APPENDIX A: Proof of proposed criteria for a measure of heterogeneity

$\hat{R}_b$ is a function of $\hat{\tau}^2$, the between-studies variance, K , the number of the studies in the meta-analysis, and v_i, the observed within-study variances of the effect estimates.

i. Dependence on the extent of heterogeneity, i.e. $\hat{R}_b$ is an increasing function of τ^2.

$$\frac{\partial\, \hat{R}_b(\hat{\tau}^2, K, v_i)}{\partial\, \hat{\tau}^2} = \frac{1}{K}\sum\nolimits_{i=1}^{K} \frac{v_i}{(v_i + \hat{\tau}^2)^2} > 0 \quad \forall\, \hat{\tau}^2$$

ii. Scale invariance: $\hat{R}_b$ is invariant to any linear transformation of the parameter space from $\mathbb{R}$ to $a + b\mathbb{R}$.

Let us consider $\hat{\beta}_i' = a + b\hat{\beta}_i$ so that $v_i{}' = b^2 v_i$ and $\hat{\tau}^{2\prime} = b^2\hat{\tau}^2$.

$$\hat{R}_b' = \frac{1}{K}\sum\nolimits_{i=1}^{K} \frac{\hat{\tau}^{2\prime}}{v_i{}' + \hat{\tau}^{2\prime}} = \frac{1}{K}\sum\nolimits_{i=1}^{K} \frac{b^2\hat{\tau}^2}{b^2(v_i + \hat{\tau}^2)} = \hat{R}_b$$

iii. Size invariance. As K increases, new v_is need to be considered. We prove the property in the case where $v_i = \sigma^2, \forall i = 1, \dots, K$.

$$\hat{R}_b = \frac{1}{K}\sum\nolimits_{i=1}^{K} \frac{\hat{\tau}^2}{\sigma^2 + \hat{\tau}^2} = \frac{\hat{\tau}^2}{\sigma^2 + \hat{\tau}^2}$$

which does not depend on the number of studies. See Appendix C, where the consistency of $\hat{R}_b$ is established, thereby proving asymptotic independence of the estimator on K.