A Simplex-Based Algorithm for 0-1 Mixed Integer Programming*

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Abstract. We present a finitely convergent cutting plane algorithm for 0-1 mixed integer programming. The algorithm is a hybrid between a strong cutting plane and a Gomory-type algorithm that generates violated facet-defining inequalities of a relaxation of the simplex tableau and uses them as cuts for the original problem. We show that the cuts can be computed in polynomial time and can be embedded in a finitely convergent algorithm.

1 Introduction

Gomory [7] in the 1950's pioneered the idea of using cutting planes to solve integer programs. In his approach, valid inequalities are generated from rows of the currently fractional simplex tableau. An advantage of this method is that a violated inequality can always be found quickly and the resulting algorithm can be proven to converge in a finite number of iterations. Although appealing conceptually, this algorithm is not effective in practice. Branch-and-cut schemes that use strong cutting planes have proven to be much more effective in solving 0-1 mixed integer programs. They proceed by relaxing the constraint set into a polytope whose structure has been studied and for which families of facet-defining inequalities (facets for short) are known. A separation procedure, usually heuristic, is called to generate a violated facet of the relaxed polytope, which is a cut for the initial problem. This idea was used by Crowder, Johnson and Padberg [5] with the theoretical foundation coming from the polyhedral studies of the 0-1 knapsack polytope by Balas [1], Balas and Zemel [3], Hammer, Johnson and Peled [11] and Wolsey [20]. Subsequently there have been many applications of this approach that imbeds strong cuts into a branch-and-bound algorithm, see surveys by Johnson, Nemhauser and Savelsvergh [12] and Marchand, Martin, Weismantel and Wolsey [13]. The main disadvantage of this approach is that the separation procedure may not return any cut, at which point partial enumeration is required. In this paper we introduce an algorithm that is a hybrid between these two approaches. We generate cuts from the simplex tableau and therefore are able to produce a violated inequality at every step of the algorithm.

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Moreover, our cuts are always facets for relaxations of the polytopes defined by the rows of the tableau and yield a finite simplex algorithm for 0-1 mixed integer programming. The cuts are generally less dense than Gomory cuts and can be computed in polynomial time, although Gomory cuts are less expensive to obtain. Our cuts are derived in Richard, de Farias and Nemhauser [18,19] and in order to understand their validity and facetial properties, it is necessary to refer to these papers.

We now introduce our basic relaxation. Let $M = \{1, \ldots, m\}$ and $N = \{1, \ldots, n\}$. Given the sets of positive integers $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_n\}$ together with the positive integer d, let

$$S = \{(x, y) \in \{0,1\}^m \times [0,1]^n \mid \sum_{j \in M} a_j x_j + \sum_{j \in N} b_j y_j \leq d\}.$$

The *mixed 0-1 knapsack polytope* is $PS = conv(S)$. Note that, since m and n are positive integers, the knaspack inequality contains both continuous and integer variables. As long as the continuous variables are bounded, it is not restrictive to choose the bounds to be 0 and 1. Also, it is not restrictive to require the coefficients a_j and b_j to be positive since we can always complement variables. Finally we may assume that the coefficients a_j and b_j are smaller than d, otherwise x_j can be fixed to 0 and y_j can be rescaled. Given these assumptions, we have

Theorem 1. *PS is full-dimensional.* □

Although we are not aware of any previous study of the mixed 0-1 knapsack polytope other than our current work in [18,19], valid and facet-defining inequalities for related polytopes have been known for quite some time. For example there are the mixed integer cuts introduced by Gomory [8], the MIR inequalities introduced by Nemhauser and Wolsey [16] and the mixed disjunctive cuts introduced by Balas, Ceria and Cornuéjols [2]. More closely related to our study is the "0-1 knapsack polytope with a single continuous variable" introduced by Marchand and Wolsey [14]. There are significant differences between the polyhedron of Marchand and Wolsey and PS, which are described in [18].

In Section 2 we propose a conceptual framework to generate cuts for 0-1 mixed integer problems, using some knowledge of PS. This technique requires an extensive use of lifting, including the lifting of continuous variables as studied in [18,19]. In Section 3 we present a family of facets for the mixed 0-1 knapsack polytope that can be used as tableau cuts for general 0-1 mixed integer problems and can be obtained in polynomial time. In Section 4 we sketch how these cuts are used in a finitely convergent algorithm.

2 Generating Cuts from the Simplex Tableau

In this section, we present a formal framework to obtain cuts from a relaxation of the simplex tableau. This procedure requires extensive use of lifting techniques and motivates the introduction of the following notation.

Let M_0, M_1 be two disjoint subsets of M, and N_0, N_1 be two disjoint subsets of N. Define

$$S(M_0, M_1, N_0, N_1) = \{ (x,y) \in S \mid x_j = 0\,\forall j \in M_0, x_j = 1\,\forall j \in M_1, \\ y_j = 0\,\forall j \in N_0,\, y_j = 1\,\forall j \in N_1\}$$

and $PS(M_0, M_1, N_0, N_1) = conv(S(M_0, M_1, N_0, N_1))$.

Consider the general 0-1 mixed-integer program

$$\max \sum_{j \in M} c_j x_j + \sum_{j \in N} c_{m+j} y_j \tag{1}$$

$$s.t. \sum_{j \in M} a_{ij} x_j + \sum_{j \in N} b_{ij} y_j \leq d_i \qquad \forall i \in H \tag{2}$$

$$x_j \in \{0,1\} \qquad \forall j \in M \tag{3}$$

$$y_j \in [0,1] \qquad \forall j \in N \tag{4}$$

where $H = \{1, \ldots, h\}$. Note that each row of (2) together with (3) and (4) has the form of S. We let Q be the set of solutions to (2)-(4) and $PQ = conv(Q)$.

Because we work with simplex tableaux, we introduce slacks and assume they are continuous. Since lower and upper bounds on the slack variables are easily obtained, they can be rescaled so that their domain is the interval $[0,1]$ or substituted out if the bounds are equal. We can therefore replace (2) by

$$\sum_{j \in M} a_{ij} x_j + \sum_{j \in N} b_{ij} y_j + u_i y_{n+i} = d_i \qquad \forall i \in H \tag{5}$$

and (4) by

$$y_j \in [0,1] \qquad \forall j \in \tilde{N} \tag{6}$$

where $\tilde{N} = N \cup \{n+1, \ldots, n+h\}$.

Now consider a solution to the LP relaxation of this problem. If none of the 0-1 variables is fractional, then the current solution is optimal. So we may assume that at least one of them is fractional. Note that nonbasic variables are either at their lower or upper bound, so that no integrality violations can occur for them. Therefore any 0-1 variable that is fractional in the current LP relaxation has to be basic. Assume the i^{th} variable of the basis is 0-1 and fractional. The i^{th} row of the simplex tableau can be written as

$$x_{B(i)} + \sum_{j \in M_0 \cup M_1} \tilde{a}_{ij} x_j + \sum_{j \in N_0 \cup N_1} \tilde{b}_{ij} y_j = f_i + \sum_{j \in M_1} \tilde{a}_{ij} + \sum_{j \in N_1} \tilde{b}_{ij} = \tilde{d}_i \tag{7}$$

where $f_i \in (0,1)$, M_0 and M_1 represent the sets of 0-1 variables that are nonbasic at lower and upper bounds respectively in the current tableau, N_0 and N_1 represent the sets of continuous variables that are nonbasic at lower and upper bounds in the current tableau, and $B(i)$ represents the index of the i^{th} basic variable. We would like to generate a strong cut from (7).

In (7) we assume that the coefficients $\tilde{a}_{ij}$ and $\tilde{b}_{ij}$ are positive. If not, we complement the corresponding variables (they will therefore switch from M_0 to M_1, N_0 to N_1 and vice-versa). Moreover, if we relax (7) to an inequality, we obtain the knapsack constraint *in standard form*

$$x_{B(i)} + \sum_{j \in M_0 \cup M_1} \tilde{a}_{ij} x_j + \sum_{j \in N_0 \cup N_1} \tilde{b}_{ij} y_j \leq \tilde{d}_i \tag{8}$$

which defines a mixed 0-1 knapsack polytope that we call PS_i. We can also relax the equality in the other direction and still obtain a mixed 0-1 knapsack polytope in standard form by complementing all the variables.

Note that we can fix to 0 all the 0-1 variables, including $x_{B(i)}$ whose coefficients in PS_i are bigger than $\tilde{d}_i$. By doing so, we detect that PQ is not full-dimensional and obtain some members of its equality set. Among the inequalities that can possibly be generated in this way, the only one that cuts off the current solution of the LP relaxation is $x_{B(i)} \leq 0$. So we will assume throughout this paper that variables for which the previous discussion applies have been substituted out of (8), that $x_{B(i)}$ is still present in (8) and that PS_i is full-dimensional. In $PS_i(M_0, M_1, N_0, N_1)$, the relaxed tableau row in standard form reads

$$x_{B(i)} \leq f_i \tag{9}$$

with $0 < f_i < 1$. Since $x_{B(i)}$ is an integer variable, the inequality

$$x_{B(i)} \leq 0 \tag{10}$$

is valid for $PS_i(M_0, M_1, N_0, N_1)$ and is clearly violated by the current solution. Note that (10) can be extended to a facet of PS_i by sequentially lifting all the nonbasic variables. Although all lifting orders provide such an inequality, some orders may lead to complex lifting schemes that will usually be expensive to carry out. So we will settle for simple schemes. Note that the mixed integer solutions of Q satisfy the constraints defining PS_i so that the inequalities we obtain are valid for PQ. Also, since all the nonbasic variables are fixed at their upper or lower bounds, the inequality obtained at the end of the lifting process will be violated and its absolute violation will be f_i. In order to implement this scheme, we need to lift both continuous and 0-1 variables. The lifting of 0-1 variables from a knapsack inequality is well-known, see for example Gu, Nemhauser and Savelsbergh [10]. For the lifting of continuous variables, we will use the results developed in [18,19]. We only need to consider the lifting of continuous variables fixed at 0 (lifting from 0) and fixed at 1 (lifting from 1) since all the continuous variables are nonbasic.

3 A Family of Tableau Cuts

In this section we show that we can choose the lifting order in a way that makes the whole lifting sequence polynomial. Consider PS_i, the mixed 0-1 knapsack polytope in standard form associated with a row of the tableau whose basic variable $x_{B(i)}$ is fractional. First define $M_0^< = \{l \in M_0 \,|\tilde{a}_{il} \leq f_i + \sum_{j\in M_1} \tilde{a}_{ij}\}$ and $M_0^> = \{l \in M_0 \,|f_i + \sum_{j\in M_1} \tilde{a}_{ij} < \tilde{a}_{il} \leq \tilde{d}_i\}$. Note that M_0 is completely covered by $M_0^<$ and $M_0^>$ since PS_i is full-dimensional. We will lift the variables in the order M_1, $M_0^<$, N_1, N_0 and $M_0^>$. We discuss the lifting problems next and illustrate them on the following example.

Example 1. Consider

$$\begin{aligned} \max \quad & x_1 + 2x_2 + 3x_3 + 4x_4 + 10y_1 \\ \text{s.t.} \quad & 10x_1 + 7x_2 + 6x_3 + 2x_4 + 2y_1 \leq 17 \\ & 6x_1 + 12x_2 + 13x_3 + 7x_4 + 13y_1 \leq 32. \end{aligned}$$

We introduce the nonnegative continuous slacks $\tilde{y}_2$ and $\tilde{y}_3$. They are bounded above by 17 and 32 respectively and below by 0. We use these bounds to rescale $\tilde{y}_2$ and $\tilde{y}_3$ so that their domains correspond to the interval $[0, 1]$. We call the scaled variables y_2 and y_3. We solve the linear relaxation of this problem using the simplex method. The optimal tableau is given by

$$\begin{aligned} \frac{96}{13} + \max \; & -\frac{5}{13}x_1 - \frac{10}{13}x_2 + \frac{31}{13}x_4 + \frac{91}{13}y_1 - \frac{96}{13}y_3 \\ \text{s.t.} \; & \frac{6}{13}x_1 + \frac{12}{13}x_2 + x_3 + \frac{7}{13}x_4 + y_1 + \frac{32}{13}y_3 = \frac{32}{13} \\ & \frac{94}{221}x_1 + \frac{19}{221}x_2 - \frac{16}{221}x_4 - \frac{52}{221}y_1 + y_2 - \frac{192}{221}y_3 = \frac{29}{221} \end{aligned}$$

where the variables x_3 and y_2 are basic, the variables x_1, x_2 and y_3 are nonbasic at lower bound, and the variables x_4 and y_1 are nonbasic at upper bound. The current objective value is $\frac{218}{13}$. For ease of exposition, we present the tableau in fractional form, although our analysis does not require it.

We derive a cut from the first row of the tableau whose basic variable $x_3 = \frac{12}{13}$. All the coefficients of this row are nonnegative, so this row is already in standard form. The polytope from which we generate our cuts is defined by the inequality

$$\frac{6}{13}x_1 + \frac{12}{13}x_2 + x_3 + \frac{7}{13}x_4 + y_1 + \frac{32}{13}y_3 \leq \frac{12}{13} + \frac{7}{13} + 1.$$

Moreover, we have that $M_0^< = \{1, 2\}$, $M_0^> = \emptyset$, $M_1 = \{4\}$, $N_0 = \{3\}$ and $N_1 = \{1\}$. □

3.1 Lifting the Members of M_1

First we lift the variables of M_1 in (10). The following theorem is a corollary of well-known results on lifting 0-1 variables.

Theorem 2. *Assume* $M_1 = \{1, \dots, s\}$ *and let* $\mu_0 = 1 - f_i$*. For* $j = 1, \dots, s$*, let* $\alpha_j = 0$ *and* $\mu_j = \mu_{j-1} - \tilde{a}_{ij}$ *when* $\tilde{a}_{ij} < \mu_{j-1}$ *and let* $\alpha_j = 1$ *and* $\mu_j = \mu_{j-1}$ *otherwise, then*

$$x_{B(i)} + \sum_{j=1}^{s} \alpha_j x_j \leq \sum_{j=1}^{s} \alpha_j \tag{11}$$

is a facet of $PS_i(M_0, \emptyset, N_0, N_1)$. □

Inequality (11) is obtained by sequentially lifting the variables of M_1 in (10) and is a cover inequality since $\alpha_j \in \{0, 1\}$. Moreover Theorem 2 yields a polynomial algorithm $O(n)$ to obtain (11). We will assume from now on that at least one variable of M_1 is lifted at value 1. The case in which no variable from M_1 yields $\alpha = 1$ is discussed in Section 3.6.

Example 1 (continued). We lift x_4 from 1 by computing $\mu = 1/13$. Since $\tilde{a}_{14} = \frac{7}{13} \geq \frac{1}{13}$, we have $\alpha_4 = 1$. So $x_3 + x_4 \leq 1$ is a facet of $PS_i(M_0, \emptyset, N_0, N_1)$. □

3.2 Lifting Members of $M_0^<$

The minimal cover obtained in Section 3.1 is next lifted with respect to the variables of $M_0^<$. There is a polynomial algorithm to perform this task, see Nemhauser and Wolsey [15] and Zemel [21]. We will not present it here, but we note that all the lifting coefficients of the members of $M_0^<$ can be computed in time $O(n^2)$ and we denote the resulting facet-defining inequality of $PS_i(M_0^>, \emptyset, N_0, N_1)$ by

$$\sum_{j \in M \backslash M_0^>} \alpha_j x_j \leq \delta. \tag{12}$$

Example 1 (continued). We lift the variables x_1 and x_2 in this order and we determine that both of these lifting coefficients are 0. Therefore $x_3 + x_4 \leq 1$ is a facet of $PS_i(M_0^>, \emptyset, N_0, N_1) = PS_i(\emptyset, \emptyset, N_0, N_1)$. □

3.3 Lifting Members of N_1

For the lifting from 1 of continuous variables in a 0-1 inequality, we have given a pseudo-polynomial lifting scheme [18] that is based on the function Λ defined, from (12), as

$$\begin{aligned} \Lambda(w) = \min & \sum_{j \in M \backslash M_0^>} \tilde{a}_{ij} x_j - (f_i + \sum_{j \in M_1} \tilde{a}_{ij}) \\ s.t. & \sum_{j \in M \backslash M_0^>} \alpha_j x_j = \delta + w \\ & x_j \in \{0, 1\} \quad \forall j \in M \backslash M_0^>. \end{aligned}$$

If, for some w, the problem defining $\Lambda(w)$ is infeasible, we define $\Lambda(w) = \infty$. We say that the function Λ is superlinear if $w^*\Lambda(w) \geq w\Lambda(w^*)$ for all $w \geq w^*$ where $w^* = \max \operatorname{argmin}\{\Lambda(w) \mid w > 0\}$. When the function Λ is superlinear, the lifting algorithm proposed in [18] can be significantly simplified. The following theorem establishes that the lifted covers obtained in Section 3.2 are superlinear with $w^* = 1$.

Theorem 3 **([19]).** *For (12) we have $\Lambda(w) \geq w\Lambda(1)$, for all $w > 0$.* □

This property leads to the following lifting theorem.

Theorem 4 **([19]).** *Assume $N_1 = \{1, \ldots, s\}$, then*

$$\sum_{j \in M \setminus M_o^>} \alpha_j x_j + \theta \sum_{j=p+1}^{s} \tilde{b}_{ij} y_j \leq \delta + \theta \sum_{j=p+1}^{s} \tilde{b}_{ij} \tag{13}$$

is a facet of $PS_i(M_0^>, \emptyset, N_0, \emptyset)$ where $p = \max\{k \in \{1, \ldots, s\} \mid \sum_{j=1}^{k} \tilde{b}_{ij} < \Lambda(1)\}$, $\theta = \frac{1}{\Lambda(1)-b}$ and $b = \sum_{j=1}^{p} \tilde{b}_{ij}$. □

This refined version of the lifting algorithm, proven in [19], is valid for lifted covers and requires only the knowledge of $\Lambda(1)$. This value can be computed in time $O(n^2)$. So, once $\Lambda(1)$ is known, Theorem 4 yields a way to compute all the lifting coefficients of variables in N_1 in time $O(n)$ independent of the lifting order. Observe that it is possible that $p = s$ in which case no continuous variable appears in (13).

Example 1 (continued). We lift from 1 the continuous variable y_1. Note that $1 > \Lambda(1) = \frac{1}{13}$. Therefore the inequality $x_3 + x_4 + 13y_1 \leq 14$ is a facet of $PS_i(\emptyset, \emptyset, N_0, \emptyset)$. □

3.4 Lifting Members of N_0

At this stage, we lift the members of N_0. We have shown in [18] that the lifting coefficient is 0 almost always. More generally,

Theorem 5 **([18]).** *Assume that (13) is a valid inequality of $PS(M_0^>, \emptyset, N_0, \emptyset)$ and that $k \in N_0$. Assume there exists $x^* \in S(M_0^>, \emptyset, N_0, \emptyset)$ that satisfies (13) at equality and the defining inequality of $PS_i(M_0^>, \emptyset, N_0, \emptyset)$ strictly at inequality, then in lifting y_k from 0, the lifting coefficient is 0.* □

For the inequality obtained in Theorem 4, the point $x^*_{B(i)} = 0$, $x^*_j = 0$ for all $j \in M_0^<$ and $x^*_j = 1$ for all $j \in M_1$ satisfies the assumption of Theorem 5. Therefore the lifting coefficients for all members of N_0 are 0. It follows that inequality (13) is a facet of $PS(M_0^>, \emptyset, \emptyset, \emptyset)$.

Example 1 (continued). We lift the continuous variable y_3. Theorem 5 implies that $x_3 + x_4 + 13y_1 \leq 14$ is a facet of $PS_i = PS_i(\emptyset, \emptyset, \emptyset, \emptyset)$. It is also a cut for our initial problem. Moreover, it can be verified that it is a facet of PQ because the points $(x_1, x_2, x_3, x_4, y_1) = (0,0,0,1,1)$, $(0,0,1,0,1)$,$(0,1,0,1,1)$, $(1,0,0,1,1)$ and $(0,0,1,1,\frac{12}{13})$ belong to Q, make the inequality tight and are affinely independent. If we add this cut and reoptimize, the solution we obtain is $(0,0,1,1,\frac{12}{13})$, which is optimal and has an objective value of $\frac{211}{13}$. □

3.5 Lifting Members of $M_0^>$

Finally we lift the members of $M_0^>$, i.e. the variables with large coefficients. First suppose the inequality we lift is a 0-1 lifted cover, i.e. $p = s$. Again, the dynamic programming algorithm presented in [15,21] can be used and all the lifting coefficients for the members of $M_0^>$ can be computed in time $O(n^2)$. Now suppose that $p < s$. There is a closed form expression for the lifting of members of $M_0^>$. This closed form expression developed in [19] for general superlinear inequalities is described in the next theorem. For $a \in \mathbb{R}$, we define $(a)^+ = \max\{a, 0\}$.

Theorem 6 ([19]). *Assume $p < s$ and (13) is a facet of $PS_i(M_0^>, \emptyset, \emptyset, \emptyset)$. Then*

$$\sum_{j \in M \setminus M_0^>} \alpha_j x_j + \theta \sum_{j=p+1}^{s} \tilde{b}_{ij} y_j + \sum_{j \in M_0^>} G(a_j) x_j \leq \delta + \theta \sum_{j=p+1}^{s} \tilde{b}_{ij} \tag{14}$$

where $G(a) = \delta + \theta(a - d_i^* - b)^+$ and $d_i^* = \tilde{d}_i - \sum_{j \in N_1} \tilde{b}_{ij}$ is a facet of PS_i. □

Theorem 6 leads to a linear time algorithm for the lifting of members of $M_0^>$ when $p < s$. Thus, the cut we add to the current LP relaxation of the 0-1 mixed integer program is either a 0-1 lifted cover or (14). The previous discussion shows that, in either case, this cut can be derived in time $O(n^2)$.

3.6 Final Remarks on Lifting

In the previous discussion, we omitted the case where all the lifting coefficients of the variables of M_1 are 0. In this case, all the lifting coefficients of members of $M_0^<$ are 0 too because the inequalities $x_j \leq 0$ for $j \in M_0^<$ would be valid in $PS_i(M_0^>, \emptyset, N_0, N_1)$ which contradicts the definition of $M_0^<$. At least one member of N_1 is lifted with a positive coefficient, otherwise $x_{B(i)} \leq 0$ would be valid for PS_i which contradicts the full-dimensionality of PS_i. The inequality we obtain is a facet of $PS_i(M_0^>, \emptyset, N_0, \emptyset)$ that can be turned into a facet of PS_i using Theorems 5 and 6. So our cut generation procedure returns either a facet of PS_i, if it is full dimensional, or, as discussed in Section 2, a member of its equality set, if it is not full dimensional.

All the cuts are generated in time $O(n^2)$ and are of the standard form

$$x_{B(i)} + \sum_{j \in M_0 \cup M_1} \alpha_j x_j + \sum_{j \in N_0 \cup N_1} \beta_j y_j \leq \sum_{j \in M_1} \alpha_j + \sum_{j \in N_1} \beta_j \tag{15}$$

with $\alpha_j \geq 0$ for $j \in M_0 \cup M_1$ and $\beta_j \geq 0$ for $j \in N_0 \cup N_1$. This simple observation leads to the following proposition that will be used to establish finite convergence of our algorithm. For convenience, we will now incorporate the upper bounds on variables in the set of constraints before using the simplex method. We refer to this variant of the simplex method as being *without upper bounds*.

Proposition 1. *If we apply the simplex method without upper bounds, the cut generated from the simplex tableau row (7) where* $0 < f_i < 1$ *is of the form*

$$x_{B(i)} + \sum_{j \in M_0} \alpha_j x_j + \sum_{j \in N_0} \beta_j y_j \leq 0 \tag{16}$$

where $\alpha_j \leq 0$ *if* $\tilde{a}_{ij} < 0$*,* $\alpha_j \geq 0$ *if* $\tilde{a}_{ij} > 0$*,* $\beta_j \leq 0$ *if* $\tilde{b}_{ij} < 0$*, and* $\beta_j \geq 0$ *if* $\tilde{b}_{ij} > 0$*.*

Proof. The tableau row (7) on which we generate the cut contains only M_0 and N_0. After all the variables with negative coefficients in (7) are complemented to fit in the standard format, we generate (15) that has only nonnegative coefficients. Since all the members of M_1 and N_1 are complemented variables, they need to be complemented back yielding (16). □

4 A Finitely Convergent Algorithm for 0-1 Mixed Integer Programming

The ability to generate a cut from every row of the simplex tableau where a basic integer variable is fractional is reminiscent of Gomory cuts. Now, as done by Gomory, we prove a finite convergence result. The approach we take is similar to the one described by Nourie and Venta [17]. Consider the mixed integer problem (1), (3), (5) and (6). For ease of notation, in this section, we denote all the variables in the 0-1 mixed integer problem by x, even if they are continuous, i.e. we define $x_{m+i} = y_i$ for $i = 1, \ldots, n+h$. We assume that every extreme point of PQ say x^q is such that $cx^q \in \mathbb{Z}$. This condition can always be met by adequately scaling the objective function. We say that $u \in \mathbb{R}^t$ is lexicographically larger than 0 ($u \succ 0$) if there exists $k \in \{1, \ldots, t\}$ such that $u_1 = \ldots = u_{k-1} = 0$ and $u_k > 0$. We say also that, for $u, v \in \mathbb{R}^t$, u is lexicographically larger than v ($u \succ v$) if $u - v \succ 0$. If the two vectors we compare are of different length, we just drop the last components of the longer one and perform the comparison on vectors of the same size.

We modify the fractional cutting plane algorithm ($FCPA$) presented in [15], p. 368-369 to handle our cuts instead of Gomory's and prove its convergence using the arguments presented in [15], p. 370-373. We recall some of the assumptions under which convergence is established.

(i) We use the simplex method without upper bounds.
(ii) The objective function is restricted to be integer. This is a valid since we know that cx^q is integer for every extreme point x^q of PQ. Therefore, an equality of the form $x_0 = cx$, where x_0 is an integer variable, is introduced in the set of constraints and the objective function is replaced by x_0.

(iii) We solve the linear relaxations in such a way that the solution we obtain is lexicographically maximum for the set of optimal solutions and we include rows of the form $x_j - x_j = 0$ in the tableaux for nonbasic variables.
(iv) We generate a single cut at every step of the algorithm and we generate it from the simplex tableau row whose basic integer variable is fractional and has the smallest index.
(v) After we add a cut, the problem must still be of the initial form. In our case, it suffices to introduce the slack and rescale it to be in the interval $[0,1]$.

The fact that the variable x_0 is a general integer variable is not a problem in the lifting steps of our algorithm since we can always keep x_0 basic. Note that when x_0 is fractional, say $x_0 = q$, the cut $x_0 \le \lfloor q \rfloor$ is valid. From the previous observation and Proposition 1 we conclude that the cut we generate from the k^{th} simplex tableau row with basic variable x_k

$$x_k + \sum_{j \in NB} \tilde{a}_{kj} x_j = \tilde{d}_k \tag{17}$$

is of the form

$$\alpha_k x_k + \sum_{j \in NB} \alpha_j x_j \le \delta_k \tag{18}$$

where $\alpha_k = 1$, $\delta_k = \lfloor \tilde{d}_k \rfloor$ and NB is the set of nonbasic variables in the current simplex tableau.

Having just presented a scheme in which we can embed our cuts, we now prove that these cuts are strong enough to yield an optimal solution in a finite number of iterations. It is not true that this property will be achieved by any family of violated inequalities. For example, as shown by Gomory and Hoffman [9], the family of cuts that require the sum of the nonbasic integer variables to be at least one, see Dantzig [6], do not necessarily lead to a convergent algorithm. They need to be improved, as described by Bowman and Nemhauser [4], to yield a convergent algorithm. A sufficient condition on the strength of cuts needed to obtain finite convergence is described in the next proposition.

Proposition 2. *Assume that all the cuts generated from simplex tableau rows (17) are of the form (18). Let f_k be the fractional part of $\tilde{d}_k$ and assume that for every $l \in NB$ such that $\alpha_l - \alpha_k \tilde{a}_{kl} < 0$ and $\tilde{a}_{kl} \ge 0$, we have*

$$f_k \alpha_l + \tilde{a}_{kl}(\alpha_k \lfloor \tilde{d}_k \rfloor - \delta_k) \ge 0. \tag{19}$$

Then FCPA converges in a finite number of iterations.

Proof. We extend the proof of Proposition 3.7 from [15]. Assume that we have already added t cuts. We work with a simplex tableau that has $v(t) = m+n+h+t$ rows of the form of (17). Assume that x^t is an optimal solution of the current relaxation and that k is the smallest index among all 0-1 variables that are currently fractional. We define $S^t = (x_0^t, \dots, x_{k-1}^t, \lfloor x_k^t \rfloor, u_{k+1}, \dots, u_m)$, where $u_{k+1}, \dots, u_m$ are upper bounds on the integer variables, i.e. 1 in our case. We have that $x^t \succ S^t$. We need to prove $x^{t+1} \preceq S^t$. Assume that, from row k of the tableau, we generate the cut $\alpha_k x_k + \sum_{j \in NB} \alpha_j x_j + u x_{v(t+1)} = \delta_k$, where NB is the set of nonbasic variables, and add it to the current formulation. Note that we introduce u (which can always be chosen to be positive) as a way to rescale the slacks since their domain has to be the interval $[0,1]$. After adding the cut, we make the slack basic and therefore obtain the basic, primal infeasible, dual optimal tableau

$$x_p + \sum_{j \in NB} \tilde{a}_{pj} x_j = \tilde{d}_p \qquad \forall p \in \{1, \dots, v(t)\}$$

$$x_{v(t+1)} + \sum_{j \in NB} \frac{\alpha_j - \alpha_k \tilde{a}_{kj}}{u} x_j = \frac{\delta_k - \alpha_k \tilde{d}_k}{u}$$

in which the column associated with x_j, $a_j \succ 0 \, \forall j \in NB$. Let $(\hat{x}_0^t, \hat{x}^t)$ be the basic solution obtained after a single dual simplex pivot and let x_l be the variable that becomes basic. Clearly $\beta_l = \frac{\alpha_l - \alpha_k \tilde{a}_{kl}}{u} < 0$, $\frac{\delta_k - \alpha_k \tilde{d}_k}{u} < 0$ and we have

$$\begin{pmatrix} \hat{x}_0^t \\ \hat{x}^t \end{pmatrix} = \begin{pmatrix} x_0^{t-1} \\ x^{t-1} \end{pmatrix} - \frac{\delta_k - \alpha_k \tilde{d}_k}{\alpha_l - \alpha_k \tilde{a}_{kl}} a_l.$$

We have that $a_l \succ 0$ and $\frac{\delta_k - \alpha_k \tilde{d}_k}{\alpha_l - \alpha_k \tilde{a}_{kl}} > 0$. Now let r be the minimum index for which $\tilde{a}_{rl} > 0$. We distinguish two cases. First $r \leq k-1$. In that case $\hat{x}_j^t = x_j^t$ for all $j < r$ and $\hat{x}_r^t < x_r^t$. Therefore $\hat{x}^t \prec S^t$ and so $x^{t+1} \prec S^t$. Now assume $r \geq k$. We have $\hat{x}_j^t = x_j^t$ for all $j < k$ and

$$\hat{x}_k^t = \lfloor \tilde{d}_k \rfloor + \frac{f_k \alpha_l + \tilde{a}_{kl}(\alpha_k \lfloor \tilde{d}_k \rfloor) - \delta_k)}{\alpha_l - \alpha_k \tilde{a}_{kl}}.$$

Now since $\tilde{a}_l \succ 0$ and $\tilde{a}_{jl} = 0$ for $j < k$, we have that $\tilde{a}_{kl} \geq 0$. Using (19), we conclude that the numerator of the fraction is nonnegative and, since its denominator is negative, that $\hat{x}_k^t \leq \lfloor \tilde{d}_k \rfloor$. It follows that $x^{t+1} \preceq \hat{x}_k^t \preceq S^t$.

□

We use Proposition 2 to prove that our algorithm is finite.

Theorem 7. *There exists a pure cutting plane algorithm, based on the cuts presented in Section 3, that solves Problem (1), (3), (5) and (6) in a finite number of iterations.*

Proof. According to (18), we have $\alpha_k = 1$ and $\delta_k = \lfloor \tilde{d}_k \rfloor$. Condition (19) becomes $f_k \alpha_l \geq 0$ for all $l \in NB$ such that $\alpha_l - \tilde{a}_{kl} < 0$ and $\tilde{a}_{kl} \geq 0$. Since $\tilde{a}_{kl} \geq 0$, we know from Proposition 1 that $\alpha_l \geq 0$ and so we conclude that condition (19) is satisfied since $f_k > 0$. □

Proposition 2 can also be used to show that Gomory cuts for integer programs yield a finitely convergent algorithm. Starting from the simplex tableau row (17), Gomory cuts are of the form $\sum_{j \in NB} f_j z_j \geq g_k$ where f_j and g_k are the fractional parts of $\tilde{a}_{kj}$ and $\tilde{d}_k$. Therefore we have that $\alpha_k = 0$, $\delta_k = -g_k$ and $\alpha_l = -f_l$. Condition (19) is then $-g_k f_l + \tilde{a}_{kl} g_k = g_k(\tilde{a}_{kl} - f_l) \geq 0$ which is satisfied when $\tilde{a}_{kl} \geq 0$ and $f_l > 0$ because g_k is positive.

5 Conclusions

The cuts we have developed are not strictly comparable to Gomory cuts. We first relax a simplex tableau row into a 0-1 mixed integer knapsack and then we find a facet of this knapsack relaxation. Thus our cuts are strong since they are facets of a good relaxation. But it is interesting that they are also robust in that they can be implemented to yield a finite pure cutting plane algorithm. Nevertheless, the practical use of these cuts is likely to come from imbedding them in a branch-and-cut algorithm which we are currently developing.

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