可约的，因为

X3 + 1 = (X + 1)(X2 − X + 1).  
x3+1=（x+1）（x2−x+1）。

The polynomial X2 − X + 1 is irreducible over R (why?). It would seem that X4 + 1 is irreducible over R, but in fact,  
多项式x2-x+1在r上是不可约的（为什么？）.似乎x4+1在r上是不可约的，但实际上，

X4 + 1 = (X2 − √2X + 1)(X2 + √2X + 1).  
x4+1=（x2−√2x+1）（x2+√2x+1）。

However, in view of the above factorization, X4 + 1 is irreducible over Q.  
然而，考虑到上述因子分解，x4+1在q上是不可约的。

It can be shown that the irreducible polynomials over R are the polynomials of degree 1, or the polynomials of degree 2 of the form aX2 + bX + c, for which b2 − 4ac < 0 (i.e., those having no real roots). This is not easy to prove! Over the complex numbers C, the only irreducible polynomials are those of degree 1. This is a version of a fact often referred to as the “Fundamental theorem of Algebra”, or, as the French sometimes say, as “d’Alembert’s theorem”!  
可以看出，R上的不可约多项式是阶1的多项式，或者是ax2+bx+c形式的阶2的多项式，其中b2−4ac<0（即那些没有实根的）。这不容易证明！在复数c上，唯一不可约多项式是阶1的多项式。这是一个事实的版本，通常被称为“代数的基本定理”，或者，正如法国人有时说的，被称为“达朗伯定理”！

We already observed that for any two nonzero polynomials f,g ∈ K[X], f divides g iff (g) ⊆ (f). In view of the definition of a maximal ideal given in Definition 29.4, we now prove that a polynomial p ∈ K[X] is irreducible iff (p) is a maximal ideal in K[X].  
我们已经观察到，对于任意两个非零多项式f，g∈k[x]，f除以g iff（g）（f）。鉴于定义29.4中给出的最大理想的定义，我们现在证明多项式p∈k[x]是不可约的，iff（p）是k[x]中的最大理想。

Proposition 29.16. A polynomial p ∈ K[X] is irreducible iff (p) is a maximal ideal in K[X].  
提案29.16。多项式p∈k[x]是不可约的，iff（p）是k[x]中的最大理想。

29.5. FACTORIZATION AND IRREDUCIBLE FACTORS IN K[X]  
29.5。K[X]中的因式分解和不可约因子

Proof. Since K[X] is an integral domain, for all nonzero polynomials p,q ∈ K[X], deg(pq) = deg(p) + deg(q), and thus, (p) =6 K[X] iff deg(p) ≥ 1. Assume that p ∈ K[X] is irreducible. Since every ideal in K[X] is a principal ideal, every ideal in K[X] is of the form (q), for some q ∈ K[X]. If (p) ⊆ (q), with deg(q) ≥ 1, then q divides p, and since p ∈ K[X] is irreducible, this implies that p = λq for some λ = 06 in K, and so, (p) = (q). Thus, (p) is a maximal ideal. Conversely, assume that (p) is a maximal ideal. Then, as we showed above, deg(p) ≥ 1, and if q divides p, with deg(q) ≥ 1, then (p) ⊆ (q), and since (p) is a maximal ideal, this implies that (p) = (q), which means that p = λq for some λ = 06 in K, and so, p is irreducible.   
证据。由于k[x]是一个积分域，对于所有非零多项式p，q∈k[x]，deg（pq）=deg（p）+deg（q），因此，（p）=6 k[x]iff deg（p）≥1。假设p∈k[x]是不可约的。因为k[x]中的每一个理想都是主理想，所以k[x]中的每一个理想都是形式（q），对于某些q∈k[x]。如果（p）（q），且deg（q）≥1，则q除以p，由于p∈k[x]是不可约的，这意味着对于一些λ=06的k，p=λq，因此，（p）=（q）。因此，（p）是最大理想。相反，假设（p）是最大理想。然后，如我们上面所示，deg（p）≥1，如果q除以p，deg（q）≥1，那么（p）（q），由于（p）是最大理想，这意味着（p）=（q），这意味着对于k中的一些λ=06，p=λq，因此p是不可约的。

Let p ∈ K[X] be irreducible. Then, for every nonzero polynomial g ∈ K[X], either p and g are relatively prime, or p divides g. Indeed, if d is any gcd of p and g, if d is a constant, then p and g are relatively prime, and if not, because p is irreducible, we have d = λp for some λ = 06 in K, and thus, p divides g. As a consequence, if p,q ∈ K[X] are both irreducible, then either p and q are relatively prime, or p = λq for some λ = 06 in K. In particular, if p,q ∈ K[X] are both irreducible monic polynomials and p =6 q, then p and q are relatively prime.  
设p∈k[x]不可约。那么，对于每一个非零多项式g∈k[x]，要么p和g是相对素数，要么p除以g。实际上，如果d是p和g的任何gcd，如果d是常数，那么p和g是相对素数，如果不是，因为p是不可约的，我们对k中的一些λ=06有d=λp，因此p除以g。作为一个常数。序列，如果p，q∈k[x]都是不可约的，那么p和q都是相对素数，或者对于k中的一些λ=06，p=λq。特别是，如果p，q∈k[x]都是不可约Monic多项式，p=6q，那么p和q都是相对素数。

We now prove the (unique) factorization of polynomials into irreducible factors. Theorem 29.17. Given any field K, for every nonzero polynomial  
我们现在证明多项式的（唯一）因式分解为不可约因子。定理29.17。对于每一个非零多项式，给定任意字段k

f = adXd + ad−1Xd−1 + ··· + a0  
f=adxd+ad−1xd−1+····+a0

of degree d = deg(f) ≥ 1 in K[X], there exists a unique set {hp1,k1i,...,hpm,kmi} such that  
当d=deg（f）≥1 in k[x]时，存在一个独特的集合hp1，k1i，…，hpm，kmi

,  
，

where the pi ∈ K[X] are distinct irreducible monic polynomials, the ki are (not necessarily distinct) integers, and m ≥ 1, ki ≥ 1.  
当pi∈k[x]是不同的不可约Monic多项式时，ki是（不一定是不同的）整数，m≥1，ki≥1。

Proof. First, we prove the existence of such a factorization by induction on d = deg(f). Clearly, it is enough to prove the result for monic polynomials f of degree d = deg(f) ≥ 1. If d = 1, then f = X + a0, which is an irreducible monic polynomial.  
证据。首先，我们通过D=deg（f）上的归纳证明了这种因子分解的存在性。显然，这足以证明D=deg（f）≥1的Monic多项式f的结果。如果d=1，则f=x+a0，这是一个不可约Monic多项式。

Assume d ≥ 2, and assume the induction hypothesis for all monic polynomials of degree < d. Consider the set S of all monic polynomials g such that deg(g) ≥ 1 and g divides f. Since f ∈ S, the set S is nonempty, and thus, S contains some monic polynomial p1 of minimal degree. Since deg(p1) ≥ 1, the monic polynomial p1 must be irreducible. Otherwise we would have p1 = g1g2, for some monic polynomials g1,g2 such that deg(p1) > deg(g1) ≥ 1 and deg(p1) > deg(g2) ≥ 1, and since p1 divide f, then g1 would divide f, contradicting the minimality of the degree of p1. Thus, we have f = p1q, for some irreducible monic polynomial p1, with q also monic. Since deg(p1) ≥ 1, we have deg(q) < deg(f), and we can apply the induction hypothesis to q. Thus, we obtain a factorization of the desired form.  
假设d≥2，并假设所有次数<d的Monic多项式的归纳假设。考虑所有Monic多项式g的集合s，使deg（g）≥1和g除以f。由于f∈s，集合s是非空的，因此s包含一些最小次数的Monic多项式p1。由于deg（p1）≥1，monic多项式p1必须是不可约的。否则我们将得到p1=g1 g2，对于一些Monic多项式g1，g2，这样deg（p1）>deg（g1）≥1和deg（p1）>deg（g2）≥1，并且由于p1除以f，那么g1将除以f，这与p1的最小程度相矛盾。因此，对于一些不可约Monic多项式p1，我们有f=p1 q，其中q也是monic。由于deg（p1）≥1，因此deg（q）<deg（f），我们可以将诱导假设应用于q，从而获得所需形式的因式分解。

We now prove uniqueness. Assume that  
我们现在证明了它的独特性。假设

,  
，

and  
和

.  
.

Thus, we have  
因此，我们

.  
.

We prove that m = n, pi = qi and hi = ki, for all i, with 1 ≤ i ≤ n.  
我们证明m=n，pi=qi，hi=ki，对于所有i，1≤i≤n。

The proof proceeds by induction on h1 + ··· + hn.  
通过对h1+·····+hn的归纳，证明了这一点。

If h1 + ··· + hn = 1, then n = 1 and h1 = 1. Then, since K[X] is an integral domain, we have  
如果h1+·····+hn=1，则n=1，h1=1。那么，既然k[x]是一个积分域，我们有

,  
，

and since q1 and the pi are irreducible monic, we must have m = 1 and p1 = q1.  
既然q1和π是不可约的Monic，我们必须有m=1和p1=q1。

If h1 + ··· + hn ≥ 2, since K[X] is an integral domain and since h1 ≥ 1, we have  
如果h1+·····+hn≥2，因为k[x]是一个积分域，并且h1≥1，我们得到



with  
具有

q = q1h1−1 ···qnhn,  
Q=Q1H1−1···QNHN，

where (h1 − 1) + ··· + hn ≥ 1 (and q1h1−1 = 1 if h1 = 1). Now, if q1 is not equal to any of the pi, by a previous remark, q1 and pi are relatively prime, and by Proposition 29.14, q1 and are relatively prime. But this contradicts the fact that q1 divides. Thus, q1 is equal to one of the pi. Without loss of generality, we can assume that q1 = p1. Then, since K[X] is an integral domain, we have  
式中（h1−1）+·····+hn≥1（如果h1=1，q1h1−1=1）。现在，如果q1不等于π中的任何一个，由前面的注释可知，q1和π是相对素数，由命题29.14可知，q1是相对素数。但这与q1分裂的事实相矛盾。因此，q1等于π之一。在不失去一般性的情况下，我们可以假设q1=p1。那么，既然k[x]是一个积分域，我们有

,  
，

where = 1, and = 1. Now, (h1 −1)+···+hn < h1 +···+hn, and we can apply the induction hypothesis to conclude that m = n, pi = qi and hi = ki, with 1 ≤ i ≤ n.   
其中=1和=1。现在，（h1−1）+·······+hn<h1······+hn），我们可以应用诱导假设得出m=n，pi=qi，hi=ki，1≤i≤n。

The above considerations about unique factorization into irreducible factors can be extended almost without changes to more general rings known as Euclidean domains. In such rings, some abstract version of the division theorem is assumed to hold.  
上述关于独特因子分解为不可约因子的考虑几乎可以扩展到更一般的环，即欧几里得域。在这样的环中，假设存在一些抽象形式的除法定理。

Definition 29.10. A Euclidean domain (or Euclidean ring) is an integral domain A such that there exists a function ϕ: A → N with the following property: For all a,b ∈ A with b = 06 , there are some q,r ∈ A such that  
定义29.10.欧几里得域（或欧几里得环）是一个积分域A，这样就存在一个具有以下性质的函数：对于所有a，b∈a和b=06，有一些q，r∈a这样

a = bq + r and ϕ(r) < ϕ(b).  
A=Bq+R和\_（R）<（B）。

29.5. FACTORIZATION AND IRREDUCIBLE FACTORS IN K[X]  
29.5。K[X]中的因式分解和不可约因子

Note that the pair (q,r) is not necessarily unique.  
注意，这对（q，r）不一定是唯一的。

Actually, unique factorization holds in principal ideal domains (PID’s), see Theorem 31.12. As shown below, every Euclidean domain is a PID, and thus, unique factorization holds for Euclidean domains.  
实际上，唯一因式分解存在于主理想域（PID），见定理31.12。如下图所示，每个欧几里德域都是一个PID，因此，欧几里德域具有唯一的因子分解。

Proposition 29.18. Every Euclidean domain A is a PID. Proof. Let I be a nonnull ideal in A. Then, the set  
提案29.18。每个欧几里得域A都是一个PID。证据。让我成为A中的一个非空理想。

{ϕ(a) | a ∈ I}  
（a）a∈i

is nonempty, and thus, has a smallest element m. Let b be any (nonnull) element of I such that m = ϕ(b). We claim that I = (b). Given any a ∈ I, we can write  
是非空的，因此有一个最小的元素m。让b是i的任何（非空）元素，这样m=\_（b）。我们声称我=（B）。给任何a∈i，我们可以写

a = bq + r  
A=Bq+R

for some q,r ∈ A, with ϕ(r) < ϕ(b). Since b ∈ I and I is an ideal, we also have bq ∈ I, and since a,bq ∈ I and I is an ideal, then r ∈ I with ϕ(r) < ϕ(b) = m, contradicting the minimality of m. Thus, r = 0 and a ∈ (b). But then,  
对于某些q，r∈a，其中（r）<（b）。既然b∈i和i是一个理想，我们也有bq∈i，既然a，bq∈i和i是一个理想，那么r∈i与（r）<（b）=m，与m的极小性相矛盾。因此，r=0和a∈（b）。但是后来，

I ⊆ (b),  
我（b）

and since b ∈ I, we get  
既然b∈i，我们得到

I = (b),  
我=（B）

and A is a PID.   
A是PID。

As a corollary of Proposition 29.18, the ring Z is a Euclidean domain (using the function ϕ(a) = |a|) and thus, a PID. If K is a field, the function ϕ on K[X] defined such that  
作为命题29.18的一个推论，环Z是一个欧几里得域（使用函数（a）=a），因此是一个PID。如果k是一个场，k[x]上的函数\_定义如下：

0 if f = 0,  
如果f=0，则为0，

f  
f

= 0,  
＝0，

shows that K[X] is a Euclidean domain.  
表明k[x]是欧几里得域。

Example 29.3. A more interesting example of a Euclidean domain is the ring Z[i] of Gaussian integers, i.e., the subring of C consisting of all complex numbers of the form a + ib, where a,b ∈ Z. Using the function ϕ defined such that  
例29.3。欧几里得域的一个更有趣的例子是高斯整数的环Z[i]，即C的子环，由A+Ib形式的所有复数组成，其中a，b∈z。使用函数，定义如下：

ϕ(a + ib) = a2 + b2,  
⑨（a+ib）=a2+b2，

we leave it as an interesting exercise to prove that Z[i] is a Euclidean domain.  
我们把它作为一个有趣的练习来证明z[i]是一个欧几里得域。

 Not every PID is a Euclidean ring.  
不是每个PID都是欧几里得环。

Remark: Given any integer d ∈ Z such thatQ(√d) is the field consisting of all complex numbersd 6= 0,1 and d does not have any square factor  
注：给定任意整数d∈z，则q（√d）是由所有复数组成的域，d 6=0,1，d不具有任何平方因子。

greater than one, the quadratic field  
大于1的二次场

is denoted byof the formwith a,b ∈ Qa. The subring of+Z[ib√√d]−. We define thed if d < 0Q, and of all the real numbers of the form(√d)ring of integersconsisting of all elements as above for whichof the field Q(√ a + b√d ifa,bd >∈ 0Z,  
用a，b∈qa表示。+Z[ib√√d]−的子框。我们定义了d<0q，和所有实数的形式（√d）环的积分，所有元素的存在，如上文所述，其中，字段q（√a+b√d ifa，bd>的∈0z，

d) as the subring of Q(√d) consisting of the following elements:  
d）作为q（√d）的子框，由以下元素组成：

1. If d ≡ 2 (mod 4) or d ≡ 3 (mod 4), then all elements of the forma + b√d (if d > 0), with a,b ∈ Z; a + ib√−d (if d < 0) or all elements of the form  
   如果d 2（mod 4）或d 3（mod 4），那么形式的所有元素+b√d（如果d>0），带有a，b∈z；a+ib√−d（如果d<0）或形式的所有元素
2. If d ≡ 1(mod4), then all elements of the forma+b√d)/2 (if d > 0), with a,b ∈ Z(aand with+ib√−d)a,b/2 (either both even or bothif d < 0) or all elements of the form ( odd.  
   如果d 1（mod4），那么所有的form a+b√d）/2元素（如果d>0），其中a，b∈z（aand with+ib√−d）a，b/2（偶数或both if d<0）或形式的所有元素（奇数）。

ZObserve that when[√ d ≡ 2(mod4) or d ≡ 3(mod4), the ring of integers of Q(√d) is equal to d].  
zobserve当[√d 2（mod4）或d 3（mod4）时，q（√d）的整数环等于d]。

It can be shown that the rings of integers of the fields Q(√−d) where d = 19, 43, 67, 163 are PID’s, but not Euclidean rings. The proof is hard and long. First, it can be shown that these rings are UFD’s (refer to Definition 31.2), see Stark [159] (Chapter 8, Theorems 8.21  
可以证明，字段q（√−d）中d=19、43、67、163的整数环是PID，但不是欧几里得环。证据很难，而且很长。首先，可以证明这些环是UFD（参考定义31.2），见Stark[159]（第8章，定理8.21

and 8.22). Then, we use the fact that the ring of integers of the field Q(√d) (with d 6= 0,1 any square-free integers) is a certain kind of integral domain called a Dedekind ring; see Atiyah-MacDonald [8] (Chapter 9, Theorem 9.5) or Samuel [139] (Chapter III, Section 3.4). Finally, we use the fact that if a Dedekind ring is a UFD then it is a PID, which follows from Proposition 31.13.  
和8.22）。然后，我们利用q（√d）（d 6=0,1任意平方自由整数）的整数环是一种称为Dedekind环的积分域的事实；见Atiyah MacDonald[8]（第9章，定理9.5）或Samuel[139]（第三章，第3.4节）。最后，我们利用这样一个事实：如果一个Dedekind环是一个UFD，那么它是一个PID，这是从命题31.13得出的。

Actually, the rings of integers of Q(√d) that are Euclidean domains are completely determined but the proof is quite difficult. It turns out that there are twenty one such rings corresponding to the integers: −11,−7,−3,−2,−1, 2,3,5,6,7,11, 13,17,19,21, 29,33,37,41,57 and 73, see Stark [159] (Chapter 8). For more on quadratic fields and their rings of integers, see Stark [159] (Chapter 8) or Niven, Zuckerman and Montgomery [128] (Chapter 9).  
实际上，欧氏域q（√d）的整数环是完全确定的，但证明是相当困难的。结果发现，有21个这样的环对应于整数：−11、−7、−3、−2、−1、2、3、5、6、7、11、13、17、19、21、29、33、37、41、57和73，见Stark[159]（第8章）。有关二次域及其整数环的更多信息，请参阅stark[159]（第8章）或niven、zuckerman和montgomery[128]（第9章）。

It is possible to characterize a larger class of rings (in terms of ideals), factorial rings (or unique factorization domains), for which unique factorization holds (see Section 31.1). We now consider zeros (or roots) of polynomials.  
可以描述一类更大的环（根据理想），阶乘环（或唯一的阶乘域），对于这些环，唯一的阶乘成立（见第31.1节）。我们现在考虑多项式的零（或根）。

## 29.6 Roots of Polynomials 29.6多项式的根

We go back to the general case of an arbitrary ring for a little while.  
我们回到一般情况下的任意环一会儿。

Definition 29.11. Given a ring A and any polynomial f ∈ A[X], we say that somef ∈) = 0.A[X1,...,Xα ∈nA],  
定义29.11.给定一个环A和任意一个多项式f∈a[x]时，我们假设somef∈）=0.a[x1，…，xα∈na]，

is a zero of f, or a root of f, if f(α) = 0. Similarly, given a polynomial we say that (α1,...,αn) ∈ An is a a zero of f, or a root of f, if f(α1,...,αn  
如果f（α）=0，则为f的零或根。同样，对于一个多项式，我们说（α1，…，αn）∈an是f的零点，或f的根，如果f（α1，…，αn

When f ∈ A[X] is the null polynomial, every α ∈ A is trivially a zero of f. This case being trivial, we usually assume that we are considering zeros of nonnull polynomials.  
当f∈a[x]是零多项式时，每个α∈a都是f的零值，这种情况是平凡的，我们通常假定我们考虑的是非零多项式的零。

Example 29.4. Considering the polynomial f(X) = X2 − 1, both +1 and −1 are zeros of f(X). Over the field of reals, the polynomial g(X) = X2 + 1 has no zeros. Over the field C of complex numbers, g(X) = X2 + 1 has two roots i and −i, the square roots of −1, which are “imaginary numbers.”  
例29.4。考虑到多项式f（x）=x2−1，+1和−1都是f（x）的零。在实数域上，多项式g（x）=x2+1没有零。在复数域C上，g（x）=x2+1有两个根i和−i，即−1的平方根，即“虚数”。

We have the following basic proposition showing the relationship between polynomial division and roots.  
下面给出了多项式除法与根的关系的基本命题。

Proposition 29.19. Let f ∈ A[X] be any polynomial and α ∈ A any element of A. If the result of dividing f by X − α is f = (X − α)q + r, then r = 0 iff f(α) = 0, i.e., α is a root of f iff r = 0.  
提案29.19。设f∈a[x]为任意多项式，α∈a为a的任意元素，如果f除以x−α的结果为f=（x−α）q+r，则r=0 iff（α）=0，即α为f iff r=0的根。

Proof. We have f = (X − α)q + r, with deg(r) < 1 = deg(X − α). Thus, r is a constant in K, and since f(α) = (α − α)q(α) + r, we get f(α) = r, and the proposition is trivial.   
证据。我们有f=（x−α）q+r，deg（r）<1=deg（x−α）。因此，r是k中的常数，因为f（α）=（α−α）q（α）+r，我们得到f（α）=r，这个命题是平凡的。

We now consider the issue of multiplicity of a root.  
我们现在考虑一个根的多重性问题。

Proposition 29.20. Let f ∈ A[X] be any nonnull polynomial and h ≥ 0 any integer. The following conditions are equivalent.  
提案29.20。设f∈a[x]为任意非零多项式，h≥0任意整数。以下条件是等效的。

1. f is divisible by (X − α)h but not by (X − α)h+1.  
   F可被（x−α）h整除，但不能被（x−α）h+1整除。
2. There is some g ∈ A[X] such that f = (X − α)hg and g(α) = 06 .  
   有一些g∈a[x]这样f=（x−α）hg和g（α）=06。

Proof. Assume (1). Then, we have f = (X − α)hg for some g ∈ A[X]. If we had g(α) = 0, by Proposition 29.19, g would be divisible by (X − α), and then f would be divisible by (X − α)h+1, contradicting (1).  
证据。假设（1）。然后，对于一些g∈a[x]，我们得到f=（x−α）hg。如果g（α）=0，根据命题29.19，g可以被（x−α）整除，那么f可以被（x−α）h+1整除，与（1）矛盾。

Assume (2), that is, f = (X − α)hg and g(α) = 06 . If f is divisible by (X − α)h+1, then we have f = (X − α)h+1g1, for some g1 ∈ A[X]. Then, we have  
假设（2），即f=（x−α）Hg和g（α）=06。如果f可被（x−α）h+1整除，那么对于某些g1∈a[x]，我们得到f=（x−α）h+1g1。那么，我们有了

(X − α)hg = (X − α)h+1g1,  
（x−α）Hg=（x−α）H+1g1，

and thus  
因此

(X − α)h(g − (X − α)g1) = 0,  
（x−α）h（g−（x−α）g1）=0，

and since the leading coefficient of (X − α)h is 1 (show this by induction), by Proposition 29.1, (X − α)h is not a zero divisor, and we get g − (X − α)g1 = 0, i.e., g = (X − α)g1, and so g(α) = 0, contrary to the hypothesis.   
由于（x−α）h的前导系数是1（通过归纳法证明），根据命题29.1，（x−α）h不是零因子，我们得到g−（x−α）g1=0，即g=（x−α）g1，因此g（α）=0，与假设相反。

As a consequence of Proposition 29.20, for every nonnull polynomial f ∈ A[X] and every α ∈ A, there is a unique integer h ≥ 0 such that f is divisible by (X − α)h but not by (X − α)h+1. Indeed, since f is divisible by (X − α)h, we have h ≤ deg(f). When h = 0, α is not a root of f, i.e., f(α) = 06 . The interesting case is when α is a root of f.  
由于29.20命题的结果，对于每一个非零多项式f∈a[x]和每一个α∈a，存在一个唯一的整数h≥0，使得f可被（x−α）h整除，但不可被（x−α）h+1整除。事实上，因为f可以被（x−α）h整除，所以h≤deg（f）。当h=0时，α不是f的根，即f（α）=06。有趣的情况是α是f的根。

Definition 29.12. Given a ring A and any nonnull polynomial f ∈ A[X], given any α ∈ A, the unique h ≥ 0 such that f is divisible by (X − α)h but not by (X − α)h+1 is called the order, or multiplicity, of α. We have h = 0 iff α is not a root of f, and when α is a root of f, if h = 1, we call α a simple root, if h = 2, a double root, and generally, a root of multiplicity h ≥ 2 is called a multiple root.  
定义29.12.给定一个环A和任意一个非零多项式f∈a[x]，给定任意α∈a，其唯一h≥0使得f可被（x−α）h整除，但不被（x−α）h+1整除，称为α的阶次或多重性。如果α不是f的根，当α是f的根时，如果h=1，我们称α为简单根，如果h=2，我们称之为双根，通常，多重性h≥2的根称为多根。

Observe that Proposition 29.20 (2) implies that if A ⊆ B, where A and B are rings, for every nonnull polynomial f ∈ A[X], if α ∈ A is a root of f, then the multiplicity of α with respect to f ∈ A[X] and the multiplicity of α with respect to f considered as a polynomial in B[X], is the same.  
观察29.20（2）命题意味着，如果a b，其中a和b是环，对于每个非零多项式f∈a[x]，如果α∈a是f的根，那么α相对于f∈a[x]的多重性和α相对于f的多重性在b[x]中被视为多项式，则为sa我。

We now show that if the ring A is an integral domain, the number of roots of a nonzero polynomial is at most its degree.  
我们现在证明，如果环A是一个积分域，则非零多项式的根的数目最多是它的次数。

Proposition 29.21. Let f,g ∈ A[X] be nonnull polynomials, let α ∈ A, and let h ≥ 0 and k ≥ 0 be the multiplicities of α with respect to f and g. The following properties hold.  
提案29.21。设f，g∈a[x]为非零多项式，设α∈a，设h≥0，k≥0为α对f和g的重数，其性质如下：

1. If l is the multiplicity of α with respect to (f + g), then l ≥ min(h,k). If h =6 k, then l = min(h,k).  
   如果L是α相对于（f+g）的重数，则L≥min（h，k）。如果h=6 k，则l=min（h，k）。
2. If m is the multiplicity of α with respect to fg, then m ≥ h + k. If A is an integral domain, then m = h + k.  
   如果m是α相对于fg的重数，则m≥h+k。如果a是积分域，则m=h+k。

Proof. (1) We have f(X) = (X − α)hf1(X), g(X) = (X − α)kg1(X), with f1(α) = 06 and g1(α) = 06 . Clearly, l ≥ min(h,k). If h =6 k, assume h < k. Then, we have f(X) + g(X) = (X − α)hf1(X) + (X − α)kg1(X) = (X − α)h(f1(X) + (X − α)k−hg1(X)), and since (f1(X) + (X − α)k−hg1(X))(α) = f1(α) = 06, we have l = h = min(h,k).  
证据。（1）我们有f（x）=（x−α）hf1（x），g（x）=（x−α）kg1（x），其中f1（α）=06和g1（α）=06。显然，l≥min（h，k）。如果h=6 k，假设h<k。那么，我们有f（x）+g（x）=（x−α）h f1（x）+（x−α）kg1（x）=（x−α）h（f1（x）+（x−α）k−hg1（x）），因为（f1（x）+（x−α）k−hg1（x））（α）=f1（α）=06，我们有l=h=min（h，k）。

(2) We have f(X)g(X) = (X − α)h+kf1(X)g1(X),  
（2）我们得到f（x）g（x）=（x−α）h+kf1（x）g1（x），

with f1(α) = 06 and g1(α) = 06 . Clearly, m ≥ h + k. If A is an integral domain, then f1(α)g1(α) = 06 , and so m = h + k.   
f1（α）=06和g1（α）=06。显然，m≥h+k。如果a是一个积分域，那么f1（α）g1（α）=06，因此m=h+k。

Proposition 29.22. Let A be an integral domain. Let f be any nonnull polynomial f ∈ A[X] and let α1,...,αm ∈ A be m ≥ 1 distinct roots of f of respective multiplicities k1,...,km.  
提案29.22。设A为积分域。设f为任意非零多项式f∈a[x]且设α1，…，αm∈a为m≥1各自多重性k 1，…，km的f的不同根。

Then, we have f(X) = (X − α1)k1 ···(X − αm)kmg(X),  
然后，我们得到f（x）=（x−α1）k1···（x−αm）kmg（x），

where g ∈ A[X] and g(αi) = 06 for all i, 1 ≤ i ≤ m.  
式中，所有i的g∈a[x]和g（αi）=06，1≤i≤m。

Proof. We proceed by induction on m. The case m = 1 is obvious in view of Definition 29.12 (which itself, is justified by Proposition 29.20). If m ≥ 2, by the induction hypothesis, we have  
证据。我们对m进行归纳，从定义29.12（其本身由29.20号提案证明）来看，案例m=1是显而易见的。如果m≥2，根据归纳假设，我们有

f(X) = (X − α1)k1 ···(X − αm−1)km−1g1(X),  
f（x）=（x−α1）k1····（x−αm−1）km−1g1（x），

where g1 ∈ A[X] and g1(αi) = 06 , for 1 ≤ i ≤ m − 1. Since A is an integral domain and αi =6 αj for i =6 j, since αm is a root of f, we have  
式中，g1∈a[x]和g1（αi）=06，对于1≤i≤m−1。因为a是一个积分域，αi=6αj是i=6 j，因为αm是f的根，我们有

0 = (αm − α1)k1 ···(αm − αm−1)km−1g1(αm),  
0=（αm−α1）k1····（αm−αm−1）km−1g1（αm），

which implies that ) = 0. Now, by Proposition 29.21 (2), since αm is not a root of the polynomial (X − α1) 1 ···(X − αm−1)km−1 and since A is an integral domain, αm must be a root of multiplicity km of g1, which means that  
这意味着=0。现在，根据命题29.21（2），由于αm不是多项式（x−α1）1···（x−αm−1）km−1的根，并且由于a是一个积分域，αm必须是g1的多重性km的根，这意味着

g1(X) = (X − αm)kmg(X),  
g1（x）=（x−αm）kmg（x）、

with g(αm) = 06 . Since g1(αi) = 06 for 1 ≤ i ≤ m − 1 and A is an integral domain, we must also have g(αi) = 06 , for 1 ≤ i ≤ m − 1. Thus, we have  
g（αm）=06。因为g1（αi）=06对于1≤i≤m−1和a是一个积分域，我们还必须有g（αi）=06，对于1≤i≤m−1。因此，我们

f(X) = (X − α1)k1 ···(X − αm)kmg(X),  
f（x）=（x−α1）k1···（x−αm）kmg（x）、

where g ∈ A[X], and g(αi) = 06 for 1 ≤ i ≤ m.   
式中，对于1≤i≤m，g∈a[x]和g（αi）=06。

As a consequence of Proposition 29.22, we get the following important result.  
根据29.22号提案，我们得到了以下重要结果。

Theorem 29.23. Let A be an integral domain. For every nonnull polynomial f ∈ A[X], if the degree of f is n = deg(f) and k1,...,km are the multiplicities of all the distinct roots of f (where m ≥ 0), then k1 + ··· + km ≤ n.  
定理29.23。设A为积分域。对于每一个非零多项式f∈a[x]，如果f的阶数为n=deg（f）和k1，…，km是f的所有不同根的重数（其中m≥0），则k1+·······+km≤n。

Proof. Immediate from Proposition 29.22.   
证据。直接引自29.22号提案。

Since fields are integral domains, Theorem 29.23 holds for nonnull polynomials over fields and in particular, for R and C. An important consequence of Theorem 29.23 is the following.  
由于域是积分域，定理29.23适用于域上的非空多项式，尤其适用于r和c。定理29.23的一个重要结论如下。

Proposition 29.24. Let A be an integral domain. For any two polynomials f,g ∈ A[X], if deg(f) ≤ n, deg(g) ≤ n, and if there are n + 1 distinct elements α1,α2,...,αn+1 ∈ A (with αi =6 αj for i =6 j) such that f(αi) = g(αi) for all i, 1 ≤ i ≤ n + 1, then f = g.  
提案29.24。设A为积分域。对于任意两个多项式f，g∈a[x]，如果deg（f）≤n，deg（g）≤n，并且如果有n+1个不同元素α1，α2，…，αn+1∈a（αi=6αj，i=6 j），那么f（αi）=g（αi）对于所有i，1≤i≤n+1，那么f=g。

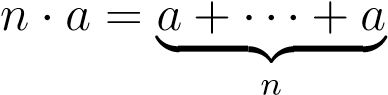
Proof. Assume f =6 g, then, (f−g) is nonnull, and since f(αi) = g(αi) for all i, 1 ≤ i ≤ n+1, the polynomial (f − g) has n + 1 distinct roots. Thus, (f − g) has n + 1 distinct roots and is of degree at most n, which contradicts Theorem 29.23.   
证据。假设f=6g，那么，（f−g）不为空，因为f（αi）=g（αi）对于所有i，1≤i≤n+1，多项式（f−g）有n+1个不同的根。因此，（f−g）有n+1个不同的根，最多有n个度，这与定理29.23相矛盾。

Proposition 29.24 is often used to show that polynomials coincide. We will use it to show some interpolation formulae due to Lagrange and Hermite. But first, we characterize the multiplicity of a root of a polynomial. For this, we need the notion of derivative familiar in analysis. Actually, we can simply define this notion algebraically.  
命题29.24常被用来证明多项式是一致的。我们将用它来表示由于拉格朗日和厄米特的一些插值公式。但首先，我们描述了多项式根的多重性。为此，我们需要分析中熟悉的导数概念。实际上，我们可以简单地用代数的方法定义这个概念。

First, we need to rule out some undesirable behaviors. Given a field K, as we saw in Example 2.8, we can define a homomorphism χ: Z → K given by  
首先，我们需要排除一些不良行为。给定一个场k，如例2.8所示，我们可以定义一个同态，由

χ(n) = n · 1,  
χ（n）=n·1，

where 1 is the multiplicative identity of K. Recall that we define n · a by  
其中1是k的乘法恒等式。回想一下，我们用



if n ≥ 0 (with 0 · a = 0) and  
如果n≥0（0·a=0），且

n · a = −(−n) · a  
N·A=−（−N）·A

if n < 0. We say that the field K is of characteristic zero if the homomorphism χ is injective. Then, for any a ∈ K with a = 06 , we have n · a = 06 for all n = 06  
如果n<0.如果同态χ是内射的，则K场的特征值为零。那么，对于a=06的任意a∈k，n=06都有n·a=06。

The fields Q, R, and C are of characteristic zero. In fact, it is easy to see that every field of characteristic zero contains a subfield isomorphic to Q. Thus, finite fields can’t be of characteristic zero.  
Q、R和C字段为特征零。事实上，很容易看出特征零点的每个场都包含一个与Q同构的子场，因此，有限场不能是特征零点。

Remark: If a field is not of characteristic zero, it is not hard to show that its characteristic, that is, the smallest n ≥ 2 such that n·1 = 0, is a prime number p. The characteristic p of K is the generator of the principal ideal pZ, the kernel of the homomorphism χ: Z → K. Thus, every finite field is of characteristic some prime p. Infinite fields of nonzero characteristic also exist.  
注：如果一个场不属于特征零点，则不难证明其特征，即n·1=0的最小n≥2是素数p，k的特征p是主理想p z的生成者，同态的核χ：z→k，因此，每一个定义Te场具有某些素数p的特征，也存在一些非零特征的无穷大的场。

Definition 29.13. Let A be a ring. The derivative f0, or Df, or D1f, of a polynomial f ∈ A[X] is defined inductively as follows:  
定义29.13.让A成为一个戒指。多项式f∈a[x]的导数f0或df或d1f归纳定义如下：

f0 = 0, if f = 0, the null polynomial, f0 = 0, if f = a, a = 06 , a ∈ A, f0 = nanXn if.  
f0=0，如果f=0，则为零多项式，f0=0，如果f=a，a=06，a∈a，f0=nanxn，如果。

If A = K is a field of characteristic zero, if deg(f) ≥ 1, the leading coefficient nan of f0 is nonzero, and thus, f0 is not the null polynomial. Thus, if A = K is a field of characteristic zero, when n = deg(f) ≥ 1, we have deg(f0) = n − 1.  
如果a=k是特征零点的一个字段，如果deg（f）≥1，f0的前导系数nan是非零的，因此f0不是零多项式。因此，如果a=k是特征零点的场，当n=deg（f）≥1时，我们得到deg（f0）=n-1。

 For rings or for fields of characteristic p ≥ 2, we could have f0 = 0, for a polynomial f of degree ≥ 1.  
对于环或特征p≥2的场，我们可以得到f0=0，对于次数≥1的多项式f。

The following standard properties of derivatives are recalled without proof (prove them as an exercise).  
以下衍生产品的标准属性被召回，但没有证据（作为练习证明）。

Given any two polynomials, f,g ∈ A[X], we have  
对于任意两个多项式，f，g∈a[x]，我们有

(f + g)0 = f0 + g0,  
（f+g）0=f0+g0，

(fg)0 = f0g + fg0.  
（fg）0=f0g+fg0。

For example, if f = (X − α)kg and k ≥ 1, we have  
例如，如果f=（x−α）kg和k≥1，我们有

f0 = k(X − α)k−1g + (X − α)kg0.  
f0=k（x−α）k−1g+（x−α）kg0。

We can now give a criterion for the existence of simple roots. The first proposition holds for any ring.  
我们现在可以给出简单根存在的标准。第一个命题适用于任何环。

Proposition 29.25. Let A be any ring. For every nonnull polynomial f ∈ A[X], α ∈ A is a simple root of f iff α is a root of f and α is not a root of f0.  
提案29.25。让A成为任何戒指。对于每一个非零多项式f∈a[x]，α∈a是f的简单根iffα是f的根，α不是f0的根。

Proof. Since α ∈ A is a root of f, we have f = (X − α)g for some g ∈ A[X]. Now, α is a simple root of f iff g(α) = 06 . However, we have f0 = g + (X − α)g0, and so f0(α) = g(α). Thus, α is a simple root of f iff f0(α) = 0.6   
证据。因为α∈a是f的根，所以对于一些g∈a[x]，我们有f=（x−α）g。现在，α是f iff g（α）=06的简单根。然而，我们有f0=g+（x-α）g0，所以f0（α）=g（α）。因此，α是f iff f0（α）=0.6的简单根。

We can improve the previous proposition as follows.  
我们可以把前面的建议改进如下。

Proposition 29.26. Let A be any ring. For every nonnull polynomial f ∈ A[X], let α ∈ A be a root of multiplicity k ≥ 1 of f. Then, α is a root of multiplicity at least k − 1 of f0. If A is a field of characteristic zero, then α is a root of multiplicity k − 1 of f0.  
提案29.26。让A成为任何戒指。对于每一个非零多项式f∈a[x]，设α∈a为f的重数k≥1的根，则α为f0的重数k-1的根。如果a是特征零的场，那么α是f0的重数k−1的根。

Proof. Since α ∈ A is a root of multiplicity k of f, we have f = (X−α)kg for some g ∈ A[X] and g(α) = 06 . Since  
证据。由于α∈a是f的多重性k的根，对于某些g∈a[x]和g（α）=06，我们得到f=（x−α）kg。自从

f0 = k(X − α)k−1g + (X − α)kg0 = (X − α)k−1(kg + (X − α)g0),  
f0=k（x−α）k−1g+（x−α）k g0=（x−α）k−1（kg+（x−α）g0），

it is clear that the multiplicity of α w.r.t. f0 is at least k−1. Now, (kg+(X−α)g0)(α) = kg(α), and if A is of characteristic zero, since g(α) = 06 , then kg(α) = 06 . Thus, α is a root of multiplicity k − 1 of f0.   
很明显，αw.r.t.f0的多重性至少为k-1。现在，（kg+（x−α）g0）（α）=kg（α），如果a为特征零，因为g（α）=06，那么kg（α）=06。因此，α是f0的重数k−1的根。

As a consequence, we obtain the following test for the existence of a root of multiplicity k for a polynomial f:  
因此，我们对多项式f的多重性k的根的存在性进行了以下测试：

Given a field K of characteristic zero, for any nonnull polynomial f ∈ K[X], any α ∈ K is a root of multiplicity k ≥ 1 of f iff α is a root of f,D1f,D2f,...,Dk−1f, but not a root of Dkf.  
给定特征零的K域，对于任意非零多项式f∈k[x]，任意α∈k是f的重数k≥1的根iffα是f，d1f，d2f，…，dk−1f的根，而不是dkf的根。

We can now return to polynomial functions and tie up some loose ends. Given a ring A, recall that every polynomial f ∈ A[X1,...,Xn] induces a function fA : An → A defined such that fA(α1,...,αn) = f(α1,...,αn), for every (α1,...,αn) ∈ An. We now give a sufficient condition for the mapping f 7→ fA to be injective.  
现在我们可以返回多项式函数，并绑定一些松散的端点。对于一个环A，回想一下，每个多项式f∈a[x1，…，xn]都会产生一个函数fa:a n→a，定义为fa（α1，…，αn）=f（α1，…，αn），对于每个（α1，…，αn）∈an。我们现在给出了映射f 7→f a是内射的充分条件。

Proposition 29.27. Let A be an integral domain. For every polynomial f ∈ A[X1,...,Xn], if A1,...,An are n infinite subsets of A such that f(α1,...,αn) = 0 for all (α1,...,αn) ∈ A1 ×···×An, then f = 0, i.e., f is the null polynomial. As a consequence, if A is an infinite integral domain, then the map f 7→ fA is injective.  
提案29.27。设A为积分域。对于每一个多项式f∈a[x1，…，xn]，如果a1，…，a n是a的n个无限子集，这样f（α1，…，αn）=0表示全部（α1，…，αn）∈a1×······×an，那么f=0，即f是零多项式。因此，如果a是一个无穷大的积分域，那么映射f 7→fa是内射的。

Proof. We proceed by induction on n. Assume n = 1. If f ∈ A[X1] is nonnull, let m = deg(f) be its degree. Since A1 is infinite and f(α1) = 0 for all α1 ∈ A1, then f has an infinite number of roots. But since f is of degree m, this contradicts Theorem 29.23. Thus, f = 0. If n ≥ 2, we can view f ∈ A[X1,...,Xn] as a polynomial  
证据。我们从n开始归纳，假设n=1。如果f∈a[x1]为非空，则m=deg（f）为其度。因为a1是无穷大的，而f（α1）=0对于所有α1∈a1，那么f有无穷多的根。但由于f是m级的，这与定理29.23相矛盾。因此，f=0。如果n≥2，我们可以把f∈a[x1，…，xn]看作一个多项式。

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，

where the coefficients gi are polynomials in A[X1,...,Xn−1]. Now, for every (α1,...,αn−1) ∈ A1 ×···×An−1, f(α1,...,αn−1,Xn) determines a polynomial h(Xn) ∈ A[Xn], and since An is infinite and h(αn) = f(α1,...,αn−1,αn) = 0 for all αn ∈ An, by the induction hypothesis, we have gi(α1,...,αn−1) = 0. Now, since A1,...,An−1 are infinite, using the induction hypothesis again, we get gi = 0, which shows that f is the null polynomial. The second part of the proposition follows immediately from the first, by letting Ai = A.   
其中，系数gi是a[x1，…，xn−1]中的多项式。现在，对于每一个（α1，…，αn-1）∈a1×·······································································现在，由于a1，…，a−1是无限的，再次使用归纳假设，我们得到gi=0，这表明f是零多项式。命题的第二部分紧接着从第一部分开始，让ai=a。

When A is an infinite integral domain, in particular an infinite field, since the map f 7→ fA is injective, we identify the polynomial f with the polynomial function fA, and we write fA simply as f.  
当a是一个无限积分域，特别是一个无限域时，由于图f 7→fa是内射的，我们用多项式函数fa来标识多项式f，并将fa简单地写为f。

The following proposition can be very useful to show polynomial identities.  
下面的命题对于证明多项式恒等式是非常有用的。

Proposition 29.28. Let A be an infinite integral domain and f,g1,...,gm ∈ A[X1,...,Xn] be polynomials. If the gi are nonnull polynomials and if  
提案29.28。设a为无穷积分域，f，g1，…，gm∈a[x1，…，xn]为多项式。如果gi是非空多项式，如果

f(α1,...,αn) = 0 whenever gi(α1,...,αn) 6= 0 for all i, 1 ≤ i ≤ m,  
当所有i的gi（α1，…，αn）6=0，1≤i≤m时，f（α1，…，αn）=0，

for every (α1,...,αn) ∈ An, then  
对于每个（α1，…，αn）∈an，那么

f = 0,  
f＝0，

i.e., f is the null polynomial.  
也就是说，F是零多项式。

Proof. If f is not the null polynomial, since the gi are nonnull and A is an integral domain, then the product fg1 ···gm is nonnull. By Proposition 29.27, only the null polynomial maps to the zero function, and thus there must be some (α1,...,αn) ∈ An, such that  
证据。如果f不是零多项式，因为gi是非零的，a是一个整型域，那么积fg1····gm是非零的。根据命题29.27，只有零多项式映射到零函数，因此必须有一些（α1，…，αn）∈an，这样

f(α1,...,αn)g1(α1,...,αn)···gm(α1,...,αn) = 06 ,  
F（α1，…，αn）g1（α1，…，αn）···gm（α1，…，αn）=06，

but this contradicts the hypothesis.   
但这与假设相矛盾。

Proposition 29.28 is often called the principle of extension of algebraic identities. Another perhaps more illuminating way of stating this proposition is as follows: For any polynomial g ∈ A[X1,...,Xn], let  
命题29.28常被称为代数恒等式的推广原理。另一种可能更具启发性的表述这个命题的方法是：对于任何多项式g∈a[x1，…，xn]，让

V (g) = {(α1,...,αn) ∈ An | g(α1,...,αn) = 0},  
v（g）=（α1，…，αn）∈an g（α1，…，αn）=0，

the set of zeros of g. Note that V (g1)∪···∪V (gm) = V (g1 ···gm). Then, Proposition 29.28 can be stated as:  
G的零点集合。注意V（g1）·····································那么，29.28号提案可以表述为：

If f(α1,...,αn) = 0 for every (α1,...,αn) ∈ An − V (g1 ···gm), then f = 0.  
如果f（α1，…，αn）=0，对于每（α1，…，αn）∈an−v（g1···gm），则f=0。

In other words, if the algebraic identity f(α1,...,αn) = 0 holds on the complement of V (g1) ∪ ··· ∪ V (gm) = V (g1 ···gm), then f(α1,...,αn) = 0 holds everywhere in An. With this second formulation, we understand better the terminology “principle of extension of algebraic identities.”  
换句话说，如果代数恒等式f（α1，…，αn）=0保留V（g1）·············································通过第二个公式，我们更好地理解术语“代数恒等式的扩展原理”。

Remark: Letting U(g) = A−V (g), the identity V (g1)∪···∪V (gm) = V (g1 ···gm) translates to U(g1) ∩ ··· ∩ U(gm) = U(g1 ···gm). This suggests to define a topology on A whose basis of open sets consists of the sets U(g). In this topology (called the Zariski topology), the sets of the form V (g) are closed sets. Also, when g1,...,gm ∈ A[X1,...,Xn] and n ≥ 2, understanding the structure of the closed sets of the form V (g1)∩···∩V (gm) is quite difficult, and it is the object of algebraic geometry (at least, its classical part).  
备注：让u（g）=a−v（g），单位v（g1）···············································这就建议定义一个拓扑，在这个拓扑的基础上，开放集由集U（G）组成。在这个拓扑（称为zariski拓扑）中，V（G）形式的集合是闭集合。另外，当g1，…，gm∈a[x1，…，xn]和n≥2时，理解形式v（g1）·v（gm）闭集的结构是相当困难的，它是代数几何的对象（至少是它的经典部分）。

 When f ∈ A[X1,...,Xn] and n ≥ 2, one should not apply Proposition 29.27 abusively. For example, let  
当f∈a[x1，…，xn]且n≥2时，不应滥用29.27号命题。例如，让

f(X,Y ) = X2 + Y 2 − 1,  
F（x，y）=x2+y 2−1，

considered as a polynomial in R[X,Y ]. Since R is an infinite field and since  
在r[x，y]中被认为是多项式。因为r是一个无限的场

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for every t ∈ R, it would be tempting to say that f = 0. But what’s wrong with the above reasoning is that there are no two infinite subsets R1,R2 of R such that f(α1,α2) = 0 for all (α1,α2) ∈ R2. For every α1 ∈ R, there are at most two α2 ∈ R such that f(α1,α2) = 0. What the example shows though, is that a nonnull polynomial f ∈ A[X1,...,Xn] where n ≥ 2 can have an infinite number of zeros. This is in contrast with nonnull polynomials in one variables over an infinite field (which have a number of roots bounded by their degree).  
对于每一个t∈r，我们都会倾向于说f=0。但上述推理的错误之处在于，R中没有两个无穷大的子集r1，r2，因此f（α1，α2）=0表示所有（α1，α2）∈r2。对于每个α1∈r，至多有两个α2∈r，使得f（α1，α2）=0。然而，示例显示，非空多项式f∈a[x1，…，xn]其中n≥2可以有无限个零。这与无限域上一个变量中的非空多项式形成对比（该变量的根数受其阶数的限制）。

We now look at polynomial interpolation.  
现在我们来看看多项式插值。

## 29.7 Polynomial Interpolation (Lagrange, Newton, Hermite) 29.7多项式插值（拉格朗日、牛顿、赫米特）

Let K be a field. First, we consider the following interpolation problem: Given a sequence  
让k成为一个场。首先，我们考虑以下插值问题：给定一个序列

(α1,...,αm+1) of pairwise distinct scalars in K and any sequence (β1,...,βm+1) of scalars in K, where the βj are not necessarily distinct, find a polynomial P(X) of degree ≤ m such that  
（α1，…，αm+1）k中的成对不定标度和k中的任何标度序列（β1，…，βm+1），其中βj不一定是不同的，求一个次数≤m的多项式p（x），这样

P(α1) = β1,...,P(αm+1) = βm+1.  
P（α1）=β1，…，P（αm+1）=βm+1。

First, observe that if such a polynomial exists, then it is unique. Indeed, this is a consequence of Proposition 29.24. Thus, we just have to find any polynomial of degree ≤ m. Consider the following so-called Lagrange polynomials:  
首先，观察这样一个多项式是否存在，那么它是唯一的。事实上，这是29.24号提案的结果。因此，我们只需要找到任何次数≤m的多项式。考虑以下所谓的拉格朗日多项式：

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Note that L(αi) = 1 and that L(αj) = 0, for all j =6 i. But then,  
注意，l（αi）=1，l（αj）=0，对于所有j=6 i，但是，

P(X) = β1L1 + ··· + βm+1Lm+1  
P（x）=β1l1+····+βm+1lm+1

is the unique desired polynomial, since clearly, P(αi) = βi. Such a polynomial is called a Lagrange interpolant. Also note that the polynomials (L1,...,Lm+1) form a basis of the vector space of all polynomials of degree ≤ m. Indeed, if we had  
是唯一的期望多项式，因为很明显，p（αi）=βi。这样的多项式称为拉格朗日插值。还需要注意的是，多项式（l1，…，lm+1）构成了度≤m的所有多项式的向量空间的基础。

λ1L1(X) + ··· + λm+1Lm+1(X) = 0,  
λ1l1（x）+·····+λm+1lm+1（x）=0，

setting X to αi, we would get λi = 0. Thus, the Li are linearly independent, and by the previous argument, they are a set of generators. We we call (L1,...,Lm+1) the Lagrange basis (of order m + 1).  
将x设为αi，我们得到λi=0。因此，li是线性独立的，根据前面的论证，它们是一组生成器。我们称（l1，…，lm+1）为拉格朗日基（m+1阶）。

It is known from numerical analysis that from a computational point of view, the Lagrange basis is not very good. Newton proposed another solution, the method of divided differences. Consider the polynomial P(X) of degree ≤ m, called the Newton interpolant,  
从数值分析可知，从计算的角度来看，拉格朗日基不是很好。牛顿提出了另一种解决方法，差分法。考虑次数≤m的多项式p（x），称为牛顿插值，

P(X) = λ0 + λ1(X − α1) + λ2(X − α1)(X − α2) + ··· + λm(X − α1)(X − α2)···(X − αm).  
p（x）=λ0+λ1（x-α1）+λ2（x-α1）（x-α2）+····+λm（x-α1）（x-α2）···（x-αm）。

Then, the λi can be determined by successively setting X to, α1,α2,...,αm+1. More precisely, we define inductively the polynomials Q(X) and Q(α1,...,αi,X), for 1 ≤ i ≤ m, as follows:  
然后，通过依次将x设置为，α1，α2，…，αm+1来确定λi。更准确地说，我们归纳地定义了1≤i≤m的多项式q（x）和q（α1，…，αi，x），如下：

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By induction on i, 1 ≤ i ≤ m − 1, it is easily verified that  
通过对i的感应，1≤i≤m−1，很容易证实

Q(X) = P(X),  
Q（x）=P（x）

Q(α1,...,αi,X) = λi + λi+1(X − αi+1) + ··· + λm(X − αi+1)···(X − αm), Q(α1,...,αm,X) = λm.  
q（α1，…，αi，x）=λi+λi+1（x−αi+1）+······+λm（x−αi+1）··（x−αm），q（α1，…，αm，x）=λm。

From the above expressions, it is clear that  
从上面的表达式可以清楚地看出

λ0 = Q(α1), λi = Q(α1,...,αi,αi+1), λm = Q(α1,...,αm,αm+1).  
λ0=q（α1），λi=q（α1，…，αi，αi+1），λm=q（α1，…，αm，αm+1）。

The expression Q(α1,α2,...,αi+1) is called the i-th difference quotient. Then, we can compute the λi in terms of β1 = P(α1),...,βm+1 = P(αm+1), using the inductive formulae for the Q(α1,...,αi,X) given above, initializing the Q(αi) such that Q(αi) = βi.  
表达式q（α1，α2，…，αi+1）称为第i个差商。然后，我们可以根据β1=p（α1），…，βm+1=p（αm+1）计算λi，使用上面给出的q（α1，…，αi，x）的归纳公式，初始化q（αi），使q（αi）=βi。

The above method is called the method of divided differences and it is due to Newton.  
上述方法称为差分法，它是牛顿的结果。

An astute observation may be used to optimize the computation. Observe that if Pi(X) is the polynomial of degree ≤ i taking the values β1,...,βi+1 at the points α1,...,αi+1, then the coefficient of Xi in Pi(X) is Q(α1,α2,...,αi+1), which is the value of λi in the Newton interpolant  
一个敏锐的观察可以用来优化计算。若皮（x）是在α1、α、αi＋1点取β1、…、βi＋1的值的多项式，则Xi在pi（x）中的系数是q（α1，α2，…，αi+1），这是牛顿插值中λi的值。

Pi(X) = λ0 + λ1(X − α1) + λ2(X − α1)(X − α2) + ··· + λi(X − α1)(X − α2)···(X − αi).  
Pi（x）=λ0+λ1（x−α1）+λ2（x−α1）（x−α2）+····+λi（x−α1）（x−α2）···（x−αi）。

As a consequence, Q(α1,α2,...,αi+1) does not depend on the specific ordering of the αj and there are better ways of computing it. For example, Q(α1,α2,...,αi+1) can be computed using  
因此，q（α1，α2，…，αi+1）不依赖于αj的特定顺序，有更好的计算方法。例如，q（α1，α2，…，αi+1）可以用

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Then, the computation can be arranged into a triangular array reminiscent of Pascal’s triangle, as follows:  
然后，可以将计算安排成一个类似于帕斯卡三角形的三角形数组，如下所示：

Initially, Q(αj) = βj, 1 ≤ j ≤ m + 1, and  
最初，q（αj）=βj，1≤j≤m+1，以及

Q(α1)  
Q（α1）

Q(α1,α2)  
Q（α1，α2）

Q(α2) Q(α1,α2,α3) Q(α2,α3) ...  
Q（α2）Q（α1，α2，α3）Q（α2，α3）……

Q(α3) Q(α2,α3,α4) Q(α3,α4) ...  
Q（α3）Q（α2，α3，α4）Q（α3，α4）……

Q(α4) ...  
Q（α4）

...  
…

In this computation, each successive column is obtained by forming the difference quotients of the preceeding column according to the formula  
在该计算中，每个连续列都是通过根据公式形成前一列的差商得到的。

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The λi are the elements of the descending diagonal.  
λi是下对角线的元素。

Observe that if we performed the above computation starting with a polynomial Q(X) of degree m, we could extend it by considering new given points αm+2, αm+3, etc. Then, from what we saw above, the (m + 1)th column consists of λm in the expression of Q(X) as a Newton interpolant and the (m + 2)th column consists of zeros. Such divided differences are used in numerical analysis.  
观察到，如果我们从m次的多项式q（x）开始进行上述计算，我们可以通过考虑新的给定点αm+2、αm+3等来扩展它。然后，从我们上面看到的，（m+1）第列由q（x）表示为牛顿插值的λm和（m+2）t组成。H列由零组成。在数值分析中使用了这种分离的差异。

Newton’s method can be used to compute the value P(α) at some α of the interpolant  
牛顿方法可以用来计算插值函数中某些α处的p（α）

P(X) taking the values β1,...,βm+1 for the (distinct) arguments α1,...,αm+1. We also mention that inductive methods for computing P(α) without first computing the coefficients of the Newton interpolant exist, for example, Aitken’s method. For this method, the reader is referred to Farin [59].  
p（x）取β1，…，βm+1作为（不同的）参数α1，…，αm+1。我们还提到了在不首先计算牛顿插值系数的情况下计算p（α）的归纳法，例如艾特肯方法。对于这种方法，读者可以参考法林[59]。

It has been observed that Lagrange interpolants oscillate quite badly as their degree increases, and thus, this makes them undesirable as a stable method for interpolation. A standard example due to Runge, is the function  
据观察，拉格朗日插值函数随着其阶数的增加而振荡得很厉害，因此，这使得拉格朗日插值函数不适合作为一种稳定的插值方法。一个标准的例子，由于梯级，是函数

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in the interval [−5, +5]. Assuming a uniform distribution of points on the curve in the interval [−5, +5], as the degree of the Lagrange interpolant increases, the interpolant shows wilder and wilder oscillations around the points x = −5 and x = +5. This phenomenon becomes quite noticeable beginning for degree 14, and gets worse and worse. For degree 22, things are quite bad! Equivalently, one may consider the function  
在区间[-5，+5]。假设曲线上点在区间[-5，+5]内的均匀分布，随着拉格朗日插值程度的增加，插值在点x=-5和x=+5周围显示出更狂野和更狂野的振荡。这种现象在14度开始时变得相当明显，并且越来越严重。对于22级，情况相当糟糕！等价地，我们可以考虑函数

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in the interval [−1, +1].  
在区间[−1，+1]。

We now consider a more general interpolation problem which will lead to the Hermite polynomials.  
我们现在考虑一个更普遍的插值问题，它将导致厄米多项式。

We consider the following interpolation problem:  
我们考虑以下插值问题：

Given a sequence (α1,...,αm+1) of pairwise distinct scalars in K, integers n1,...,nm+1 where nj ≥ 0, and m + 1 sequences () of scalars in K, letting  
给定一个k中成对不同标量的序列（α1，…，αm+1），整数n1，…，nm+1，其中nj≥0，m+1，k中标量的序列（），让

n = n1 + ··· + nm+1 + m,  
n=n1+·····+n m+1+m，

find a polynomial P of degree ≤ n, such that  
求一个多项式p的次数≤n，这样

,  
，

D1P(α1) = β11, ... D1P(αm+1) = βm1 +1,  
d1p（α1）=β11，…d1p（αm+1）=βm1+1，

...  
…

D , ... D,  
D，…D，，

...  
…

D.  
d.

Note that the above equations constitute n+1 constraints, and thus, we can expect that there is a unique polynomial of degree ≤ n satisfying the above problem. This is indeed the case and such a polynomial is called a Hermite polynomial. We call the above problem the Hermite interpolation problem.  
请注意，上述方程构成n+1约束，因此，我们可以期望有一个满足上述问题且次数≤n的唯一多项式。这确实是这样的情况，这样的多项式被称为埃尔米特多项式。我们将上述问题称为埃尔米特插值问题。

Proposition 29.29. The Hermite interpolation problem has a unique solution of degree ≤ n, where n = n1 + ··· + nm+1 + m.  
提案29.29。埃尔米特插值问题有一个唯一的解，即n=n1+·····+n m+1+m。

Proof. First, we prove that the Hermite interpolation problem has at most one solution. Assume that P and Q are two distinct solutions of degree ≤ n. Then, by Proposition 29.26 and the criterion following it, P −Q has among its roots α1 of multiplicity at least n1 +1,..., αm+1 of multiplicity at least nm+1 + 1. However, by Theorem 29.23, we should have  
证据。首先，我们证明了厄米插值问题至多只有一个解。假设p和q是两个不同的度≤n的解，那么，根据命题29.26及其后的标准，p−q在多重性的根α1中至少有n1+1，…，多重性的αm+1至少有nm+1+1。然而，根据定理29.23，我们应该

n1 + 1 + ··· + nm+1 + 1 = n1 + ··· + nm+1 + m + 1 ≤ n,  
n1+1+······+n m+1+1=n1+·····+nm+1+m+1≤n，

which is a contradiction, since n = n1 + ··· + nm+1 + m. Thus, P = Q. We are left with proving the existence of a Hermite interpolant. A quick way to do so is to use Proposition 6.13, which tells us that given a square matrix A over a field K, the following properties hold:  
这是一个矛盾，因为n=n1+·····+n m+1+m。因此，p=q。我们只剩下证明一个埃尔米特插值的存在。这样做的一个快速方法是使用命题6.13，它告诉我们，给定一个域k上的平方矩阵a，以下属性成立：

For every column vector B, there is a unique column vector X such that AX = B iff the only solution to AX = 0 is the trivial vector X = 0 iff D(A) = 0.6  
对于每个列向量b，都有一个唯一的列向量x，这样ax=b iff，ax=0的唯一解是平凡向量x=0 iff d（a）=0.6。

If we let P = y0 + y1X + ··· + ynXn, the Hermite interpolation problem yields a linear system of equations in the unknowns (y0,...,yn) with some associated (n+1)×(n+1) matrix A. Now, the system AY = 0 has a solution iff P has among its roots α1 of multiplicity at least n1 + 1,..., αm+1 of multiplicity at least nm+1 + 1. By the previous argument, since P has degree ≤ n, we must have P = 0, that is, Y = 0. This concludes the proof.   
如果p=y0+y1x+·····+ynxn，厄米特插值问题在未知（y0，…，yn）中产生一个线性方程组，其中有一些相关（n+1）×n+1）矩阵a。现在，系统ay=0有一个解，如果p在其多重性的根α1中至少有n1+1，…，αm+1，mul滴度至少为nm+1+1。根据前面的论点，由于p的度数≤n，我们必须有p=0，即y=0。这就是证据的结论。

Proposition 29.29 shows the existence of unique polynomials Hji(X) of degree ≤ n such that D ) = 1 and D ) = 0, for k =6 i or l =6 j, 1 ≤ j,l ≤ m + 1, 0 ≤ i,k ≤ nj.  
命题29.29显示了度≤n的唯一多项式hji（x）的存在，使得d=1和d=0，对于k=6 i或l=6 j，1≤j，l≤m+1，0≤i，k≤nj。

The polynomials Hji are called Hermite basis polynomials.  
多项式hji称为Hermite基多项式。

One problem with Proposition 29.29 is that it does not give an explicit way of computing the Hermite basis polynomials. We first show that this can be done explicitly in the special cases n1 = ... = nm+1 = 1, and n1 = ... = nm+1 = 2, and then suggest a method using a generalized Newton interpolant.  
29.29号命题的一个问题是，它没有给出计算厄米特基多项式的明确方法。我们首先表明，这可以在特殊情况下显式地做到n1=……=nm+1=1，n1=……=nm+1=2，然后建议使用广义牛顿插值法。

Assume that n1 = ... = nm+1 = 1. We try Hj0 = (a(X − αj) + b)L2j, and Hj1 =  
假设n1=…=nm+1=1。我们尝试hj0=（a（x−αj）+b）l2j和hj1=

(c(X − αj) + d)L2j, where Lj is the Lagrange interpolant determined earlier. Since  
（c（x−αj）+d）L2j，其中Lj是前面确定的拉格朗日插值。自从

DHj0 = aL2j + 2(a(X − αj) + b)LjDLj,  
dhj0=al2j+2（a（x−αj）+b）ljdlj，

requiring that Hj0(αj) = 1, Hj0(αk) = 0, DHj0(αj) = 0, and DHj0(αk) = 0, for k =6 j, implies b = 1 and a = −2DLj(αj). Similarly, from the requirements Hj1(αj) = 0, Hj1(αk) = 0, DHj1(αj) = 1, and DHj1(αk) = 0, k 6= j, we get c = 1 and d = 0.  
要求hj0（αj）=1，hj0（αk）=0，dhj0（αj）=0和dhj0（αk）=0，对于k=6 j，意味着b=1和a=-2dlj（αj）。同样，根据要求hj1（αj）=0，hj1（αk）=0，dhj1（αj）=1，dhj1（αk）=0，k 6=j，我们得到c=1，d=0。

Thus, we have the Hermite polynomials  
因此，我们有埃尔米特多项式

Hj0 = (1 − 2DLj(αj)(X − αj))Lj2, Hj1 = (X − αj)L2j.  
hj0=（1−2dlj（αj）（x−αj））lj2，hj1=（x−αj）l2j。

In the special case where m = 1, α1 = 0, and α2 = 1, we leave as an exercise to show that the Hermite polynomials are  
在m=1，α1=0，α2=1的特殊情况下，我们将作为一个练习来证明厄米特多项式是

H00 = 2X3 − 3X2 + 1,  
h00=2x3−3x2+1，

H10 = −2X3 + 3X2, H01 = X3 − 2X2 + X, H11 = X3 − X2.  
h10=−2x3+3x2，h01=x3−2x2+x，h11=x3−x2。

As a consequence, the polynomial P of degree 3 such that P(0) = x0, P(1) = x1, P 0(0) = m0, and P 0(1) = m1, can be written as  
因此，阶数为3的多项式p，如p（0）=x0，p（1）=x1，p 0（0）=m0，p 0（1）=m1，可以写成

P(X) = x0(2X3 − 3X2 + 1) + m0(X3 − 2X2 + X) + m1(X3 − X2) + x1(−2X3 + 3X2).  
P（x）=X0（2x3−3x2+1）+m0（x3−2x2+x）+m1（x3−x2）+x1（−2x3+3x2）。

If we want the polynomial P of degree 3 such that P(a) = x0, P(b) = x1, P 0(a) = m0, and P 0(b) = m1, where b =6 a, then we have  
如果我们想要三次多项式p，这样p（a）=x0，p（b）=x1，p 0（a）=m0，p 0（b）=m1，其中b=6a，那么我们得到

P(X) = x0(2t3 − 3t2 + 1) + (b − a)m0(t3 − 2t2 + t) + (b − a)m1(t3 − t2) + x1(−2t3 + 3t2), where  
p（x）=x0（2t3−3t2+1）+（b−a）m0（t3−2t2+t）+（b−a）m1（t3−t2）+x1（−2t3+3t2），其中

.  
.

Observe the presence of the extra factor () in front of m0 and m1, the formula would be false otherwise!  
观察m0和m1前面的额外因子（）的存在，否则公式将为假！

We now consider the case where n1 = ... = nm+1 = 2. Let us try  
我们现在考虑一下n1=的情况。=nm+1=2。让我们试试看

Hji(X) = (ai(X − αj)2 + bi(X − αj) + ci)L3j,  
hji（x）=（ai（x−αj）2+bi（x−αj）+ci）l3j，

where 0 ≤ i ≤ 2. Sparing the readers some (tedious) computations, we find:  
其中0≤i≤2。除了读者的一些（繁琐）计算，我们发现：

,  
，

Going back to the general problem, it seems to us that a kind of Newton interpolant will be more manageable. Let  
回到一般问题，在我们看来，一种牛顿插值法将更易于管理。让

P00(X) = 1,  
p00（x）=1，

Pj0(X) = (X − α1)n1+1 ···(X − αj)nj+1, 1 ≤ j ≤ m  
Pj0（x）=（x−α1）n1+1···（x−αj）nj+1，1≤j≤m

P0i(X) = (X − α1)i(X − α2)n2+1 ···(X − αm+1)nm+1+1, 1 ≤ i ≤ n1,  
p0i（x）=（x−α1）i（x−α2）n2+1····（x−αm+1）nm+1+1，1≤i≤n1，

Pji(X) = (X − α1)n1+1 ···(X − αj)nj+1(X − αj+1)i(X − αj+2)nj+2+1 ···(X − αm+1)nm+1+1,  
Pji（x）=（x−α1）n1+1···（x−αj）nj+1（x−αj+1）i（x−αj+2）nj+2+1···（x−αm+1）nm+1+1，

1 ≤ j ≤ m − 1, 1 ≤ i ≤ nj+1,  
1≤j≤m−1，1≤i≤nj+1，

Pmi (X) = (X − α1)n1+1 ···(X − αm)nm+1(X − αm+1)i, 1 ≤ i ≤ nm+1,  
pmi（x）=（x−α1）n1+1···（x−αm）nm+1（x−αm+1）i，1≤i≤nm+1，

and let  
让

j=m,i=nj+1  
j=m，i=nj+1

P(X) = X λijPji(X).  
p（x）=xλijpji（x）。

j=0,i=0  
j=0，i=0

We can think of P(X) as a generalized Newton interpolant. We can compute the derivatives D, for 1 ≤ k ≤ nj+1, and if we look for the Hermite basis polynomials ) such that DiHji(αj) = 1 and DkHji(αl) = 0, for k 6= i or l 6= j, 1 ≤ j,l ≤ m + 1, 0 ≤ i,k ≤ nj, we find that we have to solve triangular systems of linear equations. Thus, as in the simple case n1 = ... = nm+1 = 0, we can solve successively for the λij. Obviously, the computations are quite formidable and we leave such considerations for further study.  
我们可以把p（x）看作是广义牛顿插值。我们可以计算1≤k≤nj+1的导数d，如果我们寻找埃尔米特基多项式），那么dihji（αj）=1和dkhji（αl）=0，对于k 6=i或l 6=j，1≤j，l≤m+1，0≤i，k≤nj，我们发现我们必须解线性方程的三角系。因此，在简单情况下，n1=……=nm+1=0，我们可以依次求解λij。显然，这些计算是相当可怕的，我们将这些考虑留作进一步研究。

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Chapter 30  
第三十章

# Annihilating Polynomials and the Primary Decomposition 湮灭多项式与一次分解

In this chapter all vector spaces are defined over an arbitrary field K.  
在本章中，所有的向量空间都是在一个任意的k域上定义的。

In Section 6.7 we explained that if f : E → E is a linear map on a K-vector space E, then for any polynomial p(X) = a0Xd + a1Xd−1 + ··· + ad with coefficients in the field K, we can define the linear map p(f): E → E by  
在第6.7节中，我们解释了如果f:e→e是k向量空间e上的线性映射，那么对于任何多项式p（x）=a0xd+a1xd−1+·········+ad以及字段k中的系数，我们可以通过以下方式定义线性映射p（f）：e→e：

p(f) = a0fd + a1fd−1 + ··· + adid,  
p（f）=a0fd+a1fd−1+····+adid，

where fk = f ◦ ··· ◦ f, the k-fold composition of f with itself. Note that  
式中，f k=f·····f，f与其自身的k-折叠组成。注意

p(f)(u) = a0fd(u) + a1fd−1(u) + ··· + adu,  
p（f）（u）=a0fd（u）+a1fd−1（u）+·····+adu，

for every vector u ∈ E. Then we showed that if E is finite-dimensional and if χf(X) = det(XI −f) is the characteristic polynomial of f, by the Cayley–Hamilton theorem, we have  
对于每个向量uε，我们证明了如果E是有限维的，并且如果χf（x）＝DET（Xi—f）是f的特征多项式，则通过Cayley -汉密尔顿定理，我们得到了。

χf(f) = 0.  
χf（f）=0。

This fact suggests looking at the set of all polynomials p(X) such that  
这一事实表明，研究所有多项式的集合p（x），这样

p(f) = 0.  
p（f）=0.

Such polynomials are called annihilating polynomials of f, the set of all these polynomials, denoted Ann(f), is called the annihilator of f, and the Cayley-Hamilton theorem shows that it is nontrivial since it contains a polynomial of positive degree. It turns out that Ann(f) contains a polynomial mf of smallest degree that generates Ann(f), and this polynomial divides the characteristic polynomial. Furthermore, the polynomial mf encapsulates a lot of information about f, in particular whether f can be diagonalized. One of the main reasons for this is that a scalar λ ∈ K is a zero of the minimal polynomial mf if and only if λ is an eigenvalue of f.  
这类多项式称为F的湮灭多项式，所有这些多项式的集合称为F的湮灭子，Cayley-Hamilton定理表明，它包含一个正度多项式，因此是不平凡的。结果表明，ann（f）包含一个生成ann（f）的最小阶多项式mf，该多项式对特征多项式进行了划分。此外，多项式mf还封装了许多关于f的信息，特别是f是否可以对角化。其主要原因之一是当且仅当λ是f的特征值时，标量λ∈k是最小多项式mf的零。

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九百九十一

The first main result is Theorem 30.6 which states that if f : E → E is a linear map on a finite-dimensional space E, then f is diagonalizable iff its minimal polynomial m is of the form  
第一个主要结果是定理30.6，它指出，如果f:e→e是有限维空间e上的线性映射，那么f是可对角化的，如果其最小多项式m是形式

m = (X − λ1)···(X − λk),  
m=（x−λ1）···（x−λk），

where λ1,...,λk are distinct elements of K.  
其中，λ1，…，λk是k的不同元素。

One of the technical tools used to prove this result is the notion of f-conductor; see Definition 30.2. As a corollary of Theorem 30.6 we obtain results about finite commuting families of diagonalizable or triangulable linear maps.  
证明这一结果的技术工具之一是F导体的概念；见定义30.2。作为定理30.6的推论，我们得到了对角化或三角化线性映射的有限交换族的结果。

If f : E → E is a linear map and λ ∈ K is an eigenvalue of f, recall that the eigenspace Eλ associated with λ is the kernel of the linear map λid − f. If all the eigenvalues λ1 ...,λk of f are in K and if f is diagonalizable, then  
如果f:e→e是一个线性映射，而λ∈k是f的特征值，回想一下，与λ相关联的特征空间eλ是线性映射的核心，即λid−f。如果所有特征值λ1…，则f的λk都是k，如果f是可对角化的，则

E = Eλ1 ⊕ ··· ⊕ Eλk,  
e=eλ1···eλk，

but in general there are not enough eigenvectors to span E. A remedy is to generalize the notion of eigenvector and look for (nonzero) vectors u (called generalized eigenvectors) such that  
但是，一般来说，没有足够的特征向量跨越e。一种补救方法是概括特征向量的概念，并寻找（非零）向量u（称为广义特征向量），以便

(λid − f)r(u) = 0, for some r ≥ 1.  
（λid−f）r（u）=0，对于某些r≥1。

Then, it turns out that if the minimal polynomial of f is of the form  
如果F的最小多项式是

m = (X − λ1)r1 ···(X − λk)rk,  
m=（x−λ1）r1···（x−λk）Rk，

then r = ri does the job for λi; that is, if we let  
那么r=ri为λi做这个工作；也就是说，如果我们允许

Wi = Ker(λiid − f)ri,  
wi=ker（λiid−f）ri，

then  
然后

E = W1 ⊕ ··· ⊕ Wk.  
E=w1····周。

The above facts are parts of the primary decomposition theorem (Theorem 30.11). It is a special case of a more general result involving the factorization of the minimal polynomial m into its irreducible monic factors; see Theorem 30.10.  
上述事实是主分解定理（定理30.11）的一部分。它是一个更一般的结果的特例，涉及极小多项式m因式分解为其不可约Monic因子；见定理30.10。

Theorem 30.11 implies that every linear map f that has all its eigenvalues in K can be written as f = D + N, where D is diagonalizable and N is nilpotent (which means that Nr = 0 for some positive integer r). Furthermore D and N commute and are unique. This is the Jordan decomposition, Theorem 30.12.  
定理30.11表明，每一个具有k中所有特征值的线性映射f都可以写成f=d+n，其中d是可对角化的，n是幂零的（对于某个正整数r，nr=0）。此外，D和N通勤和独特。这是乔丹分解，定理30.12。

The Jordan decomposition suggests taking a closer look at nilpotent maps. We prove that for any nilpotent linear map f : E → E on a finite-dimensional vector space E of dimension n over a field K, there is a basis of E such that the matrix N of f is of the form  
乔丹的分解意味着仔细观察幂零映射。我们证明了在有限维向量空间E上的任意幂零线性映射f:e→e在k域上存在一个e的基础，使得f的矩阵n的形式为

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  γ  0 零  0 N = ... 0 N=…   γ   γ  0 0   γ  0 零 | ν1 1伏  0 零  ... …  0 零  0 零 | 0 ν2 0ω2  ... …  0 零  0 零 | ··· ·········  ···... ···…  ··· ·········  ··· ········· | 0 零  0 零  ... …  0 零  0 零 | 0  0℃  0  0℃  ... , …，  νn \_n\_  0 零 |

30.1. ANNIHILATING POLYNOMIALS AND THE MINIMAL POLYNOMIAL  
30.1。湮灭多项式与极小多项式

where νi = 1 or νi = 0; see Theorem 30.16. As a corollary we obtain the Jordan form; which involves matrices of the form  
式中，νi=1或νi=0；见定理30.16。作为推论，我们得到了乔丹形式；它涉及形式的矩阵。

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| λ γ-α  0 零   γ  . Jr(λ) = .. ②Jr（λ）=..   γ   γ  0 千分之一  0 零 | 1 λ 1兆  ... …  0 零  0 零 | 0 零  1 一  ... …  0 零  0 零 | ··· ·········  ···... ···…  ... ··· …········· | 0 0℃  0 0℃  ..., ……   γ  1 λ 1λ |

called Jordan blocks; see Theorem 30.17.  
称为约旦块；见定理30.17。

## 30.1 Annihilating Polynomials and the Minimal Polynomial 30.1湮灭多项式和最小多项式

Given a linear map f : E → E, it is easy to check that the set Ann(f) of polynomials that annihilate f is an ideal. Furthermore, when E is finite-dimensional, the Cayley–Hamilton Theorem implies that Ann(f) is not the zero ideal. Therefore, by Proposition 29.10, there is a unique monic polynomial mf that generates Ann(f). Results from Chapter 29, especially about gcd’s of polynomials, will come handy.  
给出了一个线性映射f:e→e，很容易证明湮灭f的多项式的集合ann（f）是一个理想。此外，当e是有限维时，凯莱-汉密尔顿定理意味着ann（f）不是零理想。因此，根据命题29.10，有一个唯一的monic多项式mf生成ann（f）。第29章的结果，特别是关于多项式的GCD的结果，将非常有用。

Definition 30.1. If f : E → E is a linear map on a finite-dimensional vector space E, the unique monic polynomial mf(X) that generates the ideal Ann(f) of polynomials which annihilate f (the annihilator of f) is called the minimal polynomial of f.  
定义30.1.如果f:e→e是有限维向量空间e上的一个线性映射，那么产生湮灭f（f的湮灭子）的多项式的理想ann（f）的唯一Monic多项式mf（x）称为f的最小多项式。

The minimal polynomial mf of f is the monic polynomial of smallest degree that annihilates f. Thus, the minimal polynomial divides the characteristic polynomial χf, and deg(mf) ≥ 1. For simplicity of notation, we often write m instead of mf.  
F的最小多项式Mf是最小程度的Monic多项式，将F湮没，因此，最小多项式将特征多项式χf与deg（Mf）≥1相除。为了便于记法，我们经常写m而不是mf。

If A is any n × n matrix, the set Ann(A) of polynomials that annihilate A is the set of polynomials  
如果a是任意n×n矩阵，则湮灭a的多项式的集合ann（a）是多项式的集合。

p(X) = a0Xd + a1Xd−1 + ··· + ad−1X + ad  
P（x）=a0xd+a1xd−1+····+ad−1x+ad

such that  
这样的话

a0Ad + a1Ad−1 + ··· + ad−1A + adI = 0.  
a0ad+a1ad−1+····+ad−1a+adi=0.

It is clear that Ann(A) is a nonzero ideal and its unique monic generator is called the minimal polynomial of A. We check immediately that if Q is an invertible matrix, then A and Q−1AQ have the same minimal polynomial. Also, if A is the matrix of f with respect to some basis, then f and A have the same minimal polynomial.  
很明显，ann（a）是一个非零理想，其唯一的Monic发生器称为a的极小多项式。我们立即检查，如果q是可逆矩阵，那么a和q−1aq具有相同的极小多项式。另外，如果a是关于某个基的f的矩阵，那么f和a有相同的极小多项式。

The zeros (in K) of the minimal polynomial of f and the eigenvalues of f (in K) are intimately related.  
F的最小多项式的零点（k）和F的特征值（k）是密切相关的。

Proposition 30.1. Let f : E → E be a linear map on some finite-dimensional vector space  
提案30.1.设f:e→e为有限维向量空间上的线性映射

E. Then λ ∈ K is a zero of the minimal polynomial mf(X) of f iff λ is an eigenvalue of f iff λ is a zero of χf(X). Therefore, the minimal and the characteristic polynomials have the same zeros (in K), except for multiplicities.  
e.则F iff的最小多项式mf（x）的一个零点，λ是F iff的一个特征值，λ是χf（x）的一个零点。因此，除了多重性外，最小多项式和特征多项式都有相同的零（以k为单位）。

Proof. First assume that m(λ) = 0 (with λ ∈ K, and writing m instead of mf). If so, using polynomial division, m can be factored as  
证据。首先假设m（λ）=0（用λ∈k，写m而不是mf）。如果是这样，使用多项式除法，m可以被分解为

m = (X − λ)q,  
m=（x−λ）q，

with deg(q) < deg(m). Since m is the minimal polynomial, q(f) = 06 , so there is some nonzero vector v ∈ E such that u = q(f)(v) = 06 . But then, because m is the minimal polynomial,  
度（q）<度（m）。因为m是最小多项式，q（f）=06，所以有一些非零向量v∈e，这样u=q（f）（v）=06。但是，因为m是极小多项式，

0 = m(f)(v)  
0=m（f）（v）

= (f − λid)(q(f)(v)) = (f − λid)(u),  
=（f-λid）（q（f）（v））=（f-λid）（u），

which shows that λ is an eigenvalue of f.  
这表明λ是f的特征值。

Conversely, assume that λ ∈ K is an eigenvalue of f. This means that for some u = 06 , we have f(u) = λu. Now it is easy to show that  
相反，假设λ∈k是f的特征值，这意味着对于某些u=06，我们有f（u）=λu，现在很容易证明

m(f)(u) = m(λ)u,  
m（f）（u）=m（λ）u，

and since m is the minimal polynomial of f, we have m(f)(u) = 0, so m(λ)u = 0, and since u = 06 , we must have m(λ) = 0.   
既然m是f的极小多项式，我们得到m（f）（u）=0，所以m（λ）u=0，既然u=06，我们必须得到m（λ）=0。

Proposition 30.2. Let f : E → E be a linear map on some finite-dimensional vector space E. If f diagonalizable, then its minimal polynomial is a product of distinct factors of degree  
提案30.2.设f:e→e为有限维向量空间e上的线性映射，如果f可对角化，则其最小多项式是各次因子的乘积。

1.  
1。

Proof. If we assume that f is diagonalizable, then its eigenvalues are all in K, and if λ1,...,λk are the distinct eigenvalues of f, and then by Proposition 30.1, the minimal polynomial m of f must be a product of powers of the polynomials (X − λi). Actually, we claim that  
证据。如果我们假设f是可对角化的，那么它的特征值都是k，如果λ1，…，λk是f的独特特征值，那么根据命题30.1，f的最小多项式m必须是多项式（x-λi）的幂的乘积。事实上，我们声称

m = (X − λ1)···(X − λk).  
m=（x−λ1）···（x−λk）。

For this we just have to show that m annihilates f. However, for any eigenvector u of f, one of the linear maps f − λiid sends u to 0, so  
为此，我们只需证明m会湮灭f。然而，对于f的任何特征向量u，其中一个线性映射f−λiid将u发送到0，所以

m(f)(u) = (f − λ1id) ◦ ··· ◦ (f − λkid)(u) = 0.  
m（f）（u）=（f-λ1id）······（f-λkid）（u）=0。

Since E is spanned by the eigenvectors of f, we conclude that  
由于e是由f的特征向量构成的，因此我们得出结论：

m(f) = 0.   
m（f）=0.

It turns out that the converse of Proposition 30.2 is true, but this will take a little work to establish it.  
事实证明，30.2号提案的反面是正确的，但这需要一点工作来确定它。

30.2. MINIMAL POLYNOMIALS OF DIAGONALIZABLE LINEAR MAPS  
30.2。对角化线性映射的极小多项式

## 30.2 Minimal Polynomials of Diagonalizable Linear Maps 30.2对角化线性映射的最小多项式

In this section we prove that if the minimal polynomial mf of a linear map f is of the form  
在这一节中，我们证明了如果线性映射f的最小多项式mf是

mf = (X − λ1)···(X − λk)  
Mf=（x−λ1）···（x−λk）

for distinct scalars λ1,...,λk ∈ K, then f is diagonalizable. This is a powerful result that has a number of implications. But first we need of few properties of invariant subspaces.  
对于不同的标量λ1，…，λk∈k，则f是可对角化的。这是一个强有力的结果，有很多含义。但首先我们需要一些不变子空间的性质。

Given a linear map f : E → E, recall that a subspace W of E is invariant under f if f(u) ∈ W for all u ∈ W. For example, if f : R2 → R2 is f(x,y) = (−x,y), the y-axis is invariant under f.  
给出了一个线性映射f:e→e，假设e的子空间w在f下是不变的，如果f（u）对所有u∈w都是不变的，例如，如果f:r2→r2是f（x，y）=（-x，y），y轴在f下是不变的。

Proposition 30.3. Let W be a subspace of E invariant under the linear map f : E → E (where E is finite-dimensional). Then the minimal polynomial of the restriction f | W of f to W divides the minimal polynomial of f, and the characteristic polynomial of f | W divides the characteristic polynomial of f.  
提案30.3.设w为线性映射f:e→e（其中e为有限维）下e不变量的子空间。然后F W的约束F W的最小多项式除F的最小多项式，F W的特征多项式除F的特征多项式。

Sketch of proof. The key ingredient is that we can pick a basis (e1,...,en) of E in which (e1,...,ek) is a basis of W. The matrix of f over this basis is a block matrix of the form  
证明草图。关键因素是我们可以选择e的基（e1，…，en），其中（e1，…，ek）是w的基。在这个基上，f的矩阵是形式的块矩阵。

,  
，

where B is a k × k matrix, D is an (n − k) × (n − k) matrix, and C is a k × (n − k) matrix. Then  
其中b是k×k矩阵，d是（n-k）×n-k矩阵，c是k×n-k矩阵。然后

det(XI − A) = det(XI − B)det(XI − D),  
行列式（XI）a =，

which implies the statement about the characteristic polynomials. Furthermore,  
这意味着关于特征多项式的陈述。此外，

,  
，

for some k × (n − k) matrix Ci. It follows that any polynomial which annihilates A also annihilates B and D. So the minimal polynomial of B divides the minimal polynomial of A.   
对于一些k×（n-k）矩阵Ci。由此可知，任何一个歼灭a的多项式也同时歼灭b和d，所以b的极小多项式除以a的极小多项式。

For the next step, there are at least two ways to proceed. We can use an old-fashion argument using Lagrange interpolants, or we can use a slight generalization of the notion of annihilator. We pick the second method because it illustrates nicely the power of principal ideals.  
对于下一步，至少有两种方法可以继续。我们可以用拉格朗日插值来使用一个老式的论点，或者我们可以稍微概括一下湮灭子的概念。我们选择第二种方法是因为它很好地说明了主理想的力量。

What we need is the notion of conductor (also called transporter).  
我们需要的是导体的概念（也称为运输器）。

Definition 30.2. Let f : E → E be a linear map on a finite-dimensional vector space E, let W be an invariant subspace of f, and let u be any vector in E. The set Sf(u,W) consisting of all polynomials q ∈ K[X] such that q(f)(u) ∈ W is called the f-conductor of u into W.  
定义30.2.设f:e→e为有限维向量空间e上的线性映射，设w为f的不变子空间，设u为e中的任意向量，由所有多项式q∈k[x]组成的集sf（u，w），使q（f）（u）∈w称为u到w的f导体。

Observe that the minimal polynomial mf of f always belongs to Sf(u,W), so this is a nontrivial set. Also, if W = (0), then Sf(u,(0)) is just the annihilator of f. The crucial property of Sf(u,W) is that it is an ideal.  
观察到f的最小多项式mf总是属于sf（u，w），所以这是一个非平凡集。另外，如果w=（0），那么sf（u，（0））只是f的湮灭子，sf（u，w）的关键性质是它是一个理想。

Proposition 30.4. If W is an invariant subspace for f, then for each u ∈ E, the f-conductor Sf(u,W) is an ideal in K[X].  
提案30.4.如果w是f的不变子空间，那么对于每个u∈e，f-导体sf（u，w）是k[x]中的理想。

We leave the proof as a simple exercise, using the fact that if W invariant under f, then W is invariant under every polynomial q(f) in Sf(u,W).  
我们将证明留作一个简单的练习，假设w在f下不变，那么w在sf（u，w）中的每个多项式q（f）下不变。

Since Sf(u,W) is an ideal, it is generated by a unique monic polynomial q of smallest degree, and because the minimal polynomial mf of f is in Sf(u,W), the polynomial q divides m.  
因为sf（u，w）是一个理想，它是由一个唯一的最小阶Monic多项式q生成的，并且由于f的最小多项式mf在sf（u，w）中，多项式q除以m。

Definition 30.3. The unique monic polynomial which generates Sf(u,W) is called the conductor of u into W.  
定义30.3.生成sf（u，w）的唯一Monic多项式称为u到w的导体。

Example 30.1. For example, suppose f : R2 → R2 where f(x,y) = (x,0). Observe that W = {(x,0) ∈ R2} is invariant under f. By representing, we see that mf(X) = χf(X) = X2 − X. Let u = (0,y). Then Sf(u,W) = (X) and we say X is the conductor of u into W.  
例30.1。例如，假设f:r2→r2，其中f（x，y）=（x，0）。观察w=（x，0）∈r2在f下是不变的。通过表示，我们看到mf（x）=χf（x）=x2−x。让u=（0，y）。那么sf（u，w）=（x），我们说x是u到w的导体。

Proposition 30.5. Let f : E → E be a linear map on a finite-dimensional space E and assume that the minimal polynomial m of f is of the form  
提案30.5。设f:e→e为有限维空间e上的线性映射，并假定f的最小多项式m的形式为

m = (X − λ1)r1 ···(X − λk)rk,  
m=（x−λ1）r1···（x−λk）Rk，

where the eigenvalues λ1,...,λk of f belong to K. If W is a proper subspace of E which is invariant under f, then there is a vector u ∈ E with the following properties:  
其中，f的特征值λ1，…，λk属于k。如果w是e的一个子空间，在f下不变，则有一个具有以下性质的向量u∈e：

1. u /∈ W;  
   u/∈w；
2. (f − λid)(u) ∈ W, for some eigenvalue λ of f.  
   （f-λid）（u）∈w，对于f的一些特征值λ。

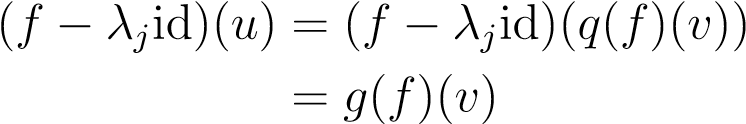
Proof. Observe that (a) and (b) together assert that the conductor of u into W is a polynomial of the form X − λi. Pick any vector v ∈ E not in W, and let g be the conductor of v into W, i.e. g(f)(v) ∈ W. Since g divides m and v /∈ W, the polynomial g is not a constant, and thus it is of the form g = (X − λ1)s1 ···(X − λk)sk,  
证据。观察（a）和（b）一起断言u到w的导体是x-λi形式的多项式，选取任意向量v∈e不在w中，并假设g是v到w的导体，即g（f）（v）∈w，因为g分m和v/∈w，多项式g不是常数，因此它是形式。g=（x−λ1）s1···（x−λk）sk，

with at least some si > 0. Choose some index j such that sj > 0. Then X − λj is a factor of g, so we can write  
至少有一些Si>0。选择一些索引j，使sj>0。那么x-λj是g的一个因子，所以我们可以写

g = (X − λj)q. (\*)  
G=（x−λj）Q.（\*）

30.2. MINIMAL POLYNOMIALS OF DIAGONALIZABLE LINEAR MAPS  
30.2。对角化线性映射的极小多项式

By definition of g, the vector u = q(f)(v) cannot be in W, since otherwise g would not be of minimal degree. However, (∗) implies that  
根据g的定义，向量u=q（f）（v）不能在w中，否则g将不具有最小程度。然而，（）意味着



is in W, which concludes the proof.   
在w中，这就是证明的结论。

We can now prove the main result of this section.  
我们现在可以证明这个部分的主要结果。

Theorem 30.6. Let f : E → E be a linear map on a finite-dimensional space E. Then f is diagonalizable iff its minimal polynomial m is of the form  
定理30.6。设f:e→e为有限维空间e上的线性映射，则f是可对角化的，只要其最小多项式m的形式为

m = (X − λ1)···(X − λk),  
m=（x−λ1）···（x−λk），

where λ1,...,λk are distinct elements of K.  
其中，λ1，…，λk是k的不同元素。

Proof. We already showed in Proposition 30.2 that if f is diagonalizable, then its minimal polynomial is of the above form (where λ1,...,λk are the distinct eigenvalues of f).  
证据。我们已经在命题30.2中证明，如果f是可对角化的，那么它的最小多项式就是上述形式（其中，λ1，…，λk是f的独特特征值）。

For the converse, let W be the subspace spanned by all the eigenvectors of f. If W =6 E, since W is invariant under f, by Proposition 30.5, there is some vector u /∈ W such that for some λj, we have  
相反，设w为f的所有特征向量所跨越的子空间。如果w=6e，由于w在f下不变，根据命题30.5，有一些向量u/∈w，因此对于一些λj，我们得到

(f − λjid)(u) ∈ W.  
（f−λjid）（u）∈w.

Let v = (f − λjid)(u) ∈ W. Since v ∈ W, we can write  
设v=（f−λjid）（u）∈w。既然v∈w，我们可以写

v = w1 + ··· + wk  
v=w1+·····+wk

where f(wi) = λiwi (either wi = 0 or wi is an eigenvector for λi), and so for every polynomial h, we have h(f)(v) = h(λ1)w1 + ··· + h(λk)wk,  
式中，f（wi）=λiwi（wi=0或wi是λi的特征向量），因此对于每个多项式h，我们有h（f）（v）=h（λ1）w1+·····+h（λk）wk，

which shows that h(f)(v) ∈ W for every polynomial h. We can write  
这表明，对于每一个多项式h，h（f）（v）∈w，我们可以写

m = (X − λj)q  
m=（x−λj）q

for some polynomial q, and also  
对于某些多项式q，以及

q − q(λj) = p(X − λj)  
q−q（λj）=p（x−λj）

for some polynomial p. We know that p(f)(v) ∈ W, and since m is the minimal polynomial of f, we have  
对于某些多项式p，我们知道p（f）（v）∈w，由于m是f的极小多项式，我们得到

0 = m(f)(u) = (f − λjid)(q(f)(u)),  
0=m（f）（u）=（f−λjid）（q（f）（u）），

which implies that q(f)(u) ∈ W (either q(f)(u) = 0, or it is an eigenvector associated with λj). However,  
这意味着q（f）（u）∈w（q（f）（u）=0，或者是与λj相关的特征向量）。然而，

q(f)(u) − q(λj)u = p(f)((f − λjid)(u)) = p(f)(v),  
q（f）（u）−q（λj）u=p（f）（（f−λjid）（u））=p（f）（v），

and since p(f)(v) ∈ W and q(f)(u) ∈ W, we conclude that q(λj)u ∈ W. But, u /∈ W, which implies that q(λj) = 0, so λj is a double root of m, a contradiction. Therefore, we must have  
由于p（f）（v）∈w和q（f）（u）∈w，我们得出q（λj）u∈w。但是，u/∈w，这意味着q（λj）=0，所以λj是m的一个矛盾的双根。因此，我们必须

W = E.   
W＝E。

Remark: Proposition 30.5 can be used to give a quick proof of Theorem 14.5.  
注：命题30.5可以用来快速证明定理14.5。

## 30.3 Commuting Families of Diagonalizable and Triangulable Maps 30.3可对角化和三角化地图的交换族

Using Theorem 30.6, we can give a short proof about commuting diagonalizable linear maps.  
利用定理30.6，我们可以给出关于交换对角化线性映射的一个简短证明。

Definition 30.4. If F is a family of linear maps on a vector space E, we say that F is a commuting family iff f ◦ g = g ◦ f for all f,g ∈ F.  
定义30.4.如果f是向量空间e上的线性映射族，我们就说f是所有f的交换族iff f g=g f，g∈f。

Proposition 30.7. Let F be a finite commuting family of diagonalizable linear maps on a vector space E. There exists a basis of E such that every linear map in F is represented in that basis by a diagonal matrix.  
提案30.7。设f为向量空间e上可对角化线性映射的有限交换族，存在e的基，使得f中的每一个线性映射都用一个对角矩阵表示。

Proof. We proceed by induction on n = dim(E). If n = 1, there is nothing to prove. If n > 1, there are two cases. If all linear maps in F are of the form λid for some λ ∈ K, then the proposition holds trivially. In the second case, let f ∈ F be some linear map in F which is not a scalar multiple of the identity. In this case, f has at least two distinct eigenvalues λ1,...,λk, and because f is diagonalizable, E is the direct sum of the corresponding eigenspaces Eλ1,...,Eλk. For every index i, the eigenspace Eλi is invariant under f and under every other linear map g in F, since for any g ∈ F and any u ∈ Eλi, because f and g commute, we have  
证据。我们在n=dim（e）上进行归纳。如果n=1，则无需证明。如果n>1，则有两种情况。如果f中的所有线性映射都是某些λ∈k的λid形式，则该命题成立。在第二种情况下，让f∈f是f中的一个线性映射，它不是恒等式的标量倍数。在这种情况下，f至少有两个不同的特征值λ1，…，λk，并且由于f是可对角化的，e是相应特征空间eλ1，…，eλk的直接和。对于每个指数i，特征空间eλi在f和f中的其他线性映射g下是不变的，因为对于任何g∈f和任何u∈eλi，因为f和g上下班，我们有

f(g(u)) = g(f(u)) = g(λiu) = λig(u)  
f（g（u））=g（f（u））=g（λiu）=λig（u）

so g(u) ∈ Eλi. Let Fi be the family obtained by restricting each f ∈ F to Eλi. By Proposition 30.3, the minimal polynomial of every linear map f | Eλi in Fi divides the minimal polynomial mf of f, and since f is diagonalizable, mf is a product of distinct linear factors, so the minimal polynomial of f | Eλi is also a product of distinct linear factors. By Theorem 30.6, the linear map f | Eλi is diagonalizable. Since k > 1, we have dim(Eλi) < dim(E) for i = 1,...,k, and by the induction hypothesis, for each i there is a basis of Eλi over which f | Eλi is represented by a diagonal matrix. Since the above argument holds for all i, by combining the bases of the Eλi, we obtain a basis of E such that the matrix of every linear map f ∈ F is represented by a diagonal matrix.   
所以g（u）∈eλi.让fi是通过限制每一个f∈f到eλi而得到的族.通过命题30.3，fi中每一个线性映射f eλi的最小多项式分割了f的最小多项式mf，由于f是可对角化的，mf是不同线性因子的乘积，所以最小多项式f eλi的omial也是不同线性因子的乘积。根据定理30.6，线性映射f eλi是可对角化的。由于k>1，对于i=1，…，k，我们有dim（eλi）<dim（e），并且通过归纳假设，对于每个i，都有eλi的基础，其中f eλi由对角矩阵表示。由于上述论点适用于所有i，通过结合eλi的基，我们得到e的基，这样每个线性映射f∈f的矩阵都由对角矩阵表示。

Remark: Proposition 30.7 also holds for infinite commuting families F of diagonalizable linear maps, because E being finite dimensional, there is a finite subfamily of linearly independent linear maps in F spanning F.  
注：命题30.7也适用于可对角化线性映射的无限交换族f，因为e是有限维的，所以f的跨度f中存在一个线性无关线性映射的有限子族。

There is also an analogous result for commuting families of linear maps represented by upper triangular matrices. To prove this we need the following proposition.  
对于用上三角矩阵表示的线性映射族，也有类似的结果。为了证明这一点，我们需要以下建议。

Proposition 30.8. Let F be a nonempty finite commuting family of triangulable linear maps on a finite-dimensional vector space E. Let W be a proper subspace of E which is invariant under F. Then there exists a vector u ∈ E such that:  
提案30.8。设f为有限维向量空间e上可三角线性映射的一个非空有限交换族。设w为e的一个子空间，该子空间在f下是不变的。然后存在一个向量u∈e，这样：

30.3. COMMUTING FAMILIES OF LINEAR MAPS  
30.3。线性映射的交换族

1. u /∈ W.  
   u/∈w.
2. For every f ∈ F, the vector f(u) belongs to the subspace W ⊕ Ku spanned by W and u.  
   对于每一个f∈f，向量f（u）属于w和u所跨越的子空间w ku。

Proof. By renaming the elements of F if necessary, we may assume that (f1,...,fr) is a basis of the subspace of End(E) spanned by F. We prove by induction on r that there exists some vector u ∈ E such that  
证据。如果必要的话，通过对f的元素进行重命名，我们可以假定（f1，…，f r）是f所跨越的end（e）的子空间的基础，通过对r的归纳，我们证明存在一些向量u∈e，这样

1. u /∈ W.  
   u/∈w.
2. (fi − αiid)(u) ∈ W for i = 1,...,r, for some scalars αi ∈ K.  
   （fi−αiid）（u）∈w表示i=1，…，r表示某些标量αi∈k。

Consider the base case r = 1. Since f1 is triangulable, its eigenvalues all belong to K since they are the diagonal entries of the triangular matrix associated with f1 (this is the easy direction of Theorem 14.5), so the minimal polynomial of f1 is of the form  
考虑基本情况r=1。因为f1是三角形的，所以它的特征值都是k，因为它们是与f1相关的三角形矩阵的对角项（这是定理14.5的简单方向），所以f1的最小多项式的形式是

m = (X − λ1)r1 ···(X − λk)rk,  
m=（x−λ1）r1···（x−λk）Rk，

where the eigenvalues λ1,...,λk of f1 belong to K. We conclude by applying Proposition  
其中，f1的特征值λ1，…，λk属于k，我们通过应用命题得出结论。

30.5.  
30.5。

Next assume that r ≥ 2 and that the induction hypothesis holds for f1,...,fr−1. Thus, there is a vector ur−1 ∈ E such that  
接下来假设r≥2，诱导假设适用于f1，…，fr−1。因此，有一个向量ur−1∈e，这样

1. ur−1 ∈/ W.  
   UR−1∈/W。
2. (fi − αiid)(ur−1) ∈ W for i = 1,...,r − 1, for some scalars αi ∈ K.  
   （fi−αiid）（ur−1）∈w表示i=1，…，r−1表示某些标量αi∈k。

Let  
让

Vr−1 = {w ∈ E | (fi − αiid)(w) ∈ W, i = 1,...,r − 1}.  
vr−1=w∈e（fi−αiid）（w）∈w，i=1，…，r−1。

Clearly, W ⊆ Vr−1 and ur−1 ∈ Vr−1. We claim that Vr−1 is invariant under F. This is because, for any v ∈ Vr−1 and any f ∈ F, since f and fi commute, we have  
显然，w vr−1和ur−1∈vr−1。我们声称，在f下，vr−1是不变的，这是因为，对于任何v∈vr−1和任何f∈f，由于f和fi交换，我们有

(fi − αiid)(f(v)) = f((fi − αiid)(v)), 1 ≤ i ≤ r − 1.  
（f i−αiid）（f（v））=f（（fi−αiid）（v）），1≤i≤r−1。

Now (fi −αiid)(v) ∈ W because v ∈ Vr−1, and W is invariant under F, so f(fi −αiid)(v)) ∈ W, that is, (fi − αiid)(f(v)) ∈ W.  
现在（fi−αiid）（v）∈w是因为v∈vr−1，而w在f下是不变的，所以f（fi−αiid）（v））是∈w，即，（fi−αiid）（f（v））是∈w。

Consider the restriction gr of fr to Vr−1. The minimal polynomial of gr divides the minimal polynomial of fr, and since fr is triangulable, just as we saw for f1, the minimal polynomial of fr is of the form  
考虑fr对vr−1的限制gr。gr的极小多项式除以fr的极小多项式，由于fr是三角的，正如我们对f1所看到的，fr的极小多项式的形式是

m = (X − λ1)r1 ···(X − λk)rk,  
m=（x−λ1）r1···（x−λk）Rk，

where the eigenvalues λ1,...,λk of fr belong to K, so the minimal polynomial of gr is of the same form. By Proposition 30.5, there is some vector ur ∈ Vr−1 such that  
其中，fr的特征值λ1，…，λk属于k，因此gr的最小多项式形式相同。根据命题30.5，有一个向量ur∈vr−1，这样

1. ur ∈/ W.  
   ur∈/w.
2. (gr − αrid)(ur) ∈ W for some scalars αr ∈ K.  
   （gr−αrid）（ur）对于某些标量αr∈k∈w。

Now since ur ∈ Vr−1, we have (fi −αiid)(ur) ∈ W for i = 1,...,r−1, so (fi −αiid)(ur) ∈ W for i = 1,...,r (since gr is the restriction of fr), which concludes the proof of the induction step. Finally, since every f ∈ F is the linear combination of (f1,...,fr), Condition (2) of the inductive claim implies Condition (2) of the proposition.   
既然Ur∈vr−1，我们有（fi−αiid）（ur）∈w代表i=1，…，r−1，所以（fi−αiid）（ur）∈w代表i=1，…，r（因为gr是fr的限制），这就结束了诱导步骤的证明。最后，由于每一个f∈f都是（f1，…，fr）的线性组合，归纳权利要求的条件（2）隐含了命题的条件（2）。

We can now prove the following result.  
我们现在可以证明以下结果。

Proposition 30.9. Let F be a nonempty finite commuting family of triangulable linear maps on a finite-dimensional vector space E. There exists a basis of E such that every linear map in F is represented in that basis by an upper triangular matrix.  
提案30.9.设f为有限维向量空间e上的可三角线性映射的非空有限交换族，存在e的基，使得f中的每一个线性映射都用上三角矩阵表示。

Proof. Let n = dim(E). We construct inductively a basis (u1,...,un) of E such that if Wi is the subspace spanned by (u1 ...,ui), then for every f ∈ F,  
证据。设n=尺寸（e）。我们归纳地构造e的基（u1，…，un），如果wi是（u1，…，ui）所跨越的子空间，那么对于每个f∈f，

,  
，

for some afij ∈ K; that is, f(ui) belongs to the subspace Wi.  
对于某些afij∈k，即f（ui）属于子空间wi。

We begin by applying Proposition 30.8 to the subspace W0 = (0) to get u1 so that for all f ∈ F,  
我们首先将30.8命题应用于子空间w0=（0），得到u1，这样对于所有f∈f，

.  
.

For the induction step, since Wi invariant under F, we apply Proposition 30.8 to the subspace Wi, to get ui+1 ∈ E such that  
对于归纳步骤，由于wi在f下是不变的，我们将命题30.8应用于子空间wi，得到ui+1∈e，这样

1. ui+1 ∈/ Wi.  
   ui+1∈/wi。
2. For every f ∈ F, the vector f(ui+1) belong to the subspace spanned by Wi and ui+1.  
   对于每个f∈f，向量f（ui+1）属于wi和ui+1所跨越的子空间。

Condition (1) implies that (u1,...,ui,ui+1) is linearly independent, and Condition (2) means that for every f ∈ F,  
条件（1）表示（u1，…，ui，ui+1）是线性独立的，条件（2）表示对于每个f∈f，

,  
，

for some, establishing the induction step. After n steps, each f ∈ F is represented by an upper triangular matrix.   
对一些人来说，建立诱导步骤。经过n步，每个f∈f用上三角矩阵表示。

Observe that if F consists of a single linear map f and if the minimal polynomial of f is of the form  
如果f由一个线性映射f组成，并且f的最小多项式的形式为

m = (X − λ1)r1 ···(X − λk)rk,  
m=（x−λ1）r1···（x−λk）Rk，

with all λi ∈ K, using Proposition 30.5 instead of Proposition 30.8, the proof of Proposition 30.9 yields another proof of Theorem 14.5.  
对于所有的λi∈k，用命题30.5代替命题30.8，命题30.9的证明给出了定理14.5的另一个证明。

## 30.4 The Primary Decomposition Theorem 30.4主分解定理

If f : E → E is a linear map and λ ∈ K is an eigenvalue of f, recall that the eigenspace Eλ associated with λ is the kernel of the linear map λid − f. If all the eigenvalues λ1 ...,λk of f are in K, it may happen that  
如果f:e→e是一个线性映射，而λ∈k是f的特征值，回想一下，与λ相关联的特征空间eλ是线性映射的核心，λid−f。如果f的所有特征值λ1…，λk都是k，则可能发生这种情况。

E = Eλ1 ⊕ ··· ⊕ Eλk,  
e=eλ1···eλk，

but in general there are not enough eigenvectors to span E. What if we generalize the notion of eigenvector and look for (nonzero) vectors u such that  
但是一般来说，没有足够的特征向量来跨越e，如果我们推广特征向量的概念，寻找（非零）向量u，那么

(λid − f)r(u) = 0, for some r ≥ 1?  
（λid−f）r（u）=0，对于某些r≥1？

It turns out that if the minimal polynomial of f is of the form  
结果表明，如果f的极小多项式是

m = (X − λ1)r1 ···(X − λk)rk,  
m=（x−λ1）r1···（x−λk）Rk，

then r = ri does the job for λi; that is, if we let  
那么r=ri为λi做这个工作；也就是说，如果我们允许

Wi = Ker(λiid − f)ri,  
wi=ker（λiid−f）ri，

then  
然后

E = W1 ⊕ ··· ⊕ Wk.  
E=w1····周。

This result is very nice but seems to require that the eigenvalues of f all belong to K. Actually, it is a special case of a more general result involving the factorization of the minimal polynomial m into its irreducible monic factors (see Theorem 29.17),  
这个结果很好，但似乎要求f的特征值都属于k。实际上，这是一个更一般的结果的特殊情况，涉及将最小多项式m因式分解为其不可约Moni因子（见定理29.17）。

,  
，

where the pi are distinct irreducible monic polynomials over K.  
其中π是K上的独特的不可约Monic多项式。

Theorem 30.10. (Primary Decomposition Theorem) Let f : E → E be a linear map on the finite-dimensional vector space E over the field K. Write the minimal polynomial m of f as  
定理30.10。（主分解定理）设f:e→e为k域上有限维向量空间e上的线性映射。将f的最小多项式m写成

,  
，

where the pi are distinct irreducible monic polynomials over K, and the ri are positive integers. Let  
式中，π是K上的独特的不可约Monic多项式，ri是正整数。让

Wi = Ker(prii(f)), i = 1,...,k.  
wi=ker（prii（f）），i=1，…，k.

Then  
然后

1. E = W1 ⊕ ··· ⊕ Wk.  
   E=w1····周。
2. Each Wi is invariant under f.  
   每个wi在f下是不变的。
3. The minimal polynomial of the restriction .  
   约束的最小多项式。

Proof. The trick is to construct projections πi using the polynomials so that the range of πi is equal to Wi. Let  
证据。诀窍是用多项式构造投影πi，使πi的范围等于wi。让

gi = m/prii = Yprjj.  
gi=m/prii=yprjj。

j6=i  
J6=

Note that  
注意



Since p1,...,pk are irreducible and distinct, they are relatively prime. Then using Proposition 29.14, it is easy to show that g1,...,gk are relatively prime. Otherwise, some irreducible polynomial p would divide all of g1,...,gk, so by Proposition 29.14 it would be equal to one of the irreducible factors pi. But that pi is missing from gi, a contradiction. Therefore, by Proposition 29.15, there exist some polynomials h1,...,hk such that  
由于p1，…，pk是不可约的和独特的，它们是相对素数。然后用29.14号命题，很容易证明g1，…，gk是相对质数。否则，一些不可约多项式p会将所有g1，…，gk除尽，所以用命题29.14，它等于不可约因子pi之一。但是π在gi中是缺失的，一个矛盾。因此，根据命题29.15，存在一些多项式h1，…，hk，使得

g1h1 + ··· + gkhk = 1.  
g1h1+····+gkhk=1.

Let qi = gihi and let πi = qi(f) = gi(f)hi(f). We have  
设qi=gi hi，设πi=qi（f）=gi（f）hi（f）。我们有

q1 + ··· + qk = 1,  
q1+····+qk=1，

and since m divides qiqj for i =6 j, we get  
因为m将qiqj除以i=6j，我们得到

π1 + ··· + πk = id πiπj = 0, i =6 j.  
π1+·····+πk=idπiπj=0，i=6 J。

(We implicitly used the fact that if p,q are two polynomials, the linear maps p(f) ◦ q(f) and q(f) ◦ p(f) are the same since p(f) and q(f) are polynomials in the powers of f, which commute.) Composing the first equation with πi and using the second equation, we get  
（我们隐式地使用了这样一个事实，即如果p，q是两个多项式，那么线性映射p（f）q（f）和q（f）p（f）是相同的，因为p（f）和q（f）是f的幂的多项式，而f的幂是交换的。）用πi组成第一个方程，然后使用第二个方程，我们得到

πi2 = πi.  
πI2=πI。

Therefore, the πi are projections, and E is the direct sum of the images of the πi. Indeed, every u ∈ E can be expressed as  
因此，πi是投影，e是πi图像的直接和。实际上，每个u∈e都可以表示为

u = π1(u) + ··· + πk(u).  
u=π1（u）+····+πk（u）。

Also, if π1(u) + ··· + πk(u) = 0,  
另外，如果π1（u）+····+πk（u）=0，

then by applying πi we get  
然后通过应用πi我们得到

0 = πi2(u) = πi(u), i = 1,...k.  
0=πi2（u）=πi（u），i=1，…k。

To finish proving (a), we need to show that  
为了完成证明（a），我们需要证明

Wi = Ker(.  
wi=ker（.

If v ∈ πi(E), then v = πi(u) for some u ∈ E, so  
如果v∈πi（e），那么对于某些u∈e，v=πi（u），那么

prii(f)(v) = prii(f)(πi(u))  
prii（f）（v）=prii（f）（πi（u））

= prii(f)gi(f)hi(f)(u)  
=prii（f）gi（f）hi（f）（u）

= hi(f)prii(f)gi(f)(u) = hi(f)m(f)(u) = 0,  
=hi（f）prii（f）gi（f）（u）=hi（f）m（f）（u）=0，

because m is the minimal polynomial of f. Therefore, v ∈ Wi.  
因为m是f的极小多项式，所以v∈wi。

Conversely, assume that v ∈ Wi = Ker(, then gjhj is divisible by, so  
相反，假设v∈wi=ker（，那么gjhj可以被除，所以

gj(f)hj(f)(v) = πj(v) = 0, j =6 i.  
gj（f）hj（f）（v）=πj（v）=0，j=6i。

Then since π1 + ··· + πk = id, we have v = πiv, which shows that v is in the range of πi. Therefore, Wi = Im(πi), and this finishes the proof of (a).  
既然π1+·····+πk=id，我们就得到了V=πiv，这表明V在πi的范围内，因此，wi=im（πi），这就完成了（a）的证明。

If prii(f)(u) = 0, then prii(f)(f(u)) = f(piri(f)(u)) = 0, so (b) holds.  
如果prii（f）（u）=0，那么prii（f）（f（u））=f（piri（f）（u））=0，那么（b）保持不变。

If we write fi = f | Wi, then prii(fi) = 0, because piri(f) = 0 on Wi (its kernel). Therefore, the minimal polynomial of fi divides. Conversely, let q be any polynomial such that q(fi) = 0 (on Wi). Since, the fact that m(f)(u) = 0 for all u ∈ E shows that  
如果我们写fi=f\_wi，那么prii（fi）=0，因为piri（f）=0在wi（它的内核）上。因此，fi的极小多项式被除。相反，假设q是任何多项式，这样q（fi）=0（wi上）。因为，m（f）（u）=0表示所有u∈e的事实表明



and thus Im(gi(f)) ⊆ Ker(prii(f)) = Wi. Consequently, since q(f) is zero on Wi,  
因此im（gi（f））ker（prii（f））=wi。因此，既然q（f）在wi上为零，

q(f)gi(f) = 0 for all u ∈ E.  
q（f）gi（f）=0表示所有u∈e。

But then qgi is divisible by the minimal polynomial, and since and gi are relatively prime, by Euclid’s proposition, must divide q. This finishes the proof that the minimal polynomial of fi is prii, which is (c).   
但是q gi可以被极小多项式整除，由于和gi是相对素数，所以必须被欧几里得命题整除q，这就证明fi的极小多项式是prii，即（c）。

To best understand the projection constructions of Theorem 30.10, we provide the following two explicit examples of the primary decomposition theorem.  
为了更好地理解定理30.10的投影构造，我们提供了下面两个主分解定理的显式示例。

Example 30.2. First let f : R3 → R3 be defined as ). In terms of the standard basis f is represented by the 3 × 3 matrix. Then a simple  
例30.2。首先让f:r3→r3定义为）。根据标准基f，用3×3矩阵表示。然后一个简单的

calculation shows that mf(x) = χf(x) = (x2 +1)(x−1). Using the notation of the preceding proof set  
计算表明，mf（x）=χf（x）=（x2+1）（x−1）。使用前面证明集的符号

m = p1p2, p1 = x2 + 1, p2 = x − 1.  
m=p1 p2，p1=x2+1，p2=x−1。

Then  
然后

.  
.

We must find h1,h2 ∈ R[x] such that g1h1 + g2h2 = 1. In general this is the hard part of the projection construction. But since we are only working with two relatively prime polynomials g1,g2, we may apply the Euclidean algorithm to discover that  
我们必须找到h1，h2∈r[x]这样g1h1+g2h2=1。一般来说，这是投影结构的硬部分。但是由于我们只研究两个相对素数多项式g1，g2，我们可以应用欧几里得算法来发现

,  
，

where while. By definition  
何时何地。按定义

id) = ,  
id）=，

and  
和

.  
.

Then R3 = W1 ⊕ W2, where  
则r3=w1 w2，其中

W1 = π1(R3) = Ker(p1(Xf)) = Ker(Xf2 + id) = Ker ,  
w1=π1（r3）=ker（p1（xf））=ker（xf2+id）=ker，

W2 = π2(R3) = Ker(p2(Xf)) = Ker(Xf − id) = Ker .  
w2=π2（r3）=ker（p2（xf））=ker（xf−id）=ker。

Example 30.3. For our second example of the primary decomposition theorem let f : R3 →  
例30.3。对于第一分解定理的第二个例子，让f:r3→

R3 be defined as f(x,y,z) = (y,−x + z,−y), with standard matrix representation Xf =  
r3定义为f（x，y，z）=（y，−x+z，−y），标准矩阵表示为xf。=

0 1 0  
0 1 1

. A simple calculation shows that mf(x) = χf(x) = x(x2 + 2). Set  
. 简单计算表明，mf（x）=χf（x）=x（x2+2）。集合

.  
.

Since gcd(g1,g2) = 1, we use the Euclidean algorithm to find  
由于gcd（g1，g2）=1，我们使用欧几里得算法来查找

,  
，

such that g1h1 + g2h2 = 1. Then  
使g1h1+g2h2=1。然后

,  
，

while  
虽然

.  
.

Although it is not entirely obvious, π1 and π2 are indeed projections since  
虽然并不十分明显，但π1和π2确实是投影，因为

,  
，

and  
和

.  
.

Furthermore observe that π1 + π2 = id. The primary decomposition theorem implies that R3 = W1 ⊕ W2 where  
此外，观察π1+π2=ID。主分解定理表明r3=w1 w2，其中

1 0 1  
10 1\_

W1 = π1(R3) = Ker(p1(f)) = Ker(X2 + 2) = Ker 0 0 0 = span{(0,1,0),(1,0,−1)},  
w1=π1（r3）=ker（p1（f））=ker（x2+2）=ker 0 0 0=span（0,1,0），（1,0，−1），

1 0 1  
1 0 1

W2 = π2(R3) = Ker(p2(f)) = Ker(X) = span{(1,0,1)}.  
w2=π2（r3）=ker（p2（f））=ker（x）=span（1,0,1）。

See Figure 30.1.  
见图30.1。

If all the eigenvalues of f belong to the field K, we obtain the following result.  
如果F的所有特征值都属于K域，则得到以下结果。

Theorem 30.11. (Primary Decomposition Theorem, Version 2) Let f : E → E be a linear map on the finite-dimensional vector space E over the field K. If all the eigenvalues λ1,...,λk of f belong to K, write  
定理30.11。（初等分解定理，第2版）让f:e→e是有限维向量空间e上k域上的线性映射。如果f的所有特征值λ1，…，则λk属于k，则写下

m = (X − λ1)r1 ···(X − λk)rk  
m=（x−λ1）r1···（x−λk）Rk

for the minimal polynomial of f,  
对于f的极小多项式，

χf = (X − λ1)n1 ···(X − λk)nk  
χf=（x-λ1）n1···（x-λk）nk

for the characteristic polynomial of f, with 1 ≤ ri ≤ ni, and let  
对于f的特征多项式，1≤ri≤ni，且let

Wi = Ker(λiid − f)ri, i = 1,...,k.  
wi=ker（λiid−f）ri，i=1，…，k.

Then  
然后

1. E = W1 ⊕ ··· ⊕ Wk.  
   E=w1····周。

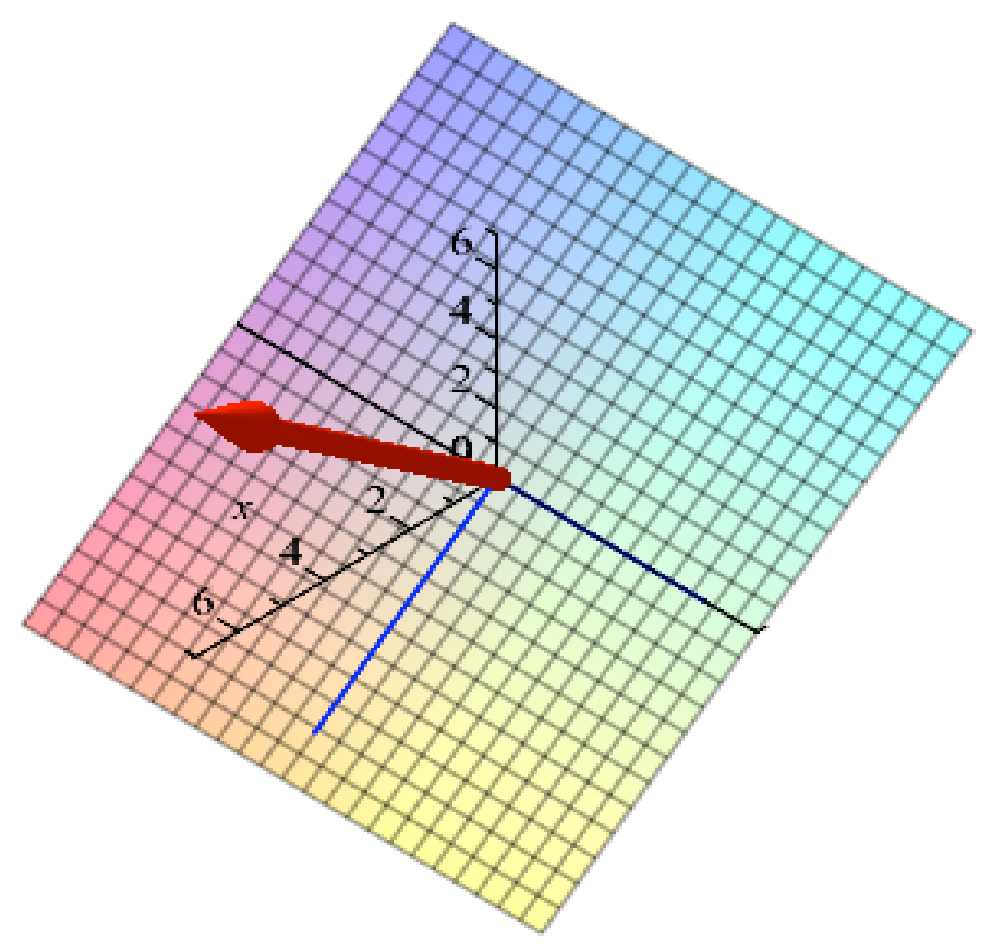
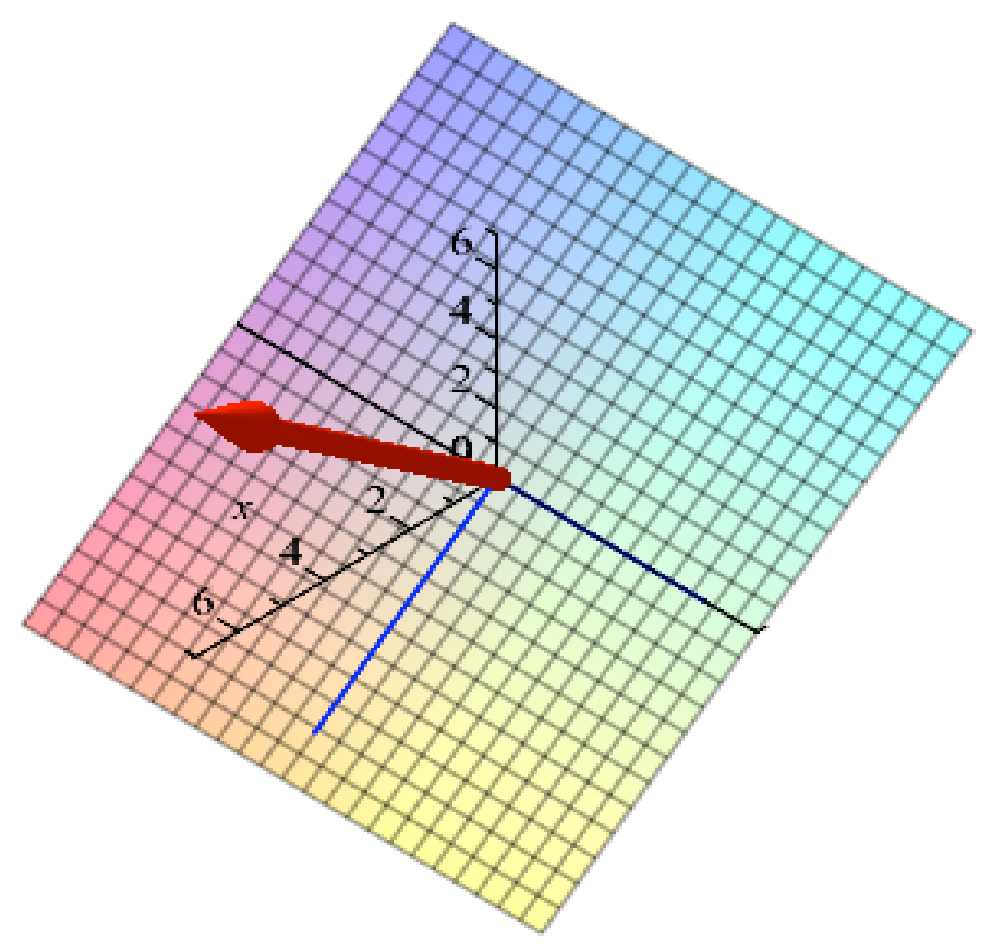


Figure 30.1: The direct sum decomposition of R3 = W1 ⊕W2 where W1 is the plane x+z = 0 and W2 is line t(1,0,1). The spanning vectors of W1 are in blue.  
图30.1:r3=w1 w2的直接和分解，其中w1是平面x+z=0，w2是直线t（1,0,1）。w1的跨度矢量为蓝色。

1. Each Wi is invariant under f.  
   每个wi在f下是不变的。
2. dim(Wi) = ni.  
   尺寸（wi）=ni。
3. The minimal polynomial of the restriction f | Wi of f to Wi is (X − λi)ri.  
   f对wi的限制f\_wi的最小多项式是（x−λi）ri。

Proof. Parts (a), (b) and (d) have already been proven in Theorem 30.10, so it remains to prove (c). Since Wi is invariant under f, let fi be the restriction of f to Wi. The characteristic polynomial χfi of fi divides χ(f), and since χ(f) has all its roots in K, so does χi(f). By Theorem 14.5, there is a basis of Wi in which fi is represented by an upper triangular matrix, and since (λiid−f)ri = 0, the diagonal entries of this matrix are equal to λi. Consequently,  
证据。第（a）、（b）和（d）部分已经在定理30.10中得到证明，因此仍需证明（c）。既然wi在f下是不变的，那么就让fi成为f对wi的限制。fi的特征多项式χfi除χ（f），由于χ（f）的根都在k中，因此χi（f）也是。根据定理14.5，Wi有一个基础，其中fi由上三角矩阵表示，由于（λiid−f）ri=0，该矩阵的对角项等于λi。因此，

χfi = (X − λi)dim(Wi),  
χfi=（x−λi）dim（wi），

and since χfi divides χ(f), we conclude hat  
由于χfi除以χ（f），我们得出：

dim(Wi) ≤ ni, i = 1,...,k.  
尺寸（wi）≤ni，i=1，…，k.

Because E is the direct sum of the Wi, we have dim(W1) + ··· + dim(Wk) = n, and since n1 + ··· + nk = n, we must have  
因为e是wi的直接和，所以我们有dim（w1）+······+dim（wk）=n，既然n1+·····+nk=n，我们必须有

dim(Wi) = ni, i = 1,...,k,  
尺寸（wi）=ni，i=1，…，k，

proving (c).   
证明（c）。

30.5. JORDAN DECOMPOSITION  
30.5。约旦分解

Definition 30.5. If λ ∈ K is an eigenvalue of f, we define a generalized eigenvector of f as a nonzero vector u ∈ E such that  
定义30.5.如果λ∈k是f的特征值，我们将f的广义特征向量定义为非零向量u∈e，这样

(λid − f)r(u) = 0, for some r ≥ 1.  
（λid−f）r（u）=0，对于某些r≥1。

The index of λ is defined as the smallest r ≥ 1 such that  
λ的指数定义为最小r≥1，因此

Ker(λid − f)r = Ker(λid − f)r+1.  
ker（λid−f）r=ker（λid−f）r+1。

It is clear that Ker(λid − f)i ⊆ Ker(λid − f)i+1 for all i ≥ 1. By Theorem 30.11(d), if λ = λi, the index of λi is equal to ri.  
很明显，对于所有i≥1，ker（λid−f）i ker（λid−f）i+1。根据定理30.11（d），如果λ=λi，则λi的指数等于ri。

## 30.5 Jordan Decomposition 30.5约旦分解

Recall that a linear map g: E → E is said to be nilpotent if there is some positive integer r such that gr = 0. Another important consequence of Theorem 30.11 is that f can be written as the sum of a diagonalizable and a nilpotent linear map (which commute). For example f : R2 → R2 be the R-linear map f(x,y) = (x,x + y) with standard matrix representation  
回想一下，如果有一个正整数r，比如gr=0，那么线性映射g:e→e就称为幂零。定理30.11的另一个重要结论是，f可以写成对角化和幂零线性映射（可交换）的和。例如：r2→r2是具有标准矩阵表示的r-线性映射f（x，y）=（x，x+y）

. A basic calculation shows that mf(x) = χf(x) = (x − 1)2. By Theorem 30.6  
. 基本计算表明，mf（x）=χf（x）=x−1）2。根据定理30.6

we know that f is not diagonalizable over R. But since the eigenvalue λ1 = 1 of f does belong to R, we may use the projection construction inherent within Theorem 30.11 to write f = D + N, where D is a diagonalizable linear map and N is a nilpotent linear map. The proof of Theorem 30.10 implies that  
我们知道f在r上不可对角化，但由于f的特征值λ1=1属于r，我们可以用定理30.11中固有的投影构造写出f=d+n，其中d是可对角化线性映射，n是幂零线性映射。定理30.10的证明表明

.  
.

Then  
然后

D = λ1π1 = id, N = f − D = f(x,y) − id(x,y) = (x,x + y) − (x,y) = (0,y),  
D=λ1π1=ID，N=F−D=F（x，y）−ID（x，y）=（x，x+y）−（x，y）=（0，y），

which is equivalent to the matrix decomposition  
相当于矩阵分解

.  
.

This example suggests that the diagonal summand of f is related to the projection constructions associated with the proof of the primary decomposition theorem. If we write  
这个例子表明f的对角和式与投影结构有关，而投影结构与主分解定理的证明有关。如果我们写信

D = λ1π1 + ··· + λkπk,  
D=λ1π1+····+λkπk，

where πi is the projection from E onto the subspace Wi defined in the proof of Theorem  
式中πi是从e到定理证明中定义的子空间wi的投影。

30.10, since  
30.10，自

π1 + ··· + πk = id,  
π1+····+πk=id，

we have f = fπ1 + ··· + fπk,  
我们有f=fπ1+····+fπk，

and so we get  
所以我们得到

N = f − D = (f − λ1id)π1 + ··· + (f − λkid)πk.  
n=f−d=（f−λ1id）π1+·····+（f−λkid）πk。

We claim that N = f − D is a nilpotent operator. Since by construction the πi are polynomials in f, they commute with f, using the properties of the πi, we get  
我们认为n=f−d是幂零算子。由于πi是f中的多项式，它们与f交换，利用πi的性质，我们得到

Nr = (f − λ1id)rπ1 + ··· + (f − λkid)rπk.  
nr=（f−λ1id）rπ1+····+（f−λkid）rπk。

Therefore, if r = max{ri}, we have (f − λkid)r = 0 for i = 1,...,k, which implies that  
因此，如果r=max r i，我们有（f−λkid）r=0，对于i=1，…，k，这意味着

Nr = 0.  
nr=0。

It remains to show that D is diagonalizable. Since N is a polynomial in f, it commutes with f, and thus with D. From  
它仍然表明d是可对角化的。因为n是f中的多项式，它与f相乘，因此与d相乘。

D = λ1π1 + ··· + λkπk,  
D=λ1π1+····+λkπk，

and  
和

π1 + ··· + πk = id,  
π1+····+πk=id，

we see that  
我们看到了

.  
.

Since the projections πj with j =6 i vanish on Wi, the above equation implies that D − λiid vanishes on Wi and that (D − λjid)(Wi) ⊆ Wi, and thus that the minimal polynomial of D is  
由于j=6i的投影πj在wi上消失，因此上述方程表明d-λiid在wi上消失，d-λjid（wi）wi也消失，因此d的最小多项式为

(X − λ1)···(X − λk).  
（x-λ1）···（x-λk）。

Since the λi are distinct, by Theorem 30.6, the linear map D is diagonalizable.  
由于λi是不同的，根据定理30.6，线性映射d是可对角化的。

In summary we have shown that when all the eigenvalues of f belong to K, there exist a diagonalizable linear map D and a nilpotent linear map N such that  
综上所述，当f的所有特征值都属于k时，存在一个对角化线性映射d和一个幂零线性映射n，从而

f = D + N DN = ND,  
f=d+n，dn=nd，

and N and D are polynomials in f.  
n和d是f中的多项式。

A decomposition of f as above is called a Jordan decomposition. In fact, we can prove more: the maps D and N are uniquely determined by f.  
如上所述，F的分解称为约旦分解。事实上，我们可以证明：图d和n是由f唯一确定的。

30.5. JORDAN DECOMPOSITION  
30.5。约旦分解

Theorem 30.12. (Jordan Decomposition) Let f : E → E be a linear map on the finitedimensional vector space E over the field K. If all the eigenvalues λ1,...,λk of f belong to  
定理30.12。（约当分解）设f:e→e为K域上有限维向量空间e上的线性映射。如果f的所有特征值λ1，…，则λk属于

K, then there exist a diagonalizable linear map D and a nilpotent linear map N such that  
k，那么存在一个对角化的线性映射d和一个幂零的线性映射n，这样

f = D + N DN = ND.  
f=d+n，dn=nd。

Furthermore, D and N are uniquely determined by the above equations and they are polynomials in f.  
此外，D和N是由上述方程唯一确定的，它们是F中的多项式。

Proof. We already proved the existence part. Suppose we also have f = D0 + N0, with D0N0 = N0D0, where D0 is diagonalizable, N0 is nilpotent, and both are polynomials in f. We need to prove that D = D0 and N = N0.  
证据。我们已经证明了存在的部分。假设我们也有f=d0+n0，其中d0是对角化的，n0是幂零的，两者都是f中的多项式，我们需要证明d=d0和n=n0。

Since D0 and N0 commute with one another and f = D0 + N0, we see that D0 and N0 commute with f. Then D0 and N0 commute with any polynomial in f; hence they commute with D and N. From  
因为d0和n0是相互通勤的，而f=d0+n0，我们看到，d0和n0是与f一起通勤的，然后，d0和n0是与f中的任何多项式一起通勤的；因此，它们是与d和n一起通勤的。

D + N = D0 + N0,  
d+n=d0+n0，

we get  
我们得到

D − D0 = N0 − N,  
d−d0=n0−n，

and D,D0,N,N0 commute with one another. Since D and D0 are both diagonalizable and commute, by Proposition 30.7, they are simultaneousy diagonalizable, so D − D0 is diagonalizable. Since N and N0 commute, by the binomial formula, for any r ≥ 1,  
D、D0、N、N0彼此通勤。由于d和d0都是可对角化的和通勤的，根据命题30.7，它们是同时可对角化的，所以d-d0是可对角化的。由于n和n0通勤，根据二项式，对于任何r≥1，

.  
.

Since both N and N0 are nilpotent, we have Nr1 = 0 and (N0)r2 = 0, for some r1,r2 > 0, so for r ≥ r1 +r2, the right-hand side of the above expression is zero, which shows that N0 −N is nilpotent. (In fact, it is easy that r1 = r2 = n works). It follows that D − D0 = N0 − N is both diagonalizable and nilpotent. Clearly, the minimal polynomial of a nilpotent linear map is of the form Xr for some r > 0 (and r ≤ dim(E)). But D − D0 is diagonalizable, so its minimal polynomial has simple roots, which means that r = 1. Therefore, the minimal polynomial of D − D0 is X, which says that D − D0 = 0, and then N = N0.   
由于n和n0都是幂零的，我们得到n r1=0和（n0）r2=0，对于一些r1，r2>0，因此对于r≥r1+r2，上述表达式的右边是零，这表明n0−n是幂零的。（事实上，很容易r1=r2=n起作用）。由此可知，d−d0=n0−n既可对角化又可幂零。很明显，幂零线性映射的最小多项式的形式是xr，对于一些r>0（和r≤dim（e））。但是d-d0是对角化的，所以它的最小多项式有简单的根，这意味着r=1。因此，d-d0的最小多项式是x，表示d-d0=0，然后n=n0。

If K is an algebraically closed field, then Theorem 30.12 holds. This is the case when K = C. This theorem reduces the study of linear maps (from E to itself) to the study of nilpotent operators. There is a special normal form for such operators which is discussed in the next section.  
如果k是代数闭场，则定理30.12成立。当k=c时就是这样。这个定理将线性映射的研究（从e到自身）简化为幂零算子的研究。这类运算符有一种特殊的正规形式，将在下一节中讨论。

## 30.6 Nilpotent Linear Maps and Jordan Form 30.6幂零线性映射和约旦形式

This section is devoted to a normal form for nilpotent maps. We follow Godement’s exposition [76]. Let f : E → E be a nilpotent linear map on a finite-dimensional vector space over a field K, and assume that f is not the zero map. There is a smallest positive integer r ≥ 1 such fr = 06 and fr+1 = 0. Clearly, the polynomial Xr+1 annihilates f, and it is the minimal polynomial of f since fr = 06 . It follows that r + 1 ≤ n = dim(E). Let us define the subspaces Ni by  
本节讨论幂零映射的正规形式。我们遵循上帝的解释[76]。设f:e→e为k域上有限维向量空间上的幂零线性映射，并假定f不是零映射。有一个最小的正整数r≥1，fr=06，fr+1=0。显然，多项式xr+1会湮灭f，这是自fr=06以来f的最小多项式。由此可知，r+1≤n=dim（e）。让我们定义子空间ni

Ni = Ker(fi), i ≥ 0.  
ni=ker（fi），i≥0。

Note that N0 = (0), N1 = Ker(f), and Nr+1 = E. Also, it is obvious that  
注意n0=（0），n1=ker（f），nr+1=e。此外，很明显

Ni ⊆ Ni+1, i ≥ 0.  
ni ni+1，i≥0。

Proposition 30.13. Given a nilpotent linear map f with fr = 06 and fr+1 = 0 as above, the inclusions in the following sequence are strict:  
提案30.13。给出了一个幂零线性映射f，fr=06，fr+1=0，如下所列的包含严格：

(0) = N0 ⊂ N1 ⊂ ··· ⊂ Nr ⊂ Nr+1 = E.  
（0）=n0 n1···nr nr+1=e。

Proof. We proceed by contradiction. Assume that Ni = Ni+1 for some i with 0 ≤ i ≤ r.  
证据。我们自相矛盾。假设某些i的ni=ni+1，其中0≤i≤r。

Since fr+1 = 0, for every u ∈ E, we have  
因为fr+1=0，对于每个u∈e，我们有

0 = fr+1(u) = fi+1(fr−i(u)),  
0=fr+1（u）=fi+1（fr−i（u）），

which shows that fr−i(u) ∈ Ni+1. Since Ni = Ni+1, we get fr−i(u) ∈ Ni, and thus fr(u) = 0. Since this holds for all u ∈ E, we see that fr = 0, a contradiction.   
即fr−i（u）∈ni+1。由于ni=ni+1，我们得到fr−i（u）∈ni，因此fr（u）=0。既然这适用于所有的u∈e，我们看到fr=0，一个矛盾。

Proposition 30.14. Given a nilpotent linear map f with fr = 06 and fr+1 = 0, for any integer i with 1 ≤ i ≤ r, for any subspace U of E, if U ∩ Ni = (0), then f(U) ∩ Ni−1 = (0), and the restriction of f to U is an isomorphism onto f(U).  
提案30.14。给定一个幂零线性映射f，f r=06，fr+1=0，对于1≤i≤r的任何整数i，对于e的任何子空间u，如果u ni=（0），那么f（u）ni−1=（0），f对u的限制是对f（u）的同构。

Proof. Pick vi(∈u) = 0f(U). Then∩ Ni−1. We haveu ∈ U ∩ Niv, so= fu(= 0u) for somesince U ∩u N∈iU= (0)and, andfi−1(vv) = 0= f(, whichu) = 0. means that f  
证据。选择vi（∈u）=0f（u）。然后ni−1。我们有u u niv，so=fu（=0u），因为u u n iu=（0），并且fi−1（vv）=0=f（，whichu）=0。意思是f

Therefore, f(U) ∩ Ni−1 = (0). The restriction of f to U is obviously surjective on f(U). Suppose that f(u) = 0 for some u ∈ U. Then u ∈ U ∩ N1 ⊆ U ∩ Ni = (0) (since i ≥ 1), so u = 0, which proves that f is also injective on U.   
因此，f（u）ni−1=（0）。f对u的限制显然是对f（u）的限制。假设f（u）=0，对于一些u∈u，那么u∈u n1 u ni=（0）（因为i≥1），那么u=0，这证明f也是对u的内射。

Proposition 30.15. Given a nilpotent linear map f with fr = 06 and fr+1 = 0, there exists a sequence of subspace U1,...,Ur+1 of E with the following properties:  
提案30.15。对于fr=06且fr+1=0的幂零线性映射f，存在一个e的子空间u1，…，ur+1序列，其性质如下：

1. Ni = Ni−1 ⊕ Ui, for i = 1,...,r + 1.  
   ni=ni−1 ui，对于i=1，…，r+1。
2. We have f(Ui) ⊆ Ui−1, and the restriction of f to Ui is an injection, for i = 2,...,r+1.  
   我们有f（ui）ui−1，而f对ui的限制是注入，对于i=2，…，r+1。

See Figure 30.2.  
见图30.2。

N

r

U

r+1

f(

U

r+1

)

f

0

E =

4

4

N

r

U

r+1

N

r-1

U

r

f(U )

r+1

f(U )

r

f

0

N

r-1

=

N

r

U

r

f(U )

r

N

r-2

U

r-1

N

r-1

U

r-1

N

r-2

=

4

U

r-1

f(

)

f

0

Figure 30.2: A schematic illustration of Ni = Ni−1⊕Ui with f(Ui) ⊆ Ui−1 for i = r+1,r,r−1.  
图30.2:ni=ni−1 ui和f（ui）ui−1（i=r+1，r，r−1）的示意图。

Proof. We proceed inductively, by defining the sequence Ur+1,Ur,...,U1. We pick Ur+1 to be any supplement of Nr in Nr+1 = E, so that  
证据。我们通过定义序列uR+1，uR，…，u1进行归纳。我们选择ur+1作为nr+1=e中nr的任何补充，因此

E = Nr+1 = Nr ⊕ Ur+1.  
E=nr+1=nr ur+1。

Since fr+1 = 0 and Nr = Ker(fr), we have f(Ur+1) ⊆ Nr, and by Proposition 30.14, as Ur+1 ∩Nr = (0), we have f(Ur+1)∩Nr−1 = (0). As a consequence, we can pick a supplement Ur of Nr−1 in Nr so that f(Ur+1) ⊆ Ur. We have  
由于fr+1=0和nr=ker（fr），我们有f（ur+1）nr，根据命题30.14，作为ur+1 nr=（0），我们有f（ur+1）nr−1=（0）。因此，我们可以在nr中选择nr−1的补充uR，以便f（ur+1）uR。我们有

Nr = Nr−1 ⊕ Ur and f(Ur+1) ⊆ Ur.  
nr=nr−1 ur和f（ur+1）ur。

By Proposition 30.14, f is an injection from Ur+1 to Ur. Assume inductively that Ur+1,...,Ui have been defined for i ≥ 2 and that they satisfy (1) and (2). Since  
根据提案30.14，F是从UR+1到UR的注入。假设u+1，…，ui被定义为i≥2，并且满足（1）和（2）。自从

Ni = Ni−1 ⊕ Ui,  
ni=ni−1 ui，

we have Ui ⊆ Ni, so fi−1(f(Ui)) = fi(Ui) = (0), which implies that f(Ui) ⊆ Ni−1. Also, since Ui ∩Ni−1 = (0), by Proposition 30.14, we have f(Ui)∩Ni−2 = (0). It follows that there is a supplement Ui−1 of Ni−2 in Ni−1 that contains f(Ui). We have  
我们有ui ni，所以fi−1（f（ui））=fi（ui）=（0），这意味着f（ui）ni−1。此外，由于ui ni−1=（0），根据命题30.14，我们得到f（ui）ni−2=（0）。因此，在ni−1中有一个包含f（ui）的ni−2的补充ui−1。我们有

Ni−1 = Ni−2 ⊕ Ui−1 and f(Ui) ⊆ Ui−1.  
ni−1=ni−2 ui−1和f（ui）ui−1。

The fact that f is an injection from Ui into Ui−1 follows from Proposition 30.14. Therefore, the induction step is proven. The construction stops when i = 1.   
事实上，f是从ui注入到ui−1的注入，这一点源于命题30.14。因此，证明了诱导步骤。当i=1时，结构停止。

Because N0 = (0) and Nr+1 = E, we see that E is the direct sum of the Ui:  
因为n0=（0）和nr+1=e，我们看到e是ui的直接和：

E = U1 ⊕ ··· ⊕ Ur+1,  
E=U1···UR+1，

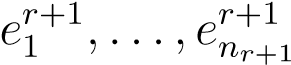
with f(Ui) ⊆ Ui−1, and f an injection from Ui to Ui−1, for i = r + 1,...,2. By a clever choice of bases in the Ui, we obtain the following nice theorem.  
对于f（ui）ui−1和f，从ui注入到ui−1，对于i=r+1，…，2。通过在用户界面中巧妙地选择基，我们得到了下面的好定理。

Theorem 30.16. For any nilpotent linear map f : E → E on a finite-dimensional vector space E of dimension n over a field K, there is a basis of E such that the matrix N of f is  
定理30.16。对于在有限维向量空间e上的任意幂零线性映射f:e→e，在k域上，有一个e的基础，使得f的矩阵n是

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| of the form 形式的 |  |  |  |  |  |
|  γ  0 零  0 千分之一   γ  N = ... N=……   γ  0 0   γ  0 零 | ν1 1伏  0 零  ... …  0 零  0 零 | 0 ν2 0ω2  ... …  0 零  0 零 | ··· ·········  ···... ···…  ··· ·········  ··· ········· | 0 零  0 零  ... …  0 零  0 零 | 0  0℃  0  0℃  ... , …，  νn \_n\_  0 零 |

where νi = 1 or νi = 0.  
式中，νi=1或νi=0。

Proof. First apply Proposition 30.15 to obtain a direct sum. Then we define a basis of E inductively as follows. First we choose a basis  
证据。首先应用30.15号提案获得直接和。然后我们归纳地定义e的一个基，如下。首先我们选择一个基础



of Ur+1. Next, for i = r + 1,...,2, given the basis  
你的+1。接下来，对于i=r+1，…，2，给出了基础



of Ui, since f is injective on Ui and f(Ui) ⊆ Ui−1, the vectors) are linearly independent, so we define a basis of Ui−1 by completing) to a basis in Ui−1:  
对于ui，因为f在ui上是内射的，而f（ui）ui-1，向量）是线性独立的，所以我们通过完成定义ui-1的基础）到ui-1的基础：



with  
具有

eij−1 = f(eij), j = 1...,ni.  
eij−1=f（eij），j=1…，ni.

Since U1 = N1 = Ker(f), we have  
因为u1=n1=ker（f），我们有

f(e1j) = 0, j = 1,...,n1.  
f（e1j）=0，j=1，…，n1.

These basis vectors can be arranged as the rows of the following matrix:  
这些基向量可以排列为以下矩阵的行：

er1+1 ··· ern+1r+1   
er1+1···ern+1r+1

  
γ

 ... ...   
…

 er1 ··· ernr+1 ernr+1+1 ··· ernr   
er1····ernr+1 ernr+1+1····ernr···

  
γ

 ... ... ... ...   
………………………

er−1 ··· ern−r+11 ern−r+11 +1 ··· ern−r 1 enr−r+11 ··· ern−r−11   
er−1···················································

 1  
1

 ... ... ... ... ... ...   ... ... ... ... ... ...   
………………………………………………………………………………………………

e11 ··· e1nr+1 e1nr+1+1 ··· e1nr e1nr+1 ··· en1r−1 ··· ··· e1n1  
E11····E1nr+1 E1nr+1+1····E1nr E1nr+1·········E1n1

Finally, we define the basis (e1,...,en) by listing each column of the above matrix from the bottom-up, starting with column one, then column two, etc. This means that we list the vectors in the following order:  
最后，我们通过自下而上列出上述矩阵的每一列来定义基础（e1，…，en），从第一列开始，然后是第二列，等等。这意味着我们按照以下顺序列出向量：

For j = 1,...,nr+1, list;  
对于j=1，…，nr+1，列表；

In general, for i = r,...,1, for j = ni+1 + 1,...,ni, list e1j,...,eij.  
一般来说，对于i=r，…，1，对于j=ni+1+1，…，ni，列出e1j，…，eij。

Then because f(e1j) = 0 and eij−1 = f(eji) for i ≥ 2, either  
然后，因为f（e1j）=0和eij−1=f（eji），对于i≥2，或者

f(ei) = 0 or f(ei) = ei−1,  
f（ei）=0或f（ei）=ei-1，

which proves the theorem.   
这证明了这个定理。

As an application of Theorem 30.16, we obtain the Jordan form of a linear map. Definition 30.6. A Jordan block is an r × r matrix Jr(λ), of the form  
作为定理30.16的一个应用，我们得到了线性映射的乔丹形式。定义30.6.约旦块是R×R矩阵Jr（λ），其形式为

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| λ γ-α  0 零   γ  . Jr(λ) = .. ②Jr（λ）=..   γ   γ  0 0  0 零 | 1 λ 1兆  ... …  0 零  0 零 | 0 零  1 一  ... …  0 零  0 零 | ··· ·········  ···... ···…  ... ··· …········· | 0 0℃  0 0℃  ..., ……   γ  1 λ 1λ |

where λ ∈ K, with J1(λ) = (λ) if r = 1. A Jordan matrix, J, is an n × n block diagonal matrix of the form  
式中，如果r=1，则λ∈k，其中j1（λ）=（λ）。约旦矩阵j是形式的n×n块对角矩阵。

,  
，

where each Jrk(λk) is a Jordan block associated with some λk ∈ K, and with r1+···+rm = n.  
其中，每个JRk（λk）是一个与一些λk∈k相关联的约旦块，并且与R1+····+Rm=n相关。

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| To simplify notation, we often write J(λ) for Jr(λ). Here is an example of a Jordan 为了简化符号，我们通常为jr（λ）编写j（λ）。下面是一个约旦的例子 | | | | | | | |
| matrix with four blocks: 四块矩阵： |  |  |  |  |  |  |  |
| λ γ-α  0 千分之一  0 0   γ  0 J = 0 0 J=0   γ   γ  0 0   γ  0 千分之一  0 零 | 1 λ 1兆  0 零  0 零  0 零  0 零  0 零  0 零 | 0 零  1 λ 1兆  0 零  0 零  0 零  0 零  0 零 | 0 0 0 0  0 λ 0兆  0 零  0 零  0 零  0 零 | 0 零  0 零  0 零  1 λ 1兆  0 零  0 零  0 零 | 0 零  0 0 0 0 0 0  0 λ 0兆  0 零  0 零 | 0 0 0 0 0 0  0 零  0 零  0 µ 千分之一  0 零 | 0 0℃  0 0℃  0 0\_  0 0. 0 0。  0 0\_  1 µ 1 \_ |

Theorem 30.17. (Jordan form) Let E be a vector space of dimension n over a field K and let f : E → E be a linear map. The following properties are equivalent:  
定理30.17。（约旦形式）让e是一个向量空间的维度n在一个领域K和让f:e→e是一个线性地图。以下属性等效：

1. The eigenvalues of f all belong to K (i.e. the roots of the characteristic polynomial χf all belong to K).  
   f的特征值都属于k（即特征多项式的根χf都属于k）。
2. There is a basis of E in which the matrix of f is a Jordan matrix.  
   有一个e的基，其中f的矩阵是约旦矩阵。

Proof. Assume (1). First we apply Theorem 30.11, and we get a direct sum, such that the restriction of gi = f − λjid to Wi is nilpotent. By Theorem 30.16, there is a basis of Wi such that the matrix of the restriction of gi is of the form  
证据。假设（1）。首先，我们应用定理30.11，得到一个直接和，使得gi=f-λjid对wi的约束是幂零的。根据定理30.16，有一个wi的基础，使得gi的约束矩阵的形式

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  γ  0 零  0 千分之一   γ  Gi = ... 吉=…   γ  0 0   γ  0 零 | ν1 1伏  0 零  ... …  0 零  0 零 | 0 ν2 0ω2  ... …  0 零  0 零 | ··· ·········  ···... ···…  ··· ·········  ··· ········· | 0 零  0 零  ... …  0 零  0 零 | 0  0℃  0  0℃  ... , …，  νni νni\_  0 零 |

where νi = 1 or νi = 0. Furthermore, over any basis, λiid is represented by the diagonal matrix Di with λi on the diagonal. Then it is clear that we can split Di + Gi into Jordan blocks by forming a Jordan block for every uninterrupted chain of 1s. By putting the bases of the Wi together, we obtain a matrix in Jordan form for f.  
式中，νi=1或νi=0。此外，在任何基础上，用对角矩阵di表示λiid，对角线上用λi表示。很明显，我们可以通过为每一个连续的1s链形成一个约旦块来将di+gi分解成约旦块，通过把wi的基部放在一起，我们得到了f的约旦形式的矩阵。

Now assume (2). If f can be represented by a Jordan matrix, it is obvious that the diagonal entries are the eigenvalues of f, so they all belong to K.   
现在假设（2）。如果F可以用约当矩阵表示，那么很明显，对角项是F的特征值，所以它们都属于K。

Observe that Theorem 30.17 applies if K = C. It turns out that there are uniqueness properties of the Jordan blocks. There are also other fundamental normal forms for linear maps, such as the rational canonical form, but to prove these results, it is better to develop more powerful machinery about finitely generated modules over a PID. To accomplish this most effectively, we need some basic knowledge about tensor products.  
注意定理30.17适用于k=c的情况。结果表明，约旦块具有唯一性。线性映射也有其他基本的正规形式，例如有理正规形式，但是为了证明这些结果，最好在PID上开发关于有限生成模块的更强大的机制。为了最有效地实现这一点，我们需要一些关于张量积的基本知识。

If a complex n × n matrix A is expressed in terms of its Jordan decomposition as A = D + N, since D and N commute, by Proposition 8.21, the exponential of A is given by  
如果一个复n×n矩阵a用它的约当分解表示为a=d+n，因为d和n乘上命题8.21，a的指数由

eA = eDeN,  
ea=伊甸园，

and since N is an n × n nilpotent matrix, Nn−1 = 0, so we obtain  
由于n是n×n的幂零矩阵，n−1=0，因此我们得到

.  
.

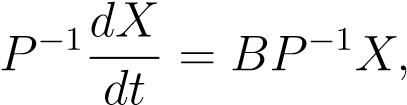
In particular, the above applies if A is a Jordan matrix. This fact can be used to solve (at least in theory) systems of first-order linear differential equations. Such systems are of the form  
特别是，如果a是jordan矩阵，则上述内容适用。这一事实可用于求解（至少在理论上）一阶线性微分方程组。这样的系统是这样的

(∗)  
（三）

where A is an n × n matrix and X is an n-dimensional vector of functions of the parameter t.  
其中a是n×n矩阵，x是参数t函数的n维向量。

It can be shown that the columns of the matrix etA form a basis of the vector space of solutions of the system of linear differential equations (∗); see Artin [7] (Chapter 4). Furthermore, for any matrix B and any invertible matrix P, if A = PBP −1, then the system  
可以看出，矩阵eta的列构成了线性微分方程组（）解的向量空间的基础；见Artin[7]（第4章）。此外，对于任何矩阵b和任何可逆矩阵p，如果a=pbp−1，则系统

(∗) is equivalent to  
（）等于

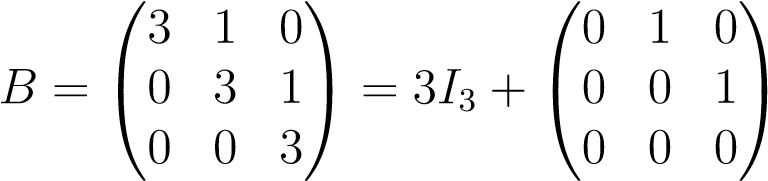


so if we make the change of variable Y = P −1X, we obtain the system  
因此，如果我们改变变量y=p−1X，我们得到系统

(∗∗)  
（）

Consequently, if B is such that the exponential etB can be easily computed, we obtain an explicit solution Y of (∗∗) , and X = PY is an explicit solution of (∗). This is the case when B is a Jordan form of A. In this case, it suffices to consider the Jordan blocks of B. Then we have and the powers Nk are easily computed.  
因此，如果b可以很容易地计算出指数ETb，我们得到（）的显式解y，x=py是（）的显式解。当b是a的约旦形式时就是这样。在这种情况下，考虑b的约旦块就足够了。然后我们就有了，并且可以很容易地计算出k的幂。

For example, if  
例如，如果



we obtain  
我们得到

.  
.

The columns of etB form a basis of the space of solutions of the system of linear differential equations  
ETB的列构成了线性微分方程组解空间的基础。

.  
.

Solving systems of first-order linear differential equations is discussed in Artin [7] and more extensively in Hirsh and Smale [91].  
一阶线性微分方程组的求解在Artin[7]中进行了讨论，在Hirsh和Smale[91]中进行了更广泛的讨论。

## 30.7 Summary 30.7总结

The main concepts and results of this chapter are listed below:  
本章的主要概念和结果如下：

* Ideals, principal ideals, greatest common divisors.  
  理想，主理想，最大公约数。
* Monic polynomial, irreducible polynomial, relatively prime polynomials.  
  Monic多项式，不可约多项式，相对素多项式。
* Annihilator of a linear map.  
  线性地图的湮灭子。
* Minimal polynomial of a linear map.  
  线性映射的极小多项式。
* Invariant subspace.  
  不变子空间。
* f-conductor of u into W; conductor of u into W.  
  F—U到W的导体；U到W的导体。
* Diagonalizable linear maps.  
  可对角化线性映射。
* Commuting families of linear maps.  
  线性地图的交换族。
* Primary decomposition.  
  初级分解。
* Generalized eigenvectors.  
  广义特征向量。
* Nilpotent linear map.  
  幂零线性映射。
* Normal form of a nilpotent linear map.  
  幂零线性映射的正规形式。
* Jordan decomposition.  
  约旦分解。
* Jordan block.  
  约旦街区。

30.8. PROBLEMS  
30.8。问题

* Jordan matrix.  
  约旦矩阵。
* Jordan normal form.  
  约旦标准形状。
* Systems of first-order linear differential equations.  
  一阶线性微分方程组。

## 30.8 Problems 30.8问题

Problem 30.1. Given a linear map f : E → E, prove that the set Ann(f) of polynomials that annihilate f is an ideal.  
问题30.1。给出了一个线性映射f:e→e，证明了湮灭f的多项式的集合ann（f）是一个理想。

Problem 30.2. Provide the details of Proposition 30.3.  
问题30.2。提供提案30.3的细节。

Problem 30.3. Prove that the f-conductor Sf(u,W) is an ideal in K[X] (Proposition 30.4).  
问题30.3。证明f导体sf（u，w）是k[x]中的理想（命题30.4）。

Problem 30.4. Prove that the polynomials g1,...,gk used in the proof of Theorem 30.10 are relatively prime.  
问题30.4。证明定理30.10证明中使用的多项式g1，…，gk是相对素数。

Problem 30.5. Find the minimal polynomial of the matrix  
问题30.5。求矩阵的极小多项式

.  
.

Problem 30.6. Find the Jordan decomposition of the matrix  
问题30.6。求矩阵的约当分解

.  
.

Problem 30.7. Let f : E → E be a linear map on a finite-dimensional vector space. Prove that if f has rank 1, then either f is diagonalizable or f is nilpotent but not both. Problem 30.8. Find the Jordan form of the matrix  
问题30.7。设f:e→e为有限维向量空间上的线性映射。证明如果f有秩1，那么f要么是对角化的，要么f是幂零的，但不是两者都是。问题30.8。找到矩阵的约旦形式

.  
.

Problem 30.9. Let N be a 3 × 3 nilpotent matrix over C. Prove that the matrix A = I + (1/2)N − (1/8)N2 satisfies the equation  
问题30.9。设n为c上的3×3幂零矩阵，证明矩阵a=i+（1/2）n−（1/8）n2满足方程

A2 = I + N.  
A2=I+N。

In other words, A is a square root of I + N.  
换句话说，a是i+n的平方根。

Generalize the above fact to any n × n nilpotent matrix N over C using the binomial series for (1 + t)1/2.  
利用（1+t）1/2的二项级数将上述事实推广到C上任意n×n的幂零矩阵n。

Problem 30.10. Let K be an algebraically closed field (for example, K = C). Prove that every 4 × 4 matrix is similar to a Jordan matrix of the following form:  
问题30.10。设k为代数闭场（例如，k=c）。证明每个4×4矩阵类似于以下形式的约旦矩阵：

λ1 0 0 0  λ 1 0 0   
λ1 0 0 0λ1 0 0

 0 λ2 0 0  0 λ 0 0   
0λ2 0 0 0λ0 0

 0 0 λ3 0 , 0 0 λ3 0 ,,  
0λ3 0，0λ3 0，，

0 0 0 λ4 0 0 0 λ4  
0 0 0λ4 0 0 0λ4

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| λ γ-α  0 0  0 0   γ  0 零 | 1 λ 1兆  0 零  0 零 | 0 零  0 µ 0℃  0 零 | 0 0℃  0, 0℃，  1 µ 1 \_ | λ γ-α  0 0  0 0   γ  0 零 | 1 λ 1兆  0 零  0 零 | 0 零  1 λ 1兆  0 零 | 0 0. 0 0。  1 λ 1λ |

Problem 30.11. In this problem the field K is of characteristic 0. Consider an (r × r)  
问题30.11。在这个问题中，字段k的特征为0。考虑a（r×r）

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Jordan block 约旦街区 |  |  |  |  |  |
|  | λ 小精灵  0 零   γ  . Jr(λ) = .. ②Jr（λ）=..   γ   γ  0 千分之一  0 零 | 1 λ 1兆  ... …  0 零  0 零 | 0 零  1 一  ... …  0 零  0 零 | ··· ·········  ···... ···…  ... ··· …········· | 0 0℃  0 0℃  .... ……   γ  1 λ 1λ |
| Prove that for any polynomial f(X), we have 证明对于任何多项式f（x），我们有 | | |

,  
，

where  
哪里

and f(k)(X) is the kth derivative of f(X).  
f（k）（x）是f（x）的kth导数。

Chapter 31  
第三十一章

# UFD’s, Noetherian Rings, Hilbert’s Basis Theorem UFD，诺特环，希尔伯特基定理

## 31.1 Unique Factorization Domains (Factorial Rings) 31.1唯一因子分解域（因子环）

We saw in Section 29.5 that if K is a field, then every nonnull polynomial in K[X] can be factored as a product of irreducible factors, and that such a factorization is essentially unique. The same property holds for the ring K[X1,...,Xn] where n ≥ 2, but a different proof is needed.  
我们在第29.5节中看到，如果k是一个字段，那么k[x]中的每个非空多项式都可以被分解为不可约因子的乘积，并且这种分解本质上是唯一的。同样的性质适用于环k[x1，…，xn]，其中n≥2，但需要不同的证明。

The reason why unique factorization holds for K[X1,...,Xn] is that if A is an integral domain for which unique factorization holds in some suitable sense, then the property of unique factorization lifts to the polynomial ring A[X]. Such rings are called factorial rings, or unique factorization domains. The first step if to define the notion of irreducible element in an integral domain, and then to define a factorial ring. If will turn out that in a factorial ring, any nonnull element a is irreducible (or prime) iff the principal ideal (a) is a prime ideal.  
唯一因式分解对k[x1，…，xn]成立的原因是，如果a是一个积分域，其中唯一因式分解在某种意义上成立，那么唯一因式分解的性质提升到多项式环a[x]。这样的环称为阶乘环，或唯一的阶乘域。第一步是在积分域中定义不可约元素的概念，然后定义阶乘环。如果在阶乘环中，任何非空元素a都是不可约的（或素数），只要主理想（a）是素数理想。

Recall that given a ring A, a unit is any invertible element (w.r.t. multiplication). The set of units of A is denoted by A∗. It is a multiplicative subgroup of A, with identity 1. Also, given a,b ∈ A, recall that a divides b if b = ac for some c ∈ A; equivalently, a divides b iff (b) ⊆ (a). Any nonzero a ∈ A is divisible by any unit u, since a = u(u−1a). The relation “a divides b,” often denoted by a | b, is reflexive and transitive, and thus, a preorder on A−{0}.  
回想一下，给定一个环A，一个单位是任何可逆元素（w.r.t.乘法）。a的一组单位用表示。它是a的乘法子群，具有标识1。另外，在给定a，b∈a的情况下，如果b=a c，则a对b进行除b，相当于a对b iff（b）（a）进行除b。任何非零a∈a可被任何单位u整除，因为a=u（u−1a）。关系“a divides b”，通常用a\_b表示，是反身的和传递的，因此是a−0\_的预订单。

Definition 31.1. Let A be an integral domain. Some element a ∈ A is irreducible if a = 06 , a /∈ A∗ (a is not a unit), and whenever a = bc, then either b or c is a unit (where b,c ∈ A).  
定义31.1.设A为积分域。如果a=06，a/∈a（a不是一个单位），某个元素a∈a是不可约的，当a=b c时，b或c是一个单位（其中b，c∈a）。

Equivalently, a ∈ A is reducible if a = 0, or a ∈ A∗ (a is a unit), or a = bc where b,c /∈ A∗  
等价地，如果a=0，或a∈a（a是一个单位）或a=b c，其中b，c/∈a a是可约的。

(a,b are both noninvertible) and b,c = 0.6  
（a，b都是不可逆的）和b，c=0.6

Observe that if a ∈ A is irreducible and u ∈ A is a unit, then ua is also irreducible. Generally, if a ∈ A, a = 06 , and u is a unit, then a and ua are said to be associated. This is the equivalence relation on nonnull elements of A induced by the divisibility preorder.  
如果a∈a是不可约的，而u∈a是一个单位，那么ua也是不可约的。一般来说，如果a∈a，a=06，u是一个单位，那么a和ua是相关联的。这是由可除性预序诱导的非空元素的等价关系。

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The following simple proposition gives a sufficient condition for an element a ∈ A to be irreducible.  
下面的简单命题给出了元素a∈a不可约的充分条件。

Proposition 31.1. Let A be an integral domain. For any a ∈ A with a = 06 , if the principal ideal (a) is a prime ideal, then a is irreducible.  
提案31.1.设A为积分域。对于任意a∈a且a=06，如果主理想（a）是素理想，则a是不可约的。

Proof. If (a) is prime, then (a) =6 A and a is not a unit. Assume that a = bc. Then, bc ∈ (a), and since (a) is prime, either b ∈ (a) or c ∈ (a). Consider the case where b ∈ (a), the other case being similar. Then, b = ax for some x ∈ A. As a consequence,  
证据。如果（a）是素数，那么（a）=6a，a不是一个单位。假设a=bc。那么，b c∈（a），由于（a）是素数，b∈（a）或c∈（a）。考虑b∈（a）的情况，另一个情况相似。那么，对于某些x∈a，b=ax。因此，

a = bc = axc,  
a=bc=axc，

and since A is an integral domain and a = 06 , we get  
因为a是一个积分域，a=06，我们得到

1 = xc,  
1=xc，

which proves that c = x−1 is a unit.   
这证明c=x−1是一个单位。

It should be noted that the converse of Proposition 31.1 is generally false. However, it holds for factorial rings, defined next.  
应当指出的是，31.1号提案的反面通常是错误的。然而，它适用于阶乘环，定义如下。

Definition 31.2. A factorial ring or unique factorization domain (UFD) (or unique factorization ring) is an integral domain A such that the following two properties hold:  
定义31.2.因式分解环或唯一因式分解域（UFD）（或唯一因式分解环）是一个积分域A，其具有以下两个性质：

1. For every nonnull a ∈ A, if a /∈ A∗ (a is not a unit), then a can be factored as a product  
   对于每一个非空a∈a，如果a/∈a（a不是一个单位），则a可以被分解为一个积。

a = a1 ···am  
A=A1···AM

where each ai ∈ A is irreducible (m ≥ 1).  
其中，每个ai∈a都是不可约的（m≥1）。

1. For every nonnull a ∈ A, if a /∈ A∗ (a is not a unit) and if  
   对于每一个非空a∈a，如果a/∈a（a不是一个单位），并且如果

a = a1 ···am = b1 ···bn  
a=a1···am=b1··bn

where ai ∈ A and bj ∈ A are irreducible, then m = n and there is a permutation σ of {1,...,m} and some units u1,...,um ∈ A∗ such that ai = uibσ(i) for all i, 1 ≤ i ≤ m.  
其中，a i∈a和bj∈a是不可约的，那么m=n，并且有一个1，…，m的置换σ和一些单位u1，…，um∈a，这样ai=uibσ（i）对于所有i，1≤i≤m。

Example 31.1. The ring Z of integers if a typical example of a UFD. Given a field K, the polynomial ring K[X] is a UFD. More generally, we will show later that every PID is a UFD (see Theorem 31.12). Thus, in particular, Z[X] is a UFD. However, we leave as an exercise to prove that the ideal (2X,X2) generated by 2X and X2 is not principal, and thus, Z[X] is not a PID.  
例31.1。整数的环Z，如果是一个典型的UFD例子。给定一个字段k，多项式环k[x]是一个ufd。更一般地说，我们稍后将展示每个PID都是一个UFD（见定理31.12）。因此，特别是z[x]是一个UFD。然而，我们作为一个练习来证明由2x和x2生成的理想（2x，x2）不是主体，因此z[x]不是PID。

First, we prove that condition (2) in Definition 31.2 is equivalent to the usual “Euclidean” condition.  
首先，我们证明了定义31.2中的条件（2）等价于通常的“欧几里德”条件。

 There are integral domains that are not UFD’s. For example, the subring√ ∈ Z[√−5] of C  
存在非UFD的积分域，例如c的子环√∈z[√−5]

consisting of the complex numbers of the form a + bi 5 where a,b Z is not a UFD. Indeed, we have  
由a+bi 5形式的复数组成，其中a、b、z不是ufd。事实上，我们有

9 = 3 · 3 = (2 + i√5)(2 − i√5),  
9=3·3=（2+i√5）（2−i√5）

and it can be shown that 3, 2+i√5, and 2√−−i√5 are irreducible, and that the units are ±1. The uniqueness condition (2) fails and Z[ 5] is not a UFD.  
可以看出，3、2+i√5和2√−−i√5是不可约的，单位为±1。唯一性条件（2）失败，Z[5]不是UFD。

Remark: For d ∈ Z with d < 0, it is known that the ring of integers of Q(√d) is a UFD iff d is one of the nine primes, d = −1,−2,−3,−7,−11,−19,−43,−67 and −163. This is a hard theorem that was conjectured by Gauss but not proved until 1966, independently by Stark and Baker. Heegner had published a proof of this result in 1952 but there was some doubt about its validity. After finding his proof, Stark reexamined Heegner’s proof and concluded that it was essentially correct after all. In sharp contrast, when d is a positive integer, the  
备注：对于d∈z且d<0，已知q（√d）的整数环是一个ufd iff d是九个素数中的一个，d=−1、−2、−3、−7、−11、−19、−43、−67和−163。这是一个困难的定理，高斯猜想，但直到1966年才被证明，斯塔克和贝克独立。海格纳在1952年发表了这一结果的证明，但对其有效性存在一些怀疑。在找到证据后，斯塔克重新检查了希格纳的证据，并得出结论，这基本上是正确的。与此形成鲜明对比的是，当d是一个正整数时，

problem of determining which of the rings of integers of Q(√d) are UFD’s, is still open. It  
确定q（√d）的整数环中哪一个是ufd的问题仍然是开放的。它

can also be shown that if√ d < 0, then the ring Z[√d] is a UFD iff d = −1 or d = −2. If  
也可以证明，如果√d<0，则环Z[√d]是UFD iff d=−1或d=−2。如果

d ≡ 1(mod4), then Z[ d] is never a UFD. For more details about these remarkable results, see Stark [159] (Chapter 8).  
D 1（mod4），那么Z[D]永远不是UFD。有关这些显著结果的更多详细信息，请参阅Stark[159]（第8章）。

Proposition 31.2. Let A be an integral domain satisfying condition (1) in Definition 31.2. Then, condition (2) in Definition 31.2 is equivalent to the following condition:  
提案31.2.设A为满足定义31.2中条件（1）的积分域。那么，定义31.2中的条件（2）等于以下条件：

(20) If a ∈ A is irreducible and a divides the product bc, where b,c ∈ A and b,c = 06 , then either a divides b or a divides c.  
（20）如果a∈a是不可约的，a将积b c除，其中b，c∈a和b，c=06，则a将b除或a将c除。

Proof. First, assume that (2) holds. Let bc = ad, where d ∈ A, d = 06 . If b is a unit, then  
证据。首先，假设（2）成立。设bc=a d，其中d∈a，d=06。如果b是一个单位，那么

c = adb−1,  
C=adb−1，

and c is divisible by a. A similar argument applies to c. Thus, we may assume that b and c are not units. In view of (1), we can write  
C可以被A整除，类似的论点也适用于C，因此我们可以假设B和C不是单位。鉴于（1），我们可以写

b = p1 ···pm and c = pm+1 ···qm+n,  
b=p1···pm，c=pm+1··qm+n，

where pi ∈ A is irreducible. Since bc = ad, a is irreducible, and b,c are not units, d cannot be a unit. In view of (1), we can write  
式中，π∈a是不可约的。因为b c=a d，a是不可约的，b，c不是单位，d不能是单位。鉴于（1），我们可以写

d = q1 ···qr,  
d=q1···qr，

where qi ∈ A is irreducible. Thus,  
其中qi∈a是不可约的。因此，

p1 ···pmpm+1 ···pm+n = aq1 ···qr,  
p1···pm pm+1···pm+n=aq1···qr，

where all the factors involved are irreducible. By (2), we must have  
所有涉及的因素都是不可约的。到（2）时，我们必须

a = ui0pi0  
A=ui0pi0

for some unit ui0 ∈ A and some index i0, 1 ≤ i0 ≤ m + n. As a consequence, if 1 ≤ i0 ≤ m, then a divides b, and if m + 1 ≤ i0 ≤ m + n, then a divides c. This proves that (20) holds. Let us now assume that (20) holds. Assume that  
对于某些单位ui0∈a和一些指数i0，1≤i0≤m+n，因此，如果1≤i0≤m，则a除以b，如果m+1≤i0≤m+n，则a除以c。这证明（20）成立。现在让我们假设（20）成立。假设

a = a1 ···am = b1 ···bn,  
a=a1···am=b1··bn，

where ai ∈ A and bj ∈ A are irreducible. Without loss of generality, we may assume that m ≤ n. We proceed by induction on m. If m = 1,  
其中，ai∈a和bj∈a是不可约的。在不失去一般性的情况下，我们可以假定m≤n。我们通过对m的归纳来进行。如果m=1，

a1 = b1 ···bn,  
a1=b1···bn，

and since a1 is irreducible, u = b1 ···bi−1bi+1bn must be a unit for some i, 1 ≤ i ≤ n. Thus, (2) holds with n = 1 and a1 = biu. Assume that m > 1 and that the induction hypothesis holds for m − 1. Since a1a2 ···am = b1 ···bn,  
由于a1是不可约的，u=b1·············································假设m>1，且诱导假设适用于m-1。由于a1a2···am=b1··bn，

a1 divides b1 ···bn, and in view of (20), a1 divides some bj. Since a1 and bj are irreducible, we must have bj = uja1, where uj ∈ A is a unit. Since A is an integral domain,  
A1划分b1···bn，从（20）来看，A1划分了一些bj。因为a1和bj是不可约的，我们必须有bj=uja1，其中uj∈a是一个单位。因为A是一个积分域，

a1a2 ···am = b1 ···bj−1uja1bj+1 ···bn  
a1a2·····am=b1····bj········1uja1bj+1···bn

implies that  
意味着

a2 ···am = (ujb1)···bj−1bj+1 ···bn,  
a2······am=（ujb1）····bj····1bj+1···bn，

and by the induction hypothesis, m − 1 = n − 1 and ai = vibτ(i) for some units vi ∈ A and some bijection τ between {2,...,m} and {1,...,j −1,j +1,...,m}. However, the bijection τ extends to a permutation σ of {1,...,m} by letting σ(1) = j, and the result holds by letting v1 = u−j 1.   
根据诱导假设，对于某些单元vi∈a和一些在2，…，m和1，…，j−1，j+1，…，m之间的双射τ，m−1=n−1和a i=vibτ（i）。然而，双射τ通过让σ（1）=j扩展到1，…，m的置换σ，结果通过让v1=u−j 1保持。

As a corollary of Proposition 31.2. we get the converse of Proposition 31.1.  
作为命题31.2的推论。我们得到了31.1号命题的逆命题。

Proposition 31.3. Let A be a factorial ring. For any a ∈ A with a = 06 , the principal ideal (a) is a prime ideal iff a is irreducible.  
提案31.3.设A为阶乘环。对于任意a∈a，当a=06时，主理想（a）是素理想，当a是不可约的。

Proof. In view of Proposition 31.1, we just have to prove that if a ∈ A is irreducible, then the principal ideal (a) is a prime ideal. Indeed, if bc ∈ (a), then a divides bc, and by Proposition 31.2, property (20) implies that either a divides b or a divides c, that is, either b ∈ (a) or c ∈ (a), which means that (a) is prime.   
证据。对于31.1号命题，我们只需要证明，如果a∈a是不可约的，那么主理想（a）就是素理想。实际上，如果b c∈（a），那么a除以bc，并且根据命题31.2，性质（20）意味着a除以b或a除以c，即b∈（a）或c∈（a），这意味着（a）是素数。

Because Proposition 31.3 holds, in a UFD, an irreducible element is often called a prime.  
因为命题31.3认为，在UFD中，不可约元素通常被称为素数。

In a UFD A, every nonzero element a ∈ A that is not a unit can be expressed as a product a = a1 ···an of irreducible elements ai, and by property (2), the number n of factors only depends on a, that is, it is the same for all factorizations into irreducible factors. We agree that this number is 0 for a unit.  
在UFD A中，每一个非零元素a∈a都可以表示为不可约元素ai的乘积a=a1··············································我们同意这个数字对于一个单位是0。

Remark: If A is a UFD, we can state the factorization properties so that they also applies to units:  
备注：如果a是ufd，我们可以说明分解性质，以便它们也适用于单位：

(1) For every nonnull a ∈ A, a can be factored as a product  
（1）对于每一个非空a∈a，a都可以被分解为一个积。

a = ua1 ···am  
A=UA1···AM

where u ∈ A∗ (u is a unit) and each ai ∈ A is irreducible (m ≥ 0). (2) For every nonnull a ∈ A, if  
其中u∈a（u是一个单位），而每个ai∈a都是不可约的（m≥0）。（2）对于每一个非空a∈a，如果

a = ua1 ···am = vb1 ···bn  
A=UA1···AM=VB1···Bn

where u,v ∈ A∗ (u,v are units) and ai ∈ A and bj ∈ A are irreducible, then m = n, and if m = n = 0 then u = v, else if m ≥ 1, then there is a permutation σ of {1,...,m} and some units u1,...,um ∈ A∗ such that ai = uibσ(i) for all i, 1 ≤ i ≤ m.  
其中u，v∈a（u，v是单位），a i∈a和bj∈a是不可约的，则m=n，如果m=n=0，则u=v，否则，如果m≥1，则存在1，…，m的置换σ，并且一些单位u1，…，um∈a，这样ai=uibσ（i）对于所有i，1≤i≤m。

We are now ready to prove that if A is a UFD, then the polynomial ring A[X] is also a UFD.  
我们现在准备证明，如果a是一个ufd，那么多项式环a[x]也是一个ufd。

First, observe that the units of A[X] are just the units of A. The fact that nonnull and nonunit polynomials in A[X] factor as products of irreducible polynomials is easier to prove than uniqueness. We will show in the proof of Theorem 31.10 that we can proceed by induction on the pairs (m,n) where m is the degree of f(X) and n is either 0 if the coefficient fm of Xm in f(X) is a unit of n is fm is the product of n irreducible elements.  
首先，观察a[x]的单位只是a的单位。事实上，在a[x]因子中的非零和非单位多项式作为不可约多项式的乘积，比唯一性更容易证明。我们将在定理31.10的证明中证明，我们可以通过对（m，n）的归纳来进行，其中m是f（x）的度数，如果f（x）中xm的系数fm是n的单位，则n是n不可约元素的积，则n是0。

For the uniqueness of the factorization, by Proposition 31.2, it is enough to prove that condition (20) holds. This is a little more tricky. There are several proofs, but they all involve a pretty Lemma due to Gauss.  
对于因式分解的唯一性，由命题31.2，足以证明条件（20）成立。这有点棘手。有几种证明，但由于高斯的缘故，它们都涉及一个漂亮的引理。

First, note the following trivial fact. Given a ring A, for any a ∈ A, a = 06 , if a divides every coefficient of some nonnull polynomial f(X) ∈ A[X], then a divides f(X). If A is an integral domain, we get the following converse.  
首先，注意下面这个小事实。给定一个环a，对于任意a∈a，a=06，如果a除以某个非空多项式f（x）∈a[x]的每一个系数，则a除以f（x）。如果A是一个积分域，我们得到如下的逆。

Proposition 31.4. Let A be an integral domain. For any a ∈ A, a = 06 , if a divides a nonnull polynomial f(X) ∈ A[X], then a divides every coefficient of f(X).  
提案31.4.设A为积分域。对于任意a∈a，a=06，如果a除以非零多项式f（x）∈a[x]，则a除以f（x）的每一个系数。

Proof. Assume that f(X) = ag(X), for some g(X) ∈ A[X]. Since a = 06 and A is an integral ring, f(X) and g(X) have the same degree m, and since for every i (0 ≤ i ≤ m) the coefficient of Xi in f(X) is equal to the coefficient of Xi in ag(x), we have fi = agi, and whenever fi = 06 , we see that a divides fi.   
证据。假设f（x）=a g（x），对于某些g（x）∈a[x]。由于a＝06和a是整环，f（x）和g（x）具有相同的度数m，并且因为对于每个i（0±i＝m），在f（x）中的Xi系数等于Ag（x）中的Xi系数，所以我们有Fi＝AGI，并且每当Fi＝06时，我们看到划分FI。

Lemma 31.5. (Gauss’s lemma) Let A be a UFD. For any a ∈ A, if a is irreducible and a divides the product f(X)g(X) of two polynomials f(X),g(X) ∈ A[X], then either a divides f(X) or a divides g(X).  
引理31.5。（高斯引理）让A成为UFD。对于任意a∈a，如果a是不可约的，a将两个多项式f（x），g（x）的积f（x）g（x）除，则a将f（x）除，或a将g（x）除。

Proof. Let f(X) = fmXm + ··· + fiXi + ··· + f0 and g(X) = gnXn + ··· + gjXj + ··· + g0. Assume that a divides neither f(X) nor g(X). By the (easy) converse of Proposition 31.4, there is some i (0 ≤ i ≤ m) such that a does not divide fi, and there is some j (0 ≤ j ≤ n) such that a does not divide gj. Pick i and j minimal such that a does not divide fi and a does not divide gj. The coefficient ci+j of Xi+j in f(X)g(X) is  
证据。设f（x）=fmxm+·····+fixi+····+f0和g（x）=gnxn+····+gjxj+····+g0。假设a既不除f（x）也不除g（x）。根据命题31.4的（简单）倒数，有一些i（0≤i≤m），这样a不除fi，有一些j（0≤j≤n），这样a不除gj。选择i和j最小值，这样a不划分fi，a不划分gj。f（x）g（x）中Xi+j的系数Ci+j

ci+j = f0gi+j + f1gi+j−1 + ··· + figj + ··· + fi+jg0  
ci+j=f0gi+j+f1gi+j−1+····+figj+···+fi+jg0

(letting fh = 0 if h > m and gk = 0 if k > n). From the choice of i and j, a cannot divide figj, since a being irreducible, by (20) of Proposition 31.2, a would divide fi or gj. However, by the choice of i and j, a divides every other nonnull term in the sum for ci+j, and since a is irreducible and divides f(X)g(X), by Proposition 31.4, a divides ci+j, which implies that a divides figj, a contradiction. Thus, either a divides f(X) or a divides g(X).   
（h>m时fh=0，k>n时gk=0）。从i和j的选择来看，a不能将fi gj分开，因为a是不可约的，除以（20）31.2，a将把fi或gj分开。然而，通过i和j的选择，a将ci+j和中的其他非空项相除，由于a是不可约的，并将f（x）g（x）除以命题31.4，a将ci+j相除，这意味着a将figj相除，这是一个矛盾。因此，要么A除以F（x），要么A除以G（x）。

As a corollary, we get the following proposition.  
作为推论，我们得到以下命题。

Proposition 31.6. Let A be a UFD. For any a ∈ A, a = 06 , if a divides the product f(X)g(X) of two polynomials f(X),g(X) ∈ A[X] and f(X) is irreducible and of degree at least 1, then a divides g(X).  
提案31.6.让A成为UFD。对于任意a∈a，a=06，如果a将两个多项式f（x）的积f（x）g（x），g（x）∈a[x]和f（x）不可约且阶数至少为1，则a将g（x）除。

Proof. The Proposition is trivial is a is a unit. Otherwise, a = a1 ···am where ai ∈ A is irreducible. Using induction and applying Lemma 31.5, we conclude that a divides g(X).   
证据。命题是平凡的是一个是一个单位。否则，a=a1·····am，其中ai∈a是不可约的。利用归纳法和引理31.5，我们得出A除以G（x）。

We now show that Lemma 31.5 also applies to the case where a is an irreducible polynomial. This requires a little excursion involving the fraction field F of A.  
我们现在证明引理31.5也适用于a是不可约多项式的情况。这需要对a的分数场f作一点偏移。

Remark: If A is a UFD, it is possible to prove the uniqueness condition (2) for A[X] directly without using the fraction field of A, see Malliavin [116], Chapter 3.  
备注：如果a是ufd，则可以直接证明[x]的唯一性条件（2），而不使用a的分数字段，见Malliavin[116]，第3章。

Given an integral domain A, we can construct a field F such that every element of F is of the form a/b, where a,b ∈ A, b = 06 , using essentially the method for constructing the field Q of rational numbers from the ring Z of integers.  
给定一个积分域A，我们可以构造一个域F，使F的每一个元素都是A/B的形式，其中A，B∈A，B=06，本质上是用整数环Z构造有理数的域Q的方法。

Proposition 31.7. Let A be an integral domain.  
提案31.7.设A为积分域。

1. There is a field F and an injective ring homomorphism i: A → F such that every element of F is of the form i(a)i(b)−1, where a,b ∈ A, b = 06 .  
   有一个场f和一个内射环同态i:a→f，这样f的每一个元素都是形式i（a）i（b）−1，其中a，b∈a，b=06。
2. For every field K and every injective ring homomorphism h: A → K, there is a (unique) field homomorphism bh: F → K such that  
   对于每个场k和每个内射环同态h:a→k，都有一个（唯一的）场同态bh:f→k，这样

bh(i(a)i(b)−1) = h(a)h(b)−1  
b h（i（a）i（b）−1）=h（a）h（b）−1

for all a,b ∈ A, b = 06 .  
对于所有a，b∈a，b=06。

1. The field F in (1) is unique up to isomorphism.  
   （1）中的F场在同构上是唯一的。

Proof. (1) Consider the binary relation ' on A × (A − {0}) defined as follows:  
证据。（1）考虑a×（a−0）上的二进制关系，定义如下：

(a,b) ' (a0,b0) iff ab0 = a0b.  
（a，b）'（a0，b0）iff ab0=a0b.

It is easily seen that ' is an equivalence relation. Note that the fact that A is an integral domain is used to prove transitivity. The equivalence class of (a,b) is denoted by a/b. Clearly, (0,b) ' (0,1) for all b ∈ A, and we denote the class of (0,1) also by 0. The equivalence class a/1 of (a,1) is also denoted by a. We define addition and multiplication on A × (A − {0}) as follows:  
很容易看出，'是一个等价关系。注意，a是一个积分域的事实被用来证明传递性。（a，b）的等价类用a/b表示，显然，（0，b）'（0，1）对于所有b∈a，我们也用0表示（0，1）的等价类。（a，1）的等价类a/1也用a表示。我们在a×（a−0）上定义了加法和乘法，如下所示：

(a,b) + (a0,b0) = (ab0 + a0b,bb0), (a,b) · (a0,b0) = (aa0,bb0).  
（a，b）+（a0，b0）=（ab0+a0b，bb0），（a，b）·（a0，b0）=（aa0，bb0）。

It is easily verified that ' is congruential w.r.t. + and ·, which means that + and · are well-defined on equivalence classes modulo '. When a,b = 06 , the inverse of a/b is b/a, and it is easily verified that F is a field. The map i: A → F defined such that i(a) = a/1 is an injection of A into F and clearly  
很容易证明“是同余的w.r.t.+和·”，这意味着+和·在等价类模上定义得很好。当a，b=06时，a/b的倒数为b/a，很容易证明f是一个场。图i:a→f定义为i（a）=a/1是a注入f的过程，并且很清楚

.  
.

1. Given an injective ring homomorphism h: A → K into a field K,  
   给定一个内射环同态H:A→K进入一个场K，

iff ab0 = a0b,  
如果ab0=a0b，

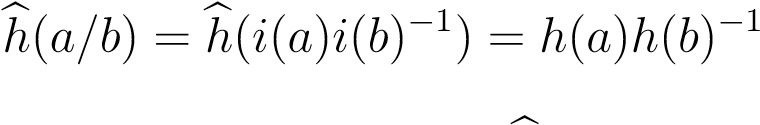
which implies that  
这意味着

h(a)h(b0) = h(a0)h(b),  
h（a）h（b0）=h（a0）h（b）

and since h is injective and b,b0 = 06 , we get  
因为h是注射剂，b，b0=06，我们得到

h(a)h(b)−1 = h(a0)h(b0)−1.  
H（a）H（b）−1=H（a0）H（b0）−1.

Thus, there is a map bh: F → K such that  
因此，有一个图bh:f→k，这样



for all a,b ∈ A, b = 06 , and it is easily checked that h is a field homomorphism. The map bh is clearly unique.  
对于所有的a，b∈a，b=06，很容易证明h是场同态。地图BH显然是独一无二的。

1. The uniqueness of F up to isomorphism follows from (2), and is left as an exercise.   
   F到同构的唯一性从（2）开始，留作练习。

The field F given by Proposition 31.7 is called the fraction field of A, and it is denoted by Frac(A).  
命题31.7给出的F场称为a的分数场，用frac（a）表示。

In particular, given an integral domain A, since A[X1,...,Xn] is also an integral domain, we can form the fraction field of the polynomial ring A[X1,...,Xn], denoted by F(X1,...,Xn), where F = Frac(A) is the fraction field of A. It is also called the field of rational functions over F, although the terminology is a bit misleading, since elements of F(X1,...,Xn) only define functions when the dominator is nonnull.  
特别是，给定一个积分域a，由于a[x1，…，xn]也是一个积分域，我们可以形成多项式环a[x1，…，xn]的分数域，用f（x1，…，xn）表示，其中f=frac（a）是a的分数域。它也被称为f上的有理函数域，尽管术语有点误导人，因为f（x1，…，xn）的元素仅在主宰符非空时定义函数。

We now have the following crucial lemma which shows that if a polynomial f(X) is reducible over F[X] where F is the fraction field of A, then f(X) is already reducible over A[X].  
我们现在有下面的关键引理，它表明如果多项式f（x）在f[x]上是可约的，其中f是a的分数域，那么f（x）已经在a[x]上是可约的。

Lemma 31.8. Let A be a UFD and let F be the fraction field of A. For any nonnull polynomial f(X) ∈ A[X] of degree m, if f(X) is not the product of two polynomials of degree strictly smaller than m, then f(X) is irreducible in F[X].  
引理31.8。设a为ufd，设f为a的分数域，对于任意非零多项式f（x）∈m的a[x]，如果f（x）不是严格小于m的两个多项式的乘积，则f（x）在f[x]中是不可约的。

Proof. Assume that f(X) is reducible in F[X] and that f(X) is neither null nor a unit. Then, f(X) = G(X)H(X),  
证据。假设f（x）在f[x]中是可约的，f（x）既不是空的也不是单位。那么，f（x）=g（x）h（x），

where G(X),H(X) ∈ F[X] are polynomials of degree p,q ≥ 1. Let a be the product of the denominators of the coefficients of G(X), and b the product of the denominators of the coefficients of H(X). Then, a,b = 06 , g1(X) = aG(X) ∈ A[X] has degree p ≥ 1, h1(X) = bH(X) ∈ A[X] has degree q ≥ 1, and  
式中g（x），h（x）∈f[x]是p，q≥1的多项式。设a为g（x）系数分母的乘积，b为h（x）系数分母的乘积。那么，a，b=06，g1（x）=ag（x）∈a[x]的阶数p≥1，h1（x）=bh（x）∈a[x]的阶数q≥1，并且

abf(X) = g1(X)h1(X).  
abf（x）=g1（x）h1（x）。

Let c = ab. If c is a unit, then f(X) is also reducible in A[X]. Otherwise, c = c1 ···cn, where ci ∈ A is irreducible. We now use induction on n to prove that  
设c=ab。如果c是一个单位，那么f（x）也可在a[x]中约化。否则，c=c1····cn，其中，ci∈a是不可约的。我们现在用n的归纳法来证明

f(X) = g(X)h(X),  
f（x）=g（x）h（x）

for some polynomials g(X) ∈ A[X] of degree p ≥ 1 and h(X) ∈ A[X] of degree q ≥ 1.  
对于某些多项式，p≥1的g（x）∈a[x]和q≥1的h（x）∈a[x]。

If n = 1, since c = c1 is irreducible, by Lemma 31.5, either c divides g1(X) or c divides h1(X). Say that c divides g1(X), the other case being similar. Then, g1(X) = cg(X) for some g(X) ∈ A[X] of degree p ≥ 1, and since A[X] is an integral ring, we get  
如果n=1，因为c=c1是不可约的，用引理31.5，c除以g1（x）或c除以h1（x）。假设c除以g1（x），另一种情况类似。然后，对于某些p≥1的g（x）∈a[x]的g1（x）=cg（x），由于a[x]是一个积分环，我们得到

f(X) = g(X)h1(X),  
f（x）=g（x）h1（x）

showing that f(X) is reducible in A[X]. If n > 1, since  
表示f（x）在a[x]中是可约化的。如果n>1，因为

c1 ···cnf(X) = g1(X)h1(X),  
c1···cnf（x）=g1（x）h1（x）

c1 divides g1(X)h1(X), and as above, either c1 divides g1(X) or c divides h1(X). In either case, we get c2 ···cnf(X) = g2(X)h2(X)  
c1除以g1（x）h1（x），如上所述，c1除以g1（x）或c除以h1（x）。在这两种情况下，我们得到c2···cnf（x）=g2（x）h2（x）

for some polynomials g2(X) ∈ A[X] of degree p ≥ 1 and h2(X) ∈ A[X] of degree q ≥ 1. By the induction hypothesis, we get  
对于某些多项式，p≥1的g2（x）∈a[x]，q≥1的h2（x）∈a[x]。通过归纳假设，我们得到

f(X) = g(X)h(X),  
f（x）=g（x）h（x）

for some polynomials g(X) ∈ A[X] of degree p ≥ 1 and h(X) ∈ A[X] of degree q ≥ 1, showing that f(X) is reducible in A[X].   
对于某些多项式，g（x）∈p≥1的a[x]和q≥1的h（x）∈a[x]，表明f（x）在a[x]中是可约的。

Finally, we can prove that (20) holds.  
最后，我们可以证明（20）成立。

Lemma 31.9. Let A be a UFD. Given any three nonnull polynomials f(X),g(X),h(X) ∈ A[X], if f(X) is irreducible and f(X) divides the product g(X)h(X), then either f(X) divides g(X) or f(X) divides h(X).  
引理31.9。让A成为UFD。给定任意三个非零多项式f（x），g（x），h（x）∈a[x]，如果f（x）是不可约的，f（x）将积g（x）h（x）除，则f（x）将g（x）或f（x）将h（x）除。

Proof. If f(X) has degree 0, then the result follows from Lemma 31.5. Thus, we may assume that the degree of f(X) is m ≥ 1. Let F be the fraction field of A. By Lemma 31.8, f(X) is also irreducible in F[X]. Since F[X] is a UFD (by Theorem 29.17), either f(X) divides g(X) or f(X) divides h(X), in F[X]. Assume that f(X) divides g(X), the other case being similar. Then, g(X) = f(X)G(X),  
证据。如果f（x）的阶数为0，那么结果来自引理31.5。因此，我们可以假设f（x）的阶数是m≥1。设f为a的分数域。由引理31.8，f（x）在f[x]中也是不可约的。因为f[x]是一个ufd（根据定理29.17），f（x）除以g（x）或f（x）除以h（x），在f[x]中。假设f（x）除以g（x），另一种情况类似。那么，g（x）=f（x）g（x），

for some G(X) ∈ F[X]. If a is the product the denominators of the coefficients of G, we have ag(X) = q1(X)f(X),  
对于某些g（x）∈f[x]。如果a是g系数分母的乘积，我们得到ag（x）=q1（x）f（x），

where q1(X) = aG(X) ∈ A[X]. If a is a unit, we see that f(X) divides g(X). Otherwise, a = a1 ···an, where ai ∈ A is irreducible. We prove by induction on n that  
式中q1（x）=ag（x）∈a[x]。如果a是一个单位，我们看到f（x）除以g（x）。否则，a=a1··································我们通过归纳法证明了

g(X) = q(X)f(X)  
g（x）=q（x）f（x）

for some q(X) ∈ A[X].  
对于某些q（x）∈a[x]。

If n = 1, since f(X) is irreducible and of degree m ≥ 1 and  
如果n=1，因为f（x）是不可约的，且m≥1，并且

a1g(X) = q1(X)f(X),  
a1g（x）=q1（x）f（x）

by Lemma 31.5, a1 divides q1(X). Thus, q1(X) = a1q(X) where q(X) ∈ A[X]. Since A[X] is an integral domain, we get g(X) = q(X)f(X),  
用引理31.5，a1除以q1（x）。因此，q1（x）=a1q（x），其中q（x）∈a[x]。因为a[x]是一个积分域，我们得到g（x）=q（x）f（x），

and f(X) divides g(X). If n > 1, from  
f（x）除以g（x）。如果n>1，从

a1 ···ang(X) = q1(X)f(X),  
a1···ang（x）=q1（x）f（x），

we note that a1 divides q1(X)f(X), and as in the previous case, a1 divides q1(X). Thus, q1(X) = a1q2(X) where q2(X) ∈ A[X], and we get  
我们注意到a1除以q1（x）f（x），和前面的例子一样，a1除以q1（x）。因此，q1（x）=a1q2（x），其中q2（x）=a[x]，我们得到

a2 ···ang(X) = q2(X)f(X).  
a2···ang（x）=q2（x）f（x）。

By the induction hypothesis, we get  
通过归纳假设，我们得到

g(X) = q(X)f(X)  
g（x）=q（x）f（x）

for some q(X) ∈ A[X], and f(X) divides g(X).   
对于一些q（x）∈a[x]，f（x）除以g（x）。

We finally obtain the fact that A[X] is a UFD when A is.  
我们最终得出这样一个事实：当a是时，a[x]是一个UFD。

Theorem 31.10. If A is a UFD then the polynomial ring A[X] is also a UFD.  
定理31.10。如果a是ufd，那么多项式环a[x]也是ufd。

Proof. As we said earlier, the factorization property (1) is easier to prove than uniqueness. Assume that f(X) has degree m and let fm be the coefficient of Xm in f(X). Either fm is a unit or it is the product of n ≥ 1 irreducible elements. If fm is a unit we set n = 0. We proceed by induction on the pair (m,n), using the well-founded ordering on pairs, i.e.,  
证据。如前所述，因子分解属性（1）比唯一性更容易证明。假设f（x）的阶数为m，fm是f（x）中xm的系数。fm要么是一个单位，要么是n≥1个不可约元素的乘积。如果fm是一个单位，我们设置n=0。我们通过对（m，n）的归纳，利用对上的有根据的排序，即：

(m,n) ≤ (m0,n0)  
（m，n）≤（m0，n0）

iff either m < m0, or m = m0 and n < n0. If f(X) is a nonnull polynomial of degree 0 which is not a unit, then f(X) ∈ A, and f(X) = fm = a1 ···an for some irreducible ai ∈ A, since A is a UFD. This proves the base case.  
如果m<m0，或m=m0和n<n0。如果f（x）是0次的非零多项式，它不是一个单位，那么f（x）∈a，f（x）=fm=a1····································这证明了基本情况。

If f(X) has degree m > 0 and f(X) is reducible, then  
如果f（x）的度数m>0且f（x）是可约化的，则

f(X) = g(X)h(X),  
f（x）=g（x）h（x）

where g(X) and h(X) have degree p,q ≤ m and are not units. There are two cases.  
其中g（x）和h（x）具有p级，q≤m且不是单位。有两种情况。

1. fm is a unit (so n = 0).  
   fm是一个单位（因此n=0）。

If so, since fm = gphq (where gp is the coefficient of Xp in g(X) and hq is the coefficient of Xq in h(X)), then gp and hq are both units. We claim that p,q ≥ 1. Otherwise, p = 0 or q = 0, but then either g(X) = g0 is a unit or h(X) = h0 is a unit, a contradiction.  
如果是这样，因为fm=gp hq（其中gp是xp的系数，单位为g（x），hq是xq的系数，单位为h（x）），那么gp和hq都是单位。我们声称p，q≥1。否则，p=0或q=0，但g（x）=g0是一个单位，或h（x）=h0是一个单位，一个矛盾。

Now, since m = p + q and p,q ≥ 1, we have p,q < m so (p,0) < (m,0) and (q,0) < (m,0), and by the induction hypothesis, both g(X) and h(X) can be written as products of irreducible factors, thus so can f(X).  
现在，由于m=p+q和p，q≥1，我们得到p，q<m so（p，0）<（m，0）和（q，0）<（m，0），根据诱导假设，g（x）和h（x）都可以写成不可约因子的乘积，因此f（x）也可以写成。

1. fm is not a unit, say fm = a1 ···an where a1,...,an are irreducible and n ≥ 1.  
   fm不是一个单位，比如fm=a1··································
   1. If p,q < m, then (p,n1) < (m,n) and (q,n2) < (m,n) where n1 is the number of irreducible factors of gp or n1 = 0 if gp is irreducible, and similarly n2 is the number of irreducible factors of hp or n2 = 0 if hp is irreducible (note that n1,n2 ≤ n and it is possible that n1 = n if hq is irreducible or n2 = n if gp is irreducible). By the induction hypothesis, g(X) and h(X) can be written as products of irreducible polynomials, thus so can f(X).  
      如果p，q<m，则（p，n1）<（m，n）和（q，n2）<（m，n），其中n1是gp的不可约因子数，如果gp不可约，n1=0；同样，n2是hp的不可约因子数，如果hp不可约，n2=0（注意n1，n2≤n，如果hq不可约，n1=n是可能的如果gp不可约，e或n2=n）。根据归纳假设，G（x）和H（x）可以写成不可约多项式的乘积，F（x）也可以写成。
   2. If p = 0 and q = m, then g(X) = gp and by hypothesis gp is not a unit. Since fm = a1 ···an = gphq and gp is not a unit, either hq is not a unit in which case, by the uniqueness of the number of irreducible elements in the decomposition of fm (since A is a UFD), hq is the product of n2 < n irreducible elements, or n2 = 0 if hq is irreducible. Since n ≥ 1, this implies that (m,n2) < (m,n), and by the induction hypothesis h(X) can be written as products of irreducible polynomials. Since gp ∈ A is not a unit, it can also be written as a product of irreducible elements, thus so can f(X).  
      如果p=0，q=m，那么g（x）=gp，假设gp不是一个单位。由于fm=a1····a n=gp hq和gp不是一个单位，在这种情况下，根据fm分解中不可约元素数量的唯一性（因为a是一个ufd），hq是n2<n不可约元素的乘积，或n2=0（如果hq是不可约元素）。由于n≥1，这意味着（m，n2）<（m，n），并且通过归纳假设h（x）可以写成不可约多项式的乘积。由于gp∈a不是一个单位，它也可以写成不可约元素的乘积，因此f（x）也可以。

The case where p = m and q = 0 is similar to the previous case.  
其中p=m和q=0的情况与前一种情况类似。

Property (20) follows by Lemma 31.9. By Proposition 31.2, A[X] is a UFD.   
属性（20）后接引理31.9。根据提案31.2，a[x]是一个UFD。

As a corollary of Theorem 31.10 and using induction, we note that for any field K, the polynomial ring K[X1,...,Xn] is a UFD.  
作为定理31.10的推论，使用归纳法，我们注意到对于任何场k，多项式环k[x1，…，xn]都是一个ufd。

For the sake of completeness, we shall prove that every PID is a UFD. First, we review the notion of gcd and the characterization of gcd’s in a PID.  
为了完整起见，我们将证明每个PID都是一个UFD。首先，我们回顾了GCD的概念和PID中GCD的特征。

Given an integral domain A, for any two elements a,b ∈ A, a,b = 06 , we say that d ∈ A  
给定一个积分域a，对于任意两个元素a，b∈a，a，b=06，我们称d∈a

(d = 0)6 is a greatest common divisor (gcd) of a and b if  
（d=0）6是a和b的最大公约数（gcd），如果

1. d divides both a and b.  
   D把A和B分开。
2. For any h ∈ A (h = 0)6 , if h divides both a and b, then h divides d.  
   对于任意h∈a（h=0）6，如果h同时除以a和b，则h又除以d。

We also say that a and b are relatively prime if 1 is a gcd of a and b.  
如果1是a和b的gcd，我们也说a和b是相对质数。

Note that a and b are relatively prime iff every gcd of a and b is a unit. If A is a PID, then gcd’s are characterized as follows.  
注意，A和B是相对主要的iff，A和B的每个gcd都是一个单位。如果A是PID，则GCD的特征如下。

Proposition 31.11. Let A be a PID.  
提案31.11.让A成为PID。

1. For any a,b,d ∈ A (a,b,d = 0)6 , d is a gcd of a and b iff  
   对于任何a，b，d∈a（a，b，d=0）6，d是a和b iff的gcd。

(d) = (a,b) = (a) + (b),  
（d）=（a，b）=（a）+（b），

i.e., d generates the principal ideal generated by a and b.  
即，D生成A和B生成的主理想。

1. (Bezout identity) Two nonnull elements a,b ∈ A are relatively prime iff there are some x,y ∈ A such that  
   （Bezout恒等式）两个非零元素a，b∈a是相对质数iff有一些x，y∈a这样

ax + by = 1.  
ax+by=1.

Proof. (1) Recall that the ideal generated by a and b is the set  
证据。（1）回想A和B产生的理想是集合

(a) + (b) = aA + bA = {ax + by | x,y ∈ A}.  
（a）+（b）=aa+ba=a x+x，y∈a。

First, assume that d is a gcd of a and b. If so, a ∈ Ad, b ∈ Ad, and thus, (a) ⊆ (d) and (b) ⊆ (d), so that  
首先，假设d是a和b的gcd，如果是，a∈ad，b∈ad，因此，（a）（d）和（b）（d），这样

(a) + (b) ⊆ (d).  
（a）+（b）（d）。

Since A is a PID, there is some t ∈ A, t = 06 , such that  
因为a是pid，所以有一些t∈a，t=06，这样

(a) + (b) = (t),  
（a）+（b）=（t）

and thus, (a) ⊆ (t) and (b) ⊆ (t), which means that t divides both a and b. Since d is a gcd of a and b, t must divide d. But then,  
因此，（a）（t）和（b）（t），这意味着t将a和b分开。由于d是a和b的gcd，t必须将d分开。但是，

(d) ⊆ (t) = (a) + (b),  
（d）（t）=（a）+（b），

and thus, (d) = (a) + (b).  
因此，（d）=（a）+（b）。

Assume now that  
现在假设

(d) = (a) + (b) = (a,b).  
（d）=（a）+（b）=（a，b）。

Since (a) ⊆ (d) and (b) ⊆ (d), d divides both a and b. Assume that t divides both a and b, so that (a) ⊆ (t) and (b) ⊆ (t). Then,  
由于（a）（d）和（b）（d），d将a和b分开。假设t将a和b分开，这样（a）（t）和（b）（t）。然后，

(d) = (a) + (b) ⊆ (t),  
（d）=（a）+（b）（t）、

which means that t divides d, and d is indeed a gcd of a and b. (2) By (1), if a and b are relatively prime, then  
这意味着t除以d，d实际上是a和b的gcd（2）乘以（1），如果a和b是相对质数，那么

(1) = (a) + (b),  
（1）=（a）+（b），

which yields the result. Conversely, if  
结果就是这样。相反，如果

ax + by = 1,  
ax+by=1，

then  
然后

(1) = (a) + (b),  
（1）=（a）+（b），

and 1 is a gcd of a and b.   
1是A和B的GCD。

Given two nonnull elements a,b ∈ A, if a is an irreducible element and a does not divide b, then a and b are relatively prime. Indeed, if d is not a unit and d divides both a and b, then a = dp and b = dq where p must be a unit, so that  
给定两个非零元素a，b∈a，如果a是不可约元素，a不除b，则a和b是相对质数。实际上，如果d不是一个单位，d将a和b分开，那么a=d p和b=dq，其中p必须是一个单位，因此

b = ap−1q,  
B=AP-1Q，

and a divides b, a contradiction.  
A分B，矛盾。

Theorem 31.12. Let A be ring. If A is a PID, then A is a UFD.  
定理31.12。给我一个电话。如果a是pid，那么a是ufd。

Proof. First, we prove that every nonnull element that is a not a unit can be factored as a product of irreducible elements. Let S be the set of nontrivial principal ideals (a) such that a = 06 is not a unit and cannot be factored as a product of irreducible elements (in particular, a is not irreducible). Assume that S is nonempty. We claim that every ascending chain in  
证据。首先，我们证明了每一个非空元素都是不可约元素的乘积。让我们做一组非平凡主理想（a），这样a=06不是一个单位，不能作为不可约元素（特别是a不是不可约元素）的乘积来考虑。假设s不是空的。我们声称每一条上升链

S is finite. Otherwise, consider an infinite ascending chain  
S是有限的。否则，考虑一个无限的上升链。

(a1) ⊂ (a2) ⊂ ··· ⊂ (an) ⊂ ··· .  
（a1）（a2）······（an）····。

It is immediately verified that  
立即证实

[  
[

(an) n≥1  
（a）n≥1

is an ideal in A. Since A is a PID,  
是A中的理想。因为A是PID，

[  
[

(an) = (a)  
（a）=（a）

n≥1  
n＝1

for some a ∈ A. However, there must be some n such that a ∈ (an), and thus,  
但是，对于某些a∈a，必须有一些n，这样a∈（an），因此，

(an) ⊆ (a) ⊆ (an),  
（an）（a）（an）、

and the chain stabilizes at (an).  
链稳定在（a）。

As a consequence, there are maximal ideals in S. Let (a) be a maximal ideal in S. Then, for any ideal (d) such that  
因此，在S中有最大理想。让（a）在S中是最大理想。然后，对于任何理想（d），这样

(a) ⊂ (d) and (a) = (6 d),  
（a）（d）和（a）=6 d，

we must have d /∈ S, since otherwise (a) would not be a maximal ideal in S. Observe that a is not irreducible, since (a) ∈ S, and thus,  
我们必须有d/∈s，否则（a）将不是s中的最大理想，观察a不是不可约的，因为（a）∈s，因此，

a = bc  
公元前

for some b,c ∈ A, where neither b nor c is a unit. Then,  
对于某些b，c∈a，其中b和c都不是一个单位。然后，

(a) ⊆ (b) and (a) ⊆ (c).  
（a）（b）和（a）（c）。

If (a) = (b), then b = au for some u ∈ A, and then  
如果（a）=（b），那么对于某些u∈a，b=au，然后

a = auc,  
A=AUC，

so that  
以便

1 = uc,  
1=UC，

since A is an integral domain, and thus, c is a unit, a contradiction. Thus, (a) = (6 b), and similarly, (a) = (6 c). But then, by a previous observation b /∈ S and c /∈ S, and since a and b are not units, both b and c factor as products of irreducible elements and so does a = bc, a contradiction. This implies that S = ∅, so every nonnull element that is a not a unit can be factored as a product of irreducible elements  
因为A是一个积分域，所以C是一个单位，一个矛盾。因此，（a）=（6 b）和类似地，（a）=（6 c）。但是，根据前面的观察b/∈s和c/∈s，由于a和b不是单位，b和c因子都是不可约元素的产物，a=bc也是一个矛盾。这意味着S=∅，因此，非单位的每个非空元素都可以作为不可约元素的乘积进行分解。

To prove the uniqueness of factorizations, we use Proposition 31.2. Assume that a is irreducible and that a divides bc. If a does not divide b, by a previous remark, a and b are relatively prime, and by Proposition 31.11, there are some x,y ∈ A such that  
为了证明因式分解的唯一性，我们使用命题31.2。假设A是不可约的，而BC是分的。如果a不除以b，通过前面的一句话，a和b是相对质数，并且通过命题31.11，有一些x，y∈a这样

ax + by = 1.  
ax+by=1.

Thus,  
因此，

acx + bcy = c,  
acx+bcy=c，

and since a divides bc, we see that a must divide c, as desired.   
由于a除以bc，我们看到a必须按需要除以c。

Thus, we get another justification of the fact that Z is a UFD and that if K is a field, then K[X] is a UFD.  
因此，我们得到了另一个理由，证明Z是一个UFD，如果K是一个场，那么K[X]是一个UFD。

It should also be noted that in a UFD, gcd’s of nonnull elements always exist. Indeed, this is trivial if a or b is a unit, and otherwise, we can write  
还应该注意的是，在UFD中，总是存在非空元素的gcd。事实上，如果a或b是一个单位，这是微不足道的，否则，我们可以写

a = p1 ···pm and b = q1 ···qn  
a=p1···pm，b=q1···qn

where pi,qj ∈ A are irreducible, and the product of the common factors of a and b is a gcd of a and b (it is 1 is there are no common factors).  
式中pi，qj∈a是不可约的，a和b的公因子的乘积是a和b的gcd（即1是没有公因子）。

We conclude this section on UFD’s by proving a proposition characterizing when a UFD is a PID. The proof is nontrivial and makes use of Zorn’s lemma (several times).  
我们通过证明当一个UFD是PID时的一个命题来结束这个关于UFD的章节。证明是不平凡的，并利用了佐恩的引理（多次）。

Proposition 31.13. Let A be a ring that is a UFD, and not a field. Then, A is a PID iff every nonzero prime ideal is maximal.  
提案31.13。让A成为一个UFD的环，而不是一个场。然后，a是PID iff，每个非零素数理想都是最大的。

Proof. Assume that A is a PID that is not a field. Consider any nonzero prime ideal, (p), and pick any proper ideal A in A such that  
证据。假设a是一个PID，而不是一个字段。考虑任何非零素数理想，（p），并在a中选择任何合适的理想a

(p) ⊆ A.  
（p）a.

Since A is a PID, the ideal A is a principal ideal, so A = (q), and since A is a proper nonzero ideal, q = 06 and q is not a unit. Since  
因为a是一个PID，理想a是一个主理想，所以a=（q），因为a是一个适当的非零理想，q=06，q不是一个单位。自从

(p) ⊆ (q),  
（p）（q）

q divides p, and we have p = qp1 for some p1 ∈ A. Now, by Proposition 31.1, since p = 06 and (p) is a prime ideal, p is irreducible. But then, since p = qp1 and p is irreducible, p1 must be a unit (since q is not a unit), which implies that  
q除以p，对于某些p1∈a，我们得到p=qp1。现在，根据命题31.1，由于p=06和（p）是素理想，p是不可约的。但是，既然p=q p1和p是不可约的，p1必须是一个单位（因为q不是一个单位），这意味着

(p) = (q);  
（p）=（q）；

that is, (p) is a maximal ideal.  
也就是说，（p）是最大理想。

Conversely, let us assume that every nonzero prime ideal is maximal. First, we prove that every prime ideal is principal. This is obvious for (0). If A is a nonzero prime ideal, then, by hypothesis, it is maximal. Since A = (0)6 , there is some nonzero element a ∈ A. Since A is maximal, a is not a unit, and since A is a UFD, there is a factorization a = a1 ···an of a into irreducible elements. Since A is prime, we have ai ∈ A for some i. Now, by Proposition 31.3, since ai is irreducible, the ideal (ai) is prime, and so, by hypothesis, (ai) is maximal. Since (ai) ⊆ A and (ai) is maximal, we get A = (ai).  
相反，假设每个非零素数理想都是最大的。首先，我们证明了每一个素理想都是主理想。这对于（0）是显而易见的。如果A是非零素数理想，那么根据假设，它是最大的。由于a=（0）6，有一些非零元素a∈a，由于a是极大的，a不是一个单位，由于a是一个ufd，所以a的因式分解a=a1·······························因为a是素数，我们有一些i的ai∈a。现在，根据命题31.3，由于ai是不可约的，理想（ai）是素数，因此，根据假设，（ai）是最大的。因为（ai）a和（ai）是最大的，所以我们得到a=（ai）。

Next, assume that A is not a PID. Define the set, F, by  
接下来，假设a不是PID。定义集合f

F = {A | A ⊆ A, A is not a principal ideal}.  
F=A A A不是主要理想。

Since A is not a PID, the set F is nonempty. Also, the reader will easily check that every chain in FSis bounded inAi is an ideal which is not principal, soF. Indeed, for any chain (ASi)ii∈∈II Aof ideals ini ∈ F. Then, by Zorn’s lemmaF it is not hard to verify that i∈I  
因为a不是PID，所以集合f是非空的。此外，读者还可以很容易地检查到fsis中的每个链都有界，inai是一个理想的，而不是主体，sof。实际上，对于任何链（asi）i i∈ii aof理想ini∈f，那么，通过zorn的lemmaf，不难证明i∈i

(Lemma B.1), the set F has some maximal element, A. Clearly, A = (0)6 is a proper ideal (since A = (1)), and A is not prime, since we just showed that prime ideals are principal. Then, by Theorem B.3, there is some maximal ideal, M, so that A ⊂ M. However, a maximal ideal is prime, and we have shown that a prime ideal is principal. Thus,  
（引理b.1），集合f有一些极大元素a。显然，a=（0）6是一个合适的理想（因为a=（1）），a不是素数，因为我们刚刚证明了素数理想是主的。然后，根据定理B.3，有一些极大理想m，因此a m。然而，极大理想是素数，我们证明了素数理想是主的。因此，

A ⊆ (p),  
A（P）、

for some p ∈ A that is not a unit. Moreover, by Proposition 31.1, the element p is irreducible. Define B = {a ∈ A | pa ∈ A}.  
对于某些p∈a，它不是一个单位。此外，根据命题31.1，元素P是不可约的。定义b=a∈a pa∈a。

Clearly, A = pB, B = (0)6 , A ⊆ B, and B is a proper ideal. We claim that A 6= B. Indeed, if A = B were true, then we would have A = pB = B, but this is impossible since p is irreducible, A is a UFD, and B = (0) (6 we get B = pmB for all m, and every element of B would be a multiple of pm for arbitrarily large m, contradicting the fact that A is a UFD). Thus, we have A ⊂ B, and since A is a maximal element of F, we must have B ∈ F/ . However, B ∈ F/ means that B is a principal ideal, and thus, A = pB is also a principal ideal, a contradiction.   
显然，a=pb，b=（0）6，a b和b是一个合适的理想。我们声称a 6=b。事实上，如果a=b是真的，那么我们将得到a=pb=b，但这是不可能的，因为p是不可约的，a是一个ufd，b=（0）（6我们得到b=pmb对于所有m，b的每个元素对于任意大的m都是pm的倍数，这与a是一个ufd的事实相矛盾。因此，我们有a b，由于a是f的极大元素，我们必须有b∈f/。然而，b∈f/意味着b是一个主理想，因此，a=pb也是一个主理想，一个矛盾。

Observe that the above proof shows that Proposition 31.13 also holds under the assumption that every prime ideal is principal.  
观察到上述证明表明，命题31.13也在每一个素数理想都是主的假设下成立。

## 31.2 The Chinese Remainder Theorem 31.2中国余数定理

In this section, which is a bit of an interlude, we prove a basic result about quotients of commutative rings by products of ideals that are pairwise relatively prime. This result has applications in number theory and in the structure theorem for finitely generated modules over a PID, which will be presented later.  
在这一部分，这是一个有点插曲的部分，我们证明了一个关于交换环商的基本结果，它是理想的两个相对素数的乘积。这个结果在数论和PID上有限生成模块的结构定理中有应用，稍后将介绍。

Given two ideals a and b of a ring A, we define the ideal ab as the set of all finite sums of the form a1b1 + ··· + akbk, ai ∈ a, bi ∈ b.  
给出了A环的两个理想a和b，我们将理想ab定义为a1b1+······+akbk，ai∈a，bi∈b形式的所有有限和的集合。

The reader should check that ab is indeed an ideal. Observe that ab ⊆ a and ab ⊆ b, so that  
读者应该检查AB确实是一个理想。观察AB A和AB B，以便

ab ⊆ a ∩ b.  
AB A B.

In general equality does not hold. However if  
一般来说，平等不成立。但是，如果

a + b = A,  
A+B=A，

then we have  
然后我们有了

ab = a ∩ b.  
AB=A B.

This is because there is some a ∈ a and some b ∈ b such that  
这是因为有一些a∈a和一些b∈b这样

a + b = 1,  
A+B=1，

so for every x ∈ a ∩ b, we have x = xa + xb,  
所以对于每一个x∈a b，我们有x=xa+xb，

which shows that x ∈ ab. Ideals a and b of A that satisfy the condition a + b = A are sometimes said to be comaximal.  
这表明满足a+b=a条件的a的x∈ab理想a和b有时被称为共轴。

We define the homomorphism ϕ: A → A/a × A/b by  
我们定义同态，用

ϕ(x) = (xa,xb),  
⑨（x）=（xa，xb）

where xa is the equivalence class of x modulo a (resp. xb is the equivalence class of x modulo b). Recall that the ideal a defines the equivalence relation ≡a on A given by  
其中x a是x模a（resp.x b是x模b的等价类）。回想一下，理想A定义了给定的

x ≡a y iff x − y ∈ a,  
x a y如果x−y∈a，

and that A/a is the quotient ring of equivalence classes xa, where x ∈ A, and similarly for A/b. Sometimes, we also write x ≡ y (mod a) for x ≡a y.  
a/a是等价类xa的商环，其中x∈a，类似于a/b，有时我们也写x y（mod a）表示x a y。

Clearly, the kernel of the homomorphism ϕ is a ∩ b. If we assume that a + b = A, then Ker(ϕ) = a∩b = ab, and because ϕ has a constant value on the equivalence classes modulo ab, the map ϕ induces a quotient homomorphism  
显然，同态的核心是a b。如果我们假设a+b=a，那么ker（\_）=a b=ab，并且由于\_在等价类模ab上有一个常量值，映射\_会产生商同态。

θ: A/ab → A/a × A/b.  
θ：A/AB→A/A×A/B。

Because Ker(ϕ) = ab, the homomorphism θ is injective. The Chinese Remainder Theorem says that θ is an isomorphism.  
因为Ker（η）=ab，同态θ是内射的。中国剩余定理认为θ是同构的。

Theorem 31.14. Given a commutative ring A, let a and b be any two ideals of A such that a + b = A. Then, the homomorphism θ: A/ab → A/a × A/b is an isomorphism.  
定理31.14。给定一个交换环a，让a和b是a的任意两个理想，这样a+b=a，那么同态θ：a/ab→a/a×a/b是同态。

Proof. We already showed that θ is injective, so we need to prove that θ is surjective. We need to prove that for any y,z ∈ A, there is some x ∈ A such that  
证据。我们已经证明θ是内射的，所以我们需要证明θ是射的。我们需要证明，对于任何y，z∈a，有一些x∈a，这样

x ≡ y (mod a) x ≡ z (mod b).  
X Y（A型）X Z（B型）。

Since a + b = A, there exist some a ∈ a and some b ∈ b such that  
由于a+b=a，存在一些a∈a和一些b∈b，因此

a + b = 1.  
A+B=1.

If we let  
如果我们让

x = az + by,  
x=az+x，

then we have x ≡a by ≡a (1 − a)y ≡a y − ay ≡a y,  
然后我们得到x a x a（1−a）y a y y a y，

and similarly x ≡b az ≡b (1 − b)z ≡b z − bz ≡b z,  
同样地，x b az b（1−b）z b z−bz b z，

which shows that x = az + by works.   
这表明x=az+是通过工程得出的。

Theorem 31.14 can be generalized to any (finite) number of ideals.  
定理31.14可以推广到任何（有限）个理想。

Theorem 31.15. (Chinese Remainder Theorem) Given a commutative ring A, let a1,...,an be any n ≥ 2 ideals of A such that ai + aj = A for all i =6 j. Then, the homomorphism θ: A/a1 ···an → A/a1 × ··· × A/an is an isomorphism.  
定理31.15。（中国剩余定理）给定交换环a，让a1，…，a是a的任意n≥2个理想，使得aI+a j=a，对于所有i=6j，同态θ：a/a1···························a/an是同态。

Proof. The map θ: A/a1 ∩ ··· ∩ an → A/a1 × ··· × A/an is induced by the homomorphism ϕ: A → A/a1 × ··· × A/an given by  
证据。图θ：a/a1····an→a/a1×··················a/an由下式给出

ϕ(x) = (xa1,...,xan).  
⑨（x）=（xa1，…，xan）。

Clearly, Ker(ϕ) = a1 ∩ ··· ∩ an, so θ is well-defined and injective. We need to prove that  
很明显，Ker（η）=A1···，因此θ定义明确且具有内射性。我们需要证明

a1 ∩ ··· ∩ an = a1 ···an  
a1····an=a1····an

and that θ is surjective. We proceed by induction. The case n = 2 is Theorem 31.14. By induction, assume that a2 ∩ ··· ∩ an = a2 ···an.  
θ是主观的。我们采用归纳法。案例n=2是定理31.14。通过归纳，假设a2·····an=a2·························

We claim that a1 + a2 ···an = A.  
我们认为a1+a2···an=a。

Indeed, since a1 + ai = A for i = 2,...,n, there exist some ai ∈ a1 and some bi ∈ ai such that  
实际上，由于a1+a i=a，对于i=2，…，n，存在一些ai∈a1和一些bi∈ai，因此

ai + bi = 1, i = 2,...,n,  
ai+bi=1，i=2，…，n，

and by multiplying these equations, we get  
通过乘以这些方程，我们得到

a + b2 ···bn = 1,  
a+b2···bn=1，

where a is a sum of terms each containing some aj as a factor, so a ∈ a1 and b2 ···bn ∈ a2 ···an, which shows that a1 + a2 ···an = A,  
式中a是一个项的总和，每个项都包含一些aj作为因子，因此a∈a1和b2·····································

as claimed. It follows that  
如要求。接下来是

a1 ∩ a2 ∩ ··· ∩ an = a1 ∩ (a2 ···an) = a1a2 ···an.  
a1 a2·····an=a1（a2·····an）=a1a2·······an。

Let us now prove that θ is surjective by induction. The case n = 2 is Theorem 31.14. Let x1,...,xn be any n ≥ 3 elements of A. First, applying Theorem 31.14 to a1 and a2 ···an, we can find y1 ∈ A such that  
现在让我们用归纳法证明θ是主观的。案例n=2是定理31.14。假设x1，…，xn是a的任意n≥3个元素。首先，将定理31.14应用于a1和a2····an，我们可以找到y1∈a，这样

y1 ≡ 1 (mod a1) y1 ≡ 0 (mod a2 ···an).  
Y1 1（A1型）Y1 0（A2型···AN型）。

By the induction hypothesis, we can find y2,...,yn ∈ A such that for all i,j with 2 ≤ i,j ≤ n,  
通过归纳假设，我们可以找到y2，…，yn∈a，这样对于所有i，j，2≤i，j≤n，

yi ≡ 1 (mod ai) yi ≡ 0 (mod aj), j =6 i.  
Yi 1（mod ai）Yi 0（mod aj），j=6 i。

We claim that  
我们声称

x = x1y1 + x2y2 + ··· + xnyn  
X=x1y1+x2y2+····+xnyn

works. Indeed, using the above congruences, for i = 2,...,n, we get  
作品。实际上，使用上面的一致性，对于i=2，…，n，我们得到

x ≡ x1y1 + xi (mod ai), (∗)  
xx1y1+Xi（mod ai），（\*）

but since a2 ···an ⊆ ai for i = 2,...,n and y1 ≡ 0 (mod a2 ···an), we have  
但是，由于a2············································

x1y1 ≡ 0 (mod ai), i = 2,...,n  
x1y1 0（mod ai），i=2，…，N

and equation (∗) reduces to  
方程式（）减至

x ≡ xi (mod ai), i = 2,...,n.  
Xi XI（mod ai），i＝2，…，n。

For i = 1, we get  
当i=1时，我们得到

x ≡ x1 (mod a1),  
x x1（A1型）

therefore  
因此

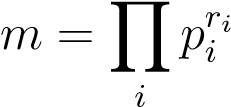
x ≡ xi (mod ai), i = 1,...,n.  
Xi XI（mod ai），i＝1，…，n。

proving surjectivity.   
证明主观臆断。

The classical version of the Chinese Remainder Theorem is the case where A = Z and where the ideals ai are defined by n pairwise relatively prime integers m1,...,mn. By the Bezout identity, since mi and mj are relatively prime whenever i =6 j, there exist some ui,uj ∈ Z such that uimi + ujmj = 1, and so miZ + mjZ = Z. In this case, we get an isomorphism  
中国剩余定理的经典版本是A=Z，理想ai由n对相对素数m1，…，mn定义的情况。根据Bezout恒等式，当i=6j时，由于mi和mj是相对素数，因此存在一些ui，uj∈z，使得uimi+ujmj=1，因此miz+mjz=z，在这种情况下，我们得到一个同构

.  
.

In particular, if m is an integer greater than 1 and  
特别是，如果m是大于1的整数，并且



is its factorization into prime factors, then  
那么它的因式分解成素因子吗？

Z/mZ ≈ YZ/priiZ.  
z/mz≈yz/priiz.

i  
我

In the previous situation where the integers m1,...,mn are pairwise relatively prime, if we write m = m1 ···mn and m0i = m/mi for i = 1...,n, then mi and m0i are relatively prime, and so has an inverse modulo mi. If ti is such an inverse, so that  
在前面的情形中，整数m1，…，m n是成对的相对素数，如果我们写m=m1····mn和m0i=m/m i表示i=1…，n，那么mi和m0i是相对素数，因此有一个反模mi。如果ti是反的，那么

,  
，

then it is not hard to show that for any a1,...,an ∈ Z,  
那么不难证明，对于任何a1，…，an∈z，



satisfies the congruences  
满足一致性

x ≡ ai (mod mi), i = 1,...,n.  
x ai（mod mi），i=1，…，n.

Theorem 31.15 can be used to characterize rings isomorphic to finite products of quotient rings. Such rings play a role in the structure theorem for torsion modules over a PID.  
定理31.15可用于描述与商环的有限积同构的环。这样的环在PID上扭转模块的结构定理中起作用。

Given n rings A1,...,An, recall that the product ring A = A1 × ··· × An is the ring in which addition and multiplication are defined componenwise. That is,  
对于n个环a1，…，an，回想一下，乘积环a=a1×··············×an是定义加法和乘法的环。也就是说，

(a1,...,an) + (b1,...,bn) = (a1 + b1,...,an + bn) (a1,...,an) · (b1,...,bn) = (a1b1,...,anbn).  
（a1，…，an）+（b1，…，bn）=（a1+b1，…，an+bn）（a1，…，an）·（b1，…，bn）=（a1b1，…，anbn）。

The additive identity is 0A = (0,...,0) and the multiplicative identity is 1A = (1,...,1). Then, for i = 1,...,n, we can define the element ei ∈ A as follows:  
加法恒等式为0a=（0，…，0），乘法恒等式为1a=（1，…，1）。然后，对于i=1，…，n，我们可以定义元素ei∈a如下：

ei = (0,...,0,1,0,...,0),  
ei=（0，…，0,1,0，…，0），

where the 1 occurs in position i. Observe that the following properties hold for all i,j = 1,...,n:  
当1出现在位置i时，观察以下属性适用于所有i，j=1，…，n：

e2i = ei  
E2i=Ei

eiej = 0, i =6 j e1 + ··· + en = 1A.  
eiej=0，i=6J e1+·····+en=1a。

Also, for any element a = (a1,...,an) ∈ A, we have  
另外，对于任何元素a=（a1，…，an）∈a，我们有

eia = (0,...,0,ai,0,...,0) = pri(a),  
EIA=（0，…，0，ai，0，…，0）=pri（a）

where pri is the projection of A onto Ai. As a consequence  
其中pri是a对ai的投影。因此

Ker(pri) = (1A − ei)A.  
ker（pri）=（1a−ei）a.

Definition 31.3. Given a commutative ring A, a direct decomposition of A is a sequence (b1,...,bn) of ideals in A such that there is an isomorphism A ≈ A/b1 × ··· × A/bn.  
定义31.3.对于交换环A，a的直接分解是a中理想的序列（b1，…，bn），因此存在一个同构a≈a/b1×····×a/bn。

The following theorem gives useful conditions characterizing direct decompositions of a ring.  
下面的定理给出了表征环的直接分解的有用条件。

Theorem 31.16. Let A be a commutative ring and let (b1,...,bn) be a sequence of ideals in A. The following conditions are equivalent:  
定理31.16。设a为交换环，设（b1，…，bn）为a中的理想序列。下列条件等效：

1. The sequence (b1,...,bn) is a direct decomposition of A.  
   序列（b1，…，bn）是a的直接分解。
2. There exist some elements e1,...,en of A such that  
   存在一些元素e1，…，en

e2i = ei  
E2i=Ei

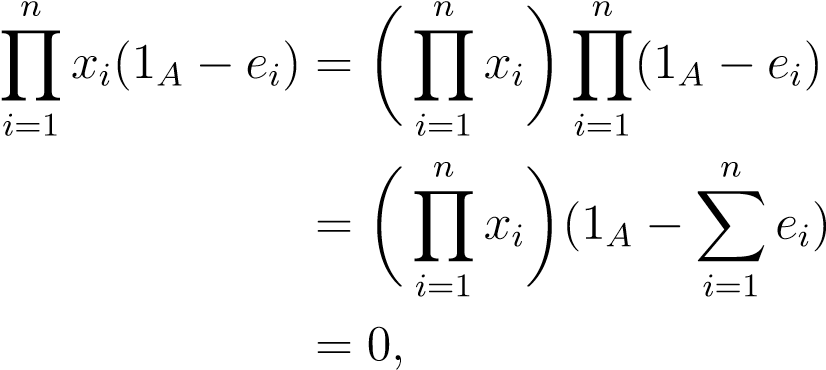
eiej = 0, i =6 j e1 + ··· + en = 1A,  
eiej=0，i=6J e1+·····+en=1a，

and bi = (1A − ei)A, for i,j = 1,...,n.  
而bi=（1a−ei）a，对于i，j=1，…，n。

1. We have bi + bj = A for all i =6 j, and b1 ···bn = (0).  
   对于所有i=6j，我们有bi+bj=a，b1···bn=（0）。
2. We have bi + bj = A for all i =6 j, and b1 ∩ ··· ∩ bn = (0).  
   对于所有i=6j，我们有bi+bj=a，b1·bn=（0）。

Proof. Assume (a). Since we have an isomorphism A ≈ A/b1 ×···× A/bn, we may identify A with A/b1 × ··· × A/bn, and bi with Ker(pri). Then, e1,...,en are the elements defined just before Definition 31.3. As noted, bi = Ker(pri) = (1A − ei)A. This proves (b).  
证据。假设（a）。由于我们有一个同构a≈a/b1×····×a/bn，我们可以用a/b1×···×a/bn来标识a，用ker（pri）来标识bi。那么，e1，…，en是在定义31.3之前定义的元素。如前所述，bi=ker（pri）=（1a−ei）a。这证明（b）。

Assume (b). Since bi = (1A − ei)A and A is a ring with unit 1A, we have 1A − ei ∈ bi for i = 1,...,n. For all i =6 j, we also have ei(1A − ej) = ei − eiej = ei, so (because bj is an ideal), ei ∈ bj, and thus, 1A = 1A − ei + ei ∈ bi + bj, which shows that bi + bj = A for all i =6 j. Furthermore, for any xi ∈ A, with 1 ≤ i ≤ n, we have  
假设（b）。既然bi=（1a−ei）a和a是一个单位为1a的环，我们有1a−ei∈bi表示i=1，…，n。对于所有i=6 j，我们也有ei（1a−ej）=ei−eiej=ei，所以（因为bj是一个理想），ei ei∈bj，因此，1a=1a−ei+ei∈bi+bj，这表明bi+bj=a表示所有i=6 j。此外，对于任何一个具有1个i i的n，我们有



which proves that b1 ···bn = (0). Thus, (c) holds.  
证明b1···bn=（0）。因此，（c）成立。

The equivalence of (c) and (d) follows from the proof of Theorem 31.15.  
（c）和（d）的等价性来源于定理31.15的证明。

The fact that (c) implies (a) is an immediate consequence of Theorem 31.15.   
（c）表示（a）的事实是定理31.15的直接结果。

Here is example of Theorem 31.16. Take the commutative ring of residue classes mod 30, namely  
这是定理31.16的例子。取剩余类mod 30的交换环，即

.  
.

Let  
让

b  
乙

b2 = 3Z/30Z := {3i}9i=0 b3 = 5Z/30Z := {5i}5i=0.  
b2=3z/30z:=3i 9i=0 b3=5z/30z:=5i 5i=0.

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Each bi is an ideal in Z/30Z. Furthermore  
每个BI都是z/30z的理想选择。

Z/30Z = (Z/30Z)/(2Z/30Z) × (Z/30Z)/(3Z/30Z) × (Z/30Z)/(5Z/30Z), where  
Z/30Z=（Z/30Z）/（2Z/30Z）×Z/30Z/（3Z/30Z）×Z/30Z/（5Z/30Z），其中

e1 = (1,0,0) → 15, e2 = (0,1,0) → 10, e3 = (0,0,1) → 6,  
e1=（1,0,0）→15，e2=（0,1,0）→10，e3=（0,0,1）→6，

since  
自从

.  
.

Note that 15 corresponds to 1 ∈ (Z/30Z)/(2Z/30Z), 10 corresponds to  
注意15对应1∈（z/30z）/（2z/30z），10对应

1 ∈ (Z/30Z)/(3Z/30Z), while 6 corresponds to 1 ∈ (Z/30Z)/(5Z/30Z).  
1∈（z/30z）/（3z/30z），6对应1∈（z/30z）/（5z/30z）。

## 31.3 Noetherian Rings and Hilbert’s Basis Theorem 31.3诺特环和希尔伯特基定理

Given a (commutative) ring A (with unit element 1), an ideal A ⊆ A is said to be finitely generated if there exists a finite set {a1,...,an} of elements from A so that A = (a1,...,an) = {λ1a1 + ··· + λnan | λi ∈ A, 1 ≤ i ≤ n}.  
给定（交换）环A（具有单位元1），如果存在a的有限集a1，…，a中的一个元素，则称理想a a是有限生成的，因此a=（a1，…，a n）=λ1a1+····+λnanλi∈a，1≤i≤n。

If K is a field, it turns out that every polynomial ideal A in K[X1,...,Xm] is finitely generated. This fact due to Hilbert and known as Hilbert’s basis theorem, has very important consequences. For example, in algebraic geometry, one is interested in the zero locus of a set of polyomial equations, i.e., the set, V (P), of n-tuples (λ1,...,λn) ∈ Kn so that  
如果k是一个域，则证明k[x1，…，xm]中的每个多项式理想a都是有限生成的。这一事实由于希尔伯特而被称为希尔伯特基本定理，具有非常重要的后果。例如，在代数几何中，人们对一组多变量方程的零轨迹感兴趣，即n-元组（λ1，…，λn）∈kn的集合v（p），因此

Pi(λ1,...,λn) = 0  
Pi（λ1，…，λn）=0

for all polynomials Pi(X1,...,Xn) in some given family, P = (Pi)i∈I. However, it is clear that  
对于某些给定族中的所有多项式p i（x1，…，xn），p=（pi）i∈i。然而，很明显

V (P) = V (A),  
V（P）=V（A）

where A is the ideal generated by P. Then, Hilbert’s basis theorem says that V (A) is actually defined by a finite number of polynomials (any set of generators of A), even if P is infinite.  
当a是p产生的理想时，希尔伯特的基定理说v（a）实际上是由有限个多项式（a的任何一组生成元）定义的，即使p是无限的。

The property that every ideal in a ring is finitely generated is equivalent to other natural properties, one of which is the so-called ascending chain condition, abbreviated a.c.c. Before proving Hilbert’s basis theorem, we explore the equivalence of these conditions.  
一个环中的每一个理想都是有限生成的，这一性质等价于其他自然性质，其中一个是所谓的上升链条件，简称A.C.C。在证明希尔伯特基定理之前，我们先探讨这些条件的等价性。

Definition 31.4. Let A be a commutative ring with unit 1. We say that A satisfies the ascending chain condition, for short, the a.c.c, if for every ascending chain of ideals  
定义31.4.设A为单位为1的交换环。我们说，A满足上升链条件，简而言之，A.C.C，如果对于每一个理想的上升链

A1 ⊆ A2 ⊆ ··· ⊆ Ai ⊆ ··· ,  
A1 A2·········

there is some integer n ≥ 1 so that  
有一个整数n≥1，所以

Ai = An for all i ≥ n + 1.  
ai=所有i的a≥n+1。

We say that A satisfies the maximum condition if every nonempty collection C of ideals in A has a maximal element, i.e., there is some ideal A ∈ C which is not contained in any other ideal in C.  
如果A中的每一个非空集合C都有一个极大元素，即在C中有一个不包含在任何其他理想中的理想A∈C，则A满足极大条件。

Proposition 31.17. A ring A satisfies the a.c.c if and only if it satisfies the maximum condition.  
提案31.17。如果且仅当A环满足最大条件时，A环满足交流。

Proof. Suppose that A does not satisfy the a.c.c. Then, there is an infinite strictly ascending sequence of ideals  
证据。假设a不满足a.c.c，那么有一个无限严格的理想上升序列。

A1 ⊂ A2 ⊂ ··· ⊂ Ai ⊂ ··· ,  
A1 A2········

and the collection C = {Ai} has no maximal element.  
集合c=ai没有最大元素。

Conversely, assume that A satisfies the a.c.c. Let C be a nonempty collection of ideals Since C is nonempty, we may pick some ideal A1 in C. If A1 is not maximal, then there is some ideal A2 in C so that  
相反，假设a满足a.c.c.假设c是理想的非空集合，因为c是非空的，我们可以在c中选择一些理想a1。如果a1不是最大的，那么在c中有一些理想a2，因此

A1 ⊂ A2.  
A1 A2.

Using this process, if C has no maximal element, we can define by induction an infinite strictly increasing sequence  
利用这个过程，如果c没有最大元素，我们可以通过归纳一个无限严格递增序列来定义。

A1 ⊂ A2 ⊂ ··· ⊂ Ai ⊂ ··· .  
A1 A2········

However, the a.c.c. implies that such a sequence cannot exist. Therefore, C has a maximal element.   
然而，交流意味着这样的序列不可能存在。因此，c有一个极大的元素。

Having shown that the a.c.c. condition is equivalent to the maximal condition, we now prove that the a.c.c. condition is equivalent to the fact that every ideal is finitely generated. Proposition 31.18. A ring A satisfies the a.c.c if and only if every ideal is finitely generated.  
在证明了交流条件等价于最大条件之后，我们现在证明了交流条件等价于每个理想都是有限生成的。提案31.18。当且仅当每个理想都是有限生成的时，A环才满足交流。

Proof. Assume that every ideal is finitely generated. Consider an ascending sequence of ideals  
证据。假设每个理想都是有限生成的。考虑理想的升序

A1 ⊆ A2 ⊆ ··· ⊆ Ai ⊆ ··· .  
A1 A2········

Observe that A = Si Ai is also an ideal. By hypothesis, A has a finite generating set {a1,...,an}. By definition of A, each ai belongs to some Aji, and since the Ai form an ascending chain, there is some m so that ai ∈ Am for i = 1,...,n. But then,  
观察a=si-ai也是一个理想。根据假设，A有一个有限的生成集A1，…，一个。根据a的定义，每个a i都属于某个aji，由于ai形成了一条上升链，所以有一些m使得ai∈am代表i=1，…，n，但是，

Ai = Am  
ai=上午

for all i ≥ m + 1, and the a.c.c. holds.  
对于所有i≥m+1，并且交流保持不变。

Conversely, assume that the a.c.c. holds. Let A be any ideal in A and consider the family C of subideals of A that are finitely generated. The family C is nonempty, since (0) is a subideal of A. By Proposition 31.17, the family C has some maximal element, say B. For  
相反，假设交流电保持不变。让a是a中的任何理想，并考虑a的子理想的C族，这些子理想是有限生成的。C族是非空的，因为（0）是A的次理想。根据命题31.17，C族有一些极大的元素，比如b。

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any a ∈ A, the ideal B + (a) (where B + (a) = {b + λa | b ∈ B, λ ∈ A}) is also finitely generated (since B is finitely generated), and by maximality, we have  
任意a∈a，理想b+（a）（其中b+（a）=b+λa b∈b，λ∈a）也是有限生成的（因为b是有限生成的），并且通过极大性，我们得到

B = B + (a).  
B=B+（A）。

So, we get a ∈ B for all a ∈ A, and thus, A = B, and A is finitely generated.   
因此，我们得到所有a∈a的a∈b，因此，a=b，a是有限生成的。

Definition 31.5. A commutative ring A (with unit 1) is called noetherian if it satisfies the a.c.c. condition. A noetherian domain is a noetherian ring that is also a domain.  
定义31.5.如果一个交换环A（带单位1）满足交流条件，它就称为非以太环。诺特域是一个诺特环，也是一个域。

By Proposition 31.17 and Proposition 31.18, a noetherian ring can also be defined as a ring that either satisfies the maximal property or such that every ideal is finitely generated. The proof of Hilbert’s basis theorem will make use the following lemma:  
通过命题31.17和命题31.18，一个诺厄特环也可以定义为一个满足最大性质或每个理想都是有限生成的环。希尔伯特基本定理的证明将使用以下引理：

Lemma 31.19. Let A be a (commutative) ring. For every ideal A in A[X], for every i ≥ 0, let Li(A) denote the set of elements of A consisting of 0 and of the coefficients of Xi in all the polynomials f(X) ∈ A which are of degree i. Then, the Li(A)’s form an ascending chain of ideals in A. Furthermore, if B is any ideal of A[X] so that A ⊆ B and if Li(A) = Li(B) for all i ≥ 0, then A = B.  
引理31.19。设A为（交换）环。对于[x]中的每个理想A，对于每i i 0，让李（a）表示由0个多项式组成的元素集合和所有i（i）x的多项式i的系数，然后，李（a）形成理想的上升链。此外，如果B是[x]的任何理想，那么Ab如果所有i≥0的li（a）=li（b），则a=b。

Proof. That Li(A) is an ideal and that Li(A) ⊆ Li+1(A) follows from the fact that if f(X) ∈ A and g(X) ∈ A, then f(X) + g(X), λf(X), and Xf(X) all belong to A. Now, let g(X) be any polynomial in B, and assume that g(X) has degree n. Since Ln(A) = Ln(B), there is some polynomial fn(X) in A, of degree n, so that g(X) − fn(X) is of degree at most n − 1. Now, since A ⊆ B, the polynomial g(X) − fn(X) belongs to B. Using this process, we can define by induction a sequence of polynomials fn+i(X) ∈ A, so that each fn+i(X) is either zero or has degree n − i, and  
证据。李（a）是一个理想，李（a）李+1（a）是从F（x）A和G（x）A出发的，那么F（x）+G（x）、λf（x）和Xf（x）都属于A，现在，假设G（x）是B中的任何多项式，假设G（x）具有n次方，因为ln（a）=ln（b），degre的a中有一些多项式fn（x）。e n，使g（x）−fn（x）至多为n−1。现在，由于a b，多项式g（x）−fn（x）属于b。利用这个过程，我们可以通过归纳定义多项式fn+i（x）∈a的序列，使每个fn+i（x）要么为零，要么为n−i，并且

g(X) − (fn(X) + fn+1(X) + ··· + fn+i(X))  
G（X）−（FN（X）+FN+1（X）+······+FN+I（X））

is of degree at most n − i − 1. Note that this last polynomial must be zero when i = n, and thus, g(X) ∈ A.   
最多为n−i−1度。注意，当i=n时，最后一个多项式必须为零，因此，g（x）∈a。

We now prove Hilbert’s basis theorem. The proof is substantially Hilbert’s original proof. A slightly shorter proof can be given but it is not as transparent as Hilbert’s proof (see the remark just after the proof of Theorem 31.20, and Zariski and Samuel [188], Chapter IV, Section 1, Theorem 1).  
我们现在证明希尔伯特基本定理。证据实质上是希尔伯特的原始证据。可以给出一个稍短的证明，但它不如希尔伯特的证明那么透明（见定理31.20证明之后的注释，以及Zariski和Samuel[188]，第四章，第1节，定理1）。

Theorem 31.20. (Hilbert’s basis theorem) If A is a noetherian ring, then A[X] is also a noetherian ring.  
定理31.20。（希尔伯特基本定理）如果A是一个非以太环，那么[X]也是一个非以太环。

Proof. Let A be any ideal in A[X], and denote by L the set of elements of A consisting of 0 and of all the coefficients of the highest degree terms of all the polynomials in A. Observe that  
证据。设a为a[x]中的任何理想，并用l表示a中由0组成的元素集和a中所有多项式的最高次数项的所有系数。观察

.  
.

Thus, L is an ideal in A (this can also be proved directly). Since A is noetherian, L is finitely generated, and let {a1,...,an} be a set of generators of L. Let f1(X),...,fn(X) be polynomials in A having respectively a1,...,an as highest degree term coefficients. These polynomials generate an ideal B. Let q be the maximum of the degrees of the fi(X)’s. Now, pick any polynomial g(X) ∈ A of degree d ≥ q, and let aXd be its term of highest degree. Since a ∈ L, we have a = λ1a1 + ··· + λnan,  
因此，L是A中的理想（这也可以直接证明）。由于a是诺特良的，l是有限生成的，并且让a1，…，a是l的一组生成元。让f1（x），…，fn（x）是a中的多项式，分别具有a1，…，作为最高阶项系数。这些多项式产生一个理想的b，设q为fi（x）的最大度数，现在选取任意一个d≥q的多项式g（x）∈a，设axd为其最高度数项。由于a∈l，我们得到a=λ1a1+·····+λnan，

for some λi ∈ A. Consider the polynomial  
对于某些λi∈a.考虑多项式

,  
，

where di is the degree of fi(X). Now, g(X) − g1(X) is a polynomial in A of degree at most d − 1. By repeating this procedure, we get a sequence of polynomials gi(X) in B, having strictly decreasing degrees, and such that the polynomial  
其中di是fi（x）的度数。现在，g（x）−g1（x）是一个次数最多为d−1的多项式。通过重复这一过程，我们得到了B中多项式gi（x）的序列，严格地具有递减的度数，这样多项式

g(X) − (g1(X) + ··· + gi(X))  
g（x）−（g1（x）+·····+gi（x））

is of degree at most d − i. This polynomial must be of degree at most q − 1 as soon as i = d − q + 1. Thus, we proved that every polynomial in A of degree d ≥ q belongs to B.  
至多为d−i的度数。只要i=d−q+1，此多项式至多为q−1的度数。由此证明了D≥Q阶A中的每一个多项式都属于B。

It remains to take care of the polynomials in A of degree at most q − 1. Since A is noetherian, each ideal Li(A) is finitely generated, and let {ai1,...,aini} be a set of generators for Li(A) (for i = 0,...,q − 1). Let fij(X) be a polynomial in A having aijXi as its highest degree term. Given any polynomial g(X) ∈ A of degree d ≤ q − 1, if we denote its term of highest degree by aXd, then, as in the previous argument, we can write  
它仍然要注意多项式的程度，最多为q-1。由于a是诺特良的，每个理想的li（a）都是有限生成的，并且让a i 1，…，aini成为li（a）的一组发生器（对于i=0，…，q−1）。让fij（x）是以aijxi为最高次数项的多项式。给定任意多项式g（x）∈a的d≤q−1，如果我们用a x d表示它的最高阶项，那么，就像前面的论点一样，我们可以写出

a = λ1ad1 + ··· + λndadnd,  
A=λ1ad1+····+λndadnd，

and we define  
我们定义

,  
，

where di is the degree of fdi(X). Then, g(X)−g1(X) is a polynomial in A of degree at most d − 1, and by repeating this procedure at most q times, we get an element of A of degree 0, and the latter is a linear combination of the f0i’s. This proves that every polynomial in A of degree at most q − 1 is a combination of the polynomials fij(X), for 0 ≤ i ≤ q − 1 and 1 ≤ j ≤ ni. Therefore, A is generated by the fk(X)’s and the fij(X)’s, a finite number of polynomials.   
式中，di是外国直接投资的程度（x）。那么，g（x）−g1（x）是至多d−1阶的多项式，通过至多q次重复这个过程，我们得到一个0阶的元素，后者是f0i的线性组合。这证明了至多q−1阶的每个多项式都是pol的组合。对于0≤i≤q−1和1≤j≤ni，ynomials fij（x）。因此，由fk（x）’和fij（x）’生成a，这是有限数量的多项式。

Remark: Only a small part of Lemma 31.19 was used in the above proof, namely, the fact that Li(A) is an ideal. A shorter proof of Theorem 31.21 making full use of Lemma 31.19 can be given as follows:  
注：上述证明中只使用了引理31.19的一小部分，即李（a）是一个理想的事实。充分利用引理31.19，定理31.21的较短证明如下：

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Proof. (Second proof) Let (Ai)i≥1 be an ascending sequence of ideals in A[X]. Consider the doubly indexed family (Li(Aj)) of ideals in A. Since A is noetherian, by the maximal property, this family has a maximal element Lp(Aq). Since the Li(Aj)’s form an ascending sequence when either i or j is fixed, we have Li(Aj) = Lp(Aq) for all i and j with i ≥ p and j ≥ q, and thus, Li(Aq) = Li(Aj) for all i and j with i ≥ p and j ≥ q. On the other hand, for any fixed i, the a.c.c. shows that there exists some integer n(i) so that Li(Aj) = Li(An(i)) for all j ≥ n(i). Since Li(Aq) = Li(Aj) when i ≥ p and j ≥ q, we may take n(i) = q if i ≥ p. This shows that there is some n0 so that n(i) ≤ n0 for all i ≥ 0, and thus, we have Li(Aj) = Li(An(0)) for every i and for every j ≥ n(0). By Lemma 31.19, we get Aj = An(0) for every j ≥ n(0), establishing the fact that A[X] satisfies the a.c.c.   
证据。（第二个证明）让（a i）i≥1是a[x]中理想的升序。考虑A中理想的双索引族（li（aj））。由于a是诺特良族，根据极大性，该族具有极大元素lp（aq）。由于当i或j固定时，li（aj）形成一个升序，所以对于i≥p和j≥q的所有i和j，我们都有li（aj）=lp（aq），因此，对于i≥p和j≥q的所有i和j，li（aq）=li（aj）。另一方面，对于任何固定i，a.c.c.表明存在一些整数n（i）。因此，对于所有j≥n（i），li（aj）=li（an（i））。由于当i≥p且j≥q时，li（aq）=li（aj），如果i≥p，我们可以取n（i）=q。这表明存在一些n0，因此对于所有i≥0，n（i）≤n0，因此，对于每个i和j≥n（0），我们都有li（aj）=li（an（0））。通过引理31.19，我们得到每j≥n（0）的aj=an（0），证明a[x]满足a.c.c。

Using induction, we immediately obtain the following important result.  
利用归纳法，我们立即得到以下重要结果。

Corollary 31.21. If A is a noetherian ring, then A[X1,...,Xn] is also a noetherian ring.  
推论31.21。如果A是一个非以太环，那么[x1，…，xn]也是一个非以太环。

Since a field K is obviously noetherian (since it has only two ideals, (0) and K), we also have:  
因为一个场k显然是非以太的（因为它只有两个理想，（0）和k），我们也有：

Corollary 31.22. If K is a field, then K[X1,...,Xn] is a noetherian ring.  
推论31.22。如果k是一个场，那么k[x1，…，xn]是一个非其他环。

## 31.4 Futher Readings 31.4进一步读数

The material of this Chapter is thoroughly covered in Lang [106], Artin [7], Mac Lane and Birkhoff [115], Bourbaki [25, 26], Malliavin [116], Zariski and Samuel [188], and Van Der Waerden [173].  
本章的材料在lang[106]、Artin[7]、Mac Lane和Birkhoff[115]、Bourbaki[25，26]、Malliavin[116]、Zariski和Samuel[188]以及van der Waerden[173]中有详细介绍。

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Chapter 32  
第三十二章

# Tensor Algebras and Symmetric Algebras 张量代数和对称代数

Tensors are creatures that we would prefer did not exist but keep showing up whenever multilinearity manifests itself.  
张量是我们希望不存在的生物，但当多能性显现出来时，张量就会不断出现。

One of the goals of differential geometry is to be able to generalize “calculus on Rn” to spaces more general than Rn, namely manifolds. We would like to differentiate functions f : M → R defined on a manifold, optimize functions (find their minima or maxima), but also to integrate such functions, as well as compute areas and volumes of subspaces of our manifold.  
微分几何的目标之一是能够将“RN上的微积分”推广到比RN更普遍的空间，即流形。我们要区分在流形上定义的函数f:m→r，优化函数（找到它们的最小值或最大值），还要集成这些函数，以及计算流形子空间的面积和体积。

The suitable notion of differentiation is the notion of tangent map, a linear notion. One of the main discoveries made at the beginning of the twentieth century by Poincar´e and Elie´ Cartan, is that the “right” approach to integration is to integrate differential forms, and not functions. To integrate a function f, we integrate the form fω, where ω is a volume form on the manifold M. The formalism of differential forms takes care of the process of the change of variables quite automatically, and allows for a very clean statement of Stokes’ formula.  
微分的适当概念是切线映射的概念，一个线性概念。Poincar'e和Elie'Cartan在二十世纪初的主要发现之一是，整合的“正确”方法是整合微分形式，而不是功能。为了积分函数f，我们将形式fω积分，其中ω是流形m上的体积形式。微分形式的形式主义完全自动地处理变量的变化过程，并允许斯托克斯公式的一个非常清晰的陈述。

Differential forms can be combined using a notion of product called the wedge product, but what really gives power to the formalism of differential forms is the magical operation d of exterior differentiation. Given a form ω, we obtain another form dω, and remarkably, the following equation holds  
微分形式可以用一个称为楔积的积的概念来组合，但真正赋予微分形式形式形式主义力量的是外部微分的神奇运算d。给定形式ω，我们得到另一个形式dω，值得注意的是，下列方程成立

ddω = 0.  
ddω=0.

As silly as it looks, the above equation lies at the core of the notion of cohomology, a powerful algebraic tool to understand the topology of manifolds, and more generally of topological spaces.  
尽管看起来很愚蠢，但上述方程是上同调概念的核心，是理解流形拓扑的强大代数工具，更普遍的是拓扑空间。

Elie Cartan had many of the intuitions that lead to the cohomology of differential forms,´ but it was George de Rham who defined it rigorously and proved some important theorems about it. It turns out that the notion of Laplacian can also be defined on differential forms using a device due to Hodge, and some important theorems can be obtained: the Hodge  
Elie Cartan有许多直觉导致了微分形式的上同调，但正是乔治·德·雷姆对其进行了严格的定义，并证明了一些重要的定理。结果表明，由于霍奇的存在，拉普拉斯的概念也可以用一个装置在微分形式上定义，并且可以得到一些重要的定理：霍奇

1045  
一千零四十五

decomposition theorem, and Hodge’s theorem about the isomorphism between the de Rham cohomology groups and the spaces of harmonic forms.  
分解定理，和霍奇关于德拉姆上同调群与调和形式空间同构的定理。

To understand all this, one needs to learn about differential forms, which turn out to be certain kinds of skew-symmetric (also called alternating) tensors.  
要理解这一切，我们需要学习微分形式，它是某种斜对称（也称为交替）张量。

If one’s only goal is to define differential forms, then it is possible to take some short cuts and to avoid introducing the general notion of a tensor. However, tensors that are not necessarily skew-symmetric arise naturally, such as the curvature tensor, and in the theory of vector bundles, general tensor products are needed.  
如果一个人的唯一目标是定义微分形式，那么有可能采取一些捷径，避免引入张量的一般概念。然而，不一定是斜对称的张量自然产生，如曲率张量，在向量束理论中，一般张量积是必要的。

Consequently, we made the (perhaps painful) decision to provide a fairly detailed exposition of tensors, starting with arbitrary tensors, and then specializing to symmetric and alternating tensors. In particular, we explain rather carefully the process of taking the dual of a tensor (of all three flavors).  
因此，我们做出了（也许是痛苦的）决定，提供了一个相当详细的张量说明，从任意张量开始，然后专门讨论对称张量和交替张量。特别是，我们非常仔细地解释了取张量（三种口味中的）的对偶的过程。

We refrained from following the approach in which a tensor is defined as a multilinear map defined on a product of dual spaces, because it seems very artificial and confusing (certainly to us). This approach relies on duality results that only hold in finite dimension, and consequently unecessarily restricts the theory of tensors to finite dimensional spaces. We also feel that it is important to begin with a coordinate-free approach. Bases can be chosen for computations, but tensor algebra should not be reduced to raising or lowering indices.  
我们避免采用张量被定义为在对偶空间乘积上定义的多行映射的方法，因为它看起来非常人为和混乱（当然对我们来说）。这种方法依赖于仅在有限维中存在的对偶结果，因此不必要地将张量理论限制在有限维空间中。我们还认为，从无坐标方法开始是很重要的。可以选择基进行计算，但张量代数不应减少到提高或降低指数。

Readers who feel that they are familiar with tensors should probably skip this chapter and the next. They can come back to them “by need.”  
那些认为自己熟悉张量的读者可能会跳过这一章和下一章。他们可以“根据需要”回到他们身边。

We begin by defining tensor products of vector spaces over a field and then we investigate some basic properties of these tensors, in particular the existence of bases and duality. After this we investigate special kinds of tensors, namely symmetric tensors and skew-symmetric tensors. Tensor products of modules over a commutative ring with identity will be discussed very briefly. They show up naturally when we consider the space of sections of a tensor product of vector bundles.  
我们首先定义场上向量空间的张量积，然后研究这些张量的一些基本性质，特别是基和对偶的存在性。在此基础上，我们研究了特殊类型的张量，即对称张量和斜对称张量。讨论了具有恒等式的交换环上模的张量积。当我们考虑向量束张量积各部分的空间时，它们自然会出现。

Given a linear map f : E → F (where E and F are two vector spaces over a field K), fwe know that if we have a basis ((ui) on the basis vectors. For a multilinear mapui)i∈I for E, thenf :fEis completely determined by its valuesn → F, we don’t know if there is such a nice property but it would certainly be very useful.  
给定一个线性映射f:e→f（其中e和f是K域上的两个向量空间），我们知道如果我们有基向量的基（（ui）。对于多行mapui）i∈i表示e，那么f:f e is完全由其值n→f决定，我们不知道是否有这样一个好的属性，但它肯定是非常有用的。

In many respects tensor products allow us to define multilinear maps in terms of their action on a suitable basis. The crucial idea is to linearize, that is, to create a new vector space  
在许多方面，张量积允许我们在适当的基础上根据它们的作用定义多行映射。关键的思想是线性化，即创建一个新的向量空间

E⊗n such that the multilinear map f : En → F is turned into a linear map f⊗ : E⊗n → F which is equivalent to f in a strong sense. If in addition, f is symmetric, then we can define a symmetric tensor power Symn(E), and every symmetric multilinear map f : En → F is turned into a linear map which is equivalent to f in a strong sense.  
e n使得多行映射f:en→f变成线性映射f：e n→f，在强意义上等价于f。另外，如果f是对称的，那么我们可以定义一个对称张量幂次对称（e），每个对称的多行映射f:en→f都变成一个强意义上等价于f的线性映射。

Similarly, if f is alternating, then we can define a skew-symmetric tensor power Vn(E), and every alternating multilinear map is turned into a linear map f∧ : Vn(E) → F which is equivalent to f in a strong sense.  
同样，如果f是交变的，那么我们可以定义一个斜对称张量幂vn（e），每个交变多线性映射都变成一个线性映射f：vn（e）→f，在强意义上等价于f。

Tensor products can be defined in various ways, some more abstract than others. We try to stay down to earth, without excess.  
张量积可以用不同的方式定义，有些比其他更抽象。我们尽量不过度地呆在地上。

Before proceeding any further, we review some facts about dual spaces and pairings. Pairings will be used to deal with dual spaces of tensors.  
在进一步讨论之前，我们回顾了一些关于双空间和配对的事实。配对将用于处理张量的对偶空间。

## 32.1 Linear Algebra Preliminaries: Dual Spaces and Pairings 32.1线性代数预备：对偶空间和对

We assume that we are dealing with vector spaces over a field K. As usual the dual space E∗ of a vector space E is defined by E∗ = Hom(E,K). The dual space E∗ is the vector space consisting of all linear maps ω: E → K with values in the field K.  
我们假设我们处理的是场k上的向量空间。通常，向量空间e的对偶空间e\_由e=hom（e，k）定义。对偶空间e是由所有线性映射构成的向量空间，ω：e→k与字段k中的值。

A problem that comes up often is to decide when a space E is isomorphic to the dual F ∗ of some other space F (possibly equal to E). The notion of pairing due to Pontrjagin provides a very clean criterion.  
一个经常出现的问题是决定空间e何时同构于其他空间f（可能等于e）的对偶f\_。Pontrjagin提出的配对概念提供了一个非常清晰的标准。

Definition 32.1. Given two vector spaces E and F over a field K, a map h−,−i: E×F → K is a nondegenerate pairing iff it is bilinear and iff hu,vi = 0 for all v ∈ F implies u = 0, and ϕhu,v: Ei = 0F ∗for alland ψu: F∈ →E Eimplies∗ defined such that for all for allv = 0. A nondegenerate pairing induces two linear mapsu ∈ E and all v ∈ F, ϕ(u) is the  
定义32.1.在一个K域上给定两个向量空间e和f，映射h−，−i:e×f→k是一个非退化的对，如果它是双线性的，并且iff h u，vi=0表示所有v∈f表示u=0，而\_hu，v:ei=0f表示all andψu:f∈→e e implies定义为all表示allv=0。非退化配对诱导两个线性mapsu∈e，且所有v∈f，ω（u）是

→  
渐次

linear form in F ∗ and ψ(v) is the linear form in E∗ given by  
f和ψ（v）中的线性形式是e中的线性形式，由

ϕ(u)(y) = hu,yi for all y ∈ F ψ(v)(x) = hx,vi for all x ∈ E.  
⑨（u）（y）=hu，yi表示所有y∈fψ（v）（x）=hx，vi表示所有x∈e。

Schematically, ϕ(u) = hu,−i and ψ(v) = h−,vi.  
式中，（u）=hu、−i和ψ（v）=h−，vi。

ϕProposition 32.1.: E → F ∗ and ψϕ::For every nondegenerate pairingFE →E∗ are linear and injective. Furthermore, ifh−,−i: E ×F → KE, the induced mapsand F are finite dimensional, then → F ∗ and ψ: F → E∗ are bijective.  
第32.1条建议：e→f和\_：：对于每一个非简并配对fe→e都是线性的和内射的。此外，如果H−、−i:e×f→ke，诱导映射和f是有限维的，那么→f和ψ:f→e是双射的。

Proof. The maps ϕ: E → F ∗ and ψ: F → E∗ are linear because u,v → h7 u,vi is bilinear.  
证据。由于u，v→h7 u，vi是双线性的，因此图\_：e→f和ψ：f→e\_是线性的。

Assume that ϕ(u) = 0. This means that ϕ(u)(y) = hu,yψ iis injective. If= 0 for all yE∈andF, and as ourF are finite pairing is nondegenerate, we must have u = 0. Similarly, dimensional, then dim(E) = dim(E∗) and dim(F) = dim(F ∗). However, the injectivity of ϕ and ψ implies that that dim(E) ≤ dim(F(E∗)) =and dimdim(F(F). Therefore, dim) ≤ dim(E∗). Consequently dim(E) = dim(F ∗) (andE) ≤ϕ dim(F) and dim(F) ≤ dim(E), so dim is bijective (and similarly dim(F) = dim(E∗) and ψ is bijective).   
假设\_（u）=0。也就是说，ψ（u）（y）=hu，yψ是注射剂。如果所有的ye∈andf都为0，并且我们的f是有限对，所以我们必须有u=0。同样，尺寸，然后dim（e）=dim（e）和dim（f）=dim（f）。然而，ω和ψ的注入率意味着dim（e）≤dim（f（e））=和dimdim（f（f）。因此，dim）≤dim（e）。因此，dim（e）=dim（f）（ande）≤dim（f）和dim（f）≤dim（e），所以dim是双射的（类似地，dim（f）=dim（e），ψ是双射的）。

Proposition 32.1 shows that when E and F are finite dimensional, a nondegenerate pairing induces canonical isomorphims ϕ: E → F ∗ and ψ: F → E∗; that is, isomorphisms that do not depend on the choice of bases. An important special case is the case where E = F and we have an inner product (a symmetric, positive definite bilinear form) on E.  
命题32.1表明，当e和f是有限维时，非简并配对产生标准同构，即不依赖于基的选择的同构。一个重要的特殊情况是E=F的情况，我们在E上有一个内积（对称的，正定双线性形式）。

Remark: When we use the term “canonical isomorphism,” we mean that such an isomorphism is defined independently of any choice of bases. For example, if E is a finite dimensional vector space and (e1,...,en) is any basis of E, we have the dual basis (  
注：当我们使用“规范同构”这个术语时，我们的意思是这样的同构是独立于任何碱基的选择而定义的。例如，如果e是一个有限维向量空间，而（e1，…，en）是e的任何基，我们就有了对偶基。（

E∗ (where,), and thus the map is an isomorphism between E and E∗.  
E（其中，），因此地图是E和E之间的同构。

This isomorphism is not canonical.  
这种同构不是典型的。

On the other hand, if h−,−i is an inner product on E, then Proposition 32.1 shows that the nondegenerate pairing h−,−i on E ×E induces a canonical isomorphism between E and E∗. This isomorphism is often denoted [: E → E∗, and we usually write u[ for [(u), with u ∈ E. Schematically, u[ = hu,−i. The inverse of [ is denoted ]: E∗ → E, and given any linear form ω ∈ E∗, we usually write ω] for ](ω). Schematically, ω = hω],−i.  
另一方面，如果h−、−i是e上的内积，那么命题32.1表明，非简并配对h−、−i在e×e上诱导e和e之间的规范同构。这种同构常被表示为[：e→e，我们通常用u[表示为[（u），用u e。用图式表示，u[=hu，−i。[表示为]：e→e的倒数，并且给定任何线性形式ωe，我们通常用ω]表示（ω）。简图上，ω=hω]，−i。

Given any basis, (e1,...,en) of E (not necessarily orthonormal), let (gij) be the n × nmatrix given by gij = hei,eji (the Gram matrix of the inner product). Recall that the dual basis consists of the coordinate forms e∗i ∈ E∗, which are characterized by the following properties: e∗i (ej) = δij, 1 ≤ i,j ≤ n.  
给定e的任何基（e1，…，e n）（不一定是正交），设（gij）为gij=hei，eji（内积的g矩阵）给出的n×n matrix。该对偶基由坐标形式e i∈e组成，其特征是：e i（e j）=δij，1≤i，j≤n。

The inverse of the Gram matrix (gij) is often denoted by (gij) (by raising the indices).  
革兰矩阵（gij）的逆矩阵通常用（gij）（通过提高指数）表示。

The tradition of raising and lowering indices is pervasive in the literature on tensors. It is indeed useful to have some notational convention to distinguish between vectors and linear forms (also called one-forms or covectors). The usual convention is that coordinates of vectors are written using superscripts, as in , and coordinates of one-forms are written using subscripts, as in. Actually, since vectors are indexed with subscripts, one-forms are indexed with superscripts, so should be written as ei.  
提高和降低指数的传统在张量文献中普遍存在。有一些符号约定来区分向量和线性形式（也称为单形或双形）确实很有用。通常的惯例是向量的坐标是用上标写的，如中所示，而一种形式的坐标是用下标写的，如中所示。实际上，因为向量是用下标索引的，所以一种形式是用上标索引的，所以应该写为ei。

The motivation is that summation signs can then be omitted, according to the Einstein summation convention. According to this convention, whenever a summation variable (such as i) appears both as a subscript and a superscript in an expression, it is assumed that it is involved in a summation. For example the sum is abbreviated as  
根据爱因斯坦的求和约定，其动机是求和符号可以省略。根据这一惯例，当求和变量（如i）在表达式中同时显示为下标和上标时，假定它与求和有关。例如，总和缩写为

uiei,  
尤伊

and the sum is abbreviated as  
和缩写为

ωiei.  
ωIEI。

In this text we will not use the Einstein summation convention, which we find somewhat confusing, and we will also write instead of ei.  
在本文中，我们将不使用爱因斯坦求和约定，我们发现这有点令人困惑，我们也将写而不是EI。

The maps [ and ] can be described explicitly in terms of the Gram matrix of the inner product and its inverse.  
图[和]可以用内积的克矩阵及其逆矩阵来明确地描述。

Proposition 32.2. For any vector space E, given a basis (e1,...,en) for E and its dual basis , for any inner product h−,−i on E, if (gij) is its Gram matrix, with gij = hei,eji, and (gij) is its inverse, then for every vector and every one-form, we have  
提案32.2.对于任何向量空间e，给定e及其对偶基的基（e1，…，en），对于任何内积h−，−i on e，如果（gij）是它的g矩阵，其中gij=hei，eji，and（gij）是它的逆矩阵，那么对于每个向量和每个形式，我们都有

, with ,  
，带有，

and  
和

, with .  
，使用。

Proof. For every, since u[(v) = hu,vi for all v ∈ E, we have  
证据。对于每一个，因为u[（v）=hu，vi对于所有v∈e，我们有

,  
，

so we get  
所以我们得到

, with .  
，使用。

If we write and, since  
如果我们写信，从那以后

, 1 ≤ i ≤ n,  
，1≤i≤n，

we get  
我们得到

,  
，

where (gij) is the inverse of the matrix (gij).   
其中（gij）是矩阵（gij）的倒数。

The map [ has the effect of lowering (flattening!) indices, and the map ] has the effect of raising (sharpening!) indices.  
地图[具有降低效果（平展！）索引和地图]具有提升效果（锐化！）指数。

Here is an explicit example of Proposition 32.2. Let (e1,e2) be a basis of E such that  
这是32.2提案的一个明确例子。设（e1，e2）为e的基础，这样

he1,e1i = 1, he1,e2i = 2, he2,e2i = 5. Then  
he1，e1i=1，he1，e2i=2，he2，e2i=5。然后

.  
.

Set u = u1e1 + u2e2 and observe that u[(e1) = hu1e1 + u2e2,e1i = he1,e1iu1 + he2,e1iu2 = g11u1 + g12u2 = u1 + 2u2 u[(e2) = hu1e1 + u2e2,e2i = he1,e2iu1 + he2,e2iu2 = g21u1 + g22u2 = 2u1 + 5u2,  
设置u=u1 e1+u2e2，观察u[（e1）=hu1e1+u2e2，e1i=he1，e1i u1+he2，e1iu2=g1u1+g12u2=u1+2u2 u[（e2）=hu1e1+u2e2，e2i=he1，e2iu1+he2，e2iu2=g21u1+g2u2=2u1+5u2，

which in turn implies that  
这反过来意味着

.  
.

Given, we calculate ω] = (ω])1e1 + (ω])2e2 from the following two linear equalities:  
给定，我们根据以下两个线性等式计算ω]=（ω]）1e1+（ω]）2e2：

ω1 = ω(e1) = hω],e1i = h(ω])1e1 + (ω])2e2,e1i  
ω1=ω（e1=hω），e1i=h（ω]）1e1+（ω]）2e2，e1i

= he1,e1i(ω])1 + he2,e1i(ω])2 = (ω])1 + 2(ω])2 = g11(ω])1 + g12(ω])2 ω2 = ω(e2) = hω],e2i = h(ω])1e1 + (ω])2e2,e2i  
=he1，e1i（ω）]1+h e2，e1i（ω）]2=（ω）]1+2（ω）]2=g11（ω）]1+g12（ω）]2ω2=ω（e2=hω），e2i=h（ω）]1e1+（ω）]2e2，e2i

= he1,e2i(ω])1 + he2,e2i(ω])2 = 2(ω])1 + 5(ω])2 = g21(ω])1 + g22(ω])2.  
=he1，e2i（ω）]1+he2，e2i（ω）]2=2（ω）]1+5（ω）]2=g21（ω）]1+g22（ω）]2.

These equalities are concisely written as  
这些等式简明地写为

.  
.

Then  
然后

,  
，

which in turn implies  
这反过来意味着

(ω])1 = 5ω1 − 2ω2, (ω])2 = −2ω1 + ω2, i.e.  
（ω）1=5ω1−2ω2，（ω）2=−2ω1+ω2，即

ω] = (5ω1 − 2ω2)e1 + (−2ω1 + ω2)e2.  
ω]=（5ω1−2ω2）e1+（−2ω1+ω2）e2。

The inner product h−,−i on E induces an inner product on E∗ denoted h−,−iE∗, and given by  
内积H−、−I on E在E∮上诱导内积，表示为H−、−IE∮，并由下式给出

, for all ω1,ω2 ∈ E∗.  
，对于所有ω1，ω2∈e。

Then we have  
然后我们有了

for all u,v ∈ E. If (e1,...,en) is a basis of E and gij = hei,eji, as  
对于所有u，v∈e，如果（e1，…，en）是e的基础，且gij=hei，eji，as

,  
，

an easy computation shows that  
一个简单的计算表明

;  
；

that is, in the basis (), the inner product on E∗ is represented by the matrix (gij), the inverse of the matrix (gij).  
也就是说，在基（）中，e上的内积由矩阵（gij）表示，即矩阵（gij）的倒数。

The inner product on a finite vector space also yields a canonical isomorphism between the space Hom(E,E;K) of bilinear forms on E, and the space Hom(E,E) of linear maps from E to itself. Using this isomorphism, we can define the trace of a bilinear form in an intrinsic manner. This technique is used in differential geometry, for example, to define the divergence of a differential one-form.  
有限向量空间上的内积也在e上双线性形式的空间hom（e，e；k）和e到其自身的线性映射的空间hom（e，e）之间产生一个规范的同构。利用这种同构，我们可以用一种内在的方式定义双线性形式的轨迹。例如，这种技术在微分几何中被用来定义微分一形式的散度。

Proposition 32.3. If h−,−i is an inner product on a finite vector space E (over a field, K), then for every bilinear form f : E × E → K, there is a unique linear map f\ : E → E  
提案32.3.如果h−，−i是有限向量空间e（在字段上，k）上的内积，那么对于每一个双线性形式f:e×e→k，都有一个唯一的线性映射f \：e→e

such that f(u,v) = hf\(u),vi, for all u,v ∈ E.  
这样f（u，v）=hf \（u），vi，对于所有u，v∈e。

The map f 7→ f\ is a linear isomorphism between Hom(E,E;K) and Hom(E,E). Proof. For every g ∈ Hom(E,E), the map given by  
图F7→F是hom（e，e；k）和hom（e，e）之间的线性同构。证据。对于每个g∈hom（e，e），由

f(u,v) = hg(u),vi, u,v ∈ E,  
f（u，v）=hg（u），vi，u，v∈e，

is clearly bilinear. It is also clear that the above defines a linear map from Hom(E,E) to Hom(E,E;K). This map is injective, because if f(u,v) = 0 for all u,v ∈ E, as h−,−i is an inner product, we get g(u) = 0 for all u ∈ E. Furthermore, both spaces Hom(E,E) and Hom(E,E;K) have the same dimension, so our linear map is an isomorphism.   
显然是双线性的。很明显，上面定义了从hom（e，e）到hom（e，e；k）的线性映射。这个映射是内射的，因为如果f（u，v）=0表示所有u，v∈e，因为h−，−i是一个内积，我们得到g（u）=0表示所有u∈e。此外，两个空间hom（e，e）和hom（e，e；k）都有相同的维数，所以我们的线性映射是同构的。

If (e1,...,en) is an orthonormal basis of E, then we check immediately that the trace of a linear map g (which is independent of the choice of a basis) is given by  
如果（e1，…，en）是e的正交基，那么我们立即检查线性映射g的迹线（与基的选择无关）是由

n  
n

tr(g) = Xhg(ei),eii,  
tr（g）=xhg（ei），eii，

i=1  
i＝1

where n = dim(E).  
其中n=尺寸（e）。

Definition 32.2. We define the trace of the bilinear form f by  
定义32.2.我们定义双线性形式f的迹线

tr(f) = tr(f\).  
Tr（f）=Tr（f\）。

From Proposition 32.3, tr(f) is given by  
根据提案32.3，Tr（f）由

n  
n

tr(f) = Xf(ei,ei),  
tr（f）=xf（ei，ei），即

i=1  
i＝1

for any orthonormal basis (e1,...,en) of E. We can also check directly that the above expression is independent of the choice of an orthonormal basis.  
对于e的任何正交基（e1，…，en），我们也可以直接检查上述表达式是否独立于正交基的选择。

We demonstrate how to calculate tr(f) where f : R2 ×R2 → R with f((x1,y1),(x2,y2)) = x1x2+2x2y1+3x1y2−y1y2. Under the standard basis for R2, the bilinear form f is represented as  
我们演示了如何计算Tr（f），其中f:r2×r2→r，其中f（（x1，y1），（x2，y2））=x1x2+2x2y1+3x1y2−y1y2。在r2的标准基础上，双线性形式f表示为

.  
.

This matrix representation shows that  
这个矩阵表示表明

,  
，

and hence  
因此

tr(.  
T.

We will also need the following proposition to show that various families are linearly independent.  
我们还需要以下的命题来证明各种族是线性独立的。

Proposition 32.4. Let E and F be two nontrivial vector spaces and let (ui)i∈I be any family of vectors ui ∈ E. The family (ui)i∈I is linearly independent iff for every family (vi)i∈I of vectors vi ∈ F, there is some linear map f : E → F so that f(ui) = vi for all i ∈ I.  
提案32.4.设e和f为两个非平凡向量空间，设（ui）i∈i为任意向量族ui∈e，向量族（ui）i∈i为每个向量族（vi）i∈i的线性独立iff，有一些线性映射f:e→f，使f（ui）=vi为所有i∈i。

Proof. Left as an exercise.   
证据。留作练习。

## 32.2 Tensors Products 32.2张量积

First we define tensor products, and then we prove their existence and uniqueness up to isomorphism.  
首先定义张量积，然后证明它们的存在性和唯一性，直至同构。

Definition 32.3. Let K be a given field, and let E1,...,En be n ≥ 2 given vector spaces. For any vector space F, a map f : E1 × ··· × En → F is multilinear iff it is linear in each of its argument; that is,  
定义32.3.设k为给定域，设e1，…，en为n≥2给定向量空间。对于任何向量空间f，映射f:e1×·······×en→f是多行的，如果它在每个参数中都是线性的，也就是说，

f(u1,...ui1,v + w,ui+1,...,un) = f(u1,...ui1,v,ui+1,...,un)  
f（u1，…ui1，v+w，ui+1，…，un）=f（u1，…ui1，v，ui+1，…，un）

+ f(u1,...ui1,w,ui+1,...,un)  
+f（u1，…ui1，w，ui+1，…，un）

f(u1,...ui1,λv,ui+1,...,un) = λf(u1,...ui1,v,ui+1,...,un),  
f（u1，…ui1，λv，ui+1，…，un）=λf（u1，…ui1，v，ui+1，…，un）

for all uj ∈ Ej (j =6 i), all v,w ∈ Ei and all λ ∈ K, for i = 1...,n.  
对于所有的uj∈ej（j=6i），所有的v，w∈ei和所有的λ∈k，对于i=1…，n。

The set of multilinear maps as above forms a vector space denoted L(E1,...,En;F) or Hom(E1,...,En;F). When n = 1, we have the vector space of linear maps L(E,F) (also denoted Hom(E,F)). (To be very precise, we write HomK(E1,...,En;F) and HomK(E,F).)  
如上所述的多行映射集形成了表示l（e1，…，en；f）或hom（e1，…，en；f）的向量空间。当n=1时，我们得到线性映射的向量空间l（e，f）（也表示hom（e，f））。（准确地说，我们写homk（e1，…，en；f）和homk（e，f）。）

Definition 32.4. A tensor product of n ≥ 2 vector spaces E1,...,En is a vector space T together with a multilinear map ϕ: E1 × ··· × En → T, such that for every vector space F and for every multilinear map f : E1×···×En → F, there is a unique linear map f⊗ : T → F with f(u1,...,un) = f⊗(ϕ(u1,...,un)),  
定义32.4.n≥2个向量空间e1，…，en的张量积是向量空间t加上一个多行映射：e1×······························································，…，un）），

for all u1 ∈ E1,...,un ∈ En, or for short  
对于所有u1∈e1，…，un∈en，或简称

f = f⊗ ◦ ϕ.  
F=F \_。

Equivalently, there is a unique linear map f⊗ such that the following diagram commutes.  
同样地，有一个独特的线性图f，这样下图就可以通勤了。

ϕ  
γ

E1 × ··· × En / T  
e1×····×en/t

NNNNNf⊗ f N& F  
nnnnn F F N&F公司

The above property is called the universal mapping property of the tensor product (T,ϕ).  
上述性质称为张量积（t，\_）的普适映射性质。

We show that any two tensor products (T1,ϕ1) and (T2,ϕ2) for E1,...,En, are isomorphic.  
我们证明e1，…，en的任何两个张量积（T1，\_）和（T2，\_）是同构的。

Proposition 32.5. Given any two tensor products (T1,ϕ1) and (T2,ϕ2) for E1,...,En, there is an isomorphism h: T1 → T2 such that  
提案32.5。对于e1，…，en，给定任意两个张量积（t1，\_）和（t2，\_），存在同构h:t1→t2，这样

ϕ2 = h ◦ ϕ1.  
\_2=H\_1.

Proof. Focusing on (T1,ϕ1), we have a multilinear map ϕ2 : E1 × ··· × En → T2, and thus there is a unique linear map (ϕ2)⊗ : T1 → T2 with  
证据。针对（T1，\_），我们有一个多行图\_:e1×······×en→t2，因此有一个独特的线性图（\_）：T1→t2

ϕ2 = (ϕ2)⊗ ◦ ϕ1  
\_=（\_）\_

as illustrated by the following commutative diagram.  
如下图所示。

ϕ1  
1

E1 × ··· × En / T1  
e1×····×en/t1

MMMMMM(ϕ2)⊗ ϕ2  
mmmmmm（\_2）\_2

M&   
M & &

T2  
T2

Similarly, focusing now on on (T2,ϕ2), we have a multilinear map ϕ1 : E1 × ··· × En → T1, and thus there is a unique linear map (ϕ1)⊗ : T2 → T1 with  
同样，现在重点放在（t2，\_）上，我们有一个多行图\_:e1×······×en→t1，因此有一个唯一的线性图（\_）：t2→t1

ϕ1 = (ϕ1)⊗ ◦ ϕ2  
\_=（\_）\_

as illustrated by the following commutative diagram.  
如下图所示。

ϕ2  
2

E1 × ··· × En / T2  
e1×····×en/t2

MMMMMM(ϕ1)⊗ ϕ1  
mmmmmm（\_1）\_1

M&   
M & &

T1  
t1

Putting these diagrams together, we obtain the commutative diagrams  
把这些图放在一起，就得到了交换图。

8 T1  
8 T1

ϕ1q qqqq(ϕ2)⊗  
q QQQ（\_）

q q  
问Q

qqqq ϕ2  
QQQ\_号

E1 × ··· × En / T2  
e1×····×en/t2

MMMMϕMMMMMMMM(ϕ1)⊗  
\_mmmm mmmm（\_1）

1 & T 1  
1&T 1

and  
和

### 8 T2 8 T2

ϕ2ppppp(ϕ1)⊗  
pppp（\_）

pp  
聚丙烯

pppp ϕ1 p  
pppp\_1p

E1 × ··· × En /T1  
e1×····×en/t1

NNNNϕN2NNNNNN&(ϕ2)⊗  
nnnn\_n2nnnnnn&（\_2）

T2,  
T2，

which means that  
也就是说

ϕ1 = (ϕ1)⊗ ◦ (ϕ2)⊗ ◦ ϕ1 and ϕ2 = (ϕ2)⊗ ◦ (ϕ1)⊗ ◦ ϕ2.  
\_=（\_）（\_）\_1和\_=（\_）（\_）\_2.

On the other hand, focusing on (T1,ϕ1), we have a multilinear map ϕ1 : E1 ×···×En → T1, but the unique linear map h: T1 → T1 with  
另一方面，聚焦于（T1，Ф1），我们有一个多行图，其中的第1行图为：e1×·································

ϕ1 = h ◦ ϕ1  
\_1=H\_1

is h = id, as illustrated by the following commutative diagram  
是h=id，如下图所示

ϕ1  
1

E1 × ··· × En / T1  
e1×····×en/t1

NNNNN id  
nnnnn标识

1. &   
   &

T1,  
T1

and since (ϕ1)⊗ ◦ (ϕ2)⊗ is linear as a composition of linear maps, we must have  
由于（\_）（\_）作为线性地图的组成部分是线性的，我们必须

(ϕ1)⊗ ◦ (ϕ2)⊗ = id.  
（\_1）（\_2）=ID.

Similarly, we have the commutative diagram  
同样，我们有交换图

ϕ2  
2

E1 × ··· ×NNENNnN / T2id  
e1×····×nnnn/t2id

1. &   
   &

T2,  
T2

and we must have  
我们必须有

(ϕ2)⊗ ◦ (ϕ1)⊗ = id.  
（\_2）（\_1）=ID.

This shows that (isomorphism betweenϕ1)⊗T1and (andϕT22).⊗ are inverse linear maps, and thus, (ϕ2)⊗ : T1 → T2 is an  
这表明（\_1）t1和（和\_t22）之间的同构）是反线性映射，因此，（\_2）：t1→t2是

Now that we have shown that tensor products are unique up to isomorphism, we give a construction that produces them. Tensor products are obtained from free vector spaces by a quotient process, so let us begin by describing the construction of the free vector space generated by a set.  
现在我们已经证明张量积在同构上是唯一的，我们给出了产生它们的构造。张量积是通过商过程从自由向量空间中得到的，所以让我们首先描述由集合生成的自由向量空间的构造。

For simplicity assume that our set I is finite, say  
为了简单起见，假设我们的集合i是有限的，比如

I = {♥,♦,♠,♣}.  
我=，♦，，。

The construction works for any field K (and in fact for any commutative ring A, in which case we obtain the free A-module generated by I). Assume that K = R. The free vector space generated by I is the set of all formal linear combinations of the form  
该结构适用于任意场k（实际上也适用于任意交换环a，在这种情况下，我们得到由i生成的自由a-模）。假设k=r，由i生成的自由向量空间是形式的所有形式线性组合的集合。

a♥ + b♦ + c♠ + d♣,  
A+B♦+C+D，

with a,b,c,d ∈ R. It is assumed that the order of the terms does not matter. For example,  
对于a，b，c，d∈r，假设条件的次序不重要。例如，

2♥ − 5♦ + 3♠ = −5♦ + 2♥ + 3♠.  
2−5♦+3=−5♦+2+3。

Addition and multiplication by a scalar are are defined as follows:  
标量的加法和乘法定义如下：

(a1♥ + b1♦ + c1♠ + d1♣) + (a2♥ + b2♦ + c2♠ + d2♣)  
（A1+B1♦+C1+D1）+（A2+B2♦+C2+D2）

= (a1 + a2)♥ + (b1 + b2)♦ + (c1 + c2)♠ + (d1 + d2)♣,  
=（a1+a2）+（b1+b2）♦+（c1+c2）+（d1+d2），

and  
和

α · (a♥ + b♦ + c♠ + d♣) = αa♥ + αb♦ + αc♠ + αd♣,  
α·（a+b♦+c+d）=αa+αb♦+αc+αd，

for all a,b,c,d,α ∈ RR. With these operations, it is immediately verified that we obtain a(I). The set I can be viewed as embedded in R(I) by the injection ι vector space denoted given by  
对于所有a，b，c，d，α∈rr。通过这些操作，我们立即证实我们获得了（i）。通过表示的注入\_矢量空间，可以将集合i视为嵌入到r（i）中

ι(♥) = 1♥, ι(♦) = 1♦, ι(♠) = 1♠, ι(♣) = 1♣.  
（）=1，（♦）=1♦，（）=1，（）=1。

Thus,case, RR(I()I)is isomorophic tocan be viewed as the vector space with the special basisR4. I = {♥,♦,♠,♣}. In our The exact same construction works for any field K, and we obtain a vector space denoted by K(I) and an injection ι: I → K(I).  
因此，案例，rr（i（）i）是同构的，视为具有特殊基础的向量空间4.i=，♦，，。在我们对任意场k的完全相同的构造中，我们得到了一个用k（i）表示的向量空间和一个注入\_：i→k（i）。

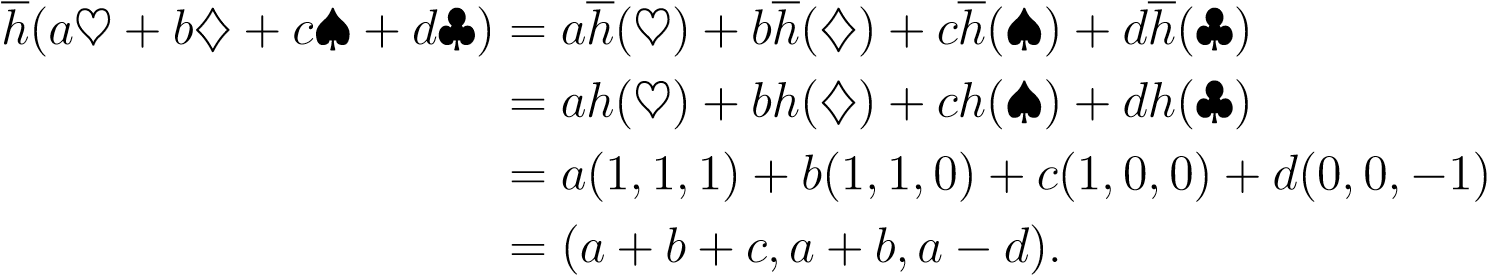
The main reason why the free vector space K(I) over a set I is interesting is that it satisfies a universal mapping property. This means that for every vector space F (over the field K), any function h: I → F, where F is considered just a set, has a unique linear  
集i上的自由向量空间k（i）有趣的主要原因是它满足一个普遍映射属性。这意味着对于每个向量空间f（在k域上），任何函数h:i→f，其中f仅被视为一个集合，具有唯一的线性。

extension h: K(I) → F. By extension, we mean that h(i) = h(i) for all i ∈ I, or more  
推广h:k（i）→f.通过推广，我们的意思是h（i）=h（i）对于所有i∈i，或更多

rigorously that h = h ◦ ι.  
严格地说，h=h。

For example, if I = {♥,♦,♠,♣}, K = R, and F = R3, the function h given by h(♥) = (1,1,1), h(♦) = (1,1,0), h(♠) = (1,0,0), h(♣) = (0,0 − 1)  
例如，如果i=，♦，，，k=r，f=r3，则h（）=（1,1,1），h（♦）=（1,1,0），h（）=（1,0,0），h（）=（0,0−1）给出的函数h。

has a unique linear extension h: R(I) → R3 to the free vector space R(I), given by  
对自由向量空间r（i）具有唯一的线性扩展h:r（i）→r3，由下式给出



To generalize the construction of a free vector space to infinite sets I, we observe that the formal linear combination a♥ + b♦ + c♠ + d♣ can be viewed as the function f : I → R given by  
为了将自由向量空间的构造推广到无穷集i，我们发现形式线性组合a+b♦+c+d可以看作是由

f(♥) = a, f(♦) = b, f(♠) = c, f(♣) = d,  
F（）=A，F（♦）=B，F（）=C，F（）=D，

where a,b,c,d ∈ R. More generally, we can replace R by any field K. If I is finite, then the set of all such functions is a vector space under pointwise addition and pointwise scalar multiplication. If I is infinite, since addition and scalar multiplication only makes sense for finite vectors, we require that our functions f : I → K take the value 0 except for possibly finitely many arguments. We can think of such functions as an infinite sequences (fi)i∈I of elements fi of K indexed by I, with only finitely many nonzero fi. The formalization of this construction goes as follows.  
其中a，b，c，d∈r，一般来说，我们可以用任意的k域代替r，如果i是有限的，那么所有这些函数的集合都是点加法和点标量乘法下的向量空间。如果我是无限的，因为加法和标量乘法只对有限向量有意义，我们要求函数f:i→k取0，除了可能有限的许多参数。我们可以把这些函数看作是一个无限序列（fi）i∈i，它是由i索引的k元素fi的i，只有有限多个非零fi。这项建设的形式化如下。

Given any set I viewed as an index set, let K(I) be the set of all functions f : I → K such that f(i) = 06 only for finitely many i ∈ I. As usual, denote such a function by (fi)i∈I; it is a family of finite support. We make K(I) into a vector space by defining addition and scalar multiplication by  
对于任何我视为索引集的集合，让k（i）是所有函数f:i→k的集合，这样f（i）=06只适用于有限多个i∈i。与通常一样，用（fi）i∈i表示该函数；它是一个有限支持族。通过定义加法和标量乘法，我们将k（i）转化为矢量空间。

(fi) + (gi) = (fi + gi) λ(fi) = (λfi).  
（fi）+（gi）=（fi+gi）λ（fi）=（λfi）。

the vector spaceThe family (ei)i∈IKis defined such that ((I), so that every w ∈ei)Kj (= 0I) can be uniquely written as a finite linearif j =6 i and (ei)i = 1. It is a basis of combination of the ei. There is also an injection ι: I → K(I) such that ι(i) = ei for every i ∈ I. Furthermore, it is easy to show that for any vector space F, and for any function h: I → F, there is a unique linear map h: K(I) → F such that h = h ◦ ι, as in the following diagram.  
向量空间家族（ei）i∈ik的定义是（（i），这样每个w∈ei）kj（=0i）都可以唯一地写为一个有限线性，如果j=6i和（ei）i=1。它是EI组合的基础。也有一个注入：i→k（i），使得（i）=ei对于每一个i∈i。此外，对于任何向量空间f，以及对于任何函数h:i→f，都有一个唯一的线性映射h:k（i）→f，这样h=h，如下图所示。

I CCChCιCCCC/ CK! (hI)  
我是CCCHC CCCC/CK！（嗨）

F  
f

Definition 32.5. The vector space (K(I),ι) constructed as above from a set I is called the free vector space generated by I (or over I). The commutativity of the above diagram is called the universal mapping property of the free vector space (K(I),ι) over I.  
定义32.5.由集合i如上构造的向量空间（k（i），称为i（或i）生成的自由向量空间。上图的交换性称为自由向量空间（k（i），）在i上的普遍映射性。

Using the proof technique of Proposition 32.5, it is not hard to prove that any two vector spaces satisfying the above universal mapping property are isomorphic.  
利用32.5命题的证明技术，不难证明满足上述普适映射性质的任意两个向量空间是同构的。

We can now return to the construction of tensor products. For simplicity consider two vector spaces E1 and E2. Whatever E1 ⊗ E2 and ϕ: E1 × E2 → E1 ⊗ E2 are, since ϕ is supposed to be bilinear, we must have  
现在我们可以回到张量积的构造。为了简单起见，考虑两个向量空间e1和e2。无论e1 e2和\_：e1×e2→e1 e2是什么，既然\_应该是双线性的，我们必须

ϕ(u1 + u2,v1) = ϕ(u1,v1) + ϕ(u2,v1) ϕ(u1,v1 + v2) = ϕ(u1,v1) + ϕ(u1,v2) ϕ(λu1,v1) = λϕ(u1,v1) ϕ(u1,µv1) = µϕ(u1,v1)  
⑨（u1+u2，v1）＝（u1，v1）＋（u2，v1）⑨（u1，v1+v2）＝（u1，v1）＋（u1，v2）⑨（λu1，v1）＝（u1，v1）（u1，\_v1）＝\_（u1，v1）

for all u1,u2 ∈ E1, all v1,v2 ∈ E2, and all λ,µ ∈ K. Since E1 ⊗E2 must satisfy the universal mapping property of Definition 32.4, we may want to define1 2 E1 ⊗E2 as the free vector space1 2 K(E ×E ) generated by I = E1 ×E2 and let ϕ be the injection of E1 ×E2 into K(E ×E ). The problem is that in K(E1×E2), vectors such that  
对于所有的u1，u2∈e1，所有的v1，v2∈e2，和所有的λ，μ∈k，由于e1 e2必须满足定义32.4的普适映射性质，我们可以将1 2 e1 e2定义为i=e1×e2生成的自由矢量空间1 2 k（e×e），并让\_为e1×e2注入k（e×e）。问题是，在k（e1×e2）中，向量

(u1 + u2,v1) and (u1,v1) + (u2,v2)  
（u1+u2，v1）和（u1，v1）+（u2，v2）

are different, when they should really be the same, since ϕ is bilinear. Since K(E1×E2) is free, there are no relations among the generators and this vector space is too big for our purpose.  
是不同的，当它们真的应该是相同的时候，因为\_是双线性的。因为k（e1×e2）是自由的，所以发电机之间没有关系，这个矢量空间对于我们来说太大了。

The remedy is simple: take the quotient of the free vector space K(E1×E2) by the subspace  
补救方法很简单：用子空间取自由向量空间k（e1×e2）的商

N generated by the vectors of the form  
n由形式的向量生成

(u1 + u2,v1) − (u1,v1) − (u2,v1)  
（u1+u2，v1）−（u1，v1）−（u2，v1）

(u1,v1 + v2) − (u1,v1) − (u1,v2)  
（U1，V1+V2）−（U1，V1）−（U1，V2）

(λu1,v1) − λ(u1,v1) (u1,µv1) − µ(u1,v1).  
（λu1，v1）−λ（u1，v1）（u1，μv1）−μ（u1，v1）。

Then, if we let E1 ⊗ E2 be the quotient space K1(E12×E2)/N and let ϕ be the quotient map, this forces ϕ to be bilinear. Checking that (K(E ×E )/N,ϕ) satisfies the universal mapping property is straightforward. Here is the detailed construction.  
那么，如果我们让e1 e2为商空间k1（e12×e2）/n，并让e2为商映射，这将强制\_为双线性。检查（k（e×e）/n，\_）是否满足通用映射属性很简单。这是详细的结构。

Theorem 32.6. Given n ≥ 2 vector spaces E1,...,En, a tensor product (E1 ⊗ ··· ⊗ En,ϕ) for E1,...,En can be constructed. Furthermore, denoting ϕ(u1,...,un) as u1 ⊗···⊗un, the tensor product E1 ⊗ ··· ⊗ En is generated by the vectors u1 ⊗ ··· ⊗ un, where u1 ∈ E1,...,un ∈ En, and for every multilinear map f : E1 × ··· × En → F, the unique linear map f⊗ : E1 ⊗ ··· ⊗ En → F such that f = f⊗ ◦ ϕ is defined by  
定理32.6。给定n≥2个向量空间e1，…，en，可构造e1，…，en的张量积（e1···en，a）。此外，将\_（u1，…，un）表示为u1················································································f=f \_定义为

f⊗(u1 ⊗ ··· ⊗ un) = f(u1,...,un)  
f（u1····un）=f（u1，…，un）

on the generators u1 ⊗ ··· ⊗ un of E1 ⊗ ··· ⊗ En.  
在发电机上，e1··················en的U1······un

Proof. First we apply the construction of a free vector space to the cartesian product I = E1×···×En, obtaining the free vector space M = K(I) on I = E1×···×En. Since every basis generator ei ∈ M is uniquely associated with some n-tuple i = (u1,...,un) ∈ E1 ×···×En, we denote ei by (u1,...,un).  
证据。首先将自由向量空间的构造应用于笛卡尔积i=e1×·········×en，得到i=e1×·······×en上的自由向量空间m=k（i）。由于每一个基生成器ei∈m都与某个n元组i（u1，…，un）∈e1×·······×en唯一相关，因此我们用（u1，…，un）来表示ei。

Next let N be the subspace of M generated by the vectors of the following type:  
接下来，n是由以下类型的向量生成的m的子空间：

(u1,...,ui + vi,...,un) − (u1,...,ui,...,un) − (u1,...,vi,...,un), (u1,...,λui,...,un) − λ(u1,...,ui,...,un).  
（u1，…，ui+vi，…，un）-（u1，…，ui，…，un）-（u1，…，vi，…，un），（u1，…，λui，…，un）-λ（u1，…，ui，…，un）。

We let E1 ⊗···⊗En be the quotient M/N of the free vector space M by N, π: M → M/N be the quotient map, and set  
我们让e1····en是自由向量空间m的商m/n乘以n，π：m→m/n是商映射，并设置

ϕ = π ◦ ι.  
⑨=π\_。

By construction, ϕ is multilinear, and since π is surjective and the ι(i) = ei generate M, the fact that each i is of the form i = (u1,...,un) ∈ E1 × ··· × En implies that ϕ(u1,...,un) generate M/N. Thus, if we denote ϕ(u1,...,un) as u1 ⊗ ··· ⊗ un, the space E1 ⊗ ··· ⊗ En is generated by the vectors u1 ⊗ ··· ⊗ un, with ui ∈ Ei.  
通过构造，\_是多行的，由于π是主观性的，且（i）=ei生成m，每个i的形式为i=（u1，…，un）∈e1×·····················································en由向量u1·····un生成，其中ui∈ei。

It remains to show that (E1 ⊗ ··· ⊗ En,ϕ) satisfies the universal mapping property. To1 n this end, we begin by proving there is a map h such that f = h ◦ ϕ1 . Sincen M = K(E ×···×E )  
仍需证明（e1····en，a）满足通用映射属性。为此，我们首先证明存在一个地图h，这样f=h\_1。sincen m=k（e×····×e）

is free on I = E1 × ··· × En, there is a unique linear map f : K(E ×···×E ) → F, such that  
i=e1×e1······×en上是自由的，有一个独特的线性图f:k（e×e）·······×e）→f，这样

f = f ◦ ι,  
F=F\_，

as in the diagram below.  
如下图所示。

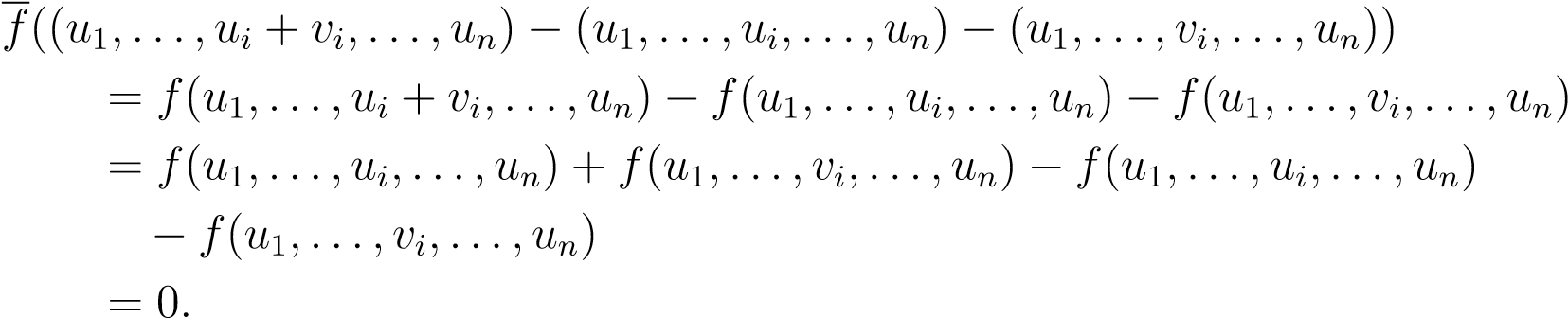
E1 × ··· × En ι / K(E1×···×En) = M  
e1×····×en/k（e1×····×en）=m

SSSSSSSSSfSSSSSSSSS)   
sssssssss）fssssssss）

Because f is multilinear, note that we must have f(w) = 0 for every w ∈ N; for example, on the generator  
因为f是多行的，请注意，对于每个w∈n，我们必须有f（w）=0；例如，在生成器上

(u1,...,ui + vi,...,un) − (u1,...,ui,...,un) − (u1,...,vi,...,un)  
（u1，…，ui+vi，…，un）-（u1，…，ui，…，un）-（u1，…，vi，…，un）

we have  
我们有



But then, f : M → F factors through M/N, which means that there is a unique linear map  
然后，通过m/n得到f:m→f因子，这意味着存在一个唯一的线性映射。

h: M/N → F such that f = h ◦ π making the following diagram commute  
h:m/n→f，使f=hπ作下表通勤

M EEEπEEEE/EEM/N" h  
m eeeπeeee/eem/n“H

f  
f

F,  
f

by defining h([z]) = f(z) for every z ∈ M, where [z] denotes the equivalence class in M/N  
通过定义h（[z]）=f（z），每个z∈m，其中[z]表示m/n中的等价类

of z ∈ M. Indeed, the fact that f vanishes on N insures that h is well defined on M/N, and  
实际上，f在n上消失的事实确保了h在m/n上定义良好，并且

it is clearly linear by definition. Since f = f ◦ ι, from the equation f = h ◦ π, by composing on the right with ι, we obtain  
根据定义，它显然是线性的。由于f=f\_，从方程式f=h\_π，通过在右边用\_组合，我们得到

f = f ◦ ι = h ◦ π ◦ ι = h ◦ ϕ,  
f=f\_=h\_π\_=h\_

as in the following commutative diagram.  
如下图所示。

E1 × ··· ×mmmnmmmmιmmmmK(E1×···×RERnRR)RRRπRRRRRKRR(( E1×···×En)/N mmm6  
e1×····················································

E  
e

RRRRRRRfRRRRRRRR( Ful lllllllhlllllll  
rrrrrrr frrrrrr（ful lllllll hlllll

We now prove the uniqueness of h. For any linear map f⊗ : E1 ⊗ ··· ⊗ En → F such that f = f⊗ ◦ ϕ, since the vectors u1 ⊗···⊗ un generate E1 ⊗···⊗ En and since ϕ(u1,...,un) = u1 ⊗ ··· ⊗ un, the map f⊗ is uniquely defined by  
我们现在证明H的唯一性。对于任何线性映射f：e1············································································

f⊗(u1 ⊗ ··· ⊗ un) = f(u1,...,un).  
f（u1····un）=f（u1，…，un）。

Since f = h ◦ ϕ, the map h is unique, and we let f⊗ = h.   
由于f=h\_，地图h是唯一的，我们让f=h。

The map ϕ from E1 × ··· × En to E1 ⊗ ··· ⊗ En is often denoted by ι⊗, so that  
从e1×····×en到e1·····en的地图通常用表示，因此

ι⊗(u1,...,un) = u1 ⊗ ··· ⊗ un.  
（u1，…，un）=u1···un.

What is important about Theorem 32.6 is not so much the construction itself but the fact that it produces a tensor product with the universal mapping property with respect to multilinear maps. Indeed, Theorem 32.6 yields a canonical isomorphism  
关于定理32.6，重要的不是构造本身，而是它产生一个张量积，关于多行映射具有普遍映射性质。事实上，定理32.6给出了一个正则同构

L(E1 ⊗ ··· ⊗ En,F) ∼= L(E1,...,En;F)  
L（e1····en，f）=L（e1，…，en；f）

between the vector space of linear maps L(E1 ⊗ ··· ⊗ En,F), and the vector space of multilinear maps L(E1,...,En;F), via the linear map − ◦ ϕ defined by h 7→ h ◦ ϕ,  
在线性映射L（e1····en，f）的矢量空间和多行映射L（e1，…，en；f）的矢量空间之间，通过H 7→H \_定义的线性映射−\_，

where h ∈ L(E1 ⊗ ··· ⊗ En,F). Indeed, h ◦ ϕ is clearly multilinear, and since by Theorem 32.6, for every multilinear map f ∈ L(E1,...,En;F), there is a unique linear map f⊗ ∈ L(E1 ⊗ ··· ⊗ En,F) such that f = f⊗ ◦ ϕ, the map − ◦ ϕ is bijective. As a matter of fact, its inverse is the map  
式中h∈l（e1····en，f）。事实上，h是明显的多行的，并且根据定理32.6，对于每一个多行映射f∈l（e1，…，en；f），都有一个唯一的线性映射f∈l（e1···en，f），这样f=f，映射−是双目标的。事实上，它的反方向是地图

f 7→ f⊗.  
F 7→F。

We record this fact as the following proposition.  
我们将这一事实记录为以下命题。

Proposition 32.7. Given a tensor product (E1 ⊗ ··· ⊗ En,ϕ), the linear map h 7→ h ◦ ϕ is a canonical isomorphism  
提案32.7。给定张量积（e1·····en，η），线性映射h 7→h \_是一个典型的同构。

L(E1 ⊗ ··· ⊗ En,F) ∼= L(E1,...,En;F)  
L（e1····en，f）=L（e1，…，en；f）

between the vector space of linear maps L(E1⊗···⊗En,F), and the vector space of multilinear maps L(E1,...,En;F).  
线性映射的向量空间L（e1·····en，f）与多行映射的向量空间L（e1，…，en；f）之间。

Using the “Hom” notation, the above canonical isomorphism is written  
使用“hom”表示法，写出上述规范同构。

Hom(E1 ⊗ ··· ⊗ En,F) ∼= Hom(E1,...,En;F).  
hom（e1····en，f）=hom（e1，…，en；f）。

Remarks:  
评论：

1. To be very precise, since the tensor product depends on the field K, we should subscript the symbol ⊗ with K and write  
   更准确地说，由于张量积依赖于K域，我们应该在符号下加上K并写

E1 ⊗K ··· ⊗K En.  
e1 k····k en.

However, we often omit the subscript K unless confusion may arise.  
然而，我们经常省略下标k，除非出现混淆。

1. For F = K, the base field, Proposition 32.7 yields a canonical isomorphism between the vector space L(E1 ⊗ ··· ⊗ En,K), and the vector space of multilinear forms L(E1,...,En;K). However, L(E1 ⊗···⊗En,K) is the dual space (E1 ⊗···⊗En)∗, and thus the vector space of multilinear forms L(E1,...,En;K) is canonically isomorphic to (E1 ⊗ ··· ⊗ En)∗.  
   对于f=k的基域，命题32.7给出了向量空间l（e1······en，k）与多行形式l（e1，…，en；k）的向量空间之间的规范同构。然而，L（e1···································································

Since this isomorphism is used often, we record it as the following proposition.  
由于这种同构常被使用，我们把它记为以下命题。

Proposition 32.8. Given a tensor product E1 ⊗···⊗En,, there is a canonical isomorphism  
提案32.8。给定张量积e1······en，存在一个正则同构。

L(E1,...,En;K) ∼= (E1 ⊗ ··· ⊗ En)∗  
L（e1，…，en；k）=（e1···en）

between the vector space of multilinear maps L(E1,...,En;K) and the dual (E1 ⊗···⊗En)∗ of the tensor product E1 ⊗ ··· ⊗ En.  
在多行映射的向量空间l（e1，…，en；k）和张量积e1············en的对偶空间之间。

The fact that the map ϕ: E1 × ··· × En → E1 ⊗ ··· ⊗ En is multilinear, can also be expressed as follows:  
图\_：e1×·······················································

u1 ⊗ ··· ⊗ (vi + wi) ⊗ ··· ⊗ un = (u1 ⊗ ··· ⊗ vi ⊗ ··· ⊗ un) + (u1 ⊗ ··· ⊗ wi ⊗ ··· ⊗ un), u1 ⊗ ··· ⊗ (λui) ⊗ ··· ⊗ un = λ(u1 ⊗ ··· ⊗ ui ⊗ ··· ⊗ un).  
U1·····················································································；ui····un）。

Of course, this is just what we wanted!  
当然，这正是我们想要的！

Definition 32.6. Tensors in E1 ⊗ ··· ⊗ En are called n-tensors, and tensors of the form u1 ⊗···⊗un, where ui ∈ Ei are called simple (or decomposable) n-tensors. Those n-tensors that are not simple are often called compound n-tensors.  
定义32.6.e1····en中的张量称为n-张量，u1·······un形式的张量，其中ui∈ei称为简单（或可分解）n-张量。那些不简单的n-张量通常称为复合n-张量。

Not only do tensor products act on spaces, but they also act on linear maps (they are functors).  
张量积不仅作用于空间，而且作用于线性映射（它们是函子）。

Proposition 32.9. Given two linear maps f : E → E0 and g: F → F 0, there is a unique linear map  
提案32.9。给定两个线性映射f:e→e0和g:f→f 0，有一个唯一的线性映射

f ⊗ g: E ⊗ F → E0 ⊗ F 0  
F G:E F→e0 F 0

such that  
这样的话

(f ⊗ g)(u ⊗ v) = f(u) ⊗ g(v),  
（f g）（u v）=f（u）g（v）

for all u ∈ E and all v ∈ F.  
对于所有u∈e和所有v∈f。

Proof. We can define h: E × F → E0 ⊗ F 0 by  
证据。我们可以通过定义h:e×f→e0 f 0

h(u, v) = f(u) ⊗ g(v).  
h（u，v）=f（u）g（v）。

It is immediately verified that h is bilinear, and thus it induces a unique linear map  
立即证明H是双线性的，从而得到一个唯一的线性映射。

f ⊗ g: E ⊗ F → E0 ⊗ F 0  
F G:E F→e0 F 0

making the following diagram commutes  
使下图成为通勤路线

ι⊗  
γ

E × F / E ⊗ F  
E×F/E F

LLLLhLLLLLL& f⊗g  
llllhllll&f G

E0 ⊗ F 0,  
e0 f 0，

such that (f ⊗ g)(u ⊗ v) = f(u) ⊗ g(v), for all u ∈ E and all v ∈ F.   
使（f g）（u v）=f（u）g（v），对于所有u∈e和所有v∈f。

Definition 32.7. The linear map f ⊗ g: E ⊗ F → E0 ⊗ F 0 given by Proposition 32.9 is called the tensor product of f : E → E0 and g: F → F 0.  
定义32.7.命题32.9给出的线性映射f g:e f→e0 f 0称为f:e→e0和g:f→f 0的张量积。

Another way to define f ⊗ g proceeds as follows. Given two linear maps f : E → E0 and g: F → F 0, the map f × g is the linear map from E × F to E0 × F 0 given by (f × g)(u,v) = (f(u),g(v)), for all u ∈ E and all v ∈ F.  
定义f g的另一种方法如下。给定两个线性映射f:e→e0和g:f→f 0，映射f×g是由（f×g）（u，v）=（f（u），g（v））给出的从e×f到e0×f 0的线性映射，对于所有u∈e和所有v∈f。

Then the map h in the proof of Proposition 32.9 is given by h = ι0 ◦ (f × g), and f ⊗ g is ⊗  
则证明32.9的地图h由h=\_0（f×g）给出，f g为

the unique linear map making the following diagram commute.  
唯一的线性地图，使以下图表通勤。

ι⊗  
γ

E × F / E ⊗ F f×gf⊗g  
E×F/E F×GF G

E0 × F 0 / E ⊗ F 0  
e0×f 0/e f 0

ι⊗  
γ

Remark: The notation f⊗g is potentially ambiguous, because Hom(E,F) and Hom(E0,F 0) are vector spaces, so we can form the tensor product Hom(E,F)⊗Hom(E0,F 0) which contains elements also denoted f ⊗ g. To avoid confusion, the first kind of tensor product of linear maps defined in Proposition 32.9 (which yields a linear map in Hom(E ⊗F,E0 ⊗F 0)) can be denoted by T(f,g). If we denote the tensor product E ⊗F by T(E,F), this notation makes it clearer that T is a bifunctor. If E,E0 and F,F 0 are finite dimensional, by picking bases it is not hard to show that the map induced by f ⊗ g 7→ T(f,g) is an isomorphism Hom(E,F) ⊗ Hom(E0,F 0) ∼= Hom(E ⊗ F,E0 ⊗ F 0).  
注：符号f g可能不明确，因为hom（e，f）和hom（e0，f 0）是向量空间，所以我们可以形成张量积hom（e，f）hom（e0，f 0），其中包含也表示f g的元素。为了避免混淆，在命题中定义的线性映射的第一类张量积32.9（以hom（e f，e0 f 0）表示的线性图）可以用t（f，g）表示。如果我们用t（e，f）表示张量积e f，这个符号可以更清楚地表明t是双算符。如果e，e0和f，f 0是有限维的，通过选取基，不难证明f g 7→t（f，g）诱导的映射是同构hom（e，f）hom（e0，f 0）=hom（e f，e0 f 0）。

Proposition 32.10. Suppose we have linear maps f : E → E0, g: F → F 0, f0 : E0 → E00 and g0 : F 0 → F 00. Then the following identity holds:  
提案32.10。假设我们有线性映射f:e→e0，g:f→f 0，f0:e0→e00和g0:f 0→f 00。然后，以下身份保持不变：

(f0 ◦ f) ⊗ (g0 ◦ g) = (f0 ⊗ g0) ◦ (f ⊗ g). (∗)  
（f0 f）（g0 g）=（f0 g0）（f g）.（）

|  |  |  |
| --- | --- | --- |
| Proof. We have the commutative diagram 证据。我们有交换图 |  |  |
|  | ι⊗ γ | / / |

E × F E ⊗ F f×gf⊗g  
e×f e f×gf g

E0 × F / E0⊗F 0 f0×g0f0⊗g0  
e0×f/e0 f 0 f0×g0f0 g0

E00F 00 / E00 ⊗ F 00,  
e00 f 00/e00 f 00，

ι⊗  
γ

|  |  |  |
| --- | --- | --- |
| and thus the commutative diagram. 因此是交换图。 |  |  |
| E F EF | ι⊗ γ | / E F f |

×⊗  
γ

(f0×g0)◦(f×g)(f0⊗g0)◦(f⊗g)  
（f0×g0）（f×g）（f0 g0）（f g）

E00F 00 / E ⊗ F 00  
e00f 00/e f 00

ι⊗  
γ

We also have the commutative diagram.  
我们也有交换图。

ι⊗  
γ

E × F / E ⊗ F  
E×F/E F

(f0◦f)×(g0◦g)(f0◦f)⊗(g0◦g)  
（f0\_f）×（g0 g）（f0\_f）（g0 g）

E00 × F ι00⊗ / E ⊗ F 00.  
e00×f\_00/e f 00.

Since we immediately verify that  
因为我们立即证实

(f0 ◦ f) × (g0 ◦ g) = (f0 × g0) ◦ (f × g),  
（f0 f）×（g0 g）=（f0×g0）（f×g）、

by uniqueness of the map between E ⊗ F and E00 ⊗ F 00 in the above diagram, we conclude that  
通过上图中E F和e00 F 00之间的映射的唯一性，我们得出：

(f0 ◦ f) ⊗ (g0 ◦ g) = (f0 ⊗ g0) ◦ (f ⊗ g),  
（f0 f）（g0 g）=（f0 g0）（f g）、

as claimed.   
如要求。

The above formula (∗) yields the following useful fact.  
上述公式（）得出以下有用的事实。

Proposition 32.11.E0 ⊗ F 0 is also an isomorphism.If f : E → E0 and g: F → F 0 are isomorphims, then f ⊗ g: E ⊗ F →  
命题32.11.e0 f 0也是同构，如果f:e→e0和g:f→f 0是同构，那么f g:e f→

Proof. If f−1 : E0 →−1E⊗ is the inverse ofg−1 : E0 ⊗ F 0 → Ef⊗: EF →is the inverse ofE0 and g−1 : Ff 0⊗→g:FE is the inverse of⊗ F → E0 ⊗ F 0, g: F → F 0 , then f which is shown as follows:  
证据。如果f−1:e0→−1e是g−1:e0 f 0→ef：ef→是e0的倒数，g−1:ff 0→g:fe是f→e0 f 0，g:f→f 0的倒数，则f如下所示：

(f ⊗ g) ◦ (f−1 ⊗ g−1) = (f ◦ f−1) ⊗ (g ◦ g−1)  
（f g）（f−1 g−1）=（f f−1）（g g−1）

= idE0 ⊗ idF0 = idE0⊗F0,  
=ide0 idf0=ide0 f0，

and  
和

(f−1 ⊗ g−1) ◦ (f ⊗ g) = (f−1 ◦ f) ⊗ (g−1 ◦ g) = idE ⊗ idF = idE⊗F .  
（f−1 g−1）（f g）=（f−1 f）（g−1 g）=ide idf=ide f。

Therefore, f ⊗ g: E ⊗ F → E0 ⊗ F 0 is an isomorphism.   
因此，f g:e f→e0 f 0是同构的。

The generalization to the tensor product f1 ⊗ ··· ⊗ fn of n ≥ 3 linear maps fi : Ei → Fi is immediate, and left to the reader.  
n≥3线性映射fi:ei→fi的张量积f1······fn的推广是直接的，留给读者。

## 32.3 Bases of Tensor Products 32.3张量积的基

We showed that E1 ⊗···⊗En is generated by the vectors of the form u1 ⊗···⊗un. However, these vectors are not linearly independent. This situation can be fixed when considering bases.  
我们发现，e1·······en是由u1······un形式的向量生成的。然而，这些向量并不是线性无关的。这种情况可以在考虑基础时加以解决。

To explain the idea of the proof, consider the case when we have two spaces E and F both of dimension 3. Given a basis (e1,e2,e3) of E and a basis (f1,f2,f3) of F, we would like to prove that  
为了解释证明的概念，考虑当我们有两个空间e和f都是维3时的情况。给定e的基（e1，e2，e3）和f的基（f1，f2，f3），我们想证明

e1 ⊗ f1, e1 ⊗ f2, e1 ⊗ f3, e2 ⊗ f1, e2 ⊗ f2, e2 ⊗ f3, e3 ⊗ f1, e3 ⊗ f2, e3 ⊗ f3  
e1 f1，e1 f2，e1 f3，e2 f1，e2 f2，e2 f3，e3 f1，e3 f2，e3 f3

are linearly independent. To prove this, it suffices to show that for any vector space G, if w11,w12,w13,w21,w22,w23,w31,w32,w33 are any vectors in G, then there is a bilinear map h: E × F → G such that h(ei,ej) = wij, 1 ≤ i,j ≤ 3.  
线性无关。为了证明这一点，可以证明对于任何向量空间g，如果w11、w12、w13、w21、w22、w23、w31、w32、w33是g中的任何向量，则存在一个双线性映射h:e×f→g，使得h（e i，e j）=wij，1≤i，j≤3。

Because h yields a unique linear map h⊗ : E ⊗ F → G such that  
因为h产生一个独特的线性映射h：e f→g，这样

h⊗(ei ⊗ ej) = wij, 1 ≤ i,j ≤ 3,  
h（ei ej）=wij，1≤i，j≤3，

and by Proposition 32.4, the vectors  
根据命题32.4，向量

e1 ⊗ f1, e1 ⊗ f2, e1 ⊗ f3, e2 ⊗ f1, e2 ⊗ f2, e2 ⊗ f3, e3 ⊗ f1, e3 ⊗ f2, e3 ⊗ f3  
e1 f1，e1 f2，e1 f3，e2 f1，e2 f2，e2 f3，e3 f1，e3 f2，e3 f3

are linearly independent. This suggests understanding how a bilinear function f : E×F → G is expressed in terms of its values f(ei,fj) on the basis vectors (e1,e2,e3) and (f1,f2,f3), and this can be done easily. Using bilinearity we obtain  
线性无关。这意味着要理解双线性函数f:e×f→g是如何在基向量（e1、e2、e3）和（f1、f2、f3）的基础上用其值f（ei、fj）表示的，并且这很容易做到。利用双线性我们得到

f(u1e1 + u2e2 + u3e3,v1f1 + v2f2 + v3f3) = u1v1f(e1,f1) + u1v2f(e1,f2) + u1v3f(e1,f3) + u2v1f(e2,f1) + u2v2f(e2,f2) + u2v3f(e2,f3)  
f（u1e1+u2e2+u3e3，v1f1+v2f2+v3f3）=u1v1f（e1，f1）+u1v2f（e1，f2）+u1v3f（e1，f3）+u2v1f（e2，f1）+u2v2f（e2，f2）+u2v3f（e2，f3）

+ u3v1f(e3,f1) + u3v2f(e3,f2) + u3v3f(e3,f3). Therefore, given w11,w12,w13,w21,w22,w23,w31,w32,w33 ∈ G, the function h given by  
+U3V1F（E3，F1）+U3V2F（E3，F2）+U3V3F（E3，F3）。因此，给定w11、w12、w13、w21、w22、w23、w31、w32、w33∈g，由

h(u1e1 + u2e2 + u3e3,v1f1 + v2f2 + v3f3) = u1v1w11 + u1v2w12 + u1v3w13  
h（u1e1+u2e2+u3e3，v1f1+v2f2+v3f3）=u1v1w11+u1v2w12+u1v3w13

+ u2v1w21 + u2v2w22 + u2v3w23  
+u2v1w21+u2v2w22+u2v3w23

+ u3v1w31 + u3v2w33 + u3v3w33  
+U3V1W31+U3V2W33+U3V3W33

is clearly bilinear, and by construction h(ei,fj) = wij, so it does the job.  
很明显是双线性的，并且通过构造h（ei，fj）=wij，所以它完成了工作。

The generalization of this argument to any number of vector spaces of any dimension (even infinite) is straightforward.  
把这个论点推广到任何维（甚至无穷大）的任意数量的向量空间是很简单的。

Proposition 32.12. Given n ≥ 2 vector spaces is a basis for Ek,  
提案32.12。如果n≥2个向量空间是Ek的基础，

1 ≤ k ≤ n, then the family of vectors  
1≤k≤n，则向量族



is a basis of the tensor product E1 ⊗ ··· ⊗ En.  
是张量积e1·····en的基础。

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Proof. For each k, 1 ≤ k ≤ n, every vk ∈ Ek can be written uniquely as  
证据。对于每个k，1≤k≤n，每个vk∈ek可以唯一地写为

vk = Xvjkukj ,  
VK=XVJkukj，

j∈Ik  
杰克

for some family of scalars (vjk)j∈Ik. Let F be any nontrivial vector space. We show that for every family  
对于某些scalars家族（vjk）j∈ik。设f为任意非平凡向量空间。我们为每个家庭展示这一点

(wi1,...,in)(i1,...,in)∈I1×...×In,  
（wi1，…，in）（i1，…，in）∈i1×…×in，

of vectors in F, there is some linear map h: E1 ⊗ ··· ⊗ En → F such that  
在f中，有一些线性映射h:e1···en→f，这样

.  
.

Then by Proposition 32.4, it follows that  
然后，根据32.4号提案，它遵循了



is linearly independent. However, since ( is a basis for Ek, the also generate E1 ⊗ ··· ⊗ En, and thus, they form a basis of E1 ⊗ ··· ⊗ En.  
是线性无关的。但是，由于（是EK的基础，也会生成e1······en，因此它们构成e1·····en的基础。

We define the function f : E1 × ··· × En → F as follows: For any n nonempty finite subsets J1,...,Jn such that Jk ⊆ Ik for k = 1,...,n,  
我们定义函数f:e1×································································································

.  
.

It is immediately verified that f is multilinear. By the universal mapping property of the tensor product, the linear map f⊗ : E1 ⊗ ··· ⊗ En → F such that f = f⊗ ◦ ϕ, is the desired map h.   
立即证实f是多行的。根据张量积的普适映射性质，线性映射f：e1····en→f使得f=f \_，是所需的映射h。

In particular, when each Ik is finite and of size mk = dim(Ek), we see that the dimension of the tensor product E1 ⊗ ··· ⊗ En is m1 ···mn. As a corollary of Proposition 32.12, if (uki )i∈Ik is a basis for Ek, 1 ≤ k ≤ n, then every tensor z ∈ E1 ⊗ ··· ⊗ En can be written in a unique way as  
特别是，当每个ik都是有限的，并且其尺寸为mk=dim（ek）时，我们可以看到张量积e1································作为32.12命题的推论，如果（uki）i∈i k是ek的基础，1≤k≤n，那么每个张量z∈e1····en都可以用一种独特的方式写成

z = X λi1,...,in u1i1 ⊗ ··· ⊗ unin,  
z=xλi1，…，单位：u1i1··uni，

(i1,...,in) ∈ I1×...×In  
（I1，…

for some unique family of scalars λi1,...,in ∈ K, all zero except for a finite number.  
对于某些唯一的标量族，在∈k中，除一个有限数外，其余均为零。

## 32.4 Some Useful Isomorphisms for Tensor Products 32.4张量积的一些有用同构

Proposition 32.13. Given three vector spaces E,F,G, there exists unique canonical isomorphisms  
提案32.13。给定三个向量空间e，f，g，存在唯一的正则同构。

1. E ⊗ F ∼= F ⊗ E  
   E F=F E
2. (E ⊗ F) ⊗ G ∼= E ⊗ (F ⊗ G) ∼= E ⊗ F ⊗ G  
   （E F）G=E（F G）=E F G
3. (E ⊕ F) ⊗ G ∼= (E ⊗ G) ⊕ (F ⊗ G)  
   （e\_f）g～=（e g）（f g）
4. K ⊗ E ∼= E  
   K E～E=E

such that respectively  
这样分别

* 1. u ⊗ v 7→ v ⊗ u  
     U V 7→V U
  2. (u ⊗ v) ⊗ w →7 u ⊗ (v ⊗ w) 7→ u ⊗ v ⊗ w  
     （u v）w→7 u（v w）7→u v w
  3. (u, v) ⊗ w →7 (u ⊗ w, v ⊗ w) (d) λ ⊗ u 7→ λu.  
     （u，v）w→7（u w，v w）（d）λu 7→λu。

Proof. Except for (3), these isomorphisms are proved using the universal mapping property of tensor products.  
证据。除（3）外，利用张量积的普遍映射性质证明了这些同构。

1. The map from E × F to F ⊗ E given by (u,v) 7→ v ⊗ u is clearly bilinear, thus it induces a unique linear α: E ⊗ F → F ⊗ E making the following diagram commute  
   由（u，v）7→v u给出的从e×f到f e的图显然是双线性的，因此它产生了一个独特的线性α：e f→f e，使下面的图表通勤

ι  
γ

× FLLLLL⊗LLLL/ LE% ⊗ αF  
×flllll llll/le%αf

⊗ E,  
e，

such that  
这样的话

α(u ⊗ v) = v ⊗ u, for all u ∈ E and all v ∈ F.  
α（u v）=v u，表示所有u∈e和所有v∈f。

Similarly, the map frominduces a unique linear βF: F× ⊗E Eto→E E⊗ ⊗F Fgiven by (making the following diagram commutev,u) 7→ u ⊗ v is clearly bilinear, thus it  
同样地，图中通过（制作下图commentv，u）7→u v明显是双线性的，从而产生了一个独特的线性βf:f×e to→e e e f fgiven，因此

ι  
γ

F × ELLLLL⊗LLLL/ LF% ⊗ βE  
F×elllll lll/lf%βe

E ⊗ F,  
E F，

such that  
这样的话

β(v ⊗ u) = u ⊗ v, for all u ∈ E and all v ∈ F.  
β（v u）=u v，表示所有u∈e和所有v∈f。

It is immediately verified that  
立即证实

(β ◦ α)(u ⊗ v) = u ⊗ v and (α ◦ β)(v ⊗ u) = v ⊗ u  
（βα）（u v）=u v和（αβ）（v u）=v u

for all u ∈ E and all v ∈ F. Since the tensors of the form u ⊗ v span E ⊗ F and similarly the tensors of the form v ⊗ u span F ⊗F E⊗, the mapE, so α andβ ◦βαare isomorphisms.is actually the identity on E ⊗ F, and similarly α ◦ β is the identity on  
对于所有u e和所有v f，由于形式u v SPAN e f的张量和形式v u SPAN f f e f的张量相似，因此，α和βα是同构的，实际上是e f上的恒等式，同样αβ是上的恒等式。

32.4. SOME USEFUL ISOMORPHISMS FOR TENSOR PRODUCTS  
32.4。张量积的一些有用同构

1. Fix some w ∈ G. The map  
   固定一些w∈g.地图

(u, v) 7→ u ⊗ v ⊗ w  
（u，v）7→u v w

from E ×F to E ⊗F ⊗G is bilinear, and thus there is a linear map fw : E ⊗F → E ⊗F ⊗G making the following diagram commute  
从E×F到E F G是双线性的，因此有一个线性图fw:E F→E F G，使下面的图表成为通勤图

ι⊗  
γ

E × FOOOOOOOOOO/O'E ⊗ fwF  
e×foooooooo/o'e前

E ⊗ F ⊗ G,  
E F G，

with fw(u ⊗ v) = u ⊗ v ⊗ w.  
当fw（u v）=u v w时。

Next consider the map  
接下来看地图

(z, w) 7→ fw(z),  
（z，w）7→fw（z）

from (linear mapE ⊗ Ff : () ×EG⊗intoF) ⊗EG⊗→FE⊗⊗GF. It is easily seen to be bilinear, and thus it induces a⊗ G making the following diagram commute  
自（Linear Mape ff：（）×eg intof）eg→fe gf。很容易看出它是双线性的，因此它诱导a g，使下面的图表变为通勤

ι⊗  
γ

(E ⊗ F) ×RGRRRRRRRR/RR(RER( ⊗ Ff) ⊗ G  
（e f）×rgrrrrrrrr/rr（rr（ff）g

E ⊗ F ⊗ G,  
E F G，

with f((u ⊗ v) ⊗ w) = u ⊗ v ⊗ w.  
f（（u v）w）=u v w。

Also consider the map  
还要考虑地图

(u, v,w) 7→ (u ⊗ v) ⊗ w  
（u，v，w）7→（u v）w

fromE E×) ⊗FG×Gmaking the following diagram commuteto (E⊗F)⊗G. It is trilinear, and thus there is a linear map g: E⊗F ⊗G →  
frome e×）f g×gma在下图中绘制通勤（e f）g。它是三线的，因此有一个线性图g:e f g→

( ⊗ F  
（F

ι⊗  
γ

E × F ×QGQQQQQQQQQQ/ QEQ( ⊗ F g⊗ G  
E×F×QGQQQQQQQ/QEQ（F G G

(E ⊗ F) ⊗ G,  
（E F）G，

with g(u ⊗ v ⊗ w) = (u ⊗ v) ⊗ w. Clearly, f ◦ g and g ◦ f are identity maps, and thus f and g are isomorphisms. The other case is similar.  
当g（u v w）=（u v）w时，显然，f g和g f是身份图，因此f和g是同构的。另一种情况类似。

1. Given a fixed vector space G, for any two vector spaces M and N and every linear map f : M → N, let τG(f) = f ⊗idG be the unique linear map making the following diagram commute.  
   给定一个固定的向量空间g，对于任意两个向量空间m和n以及每一个线性映射f:m→n，让τg（f）=f idg成为唯一的线性映射，从而使下表通勤。

ιM⊗  
米尔

* + - * 1. × G / M ⊗ G  
           ×g/m g

f×idGf⊗idG  
f×idgf idg

* + - * 1. × G ιN⊗ / N ⊗ G  
           ×g\_n/n\_g

The identity (∗) proved in Proposition 32.10 shows that if g: N → P is another linear map, then  
命题32.10中证明的恒等式（）表明，如果g:n→p是另一个线性映射，那么

τG(g) ◦ τG(f) = (g ⊗ idG) ◦ (f ⊗ idG) = (g ◦ f) ⊗ (idG ◦ idG) = (g ◦ f) ⊗ idG = τG(g ◦ f). Clearly, τG(0) = 0, and a direct computation on generators also shows that  
τg（g）τg（f）=（g idg）（f idg）=（g f）（idg idg）=（g f）idg=τg（g f）。显然，τg（0）=0，对发电机的直接计算也表明

τG(idM) = (idM ⊗ idG) = idM⊗G,  
τg（idm）=（idm idg）=idm g，

and that if f0 : M → N is another linear map, then  
如果f0:m→n是另一个线性映射，那么

τG(f + f0) = τG(f) + τG(f0).  
τg（f+f0）=τg（f）+τg（f0）。

In fancy terms, τG is a functor. Now, if E ⊕ F is a direct sum, it is a standard fact of linear algebra that if πE : E ⊕ F → E and πF : E ⊕ F → F are the projection maps, then  
用花哨的术语来说，τg是一个函数。现在，如果e f是一个直和，那么线性代数的标准事实是，如果πe:e f→e和πf:e f→f是投影图，那么

πE ◦ πE = πE πF ◦ πF = πF πE ◦ πF = 0 πF ◦ πE = 0 πE + πF = idE⊕F .  
πeπe=πeπfπf=πfπeπf=0πfπe=0πe+πf=ide f。

If we apply τG to these identites, we get  
如果我们把τg应用到这些标识上，我们得到

τG(πE) ◦ τG(πE) = τG(πE) τG(πF ) ◦ τG(πF ) = τG(πF ) τG(πE) ◦ τG(πF ) = 0 τG(πF ) ◦ τG(πE) = 0 τG(πE) + τG(πF ) = id(E⊕F)⊗G.  
τg（πe）τg（πe）=τg（πe）τg（πf）τg（πf）=τg（πf）τg（πe）τg（πf）=0τg（πf）τg（πe）=0τg（πe）+τg（πf）=id（E f）g。

Observe that τG(πE) = πE ⊗idG is a map from (E ⊕F)⊗G onto E ⊗G and that τG(πF ) = πF ⊗idG is a map from (E ⊕F)⊗G onto F ⊗G, and by linear algebra, the above equations mean that we have a direct sum  
观察τg（πe）=πe idg是从（e f）g到e g的映射，τg（πf）=πf idg是从（e f）g到f g的映射，通过线性代数，上述方程意味着我们有一个直接和

(E ⊗ G) ⊕ (F ⊗ G) ∼= (E ⊕ F) ⊗ G.  
（e g）（f g）（e f）g.

(4) We have the linear mapgiven by  
（4）我们得到的线性映射由

for all u ∈ E.  
对于所有的u∈e。

The map (λ,u) 7→ λu from K × E to E is bilinear, so it induces a unique linear map η: K ⊗ E → E making the following diagram commute  
从k×e到e的映射（λ，u）7→λu是双线性的，因此它归纳出一个唯一的线性映射η：k e→e，使下面的图表通勤

ι⊗  
γ

K × E / K ⊗ E  
K×E/K E

LLLLLLLLLLL% η  
lllllllll%η

E,  
E

such that η(λ ⊗ u) = λu, for all λ ∈ K and all u ∈ E. We have  
使得η（λu）=λu，对于所有的λ∈k和所有的u∈e，我们得到



and  
和



which shows that both are the identity, soare isomorphisms.   
这说明两者都是同一性的，翱翔同构。

Remark: The isomorphism (3) can be generalized to finite and even arbitrary direct sums Li∈I Ei of vector spaces (where I is an arbitrary nonempty index set). We have an isomorphism  
注：同构（3）可推广到向量空间的有限甚至任意直和li∈i ei（其中i是任意非空索引集）。我们有同构

.  
.

This isomorphism (with isomorphism (1)) can be used to give another proof of Proposition 32.12 (see Bertin [15], Chapter 4, Section 1) or Lang [106], Chapter XVI, Section 2).  
这种同构（与同构（1））可用于证明命题32.12（见Bertin[15]，第4章，第1节）或Lang[106]，第十六章，第2节）。

Proposition 32.14. Given any three vector spaces E,F,G, we have the canonical isomorphism  
提案32.14。给定任意三个向量空间e，f，g，我们有正则同构

Hom(E,F;G) ∼= Hom(E,Hom(F,G)).  
hom（e，f；g）=hom（e，hom（f，g））。

Proof. Any bilinear map f : E × F → G gives the linear map ϕ(f) ∈ Hom(E,Hom(F,G)), where ϕ(f)(u) is the linear map in Hom(F,G) given by  
证据。任何双线性图f:e×f→g给出线性图（f）∈hom（e，hom（f，g）），其中（f）（u）是hom（f，g）中的线性图，由下式给出

ϕ(f)(u)(v) = f(u,v).  
⑨（f）（u）（v）=f（u，v）。

Conversely, given a linear map g ∈ Hom(E,Hom(F,G)), we get the bilinear map ψ(g) given by ψ(g)(u,v) = g(u)(v),  
反之，给定线性映射g∈hom（e，hom（f，g）），得到由ψ（g）（u，v）=g（u）（v）给出的双线性映射ψ（g）。

and it is clear that ϕ and ψ and mutual inverses.   
很明显，ψ和ψ和相互反比。

Since by Proposition 32.7 there is a canonical isomorphism  
因为在32.7命题中，有一个典型的同构

Hom(E ⊗ F,G) ∼= Hom(E,F;G),  
hom（e f，g）=hom（e，f；g），

together with the isomorphism  
以及同构

Hom(E,F;G) ∼= Hom(E,Hom(F,G))  
hom（e，f；g）=hom（e，hom（f，g））

given by Proposition 32.14, we obtain the important corollary:  
由32.14号命题给出，我们得到了重要的推论：

Proposition 32.15. For any three vector spaces E,F,G, we have the canonical isomorphism  
提案32.15。对于任意三个向量空间e，f，g，我们都有正则同构。

Hom(E ⊗ F,G) ∼= Hom(E,Hom(F,G)).  
hom（e f，g）=hom（e，hom（f，g））。

## 32.5 Duality for Tensor Products 32.5张量积的对偶性

In this section all vector spaces are assumed to have finite dimension, unless specified otherwise. Let us now see how tensor products behave under duality. For this, we define a pairing between and E1⊗···⊗En as follows: For any fixed (, we have the multilinear map  
在本节中，除非另有规定，否则假设所有向量空间都有有限维。现在让我们看看张量积在二元性下的行为。为此，我们定义了和e1·····en之间的配对，如下所示：对于任何固定的（，我们都有多行映射



from E1 × ··· × En to K. The map extends uniquely to a linear map making the following diagram commute.  
从e1×······································

ι⊗  
γ

E1 × ··· × En / E1 ⊗ ··· ⊗ En  
e1×····×en/e1····en

SSSSSSSSSSSSSSSS) KLv1∗,...,vn∗  
ssssssssssss）klv1，…，vn



We also have the multilinear map  
我们还有多行地图



fromto Hom(E1 ⊗···⊗En,K), which extends to a unique linear map L from to Hom(E1 ⊗ ··· ⊗ En,K) making the following diagram commute.  
From to Hom（e1·················································

/  
/

UUULUvU1∗U,...,vUUUUn∗UUUUUU\* L  
Uuuluvu1 u，…，Vuuuuun Uuuuuu\*l

Hom(E1 ⊗ ··· ⊗ En;K)  
hom（e1···en；k）

However, in view of the isomorphism  
然而，鉴于同构

Hom(U ⊗ V,W) ∼= Hom(U,Hom(V,W))  
hom（u v，w）=hom（u，hom（v，w））。

given by Proposition 32.15, with and W = K, we can view L as a linear map  
由命题32.15给出，当且w=k时，我们可以把l看作一个线性映射。



which corresponds to a bilinear map  
对应双线性图

(††)  
（††）

via the isomorphism (U ⊗ V )∗ ∼= Hom(U,V ;K) given by Proposition 32.8. This pairing is given explicitly on generators by  
通过32.8号命题给出的同构（u v）～=hom（u，v；k）。这种配对在发电机上是通过

.  
.

This pairing is nondegenerate, as proved below.  
如下文所述，这种配对是非退化的。

Proof. If () are bases for E1,...,En, then for every basis element  
证据。如果（）是e1，…，en的基，则表示每个基元素

, and any basis element,  
以及任何基本元素，

we have  
我们有

··· ,  
···我是说，

where δij is Kronecker delta, defined such that δij = 1 if i = j, and 0 otherwise. Given any , assume that hα,βi = 0 for all β ∈ E1 ⊗···⊗En. The vector α is a finite linear combination, for some unique λi1,...,in ∈ K. If we choose, then we get  
式中，δi j为Kronecker delta，定义如下：如果i=j，δij=1，否则为0。假设hα，βi=0代表所有β∈e1·····en。向量α是一个有限线性组合，对于一些唯一的λi1，…，在∈k中，如果我们选择，那么我们得到

0 = hα,e1i1 ⊗ ··· ⊗ enini = DXλi1,...,in(e1i1)∗ ⊗ ··· ⊗ (einn)∗,ei11 ⊗ ··· ⊗ einnE  
0=Hα，e1i1···enini=dxλi1，…，in（e1i1）···（einn），ei11···einne

= Xλi1,...,inh(e1i1)∗ ⊗ ··· ⊗ (einn)∗,e1i1 ⊗ ··· ⊗ enini  
=xλi1，…，inh（e1i1）·····（einn），e1i1····enini

= λi1,...,in.  
=λi1，…，英寸

Therefore, α = 0,  
因此α=0，

Conversely, given any β ∈ E1⊗···⊗En, assume that hα,βi = 0, for all. The vector β is a finite linear combination, for some unique λi1,...,in ∈ K. If we choose, then we get  
相反，假设任何β∈e1·····en，假设hα，βi=0。向量β是一个有限线性组合，对于一些唯一的λi1，…，在∈k中，如果我们选择，那么我们得到

0 = h(e1i1)∗ ⊗ ··· ⊗ (enin)∗,βi = D(ei11)∗ ⊗ ··· ⊗ (einn)∗,Xλi1,...,ine1i1 ⊗ ··· ⊗ eninE = Xλi1,...,inh(e1i1)∗ ⊗ ··· ⊗ (einn)∗,e1i1 ⊗ ··· ⊗ enini  
0=H（e1i1）·······（enin），βi=D（ei11）·········（einn），xλi1，…，ine1i1····················································

= λi1,...,in.  
=λi1，…，英寸

Therefore, β = 0.   
因此，β=0。

By Proposition 32.1, we have a canonical isomorphism  
根据命题32.1，我们有一个规范同构

.  
.

Here is our main proposition about duality of tensor products. Proposition 32.16. We have canonical isomorphisms  
这是我们关于张量积二元性的主要命题。提案32.16。我们有规范同构

,  
，

and  
和

Hom(E1,...,En;K).  
hom（e1，…，en；k）。

Proof. The second isomorphism follows from the isomorphism ( together with the isomorphism Hom(E1,...,En;K) ∼= (E1 ⊗···⊗En)∗ given by Proposition  
证据。第二个同构来自同构（连同同构hom（e1，…，en；k）=（e1···en）由命题给出。

32.8.   
32.8。

Remarks:  
评论：

1. The isomorphism Hom(E1,...,En;K) can be described explicitly as the linear extension toof the map given by  
   同构hom（e1，…，en；k）可以明确地描述为映射的线性延伸

.  
.

1. The canonical isomorphism of Proposition 32.16 holds under more general conditions. Namely, that K is a commutative ring with identity and that the Ei are finitelygenerated projective K-modules (see Definition 34.7). See Bourbaki, [25] (Chapter III, §11, Section 5, Proposition 7).  
   命题32.16的规范同构在更一般的条件下成立。也就是说，k是一个具有同一性的交换环，ei是有限生成的射影k模（见定义34.7）。见Bourbaki，[25]（第三章，第11节，第5节，提案7）。

We prove another useful canonical isomorphism that allows us to treat linear maps as tensors.  
我们证明了另一个有用的正则同构，它允许我们把线性映射看作张量。

Let E and F be two vector spaces and let α: E∗ × F → Hom(E,F) be the map defined such that α(u∗,f)(x) = u∗(x)f,  
设e和f为两个向量空间，设α：e×f→hom（e，f）为定义的映射，α（u，f）（x）=u（x）f，

for all u∗ ∈ E∗, f ∈ F, and x ∈ E. This map is clearly bilinear, and thus it induces a linear map α⊗ : E∗ ⊗ F → Hom(E,F) making the following diagram commute  
对于所有u e，f f，和x e，该图显然是双线性的，因此它归纳出一个线性图α：e f→hom（e，f），使下面的图通勤

E∗ × F ι⊗ / E∗ ⊗ F  
E×F/E F

OOOOOαOOOOOO' α⊗  
OOOOOαOOOOOα\_

Hom(E,F),  
霍姆（E，F）

such that α⊗(u∗ ⊗ f)(x) = u∗(x)f.  
这样α（u f）（x）=u（x）f.

Proposition 32.17. If E and F are vector spaces (not necessarily finite dimensional), then the following properties hold:  
提案32.17。如果e和f是向量空间（不一定是有限维），则以下属性成立：

1. The linear map α⊗ : E∗ ⊗ F → Hom(E,F) is injective.  
   线性映射α：e f→hom（e，f）是内射的。
2. If E is finite-dimensional, then α⊗ : E∗ ⊗F → Hom(E,F) is a canonical isomorphism.  
   如果e是有限维，那么α：e f→hom（e，f）是规范同构。
3. If F is finite-dimensional, then α⊗ : E∗ ⊗F → Hom(E,F) is a canonical isomorphism.  
   如果f是有限维，那么α：e f→hom（e，f）是规范同构。

Proof. (1) Let (e∗i )i∈I be a basis of E∗ and let (fj)j∈J be a basis of F. Then we know that (e∗i ⊗fj)i∈I,j∈J is a basis of E∗ ⊗F. To prove that α⊗ is injective, let us show that its kernel is reduced to (0). For any vector  
证据。（1）让（e i）i i为e的基，让（f j）j j为f的基，则我们知道（e i fj）i i，j j是e f的基，为了证明α是内射的，让我们证明它的核是（0）的。对于任何向量

ω = X λij e∗i ⊗ fj  
ω=xλij e i fj

i∈I0,j∈J0  
i∈i0，j∈j0

in E∗ ⊗ F, with I0 and J0 some finite sets, assume that α⊗(ω) = 0. This means that for every x ∈ E, we have α⊗(ω)(x) = 0; that is,  
在e f中，对于i0和j0一些有限集，假设α（ω）=0。这意味着，对于每一个x∈e，我们有α（ω）（x）=0；也就是说，

.  
.

Since (fj)j∈J is a basis of F, for every j ∈ J0, we must have  
因为（f j）j∈j是f的基础，对于每一个j∈j0，我们必须

Xλije∗i (x) = 0, for all x ∈ E.  
xλije i（x）=0，对于所有x∈e。

i∈I0  
我喜欢

But then (e∗i )i∈I0 would be linearly dependent, contradicting the fact that (e∗i )i∈I is a basis of E∗, so we must have  
但是（e i）i i0是线性相关的，这与（e i）i i是e i的基础这一事实相矛盾，因此我们必须

λij = 0, for all i ∈ I0 and all j ∈ J0,  
λi j=0，对于所有i∈i0和所有j∈j0，

which shows that ω = 0. Therefore, α⊗ is injective.  
这表明ω=0。因此，α是注射剂。

(2) Let (ej)1≤j≤n be a finite basis of E, and as usual, let e∗j ∈ E∗ be the linear form defined by  
（2）让（e j）1≤j≤n是e的有限基，通常，让e j∈e是由

e∗j(ek) = δj,k,  
e j（Ek）=δj，k，

where δj,k = 1 iff j = k and 0 otherwise. We know that (e∗j)1≤j≤n is a basis of E∗ (this is where we use the finite dimension of E). For any linear map f ∈ Hom(E,F), for every x = x1e1 + ··· + xnen ∈ E, we have  
式中，δj，k=1 iff j=k，否则为0。我们知道（e j）1≤j≤n是e的基础（这是我们使用e的有限维的地方）。对于任何线性映射f∈hom（e，f），对于每个x=x1e1+·····+xnen∈e，我们有

.  
.

Consequently, every linear map f ∈ Hom(E,F) can be expressed as  
因此，每个线性映射f∈hom（e，f）可以表示为

,  
，

for some fi ∈ F. Furthermore, if we apply f to ei, we get f(ei) = fi, so the fi are unique. Observe that  
对于某些fi∈f，而且，如果将f应用于ei，我们得到f（ei）=fi，因此fi是唯一的。注意

.  
.

Thus, α⊗ is surjective, so α⊗ is a bijection.  
因此，α是可预测的，因此α是双射。

(3) Let (f1,...,fm) be a finite basis of F, and let () be its dual basis. Given any linear map h: E → F, for all u ∈ E, since  
（3）设（f1，…，fm）为f的有限基，设（）为其对偶基。给定任意线性映射h:e→f，对于所有u∈e，因为

.  
.

If  
如果

for all u ∈ E (∗)  
对于所有u∈e（）

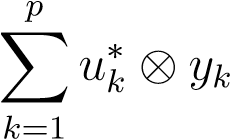
for some linear forms (, then  
对于一些线性形式（，那么

) for all u ∈ E,  
）对于所有u∈e，

which shows that vi∗ = fi∗ ◦ h for i = 1,...,m. This means that h has a unique expression in terms of linear forms as in (∗). Define the map α from (E∗)m to Hom(E,F) by  
这表明，对于i=1，…，m，vi=fi h。这意味着h在线性形式方面有一个独特的表达式，如（）所示。定义从（e）m到hom（e，f）的映射α

for all u ∈ E.  
对于所有的u∈e。

This map is linear. For any h ∈ Hom(E,F), we showed earlier that the expression of h in (∗) is unique, thus α is an isomorphism. Similarly, E∗ ⊗ F is isomorphic to (E∗)m. Any tensor ω ∈ E∗ ⊗ F can be written as a linear combination  
这张地图是线性的。对于任何h∈hom（e，f），我们之前已经证明h在（）中的表达是唯一的，因此α是同构的。同样，e f同构于（e）m。任何张量ω∈e f都可以写成线性组合。



for some and some yk ∈ F, and since (f1,...,fm) is a basis of F, each yk can be written as a linear combination of (f1,...,fm), so ω can be expressed as  
对于一些和一些yk∈f，由于（f1，…，fm）是f的基础，每个yk可以写成（f1，…，fm）的线性组合，因此ω可以表示为

, (†)  
，（？）

for some linear forms which are linear combinations of the. If we pick a basis  
对于一些线性形式，它们是的线性组合。如果我们选择一个基础

(this implies that thewi∗)i∈I for E∗, then we know that the family (vi∗ in (†) are unique. Define the linear mapwi∗ ⊗ fj)i∈I,1≤j≤mβis a basis offrom (E∗)m toE∗E⊗∗ ⊗F, andF by  
（这意味着wi）i i代表e，那么我们知道家族（vi in（†）是独一无二的。定义线性映射wi f j）i∈i，1≤j≤mβ是（e）m toe e f的基

.  
.

Since every tensor ω ∈ E∗ ⊗ F can be written in a unique way as in (†), this map is an isomorphism.   
由于每一张量ω∈e f都可以用与（†）中一样的独特方式书写，因此该图是同构的。

Note that in Proposition 32.17, we have an isomorphism if either E or F has finite dimension. The following proposition allows us to view a multilinear as a tensor product.  
注意，在命题32.17中，如果e或f有有限维，我们有同构。下面的命题允许我们将多行视为张量积。

Proposition 32.18. If the E1,...En are finite-dimensional vector spaces and F is any vector space, then we have the canonical isomorphism Hom(  
提案32.18。如果e1，…（

Proof. In view of the canonical isomorphism  
证据。从规范同构看

Hom(E1,...,En;F) ∼= Hom(E1 ⊗ ··· ⊗ En,F)  
hom（e1，…，en；f）=hom（e1··en，f）

given by Proposition 32.7 and the canonical isomorphism (  
由32.7命题和正则同构给出（

given by Proposition 32.16, if the Ei’s are finite-dimensional, then Proposition 32.17 yields the canonical isomorphism  
由命题32.16给出，如果ei是有限维的，那么命题32.17产生正则同构。

Hom(  
霍姆

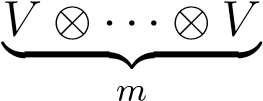
as claimed.   
如要求。

## 32.6 Tensor Algebras 32.6张量代数

Our goal is to define a vector space T(V ) obtained by taking the direct sum of the tensor products  
我们的目标是定义一个向量空间t（v），该向量空间t（v）是通过取张量积的直接和得到的。

,  
，

and to define a multiplication operation on T(V ) which makes T(V ) into an algebraic structure called an algebra. The algebra T(V ) satisfies a universal property stated in Proposition 32.19, which makes it the “free algebra” generated by the vector space V . Definition 32.8. The tensor product  
在t（v）上定义一个乘法运算，使t（v）成为称为代数的代数结构。代数t（v）满足命题32.19中所述的一个普适性，这使得它成为向量空间v生成的“自由代数”。定义32.8.张量积



is also denoted as  
也表示为

or V ⊗m  
或V m

and is called the m-th tensor power of V (with V ⊗1 = V , and V ⊗0 = K).  
称为v的m次张量幂（v 1=v，v 0=k）。

We can pack all the tensor powers of V into the “big” vector space  
我们可以把V的张量幂压缩到“大”向量空间中。

T(V ) = M V ⊗m,  
t（v）=m v m，

m≥0  
m 0

denoted T •(V ) or NV to avoid confusion with the tangent bundle.  
表示t•（v）或nv，以避免与切线束混淆。

This is an interesting object because we can define a multiplication operation on it which makes it into an algebra.  
这是一个有趣的对象，因为我们可以在它上面定义一个乘法运算，使它成为一个代数。

When V is of finite dimension n, we can pick some basis (e1 ...,en) of V , and then every tensor ω ∈ T(V ) can be expressed as a linear combination of terms of the form ei1 ⊗···⊗eik, where (i1,...,ik) is any sequence of elements from the set {1,...,n}. We can think of the tensors ei1 ⊗···⊗eik as monomials in the noncommuting variables e1,...,en. Thus the space T(V ) corresponds to the algebra of polynomials with coefficients in K in n noncommuting variables.  
当v为有限维n时，我们可以选取v的一些基（e1…，en），然后每个张量ω∈t（v）可以表示为ei1·······eik形式的项的线性组合，其中（i1，…，ik）是集合1，…，n中的任何元素序列。我们可以把张量Ei1·····Eik看作非定变量e1，…，en中的单项式。因此，空间t（v）对应于n个非交换变量中系数为k的多项式代数。

Let us review the definition of an algebra over a field. Let K denote any (commutative) field, although for our purposes, we may assume that K = R (and occasionally, K = C). Since we will only be dealing with associative algebras with a multiplicative unit, we only define algebras of this kind.  
让我们回顾一个领域上代数的定义。让k表示任何（交换）场，尽管为了我们的目的，我们可以假设k=r（偶尔，k=c）。因为我们只处理带乘法单位的结合代数，所以我们只定义这种代数。

Definition 32.9. Given a field K, a K-algebra is a K-vector space A together with a bilinear operation ·: A × A → A, called multiplication, which makes A into a ring with unity 1 (or 1A, when we want to be very precise). This means that · is associative and that there is a multiplicative identity element 1 so that 1 · a = a · 1 = a, for all a ∈ A. Given two K-algebras A and B, a K-algebra homomorphism h: A → B is a linear map that is also a ring homomorphism, with h(1A) = 1B; that is,  
定义32.9.对于一个k域，k-代数是一个k-向量空间a加上一个双线性运算·：a×a→a，称为乘法，它使a变成一个单位为1的环（当我们想非常精确的时候，也可以称为1a）。这意味着·是关联的，并且有一个乘法恒等元1，因此1·a=a·1=a，对于所有a∈a。给定两个k-代数a和b，k-代数同态h:a→b是一个线性映射，也是一个环同态，h（1a）=1b；也就是说，

h(a1 · a2) = h(a1) · h(a2) for all a1,a2 ∈ A h(1A) = 1B.  
h（a1·a2）=h（a1）·h（a2）表示所有a1，a2∈a h（1a）=1b。

The set of K-algebra homomorphisms between A and B is denoted Homalg(A,B).  
在a和b之间的k-代数同态集合称为homalg（a，b）。

For example, the ring Mn(K) of all n × n matrices over a field K is a K-algebra.  
例如，域k上所有n×n矩阵的环mn（k）是k-代数。

There is an obvious notion of ideal of a K-algebra.  
有一个关于k-代数理想的明显概念。

Definition 32.10. Let A be a K-algebra. An ideal A ⊆ A is a linear subspace of A that is also a two-sided ideal with respect to multiplication in A; this means that for all a ∈ A and all α,β ∈ A, we have αaβ ∈ A.  
定义32.10。让a成为k-代数。理想a a是a的线性子空间，也是a中关于乘法的双面理想；这意味着对于所有a∈a和所有α，β∈a，我们都有αaβ∈a。

If the field K is understood, we usually simply say an algebra instead of a K-algebra.  
如果能理解k域，我们通常只说代数而不是k代数。

We would like to define a multiplication operation on T(V ) which makes it into a Kalgebra. As  
我们要在t（v）上定义一个乘法运算，使它成为一个Kalgebra。AS

T(V ) = M V ⊗i,  
t（v）=m v i，

i≥0  
我超过0岁

for every i ≥ 0, there is a natural injection ιn : V ⊗n → T(V ), and in particular, an injection ι0 : K → T(V ). The multiplicative unit 1 of T(V ) is the image ι0(1) in T(V ) of the unit 1 of the field K. Since every v ∈ T(V ) can be expressed as a finite sum  
对于每一个i≥0，有一个自然的注入n:v n→t（v），特别是注入0:k→t（v）。t（v）的乘性单位1是k场的单位1的t（v）中的图像\_0（1）。因为每个v∈t（v）都可以表示为一个有限和。

v = ιn1(v1) + ··· + ιnk(vk),  
V=\_n1（v1）+····+nk（vk）、

where vi ∈ V ⊗ni and the ni are natural numbers with ni =6 nj if i =6 j, to define multiplication in T(V ), using bilinearity, it is enough to define multiplication operations  
其中，v i∈v ni和ni是自然数，ni=6 nj，如果i=6 j，用t（v）定义乘法，用双线性定义乘法运算就足够了。

·: V ⊗m × V ⊗n −→ V ⊗(m+n), which, using the isomorphisms V ⊗n ∼= ιn(V ⊗n), yield multiplication operations ·: ιm(V ⊗m) × ιn(V ⊗n) −→ ιm+n(V ⊗(m+n)). First, for ω1 ∈ V ⊗m and ω2 ∈ V ⊗n, we let ω1 · ω2 = ω1 ⊗ ω2.  
·：v m×v n−→v（m+n），使用同构v n=n（v n），产生乘法运算·：m（v m）×n（v n）−→m+n（v（m+n））。首先，对于ω1∈v m和ω2∈v n，我们假设ω1·ω2=ω1ω2。

This defines a bilinear map so it defines a multiplication V ⊗m × V ⊗n −→ V ⊗m ⊗ V ⊗n. This is not quite what we want, but there is a canonical isomorphism  
这定义了一个双线性映射，因此它定义了一个乘法v m×v n−→v m v n。这不是我们想要的，但有一个规范的同构

V ⊗m ⊗ V ⊗n ∼= V ⊗(m+n)  
V m V n=V（m+n）

which yields the desired multiplication ·: V ⊗m × V ⊗n −→ V ⊗(m+n).  
得到所需的乘法·：v m×v n−→v（m+n）。

The isomorphism V ⊗m ⊗ V ⊗n ∼= V ⊗(m+n) can be established by induction using the isomorphism (E ⊗ F) ⊗ G ∼= E ⊗ F ⊗ G. First we prove by induction on m ≥ 2 that  
同构v m v n=v（m+n）可通过使用同构（e f）g=e f g的诱导建立。首先，我们通过在m≥2上的诱导证明

V ⊗(m−1) ⊗ V ∼= V ⊗m,  
V（m-1）V=V m，

and then by induction on n ≥ 1 than  
然后在n≥1上感应

V ⊗m ⊗ V ⊗n ∼= V ⊗(m+n).  
v m v n=v（m+n）。

In summary the multiplication V ⊗m × V ⊗n −→ V ⊗(m+n) is defined so that  
总之，乘法v m×v n−→v（m+n）的定义是：

(v1 ⊗ ··· ⊗ vm) · (w1 ⊗ ··· ⊗ wn) = v1 ⊗ ··· ⊗ vm ⊗ w1 ⊗ ··· ⊗ wn.  
（v1····vm）·（w1·············································

(This has to be made rigorous by using isomorphisms involving the associativity of tensor products, for details, see Jacobson [95], Section 3.9, or Bertin [15], Chapter 4, Section 2.)  
（这必须通过使用涉及张量积关联性的同构来严格，有关详细信息，请参见Jacobson[95]，第3.9节或Bertin[15]，第4章，第2节。）

Definition 32.11. Given a K-vector space V (not necessarily finite dimensional), the vector space  
定义32.11.给定一个k向量空间v（不一定是有限维），向量空间

T(V ) = M V ⊗m  
t（v）=m v m

m≥0  
m 0

denoted T •(V ) or NV equipped with the multiplication operations V ⊗m×V ⊗n −→ V ⊗(m+n) defined above is called the tensor algebra of V .  
表示t•（v）或nv，配上上面定义的乘法运算v m×v n−→v（m+n），称为v的张量代数。

Remark: It is important to note that multiplication in T(V ) is not commutative. Also, in all rigor, the unit 1 of T(V ) is not equal to 1, the unit of the field K. However, in view of the injection ι0 : K → T(V ), for the sake of notational simplicity, we will denote 1 by 1. More generally, in view of the injections ιn : V ⊗n → T(V ), we identify elements of V ⊗n with their images in T(V ).  
注：重要的是要注意t（v）中的乘法不是交换的。同样，严格地说，t（v）的单位1不等于场k的单位1。然而，考虑到注入\_0:k→t（v），为了便于记法，我们将用1表示1。一般来说，考虑到注入n:v n→t（v），我们用t（v）中的图像识别v n的元素。

The algebra T(V ) satisfies a universal mapping property which shows that it is unique up to isomorphism. For simplicity of notation, let i: V → T(V ) be the natural injection of V into T(V ).  
代数t（v）满足一个普适映射性质，表明它在同构方面是唯一的。为了便于记法，让i:v→t（v）是v自然地注入t（v）。

Proposition 32.19. Given any K-algebra A, for any linear map f : V → A, there is a  
提案32.19。对于任何k-代数a，对于任何线性映射f:v→a，有一个

unique K-algebra homomorphism f : T(V ) → A so that  
唯一的k-代数同态f:t（v）→a，因此

f = f ◦ i,  
f=f\_i，

as in the diagram below.  
如下图所示。

V EEEfEEiEE/EET" (Vf )  
v eeefeeee/eet（VF）

A  
一

Proof. Left an an exercise (use Theorem 32.6). A proof can be found in Knapp [102] (Appendix A, Proposition A.14) or Bertin [15] (Chapter 4, Theorem 2.4).   
证据。留下一个练习（使用定理32.6）。可在Knapp[102]中（附录A，提案A.14）或Bertin[15]中找到证据（第4章，定理2.4）。

Proposition 32.19 implies that there is a natural isomorphism  
命题32.19意味着存在一个自然同构

Homalg(T(V ),A) ∼= Hom(V,A),  
homalg（t（v），a）=hom（v，a），

where the algebra A on the right-hand side is viewed as a vector space. Proposition 32.19 also has the following corollary.  
右边的代数A被视为向量空间。命题32.19也有以下推论。

Proposition 32.20. Given a linear map h: V1 → V2 between two vectors spaces V1,V2 over a field K, there is a unique K-algebra homomorphism ⊗h: T(V1) → T(V2) making the following diagram commute.  
提案32.20。在两个向量空间v1，v2之间的线性映射h:v1→v2在一个域k上，有一个独特的k-代数同态h:t（v1）→t（v2），使得下面的图变为通勤图。

i1  
I1

V / T(V1)  
V/T（v1）

h⊗h   
H·H

V2 / T(V2).  
V2/T（V2）。

Most algebras of interest arise as well-chosen quotients of the tensor algebra T(V ). This is true for the exterior algebra V(V ) (also called Grassmann algebra), where we take the quotient of T(V ) modulo the ideal generated by all elements of the form v ⊗ v, where v ∈ V ,and for the symmetric algebra Sym(V ), where we take the quotient of T(V ) modulo the ideal generated by all elements of the form v ⊗ w − w ⊗ v, where v,w ∈ V .  
大多数感兴趣的代数都出现在张量代数t（v）的商中。对于外部代数v（v）（也称为格拉斯曼代数），我们取t（v）模的商为形式v v的所有元素生成的理想，其中v∈v；对于对称代数sym（v），我们取t（v）模的商为形式v v的理想。形式为v w w v的l元素，其中v，w∈v。

Algebras such as T(V ) are graded in the sense that there is a sequence of subspaces  
像t（v）这样的代数在存在一系列子空间的意义上是分级的。

V ⊗n ⊆ T(V ) such that  
v n t（v）使得

T(V ) = MV ⊗n,  
T（V）=mV N，

k≥0  
K＝0

and the multiplication ⊗ behaves well w.r.t. the grading, i.e., ⊗: V ⊗m × V ⊗n → V ⊗(m+n).  
乘法表现良好，即：v m×v n→v（m+n）。

Definition 32.12. A K-algebra E is said to be a graded algebra iff there is a sequence of subspaces En ⊆ E such that  
定义32.12.一个k-代数e被称为一个等级代数，如果有一个子空间序列en e，那么

E = MEn,  
E=男性，

k≥0  
K＝0

(with E0 = K) and the multiplication · respects the grading; that is, ·: Em × En → Em+n. Elements in En are called homogeneous elements of rank (or degree) n.  
（e0=k），乘法表示等级，即：em×en→em+n。en中的元素称为秩（或度）n的齐次元素。

In differential geometry and in physics it is necessary to consider slightly more general tensors.  
在微分几何和物理学中，有必要考虑稍微更一般的张量。

Definition 32.13. Given a vector space V , for any pair of nonnegative integers (r,s), the tensor space Tr,s(V ) of type (r,s) is the tensor product  
定义32.13.给定向量空间v，对于任意一对非负整数（r，s），（r，s）类型的张量空间tr，s（v）是张量积。

,  
，

with T0,0(V ) = K. We also define the tensor algebra T •,•(V ) as the direct sum (coproduct)  
当t0,0（v）=k时，我们也将张量代数t•，•（v）定义为直接和（副积）

T •,•(V ) = M Tr,s(V ).  
t•，•（v）=m tr，s（v）。

r,s≥0  
r，s≥0

Tensors in Tr,s(V ) are called homogeneous of degree (r,s).  
在tr，s（v）中的张量称为度的齐次（r，s）。

Note that tensors in Tr,0(V ) are just our “old tensors” in V ⊗r. We make T •,•(V ) into an algebra by defining multiplication operations  
请注意，t r，0（v）中的张量只是v r中的“旧张量”。我们通过定义乘法运算将t•，•（v）转化为代数。

Tr1,s1(V ) × Tr2,s2(V ) −→ Tr1+r2,s1+s2(V )  
tr1，s1（v）×tr2，s2（v）−→tr1+r2，s1+s2（v）

in the usual way, namely: For and  
以通常的方式，即：对于和

, let  
让

.  
.

Denote by Hom() the vector space of all multilinear maps from V r × (V ∗)s to W. Then we have the universal mapping property which asserts that there is a canonical isomorphism  
用hom（）表示从v r×（v）s到w的所有多行映射的向量空间，然后我们得到一个普适映射性质，即存在一个规范同构。

Hom(Tr,s(V ),W) ∼= Hom(.  
hom（tr，s（v），w）=hom（.

In particular,  
特别地，

(Tr,s(V ))∗ ∼= Hom(.  
（tr，s（v））～=hom（.

For finite dimensional vector spaces, the duality of Section 32.5 is also easily extended to the tensor spaces Tr,s(V ). We define the pairing  
对于有限维向量空间，第32.5节的对偶性也很容易扩展到张量空间tr，s（v）。我们定义了配对

Tr,s(V ∗) × Tr,s(V ) −→ K  
Tr，S（V）×Tr，S（V）−→K

as follows: if  
如下：如果



and  
和

,  
，

then  
然后

.  
.

This is a nondegenerate pairing, and thus we get a canonical isomorphism (Tr,s(V ))∗ ∼= Tr,s(V ∗).  
这是一个非退化配对，因此我们得到一个正则同构（tr，s（v））～=tr，s（v）。

Consequently, we get a canonical isomorphism  
因此，我们得到了一个正则同构。

Tr,s(V ∗) ∼= Hom(.  
tr，s（v）=hom（.

We summarize these results in the following proposition.  
我们将这些结果概括为以下命题。

Proposition 32.21. Let V be a vector space and let  
提案32.21。设V为向量空间，设

.  
.

We have the canonical isomorphisms  
我们有规范同构

(Tr,s(V ))∗ ∼= Tr,s(V ∗),  
（tr，s（v））～=tr，s（v），

and  
和

Tr,s(V ∗) ∼= Hom(.  
tr，s（v）=hom（.

Remark: The tensor spaces, Tr,s(V ) are also denoted ). A tensor α ∈ Tr,s(V ) is said to be contravariant in the first r arguments and covariant in the last s arguments.  
注：张量空间tr，s（v）也表示。张量α∈tr，s（v）在第一个r变元中是反变的，在最后一个s变元中是协变的。

This terminology refers to the way tensors behave under coordinate changes. Given a basis  
这个术语指的是张量在坐标变化下的行为方式。有根据的

) denotes the dual basis, then every tensor α ∈ Tr,s(V ) is given  
）表示对偶基，然后给出每个张量α∈tr，s（v）

by an expression of the form  
通过形式的表达

α = X aij11,...,i,...,jrsei1 ⊗ ··· ⊗ eir ⊗ e∗j1 ⊗ ··· ⊗ e∗js.  
α=x aij11，…，i，…，jrsei1··············································

i1,...,ir j1,...,js  
I1，…，IR J1，…，JS

The tradition in classical tensor notation is to use lower indices on vectors and upper indices on linear forms and in accordance to Einstein summation convention (or Einstein notation) the position of the indices on the coefficients is reversed. Einstein summation convention (already encountered in Section 32.1) is to assume that a summation is performed for all values of every index that appears simultaneously once as an upper index and once as a lower index. According to this convention, the tensor α above is written  
经典张量记法的传统是在向量上使用低指数，在线性形式上使用高指数，并且根据爱因斯坦求和约定（或爱因斯坦记法），指数在系数上的位置是相反的。爱因斯坦求和约定（已在第32.1节中遇到）假设对同时出现一次作为上索引和一次作为下索引的每个索引的所有值进行求和。根据这个惯例，上面的张量α是写的

.  
.

An older view of tensors is that they are multidimensional arrays of coefficients,  
张量的一个古老观点是它们是系数的多维数组，

,  
，

subject to the rules for changes of bases.  
以基地变更规则为准。

Another operation on general tensors, contraction, is useful in differential geometry.  
对一般张量收缩的另一个运算在微分几何中很有用。

Definition 32.14. For all r,s ≥ 1, the contraction ci,j : Tr,s(V ) → Tr−1,s−1(V ), with 1 ≤ i ≤ r and 1 ≤ j ≤ s, is the linear map defined on generators by  
定义32.14.对于所有r，s≥1，收缩ci，j:tr，s（v）→tr−1，s−1（v），其中1≤i≤r和1≤j≤s，是发电机上定义的线性映射

ci,j(u1 ⊗ ··· ⊗ ur ⊗ v1∗ ⊗ ··· ⊗ vs∗)  
Ci，J（U1·······v1···vs）

= vj∗(ui)u1 ⊗ ··· ⊗ ubi ⊗ ··· ⊗ ur ⊗ v1∗ ⊗ ··· ⊗ vbj∗ ⊗ ··· ⊗ vs∗,  
=vj（ui）u1······························································

where the hat over an argument means that it should be omitted.  
在一个论点上的帽子意味着它应该被省略。

Let us figure our what is c1,1 : T1,1(V ) → R, that is c1,1 : V ⊗ V ∗ → R. If (e1,...,en) is a basis of V and () is the dual basis, by Proposition 32.17 every h ∈ V ⊗ V ∗ ∼= Hom(V,V ) can be expressed as  
让我们来计算我们的c1,1:t1,1（v）→r，即c1,1:v v→r。如果（e1，…，en）是v的基，并且（）是对偶基，根据命题32.17，每个h∈v v=hom（v，v）可以表示为

.  
.

As  
AS

,  
，

we get  
我们得到

,  
，

where tr(h) is the trace of h, where h is viewed as the linear map given by the matrix, (aij). Actually, since c1,1 is defined independently of any basis, c1,1 provides an intrinsic definition of the trace of a linear map h ∈ Hom(V,V ).  
其中，tr（h）是h的迹线，其中h是矩阵（aij）给出的线性映射。实际上，由于c1,1是独立于任何基础定义的，c1,1提供了线性映射H∈hom（v，v）轨迹的内在定义。

Remark: Using the Einstein summation convention, if  
备注：如果

,  
，

then  
然后

.  
.

If E and F are two K-algebras, we know that their tensor product E ⊗ F exists as a vector space. We can make E ⊗ F into an algebra as well. Indeed, we have the multilinear map  
如果e和f是两个k-代数，我们知道它们的张量积e f作为向量空间存在。我们也可以把e f变成代数。实际上，我们有多行地图

E × F × E × F −→ E ⊗ F  
E×F×E×F−→E F

given by (a,b,c,d) 7→F(. By the universal mapping property, we get a linear map,ac) ⊗ (bd), where ac is the product of a and c in E and bd is the  
由（a，b，c，d）7→f（.根据通用映射性质，我们得到一个线性映射，a c）（bd），其中ac是e中a和c的乘积，bd是

product of b and d in  
B和D的乘积

E ⊗ F ⊗ E ⊗ F −→ E ⊗ F.  
E F E F−→E F.

Using the isomorphism  
使用同构

E ⊗ F ⊗ E ⊗ F ∼= (E ⊗ F) ⊗ (E ⊗ F),  
E F E F～=（E F）（E F），

we get a linear map  
我们得到一张线性地图

(E ⊗ F) ⊗ (E ⊗ F) −→ E ⊗ F,  
（E F）（E F）−→E F，

and thus a bilinear map,  
因此双线性图，

(E ⊗ F) × (E ⊗ F) −→ E ⊗ F  
（E F）×E F−→E F

which is our multiplication operation in E ⊗ F. This multiplication is determined by  
这是我们在e f中的乘法运算。这个乘法是由

(a ⊗ b) · (c ⊗ d) = (ac) ⊗ (bd).  
（a b）·（c d）=（ac）（bd）。

In summary we have the following proposition.  
总之，我们有以下建议。

Proposition 32.22. Given two K-algebra E and F, the operation on E ⊗ F defined on generators by  
提案32.22。给定两个k-代数e和f，在e f上的运算由

(a ⊗ b) · (c ⊗ d) = (ac) ⊗ (bd)  
（a b）·（c d）=（ac）（bd）

makes E ⊗ F into a K-algebra.  
使e f变成k-代数。

We now turn to symmetric tensors.  
现在我们来讨论对称张量。

## 32.7 Symmetric Tensor Powers 32.7对称张量幂

Our goal is to come up with a notion of tensor product that will allow us to treat symmetric multilinear maps as linear maps. Note that we have to restrict ourselves to a single vector space E, rather then n vector spaces E1,...,En, so that symmetry makes sense. Definition 32.15. A multilinear map f : En → F is symmetric iff  
我们的目标是提出张量积的概念，使我们能够将对称多线性映射视为线性映射。注意，我们必须把自己限制在一个向量空间e，而不是n个向量空间e1，…，en，这样对称才有意义。定义32.15。多行映射f:en→f是对称iff

f(uσ(1),...,uσ(n)) = f(u1,...,un),  
f（uσ（1），…，uσ（n））=f（u1，…，un）

for all ui ∈ E and all permutations, σ: {1,...,n} → {1,...,n}. The group of permutations on {1,...,n} (the symmetric group) is denoted Sn. The vector space of all symmetric multilinear maps f : En → F is denoted by Symn(E;F) or Homsymlin(En,F). Note that Sym1(E;F) = Hom(E,F).  
对于所有ui∈e和所有置换，σ：1，…，n→1，…，n。在1，…，n（对称群）上的排列群表示sn。所有对称多行映射f:en→f的向量空间用symn（e；f）或homsymlin（en，f）表示。注意sym1（e；f）=hom（e，f）。

We could proceed directly as in Theorem 32.6 and construct symmetric tensor products from scratch. However, since we already have the notion of a tensor product, there is a more economical method. First we define symmetric tensor powers.  
我们可以像定理32.6那样直接进行，从头开始构造对称张量积。然而，由于我们已经有了张量积的概念，所以有一种更经济的方法。首先我们定义对称张量幂。

Definition 32.16. An n-th symmetric tensor power of a vector space E, where n ≥ 1, is a vector space S together with a symmetric multilinear map ϕ: En → S such that, for every vector space F and for every symmetric multilinear map f : En → F, there is a unique linear map, with  
定义32.16。向量空间e的第n次对称张量幂，其中n≥1，是向量空间s加上对称多行映射，因此，对于每个向量空间f和每个对称多行映射f:en→f，都有一个唯一的线性映射，其中

,  
，

for all u1,...,un ∈ E, or for short  
对于所有的u1，…，un∈e，或简称



Equivalently, there is a unique linear map f such that the following diagram commutes.  
同样地，有一个独特的线性图f，如下图表通勤。

En ϕ / S  
\_/s

CCCCCCCC! ff  
中交委！FF

F  
f

The above property is called the universal mapping property of the symmetric tensor power (S,ϕ).  
上述性质称为对称张量幂（S，\_）的普适映射性质。

We next show that any two symmetric n-th tensor powers (S1,ϕ1) and (S2,ϕ2) for E are isomorphic.  
接下来我们证明任意两个对称的n阶张量幂（s1，\_）和（s2，\_）对于e是同构的。

Proposition 32.23. Given any two symmetric n-th tensor powers (S1,ϕ1) and (S2,ϕ2) for E, there is an isomorphism h: S1 → S2 such that  
提案32.23。对于e，给定任意两个对称n阶张量幂（s1，\_）和（s2，\_），存在同构h:s1→s2，这样

ϕ2 = h ◦ ϕ1.  
\_2=H\_1.

32.7. SYMMETRIC TENSOR POWERS  
32.7。对称张量幂

Proof. Replace tensor product by n-th symmetric tensor power in the proof of Proposition  
证据。在命题证明中用n次对称张量幂代替张量积

32.5.   
32.5。

We now give a construction that produces a symmetric n-th tensor power of a vector space E.  
我们现在给出一个构造，它产生一个向量空间e的对称n次张量幂。

Theorem 32.24. Given a vector space E, a symmetric n-th tensor power (Sn(E),ϕ) for  
定理32.24。给定一个向量空间e，对称n阶张量幂（sn（e），a），用于

E can be constructed (n ≥ 1). Furthermore, denoting , the symmetric tensor power Sn(E) is generated by the vectors, where u1,...,un ∈ E, and for every symmetric multilinear map f : En → F, the unique linear map such that is defined by  
e可构造（n≥1）。此外，表示对称张量幂sn（e）是由向量生成的，其中，u1，…，un∈e，对于每个对称多行映射f:en→f，唯一的线性映射，其定义如下：



on the generators of Sn(E).  
在sn（e）的发电机上。

Proof. The tensor power E⊗n is too big, and thus we define an appropriate quotient. Let C be the subspace of E⊗n generated by the vectors of the form  
证据。张量幂E n太大，因此我们定义了一个适当的商。设c为由形式向量生成的e n的子空间。

u1 ⊗ ··· ⊗ un − uσ(1) ⊗ ··· ⊗ uσ(n),  
U1···Un−Uσ（1）···Uσ（n）、

for all ui ∈ E, and all permutations σ: {1,...,n} → {1,...,n}. We claim that the quotient space (E⊗n)/C does the job.  
对于所有的ui∈e，和所有的置换σ：1，…，n→1，…，n。我们声称商空间（e n）/c起作用。

Let p: E⊗n → (E⊗n)/C be the quotient map, and let ϕ: En → (E⊗n)/C be the map given by  
设p:e n→（e n）/c为商映射，设a:en→（e n）/c为下式给出的映射

ϕ = p ◦ ϕ0,  
\_=P\_\_0，

where ϕ0 : En → E⊗n is the injection given by ϕ0(u1,...,un) = u1 ⊗ ··· ⊗ un.  
式中，ω0:e n→e n为ω0（u1，…，un）=u1··un给出的注入量。

Let us denote. It is clear that ϕ is symmetric. Since the vectors u1 ⊗ ··· ⊗ un generate E⊗n, and p is surjective, the vectors generate (E⊗n)/C.  
让我们来说明一下。很明显，Ф是对称的。由于向量u1····un生成e n，而p是可预测的，因此向量生成（e n）/c。

It remains to show that ((E⊗n)/C,ϕ) satisfies the universal mapping property. To this end we begin by proving that there is a map h such that f = h ◦ ϕ. Given any symmetric multilinear map f : En → F, by Theorem 32.6 there is a linear map f⊗ : E⊗n → F such that f = f⊗ ◦ ϕ0, as in the diagram below.  
仍需证明（（e n）/c，η）满足通用映射属性。为此，我们首先证明有一个地图h，这样f=h\_。根据定理32.6，对于任何对称多行映射f:e n→f，都有一个线性映射f：e n→f，这样f=f \_0，如下图所示。

En ϕ0 / E⊗n  
\_0/e n

FFFFFFFFF# f⊗ f  
fffffffff f f

F  
f

However, since f is symmetric, we haven)/C → F making the following diagram commute.f⊗(z) = 0 for every z ∈ C. Thus, we get an induced linear map h: (E⊗  
然而，由于f是对称的，所以我们得到了/c→f，使下面的图表通勤。f（z）=0，对于每个z∈c，我们得到了一个诱导线性映射h：（e

E<⊗nK yyϕy0yyyyyy⊗KKKKKpKKKK% n)/C  
E<NK YY\_Y0YYYY KKKKK PKKKK%n）/C

EnEEEfEEEEEE"f yrrrrrrhrrr(rE⊗  
Eneefeeee“f yrrrrhrrr（re\_

F  
f

If we define h([z]) = f⊗(z) for every z ∈ E⊗n, where [z] is the equivalence class in (E⊗n)/C of z ∈ E⊗n, the above diagram shows that f = h ◦ p ◦ ϕ0 = h ◦ ϕ. We now prove the uniqueness of h. For any linear map such that , since  
如果我们定义h（[z]）=f（z）为每个z∈e n，其中[z]是z∈e n的（e n）/c中的等价类，上图显示f=h p \_0=h \_。我们现在证明了H.对于任何线性映射的唯一性，因为

and the vectors generate (E⊗n)/C, the map f  
向量生成（e n）/c，图f

is uniquely defined by  
由唯一定义

.  
.

Since f = h ◦ ϕ, the map h is unique, and we let . Thus, Sn(E) = (E⊗n)/C and ϕ constitute a symmetric n-th tensor power of E.   
由于f=h\_，地图h是唯一的，我们让。因此，sn（e）=（e n）/c和ξ构成e的对称n阶张量幂。

The map ϕ from En to Sn(E) is often denoted ι, so that  
从en到sn（e）的图\_通常表示为\_，因此

.  
.

Again, the actual construction is not important. What is important is that the symmetric n-th power has the universal mapping property with respect to symmetric multilinear maps.  
同样，实际施工也不重要。重要的是对称n次幂对对称多行映射具有普遍的映射性质。

Remark: The notation for the commutative multiplication of symmetric tensor powers is not standard. Another notation commonly used is ·. We often abbreviate “symmetric tensor power” as “symmetric power.” The symmetric power Sn(E) is also denoted SymnE but we prefer to use the notation Sym to denote spaces of symmetric multilinear maps. To be consistent with the use of , we could have used the notation Jn E. Clearly, S1(E) ∼= E and it is convenient to set S0(E) = K.  
注：对称张量幂的交换乘法符号不规范。另一个常用的符号是·。我们通常把“对称张量幂”简称为“对称幂”，对称幂sn（e）也表示为symne，但我们更喜欢用sym表示对称多线性映射的空间。为了与的用法一致，我们可以使用符号Jn e。很明显，s1（e）=e，并且设置S0（e）=k很方便。

The fact that the map ϕ: En → Sn(E) is symmetric and multilinear can also be expressed as follows:  
图\_：en→sn（e）是对称的，多行也可以表示为：

,  
，

for all permutations σ ∈ Sn.  
对于所有置换，σ∈sn。

The last identity shows that the “operation” is commutative. This allows us to view the symmetric tensor as an object called a multiset.  
最后一个恒等式表明“运算”是交换的。这允许我们将对称张量看作一个称为多重集的对象。

32.7. SYMMETRIC TENSOR POWERS  
32.7。对称张量幂

Given a set A, a multiset with elements from A is a generalization of the concept of a set that allows multiple instances of elements from A to occur. For example, if A = {a,b,c,d}, the following are multisets:  
给定一个集合A，包含a元素的多集是集合概念的一个推广，该集合允许出现a元素的多个实例。例如，如果A=A、B、C、D，则以下是多集：

M1 = {a,a,b}, M2 = {a,a,b,b,c}, M3 = {a,a,b,b,c,d,d,d}.  
M1=A，A，B，M2=A，A，B，B，C，M3=A，A，B，B，C，D，D，D。

Here is another way to represent multisets as tables showing the multiplicities of the elements in the multiset:  
下面是另一种将多集表示为显示多集中元素多重性的表的方法：

.  
.

The above are just graphs of functions from the set A = {a,b,c,d} to N. This suggests the following definition.  
以上只是从集合A=A、B、C、D到N的函数图。这表明了以下定义。

Definition 32.17. A finite multiset M over a set A is a function M : A → N such that M(a) = 06 for finitely many a ∈ A. The multiplicity of an element a ∈ A in M is M(a). The set of all multisets over A is denoted by N(A), and we let dom(M) = {a ∈ A | M(a) = 06 }, which is a finite set. The set dom(M) is the set of elements in A that actually occur in  
定义32.17。在集合A上的有限多集M是一个函数m:a→n，这样m（a）=06表示有限多a∈a。在m中，元素a∈a的多重性是m（a）。a上所有多集的集合用n（a）表示，我们让dom（m）=a∈a m（a）=06，这是一个有限集。set dom（m）是a中实际发生的元素集。

MmultisetP. For any multiseta∈dom(AA) Mand is called the(a), and domM ∈(MsizeN)(A)is finite; this sum is the total number of elements in the, note thatof M. Let |MP|a∈=APMa(∈aA)Mmakes sense, since(a). Pa∈A M(a) =  
多集。对于任意多集合∈dom（a a）m and称为（a），而dom∈（msizen）（a）是有限的；这个和是m中的元素总数，注意m的元素总数。让mp a∈=apma（∈aa）mmakes sense，因为（a）.pa∈a m（a）=

Going back to our symmetric tensors, we can view the tensors of the form as multisets of size n over the set E.  
回到对称张量，我们可以把形式的张量看作是集合e上大小为n的多集合。

Theorem 32.24 implies the following proposition.  
定理32.24包含以下命题。

Proposition 32.25. There is a canonical isomorphism  
提案32.25。有一个典型的同构

Hom(Sn(E),F) ∼= Symn(E;F),  
hom（sn（e），f）=symn（e；f），

between the vector space of linear maps Hom(Sn(E),F) and the vector space of symmetric multilinear maps Symn(E;F) given by the linear map − ◦ ϕ defined by h 7→ h ◦ ϕ, with h ∈ Hom(Sn(E),F).  
在线性映射的向量空间hom（sn（e），f）和对称多线性映射的向量空间symn（e；f）之间，由h 7→h\_定义的线性映射−\_给出，其中h∈hom（sn（e），f）。

Proof. The map h ◦ ϕ is clearly symmetric multilinear. By Theorem 32.24, for every symmetric multilinear map f ∈ Symn(E;F) there is a unique linear map Hom(Sn(E),F) such that, so the map − ◦ ϕ is bijective. Its inverse is the map.   
证据。地图H\_是清晰对称的多行。根据定理32.24，对于每一个对称多行映射f∈symn（e；f），都有一个唯一的线性映射hom（sn（e），f），因此该映射−是双射的。与之相反的是地图。

In particular, when F = K, we get the following important fact. Proposition 32.26. There is a canonical isomorphism  
特别是，当f=k时，我们得到以下重要事实。提案32.26。有一个典型的同构

(Sn(E))∗ ∼= Symn(E;K).  
（sn（e））～=symn（e；k）。

Definition 32.18. Symmetric tensors in Sn(E) are called symmetric n-tensors, and tensors of the form, where ui ∈ E, are called simple (or decomposable) symmetric ntensors. Those symmetric n-tensors that are not simple are often called compound symmetric n-tensors.  
定义32.18。sn（e）中的对称张量称为对称n张量，其中ui∈e形式的张量称为简单（或可分解）对称张量。那些不简单的对称n-张量通常称为复合对称n-张量。

Given two linear maps f : E → E0 and g: E → E0, since the map ) is bilinear and symmetric, there is a unique linear map making the following diagram commute.  
给定两个线性映射f:e→e0和g:e→e0，由于该映射）是双线性和对称的，因此有一个唯一的线性映射使下面的图变为通勤图。

/  
/

f×g   
F×G

/ S2(E0).  
/s2（e0）。

Observe thatis determined by  
观察这是由

.  
.

Proposition 32.27. Given any linear maps f : E → E0, g: E → E0, f0 : E0 → E00, and g0 : E0 → E00, we have  
提案32.27。对于任何线性映射f:e→e0，g:e→e0，f0:e0→e00和g0:e0→e00，我们有

.  
.

The generalization to the symmetric tensor product 3 linear maps fi : E → E0 is immediate, and left to the reader.  
对称张量积3线性映射fi:e→e0的推广是直接的，留给读者。

## 32.8 Bases of Symmetric Powers 32.8对称幂的基

The vectorswhere u1,...,um ∈ E generate Sm(E), but they are not linearly independent. We will prove a version of Proposition 32.12 for symmetric tensor powers using multisets.  
其中U1，…，Um∈e产生Sm（e），但它们不是线性无关的。我们将证明一个关于使用多重集的对称张量幂的32.12命题的版本。

Recall that a (finite) multiset over a set I is a function M : I → N, such that M(i) = 06 for finitely many i ∈ I. The set of all multisets over I is denoted as N(I) and we let dom(M) = {i ∈ I | M(i) = 06 }, the finite set of elements in I that actually occur in M. The size of the multiset M is |M| = Pa∈A M(a).  
回想一个集合i上的（有限）多集是一个函数m:i→n，这样m（i）=06表示有限多i∈i。i上所有多集的集合表示为n（i），我们让dom（m）=i∈i m（i）=06，i中实际出现的有限元素集m。多集m的大小是m=pa∈a m（a）。

To explain the idea of the proof, consider the case when m = 2 and E has dimension 3. Given a basis (e1,e2,e3) of E, we would like to prove that  
为了解释证明的概念，考虑当m=2和e的维数为3时的情况。考虑到e的基础（e1，e2，e3），我们想证明



are linearly independent. To prove this, it suffices to show that for any vector space F, if w11,w12,w13,w22,w23,w33 are any vectors in F, then there is a symmetric bilinear map h: E2 → F such that h(ei,ej) = wij, 1 ≤ i ≤ j ≤ 3.  
线性无关。为了证明这一点，可以证明对于任何向量空间f，如果w11、w12、w13、w22、w23、w33是f中的任何向量，则存在对称双线性映射h:e2→f，这样h（ei，ej）=wij，1≤i≤j≤3。

32.8. BASES OF SYMMETRIC POWERS  
32.8。对称幂的基

Because h yields a unique linear map such that  
因为H生成一个唯一的线性映射，这样

,  
，

by Proposition 32.4, the vectors  
根据命题32.4，向量



are linearly independent. This suggests understanding how a symmetric bilinear function f : E2 → F is expressed in terms of its values f(ei,ej) on the basis vectors (e1,e2,e3), and this can be done easily. Using bilinearity and symmetry, we obtain  
线性无关。这表明理解对称双线性函数f:e2→f是如何在基向量（e1，e2，e3）上用其值f（ei，ej）表示的，这很容易做到。利用双线性和对称性，我们得到

f(u1e1 + u2e2 + u3e3,v1e1 + v2e2 + v3e3) = u1v1f(e1,e1) + (u1v2 + u2v1)f(e1,e2) + (u1v3 + u3v1)f(e1,e3) + u2v2f(e2,e2)  
f（u1e1+u2e2+u3e3，v1e1+v2e2+v3e3）=u1v1f（e1，e1）+（u1v2+u2v1）f（e1，e2）+（u1v3+u3v1）f（e1，e3）+u2v2f（e2，e2）

+ (u2v3 + u3v2)f(e2,e3) + u3v3f(e3,e3). Therefore, given w11,w12,w13,w22,w23,w33 ∈ F, the function h given by  
+（u2v3+u3v2）f（e2，e3）+u3v3f（e3，e3）。因此，给定w11、w12、w13、w22、w23、w33∈f，函数h由

h(u1e1 + u2e2 + u3e3,v1e1 + v2e2 + v3e3) = u1v1w11 + (u1v2 + u2v1)w12  
h（u1e1+u2e2+u3e3，v1e1+v2e2+v3e3）=u1v1w11+（u1v2+u2v1）w12

+ (u1v3 + u3v1)w13 + u2v2w22  
+（U1v3+U3v1）w13+u2v2w22

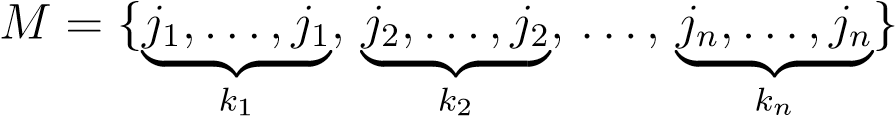
+ (u2v3 + u3v2)w23 + u3v3w33  
+（u2v3+u3v2）w23+u3v3w33

is clearly bilinear symmetric, and by construction h(ei,ej) = wij, so it does the job.  
显然是双线性对称的，并且通过构造h（ei，ej）=wij，所以它完成了任务。

The generalization of this argument to any m ≥ 2 and to a space E of any dimension (even infinite) is conceptually clear, but notationally messy. If dim(E) = n and if (e1,...,en) is a basis of E, for any m vectors, for any symmetric multilinear map f : Em → F, we have  
这个论点对任何m≥2和任何维（甚至无穷大）的空间e的推广在概念上是清楚的，但在本质上是混乱的。如果dim（e）=n并且if（e1，…，en）是e的基础，对于任意m向量，对于任意对称多行映射f:em→f，我们有

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.

Definition 32.19. Given any set J of n ≥ 1 elements, say J = {j1,...,jn}, and given any m ≥ 2, for any sequence (k1 ...,kn) of natural numbers ki ∈ N such that k1 + ··· + kn = m, the multiset M of size m  
定义32.19。给定任意一组n≥1个元素的j，如j=j1，…，jn，并给定任意m≥2，对于任意自然数序列（k1…，kn），ki∈n，使得k1+······+kn=m，大小为m的多集m

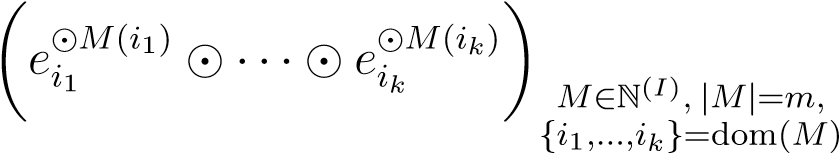


is denoted by M(m,J,k1,...,kn). Note that M(ji) = ki, for i = 1,...,n. Given any k ≥ 1, and any u ∈ E, we denote.  
用m（m，j，k1，…，kn）表示。注意，m（ji）=k i，对于i=1，…，n.给定任意k≥1，且任意u∈e，我们表示。

We can now prove the following proposition.  
我们现在可以证明以下命题。

Proposition 32.28. Given a vector space E, if (ei)i∈I is a basis for E, then the family of  
提案32.28。给定一个向量空间e，如果（e i）i∈i是e的基础，那么

vectors  
向量



is a basis of the symmetric m-th tensor power Sm(E).  
是对称m-th张量幂sm（e）的基础。

Proof. The proof is very similar to that of Proposition 32.12. First assume that E has finite dimension n. In this case I = {1,...,n}, and any multiset M ∈ N(I) of size |M| = m is of the form M(m,{1,...,n},k1,...,kn), with ki = M(i) and k1 + ··· + kn = m. For any nontrivial vector space F, for any family of vectors  
证据。证据与32.12号提案的证据非常相似。首先假设e有有限维n，在这种情况下，i=1，…，n，任何尺寸为m=m的多集m∈n（i）的形式为m（m，1，…，n，k1，…，kn），其中ki=m（i）和k1+·····+kn=m。对于任何非平凡向量空间f，对于任何向量族

(wM)M∈N(I), |M|=m,  
（wm）m∈n（i），m=m，

we show the existence of a symmetric multilinear map h: Sm(E) → F, such that for every M ∈ N(I) with |M| = m, we have  
我们证明了对称多行映射h:sm（e）→f的存在性，这样对于每一个m∈n（i）和m=m，我们得到

,  
，

where {i1,...,ik} = dom(M). We define the map f : Em → F as follows: for any m vectors  
其中i1，…，ik=dom（m）。我们将映射f:em→f定义为：对于任何m向量

v1,...,vm ∈ E we can writeand we set  
v1，…，vm∈e我们可以写，我们可以设置

f(v1,...,vm)  
F（v1，…，vm）

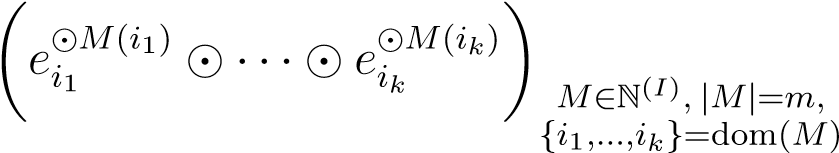
! !!  
！！！

= X X Y u1,i1 ··· Y un,in wM(m,{1,...,n},k1,...,kn).  
=x x y u1，i1···y un，单位为wm（m，1，…，n，k1，…，kn）。

k1+···+kn=m IIi1∩∪···∪Ij=∅I, in=6={j,1|,...,mIj|=k}j i1∈I1 in∈In  
k1+······························································

It is not difficult to verify that f is symmetric and multilinear. By the universal mapping property of the symmetric tensor product, the linear map such that  
不难证明f是对称的和多行的。由对称张量积的普遍映射性质，使线性映射

, is the desired map h. Then by Proposition 32.4, it follows that the family  
，是所需的地图h。然后，根据32.4号提案，它遵循了家庭



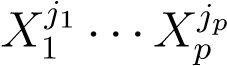
is linearly independent. Using the commutativity of , we can also show that these vectors generate Sm(E), and thus, they form a basis for Sm(E).  
是线性无关的。利用的交换性，我们还可以证明这些向量产生sm（e），因此它们构成了sm（e）的基础。

If I is infinite dimensional, then for any m vectors v1,...,vm ∈ F there is a finite subset J of I such that vk = Pj∈J uj,kej for k = 1,...,m, and if we write n = |J|, then the formula for f(v1,...,vm) is obtained by replacing the set {1,...,n} by J. The details are left as an exercise.   
如果i是无限维，那么对于任意m向量v1，…，vm∈f，i有一个有限的子集j，使得vk=pj∈j uj，kej为k=1，…，m，如果我们写n=j，那么用j替换集合1，…，n得到f（v1，…，vm）的公式。细节留作练习。

32.9. SOME USEFUL ISOMORPHISMS FOR SYMMETRIC POWERS  
32.9。对称幂的一些有用同构

As a consequence, when I is finite, say of size p = dim(E), the dimension of Sm(E) is the number of finite multisets (j1,...,jp), such that j1 + ··· + jp = m, jk ≥ 0. We leave as an exercise to show that this number is m , then the dimension of S. Compare with the dimension of E⊗m, which is pm. In particular, when p = 2, the dimension of Sm(E) is m + 1. This can also be seen directly.  
因此，当我是有限的时，比如说尺寸p=dim（e），sm（e）的维数是有限多集（j1，…，jp）的个数，因此j1+·····+jp=m，jk≥0。作为练习，我们将这个数字表示为m，然后是s的维数，与e m的维数（即pm）进行比较。特别是当p=2时，sm（e）的维数为m+1。这也可以直接看到。

Remark: The number is also the number of homogeneous monomials  
注：该数也是齐次单项式的个数。

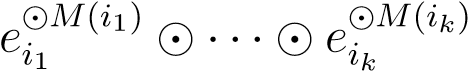


of total degree m in p variables (we have j1 +···+jp = m). This is not a coincidence! Given a vector space E and a basis (ei)i∈I for E, Proposition 32.28 shows that every symmetric tensor z ∈ Sm(E) can be written in a unique way as  
在P变量中的总度数m（我们有j1+····+jp=m）。这不是巧合！在给定向量空间e和基（e i）i∈i for e的情况下，命题32.28表明，每一个对称张量z∈sm（e）都可以用一种独特的方式写成

,  
，

for some unique family of scalars λM ∈ K, all zero except for a finite number.  
对于某些唯一的标量族，除有限数外，其余均为零。

This looks like a homogeneous polynomial of total degree m, where the monomials of total degree m are the symmetric tensors  
这看起来像是总次数m的齐次多项式，其中总次数m的单项式是对称张量。



in the “indeterminates” ei, where i ∈ I (recall that M(i1) + ··· + M(ik) = m) and implies that polynomials can be defined in terms of symmetric tensors.  
在“不确定”ei中，其中i∈i（回想一下m（i1）+·····+m（ik）=m），并暗示多项式可以用对称张量来定义。

## 32.9 Some Useful Isomorphisms for Symmetric Powers 32.9对称幂的一些有用同构

We can show the following property of the symmetric tensor product, using the proof technique of Proposition 32.13 (3).  
利用命题32.13（3）的证明技术，我们可以证明对称张量积的以下性质。

Proposition 32.29. We have the following isomorphism:  
提案32.29。我们有以下同构：

n  
n

Sn(E ⊕ F) ∼= MSk(E) ⊗ Sn−k(F).  
sn（e\_f）=msk（e）sn−k（f）。

k=0  
K＝0

## 32.10 Duality for Symmetric Powers 32.10对称功率的对偶性

In this section all vector spaces are assumed to have finite dimension over a field of characteristic zero. We define a nondegenerate pairing Sn(E∗)×Sn(E) −→ K as follows: Consider the multilinear map  
在本节中，假设所有向量空间在特征为零的场上都有有限维。我们定义了一个非退化配对sn（e）×sn（e）−→k如下：考虑多行映射

(E∗)n × En −→ K  
（e）n×en−→k

given by  
给出者

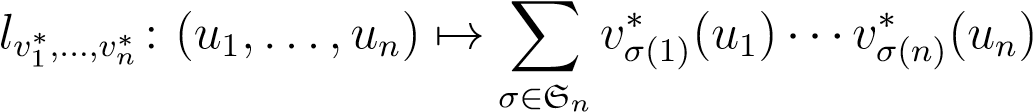
(v1∗,...,vn∗,u1,...,un) 7→ X vσ∗(1)(u1)···vσ∗(n)(un).  
（v1，…，v n，u1，…，un）7→x vσ（1）（u1）···vσ（n）（un）。

σ∈Sn  
σ∈sn

Note that the expression on the right-hand side is “almost” the determinant det(vj∗(ui)), except that the sign sgn(σ) is missing (where sgn(σ) is the signature of the permutation σ; that is, the parity of the number of transpositions into which σ can be factored). Such an expression is called a permanent.  
注意，右边的表达式“几乎”是行列式det（vj（ui）），只是缺少符号sgn（σ）（其中sgn（σ）是置换σ的签名；也就是说，可以将σ分解成因子的转置数的奇偶性）。这样的表达式称为永久表达式。

It can be verified that this expression is symmetric w.r.t. the ui’s and also w.r.t. the vj∗.  
可以验证该表达式是对称的w.r.t.UI和w.r.t.vj。

For any fixed (, we get a symmetric multilinear map  
对于任何固定的（，我们得到一个对称的多行映射



from En to K. The map extends uniquely to a linear map making the following diagram commute:  
从en到k。地图独特地延伸到线性地图，使以下图表通勤：

/  
/

lv1∗,...,vGGGGn∗GGGGG#  
Lv1，…，vgggn ggg#

K. We also have the symmetric multilinear map  
k.我们还有对称多行图



from (E∗)n to Hom(Sn(E),K), which extends to a linear map L from Sn(E∗) to Hom(Sn(E),K) making the following diagram commute:  
从（e）n到hom（sn（e），k），这延伸到从sn（e）到hom（sn（e），k）的线性图l，使以下图表通勤：

/ Sn(E∗)  
/序号（E）

PPPPPPPPPPPP' L  
pppppppppppp'l

Hom(Sn(E),K).  
HOM（序列号（E），K）。

However, in view of the isomorphism  
然而，鉴于同构

Hom(U ⊗ V,W) ∼= Hom(U,Hom(V,W)),  
hom（u v，w）=hom（u，hom（v，w）），

with U = Sn(E∗), V = Sn(E) and W = K, we can view L as a linear map  
当u=sn（e）、v=sn（e）和w=k时，我们可以将l视为线性映射。

L: Sn(E∗) ⊗ Sn(E) −→ K,  
L:锡（E）锡（E）−→K，

which by Proposition 32.8 corresponds to a bilinear map  
根据命题32.8，它对应于双线性图。

h−,−i: Sn(E∗) × Sn(E) −→ K. (∗)  
H−、−I:锡（E）×锡（E）−→K.（）

32.10. DUALITY FOR SYMMETRIC POWERS  
第32.10条。对称幂的对偶性

This pairing is given explicitly on generators by  
这种配对在发电机上是通过

.  
.

Now this pairing in nondegenerate. This can be shown using bases. If (e1,...,em) is a basis of E, then for every basis element (of Sn(E∗), with n1+···+nk = n, we have  
现在这个配对是非退化的。这可以用碱来表示。如果（e1，…，em）是e的基，那么对于每个基元（sn（e），对于n1+····+nk=n，我们有

,  
，

and  
和



if (  
如果（

n1 nk  
N1 NK

If the field K has characteristic zero, then n1!···nk! = 06 . We leave the details as an exercise to the reader. Therefore we get a canonical isomorphism  
如果场k的特征为零，则n1！···NK！=06。我们把细节留给读者作为练习。因此我们得到了一个正则同构

(Sn(E))∗ ∼= Sn(E∗).  
（sn（e））～=sn（e）。

The following proposition summarizes the duality properties of symmetric powers.  
下面的命题总结了对称幂的对偶性。

Proposition 32.30. Assume the field K has characteristic zero. We have the canonical isomorphisms  
提案32.30。假设场k的特征为零。我们有规范同构

(Sn(E))∗ ∼= Sn(E∗)  
（sn（e））～=sn（e）

and  
和

Sn(E∗) ∼= Symn(E;K) = Homsymlin(En,K),  
sn（e））=symn（e；k）=homsymlin（en，k）、

which allows us to interpret symmetric tensors over E∗ as symmetric multilinear maps.  
这使得我们可以将E上的对称张量解释为对称多线性映射。

Proof. The isomorphism µ: Sn(E∗) ∼= Symn(E;K)  
证据。同构：sn（e））=symn（e；k）

follows from the isomorphisms (Sn(E))∗ ∼= Sn(E∗) and (Sn(E))∗ ∼= Symn(E;K) given by Proposition 32.26.   
从32.26号提案给出的同构（sn（e））～=sn（e）和（sn（e））～=symn（e；k）得出。

Remarks:  
评论：

1. The isomorphism µ: Sn(E∗) ∼= Symn(E;K) discussed above can be described explicitly as the linear extension of the map given by  
   上述讨论的同构（e））=symn（e；k）可以明确地描述为图的线性延伸

.  
.

If (e1,...,em) is a basis of E, then for every basis element ( of  
如果（e1，…，em）是e的基，那么对于

Sn(E∗), with n1 + ··· + nk = n, we have  
sn（e），n1+·····+nk=n，我们有

,  
，

If the field K has positive characteristic, then it is possible that n1!···nk! = 0, and this is why we required K to be of characteristic 0 in order for Proposition 32.30 to hold.  
如果场k具有正特性，则可能是n1！···NK！=0，这就是为什么我们要求k的特征为0，以保持命题32.30。

1. The canonical isomorphism of Proposition 32.30 holds under more general conditions. Namely, that K is a commutative algebra with identity over Q, and that the E is a finitely-generated projective K-module (see Definition 34.7). See Bourbaki, [25] (Chapter III, §11, Section 5, Proposition 8).  
   在更一般的条件下，32.30命题的规范同构成立。也就是说，k是一个恒等式超过q的交换代数，e是一个有限生成的射影k模（见定义34.7）。见Bourbaki，[25]（第三章，第11节，第5节，提案8）。

The map from En to Sn(E) given by ( yields a surjection π: E⊗n → Sn(E). Because we are dealing with vector spaces, this map has some section; that is, there is some injection η: Sn(E) → E⊗n with π◦η = id. Since our field K has characteristic 0, there is a special section having a natural definition involving a symmetrization process defined as follows: For every permutation σ, we have the map rσ : En → E⊗n given by  
从e n到sn（e）的映射由（产生一个推测π：e n→sn（e）。因为我们处理的是向量空间，所以这张图有一个部分，也就是说，有一些注入η：sn（e）→e n，πη=id。由于我们的场k有特征0，所以有一个特殊的部分有一个自然定义，涉及一个对称化过程，定义如下：对于every排列σ，我们得到了图rσ：e n→e n，由

rσ(u1,...,un) = uσ(1) ⊗ ··· ⊗ uσ(n).  
rσ（u1，…，u n）=uσ（1）··uσ（n）。

As rσ is clearly multilinear, rσ extends to a linear map (rσ)⊗ : E⊗n → E⊗n making the following diagram commute  
由于rσ显然是多行的，rσ延伸到一个线性图（rσ）：e n→e n，使得下图中的通勤

EnFFrFσFιF⊗FFFF/ " E⊗ (rnσ)⊗  
Enfrfσf\_f ffff/“e（rnσ）

E⊗n,  
埃恩，

and we get a map Sn × E⊗n −→ E⊗n, namely  
我们得到一个图sn×e n−→e n，即

σ · z = (rσ)⊗(z).  
σ·z=（rσ）（z）。

It is immediately checked that this is a left action of the symmetric group Sn on E⊗n, and the tensors z ∈ E⊗n such that  
立即检查这是对称群sn在e n上的左作用，张量z∈e n使得

σ · z = z, for all σ ∈ Sn  
σ·z=z，对于所有的σ∈sn

are called symmetrized tensors.  
称为对称张量。

We define the map η: En → E⊗n by  
我们定义了图η：e n→e n

.  
.

32.11. SYMMETRIC ALGEBRAS  
32.11.对称代数

As the right hand side is clearly symmetric, we get a linear map making the following diagram commute.  
由于右手边是明显对称的，所以我们得到了一个线性地图，使下面的图表成为通勤路线。

/  
/

GGGηGGGGGG# η  
gggηgggggη

E⊗n  
埃恩

Clearly, )) is the set of symmetrized tensors in E⊗n. If we consider the map where π is the surjection π: E⊗n → Sn(E), it is easy to check  
很明显，）是e n中对称张量的集合。如果我们考虑π是投影π：e n→sn（e）的映射，很容易检查

that S ◦ S = S. Therefore, S is a projection, and by linear algebra, we know that  
S S=S。因此，S是一个投影，根据线性代数，我们知道



It turns out that KerS = E⊗n ∩I = Ker π, where I is the two-sided ideal of T(E) generated by all tensors of the form u ⊗ v − v ⊗ u ∈ E⊗2 (for example, see Knapp [102], Appendix A).  
结果表明，Kers=e n i=Kerπ，其中i是由形式为u v v u e 2的所有张量产生的t（e）的双面理想（例如，见Knapp[102]，附录A）。

Therefore, η is injective,  
因此，η是注射剂，



and the symmetric tensor power Sn(E) is naturally embedded into E⊗n.  
对称张量幂sn（e）自然嵌入e n中。

## 32.11 Symmetric Algebras 32.11对称代数

As in the case of tensors, we can pack together all the symmetric powers Sn(V ) into an algebra.  
在张量的情况下，我们可以将所有对称幂sn（v）组合成一个代数。

Definition 32.20. Given a vector space V , the space  
定义32.20。给定向量空间v，空间

S(V ) = M Sm(V ),  
s（v）=m sm（v）、

m≥0  
m 0

is called the symmetric tensor algebra of V .  
称为V的对称张量代数。

We could adapt what we did in Section 32.6 for general tensor powers to symmetric tensors but since we already have the algebra T(V ), we can proceed faster. If I is the two-sided ideal generated by all tensors of the form u ⊗ v − v ⊗ u ∈ V ⊗2, we set  
我们可以将第32.6节中关于一般张量幂的内容修改为对称张量，但是由于我们已经有了代数t（v），我们可以更快地进行。如果i是由u v v u∈v 2形式的所有张量产生的双面理想，我们将

S•(V ) = T(V )/I.  
S•（V）=T（V）/I。

Observe that since the ideal I is generated by elements in V ⊗2, every tensor in I is a linear combination of tensors of the form ω1 ⊗(u⊗v −v ⊗u)⊗ω2, with ω1 ∈ V ⊗n1 and ω2 ∈ V ⊗n2 for some n1,n2 ∈ N, which implies that  
观察到，由于理想i是由v 2中的元素生成的，i中的每个张量都是形式为ω1（u v u）ω2的张量的线性组合，其中ω1∈v n1和ω2∈v n2对于某些n1，n2∈n，这意味着

I = M (I ∩ V ⊗m).  
i=m（i v m）。

m≥0  
m 0

Then, S•(V ) automatically inherits a multiplication operation which is commutative, and since T(V ) is graded, that is  
然后，s•（v）自动继承一个可交换的乘法运算，因为t（v）是分级的，即

T(V ) = M V ⊗m,  
t（v）=m v m，

m≥0  
m 0

we have  
我们有

S•(V ) = M V ⊗m/(I ∩ V ⊗m).  
S•（V）=M V M/（I V M）。

m≥0  
m 0

However, it is easy to check that  
但是，很容易检查

Sm(V ) ∼= V ⊗m/(I ∩ V ⊗m),  
sm（v）=v m/（i v m）、

so  
所以

S•(V ) ∼= S(V ).  
S•（V）=S（V）。

When V is of finite dimension n, S(V ) corresponds to the algebra of polynomials with coefficients in K in n variables (this can be seen from Proposition 32.28). When V is of infinite dimension and (ui)i∈I is a basis of V , the algebra S(V ) corresponds to the algebra of polynomials in infinitely many variables in I. What’s nice about the symmetric tensor algebra S(V ) is that it provides an intrinsic definition of a polynomial algebra in any set of I variables.  
当v为有限维n时，s（v）对应于n个变量中系数为k的多项式代数（这可以从命题32.28中看出）。当v是无穷维且（ui）i∈i是v的基础时，代数s（v）对应于i中无穷多变量中的多项式代数。对称张量代数s（v）的优点在于它提供了任意i v集合中多项式代数的内在定义。咏叹调。

It is also easy to see that S(V ) satisfies the following universal mapping property.  
也很容易看出S（V）满足以下通用映射属性。

Proposition 32.31. Given any commutative K-algebra A, for any linear map f : V → A, there is a unique K-algebra homomorphism f : S(V ) → A so that  
提案32.31。对于任意一个交换的k-代数a，对于任何线性映射f:v→a，都有一个唯一的k-代数同态f:s（v）→a，因此

f = f ◦ i,  
f=f\_i，

as in the diagram below.  
如下图所示。

V EEEfEEiEE/EES(" V f )  
电子设备/电子设备（“V F）

A  
一

Remark: If E is finite-dimensional, recall the isomorphism µ: Sn(E∗) −→ Symn(E;K) defined as the linear extension of the map given by  
备注：如果e为有限维，则回忆同构：sn（e）−→symn（e；k），定义为由

.  
.

Now we have also a multiplication operation Sm(E∗)×Sn(E∗) −→ Sm+n(E∗). The following question then arises:  
现在我们还有一个乘法运算sm（e）×sn（e）－→sm+n（e）。然后出现以下问题：

32.11. SYMMETRIC ALGEBRAS  
32.11.对称代数

Can we define a multiplication Symm(E;K) × Symn(E;K) −→ Symm+n(E;K) directly on symmetric multilinear forms, so that the following diagram commutes?  
我们可以直接在对称多行形式上定义一个乘法符号（e；k）×symn（e；k）－→sym+n（e；k），以便下面的图表转换吗？

S /  
S/

µm×µn µm+n  
礹m×礹n礹m+n

Symm(E;K) × Symn(E;K) · / Symm+n (E;K)  
symm（e；k）×symm（e；k）·/symm+n（e；k）

The answer is yes! The solution is to define this multiplication such that for f ∈ Symm(E;K) and g ∈ Symn(E;K),  
答案是肯定的！解决方法是定义这个乘法，这样对于f∈symm（e；k）和g∈symn（e；k），就

(f · g)(u1,...,um+n) = X f(uσ(1),...,uσ(m))g(uσ(m+1),...,uσ(m+n)), (∗)  
（f·g）（u1，…，u m+n）=x f（uσ（1），…，uσ（m））g（uσ（m+1），…，uσ（m+n）），（）

σ∈shuffle(m,n)  
σ∈洗牌（m，n）

where shuffle(m,n) consists of all (m,n)-“shuffles;” that is, permutations σ of {1,...m+n} such that σ(1) < ··· < σ(m) and σ(m + 1) < ··· < σ(m + n). Observe that a (m,n)-shuffle is completely determined by the sequence σ(1) < ··· < σ(m).  
式中，shuffle（m，n）由所有（m，n）-“shuffles”组成；即，置换σof 1，…m+n，这样，σ（1）<······<σ（m）和σ（m+1）<······<σ（m+n）。观察到a（m，n）-洗牌完全由序列σ（1）<······<σ（m）决定。

For example, suppose m = 2 and n = 1. Given , the multiplication structure on S(E∗) implies that (). Furthermore, for u1,u2,u3,∈ E,  
例如，假设m=2，n=1。给定，s（e）上的乘法结构意味着（）。此外，对于U1，U2，U3，∈e，

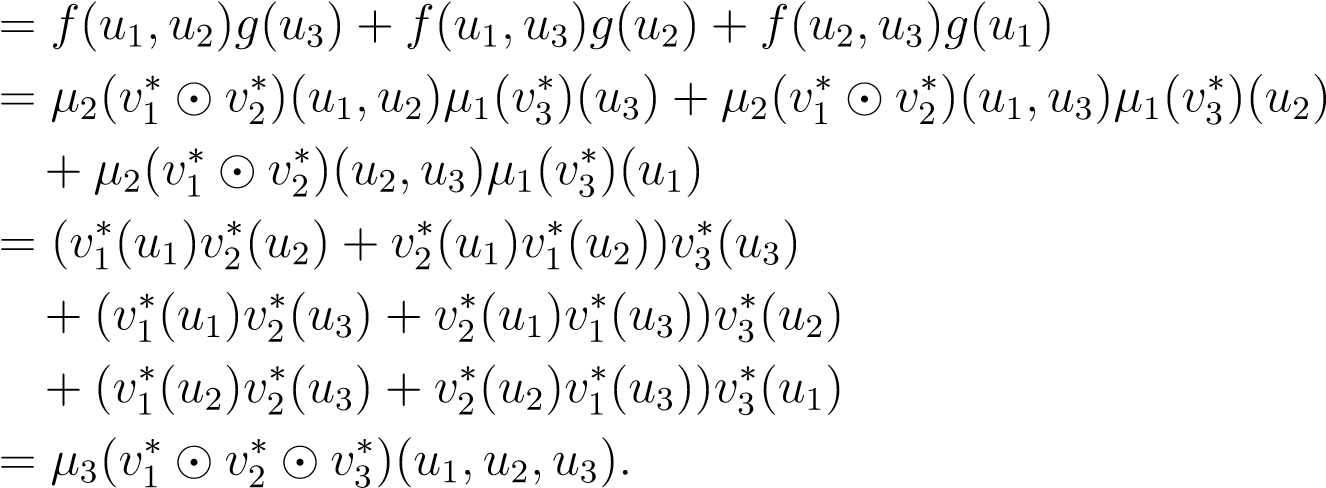
.  
.

Now the (2,1)- shuffles of {1,2,3} are the following three permutations, namely  
现在1,2,3的（2,1）-洗牌是以下三种排列，即

, , .  
、、。

If) and), then (∗) implies that  
如果）和），那么（）意味着

(f f(uσ(1),uσ(2))g(uσ(3))  
（f f（uσ（1），uσ（2））g（uσ（3））



We leave it as an exercise for the reader to verify Equation (∗) for arbitrary nonnegative integers m and n.  
我们把它作为一个练习，让读者验证任意非负整数m和n的方程（）。

Another useful canonical isomorphism (of K-algebras) is given below.  
另一个有用的正则同构（k-代数）如下。

Proposition 32.32. For any two vector spaces E and F, there is a canonical isomorphism  
提案32.32。对于任意两个向量空间e和f，都存在一个正则同构。

(of K-algebras)  
（关于k-代数）

S(E ⊕ F) ∼= S(E) ⊗ S(F).  
s（e\_f）=s（e）s（f）。

## 32.12 Problems 32.12问题

Problem 32.1. Prove Proposition 32.4.  
问题32.1。证明32.4号提案。

Problem 32.2. Given two linear maps f : E → E0 and g: F → F 0, we defined the unique linear map  
问题32.2。给定两个线性映射f:e→e0和g:f→f 0，我们定义了唯一的线性映射

f ⊗ g: E ⊗ F → E0 ⊗ F 0  
F G:E F→e0 F 0

by  
通过

(f ⊗ g)(u ⊗ v) = f(u) ⊗ g(v),  
（f g）（u v）=f（u）g（v）

for all u ∈ E and all v ∈ F. See Proposition 32.9. Thus f ⊗ g ∈ Hom(E ⊗ F,E0 ⊗ F 0). If we denote the tensor product E ⊗ F by T(E,F), and we assume that E,E0 and F,F 0 are finite dimensional, pick bases and show that the map induced by f ⊗ g →7 T(f,g) is an isomorphism  
关于所有u∈e和所有v∈f，见命题32.9。因此f g∈hom（e f，e0 f 0）。如果我们用t（e，f）表示张量积e f，并且假定e，e0和f，f 0是有限维的，那么选取基并证明由f g→7t（f，g）引起的映射是同构的。

Hom(E,F) ⊗ Hom(E0,F 0) ∼= Hom(E ⊗ F,E0 ⊗ F 0).  
hom（e，f）hom（e0，f 0）=hom（e f，e0 f 0）。

Problem 32.3. Adjust the proof of Proposition 32.13 (2) to show that  
问题32.3。调整建议32.13（2）的证明，以证明

E ⊗ (F ⊗ G) ∼= E ⊗ F ⊗ G,  
E（F G）=E F G，

whenever E, F, and G are arbitrary vector spaces.  
当e、f和g是任意向量空间时。

Problem 32.4. Given a fixed vector space G, for any two vector spaces M and N and every linear map f : M → N, we defined τG(f) = f ⊗idG to be the unique linear map making the following diagram commute.  
问题32.4。给定一个固定的向量空间g，对于任意两个向量空间m和n以及每一个线性映射f:m→n，我们将τg（f）=f idg定义为唯一的线性映射，使下表通勤。

ιM⊗  
米尔

1. × G / M ⊗ G  
   ×g/m g

f×idGf⊗idG  
f×idgf idg

1. × G ιN⊗ / N ⊗ G  
   ×g\_n/n\_g

See the proof of Proposition 32.13 (3). Show that  
见提案32.13（3）的证明。展示一下

1. τG(0) = 0,  
   τg（0）=0，
2. τG(idM) = (idM ⊗ idG) = idM⊗G,  
   τg（idm）=（idm idg）=idm g，
3. If f0 : M → N is another linear map, then τG(f + f0) = τG(f) + τG(f0).  
   如果f0:m→n是另一个线性映射，则τg（f+f0）=τg（f）+τg（f0）。

32.12. PROBLEMS  
32.12条。问题

Problem 32.5. Induct on m ≥ 2 to prove the canonical isomorphism  
问题32.5。引入m≥2证明正则同构

V ⊗m ⊗ V ⊗n ∼= V ⊗(m+n).  
v m v n=v（m+n）。

Use this isomorphism to show that ·: V ⊗m × V ⊗n −→ V ⊗(m+n) defined as  
用这个同构表示·：v m×v n−→v（m+n）定义为

(v1 ⊗ ··· ⊗ vm) · (w1 ⊗ ··· ⊗ wn) = v1 ⊗ ··· ⊗ vm ⊗ w1 ⊗ ··· ⊗ wn.  
（v1····vm）·（w1·············································

induces a multiplication on T(V ).  
诱导t（v）上的乘法。

Hint. See Jacobson [95], Section 3.9, or Bertin [15], Chapter 4, Section 2.).  
暗示。见Jacobson[95]第3.9节或Bertin[15]第4章第2节。

Problem 32.6. Prove Proposition 32.19.  
问题32.6。证明32.19号提案。

Hint. See Knapp [102] (Appendix A, Proposition A.14) or Bertin [15] (Chapter 4, Theorem  
暗示。参见Knapp[102]（附录A，提案A.14）或Bertin[15]（第4章，定理

2.4).  
2.4）。

Problem 32.7. Given linear maps f0 : E0 → E00 and g0 : E0 → E00, show that  
问题32.7。给定线性映射f0:e0→e00和g0:e0→e00，显示

.  
.

Problem 32.8. Complete the proof of Proposition 32.28 for the case of an infinite dimensional vector space E.  
问题32.8。对于无穷维向量空间e，完成32.28命题的证明。

Problem 32.9. Let I be a finite index set of cardinality p. Let m be a nonnegative integer. Show that the number of multisets over I with cardinality.  
问题32.9。设为基数p的有限索引集，设m为非负整数。用基数显示多集在i上的数目。

Problem 32.10. Prove Proposition 32.29.  
问题32.10。证明32.29号提案。

Problem 32.11. Using bases, show that the bilinear map at (∗) in Section 32.10 produces a nondegenerate pairing.  
问题32.11。使用碱基，显示第32.10节中（）处的双线性映射产生非退化配对。

Problem 32.12. Let I be the two-sided ideal generated by all tensors of the form u ⊗ v − v ⊗ u ∈ V ⊗2. Prove that Sm(V ) ∼= V ⊗m/(I ∩ V ⊗m).  
问题32.12。我是由u v v u∈v 2形式的所有张量产生的双面理想。证明sm（v）=v m/（i v m）。

Problem 32.13. Verify Equation (∗) of Section 32.11 for arbitrary nonnegative integers m and n.  
问题32.13。验证第32.11节中任意非负整数m和n的方程（）。

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Chapter 33  
第三十三章

# Exterior Tensor Powers and Exterior Algebras 外张量幂与外代数

## 33.1 Exterior Tensor Powers 33.1外部张量幂

In this chapter we consider alternating (also called skew-symmetric) multilinear maps and exterior tensor powers (also called alternating tensor powers), denoted Vn(E). In many respects alternating multilinear maps and exterior tensor powers can be treated much like symmetric tensor powers, except that sgn(σ) needs to be inserted in front of the formulae valid for symmetric powers.  
在本章中，我们考虑交替（也称为斜对称）多线性映射和外部张量幂（也称为交替张量幂），表示vn（e）。在许多方面，交替的多线性映射和外部张量幂可以被看作是对称张量幂，除了sgn（σ）需要插入到对称幂的有效公式前面。

Roughly speaking, we are now in the world of determinants rather than in the world of permanents. However, there are also some fundamental differences, one of which being that the exterior tensor power Vn(E) is the trivial vector space (0) when E is finite-dimensional and when n > dim(E). This chapter provides the firm foundations for understanding differential forms.  
粗略地说，我们现在处于行列式的世界，而不是永久性的世界。然而，也有一些基本的区别，其中一个是，当e为有限维和n>dim（e）时，外张量幂Vn（e）是平凡向量空间（0）。本章为理解微分形式提供了坚实的基础。

As in the case of symmetric tensor powers, since we already have the tensor algebra T(V ), we can proceed rather quickly. But first let us review some basic definitions and facts.  
在对称张量幂的情况下，由于我们已经有了张量代数t（v），我们可以很快地进行。但首先让我们回顾一些基本的定义和事实。

Definition 33.1. Let f : En → F be a multilinear map. We say that f alternating iff for all ui ∈ E, f(u1,...,un) = 0 whenever ui = ui+1, for some i with 1 ≤ i ≤ n − 1; that is, f(u1,...,un) = 0 whenever two adjacent arguments are identical. We say that f is skew-symmetric (or anti-symmetric) iff  
定义33.1.设f:en→f为多行地图。我们说，当ui=ui+1时，f（u1，…，un）=0，对于某些i，当1≤i≤n-1时，f（u1，…，un）=0，即当两个相邻参数相同时，f（u1，…，un）=0。我们说f是斜对称（或反对称）iff

f(uσ(1),...,uσ(n)) = sgn(σ)f(u1,...,un),  
f（uσ（1），…，uσ（n））=sgn（σ）f（u1，…，un）

for every permutation σ ∈ Sn, and all ui ∈ E.  
对于每一个置换，σ∈sn，和所有ui∈e。

For n = 1, we agree that every linear map f : E → F is alternating. The vector space of all multilinear alternating maps f : En → F is denoted Altn(E;F). Note that Alt1(E;F) = Hom(E,F). The following basic proposition shows the relationship between alternation and skew-symmetry.  
对于n=1，我们同意每个线性映射f:e→f是交替的。所有多行交替映射f:en→f的向量空间表示为altn（e；f）。注意alt1（e；f）=hom（e，f）。下面的基本命题说明了交替对称与斜对称的关系。

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Proposition 33.1. Let f : En → F be a multilinear map. If f is alternating, then the following properties hold:  
提案33.1.设f:en→f为多行地图。如果f是交替的，则以下属性保持不变：

1. For all i, with 1 ≤ i ≤ n − 1,  
   对于所有i，1≤i≤n-1，

f(...,ui,ui+1,...) = −f(...,ui+1,ui,...).  
f（…，ui，ui+1，…）=-f（…，ui+1，ui，…）。

1. For every permutation σ ∈ Sn,  
   对于每一个置换σ∈sn，

f(uσ(1),...,uσ(n)) = sgn(σ)f(u1,...,un).  
f（uσ（1），…，uσ（n））=sgn（σ）f（u1，…，un）。

1. For all i,j, with 1 ≤ i < j ≤ n,  
   对于所有i，j，1≤i<j≤n，

f(...,ui,...uj,...) = 0 whenever ui = uj.  
当ui=uj时，f（…，ui，…uj，…）=0。

Moreover, if our field K has characteristic different from 2, then every skew-symmetric multilinear map is alternating.  
此外，如果我们的场k的特征不同于2，那么每个斜对称多行映射都是交替的。

Proof. (1) By multilinearity applied twice, we have  
证据。（1）通过应用两次多语种，我们已经

f(...,ui + ui+1,ui + ui+1,...) = f(...,ui,ui,...) + f(...,ui,ui+1,...)  
f（…，ui+ui+1，ui+ui+1，…）=f（…，ui，ui，…）+f（…，ui，ui+1，…）

+ f(...,ui+1,ui,...) + f(...,ui+1,ui+1,...).  
+f（…，ui+1，ui，…）+f（…，ui+1，ui+1，…）。

Since f is alternating, we get  
因为f是交替的，我们得到

0 = f(...,ui,ui+1,...) + f(...,ui+1,ui,...);  
0=f（…，ui，ui+1，…）+f（…，ui+1，ui，…）；

that is, f(...,ui,ui+1,...) = −f(...,ui+1,ui,...).  
也就是说，f（…，ui，ui+1，…）=f（…，ui+1，ui，…）。

1. Clearly, the symmetric group, Sn, acts on Altn(E;F) on the left, via  
   显然，对称群sn作用于左边的altn（e；f），via

σ · f(u1,...,un) = f(uσ(1),...,uσ(n)).  
σ·f（u1，…，u n）=f（uσ（1），…，uσ（n））。

Consequently, as Sn is generated by the transpositions (permutations that swap exactly two elements), since for a transposition, (2) is simply (1), we deduce (2) by induction on the number of transpositions in σ.  
因此，由于sn是由换位（正好交换两个元素的换位）生成的，因为对于换位，（2）是简单的（1），我们通过诱导σ中换位的数量来推导（2）。

1. There is a permutation σ that sends ui and uj respectively to u1 and u2. By hypothesis ui = uj, so we have uσ(1) = uσ(2), and as f is alternating we have  
   有一个排列σ，它将ui和uj分别发送到u1和u2。假设ui=uj，那么我们有uσ（1）=uσ（2），当f是交替的，我们有

f(uσ(1),...,uσ(n)) = 0.  
f（uσ（1），…，uσ（n））=0.

However, by (2),  
然而，在（2）中，

f(u1,...,un) = sgn(σ)f(uσ(1),...,uσ(n)) = 0.  
f（u1，…，u n）=sgn（σ）f（uσ（1），…，uσ（n））=0。

Now when f is skew-symmetric, if σ is the transposition swapping ui and ui+1 = ui, as sgn(σ) = −1, we get  
当f为斜对称时，如果σ为换位交换ui且ui+1=ui，当sgn（σ）=1时，我们得到

f(...,ui,ui,...) = −f(...,ui,ui,...),  
f（…，ui，ui，…）=-f（…，ui，ui，…），

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so that  
以便

2f(...,ui,ui,...) = 0,  
2f（…，ui，ui，…）=0，

and in every characteristic except 2, we conclude that f(...,ui,ui,...) = 0, namely f is alternating.   
在除2之外的所有特征中，我们得出f（…，ui，ui，…）=0，即f是交替的。

Proposition 33.1 shows that in every characteristic except 2, alternating and skewsymmetric multilinear maps are identical. Using Proposition 33.1 we easily deduce the following crucial fact.  
命题33.1表明，除2外，在每个特征中，交替和偏对称多行映射是相同的。利用33.1号提案，我们很容易推断出以下关键事实。

Proposition 33.2. Let f : En → F be an alternating multilinear map. For any families of vectors, (u1,...,un) and (v1,...,vn), with ui,vi ∈ E, if  
提案33.2.设f:en→f为交替多行地图。对于任何向量族，（u1，…，un）和（v1，…，vn），带有ui，vi∈e，if

, 1 ≤ j ≤ n,  
，1≤j≤n，

then  
然后

,  
，

where A is the n × n matrix, A = (aij).  
其中a是n×n矩阵，a=（aij）。

Proof. Use Property (ii) of Proposition 33.1.   
证据。使用提案33.1的财产（ii）。

We are now ready to define and construct exterior tensor powers.  
我们现在已经准备好定义和构造外部张量幂。

Definition 33.2. An n-th exterior tensor power of a vector space E, where n ≥ 1, is a vector space A together with an alternating multilinear map ϕ: En → A, such that for every vector space F and for every alternating multilinear map f : En → F, there is a unique linear map f∧ : A → F with  
定义33.2.向量空间e的第n次外张量幂，其中n≥1，是向量空间a加上交替的多行映射，使得对于每个向量空间f和每个交替的多行映射f:en→f，都有一个唯一的线性映射f：a→f，其中

f(u1,...,un) = f∧(ϕ(u1,...,un)),  
f（u1，…，un）=f（（u1，…，un）），

for all u1,...,un ∈ E, or for short  
对于所有的u1，…，un∈e，或简称

f = f∧ ◦ ϕ.  
F=F \_。

Equivalently, there is a unique linear map f∧ such that the following diagram commutes:  
同样地，有一个独特的线性图f，这样下图就可以通勤：

EnDDDDϕDDDD!/ A f∧  
endddd\_dddd！/A F\_

f  
f

F.  
f.

The above property is called the universal mapping property of the exterior tensor power (A,ϕ).  
上述性质称为外张量幂（A，\_）的普适映射性质。

We now show that any two n-th exterior tensor powers (A1,ϕ1) and (A2,ϕ2) for E are isomorphic.  
我们现在证明，e的任意两个n次外部张量幂（a1，\_）和（a2，\_）是同构的。

Proposition 33.3. Given any two n-th exterior tensor powers (A1,ϕ1) and (A2,ϕ2) for E, there is an isomorphism h: A1 → A2 such that  
提案33.3.考虑到e的任意两个n阶外张量幂（a1，\_）和（a2，\_），存在同构h:a1→a2，这样

ϕ2 = h ◦ ϕ1.  
\_2=H\_1.

Proof. Replace tensor product by n-th exterior tensor power in the proof of Proposition  
证据。在命题证明中用第n次外张量幂代替张量积

32.5.   
32.5。

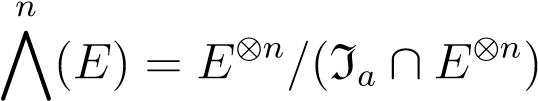
We next give a construction that produces an n-th exterior tensor power of a vector space E.  
接下来我们给出一个构造，它产生向量空间e的第n次外张量幂。

Theorem 33.4. Given a vector space E, an n-th exterior tensor power (Vn(E),ϕ) for E can be constructed (n ≥ 1). Furthermore, denoting ϕ(u1,...,un) as u1 ∧···∧un, the exterior tensor power Vn(E) is generated by the vectors u1 ∧ ··· ∧ un, where u1,...,un ∈ E, and for every alternating multilinear map f : En → F, the unique linear map f∧ : Vn(E) → F such that f = f∧ ◦ ϕ is defined by  
定理33.4。给定向量空间e，可构造e的第n次外张量幂（vn（e），a）（n≥1）。此外，将\_（u1，…，un）表示为u1····un，外部张量幂Vn（e）由矢量u1··············un生成，其中u1，…，un∈e，对于每个交替的多线性映射f:en→f，唯一的线性映射f：vn（e）→f，这样f=f

f∧(u1 ∧ ··· ∧ un) = f(u1,...,un)  
f（u1····un）=f（u1，…，un）

on the generators u1 ∧ ··· ∧ un of Vn(E).  
在VN（E）的U1····Un发电机上。

Proof sketch. We can give a quick proof using the tensor algebra T(E). Let Ia be the two-sided ideal of T(E) generated by all tensors of the form u ⊗ u ∈ E⊗2. Then let  
证明草图。我们可以用张量代数t（e）给出一个快速的证明。假设Ia是由u u∈e 2形式的所有张量产生的t（e）的双面理想。那就让



and let π be the projection π: E⊗n → Vn(E). If we let u1 ∧ ··· ∧ un = π(u1 ⊗ ··· ⊗ un), it is easy to check that (Vn(E),∧) satisfies the conditions of Theorem 33.4.   
设π为投影π：e n→vn（e）。如果我们让u1·····un=π（u1·····un），很容易检查（vn（e），）满足定理33.4的条件。

Remark: We can also define  
备注：我们也可以定义

,  
，

the exterior algebra of E. This is the skew-symmetric counterpart of S(E), and we will study it a little later.  
E的外代数，这是S（E）的斜对称对应物，稍后我们将研究它。

For simplicity of notation, we may write Vn E for Vn(E). We also abbreviate “exterior tensor power” as “exterior power.” Clearly, V1(E) ∼= E, and it is convenient to set V0(E) = K.  
为了便于记法，我们可以为vn（e）写vn e。我们还将“外张量幂”缩写为“外张量幂”。显然，v1（e）=e，并且设置v0（e）=k很方便。

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The fact that the map ϕ: En → Vn(E) is alternating and multilinear can also be expressed as follows:  
图\_：en→vn（e）是交替的，多行也可以表示为：

|  |  |  |
| --- | --- | --- |
| u1 ∧ ··· ∧ (ui + vi) ∧ ··· ∧ un U1·····（UI+VI）·········UN | = = | (u1 ∧ ··· ∧ ui ∧ ··· ∧ un) （U1·········Ui····Un）  + (u1 ∧ ··· ∧ vi ∧ ··· ∧ un), +（U1··············联合国） |
| u1 ∧ ··· ∧ (λui) ∧ ··· ∧ un U1···（λui）····un | = = | λ(u1 ∧ ··· ∧ ui ∧ ··· ∧ un), λ（u1·········uI···un） |
| uσ(1) ∧ ··· ∧ uσ(n) uσ（1）···uσ（n）  for all σ ∈ Sn. 对于所有的σ∈sn。 | = = | sgn(σ)u1 ∧ ··· ∧ un, sgn（σ）u1····un， |

The map ϕ from En to Vn(E) is often denoted ι∧, so that  
从en到vn（e）的图\_通常表示为，因此

ι∧(u1,...,un) = u1 ∧ ··· ∧ un.  
（u1，…，un）=u1···un.

Theorem 33.4 implies the following result. Proposition 33.5. There is a canonical isomorphism  
定理33.4包含以下结果。提案33.5。有一个典型的同构

n  
n

Hom(^(E),F) ∼= Altn(E;F)  
hom（^（e），f）=altn（e；f）

between the vector space of linear maps Hom(Vn(E),F) and the vector space of alternating multilinear maps Altn(E;F), given by the linear map − ◦ ϕ defined by →7 h ◦ ϕ, with h ∈ Hom(Vn(E),F). In particular, when F = K, we get a canonical isomorphism  
在线性映射的向量空间hom（vn（e），f）和交替多线性映射的向量空间altn（e；f）之间，由线性映射−\_定义为→7 h \_，其中h∈hom（vn（e），f）给出。特别是，当f=k时，我们得到一个正则同构。

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Definition 33.3. Tensors α ∈ Vn(E) are called alternating n-tensors or alternating tensors of degree n and we write deg(α) = n. Tensors of the form u1∧···∧un, where ui ∈ E, are called simple (or decomposable) alternating n-tensors. Those alternating n-tensors that are not simple are often called compound alternating n-tensors. Simple tensors u1∧···∧un ∈ Vn(E) are also called n-vectors and tensors in Vn(E∗) are often called (alternating) n-forms.  
定义33.3.张量α∈vn（e）称为n阶交替张量或n阶交替张量，我们写出deg（α）=n。形式为u1·····un的张量，其中ui∈e称为简单（或可分解）交替n张量。那些不简单的交替n-张量通常称为复合交替n-张量。简单张量u1·····un∈vn（e）也称为N矢量，vn（e）中的张量常称为（交替）N型。

Given two linear maps f : E → E0 and g: E → E0, since the map) is bilinear and alternating, there is a unique linear map f ∧ g: V2(E) → V2(E0) making the following diagram commute:  
给定两个线性映射f:e→e0和g:e→e0，由于映射）是双线性且交替的，因此有一个唯一的线性映射f g:v2（e）→v2（e0），使以下图表成为通勤图：

E2 ι∧ / V2(E)  
e2/v2（e）

f×gf∧g  
f×gf g

0

2

V

2

(E ) 0 / (E0).  
（e）0/（e0）。

ι∧  
γ射线

The map f ∧ g: V2(E) → V2(E0) is determined by  
图f g:v2（e）→v2（e0）由

(f ∧ g)(u ∧ v) = f(u) ∧ g(u).  
（f\_g）（u\_v）=f（u）g（u）。

Proposition 33.6. Given any linear maps f : E → E0, g: E → E0, f0 : E0 → E00 and g0 : E0 → E00, we have  
提案33.6.对于任何线性映射f:e→e0，g:e→e0，f0:e0→e00和g0:e0→e00，我们有

(f0 ◦ f) ∧ (g0 ◦ g) = (f0 ∧ g0) ◦ (f ∧ g).  
（f0 f）（g0 g）=（f0 g0）（f g）。

The generalization to the alternating product f1∧···∧fn of n ≥ 3 linear maps fi : E → E0 is immediate, and left to the reader.  
n≥3个线性映射fi:e→e0的交替积f1······fn的推广是直接的，留给读者。

## 33.2 Bases of Exterior Powers 33.2外部权力基础

Definition 33.4. Let E be any vector space. For any basis (ui)i∈Σ for E, we assume that some total ordering ≤ on the index set Σ has been chosen. Call the pair ((ui)i∈Σ,≤) an ordered basis. Then for any nonempty finite subset I ⊆ Σ, let  
定义33.4.设e为任意向量空间。对于任意基（ui）i∈∑对于e，我们假设在索引集∑上选择了一些总序。称对（（ui）i∈∑，≤）为有序基。那么对于任何非空有限子集i∑，让

uI = ui1 ∧ ··· ∧ uim,  
ui=ui1···uim，

where I = {i1,...,im}, with i1 < ··· < im.  
式中，i=i1，…，im，其中i1<······<im.

Since Vn(E) is generated by the tensors of the form v1 ∧···∧vn, with vi ∈ E, in view of skew-symmetry, it is clear that the tensors uI with |I| = n generate Vn(E) (where ((ui)i∈Σ,≤) is an ordered basis). Actually they form a basis. To gain an intuitive understanding of this statement, let m = 2 and E be a 3-dimensional vector space lexicographically ordered basis {e1,e2,e3}. We claim that  
由于vn（e）是由v1····vn形式的张量生成的，因此，从斜对称性的角度来看，i=n形式的张量ui生成vn（e）（其中（（ui）i∈∑，≤）是有序基）。实际上，它们构成了一个基础。为了直观地理解这句话，让m=2和e是一个三维向量空间，在词典上是有序的基e1，e2，e3。我们声称

e1 ∧ e2, e1 ∧ e3, e2 ∧ e3  
e1 e2、e1 e3、e2 e3

form a basis for V2(E) since they not only generate V2(E) but are linearly independent. The linear independence is argued as follows: given any vector space F, if w12,w13,w23 are any vectors in F, there is an alternating bilinear map h: E2 → F such that  
形成V2（e）的基础，因为它们不仅生成V2（e），而且是线性独立的。线性独立性的论证如下：给定任意向量空间f，如果w12、w13、w23是f中的任意向量，则存在一个交替的双线性映射h:e2→f，从而

h(e1,e2) = w12, h(e1,e3) = w13, h(e2,e3) = w23.  
h（e1，e2）=w12，h（e1，e3）=w13，h（e2，e3）=w23。

Because h yields a unique linear map h∧ : V2 E → F such that  
因为h生成一个唯一的线性映射h：v2 e→f这样

h∧(ei ∧ ej) = wij, 1 ≤ i < j ≤ 3,  
h（ei ej）=wij，1≤i<j≤3，

by Proposition 32.4, the vectors  
根据命题32.4，向量

e1 ∧ e2, e1 ∧ e3, e2 ∧ e3  
e1 e2、e1 e3、e2 e3

are linearly independent. This suggests understanding how an alternating bilinear function f : E2 → F is expressed in terms of its values f(ei,ej) on the basis vectors (e1,e2,e3). Using bilinearity and alternation, we obtain  
线性无关。这意味着要理解交替双线性函数f:e2→f是如何在基向量（e1，e2，e3）上用其值f（ei，ej）表示的。利用双线性和交替，我们得到

f(u1e1 + u2e2 + u3e3,v1e1 + v2e2 + v3e3) = (u1v2 − u2v1)f(e1,e2) + (u1v3 − u3v1)f(e1,e3) + (u2v3 − u3v2)f(e2,e3).  
f（u1e1+u2e2+u3e3，v1e1+v2e2+v3e3）=（u1v2−u2v1）f（e1，e2）+（u1v3−u3v1）f（e1，e3）+（u2v3−u3v2）f（e2，e3）。

33.2. BASES OF EXTERIOR POWERS  
33.2。外部权力基础

Therefore, given w12,w13,w23 ∈ F, the function h given by h(u1e1 + u2e2 + u3e3,v1e1 + v2e2 + v3e3) = (u1v2 − u2v1)w12 + (u1v3 − u3v1)w13  
因此，给定w12，w13，w23∈f，h给出的函数h（u1e1+u2e2+u3e3，v1e1+v2e2+v3e3）=（u1v2−u2v1）w12+（u1v3−u3v1）w13

+ (u2v3 − u3v2)w23  
+（u2v3−u3v2）w23

is clearly bilinear and alternating, and by construction h(ei,ej) = wij, with 1 ≤ i < j ≤ 3 does the job.  
明显是双线性和交替的，并且通过构造h（ei，ej）=wij，1≤i<j≤3来完成工作。

We now prove the assertion that tensors uI with |I| = n generate Vn(E) for arbitrary n.  
我们现在证明了张量ui with i=n生成vn（e）用于任意n的断言。

Proposition 33.7. Given any vector space E, if E has finite dimension d = dim(E), then for all n > d, the exterior power Vn(E) is trivial; that is Vn(E) = (0). If n ≤ d or if E is infinite dimensional, then for every ordered basisVn(E), where I ranges over finite nonempty subsets of((uΣi)i∈of sizeΣ,≤), the family|I| = n. (uI) is basis of  
提案33.7。对于任何向量空间e，如果e的有限维d=dim（e），那么对于所有n>d，外幂Vn（e）都是微不足道的；即Vn（e）=0。如果n≤d或者e是无限维，那么对于每一个有序的basisvn（e），其中i的范围在（（u∑i）i∈的有限非空子集上，≤），族i=n（ui）是

Proof. First assume that E has finite dimension d = dim(E) and that n > d. We know that Vn(E) is generated by the tensors of the form v1 ∧ ··· ∧ vn, with vi ∈ E. If u1,...,ud is a basis of E, as every vi is a linear combination of the uj, when we expand v1 ∧ ··· ∧ vn using multilinearity, we get a linear combination of the form  
证据。首先假设e有有限维d=dim（e）和n>d，我们知道vn（e）是由形式v1·····vn的张量生成的，其中vi∈e。如果u1，…，ud是e的基础，因为当我们使用多线性展开v1······vn时，每个vi都是uj的线性组合，我们得到了形式的线性组合

v1 ∧ ··· ∧ vn = X λ(j1,...,jn) uj1 ∧ ··· ∧ ujn,  
v1···vn=xλ（j1，…，jn）uj1···ujn，

(j1,...,jn)  
（j1，…，jn）

where each (j1,...,jn) is some sequence of integers jk ∈ {1,...,d}. As n > d, each sequence (j1,...,jn) must contain two identical elements. By alternation, uj1 ∧ ··· ∧ ujn = 0, and so v1 ∧ ··· ∧ vn = 0. It follows that Vn(E) = (0).  
其中，每个（j1，…，jn）是一些整数序列jk∈1，…，d。由于n>d，每个序列（j1，…，jn）必须包含两个相同的元素。通过交替，uj1····ujn=0，因此v1···vn=0。由此得出Vn（e）=（0）。

Now assume that either dim(E) = d and n ≤ d, or that E is infinite dimensional. The argument below shows that the uI are nonzero and linearly independent. As usual, let be the linear form given by  
现在假设dim（e）=d和n≤d，或者e是无限维。下面的参数显示用户界面是非零的，并且是线性独立的。和往常一样，假设线性形式由

.  
.

For any nonempty subset I = {i1,...,in} ⊆ Σ with i1 < ··· < in, for any n vectors v1,...,vn ∈ E, let  
对于任何非空子集i=i1，…，i n∑with i1<····<in，对于任意n向量v1，…，vn∈e，let

.  
.

If we let the n-tuple (v1,...,vn) vary we obtain a map lI from En to K, and it is easy to check that this map is alternating multilinear. Thus lI induces a unique linear map LI : Vn(E) → K making the following diagram commute.  
如果让n元组（v1，…，vn）发生变化，我们得到一个从en到k的映射li，并且很容易检查该映射是否是交替多行的。因此，li归纳出一个独特的线性图li:vn（e）→k，使下面的图表通勤。

EnHHHιlH∧IHHH/HHVH$ n( LEI )  
ehh lh ihh/hhvh$N（LEI）

K  
K

Observe that for any nonempty finite subset J ⊆ Σ with |J| = n, we have  
注意，对于任何非空有限子集j∑，j=n，我们有

.  
.

Note that when dim(E) = d and n ≤ d, or when E is infinite-dimensional, the forms are all distinct, so the above does hold. Since LI(uI) = 1, we conclude that  
请注意，当dim（e）=d和n≤d时，或者当e是无限维时，这些形式都是不同的，因此上面的内容是成立的。由于li（ui）=1，我们得出结论：

uI = 06 . If we have a linear combination  
ui=06。如果我们有一个线性组合

,  
，

where the above sum is finite and involves nonempty finite subset I ⊆ Σ with |I| = n, for every such I, when we apply LI we get λI = 0, proving linear independence.   
当上面的和是有限的并且涉及非空的有限子集i∑with i=n时，对于每一个这样的i，当我们应用li时，我们得到λi=0，证明线性独立性。

As a corollary, if E is finite dimensional, say dim(E) = d, and if 1 ≤ n ≤ d, then we have  
作为推论，如果e是有限维的，比如dim（e）=d，如果1≤n≤d，那么我们有

dim(,  
昏暗的

and if n > d, then dim(Vn(E)) = 0.  
如果n>d，则dim（vn（e））=0。

Remark: When n = 0, if we set u∅ = 1, then (u∅) = (1) is a basis of V0(V ) = K.  
注：当n=0时，如果设u∅=1，则（u∅）=（1）为v0（v）=k的基础。

It follows from Proposition 33.7 that the family (uI)I where I ⊆ Σ ranges over finite subsets of Σ is a basis of V(V ) = Ln≥0 Vn(V ).  
由命题33.7得出，i∑在∑的有限子集上的族（ui）i是v（v）=ln≥0 vn（v）的基础。

As a corollary of Proposition 33.7 we obtain the following useful criterion for linear independence.  
作为33.7号命题的推论，我们得到了以下关于线性独立性的有用准则。

Proposition 33.8. For any vector space E, the vectors u1,...,un ∈ E are linearly independent iff u1 ∧ ··· ∧ un = 06 .  
提案33.8。对于任意向量空间e，向量u1，…，un∈e是线性无关的iff u1···un=06。

Proof. If u1 ∧ ··· ∧ un = 06 , then u1,...,un must be linearly independent. Otherwise, some ui would be a linear combination of the other uj’s (with j =6 i), and then, as in the proof of Proposition 33.7, u1 ∧···∧un would be a linear combination of wedges in which two vectors are identical, and thus zero.  
证据。如果U1····un=06，那么U1，…，un必须是线性无关的。否则，一些ui将是另一个uj的线性组合（j=6i），然后，如在命题33.7的证明中，u1······un将是两个向量相同，因而为零的楔形的线性组合。

Conversely, assume that u1,...,un are linearly independent. Then we have the linear forms such that  
相反，假设u1，…，un是线性无关的。那么我们有这样的线性形式

1 ≤ i,j ≤ n.  
1≤i，j≤n。

As in the proof of Proposition 33.7, we have a linear map Lu1,...,un : Vn(E) → K given by  
如在33.7号命题的证明中，我们有一个线性图lu1，…，un:vn（e）→k，由

,  
，

for all v1 ∧···∧vn ∈ Vn(E). As Lu1,...,un(u1 ∧···∧un) = 1, we conclude that u1 ∧···∧un 6=  
对于所有v1····vn∈vn（e）。作为lu1，…，un（u1·····un）=1，我们得出u1·····un 6=

0.   
0。

33.3. SOME USEFUL ISOMORPHISMS FOR EXTERIOR POWERS  
小精灵。关于外部力量的一些有用同构

Proposition 33.8 shows that geometrically every nonzero wedge u1 ∧ ··· ∧ un corresponds to some oriented version of an n-dimensional subspace of E.  
命题33.8表明，几何上每一个非零楔形U1····Un对应于E的N维子空间的一些定向版本。

## 33.3 Some Useful Isomorphisms for Exterior Powers 33.3外部力量的一些有用同构

We can show the following property of the exterior tensor product, using the proof technique of Proposition 32.13.  
利用32.13命题的证明技术，我们可以证明外张量积的以下性质。

Proposition 33.9. We have the following isomorphism:  
提案33.9.我们有以下同构：

^n Mn ^k n^−k  
^ N mn^k N^−k

(E ⊕ F) ∼= (E) ⊗ (F).  
（e\_f）=（e）（f）。

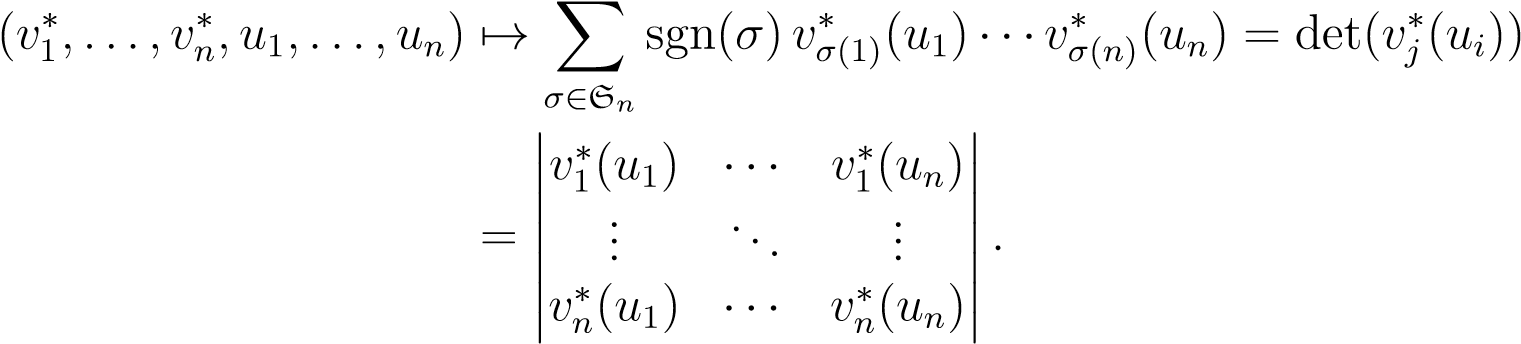
k=0  
K＝0

## 33.4 Duality for Exterior Powers 33.4外部权力的双重性

In this section all vector spaces are assumed to have finite dimension. We define a nondegenerate pairing Vn(E∗) × Vn(E) −→ K as follows: Consider the multilinear map  
在本节中，假设所有向量空间都有有限维。我们定义了一个非退化配对vn（e）×vn（e）−→k如下：考虑多行映射

(E∗)n × En −→ K  
（e）n×en−→k

given by  
给出者



It is easily checked that this expression is alternating w.r.t. the ui’s and also w.r.t. the.  
很容易检查该表达式是否交替使用了UI和W.R.T.。

For any fixed (, we get an alternating multilinear map  
对于任何固定的（，我们得到一个交替的多行映射



from En to K. The map extends uniquely to a linear map making the following diagram commute:  
从en到k。地图独特地延伸到线性地图，使以下图表通勤：

EnHHHιH∧HHHH/ HVH$ n( E)  
n h h\_h\_hhh/hvh$N（E）



K.  
K

We also have the alternating multilinear map  
我们还有交替的多行地图



from (E∗)n to Hom(Vn(E),K), which extends to a linear map L from Vn(E∗) to Hom(Vn(E),K) making the following diagram commute:  
从（e）n到hom（vn（e），k），它延伸到从vn（e）到hom（vn（e），k的线性图l，使以下图表通勤：

(E∗)n ι∧∗ / Vn(E∗)  
（E）N/VN（E）

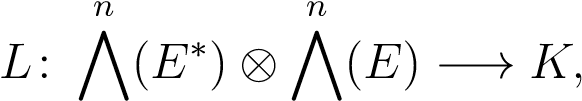
L  
L

' Vn (E),K). Hom(  
‘VN（e），k）。霍姆

However, in view of the isomorphism  
然而，鉴于同构

Hom(U ⊗ V,W) ∼= Hom(U,Hom(V,W)),  
hom（u v，w）=hom（u，hom（v，w）），

with U = Vn(E∗), V = Vn(E) and W = K, we can view L as a linear map  
当u=vn（e）、v=vn（e）和w=k时，我们可以将l视为线性映射。



which by Proposition 32.8 corresponds to a bilinear map  
根据命题32.8，它对应于双线性图。

(∗)  
（三）

This pairing is given explicitly in terms of generators by  
这种配对是根据发电机明确给出的

.  
.

Now this pairing in nondegenerate. This can be shown using bases. Given any basis (e1,...,em) of E, for every basis element ) (with 1 ≤ i1 < ··· < in ≤ m), we have  
现在这个配对是非退化的。这可以用碱来表示。给定e的任何基（e1，…，e m），对于每个基元（1≤i1<·······<in≤m），我们有

(  
（

he∗i1 ∧ ··· ∧ e∗in,ej1,...,ejni = 1,...,jn) = (i1,...,in) 1 if (j  
He i1·····e in，e j 1，…，ejni=1，…，jn）=（i1，…，in）1如果（j

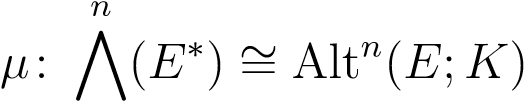
0 otherwise.  
否则为0。

We leave the details as an exercise to the reader. As a consequence we get the following canonical isomorphisms.  
我们把细节留给读者作为练习。因此，我们得到了以下规范同构。

Proposition 33.10. There is a canonical isomorphism  
提案33.10。有一个典型的同构

.  
.

There is also a canonical isomorphism  
还有一个典型的同构



which allows us to interpret alternating tensors over E∗ as alternating multilinear maps.  
这使得我们可以把E上的交替张量解释为交替的多行映射。

33.4. DUALITY FOR EXTERIOR POWERS  
33.4。外部力量的二重性

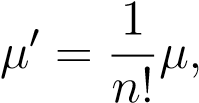
Proof. The second isomorphism follows from the canonical isomorphism (Vn(E))∗ ∼= Vn(E∗) and the canonical isomorphism (Vn(E))∗ ∼= Altn(E;K) given by Proposition 33.5.   
证据。第二个同构来自33.5号命题给出的规范同构（vn（e））～=vn（e）和规范同构（vn（e））～=altn（e；k）。

Remarks:  
评论：

1. The isomorphism µ: Vn(E∗) ∼= Altn(E;K) discussed above can be described explicitly as the linear extension of the map given by  
   上述讨论的同构（e））=altn（e；k）可以明确地描述为图的线性延伸

.  
.

1. The canonical isomorphism of Proposition 33.10 holds under more general conditions. Namely, that K is a commutative ring with identity and that E is a finitely-generated projective K-module (see Definition 34.7). See Bourbaki, [25] (Chapter III, §11, Section 5, Proposition 7).  
   33.10命题的规范同构在更一般的条件下成立。也就是说，k是一个具有恒等式的交换环，e是一个有限生成的射影k模（见定义34.7）。见Bourbaki，[25]（第三章，第11节，第5节，提案7）。
2. Variants of our isomorphism µ are found in the literature. For example, there is a version µ0, where  
   文献中发现了我们的同构μ的变体。例如，有一个版本祄0，其中



with the factor added in front of the determinant. Each version has its its own merits and inconveniences. Morita [125] uses µ0 because it is more convenient than µ when dealing with characteristic classes. On the other hand, µ0 may not be defined for a field with positive characteristic, and when using µ0, some extra factor is needed in defining the wedge operation of alternating multilinear forms (see Section 33.5) and for exterior differentiation. The version µ is the one adopted by Warner [180], Knapp [102], Fulton and Harris [69], and Cartan [34, 35].  
在行列式前面加上因子。每个版本都有自己的优点和不便。森田[125]使用了μ0，因为它在处理特征类时比μ更方便。另一方面，对于具有正特性的场，可以不定义μ0，当使用μ0时，在定义交替多行形式的楔形操作（见第33.5节）和外部差异时，需要一些额外的因素。版本μ是华纳[180]、纳普[102]、富尔顿和哈里斯[69]以及卡坦[34，35]采用的版本。

If f : E → F is any linear map, by transposition we get a linear map f> : F ∗ → E∗ given by  
如果f:e→f是任何线性映射，通过换位我们得到一个线性映射f>：f→e，由

f>(v∗) = v∗ ◦ f, v∗ ∈ F ∗.  
f>（v）=v f，v∈f。

Consequently, we have  
因此，我们

f>(v∗)(u) = v∗(f(u)), for all u ∈ E and all v∗ ∈ F ∗.  
f>（v）（u）=v（f（u）），对于所有u e和所有v∈f。

For any p ≥ 1, the map  
对于任何p≥1，图

(u1,...,up) 7→ f(u1) ∧ ··· ∧ f(up)  
（U1，…，向上）7→F（U1）···F（向上）

from Ep to Vp F is multilinear alternating, so it induces a unique linear map Vp f : Vp E → Vp F making the following diagram commute  
从ep到vp f是多行交替的，因此它会产生一个独特的线性映射vp f:vp e→vp f，从而使下图变为通勤图。

Ep ι∧ / Vp E  
ep/vp e

EEEEEEEEEV" p VF,p f  
eeeeeeeee v“p v f，p f

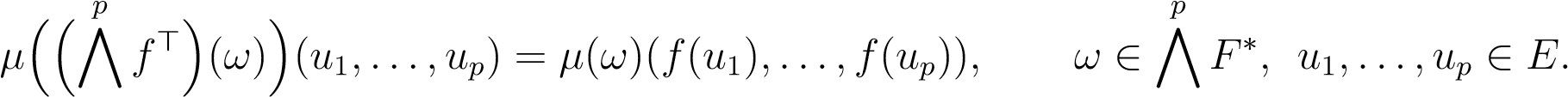
and defined on generators by  
在发电机上定义

.  
.

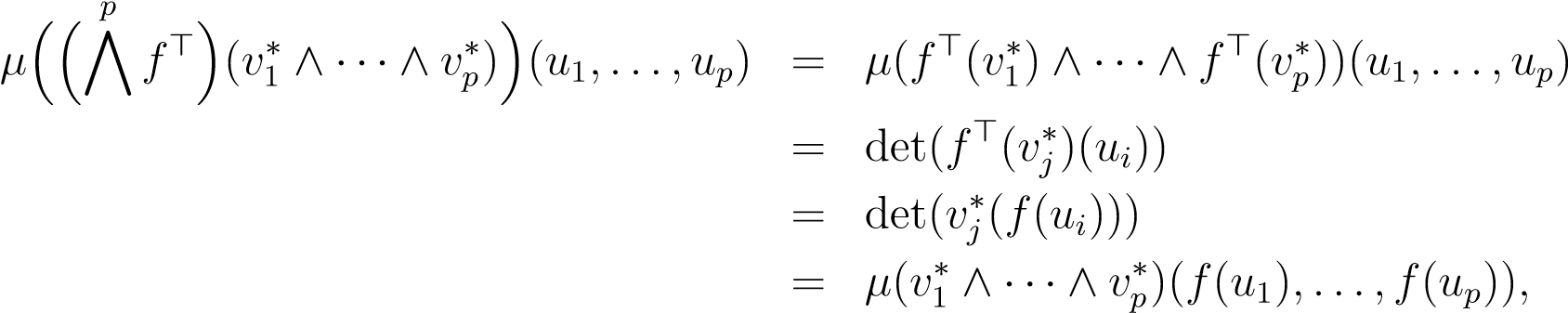
Combining Vp and duality, we get a linear map Vp f> : Vp F ∗ → Vp E∗ defined on generators by  
结合vp和对偶性，我们得到一个线性映射vp f>：vp f→vp e定义在生成器上

.  
.

Proposition 33.11. If f : E → F is any linear map between two finite-dimensional vector spaces E and F, then  
提案33.11.如果f:e→f是有限维向量空间e和f之间的任何线性映射，则



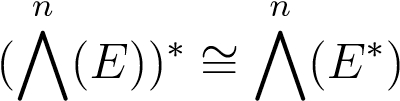
Proof. It is enough to prove the formula on generators. By definition of µ, we have  
证据。在发电机上证明这个公式已经足够了。根据μ的定义，我们有



as claimed.   
如要求。

Remark: The map Vp f> is often denoted f∗, although this is an ambiguous notation since p is dropped. Proposition 33.11 gives us the behavior of Vp f> under the identification of Vp E∗ and Altp(E;K) via the isomorphism µ.  
备注：map vp f>通常表示为f，尽管这是一个不明确的符号，因为p被删除了。命题33.11通过同构μ给出了vp e和altp（e；k）标识下vp f>的行为。

As in the case of symmetric powers, the map from En to Vn(E) given by (u,...,un) 7→ u1 ∧···∧ un yields a surjection π: E⊗n → Vn(E). Now this map has some section, so there is some injection η: Vn(E) → E⊗n with π ◦ η = id. As we saw in Proposition 33.10 there is a canonical isomorphism  
在对称幂的情况下，由（u，…，u n）7→u1·······un给出的从en到vn（e）的映射产生一个投影π：e n→vn（e）。现在这个图有一个部分，所以有一些注入η：vn（e）→e n，πη=id。正如我们在命题33.10中看到的，有一个规范的同构。



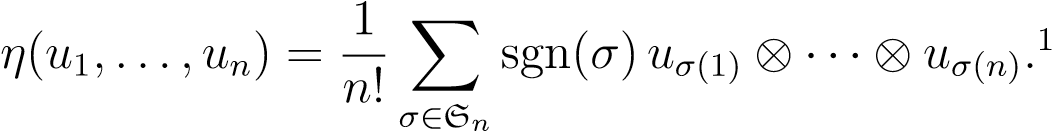
for any field K, even of positive characteristic. However, if our field K has characteristic 0, then there is a special section having a natural definition involving an antisymmetrization process.  
对于任何场k，即使是正特性。然而，如果我们的场k具有0的特征，那么有一个特殊的部分具有涉及反对称化过程的自然定义。

Recall, from Section 32.10 that we have a left action of the symmetric group Sn on E⊗n.  
回想一下，在第32.10节中，对称群sn在e n上有一个左作用。

The tensors z ∈ E⊗n such that  
张量z∈e n使得

σ · z = sgn(σ)z, for all σ ∈ Sn  
σ·z=sgn（σ）z，所有σ∈sn

are called antisymmetrized tensors. We define the map η: En → E⊗n by  
称为反对称张量。我们定义了图η：e n→e n



33.5. EXTERIOR ALGEBRAS  
33.5。外部代数

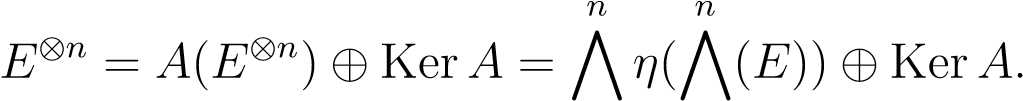
As the right hand side is an alternating map, we get a unique linear map Vn η: Vn(E) → E⊗n making the following diagram commute.  
由于右手边是一个交替的地图，我们得到一个独特的线性地图vnη：vn（e）→e n，使下面的图表通勤。

EnHHHιηH∧HHH/HHV# n( VEn)η  
Enhh\_ηh\_hh/hhv\_n（ven）η

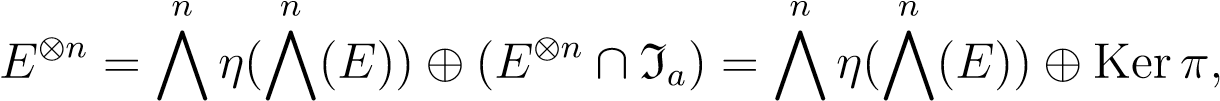
E⊗n.  
埃恩

Clearly, Vn η(Vn(E)) is the set of antisymmetrized tensors in E⊗n. If we consider the map A = (Vn η) ◦ π: E⊗n −→ E⊗n, it is easy to check that A ◦ A = A. Therefore, A is a  
显然，vnη（vn（e））是e n中反对称张量的集合。如果我们考虑映射a=（vnη）π：e n−→e n，很容易检查a a=a。因此，a是

projection, and by linear algebra, we know that  
投影，通过线性代数，我们知道



It turns out that KerA = E⊗n ∩ Ia = Ker π, where Ia is the two-sided ideal of T(E) generated by all tensors of the form u ⊗ u ∈ E⊗2 (for example, see Knapp [102], Appendix A). Therefore, Vn η is injective,  
结果表明，ker a=e n ia=kerπ，其中ia是由u u∈e 2形式的所有张量产生的t（e）的双面理想（例如，见Knapp[102]，附录A）。因此，vnη是内射的，

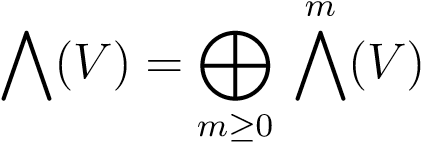


and the exterior tensor power Vn(E) is naturally embedded into E⊗n.  
外张量功率Vn（e）自然嵌入E n中。

## 33.5 Exterior Algebras 33.5外部代数

As in the case of symmetric tensors, we can pack together all the exterior powers Vn(V ) into an algebra.  
在对称张量的情况下，我们可以把所有的外幂vn（v）组合成一个代数。

Definition 33.5. Gieven any vector space V , the vector space  
定义33.5.即使任何向量空间v，向量空间



is called the exterior algebra (or Grassmann algebra) of V .  
称为V的外部代数（或格拉斯曼代数）。

To make V(V ) into an algebra, we mimic the procedure used for symmetric powers. If Ia is the two-sided ideal generated by all tensors of the form u ⊗ u ∈ V ⊗2, we set  
为了使V（V）变成代数，我们模拟了对称幂的过程。如果ia是由u u∈v 2形式的所有张量产生的双面理想，我们将

.  
.

Then V•(V ) automatically inherits a multiplication operation, called wedge product, and since T(V ) is graded, that is  
然后v•（v）自动继承一个称为楔形积的乘法运算，因为t（v）是分级的，也就是说

T(V ) = M V ⊗m,  
t（v）=m v m，

m≥0  
m 0

we have  
我们有

.  
.

However, it is easy to check that  
但是，很容易检查

,  
，

so  
所以

.  
.

When V has finite dimension d, we actually have a finite direct sum (coproduct)  
当v有有限维d时，我们实际上有一个有限直和（副积）

,  
，

and since each Vm(V ) has dimension , we deduce that  
因为每个vm（v）都有维，我们推断

dim(^(V )) = 2d = 2dim(V ).  
尺寸（^（V））=2d=2dim（V）。

The multiplication, ∧: Vm(V )×Vn(V ) → Vm+n(V ), is skew-symmetric in the following precise sense:  
乘法，：vm（v）×v n（v）→vm+n（v），在以下精确意义上是斜对称的：

Proposition 33.12. For all α ∈ Vm(V ) and all β ∈ Vn(V ), we have  
提案33.12。对于所有α∈vm（v）和所有β∈vn（v），我们有

β ∧ α = (−1)mnα ∧ β.  
βα=（-1）mNαβ。

Proof. Since v ∧ u = −u ∧ v for all u,v ∈ V , Proposition 33.12 follows by induction.   
证据。既然v u=−u v对于所有u，v v，命题33.12随后是归纳法。

Since α ∧ α = 0 for every simple (also called decomposable) tensor α = u1 ∧ ··· ∧ un, it seems natural to infer that α ∧ α = 0 for every tensor α ∈ V(V ). If we consider the case where dim(V ) ≤ 3, we can indeed prove the above assertion. However, if dim(V ) ≥ 4, the above fact is generally false! For example, when dim(V ) = 4, if (u1,u2,u3,u4) is a basis for V , for α = u1 ∧ u2 + u3 ∧ u4, we check that  
由于αα=0表示每个简单张量α=u1······un，因此很自然地推断αα=0表示每个张量α∈v（v）。如果我们考虑dim（v）≤3的情况，我们确实可以证明上述观点。然而，如果dim（v）≥4，上述事实通常是错误的！例如，当dim（v）=4时，如果（u1，u2，u3，u4）是v的基础，对于α=u1 u2+u3 u4，我们检查

α ∧ α = 2u1 ∧ u2 ∧ u3 ∧ u4,  
αα=2u1 u2 u3 u4，

which is nonzero. However, if α ∈ Vm E with m odd, since m2 is also odd, we have  
这不是零。然而，如果α∈vm e与m奇数，由于m2也是奇数，我们有

α ∧ α = (−1)m2α ∧ α = −α ∧ α,  
αα=−1）m2αα=−αα，

so indeed α ∧ α = 0 (if K is not a field of characteristic 2).  
因此，实际上αα=0（如果k不是特征2的场）。

33.5. EXTERIOR ALGEBRAS  
33.5。外部代数

The above discussion suggests that it might be useful to know when an alternating tensor is simple (decomposable). We will show in Section 33.7 that for tensors α ∈ V2(V ), α∧α = 0 iff α is simple.  
上述讨论表明，知道交变张量何时是简单的（可分解的）可能是有用的。我们将在第33.7节中说明，对于张量α∈v2（v），α\_α=0 iffα是简单的。

A general criterion for decomposability can be given in terms of some operations known as left hook and right hook (also called interior products); see Section 33.7.  
可分解性的一般标准可根据一些被称为左钩和右钩（也称为内部产品）的操作给出；见第33.7节。

It is easy to see that V(V ) satisfies the following universal mapping property.  
很容易看出，V（V）满足以下通用映射性质。

Proposition 33.13. Given any K-algebra A, for any linear map f : V → A, if (f(v))2 = 0  
提案33.13。给定任意k-代数a，对于任意线性映射f:v→a，如果（f（v））2=0

for all v ∈ V , then there is a unique K-algebra homomorphism f : V(V ) → A so that  
对于所有的v∈v，则有一个唯一的k-代数同态f:v（v）→a，因此

f = f ◦ i,  
f=f\_i，

as in the diagram below.  
如下图所示。

V FFFfFFiFF/ FVF" (Vf )  
v ffffff/fvf”（vf）

A  
一

When E is finite-dimensional, recall the isomorphism µ: Vn(E∗) −→ Altn(E;K), defined as the linear extension of the map given by  
当e是有限维时，回想同构：vn（e）−→altn（e；k），定义为由

.  
.

Now, we have also a multiplication operation Vm(E∗) × Vn(E∗) −→ Vm+n(E∗). The following question then arises:  
现在，我们还有一个乘法运算vm（e）×vn（e）−→vm+n（e）。然后出现以下问题：

Can we define a multiplication Altm(E;K) × Altn(E;K) −→ Altm+n(E;K) directly on alternating multilinear forms, so that the following diagram commutes?  
我们能在交替的多行形式上直接定义一个乘法altm（e；k）×altn（e；k）－→altm+n（e；k），以便下面的图表可以转换吗？

Vm(E∗) × Vn(E∗) ∧ / Vm+n(E∗)  
vm（e）×vn（e）/vm+n（e）

µm×µn µm+n  
礹m×礹n礹m+n

Altm(E;K) × Altn(E;K) ∧ / Altm+n (E;K)  
altm（e；k）×altn（e；k）/altm+n（e；k）

As in the symmetric case, the answer is yes! The solution is to define this multiplication such that, for f ∈ Altm(E;K) and g ∈ Altn(E;K),  
和对称情况一样，答案是肯定的！解决方法是定义这个乘法，这样，对于f∈altm（e；k）和g∈altn（e；k），就

(f ∧ g)(u1,...,um+n) = X sgn(σ)f(uσ(1),...,uσ(m))g(uσ(m+1),...,uσ(m+n)), (∗∗)  
（f g）（u1，…，u m+n）=x sgn（σ）f（uσ（1），…，uσ（m））g（uσ（m+1），…，uσ（m+n）），（）

σ∈shuffle(m,n)  
σ∈洗牌（m，n）

where shuffle(m,n) consists of all (m,n)-“shuffles;” that is, permutations σ of {1,...m+n} such that σ(1) < ··· < σ(m) and σ(m+1) < ··· < σ(m+n). For example, when m = n = 1, we have  
式中，shuffle（m，n）由所有（m，n）-“shuffles”组成；即，置换σof 1，…m+n，这样，σ（1）<······<σ（m）和σ（m+1）<······<σ（m+n）。例如，当m=n=1时，我们有

(f ∧ g)(u,v) = f(u)g(v) − g(u)f(v).  
（f g）（u，v）=f（u）g（v）−g（u）f（v）。

When m = 1 and n ≥ 2, check that  
当m=1且n≥2时，检查

,  
，

where the hat over the argument ui means that it should be omitted.  
其中，参数ui上的帽子意味着应该省略它。

Here is another explicit example. Suppose m = 2 and n = 1. Given , the multiplication structure on V(E∗) implies that (  
这是另一个明确的例子。假设m=2，n=1。给定，v（e）上的乘法结构意味着（

Furthermore, for u1,u2,u3,∈ E,  
此外，对于U1，U2，U3，∈e，

µ3(v1∗ ∧ v2∗ ∧ v3∗)(u1,u2,u3) = X sgn(σ)vσ∗(1)(u1)vσ∗(2)(u2)vσ∗(3)(u3)  
μ3（v1 v2 v3）（u1，u2，u3）=x sgn（σ）vσ（1）（u1）vσ（2）（u2）vσ（3）（u3）

σ∈S3  
σ∈S3

= v1∗(u1)v2∗(u2)v3∗(u3) − v1∗(u1)v3∗(u2)v2∗(u3) − v2∗(u1)v1∗(u2)v3∗(u3) + v2∗(u1)v3∗(u2)v1∗(u3)  
=v1（u1）v2（u2）v3（u3）−v1（u1）v3（u2）v2（u3）−v2（u1）v1（u2）v3（u3）+v2（u1）v3（u2）v1（u3）

+ v3∗(u1)v1∗(u2)v2∗(u3) − v3∗(u1)v2∗(u2)v1∗(u3). Now the (2,1)- shuffles of {1,2,3} are the following three permutations, namely  
+v3（u1）v1（u2）v2（u3）−v3（u1）v2（u2）v1（u3）。现在1,2,3的（2,1）-洗牌是以下三种排列，即

, , .  
、、。

If) and), then (∗∗) implies that  
如果）和），那么（）意味着

(f · g)(u1,u2,u3) = X sgn(σ)f(uσ(1),uσ(2))g(uσ(3))  
（f·g）（u1，u2，u3）=x sgn（σ）f（uσ（1），uσ（2））g（uσ（3））

σ∈shuffle(2,1)  
σ∈洗牌（2,1）

= f(u1,u2)g(u3) − f(u1,u3)g(u2) + f(u2,u3)g(u1)  
=f（u1，u2）g（u3）−f（u1，u3）g（u2）+f（u2，u3）g（u1）

= µ2(v1∗ ∧ v2∗)(u1,u2)µ1(v3∗)(u3) − µ2(v1∗ ∧ v2∗)(u1,u3)µ1(v3∗)(u2)  
=μ2（v1 v2）（u1，u2）μ1（v3）（u3）−μ2（v1 v2）（u1，u3）μ1（v3）（u2）

+ µ2(v1∗ ∧ v2∗)(u2,u3)µ1(v3∗)(u1)  
+μ2（v1 v2）（u2，u3）μ1（v3）（u1）

= (v1∗(u1)v2∗(u2) − v2∗(u1)v1∗(u2))v3∗(u3)  
=（v1（u1）v2（u2）−v2（u1）v1（u2））v3（u3）

− (v1∗(u1)v2∗(u3) − v2∗(u1)v1∗(u3))v3∗(u2)  
−（v1（u1）v2（u3）−v2（u1）v1（u3））v3（u2）

+ (v1∗(u2)v2∗(u3) − v2∗(u2)v1∗(u3))v3∗(u1) = µ3(v1∗ ∧ v2∗ ∧ v3∗)(u1,u2,u3).  
+（v1（u2）v2（u3）−v2（u2）v1（u3））v3（u1）=（v1 v2 v3）（u1、u2、u3）。

As a result of all this, the direct sum  
因此，直接和

Alt(E) = MAltn(E;K)  
alt（e）=maltn（e；k）

n≥0  
n＝0

is an algebra under the above multiplication, and this algebra is isomorphic to V(E∗). For the record we state  
是上面乘法下的代数，这个代数同构于v（e）。为了我们的记录

33.6. THE HODGE ∗-OPERATOR  
小精灵。霍奇-操作员

Proposition 33.14. When E is finite dimensional, the maps µ: Vn(E∗) −→ Altn(E;K) induced by the linear extensions of the maps given by  
提案33.14.当e为有限维时，图的μ：vn（e）−→altn（e；k）由以下公式给出的图的线性扩展所诱导：



yield a canonical isomorphism of algebras µ: V(E∗) −→ Alt(E), where the multiplication in Alt(E) is defined by the maps ∧: Altm(E;K) × Altn(E;K) −→ Altm+n(E;K), with  
生成代数的典型同构：v（e）−→alt（e），其中alt（e）中的乘法由映射：altm（e；k）×alt n（e；k）−→altm+n（e；k）定义，其中

(f ∧ g)(u1,...,um+n) = X sgn(σ)f(uσ(1),...,uσ(m))g(uσ(m+1),...,uσ(m+n)),  
（f g）（u1，…，u m+n）=x sgn（σ）f（uσ（1），…，uσ（m））g（uσ（m+1），…，uσ（m+n）），

σ∈shuffle(m,n)  
σ∈洗牌（m，n）

where shuffle(m,n) consists of all (m,n)-“shuffles,” that is, permutations σ of {1,...m+n} such that σ(1) < ··· < σ(m) and σ(m + 1) < ··· < σ(m + n).  
其中shuffle（m，n）由所有（m，n）-“shuffles”组成，即1，…m+n的置换σ，使得σ（1）<······<σ（m）和σ（m+1）<······<σ（m+n）。

Remark: The algebra V(E) is a graded algebra. Given two graded algebras E and F, we can make a new tensor product E ⊗b F, where E ⊗b F is equal to E ⊗ F as a vector space, but with a skew-commutative multiplication given by  
注：代数V（E）是一个等级代数。给定两个分次代数e和f，我们可以得到一个新的张量积e b f，其中e b f等于e f作为向量空间，但用下式给出的一个斜交换乘法

(a ⊗ b) ∧ (c ⊗ d) = (−1)deg(b)deg(c)(ac) ⊗ (bd),  
（a b）（c d）=（-1）deg（b）deg（c）（ac）（bd）、

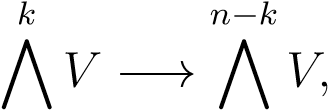
where a ∈ Em,b ∈ Fp, c ∈ En,d ∈ Fq. Then, it can be shown that  
其中a∈em，b∈fp，c∈en，d∈fq。那么，可以看出

^(E ⊕ F) ∼= ^(E) ⊗b^(F).  
^（e\_f）b（f）。

## 33.6 The Hodge ∗-Operator 33.6 Hodge-操作员

In order to define a generalization of the Laplacian that applies to differential forms on a  
为了定义拉普拉斯的推广，它适用于

Riemannian manifold, we need to define isomorphisms  
黎曼流形，我们需要定义同构



for any Euclidean vector space V of dimension n and any k, with 0 ≤ k ≤ n. If h−,−i denotes the inner product on V , we define an inner product on Vk V , denoted h−,−i∧, by setting hu1 ∧ ··· ∧ uk,v1 ∧ ··· ∧ vki∧ = det(hui,vji),  
对于尺寸n的任何欧几里得向量空间v和任何k，0≤k≤n。如果h−，−i表示v上的内积，我们在vk v上定义内积，表示h−，−i，通过设置hu1············································

for all ui,vi ∈ V , and extending h−,−i∧ by bilinearity.  
对于所有用户界面，v i∈v，并通过双线性扩展h−、−i。

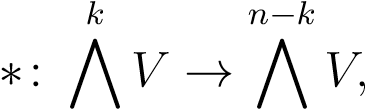
It is easy to show that if (e1,...,en) is an orthonormal basis of V , then the basis of Vk V consisting of the eI (where I = {i1,...,ik}, with 1 ≤ i1 < ··· < ik ≤ n) is an orthonormal basis of Vk V . Since the inner product on V induces an inner product on V ∗ (recall that , for all ω1,ω2 ∈ V ∗), we also get an inner product on Vk V ∗.  
很容易看出，如果（e1，…，en）是v的正态基，那么由ei组成的vk v的基（其中i=i1，…，ik，其中1≤i1<······<ik≤n）是vk v的正态基。由于v上的内积在v上诱导内积（回想一下，对于所有ω1，ω2∈v），我们也得到vk v上的内积。

Definition 33.6. An orientation of a vector space V of dimension n is given by the choice of some basis (e1,...,en). We say that a basis (u1,...,un) of V is positively oriented iff det(u1,...,un) > 0 (where det(u1,...,un) denotes the determinant of the matrix whose jth column consists of the coordinates of uj over the basis (e1,...,en)), otherwise it is negatively oriented. An oriented vector space is a vector space V together with an orientation of V .  
定义33.6.通过选择一些基（e1，…，en），给出了n维向量空间v的方向。我们说V的基（u1，…，un）是正向的，iff det（u1，…，un）>0（其中det（u1，…，un）表示矩阵的行列式，其jth列由基（e1，…，en）上的uj坐标组成），否则它是负向的。有向向量空间是向量空间V加上V的方向。

If V is oriented by the basis (e1,...,en), then V ∗ is oriented by the dual basis (  
如果V由基（e1，…，en）定向，则V由双基定向（

If σ is any permutation of {1,...,n}, then the basis (eσ(1),...,eσ(n)) has positive orientation iff the signature sgn(σ) of the permutation σ is even.  
如果σ是1，…，n的任意置换，则基（eσ（1），…，eσ（n））具有正方向，如果置换σ的签名sgn（σ）是偶数。

If V is an oriented vector space of dimension n, then we can define a linear isomorphism  
如果v是一个维数为n的定向向量空间，那么我们可以定义一个线性同构。



called the Hodge ∗-operator. The existence of this operator is guaranteed by the following proposition.  
称为hodge-运算符。这个算符的存在性由以下命题来保证。

Proposition 33.15. Let V be any oriented Euclidean vector space whose orientation is given by some chosen orthonormal basis (e1,...,en). For any alternating tensor α ∈ Vk V , there is a unique alternating tensor ∗α ∈ Vn−k V such that  
提案33.15。设V为任意定向欧几里得向量空间，其方向由选定的正交基（e1，…，en）给出。对于任何交变张量α∈v k v，都有一个唯一的交变张量α∈vn−k v，这样

α ∧ β = h∗α,βi∧ e1 ∧ ··· ∧ en  
αβ=Hα，βi e1···en

for all β ∈ Vn−k V . The alternating tensor ∗α is independent of the choice of the positive orthonormal basis (e1,...,en).  
对于所有β∈VN−K V。交替张量α与正正交基（e1，…，en）的选择无关。

Proof. Since Vn V has dimension 1, the alternating tensor e1 ∧ ··· ∧ en is a basis of Vn V . It follows that for any fixed α ∈ Vk V , the linear map λα from Vn−k V to Vn V given by  
证据。由于vn v的维数为1，交变张量e1····en是vn v的基础。由此可知，对于任何固定α∈v k v，从vn−k v到vn v的线性映射为

λα(β) = α ∧ β  
λα（β）=αβ

is of the form λα(β) = fα(β)e1 ∧ ··· ∧ en  
形式为λα（β）=fα（β）e1···en

for some linear form. But then, by the duality induced by the inner product h−,−i on Vn−k V , there is a unique vector ∗α ∈ Vn−k V such that  
对于某种线性形式。但是，由于内积h−、−i在vn−k v上诱导的对偶性，存在一个唯一的向量∮α∈vn−k v，这样

fλ(β) = h∗α,βi∧ for all β ∈ Vn−k V ,  
fλ（β）=hα，βi对于所有β∈vn−k v，

which implies that  
这意味着

α ∧ β = λα(β) = fα(β)e1 ∧ ··· ∧ en = h∗α,βi∧ e1 ∧ ··· ∧ en,  
αβ=λα（β）=fα（β）e1·····en=hα，βi e1····en，

as claimed. If () is any other positively oriented orthonormal basis, by Proposition  
如要求。如果（）是任何其他正方向正交基，通过命题

, since det(P) = 1 where P is the  
，因为det（p）=1，其中p是

change of basis from () and both bases are positively oriented.   
基础从（）开始改变，两个基础都是正向的。

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Definition 33.7. The operator ∗ from Vk V to Vn−k V defined by Proposition 33.15 is called the Hodge ∗-operator.  
定义33.7.由命题33.15定义的从v k v到vn−k v的算符称为hodge算符。

Obseve that the Hodge ∗-operator is linear.  
注意Hodge-运算符是线性的。

The Hodge ∗-operator is defined in terms of the orthonormal basis elements of VV as follows: For any increasing sequence (i1,...,ik) of elements ip ∈ {1,...,n}, if (j1,...,jn−k) is the increasing sequence of elements jq ∈ {1,...,n} such that  
Hodge-算符是根据vv的正交基元定义的，如下所示：对于元素ip 1，…，n的任何递增序列（i1，…，ik），如果（j1，…，jn−k）是元素jq 1，…，n的递增序列，这样

{i1,...,ik} ∪ {j1,...,jn−k} = {1,...,n},  
i1，…，ik j1，…，jn−k=1，…，n，

then  
然后

∗(ei1 ∧ ··· ∧ eik) = sign(i1,...ik,j1,...,jn−k)ej1 ∧ ··· ∧ ejn−k.  
（ei1···eik）=符号（i1，…ik，j1，…，jn−k）ej1···ejn−k。

In particular, for k = 0 and k = n, we have  
特别是，对于k=0和k=n，我们有

∗(1) = e1 ∧ ··· ∧ en ∗(e1 ∧ ··· ∧ en) = 1.  
（1）=e1······（e1····en）=1.

For example, if n = 3, we have  
例如，如果n=3，我们有

∗e1 = e2 ∧ e3  
e1=e2 e3

∗e2 = −e1 ∧ e3  
e2=−e1 e3

∗e3 = e1 ∧ e2  
e3=e1 e2

∗(e1 ∧ e2) = e3  
（e1 e2）=e3

∗(e1 ∧ e3) = −e2 ∗(e2 ∧ e3) = e1.  
（e1 e3）=-e2（e2 e3）=e1.

The Hodge ∗-operators ∗: Vk V → Vn−k V induce a linear map ∗: V(V ) → V(V ). We also have Hodge ∗-operators ∗: Vk V ∗ → Vn−k V ∗.  
Hodge-运算符：v k v→vn−k v诱导线性映射：v（v）→v（v）。我们还有hodge-operators：v k v→vn−k v。

The following proposition shows that the linear map ∗: V(V ) → V(V ) is an isomorphism.  
下面的命题表明线性映射：v（v）→v（v）是同构的。

Proposition 33.16. If V is any oriented vector space of dimension n, for every k with  
提案33.16。如果v是维度n的任何定向向量空间，对于每个k

0 ≤ k ≤ n, we have  
0≤k≤n，我们有

1. ∗∗ = (−id)k(n−k).  
   =（−ID）K（N−K）。
2. hx,yi∧ = ∗(x ∧ ∗y) = ∗(y ∧ ∗x), for all x,y ∈ Vk V .  
   hx，yi=（x y）=（y x），对于所有x，y∈vk v。

Proof. (1) Let ( is an orthonormal basis of V . It is enough to check the identity on basis elements. We have  
证据。（1）Let（是v的正交基。只需根据基本元素检查身份即可。我们有

∗(ei1 ∧ ··· ∧ eik) = sign(i1,...ik,j1,...,jn−k)ej1 ∧ ··· ∧ ejn−k  
（ei1···eik）=符号（i1，…ik，j1，…，jn−k）ej1···ejn−k

and  
和

∗∗(ei1 ∧ ··· ∧ eik) = sign(i1,...ik,j1,...,jn−k) ∗(ej1 ∧ ··· ∧ ejn−k)  
（ei1···eik）=符号（i1，…ik，j1，…，jn−k）（ej1····ejn−k）

= sign(i1,...ik,j1,...,jn−k)sign(j1,...,jn−k,i1,...ik)ei1 ∧ ··· ∧ eik.  
=符号（i1，…ik，j1，…，jn−k）符号（j1，…，jn−k，i1，…ik）ei1···eik。

It is easy to see that  
很容易看到

sign(i1,...ik,j1,...,jn−k)sign(j1,...,jn−k,i1,...ik) = (−1)k(n−k),  
符号（i1，…ik，j1，…，jn−k）符号（j1，…，jn−k，i1，…ik）=（−1）k（n−k），

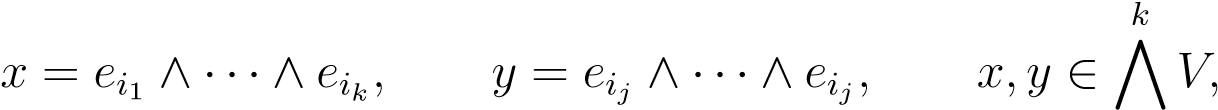
which yields  
会产生

∗∗(ei1 ∧ ··· ∧ eik) = (−1)k(n−k) ei1 ∧ ··· ∧ eik,  
（ei1····eik）=（−1）k（n−k）ei1····eik，

as claimed.  
如要求。

(ii) These identities are easily checked on basis elements; see Jost [98], Chapter 2, Lemma  
（ii）这些标识很容易根据元素进行检查；见Jost[98]，第2章，引理

2.1.1. In particular let  
2.1.1.尤其是让



xwhere (not equal to any is an orthonormal basis ofejq of by the orthonormality of the basis, this means theV . If x =6 y, hx,yi∧ = 0 since there is someth row ofeip of y p  
式中（不等于任何）是ejq的正交基，由基的正交性决定，这意味着v。如果x=6y，hx，yi=0，因为y p的某一行EIP

(heil,ejsi) consists entirely of zeroes. Also x =6 y implies that y ∧ ∗x = 0 since  
（heil，ejsi）完全由零组成。同样，x=6y意味着y x=0，因为

∗x = sign(i1,...ik,l1,...,ln−k)el1 ∧ ··· ∧ eln−k,  
X=符号（I1，…Ik，L1，…，Ln−K）El1···Eln−K，

where els is the same as some ep in y. A similar argument shows that if x =6 y, x ∧ ∗y = 0.  
其中，els与y中的某个ep相同。类似的论点表明，如果x=6y，x y=0。

So now assume x = y. Then  
所以现在假设x=y。

∗(ei1 ∧ ··· ∧ eik ∧ ∗(ei1 ∧ ··· ∧ eik)) = ∗(e1 ∧ e2 ··· ∧ en) = 1 = hx,xi∧.   
（E1）····εEk（1）＝HX，（X1）。

It is possible to express ∗(1) in terms of any basis (not necessarily orthonormal) of V .  
可以用V的任何基（不一定是正交）来表示（1）。

Proposition 33.17. If V is any finite-dimensional oriented vector space, for any basis (v!,...,vn) of V , we have  
提案33.17。如果v是任何有限维的定向向量空间，对于任何基（v！，…，Vn）的，我们有

.  
.

Proof. If (e1,...,en) is an orthonormal basis of V and (v1,...,vn) is any other basis of V , then hv1 ∧ ··· ∧ vn,v1 ∧ ··· ∧ vni∧ = det(hvi,vji),  
证据。如果（e1，…，en）是v的正交基，（v1，…，vn）是v的任何其他基，那么hv1·············································

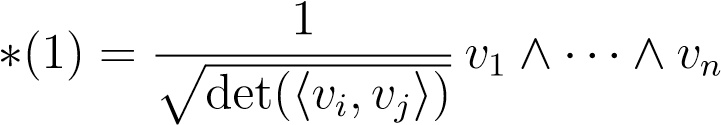
and since  
从那以后

v1 ∧ ··· ∧ vn = det(A)e1 ∧ ··· ∧ en  
v1···vn=Det（a）e1····en

where A is the matrix expressing the vj in terms of the ei, we have hv1 ∧ ··· ∧ vn,v1 ∧ ··· ∧ vni∧ = det(A)2he1 ∧ ··· ∧ en,e1 ∧ ··· ∧ eni = det(A)2. As a consequence, det(A) = pdet(hvi,vji), and  
式中，a是用ei表示vj的矩阵，我们有hv1··································································因此，DET（A）=PDET（HVI、VJI），以及

q v1 ∧ ··· ∧ vn = det(hvi,vji)e1 ∧ ··· ∧ en,  
q v1····vn=Det（hvi，vji）e1····en，

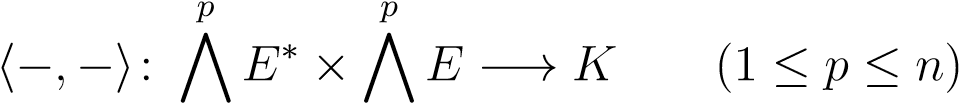
from which it follows that  
从中可以看出



(see Jost [98], Chapter 2, Lemma 2.1.3).   
（见Jost[98]，第2章，Lemma 2.1.3）。

## 33.7 Left and Right Hooks ~ 33.7左右挂钩~

In this section all vector spaces are assumed to have finite dimension. Say dim(E) = n. Using our nonsingular pairing  
在本节中，假设所有向量空间都有有限维。假设dim（e）=n.使用我们的非奇异配对



defined on generators by  
在发电机上定义

,  
，

we define various contraction operations (partial evaluation operators)  
我们定义了各种收缩操作（部分评估运算符）

y : (left hook)  
Y：（左钩）

and  
和

x : (right hook),  
X：（右钩）

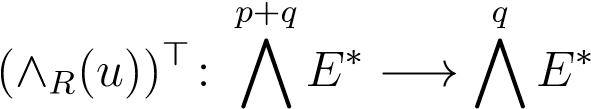
as well as the versions obtained by replacing E by E∗ and E∗∗ by E. We begin with the left interior product or left hook, y.  
以及用e替换e得到的版本，用e替换e获得的版本。我们从左侧内饰产品或左侧挂钩y开始。

Let u ∈ Vp E. For any q such that p + q ≤ n, multiplication on the right by u is a linear map given by v 7→ v ∧ u  
设u∈v p e，对于任意q，如果p+q≤n，右边乘以u是由v 7→v\_u给出的线性映射。

where v ∈ Vq E. The transpose of ∧R(u) yields a linear map  
式中v∈vq e，r（u）的转置产生一个线性映射。

,  
，

which, using the isomorphisms and , can be viewed as a map  
它使用同构和，可以被看作是一个地图



given by  
给出者

z∗ 7→ z∗ ◦ ∧R(u),  
Z 7→Z R（U）

uwherey of z∧∗R∈(uV) defined byp+q E∗. We denote z∗ ◦ ∧R(u) by u y z∗. In terms of our pairing, the adjoint  
由p+q e定义的z r（uv）的公式。我们用u y z\_表示z\_\_r（u）。就我们的配对而言，

;  
；

this in turn leads to the following definition.  
这反过来导致了以下定义。

Definition 33.8. Let u ∈ Vp E and z∗ ∈ Vp+q E∗. We define u y z∗ ∈ Vq E∗ to be q-vector uniquely determined by  
定义33.8.设u vp e和z vp+q e。我们将u y z∈vq e定义为唯一由

hu y z∗,vi = hz∗,v ∧ ui, for all v ∈ Vq E.  
对于所有v∈vq e，hu y z，vi=hz，v ui。

Remark: Note that to be precise the operator  
备注：请注意，准确地说，操作员

y :   
Y：

depends of p,q, so we really defined a family of operators y p,q. This family of operators y p,q induces a map  
依赖于p，q，所以我们真的定义了一个算子家族y，p，q。这个算子家族y，p，q产生了一个映射

y : ^E × ^E∗ −→ ^E∗,  
Y：^E×E−→^E，

with y  
用Y

as defined before. The common practice is to omit the subscripts of y .  
如前所述。通常的做法是省略y的下标。

It is immediately verified that  
立即证实

(u ∧ v) y z∗ = u y (v y z∗),  
（u v）y z=u y（v y z），

for all u ∈ Vk E,v ∈ Vp−k E, z∗ ∈ Vp+q E∗ since h(u ∧ v) y z∗,wi = hz∗,w ∧ u ∧ vi = hv y z∗,w ∧ ui = hu y (v y z∗),wi,  
对于所有u v k e，v vp k e，z vp+q e，因为h（u v）y z，wi=hz，w u vi=hv y z，w ui=hu y（v y z），wi，

whenever w ∈ Vq E. This means that  
每当w∈vq e，这意味着

y : ^E × ^E∗ −→ ^E∗  
Y：^E×E−→^E

is a left action of the (noncommutative) ring VE with multiplication ∧ on VE∗, which makes VE∗ into a left VE-module.  
是（非交换）环ve的左操作，在ve上乘，使ve成为左ve模块。

By interchanging E and E∗ and using the isomorphism  
通过交换e和e并使用同构

,  
，

we can also define some maps  
我们也可以定义一些地图

y :   
Y：

and make the following definition.  
并做出以下定义。

Definition 33.9. Let u∗ ∈ Vp E∗, and z ∈ Vp+q E. We define u∗ y z ∈ Vq as the q-vector uniquely defined by  
定义33.9.设u vp e，z vp+q e，定义u y z vq为唯一由

hv∗ ∧ u∗,zi = hv∗,u∗ y zi, for all v∗ ∈ Vq E∗.  
hv u，zi=hv，u y zi，对于所有v∈vq e。

As for the previous version, we have a family of operators y p,q which define an operator  
对于前一个版本，我们有一个运算符Y p，Q系列，它定义了一个运算符

y : ^E∗ × ^E −→ ^E.  
Y：^E×^E−→^E.

We easily verify that  
我们很容易证实

(u∗ ∧ v∗) y z = u∗ y (v∗ y z),  
（u v）y z=u y（v y z），

whenever u∗ ∈ Vk E∗, v∗ ∈ Vp−k E∗, and z ∈ Vp+q E; so this version of y is a left action of the ring VE∗ on VE which makes VE into a left VE∗-module.  
当u∈v k e，v∈vp−k e，z∈vp+q e时，y的这个版本是环ve on ve的左作用，使ve成为左ve模件。

In order to proceed any further we need some combinatorial properties of the basis of Vp E constructed from a basis (e1,...,en) of E. Recall that for any (nonempty) subset I ⊆ {1,...,n}, we let eI = ei1 ∧ ··· ∧ eip,  
为了进一步，我们需要从e的基（e1，…，e n）构造出vp e的基的一些组合性质。回想一下，对于任何（非空）子集i 1，…，n，我们让ei=e i 1····eip，

where I = {i1,...,ip} with i1 < ··· < ip. We also let e∅ = 1.  
其中，i=i1，…，ip with i1<·······<ip.我们也让e∅=1。

Given any two nonempty subsets H,L ⊆ {1,...,n} both listed in increasing order, say H = {h1 < ... < hp} and L = {`1 < ... < `q}, if H and L are disjoint, let H ∪ L be union of H and L considered as the ordered sequence  
考虑到任意两个非空子集h，l 1，…，n都按递增顺序列出，例如h=h1<…<hp and l=`1<…<`Q，如果h和l是不相交的，那么h l是h和l的并集，被认为是有序序列。

(h1,...,hp,`1,...,`q).  
（h1，…，hp，`1，…`q）。

Then let  
那就让

,  
，

ρH,L (−1)ν if H ∩ L = ∅,  
ρh，l（−1）ν，如果h l=∅，

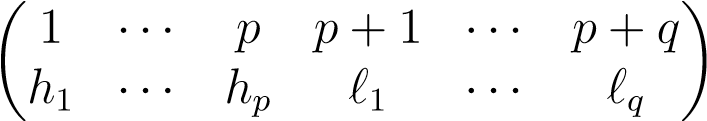
where ν = |{(h,l) | (h,l) ∈ H × L, h > l}|.  
式中，ν=（h，l）（h，l）∈h×l，h>l。

Observe that when H∩L = ∅, |H| = p and |L| = q, the number ν is the number of inversions of the sequence  
观察H L=∅、H=P、L=Q时，数濉为序列的倒转数。

(h1, ··· , hp, `1, ··· , `q),  
（h1，···，hp，`1，···，`Q）

where an inversion is a pair (hi,`j) such that hi > `j.  
其中一个倒转是一对（hi，`j），这样hi>`j。

 Unless p + q = n, the function whose graph is given by  
除非p+q=n，否则其图由下式给出的函数



is not a permutation of {1,...,n}. We can view ν as a slight generalization of the notion of the number of inversions of a permutation.  
不是1，…，n的排列。我们可以把π看作是置换倒数概念的一个小小的推广。

Proposition 33.18. For any basis (e1,...,en) of E the following properties hold: (1) If H ∩ L = ∅, |H| = p, and |L| = q, then  
提案33.18。对于e的任何基（e1，…，en），以下属性保持不变：（1）如果h l=∅，h=p，l=q，则

ρH,LρL,H = (−1)ν(−1)pq−ν = (−1)pq.  
ρh，lρl，h=−1）ν（−1）pq−ν=−1）pq。

1. For H,L ⊆ {1,...,m} listed in increasing order, we have  
   对于H，L 1，…，M按递增顺序列出，我们有

eH ∧ eL = ρH,LeH∪L.  
eh el=ρh，leh l.

Similarly, e∗H ∧ e∗L = ρH,Le∗H∪L.  
同样，e h e l=ρh，le h l。

1. For the left hook  
   对于左钩子

y : ,  
Y：

we have  
我们有

eH y e∗L = 0 if H 6⊆ L eH y e∗L = ρL−H,He∗L−H if H ⊆ L.  
e h y e l=0如果h 6 l eh y e l=ρl−h，he l−h如果h l。

1. For the left hook y :   
   对于左钩Y：

we have  
我们有

e∗H y eL = 0 if H 6⊆ L e∗H y eL = ρL−H,HeL−H if H ⊆ L.  
如果h 6 l e h y el=ρl−h，则e h y el=0；如果h l，则hel−h。

Proof. These are proved in Bourbaki [25] (Chapter III, §11, Section 11), but the proofs of (3) and (4) are very concise. We elaborate on the proofs of (2) and (4), the proof of (3) being similar.  
证据。这些在Bourbaki[25]中得到了证明（第三章第11节），但是（3）和（4）的证明非常简洁。我们详细阐述了（2）和（4）的证明，其中（3）的证明是相似的。

In (2) if H∩L =6 ∅, then eH ∧eL contains some vector twice and so eH ∧eL = 0. Otherwise, eH ∧ eL consists of eh1 ∧ ··· ∧ ehp ∧ e`1 ∧ ··· ∧ e`q,  
在（2）中，如果h l=6∅，那么eh el包含一些向量两次，因此eh el=0。否则，eh el由eh1····ehp e`1···e`q组成，

and to order the sequence of indices in increasing order we need to transpose any two indices (hi,`j) corresponding to an inversion, which yields ρH,LeH∪L.  
为了使指数序列按递增顺序排列，我们需要将对应于反演的任意两个指数（hi，`j）转置，得到ρh，leh l。

Let us now consider (4). We have |L| = p + q and |H| = p, and the q-vector is  
现在让我们考虑（4）。我们有l=p+q和h=p，q矢量是

characterized by  
特点是



for all v∗ ∈ Vq E∗. There are two cases.  
对于所有v∈vq e。有两种情况。

Case 1: H 6⊆ L. If so, no matter what v∗ ∈ Vq E∗ is, since H contains some index h not in L, the hth row ( )) of the determinant must be zero, so hv∗ ∧ e∗H,eLi = 0 for all v∗ ∈ Vq E∗, and since the pairing is nongenerate, we must have e∗H y eL = 0.  
例1:h 6 l。如果是，那么不管v∈vq e是什么，因为h包含一些不在l中的指数h，行列式的hth行（））必须为零，所以h v e h，eli=0代表所有v∈vq e，并且由于配对是非一般的，我们必须有e h y e l=0。

Case 2: H ⊆ L. In this case, for v∗ = e∗L−H, by (2) we have  
案例2：H L。在这种情况下，对于V=E L−H，通过（2）我们得到

,  
，

which yields  
会产生

he∗L−H,e∗H y eLi = ρL−H,H.  
h e l−h，e h y eli=ρl−h，h.

The q-vector can be written as a linear combinationwith |J| = q so  
q矢量可以写成一个线性组合，其中j=q so

.  
.

By definition of the pairing, = 0 unless J = L − H, which means that  
根据配对的定义，=0，除非j=l−h，这意味着

,  
，

so λL−H = ρL−H,H, as claimed.   
因此，如权利要求所述，λl−h=ρl−h，h。

Using Proposition 33.18, we have the Proposition 33.19. For the left hook  
利用33.18号提案，我们得到33.19号提案。对于左钩子

y :,  
Y：

for every u ∈ E, x∗ ∈ Vq+1−s E∗, and y∗ ∈ Vs E∗, we have  
对于每一个u∈e，x∈vq+1−s e，和y∈vs e，我们有

u y (x∗ ∧ y∗) = (−1)s(u y x∗) ∧ y∗ + x∗ ∧ (u y y∗).  
u y（x y）=（-1）s（u y x）y+x（u y）。

Proof. We can prove the above identity assuming that x∗ and y∗ are of the form e∗I and e∗J using Proposition 33.18 and leave the details as an exercise for the reader.   
证据。我们可以用命题33.18证明上述身份，假设x和y为e i和e j的形式，并将细节留给读者作为练习。

Thus, y : E×Vq+1 E∗ −→ Vq E∗ is almost an anti-derivation, except that the sign (−1)s is applied to the wrong factor.  
因此，y:e×vq+1 e−→vq e几乎是反派生的，除了符号（−1）s应用于错误的因子。

We have a similar identity for the other version of the left hook  
对于另一个版本的左钩子，我们有类似的身份

y :   
Y：

namely u∗ y (x ∧ y) = (−1)s(u∗ y x) ∧ y + x ∧ (u∗ y y)  
即u y（x y）=（−1）s（u y x）y+x（u y）

for every u∗ ∈ E∗, x ∈ Vq+1−s E, and y ∈ Vs E.  
对于每一个u e，x vq+1 s e，y vs e。

An application of this formula when q = 3 and s = 2 yields an interesting equation. In this case, u∗ ∈ E∗ and x,y ∈ V2 E, so we get  
当q=3，s=2时，这个公式的应用产生了一个有趣的方程。在这种情况下，u∈e和x，y∈v2 e，我们得到

u∗ y (x ∧ y) = (u∗ y x) ∧ y + x ∧ (u∗ y y).  
u y（x y）=（u y x）y+x（u y）。

(In particular, foru∗ y x) ∧ x = x ∧x(u=∗ yyx, since), and we obtainx ∈ V2 E and u∗ y x ∈ E, Proposition 33.12 implies that  
（特别是，对于u y x）x=x x（u y x，since），我们得到x v2 e和u y x e，命题33.12暗示

u∗ y (x ∧ x) = 2((u∗ y x) ∧ x). (†)  
u y（x x）=2（（u y x）x）。（†）

As a consequence, (u∗ y x) ∧ x = 0 iff u∗ y (x ∧ x) = 0. We will use this identity together with Proposition 33.25 to prove that a 2-vector x ∈ V2 E is decomposable iff x ∧ x = 0.  
因此，（u y x）x=0 iff u y（x x）=0。我们将利用这个恒等式和33.25命题来证明2向量x∈v2 e是可分解的iff x x=0。

It is also possible to define a right interior product or right hook x, using multiplication on the left rather than multiplication on the right. Then we use the maps  
也可以使用左边的乘法而不是右边的乘法来定义右边的内部积或右边的钩子x。然后我们用地图

x :   
X：

to make the following definition.  
做出以下定义。

Definition 33.10. Let u ∈ Vp E and z∗ ∈ Vp+q E∗. We define z∗ x u ∈ Vq E∗ to be the q-vector uniquely defined as  
定义33.10.设u vp e和z vp+q e。我们将z x u∈vq e定义为唯一定义为

hz∗ x u,vi = hz∗,u ∧ vi, for all v ∈ Vq E.  
hz x u，vi=hz，u vi，对于所有v vq e。

This time we can prove that  
这次我们可以证明

z∗ x (u ∧ v) = (z∗ x u) x v,  
Z X（U V）=（Z X U）X V，

so the family of operators x p,q defines a right action  
因此，操作符x p，q系列定义了一个正确的动作。

x : ^E∗ × ^E −→ ^E∗  
x：^e×e−→^e

of the ring VE on VE∗ which makes VE∗ into a right VE-module. Similarly, we have maps  
把ve变成一个正确的ve模件。同样，我们有地图

x :   
X：

which in turn leads to the following dual formation of the right hook.  
这反过来又导致了右钩子的以下双重形成。

Definition 33.11. Let u∗ ∈ Vp E∗ and z ∈ Vp+q E. We define zxu∗ ∈ Vq to be the q-vector uniquely defined by  
定义33.11.设u∈vp e和z∈vp+q e，定义zxu∈vq为

hu∗ ∧ v∗,zi = hv∗,z x u∗i, for all v∗ ∈ Vq E∗.  
胡V，z i=hv，z x u i，对于所有的V∈vq e。

We can prove that  
我们可以证明

z x (u∗ ∧ v∗) = (z x u∗) x v∗,  
z x（u v）=（z x u）x v，

so the family of operators x p,q defines a right action  
因此，操作符x p，q系列定义了一个正确的动作。

x : ^E × ^E∗ −→ ^E  
x：^e×e−→^e

of the ring VE∗ on VE which makes VE into a right VE∗-module.  
在使ve成为一个右ve模块的ve环上。

Since the left hook y : Vp E × Vp+q E∗ −→ Vq E∗ is defined by hu y z∗,vi = hz∗,v ∧ ui, for all u ∈ Vp E, v ∈ Vq E and z∗ ∈ Vp+q E∗,  
由于左钩y:vp e×vp+q e−→vq e由hu y z，vi=hz，v ui定义，对于所有u∈vp e，v∈vq e和z∈vp+q e，

the right hook x :   
右钩X：

by hz∗ x u,vi = hz∗,u ∧ vi, for all u ∈ Vp E, v ∈ Vq E, and z∗ ∈ Vp+q E∗, and v ∧ u = (−1)pqu ∧ v, we conclude that  
通过hz x u，vi=hz，u vi，对于所有u vp e，v v q e，和z vp+q e，和v u=−1）pqu v，我们得出：

z∗ x u = (−1)pq u y z∗.  
Z X U=（-1）PQ U Y Z。

Similarly, since  
同样，因为

hv∗∗ ∧∧uv∗∗,z,zii == hhvv∗∗,u,z∗xyuz∗ii,, for allfor all uu∗∗ ∈∈ VVpp EE∗∗,, vv∗∗ ∈∈ VVqq EE∗∗, andand zz∈∈VVpp++qqEE,  
hv uv，z，z ii==hhvv，u，z xyuz ii，，对于所有uu∈vvpp ee，vv∈vqq ee，and zz∈vvpp++qee，

hu and v∗ ∧ u∗ = (−1)pqu∗ ∧ v∗, we have  
Hu和V U=（-1）PQU V，我们有

z x u∗ = (−1)pq u∗ y z.  
Z x U=（−1）PQ U Y Z.

We summarize the above facts in the following proposition.  
我们将上述事实概括为以下命题。

Proposition 33.20. The following identities hold:  
提案33.20。以下标识保留：

z∗ x u = (−1)pq u y z∗ for all u ∈ Vp E and all z∗ ∈ Vp+q E∗ z x u∗ = (−1)pq u∗ y z for all u∗ ∈ Vp E∗ and all z ∈ Vp+q E.  
z x u=−1）pq u y z对于所有u vp e和所有z vp+q e z x u=（-1）pq u y z对于所有u vp e和所有z vp+q e。

Therefore the left and right hooks are not independent, and in fact each one determines the other. As a consequence, we can restrict our attention to only one of the hooks, for example the left hook, but there are a few situations where it is nice to use both, for example in Proposition 33.23.  
因此，左钩和右钩不是独立的，事实上，每一个都决定着另一个。因此，我们只能将注意力限制在其中一个钩子上，例如左钩子，但在一些情况下，最好同时使用这两个钩子，例如在33.23号提案中。

A version of Proposition 33.18 holds for right hooks, but beware that the indices in ρL−H,H are permuted. This permutation has to do with the fact that the left hook and the right hook are related via a sign factor.  
建议33.18的一个版本适用于右钩，但要注意，ρl−h，h中的指数是排列的。这种排列与一个事实有关，即左钩子和右钩子是通过一个符号因子关联的。

Proposition 33.21. For any basis (e1,...,en) of E the following properties hold:  
提案33.21。对于e的任何基础（e1，…，en），以下属性保持不变：

1. For the right hook x :   
   对于右挂钩X：

we have  
我们有

eL x e∗H = 0 if H 6⊆ L eL x e∗H = ρH,L−HeL−H if H ⊆ L.  
如果h 6 l，则el x e h=0；如果h l，则l−hel−h。

1. For the right hook x :   
   对于右挂钩X：

we have  
我们有

e∗L x eH = 0 if H 6⊆ L e∗L x eH = ρH,L−He∗L−H if H ⊆ L.  
如果h 6 l e l x eh=ρh，则e l x eh=0；如果h l，则l−he l−h。

Remark: Our definition of left hooks as left actions y : Vp E × Vp+q E∗ −→ Vq E∗ and y : Vp E∗×Vp+q E −→ Vq E and right hooks as right actions x : Vp+q E∗×Vp E −→ Vq E∗ and x: Vp+q E×Vp E∗ −→ Vq E is identical to the definition found in Fulton and Harris [69] (Appendix B). However, the reader should be aware that this is not a universally accepted notation. In fact, the left hook u∗ y z defined in Bourbaki [25] is our right hook z x u∗, up to the sign (−1)p(p−1)/2. This has to do with the fact that Bourbaki uses a different pairing which also involves an extra sign, namely  
备注：我们将左钩定义为左作用y:vp e×vp+q e−→vq e和y:vp e×vp+q e−→vq e和右钩定义为右作用x:vp+q e×vp e−→vq e和x:vp+q e×vp e−→vq e与Fulton和Harris[69]中的定义相同（附加九b）。然而，读者应该知道，这不是一个普遍接受的符号。实际上，在Bourbaki[25]中定义的左钩子u\_y z是我们的右钩子z x u，直到符号（−1）p（p−1）/2。这是因为Bourbaki使用了一个不同的配对，它还包含一个额外的符号，即

hv∗,u∗ y zi = (−1)p(p−1)/2hu∗ ∧ v∗,zi.  
hv，u y zi=（-1）p（p−1）/2hu v，zi.

One of the side-effects of this choice is that Bourbaki’s version of Formula (4) of Proposition 33.18 (Bourbaki [25], Chapter III, page 168) is  
这种选择的一个副作用是，布尔巴基对33.18号提案（布尔巴基[25]，第三章，第168页）公式（4）的版本是

e∗H y eL = 0 if H ⊆6 L  
e h y e l=0，如果h 6 l

e∗H y eL = (−1)p(p−1)/2ρH,L−HeL−H if H ⊆ L,  
e h y e l=（−1）p（p−1）/2ρh，l−hel−h，如果h l，

where |H| = p and |L| = p + q. This correspond to Formula (1) of Proposition 33.21 up to the sign factor (−1)p(p−1)/2, which we find horribly confusing. Curiously, an older edition of Bourbaki (1958) uses the same pairing as Fulton and Harris [69]. The reason (and the advantage) for this change of sign convention is not clear to us.  
式中h=p和l=p+q。这符合命题33.21的公式（1），直到符号因子（−1）p（p−1）/2，我们发现这令人非常困惑。奇怪的是，一个旧版本的Bourbaki（1958）使用了与Fulton和Harris相同的配对[69]。我们不清楚这种标志公约变更的原因（以及优势）。

We also have the following version of Proposition 33.19 for the right hook. Proposition 33.22. For the right hook  
我们还为右钩子提供了33.19号提案的以下版本。提案33.22。对于右钩子

x : ,  
X：

for every u ∈ E, x∗ ∈ Vr E∗, and y∗ ∈ Vq+1−r E∗, we have  
对于每个u∈e，x∈vr e，和y∈vq+1−r e，我们有

(x∗ ∧ y∗) x u = (x∗ x u) ∧ y∗ + (−1)rx∗ ∧ (y∗ x u).  
（x y）x u=（x x u）y+（-1）rx（y x u）。

Proof. A proof involving determinants can be found in Warner [180], Chapter 2.   
证据。华纳[180]第2章提供了一个涉及行列式的证明。

Thus, x : Vq+1 E∗ × E −→ Vq E∗ is an anti-derivation. A similar formula holds for the the right hook x : Vq+1 E × E∗ −→ Vq E, namely  
因此，x:vq+1e×e−→vq e是反派生的。右钩子x:vq+1 e×e−→vq e的类似公式成立，即

(x ∧ y) x u∗ = (x x u∗) ∧ y + (−1)rx ∧ (y x u∗),  
（x y）x u=（x x u）y+（-1）rx（y x u），

for every u∗ ∈ E, ∈ Vr E, and y ∈ Vq+1−r E. This formula is used by Shafarevitch [153] to define a hook, but beware that Shafarevitch use the left hook notation u∗ y x rather than the right hook notation. Shafarevitch uses the terminology convolution, which seems very unfortunate.  
对于每一个u e，vr e，和y vq+1−r e，shafarevitch[153]使用这个公式来定义一个钩子，但是要注意shafarevitch使用左钩子符号u y x而不是右钩子符号。Shafarevitch使用了卷积这个术语，这看起来很不幸。

For u ∈ E, the right hook z∗ x u is also denoted i(u)z∗, and called insertion operator or interior product. This operator plays an important role in differential geometry.  
对于u∈e，右钩z x u也表示i（u）z，称为插入运算符或内部积。这个算符在微分几何中起着重要作用。

Definition 33.12. Let u ∈ E and z∗ ∈ Vn+1(E∗). If we view z∗ as an alternating multilinear map in Altn+1(E;K), then we define i(u)z∗ ∈ Altn(E;K) as given by  
定义33.12。设u e和z vn+1（e）。如果我们把z看作Altn+1（e；k）中的交替多行映射，那么我们定义i（u）z∈Altn（e；k），如下所示：

(i(u)z∗)(v1,...,vn) = z∗(u,v1,...,vn).  
（i（u）z）（v1，…，vn）=z（u，v1，…，vn）。

Using the left hook y and the right hook x we can define two linear maps γ: Vp E → Vn−p E∗ and δ: Vp E∗ → Vn−p E as follows:  
使用左钩Y和右钩X，我们可以定义两个线性映射γ：vp e→vn−p e和δ：vp e→vn−p e，如下所示：

Definition 33.13. For any basis (e1,...,en) of E, if we let M = {1,...,n}, e = e1 ∧···∧en, and, define γ: Vp E → Vn−p E∗ and δ: Vp E∗ → Vn−p E as  
定义33.13.对于e的任何基（e1，…，e n），如果我们让m=1，…，n，e=e1···en，并定义γ：vp e→vn−p e和δ：vp e→vn−p e为

γ(u) = u y e∗ and δ(v∗) = e x v∗,  
γ（u）=u y e和δ（v）=e x v，

for all u ∈ Vp E and all v∗ ∈ Vp E∗.  
对于所有u vp e和所有v vp e。

Proposition 33.23. The linear maps γ: Vp E → Vn−p E∗ and δ: Vp E∗ → Vn−p E are isomorphims, and γ−1 = δ. The isomorphisms γ and δ map decomposable vectors to decomposable vectors. Furthermore, if z ∈ Vp E is decomposable, say z = u1 ∧ ··· ∧ up for some ui ∈ E, then for some , and for all i,j. A similar property holds for is any other basis of E and γ0 : Vp E → Vn−p E∗ and δ0 : Vp E∗ → Vn−p E are the corresponding isomorphisms, then γ0 = λγ and δ0 = λ−1δ for some nonzero λ ∈ K.  
提案33.23。线性映射γ：vp e→vn−p e和δ：vp e→vn−p e是同构的，γ−1=δ。同构γ和δ映射可分解向量到可分解向量。此外，如果z∈vp e是可分解的，假设z=u1······向上表示某些ui∈e，那么对于一些i，j，对于所有i，j，一个相似的性质持有的是e的任何其他基，而γ0:vp e→vn−p e和δ0:vp e→vn−p e是相应的同构，那么γ0=γ和δ对于一些非零的λ∈k，0=λ−1δ。

Proof. Using Propositions 33.18 and 33.21, for any subset J ⊆ {1,...,n} = M such that |J| = p, we have  
证据。使用命题33.18和33.21，对于任何子集j 1，…，n=m，这样j=p，我们

and .  
而且。

Thus, δ ◦ γ(eJ) = ρM−J,JρM−J,JeJ = eJ,  
因此，δγ（ej）=ρm−j，jρm−j，jej=ej，

since ρM−J,J = ±1. A similar result holds for γ ◦ δ. This implies that  
因为ρm−j，j=±1.同样的结果也适用于γδ。这意味着

δ ◦ γ = id and γ ◦ δ = id.  
δγ=id和γδ=id。

Thus, γ and δ are inverse isomorphisms.  
因此，γ和δ是逆同构。

If z ∈ Vp E is decomposable, then z = u1 ∧ ··· ∧ up where u1,...,up are linearly independent since z = 06 , and we can pick a basis of E of the form (u1,...,un). Then the above formulae show that  
如果z∈vp e是可分解的，那么z=u1·····up，其中u1，…，up自z=06以来是线性独立的，我们可以选取形式（u1，…，un）的e的基。上述公式表明

.  
.

Since () is the dual basis of (u1,...,un), we have) is any other basis of E, because Vn E has dimension 1, we have  
因为（）是（u1，…，un）的对偶基，我们有）是e的任何其他基，因为vn e有维1，我们有



for some nonzero λ ∈ K, and the rest is trivial.   
对于一些非零的λ∈k，其余的都是平凡的。

Applying Proposition 33.23 to the case where p = n − 1, the isomorphism γ: Vn−1 E → V1 E∗ maps indecomposable vectors in Vn−1 E to indecomposable vectors in V1 E∗ = E∗. But every vector in E∗ is decomposable, so every vector in Vn−1 E is decomposable.  
将命题33.23应用于p=n-1的情况，同构γ：vn-1e→v1e将vn-1e中的不可分解矢量映射到v1e=e中的不可分解矢量。但e中的每一个向量都是可分解的，所以vn-1e中的每一个向量都是可分解的。

Corollary 33.24. If E is a finite-dimensional vector space, then every vector in Vn−1 E is decomposable.  
推论33.24。如果e是一个有限维向量空间，那么vn-1e中的每个向量都是可分解的。

33.8. TESTING DECOMPOSABILITY ~  
33.8。测试可分解性~

## 33.8 Testing Decomposability ~ 33.8测试可分解性~

We are now ready to tackle the problem of finding criteria for decomposability. Such criteria will use the left hook. Once again, in this section all vector spaces are assumed to have finite dimension. But before stating our criteria, we need a few preliminary results.  
我们现在已经准备好解决寻找可分解性标准的问题。这样的标准将使用左钩子。再一次，在这一节中，假设所有向量空间都有有限维。但在说明我们的标准之前，我们需要一些初步的结果。

Proposition 33.25. Given z ∈ Vp E with z = 06 , the smallest vector space W ⊆ E such that z ∈ Vp W is generated by the vectors of the form  
提案33.25。给定z∈vp e，z=06，最小向量空间w e，使z∈vp w由形式的向量生成。

u∗ y z, with u∗ ∈ Vp−1 E∗.  
u y z，其中u∈vp−1 e。

Proof. First let W be any subspace such that z ∈ Vp(W) and let (e1,...,er,er+1,...,en) be a basis of E such that (e1,...,er) is a basis of W. Then,   
证据。首先让w是任何子空间，使得z∈vp（w）和let（e1，…，er，er+1，…，en）是e的基础，这样（e1，…，er）是w的基础。

and |I| = p − 1, and z = PJ µJeJ, where J ⊆ {1,...,r} and |J| = p ≤ r. It follows immediately from the formula of Proposition 33.18 (4), namely  
和i=p−1，z=p j\_jej，其中j 1，…，r和j=p≤r。它立即从命题33.18（4）的公式中得出，即

e∗I y eJ = ρJ−I,JeJ−I,  
e i y ej=ρj−i，jej−i，

that u∗ y z ∈ W, since J − I ⊆ {1,...,r}.  
u y z w，因为j 1，…，r。

Next we prove that if W is the smallest subspace of E such that z ∈ Vp(W), then W is generated by the vectors of the form u∗ y z, where u∗ ∈ Vp−1 E∗. Suppose not. Then the vectors u∗ y z with u∗ ∈ Vp−1 E∗ span a proper subspace U of W. We prove that for every subspace W 0 of W with dim(W 0) = dim(W) − 1 = r − 1, it is not possible that u∗ y z ∈ W 0 for all u∗ ∈ Vp−1 E∗. But then, as U is a proper subspace of W, it is contained in some subspace W 0 with dim(W 0) = r − 1, and we have a contradiction.  
然后证明，如果w是e的最小子空间，使得z∈vp（w），那么w是由形式为u y z的向量生成的，其中u∈vp−1e。假设不是。然后，带u的向量u y z∈vp−1 e跨越W的一个合适的子空间u，我们证明了对于带dim（w 0）=dim（w）−1=r−1的W的每个子空间w 0，u y z∈w 0不可能代表所有u∈vp−1 e。但是，由于u是w的一个合适的子空间，它包含在一些子空间w 0中，其中dim（w0）=r-1，我们有一个矛盾。

AnyLetz ∈wV∈p(WW−) can be written asW 0 and pick a basis ofz = z0W+ formed by a basis (w ∧ z00, where z0 ∈ eV1,...,ep W 0 andr−1)zof00 ∈WV0 pand−1 Ww0., and since W is the smallest subspace containing z, we have z00 = 06 . Consequently, if we write z00 = PI λIeI in terms of the basis (e1,...,er−1) of W 0, there is some eI, with I ⊆ { } I| 1, so that the coefficient λI is nonzero. Now, using any basis of E 1,...,er−1,w), by Proposition 33.18 (4), we see that  
Anyletz∈wv∈p（w w−）可以写成w 0，并选取由基（w z00，其中z0∈ev1，…，ep w 0 andr−1）形成的z=z0w+的基，zof00∈wv0 p and−1 ww0，由于w是包含z的最小子空间，所以z00=06。因此，如果我们根据w 0的基（e1，…，er−1）写出z00=piλi ei，则存在一些ei，其中i i 1，因此系数λi不为零。现在，根据命题33.18（4），使用e 1，…，er-1，w的任何基础，我们看到

e∗I y (w ∧ eI) = λw, λ = ±1.  
e i y（w ei）=λw，λ=±1.

It follows that  
接下来是



with, which shows that . Therefore, W is indeed generated by the vectors of the form u∗ y z, where u∗ ∈ Vp−1 E∗.   
有，这就说明了。因此，w确实是由形式为u y z的向量生成的，其中u∈vp−1 e。

To help understand Proposition 33.25, let E be the vector space with basis {e1,e2,e3,e4} and z = e1 ∧ e2 + e2 ∧ e3. Note that z ∈ V2 E. To find the smallest vector space W ⊆ E such that z ∈ V2 W, we calculate u∗ y z, where u∗ ∈ V1 E∗. The multilinearity of y implies it is enough to calculate u∗ y z for . Proposition 33.18 (4) implies that  
为了有助于理解33.25号提案，让e作为基e1、e2、e3、e4和z=e1 e2+e2 e3的向量空间。注意z∈v2 e。为了找到最小的向量空间w e，使z∈v2 w，我们计算u y z，其中u∈v1e。y的多重性意味着它足以计算u y z。提案33.18（4）意味着

e∗1 y z = e∗1 y (e1 ∧ e2 + e2 ∧ e3) = e1∗ y e1 ∧ e2 = −e2 e∗2 y z = e∗2 y (e1 ∧ e2 + e2 ∧ e3) = e1 − e3 e∗3 y z = e∗3 y (e1 ∧ e2 + e2 ∧ e3) = e3∗ y e2 ∧ e3 = e2 e∗4 y z = e∗4 y (e1 ∧ e2 + e2 ∧ e3) = 0.  
e 1 y z=e 1 y（e1 e2+e2 e3）=e1 y e1 e2；2 y z=e 2 y（e1 e2+e2；e3；3 y z=e 3 y（e1；e2+e2 e3）=e3 3 y（e1 e2+e2+e2 4 y z=e 4 y（e1 e2+e2 e3）=0.

Thus W is the two-dimensional vector space generated by the basis {e2,e1 − e3}. This is not surprising since z = −e2 ∧ (e1 − e3) and is in fact decomposable. As this example demonstrates, the action of the left hook provides a way of extracting a basis of W from z. Proposition 33.25 implies the following corollary.  
因此w是基e2，e1−e3生成的二维矢量空间。这并不奇怪，因为z=−e2（e1−e3）实际上是可分解的。如本例所示，左钩子的作用提供了一种从z中提取w的基础的方法。命题33.25暗示了以下推论。

Corollary 33.26. Any nonzero z ∈ Vp E is decomposable iff the smallest subspace W of E such that z ∈ Vp W has dimension p. Furthermore, if z = u1 ∧···∧up is decomposable, then (u1,...,up) is a basis of the smallest subspace W of E such that z ∈ Vp W  
推论33.26。任何非零的z∈vp e都是可分解的，只要e的最小子空间w使z∈vp w具有维数p，而且，如果z=u1·····up是可分解的，那么（u1，…，up）是e的最小子空间w的基础，这样z∈vp w

Proof. If dim(W) = p, then for any basis (e1,...,ep) of W we know that Vp W has e1∧···∧ep has a basis, and thus has dimension 1. Since z ∈ Vp W, we have z = λe1 ∧ ··· ∧ ep for some nonzero λ, so z is decomposable.  
证据。如果dim（w）=p，那么对于w的任何基（e1，…，ep），我们知道vp w具有e1·······························由于z∈vp w，对于一些非零的λ，我们有z=λe1···ep，因此z是可分解的。

Conversely assume that z ∈ Vp W is nonzero and decomposable. Then, z = u1 ∧···∧up, and since z = 06 , by Proposition 33.8 (u1,...,up) are linearly independent. Then for any  
相反地，假设z∈vp w为非零且可分解。然后，z=u1······向上，由于z=06，由33.8（u1，…，向上）命题是线性独立的。那么对于任何

(where u∗i is omitted), we have  
（省略u i），我们有

,  
，

so by Proposition 33.25 we have ui ∈ W for i = 1,...,p. This shows that dim(W) ≥ p, but since z = u1 ∧ ··· ∧ up, we have dim(W) = p, which means that (u1,...,up) is a basis of W.   
因此，在33.25号命题中，我们有i=1，…，p的ui∈w，这表明dim（w）≥p，但由于z=u1······up，我们有dim（w）=p，这意味着（u1，…，up）是w的基础。

Finally we are ready to state and prove the criterion for decomposability with respect to left hooks.  
最后，我们准备说明并证明左钩子的可分解性准则。

Proposition 33.27. Any nonzero z ∈ Vp E is decomposable iff  
提案33.27。任何非零z∈vp e都是可分解的iff

(u∗ y z) ∧ z = 0, for all u∗ ∈ Vp−1 E∗.  
（u y z）z=0，对于所有u∈vp−1 e。

Proof. First assume that z ∈ Vp E is decomposable. If so, by Corollary 33.26, the smallest subspace W of E such that z ∈ Vp W has dimension p, so we have z = e1 ∧ ··· ∧ ep where e1,...,ep form a basis of W. By Proposition 33.25, for every u∗ ∈ Vp−1 E∗, we have u∗ y z ∈ W, so each u∗ y z is a linear combination of the ei’s, say  
证据。首先假设z∈vp e是可分解的。如果是这样，由推论33.26，e的最小子空间w，使z∈vp w具有维数p，那么我们得到z=e1···ep，其中e1，…，ep构成w的基础。由命题33.25，对于每个u∈vp−1e，我们得到u y z∈w，因此每个u y z是ei的线性组合，说

u∗ y z = α1e1 + ··· + αpep,  
u y z=α1e1+····+αpep，

33.8. TESTING DECOMPOSABILITY ~  
33.8。测试可分解性~

and p  
和P

(u∗ y z) ∧ z = Xαiei ∧ e1 ∧ ··· ∧ ei ∧ ··· ∧ ep = 0.  
（u y z）z=xαiei e1····ei···ep=0.

i=1  
i＝1

Now assume that (u∗ yz)∧z = 0 for all u∗ ∈ Vp−1 E∗, and that dim(W) = m > p, where W is the smallest subspace of E such that z ∈ Vp W If e1,...,em is a basis of W, then we have z = PI λIeI, where I ⊆ {1,...,m} and |I| = p. Recall that z = 06 , and so, some λI is nonzero. By Proposition 33.25, each ei can be written as u∗ y z for some u∗ ∈ Vp−1 E∗, and since (u∗ y z) ∧ z = 0 for all u∗ ∈ Vp−1 E∗, we get  
现在假设（u yz）z=0表示所有u∈vp−1 e，并且dim（w）=m>p，其中w是e的最小子空间，因此z∈vp w如果e1，…，em是w的基础，那么我们有z=p iλiei，其中i 1，…，m和i p。回想一下z=06，因此，一些λi是非零的。根据命题33.25，对于某些u∈vp−1 e，每个ei可以写成u y z，由于（u y z）z=0对于所有u∈vp−1 e，我们得到

ej ∧ z = 0 for j = 1,...,m.  
j=1，…，m时，ej z=0。

By wedging z = PI λIeI with each ej, as m > p, we deduce λI = 0 for all I, so z = 0, a contradiction. Therefore, m = p and Corollary 33.26 implies that z is decomposable.   
通过将z=p iλiei与每个ej楔入，作为m>p，我们推导出所有i的λi=0，因此z=0，这是一个矛盾。因此，m=p和推论33.26意味着z是可分解的。

As a corollary of Proposition 33.27 we obtain the following fact that we stated earlier without proof.  
作为第33.27号命题的推论，我们得出了以下事实，即我们在前面陈述时没有证据。

Proposition 33.28. Given any vector space E of dimension n, a vector x ∈ V2 E is decomposable iff x ∧ x = 0.  
提案33.28。在任意维为n的向量空间e下，向量x∈v2 e是可分解的iff x x=0。

Proof. Recall that as an application of Proposition 33.19 we proved the formula (†), namely  
证据。回想一下，作为命题33.19的一个应用，我们证明了公式（†），即

u∗ y (x ∧ x) = 2((u∗ y x) ∧ x)  
u y（x x）=2（（u y x）x）

for all x ∈ V2 E and all u∗ ∈ E∗. As a consequence, (u∗ y x) ∧ x = 0 iff u∗ y (x ∧ x) = 0. By Proposition 33.27, the 2-vector x is decomposable iff u∗ y (x ∧ x) = 0 for all u∗ ∈ E∗ iff x ∧ x = 0. Therefore, a 2-vector x is decomposable iff x ∧ x = 0.   
对于所有x v2 e和所有u e。因此，（u y x）x=0 iff u y（x x）=0。根据命题33.27，对于所有u∈e iff x x=0，2向量x是可分解的。因此，2向量x是可分解的iff x x=0。

As an application of Proposition 33.28, assume that dim(E) = 3 and that (e1,e2,e3) is a basis of E. Then any 2-vector x ∈ V2 E is of the form  
作为33.28号命题的一个应用，假设dim（e）=3，并且（e1，e2，e3）是e的基础，那么任何2向量x∈v2 e的形式

x = αe1 ∧ e2 + βe1 ∧ e3 + γe2 ∧ e3.  
x=αe1 e2+βe1 e3+γe2 e3。

We have x ∧ x = (αe1 ∧ e2 + βe1 ∧ e3 + γe2 ∧ e3) ∧ (αe1 ∧ e2 + βe1 ∧ e3 + γe2 ∧ e3) = 0,  
我们得到x x=（αe1 e2+βe1 e3+γe2 e3）（αe1 e2+βe1 e3+γe2 e3）=0，

because all the terms involved are of the form cei1 ∧ei2 ∧ei3 ∧ei4 with i1,i2,i3,i4 ∈ {1,2,3}, and so at least two of these indices are identical. Therefore, every 2-vector x = αe1 ∧ e2 + βe1 ∧ e3 + γe2 ∧ e3 is decomposable, although this not obvious at first glance. For example,  
因为所有涉及的术语都是CEI1 EI2 EI3 EI4的形式，其中I1、I2、I3、I4 1、2、3，因此至少有两个指数是相同的。因此，每2个向量x=αe1 e2+βe1 e3+γe2 e3都是可分解的，尽管乍一看这并不明显。例如，

e1 ∧ e2 + e1 ∧ e3 + e2 ∧ e3 = (e1 + e2) ∧ (e2 + e3).  
e1 e2+e1 e3+e2 e3=（e1+e2）（e2+e3）。

We now show that Proposition 33.27 yields an equational criterion for the decomposability of an alternating tensor z ∈ Vp E.  
我们现在证明，命题33.27给出了交替张量z∈vp e可分解性的一个等式准则。

## 33.9 The Grassmann-Plu¨cker’s Equations and Grassmannian Manifolds ~ 33.9格拉斯曼PLU–克尔方程和格拉斯曼流形~

We follow an argument adapted from Bourbaki [25] (Chapter III, §11, Section 13).  
我们遵循一个改编自Bourbaki[25]的论点（第三章，第11节，第13节）。

Let E be a vector space of dimensions n, let (e1,...,en) be a basis of E, and let (  
设e为n维的向量空间，设（e1，…，en）为e的基，设（

be its dual basis.decomposable. By Proposition 33.27, the vectorOur objective is to determine whether a nonzero vectorz is decomposable iff (u∗ yz) zz ∈= 0Vfor allp E is u∗ ∈ Vp−1 E∗. We can let u∗ range over a basis of Vp−1 E∗, and then the conditions are∧  
是它的双重基础。可分解。根据命题33.27，向量机的目标是确定非零向量是否可分解iff（u yz）zz∈=0vforallp e是u∈vp−1e。我们可以让u范围在vp−1 e的基础上，然后条件是



for all H ⊆ {1,...,n}, with |H| = p − 1. Since (e∗H y z) ∧ z ∈ Vp+1 E, this is equivalent to  
对于所有H 1，…，N，其中H=P−1。由于（e h y z）z∈vp+1e，这相当于



for all H,J ⊆ {0| =1,...,np, Formulae (2) and (4) of Proposition 33.18 show that}, with |H| = p − 1 and |J| = p + 1. Then, for all I,I0 ⊆ {1,...,n} with |I| = |I  
对于所有h，j 0=1，…，np，命题33.18的公式（2）和（4）表明，其中h=p−1和j=p+1。那么，对于所有的i，i0 1，…，n with i=i

he∗J,(e∗H y eI) ∧ eI0i = 0,  
他j，（e h y ei）ei0i=0，

unless there is some i ∈ {1,...,n} such that  
除非有一些i∈1，…，n这样

I − H = {i}, J − I0 = {i}.  
I−H I，J−I0 I。

In this case, I = H ∪ {i} and I0 = J − {i}, and using Formulae (2) and (4) of Proposition 33.18, we have  
在这种情况下，i=h i和i0=j−i，利用33.18号提案的公式（2）和（4），我们得出

.  
.

If we let  
如果我们让

,  
，

we have = +1 if the parity of the number of j ∈ J such that1 otherwise.j < i is the same as the parity of the number of h ∈ H such that h < i, and  
如果j∈j的个数的奇偶性为1，则为=+1，否则为1。j<i与h∈h的个数的奇偶性相同，因此h<i，并且

Finally we obtain the following criterion in terms of quadratic equations (Plu¨cker’s equations) for the decomposability of an alternating tensor.  
最后，我们用二次方程（plu–cker方程）得到了交变张量可分解性的以下准则。

Proposition 33.29.ditions for z 6= 0 to be decomposable are(Grassmann-Plu¨cker’s Equations) For z = PI λIeI ∈ Vp E, the con-  
命题33.29.z 6=0可分解的条件是（格拉斯曼-普卢克方程）对于z=piλiei∈vp e，the con-

,  
，

with, for all H,J ⊆ {1,...,n} such that |H| = p−1, |J| = p+1, and all i ∈ J − H.  
对于所有h，j 1，…，n使得h=p−1，j=p+1，并且所有i∈j−h。

33.9. THE GRASSMANN-PLUCKER’S EQUATIONS AND GRASSMANNIANS¨ ~  
33.9。格拉斯曼-普勒方程与格拉斯曼方程~

Using the above criterion, it is a good exercise to reprove that if dim(E) = n, then every tensor in Vn−1(E) is decomposable. We already proved this fact as a corollary of Proposition 33.23.  
利用上述标准，我们可以很好地证明，如果dim（e）=n，那么vn−1（e）中的每个张量都是可分解的。我们已经证明了这一事实，作为33.23号命题的推论。

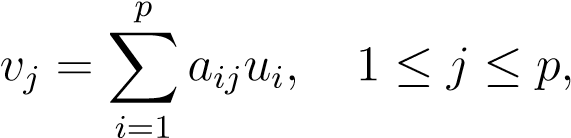
Given any z = PI λIeI ∈ Vp E where dim(E) = n, the family of scalars (λI) (with I = < ip} ⊆ {1,...,n} listed in increasing order) is called the Plu¨cker coordinates of .The Grassmann-Plu¨cker’s equations give necessary and sufficient conditions for any nonzero z to be decomposable.  
给定任意z=piλi e i∈vp e，其中dim（e）=n，标量族（λi）（i=<ip 1，…，n按递增顺序列出）称为的plu cker坐标。格拉斯曼plu¨cker方程给出了任何非零z可分解的必要和充分条件。

For example, when dim(E) = n = 4 and p = 2, these equations reduce to the single equation λ12λ34 − λ13λ24 + λ14λ23 = 0.  
例如，当dim（e）=n=4和p=2时，这些方程简化为单个方程λ12λ34−λ13λ24+λ14λ23=0。

However, it should be noted that the equations given by Proposition 33.29 are not independent in general.  
然而，应当注意的是，命题33.29给出的方程一般不独立。

We are now in the position to prove that the Grassmannian G(p,n) can be embedded in the projective space,  
我们现在可以证明格拉斯曼G（p，n）可以嵌入到射影空间中，

For any n ≥ 1 and any k with 1 ≤ p ≤ n, recall that the Grassmannian G(p,n) is the set of all linear p-dimensional subspaces of Rn (also called p-planes). Any p-dimensional subspace U of Rn is spanned by p linearly independent vectors u1,...,up in Rn; write U = span(u1,...,uk). By Proposition 33.8, (u1,...,up) are linearly independent iff u1 ∧···∧up =6 0. If (v1,...,vp) are any other linearly independent vectors spanning U, then we have  
对于任何n≥1和任何k（1≤p≤n），回想格拉斯曼尼亚g（p，n）是RN的所有线性p维子空间（也称为p平面）的集合。RN的任何p维子空间u都由p线性无关向量u1，…，向上在rn中表示；写u=span（u1，…，uk）。根据命题33.8，（u1，…，up）是线性独立的iff u1····up=6 0。如果（v1，…，vp）是跨越u的任何其他线性无关向量，那么我们有



for some aij ∈ R, and by Proposition 33.2  
对于某些aij∈r，并通过命题33.2

v1 ∧ ··· ∧ vp = det(A)u1 ∧ ··· ∧ up,  
v1···vp=Det（a）U1····向上，

where A = (aij). As a consequence, we can define a map such that for any k-plane U, for any basis (u1,...,up) of U,  
其中a=（aij）。因此，我们可以定义一个图，对于任何k平面u，对于u的任何基（u1，…，向上）。

iG(U) = [u1 ∧ ··· ∧ up],  
ig（u）=[u1·····向上]，

the point of given by the one-dimensional subspace of spanned by u1 ∧···∧up.  
由u1·····向上跨越的一维子空间给出的点。

Proposition 33.30. The map is injective.  
提案33.30。地图是注射剂。

Proof. Let U and V be any two p-planes and assume that iG(U) = iG(V ). This means that there is a basis (u1,...,up) of U and a basis (v1,...,vp) of V such that  
证据。设u和v为任意两个p平面，并假定ig（u）=ig（v）。这意味着有一个u的基（u1，…，up）和一个v的基（v1，…，vp），这样

v1 ∧ ··· ∧ vp = cu1 ∧ ··· ∧ up  
v1···vp=cu1·····向上

for some nonzero c ∈ R. The above implies that the smallest subspaces W and W 0 of Rn such that u1 ∧ ··· ∧ up ∈ Vp W and v1 ∧ ··· ∧ vp ∈ Vp W 0 are identical, so W = W 0. By  
对于一些非零c∈r，上面暗示了RN的最小子空间w和w 0，使得u1··························vp w 0和v1··········vp w 0是相同的，所以w=w 0。通过

Corollary 33.26, this smallest subspace W has both (u1,...,up) and (v1,...,vp) as bases, so the vj are linear combinations of the ui (and vice-versa), and U = V . Since any nonzero z ∈ Vp Rn can be uniquely written as  
推论33.26，这个最小的子空间w以（u1，…，up）和（v1，…，vp）为基，所以vj是ui的线性组合（反之亦然），u=v。因为任何非零z∈vp rn可以唯一地写为

z = XλIeI  
Z=xλiei

I  
我

in terms of its Plu¨cker coordinates (λI), every point of is defined by the Plu¨cker coordinates (λI) viewed as homogeneous coordinates. The points of corresponding to one-dimensional spaces associated with decomposable alternating p-tensors are the points whose coordinates satisfy the Grassmann-Plu¨cker’s equations of Proposition 33.29. Therefore, the map iG embeds the Grassmannian G(p,n) as an algebraic variety in defined by equations of degree 2.  
根据其PLU–KER坐标（λi），每个点都由被视为齐次坐标的PLU–KER坐标（λi）定义。对应于与可分解交替p张量相关的一维空间的点是坐标满足格拉斯曼-普卢克命题33.29方程的点。因此，map ig将grassmannian g（p，n）嵌入到由2次方程定义的代数变量中。

We can replace the field R by C in the above reasoning and we obtain an embedding of the complex Grassmannian GC(p,n) as an algebraic variety in defined by equations of degree 2.  
在上述推理中，我们可以用C来代替R域，得到了复数格拉斯曼GC（P，N）的嵌入，它是由2次方程定义的代数变量。

In particular, if n = 4 and p = 2, the equation  
特别是，如果n=4，p=2，则方程式

λ12λ34 − λ13λ24 + λ14λ23 = 0  
λ12λ34−λ13λ24+λ14λ23=0

is the homogeneous equation of a quadric in CP5 known as the Klein quadric. The points on this quadric are in one-to-one correspondence with the lines in CP3.  
是CP5中二次曲面的齐次方程，称为克莱恩二次曲面。二次曲面上的点与CP3中的直线一一对应。

There is also a simple algebraic criterion to decide whether the smallest subspaces U and V associated with two nonzero decomposable vectors u1 ∧ ··· ∧ up and v1 ∧ ··· ∧ vq have a nontrivial intersection.  
也有一个简单的代数准则来决定最小的子空间u和v是否与两个非零可分解向量u1··········································

Proposition 33.31. Let E be any n-dimensional vector space over a field K, and let U and V be the smallest subspaces of E associated with two nonzero decomposable vectors u = u1 ∧ ··· ∧ up ∈ Vp U and v = v1 ∧ ··· ∧ vq ∈ Vq V . The following properties hold:  
提案33.31。设e为k域上的任意n维向量空间，设u和v为e的最小子空间，与两个非零可分解向量u=u1································vq V相关。以下属性保留：

1. *We have U* ∩ *V* = (0) *iff u* ∧ *v* = 06 *.*
2. *If U* ∩ *V* = (0)*, then U* + *V is the least subspace associated with u* ∧ *v.*

*Proof.* Assume *U* ∩ *V* = (0). We know by Corollary 33.26 that (*u*1*,...,up*) is a basis of *U* and (*v*1*,...,vq*) is a basis of *V* . Since *U* ∩*V* = (0), (*u*1*,...,up,v*1*,...,vq*) is a basis of *U* +*V* , and by Proposition 33.8, we have

*u* ∧ *v* = *u*1 ∧ ··· ∧ *up* ∧ *v*1 ∧ ··· ∧ *vq* = 06 *.*

This also proves (2).