APPENDIX B: Relationship between $\widehat{R}_b$, $\widehat{R}_I$, and I^2

i. $\hat{R}_b \leq \hat{R}_I \;\; \forall \hat{\tau}^2$

ii. $I^2 \leq \hat{R}_I \;\; \forall \hat{\tau}^2$

i. From the definition of the two measures, the first inequality can be written as

$$\frac{\hat{\tau}^2}{K\, Var\left(\overline{\beta}_{re}\right)} \leq \frac{\hat{\tau}^2}{\hat{\tau}^2 + K\, Var\left(\overline{\beta}_{fe}\right)} \quad \forall \hat{\tau}^2$$

We need to prove that

$$K\, Var\left(\overline{\beta}_{re}\right) \geq \hat{\tau}^2 + K\, Var\left(\overline{\beta}_{fe}\right) \quad \forall \tau^2$$

Or, as a function of $\hat{\tau}^2$, v_i, and K

$$f(\hat{\tau}^2, v_i, K) = Var\left(\overline{\beta}_{re}\right) - Var\left(\overline{\beta}_{fe}\right) - \hat{\tau}^2/K \geq 0$$

Substituting the variances with their expressions

$$f(\hat{\tau}^2, v_i, K) = \frac{1}{\sum_{i=1}^{K} 1/(v_i + \hat{\tau}^2)} - \frac{1}{\sum_{i=1}^{K} 1/v_i} - \hat{\tau}^2/K$$

$$\frac{\partial f(\hat{\tau}^2, v_i, K)}{\partial \hat{\tau}^2} = \frac{\sum_{i=1}^{K} \frac{1}{(v_i + \hat{\tau}^2)^2}}{\left(\sum_{i=1}^{K} \frac{1}{(v_i + \hat{\tau}^2)}\right)^2} - \frac{1}{K}$$

We note that $f(\hat{\tau}^2 = 0, v_i, K) = 0$ and that $\frac{\partial f(\hat{\tau}^2, v_i, K)}{\partial \hat{\tau}^2} \geq 0 \;\; \forall\, \hat{\tau}^2$ using the Cauchy-Schwartz inequality $K \sum_{i=1}^{K} w_i^{*2} \geq \left(\sum_{i=1}^{K} w_i^*\right)^2$.

ii. From the definition of the two measures, the latter inequality can be written as

$$\frac{(K-1)\sum_{i=1}^{K} w_i}{(\sum_{i=1}^{K} w_i)^2 - \frac{\sum_{i=1}^{K} w_i{}^2}{\sum_{i=1}^{K} w_i}} \geq \frac{K}{\sum_{i=1}^{K} w_i}$$

$$\frac{(K-1)}{1 - \frac{\sum_{i=1}^{K} w_i{}^2}{(\sum_{i=1}^{K} w_i)^2}} \geq K$$

$$K\frac{\sum_{i=1}^{K} w_i{}^2}{(\sum_{i=1}^{K} w_i)^2} \geq 1$$

$$K\sum_{i=1}^{K} w_i{}^2 \geq \left(\sum_{i=1}^{K} w_i\right)^2$$

The last inequality is always true as a partial case of Cauchy-Shwartz inequality.

Another way to explore the relationship between I^2 and R_I, rewrite I^2 and R_I as

$$I^2 = \frac{Q-K+1}{Q} \qquad R_I = \frac{Q-K+1}{Q-\ CV_{w_i}^2}$$

where $CV_{w_i}^2 = \mathrm{Var}[w_i]/(E[w_i]^2) = K\sum_{i=1}^{K} w_i^2/(\sum_{i=1}^{K} w_i)^2 - 1$

is the coefficient of variation for the fixed effects weights, w_i. The two quantities differ only in the subtraction of $CV_{w_i}^2$ from the denominator of R_I. Using the Cauchy-Schwartz inequality, $K\sum_{i=1}^{K} w_i{}^2 \geq\ (\sum_{i=1}^{K} w_i)^2$. Therefore, $R_I \geq I^2$, with equality when $CV_{w_i}^2 = 0$, i.e. when all w_i $i = 1,\ldots,K$ are equal.

iii.

$\hat{R}_b \leq I^2$ if

$$\frac{K}{\sum_{i=1}^{K}\frac{1}{\hat{\tau}^2 + v_i}} \geq \hat{\tau}^2 + \frac{(K-1)\sum_{i=1}^{K} w_i}{(\sum_{i=1}^{K} w_i)^2 - \sum_{i=1}^{K} w_i^2},$$

otherwise, $\hat{R}_b > I^2$. The inequality above is not correct when $s_1^2 \ll s_2^2$, where s_1^2 and s_2^2 are the estimators of the within-study variance given in Section 2.2. In our experience, the relationship $\hat{R}_b \leq I^2$ is usually observed in meta-analyses, but some configurations of $\hat{\tau}^2$ and v_i can produce the opposite inequality, as in the following example: assume $\hat{\tau}^2 = 0.03$ and $K = 10$ studies with the observed within-study variance vector v = (2.58, 2.00, 1.92, 1.87, 1.87, 1.83, 1.54, 0.70, 0.34, 0.03). In this case, $s_1^2 = 0.24$ and $s_2^2 = 0.63$, hence $\hat{R}_b$= 7.3% is greater than I^2 = 4.5%.