Option Pricing and Stochastic Calculus FRE6233, Spring 2017

Homework 6

Remark for all Exercises :

In all exercises, W_t is a standard Brownian motion under the probability measure $\mathbb{P}$ and $\mathcal{F}_t$ the filtration it generates, and $\widetilde{W}_t$ is a Brownian motion under the risk neutral measure $\widetilde{\mathbb{P}}$.

Exercise 1: American options in discrete time

Consider a stock S in discrete time; $S_0 = 100$ and between time $t = n\delta t$ and $t + \delta t = (n+1)\delta t$ the stock can either go up by a factor $u = 2$ or down by a factor $d = 1/2$, the risk free rate r is chosen such that $e^{\delta t r} = 3/2$.

(a) Show that the risk neutral probabilities q for the up and down states are $q_{up} = 2/3$ and $q_{down} = 1/3$.

(b) Write a code to price an American put with strike $K = 100$ and maturity $T = N\delta t$, by using backward pricing; $V_n^i = \max\left((K - S_n^i)_+, e^{-r\delta t}(q_{up}V_{n+1}^{i+1} + q_{down}V_{n+1}^i)\right)$ for the i^{th} node at time n. Give the value of the option if $N = 10$.

Exercise 2: American Call with no dividends

Let S_t be a Geometric Brownian Motion with risk neutral dynamics

$$dS_t = rS_t dt + \sigma S_t d\widetilde{W}_t$$

(a) Let $K > 0$. Show that $e^{-rt}(S_t - K)$ is a submartingale, i.e.

$$\widetilde{\mathbb{E}}\left[e^{-rt}(S_t - K)|\mathcal{F}_s\right] \geq e^{-rs}(S_s - K)$$

for $s \leq t$.

(b) Show that if g is a convex and increasing function, $g(S_t)$ is still a submartingale.

(c) Using Doob's optional stopping theorem, show that we have

$$\widetilde{\mathbb{E}}\left[g(S_\tau)\right] \leq \widetilde{\mathbb{E}}\left[g(S_T)\right]$$

for any stopping time τ.

(d) Conclude that the price of an American call with an underlying that pays no dividends is given by $\widetilde{\mathbb{E}}\left[e^{-rT}(S_T - K)_+\right]$, i.e. the price of a European call. Why doesn't this work when adding dividends ?

Exercise 3: American call with dividends

Let S_t be a Geometric Brownian Motion with risk neutral dynamics

$$dS_t = (r - d)S_t dt + \sigma S_t d\widetilde{W}_t$$

where d is the dividends rate, r the interest rate and σ the volatility, and $S_t = x$.

(a) What is the value of an American call on the underlying S defined above ?

(b) Write the variational inequalities that would allow you to solve the prove. I am not asking you to solve it.

(c) Assume that you were told that the optimal stopping time is of the type $\tau_L = \inf\{s \geq t | S_s = L\}$ for some L. What would the variational inequalities become ? Write the PDE to solve.

(d) What condition allows you to find L ?