Option Pricing and Stochastic Calculus FRE6233, Spring 2017

Homework 5

Remark for all Exercises :

In all exercises, W_t is a standard Brownian motion under the probability measure $\mathbb{P}$ and $\mathcal{F}_t$ the filtration it generates, and $\widetilde{W}_t$ is a Brownian motion under the risk neutral measure $\widetilde{P}$.

Exercise : On different methods to price a down-and-out call

In this exercise S_t represents the price of a stock. It is assumed that S_t has a continuous trajectory.
We would like to price a down-and-out call; Given a maturity time T, a strike K and a barrier B, the option that pays $(S_T - K)_+$ only if $S_u \geq B$ for all $u \in (t, T)$.

Using no arbitrage arguments

(a) In this question we do **not** assume the dynamics of the stock price; in particular we do **not** know if it follows a Geometric Brownian Motion or any other type of Ito diffusion. We only know that the trajectories are continuous.
Assume that we start at $S_t = 100\$$, and that the barrier and strike are the same $B = K = 80\$$. Also assume that the interest rate $r = 0$ for simplicity.
Find a price V_t of the option by constructing a replicating portfolio. Your reasoning should only use simple no arbitrage arguments and the hedging strategy should be very simple.
Would this reasoning still work if $K \neq B$?

Let's return to the general case; K, B are general and might not be the same, the constant interest rate r is not necessarily 0, S_t starts at $x > B$ and follows a Geometric Brownian Motion

$$dS_t = rS_t dt + \sigma S_t d\widetilde{W}_t$$

The following questions will present the two main numerical approaches to price options.

In all the numerical experiments, we will take $t = 0$, $T = 1$ year, $r = 0.05$ (5% per year) and $\sigma = 0.20$ (20% per year), $x = 100\$$, $B = 80\$$ and $K = 110\$$.

Using Monte Carlo

We will start by using a Monte-Carlo approach to price the option:

(b) Explain that the value of the option should be

$$v_t = e^{-r(T-t)}\widetilde{\mathbb{E}}[(S_T - K)_+ \mathbb{1}_{S_u \geq B, \forall u \in (t,T)} | \mathcal{F}_t]$$

(c) Using the SDE, write a program that computes one trajectory of S_t for $t < T$; it should return a list of values $S_0, S_{\Delta t}, S_{2\Delta t} ..., S_T$ for $\Delta T = \frac{T}{N}$ where N is the number of points (you can choose $N = 100$ for example).

(d) Write a program that takes the list of values computed in the previous question and returns the payoff of the option $\phi(S) = (S_T - K)_+$ if all $S_{k\Delta t} > B$, and 0 otherwise.

(e) Using your previous code, write a program that generates 1000 trajectories of S, and computes the average

$$\frac{e^{-r(T-t)}}{1000} \sum_{j=1}^{1000} \phi(S^{(j)})$$

where $S^{(j)}$ is the j^{th} trajectory. Deduce the price of the option.

Using the PDE

Now we will do the same using a PDE approach. As it was hastily explained during the class, I will make you go through the derivation again step by step.

(f) Define the stopping time

$$\tau = \min\{\inf\{u \geq t | S_u = B\}, T\}$$

which is the first time you hit the barrier if it is less than T, and T otherwise. Also define the function ϕ to be

$$\phi(y, s) = \begin{cases} (y - K)_+ \text{ if } s = T \\ 0 \text{ if } s < T \end{cases}$$

Explain (without proof) that the price of the option is given by

$$\widetilde{\mathbb{E}}[e^{-r(\tau - t)} \phi(S_\tau, \tau) | S_t = x]$$

Let's prove that this price solves a PDE in the following questions;

(g) Assume that we know that there is a function v that solves the following PDE:

$$-rv(t, x) + v_t(t, x) + rxv_x(t, x) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t, x) = 0$$

for $t < T$ and $B < x$, with the final time condition $v(T, x) = (x - K)_+$ for all $x > B$ and the boundary condition $v(t, B) = 0$ for all $t < T$.
Apply Ito's lemma to $d(e^{-rt} v(t, S_t))$ to deduce that

$$d(e^{-rt} v(t, S_t)) = e^{-rt} \sigma S_t v_x(t, S_t) d\widetilde{W}_t$$

(h) Integrate the SDE above between time t and τ to deduce that

$$e^{-r\tau} v(\tau, S_\tau) - e^{-rt} v(t, S_t) = \int_t^\tau \sigma e^{-rs} S_s v_x(s, S_s) d\widetilde{W}_s$$

(i) By using the following formula (called Dynkin's theorem, or just an application of Doob's optional stopping time theorem):

$$\widetilde{\mathbb{E}}\left[\int_t^\tau \sigma e^{-rs} S_s v_x(s, S_s) d\widetilde{W}_s \middle| S_t = x\right] = 0$$

prove that we necessarily have

$$v(t, x) = \widetilde{\mathbb{E}}[e^{-r(\tau - t)} \phi(S_\tau, \tau) | S_t = x]$$

This reasoning show that if a function v solves the PDE, it is necessarily equal to $\widetilde{\mathbb{E}}[e^{-r(\tau-t)}\phi(S_\tau,\tau)|S_t=x]$. PDE theory tells us that the solution of this specific PDE exists and is unique, so we also have the converse statement so $\widetilde{\mathbb{E}}[e^{-r(\tau-t)}\phi(S_\tau,\tau)|S_t=x]$ solves the PDE given above. This concludes the proof (that is very general) that even in the case where we have a stopping time, we can still write a PDE version for the price of the option;

$$\begin{aligned}
&-rv(t,x)+v_t(t,x)+rxv_x(t,x)+\frac{1}{2}\sigma^2x^2v_{xx}(t,x)=0, \quad \text{for } t<T \text{ and } B<x\\
&v(T,x)=(x-K)_+, \quad \text{for } B<x\\
&v(t,B)=0, \quad \text{for } t<T
\end{aligned}$$

There is technically another growth condition that is

$$\lim_{x\to+\infty} v(t,x)-(x-e^{-r(T-t)}K)=0$$

(j) Explain that growth condition (you can get inspired by what we did in lecture 3 for the boundary conditions for the Black-Scholes PDE)

Now we will turn to option pricing using the PDE version. We can solve this PDE analytically as shown in class, but we won't purse this route.

Numerical solution of the PDE

We will sove the PDE numerically instead.
We will solve only for $x\in(B,R)$ for some big constant $R=300$. Let's choose $N_x=1000$ to be the number of points of x and $N_t=100$ to be the number of time points.
Define $\Delta x=\frac{R-B}{N_x}$, $\Delta t=\frac{T-t}{N_t}$, $x_k=B+j\Delta x$ for $j=0,..,N_x$ and $t_j=t+j\Delta t$ for $j=0,..,N_t$.
In the following, we will define $u(t_j,x_k)=u_j^k$ for any function u.

(k) Check that the final time and boundary conditions of the PDE can be numerically written as

$$\begin{aligned}
&v_{N_t}^k=(x_k-K)_+, \quad \text{for } k=0,..,N_x\\
&v_j^0=0, \quad \text{for } j=0,..,N_t\\
&v_j^{N_x}=R-e^{-r(T-t_j)}K, \quad \text{for } j=0,..,N_t
\end{aligned}$$

Notice that there is no discontinuity at $t=T, x=B$ or $t=T, x=R$ (all the values specified are consistent).

(l) Show that after discretization of the PDE we get

$$v_{j-1}^k=\left(1-r\Delta t-\sigma^2\frac{\Delta t}{(\Delta x)^2}x_k^2\right)v_j^k+\Delta t\left(\frac{rx_k}{2\Delta x}+\frac{1}{2(\Delta x)^2}\sigma^2x_k^2\right)v_j^{k+1}+\Delta t\left(-\frac{rx_k}{2\Delta x}+\frac{1}{2(\Delta x)^2}\sigma^2x_k^2\right)v_j^{k-1}$$

for all $k=1,..,N_x-1$ and $j=0,..,N_t-1$.
(Hint : Use a symmetric difference approximation for the first order derivative in x; $v_x\approx\frac{v_j^{k+1}-v_j^{k-1}}{2\Delta x}$)
Define $m_{k,k}=\left(1-r\Delta t-\sigma^2\frac{\Delta t}{(\Delta x)^2}x_k^2\right)$, $m_{k,k+1}=\Delta t\left(\frac{rx_k}{2\Delta x}+\frac{1}{2(\Delta x)^2}\sigma^2x_k^2\right)$ and $m_{k,k-1}=\Delta t\left(-\frac{rx_k}{2\Delta x}+\frac{1}{2(\Delta x)^2}\sigma^2x_k^2\right)$ so that the equation above becomes

$$v_{j-1}^k=m_{k,k}v_j^k+m_{k,k+1}v_j^{k+1}+m_{k,k-1}v_j^{k-1}$$

Note that the coefficients m do not depend on j here because there was no explicit time dependence in the coefficients of the SDE.

(m) Define the $(N_x+1)\times 1$ vectors

$$V_j = \begin{pmatrix} v_j^0 \\ \vdots \\ v_j^{N_x} \end{pmatrix}, \quad C_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ R - e^{-r(T-t_j)}K \end{pmatrix}$$

and the $(N_x-1)\times(N_x-1)$ matrix $\tilde{M}$

$$\tilde{M} = \begin{pmatrix} m_{1,1} & m_{1,2} & & & \\ m_{2,1} & m_{2,2} & m_{2,3} & & \\ & \ddots & \ddots & \ddots & \\ & & m_{k,k-1} & m_{k,k} & m_{k,k+1} \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

So all entries of $\tilde{M}$ are 0's except $\tilde{M}_{k,k-1} = m_{k,k-1}$ for $k = 2, .., N_x - 1$,$\tilde{M}_{k,k} = m_{k,k}$ for $k = 1, .., N_x - 1$ and $\tilde{M}_{k,k+1} = m_{k,k+1}$ for for $k = 1, .., N_x - 2$.
Define the $(N_x+1)\times(N_x+1)$ matrix M to be

$$M = \begin{pmatrix} 0 & \cdots & & \cdots & 0 \\ m_{1,0} & & & & \\ & & \tilde{M} & & \\ & & & & m_{N_x-1,N_x} \\ 0 & \cdots & & \cdots & 0 \end{pmatrix}$$

so all entries of M are zeros except the entry $M_{2,1} = m_{1,0}$, the entry $M_{N_x,N_x+1} = m_{N_x-1,N_x}$ and all entries of M for lines 2 to N_x and columns 2 to N_x which are replaced by the entries of $\tilde{M}$.

Show that the discretized PDE and boundary conditions can be rewritten in matrix form as

$$V_{j-1} = C_{j-1} + MV_j$$

(n) Given the final time vector V_{N_t} (which is the final payoff, with entries $(x_k - K)_+$), write a program that computes V_0 and deduce the price of the option we initially considered. In particular, explain that only the entries $k = 90$ or $k = 91$ of the vector V_0 is of interest if the initial price is 100$.

Analytical solution of the PDE

(o) If we solved the PDE analytically instead of numerically, we would have obtained the formula:

$$v(t,x) = c_K(t,x) - \left(\frac{x}{B}\right)^{2\alpha} c_K\left(\frac{B^2}{x}, t\right)$$

where $c_K(t,x)$ is the Black-Scholes value of a European call of strike K and maturity T if $S_t = x$ and if the interest rate is r, and $\alpha = \frac{1}{2}(1 - \frac{2r}{\sigma^2})$. Check that this formula solves the PDE and the boundary/final time conditions given above question (j) .
(Hint : Use the Black-Scholes PDE and boundary/final time conditions we saw in lecture 3)

(p) Compare the price obtained numerically by Monte Carlo and the PDE method, as well as with theoretical price from the PDE.