APPENDIX C: Derivation of the consistency, asymptotic normality and $\boldsymbol{Var(\hat{R}_b)}$

As a method of moments estimator, $\hat{\tau}^2 \xrightarrow[K\to\infty]{P} \tau^2$ [7]. By Slutsky's theorem, for each study, $\frac{\hat{\tau}^2}{\hat{\tau}^2+v_i} \xrightarrow[K\to\infty]{P} \frac{\tau^2}{\tau^2+v_i}$ and then by the weak law of large numbers, $\hat{R}_b = \frac{1}{K}\sum_{i=1}^{K}\frac{\hat{\tau}^2}{\hat{\tau}^2+v_i} \xrightarrow[K\to\infty]{P} E\left(\frac{\tau^2}{\tau^2+v}\right)$, where the study-specific variances, $v_1, v_2, \dots, v_K$, , are realizations of the random variable, v.

Also, by the Central Limit Theorem, as $K \to \infty$, $\hat{R}_b$ has a normal distribution with mean value equal to its expectation, i.e. $E\left(\frac{\tau^2}{\tau^2+v}\right)$.

The asymptotic $Var(\hat{R}_b)$ was derived using the delta method using the following relation between $\hat{R}_b$ and Q

$$\hat{R}_b = \frac{1}{K}\sum_{i=1}^{K}\frac{Q-(K-1)}{Q+\ a_i-(K-1)}$$

where $a_i = \ v_i\left(\sum_{i=1}^{K} w_i^* - \sum_{i=1}^{K} {w_i^*}^2 / \sum_{i=1}^{K} w_i^*\right)$.

$$Var(\hat{R}_b) \approx \left(\frac{1}{K}\sum_{i=1}^{K}\frac{a_i}{(Q+\ a_i-(K-1))^2}\right)^2 Var(Q)$$

The formula for $Var(Q)$ was provided by Biggerstaff and Tweedie [12] as

$$Var(Q) = 2(k-1) +\ 4\left(S_1 - \frac{S_2}{S_1}\right)\tau^2 + 2\left(S_2 - \frac{2S_3}{S_1} + \frac{S_2^2}{S_1^2}\right)\tau^4$$

with $S_r = \sum_{i=1}^{K} w_i^r$.