수학기반 인공지능프로그래밍

(44630-01)

과제 5

제출일자: 11월 13일 수요일

담당교수: 이덕우 (dwoolee@kmu.ac.kr)

유의: 과제는 개별로 수행합니다. 제출일자 이후에 과제를 제출할 경우 획득점수의 50%를 부여합니다.

5장 연습문제

1. 다음과 같은 행렬의 고윳값을 구하라. (5-1)

5.1

A)

$$A = \begin{bmatrix} 2 & 8 \\ 4 & 6 \end{bmatrix}$$

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda-2 & -8 \\ -4 & \lambda-6 \end{bmatrix}$$

$(\lambda-2)(\lambda-6)-32=0$

$\lambda^2-8\lambda+12-32=0$

$\lambda^2-8\lambda-20=0$

$(\lambda-10)(\lambda+2)=0$

$\therefore \lambda=10$ or $\lambda=-2$

B)

$$B = \begin{bmatrix} 3 & 0 \\ 2 & 4 \end{bmatrix}$$

$$\det(\lambda I - B) = \det\begin{bmatrix} \lambda-3 & 0 \\ -2 & \lambda-4 \end{bmatrix}$$

$(\lambda-3)(\lambda-4)-0=0$

$\therefore \lambda=3$ or $\lambda=4$

2. 다음과 같은 행렬 A의 고윳값을 구하고, 갓 고윳값에 대한 고유공간을 구하라. (5-3)

5.3

$$A = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 7 & 0 \\ 5 & 0 & 4 \end{bmatrix}$$

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda-1 & 0 & -5 \\ 0 & \lambda-7 & 0 \\ -5 & 0 & \lambda-4 \end{bmatrix}$$

$$(\lambda-1)\det\begin{bmatrix} \lambda-7 & 0 \\ 0 & \lambda-4 \end{bmatrix} - 5\cdot\det\begin{bmatrix} 0 & \lambda-7 \\ -5 & 0 \end{bmatrix} = 0$$

$$\Rightarrow (\lambda-1)(\lambda-7)(\lambda-4) - 25(\lambda-7) = 0$$

$$\Rightarrow (\lambda-7)\{(\lambda-1)(\lambda-4) - 25\} = 0$$

$$\Rightarrow (\lambda-7)\{\lambda^2-5\lambda+4-25\} = 0$$

$$\Rightarrow (\lambda-7)(\lambda^2-5\lambda-21) = 0$$

$$\therefore \lambda = 7 \text{ or } \lambda = \frac{5 \pm \sqrt{109}}{2}$$

if) $\lambda = 7$

$$\left[\begin{array}{ccc|c} 7-1 & 0 & -5 & 0 \\ 0 & 7-7 & 0 & 0 \\ -5 & 0 & 7-4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 6 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 \\ -5 & 0 & 3 & 0 \end{array}\right]$$

$6x_1 - 5x_3 = 0 \quad \therefore x_1 = 0,\ x_3 = 0$

$-5x_1 + 3x_3 = 0$

$$\therefore E_1 = \text{span}\left\{\begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix}\right\}$$

5.3

if) $\lambda = \frac{\sqrt{109}+5}{2}$

$$\left[\begin{array}{ccc|c} \frac{\sqrt{109}+3}{2} & 0 & -5 & 0 \\ 0 & \frac{\sqrt{109}-9}{2} & 0 & 0 \\ -5 & 0 & \frac{\sqrt{109}-3}{2} & 0 \end{array}\right]$$

$$\frac{\sqrt{109}+3}{2}x_1 - 5x_3 = 0$$

$$\frac{\sqrt{109}-9}{2}x_2 = 0$$

$$-5x_1 + \frac{\sqrt{109}-3}{2}x_3 = 0 \Rightarrow x_1 = \frac{\sqrt{109}-3}{10}x_3$$

5.3

if) $\lambda = \frac{-\sqrt{109}+5}{2}$

$$\left[\begin{array}{ccc|c} \frac{-\sqrt{109}+3}{2} & 0 & -5 & 0 \\ 0 & \frac{-\sqrt{109}-9}{2} & 0 & 0 \\ -5 & 0 & \frac{-\sqrt{109}-3}{2} & 0 \end{array}\right]$$

$$\frac{-\sqrt{109}+3}{2}x_1 - 5x_3 = 0$$

$$\frac{-\sqrt{109}-9}{2}x_2 = 0$$

$$-5x_1 + \frac{-\sqrt{109}-3}{2}x_3 = 0 \Rightarrow x_1 = \frac{-\sqrt{109}-3}{10}x_3$$

$$\therefore E_2 = \text{Span}\left\{\begin{bmatrix} \frac{-\sqrt{109}-3}{10}x_3 \\ 0 \\ x_3 \end{bmatrix}\right\}$$

3. 다음과 같은 선형변환 L의 표준행렬에 대하여 고윳값과 고유공간을 구하라. (5-5)

A. L(x, y) = (x – 4y, -2x - 3y)

5.5

a) $L(x,y) = (x-4y, -2x-3y)$ $\begin{bmatrix} 1 & -4 \\ -2 & -3 \end{bmatrix}$

$$\det(\lambda I - L) = \det\begin{bmatrix} \lambda-1 & 4 \\ 2 & \lambda+3 \end{bmatrix}$$

$(\lambda-1)(\lambda+3)-8=0$

$\lambda^2+2\lambda-3-8=0$

$\lambda^2+2\lambda-11=0$

$\therefore \lambda = \dfrac{-2\pm\sqrt{48}}{2} = -1\pm2\sqrt{3}$

if) $\lambda = -1+2\sqrt{3}$

$$\left[\begin{array}{cc|c} 2\sqrt{3}-2 & 4 & 0 \\ 2 & 2+2\sqrt{3} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} \sqrt{3}-1 & 2 & 0 \\ 1 & 1+\sqrt{3} & 0 \end{array}\right]$$

$(\sqrt{3}-1)x_1 + 2x_2 = 0$

$x_1 = \dfrac{-2}{\sqrt{3}-1}x_2 = \dfrac{-2(\sqrt{3}+1)}{2}x_2 = -(\sqrt{3}+1)x_2$

$$E_1 = \mathrm{span}\left\{\begin{bmatrix} -(\sqrt{3}+1)x_2 \\ x_2 \end{bmatrix}\right\}$$

ii) $\lambda = -1-2\sqrt{3}$

$$\left[\begin{array}{cc|c} -2-2\sqrt{3} & 4 & 0 \\ 2 & 2-2\sqrt{3} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -1-\sqrt{3} & 2 & 0 \\ 1 & 1-\sqrt{3} & 0 \end{array}\right]$$

$$\begin{cases} -(1+\sqrt{3})x_1 + 2x_2 = 0 \\ x_1 + (1-\sqrt{3})x_2 = 0 \end{cases}$$

$x_1 = \dfrac{-2x_2}{-(1+\sqrt{3})} = \dfrac{2}{1+\sqrt{3}}x_2 = \dfrac{2(\sqrt{3}-1)}{2}x_2$

$x_1 = (\sqrt{3}-1)x_2$

$\therefore E_2 = span\left\{\begin{bmatrix} (\sqrt{3}-1)x_2 \\ x_2 \end{bmatrix}\right\}$

B. L(x, y, z) = (2x – y – z, - x + z, z - y - 2z)

5.5

b) $L(x,y,z) = (2x-y-z, -x+z, x-y-2z)$

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & -2 \end{bmatrix}$$

$$\det(\lambda I - L) = \det\begin{bmatrix} \lambda-2 & 1 & 1 \\ 1 & \lambda & -1 \\ -1 & 1 & \lambda+2 \end{bmatrix}$$

$$(\lambda-2)\det\begin{bmatrix} \lambda & -1 \\ 1 & \lambda+2 \end{bmatrix} - \det\begin{bmatrix} 1 & -1 \\ -1 & \lambda+2 \end{bmatrix} + \det\begin{bmatrix} 1 & \lambda \\ -1 & 1 \end{bmatrix}$$

$$(\lambda-2)\{\lambda(\lambda+2)+1\} - \{(\lambda+2)-1\} + (1+\lambda)$$

$$= (\lambda-2)(\lambda^2+2\lambda+1) - (\lambda+1) + (\lambda+1)$$

$$= (\lambda-2)(\lambda^2+2\lambda+1)$$

$\therefore \lambda = 2$ or $\lambda = -1$

if) $\lambda = -1$

$$\left[\begin{array}{ccc|c} -3 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ -1 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ -1 & 1 & 1 & 0 \end{array}\right]$$

$$\begin{cases} -2x_1 = 0 \\ x_2 = -x_3 \end{cases}$$

$$\therefore E_1 = \mathrm{span}\left\{\begin{bmatrix} 0 \\ x_2 \\ -x_3 \end{bmatrix}\right\}$$

5.5

b)

if) $\lambda = 2$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 2 & -1 & 0 \\ -1 & 1 & 4 & 0 \end{array}\right] \qquad \begin{cases} x_2 + x_3 = 0 \\ x_1 + 3x_2 = 0 \end{cases} \Rightarrow \begin{array}{l} x_3 = -x_2 \\ x_1 = -3x_2 \end{array}$$

$$E_2 = \text{span}\left\{\begin{bmatrix} -3x_2 \\ x_2 \\ -x_2 \end{bmatrix}\right\}$$

$$x_2 = -1 \Rightarrow E_2 = \text{span}\left\{\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}\right\}$$

4. 다음과 같은 행렬 A의 고윳값을 구하라. (5-8)

5.8

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k^3 & -4k^2 & 4k \end{bmatrix}$$

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -k^3 & 4k^2 & \lambda-4k \end{bmatrix}$$

$$\lambda \det\begin{bmatrix} \lambda & -1 \\ 4k^2 & \lambda-4k \end{bmatrix} + \det\begin{bmatrix} 0 & -1 \\ -k^3 & \lambda-4k \end{bmatrix}$$

$$\lambda\{\lambda(\lambda-4k)+4k^2\} - k^3$$

$$\lambda(\lambda^2-4k\lambda+4k^2) - k^3$$

$$\lambda^3 - 4\lambda^2 k + 4k^2\lambda - k^3$$

$$= (\lambda-k)(\lambda^2-3k\lambda+k^2)$$

$$\therefore \lambda = k, \quad \lambda = \frac{3\pm\sqrt{9-4}}{2}k = \frac{3\pm\sqrt{5}}{2}k$$

5. 벡터공간 V에서 정의된 다음과 같은 선형변환 L에 대하여, L의 행렬식과 L의 특성다항식을 구하라. (5-9)

5.9

a) $V = R^2$, $L\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a - b \\ 5a + 3b \end{bmatrix}$

L의 행렬식 $= \begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix}$ $\boxed{\det\begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix} = 6+5 = 11}$

$\det(\lambda I - A) = \det\begin{bmatrix} \lambda-2 & 1 \\ -5 & \lambda-3 \end{bmatrix}$

$(\lambda-2)(\lambda-3)+5$

$= \lambda^2 - 5\lambda + 6 + 5$

$\boxed{= \lambda^2 - 5\lambda + 11}$

b) $V = R^3$ $L\begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 2 \\ -2 & 1 & 1 \\ 4 & 0 & -1 \end{bmatrix}$

$\det\begin{bmatrix} \lambda-1 & 3 & -2 \\ 2 & \lambda-1 & -1 \\ -4 & 0 & \lambda+1 \end{bmatrix}$

$(\lambda-1)\det\begin{bmatrix} \lambda-1 & -1 \\ 0 & \lambda+1 \end{bmatrix} - 3\det\begin{bmatrix} 2 & -1 \\ -4 & \lambda+1 \end{bmatrix} - 2\det\begin{bmatrix} 2 & \lambda-1 \\ -4 & 0 \end{bmatrix}$

$(\lambda-1)(\lambda-1)(\lambda+1) - 3\{2(\lambda+1) - 4\} - 2\cdot 4(\lambda-1)$

$= (\lambda-1)(\lambda-1)(\lambda+1) - 6(\lambda+1) + 12 - 8(\lambda-1)$

$= (\lambda^2 - 2\lambda + 1)(\lambda+1) - 6\lambda - 6 + 12 - 8\lambda + 8$

$= \lambda^3 - 2\lambda^2 + \lambda + \lambda^2 - 2\lambda + 1 - 6\lambda - 6 + 12 - 8\lambda + 8$

$= \lambda^3 - \lambda^2 - 15\lambda + 15$

6. 행렬 A에 대하여 다음 물음에 답하라 (5-10)

A. 고윳값, 고유벡터와 특성 다항식을 구하라

5.10

a)

$$\det(\lambda I - A) = \begin{bmatrix} \lambda-1 & 0 & -1 \\ 0 & \lambda-2 & 0 \\ 0 & 0 & \lambda-2 \end{bmatrix}$$

$$(\lambda-1)\det\begin{bmatrix} \lambda-2 & 0 \\ 0 & \lambda-2 \end{bmatrix} - \det\begin{bmatrix} 0 & \lambda-2 \\ 0 & 0 \end{bmatrix}$$

$$= (\lambda-1)(\lambda-2)(\lambda-2)$$

$\lambda = 1$ or $\lambda = 2$

$$(\lambda-1)(\lambda^2-4\lambda+4) = 0$$

$$= \lambda^3 - 4\lambda^2 + 4\lambda - \lambda^2 + 4\lambda - 4$$

$$= \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

if) $\lambda = 1$

$$\left[\begin{array}{ccc|c} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$\therefore E = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

if) $\lambda = 2$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore E = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

B. 케일리-해밀턴 정리를 이용하여 A^{10} 과 A^{-1} 를 구하라.

$$\lambda^3 = 5\lambda^2 - 8\lambda + 4$$

$$A^{10} = 5A^9 - 8A^8 + 4A^4$$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 5 & 0 & 15 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix} - \begin{bmatrix} 8 & 0 & 8 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 5 & 0 & 35 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{bmatrix} - \begin{bmatrix} 8 & 0 & 24 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 4 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 15 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{bmatrix}$$

$$A^n = \begin{bmatrix} 1 & 0 & 2^n-1 \\ 0 & 2^n & 0 \\ 0 & 0 & 2^n \end{bmatrix}$$

$$A^{10} = \begin{bmatrix} 1 & 0 & 1023 \\ 0 & 1024 & 0 \\ 0 & 0 & 1024 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

5장 실습문제

7. 다음 행렬 A, B, C의 고윳값과 고유벡터를 구하라

   A. 코드

```python
import numpy as np

# 행렬 A 의 고윳값과 고유벡터 계산
A = np.array([[1, -4, 1], [3, 2, 3], [1, 1, 3]])
w1, V1 = np.linalg.eig(A)
print("A 의 고윳값 = \n", w1)
print("A 의 고유벡터 = \n", V1)

# 행렬 B 의 고윳값과 고유벡터 계산
B = np.array([[0,1,0,1], [1,0,1,0], [0,1,0,1], [1,0,1,0]])
w2, V2 = np.linalg.eig(B)
print("\nB 의 고윳값 = ", w2)
print("B 의 고유벡터 = ", V2)

# 행렬 C 의 고윳값과 고유벡터 계산
C = np.array([[2,-1,0,-1], [-1,2,-1,0], [0,-1,2,-1], [-1,0,-1,2]])
w3, V3 = np.linalg.eig(C)
print("\nC 의 고윳값 = \n", w3)
print("C 의 고유벡터 = \n", V3)
```

   B. 결과

```
[Running] python -u "c:\Users\man25\Desktop\AI_Math-with-python\work4\work4-5-1.py"
A의 고윳값 =
 [1.60629927+2.73221536j 1.60629927-2.73221536j 2.78740146+0.j        ]
A의 고유벡터 =
 [[ 0.77197967+0.j          0.77197967-0.j         -0.57870193+0.j        ]
 [-0.1789124 -0.54999483j -0.1789124 +0.54999483j  0.43158808+0.j        ]
 [-0.24759891-0.09076462j -0.24759891+0.09076462j  0.69197963+0.j        ]]

B의 고윳값 =  [-2.00000000e+00  6.59737022e-17  2.00000000e+00  0.00000000e+00]
B의 고유벡터 =  [[ 5.00000000e-01 -7.07106781e-01  5.00000000e-01  0.00000000e+00]
 [-5.00000000e-01 -4.63633825e-17  5.00000000e-01 -7.07106781e-01]
 [ 5.00000000e-01  7.07106781e-01  5.00000000e-01  0.00000000e+00]
 [-5.00000000e-01 -3.76905605e-17  5.00000000e-01  7.07106781e-01]]

C의 고윳값 =
 [-6.66133815e-16  2.00000000e+00  4.00000000e+00  2.00000000e+00]
C의 고유벡터 =
 [[ 5.00000000e-01  7.07106781e-01 -5.00000000e-01  5.55111512e-17]
 [ 5.00000000e-01 -2.74466000e-16  5.00000000e-01 -7.07106781e-01]
 [ 5.00000000e-01 -7.07106781e-01 -5.00000000e-01  2.58507341e-16]
 [ 5.00000000e-01  3.08103393e-17  5.00000000e-01  7.07106781e-01]]
```

8. 다음 행렬 A, B, C에 대하여 케일리-해밀턴 정리가 성립함을 보여라

A. 코드

```
import numpy as np
import sympy as sp

# 행렬 A 의 케일리-해밀턴 정리 만족 여부 확인
print("문제 (a)")
A = sp.Matrix([[1, -4, 1], [3, 2, 3], [1, 1, 3]])
print("A = ", A)
lamda = sp.symbols('lamda')
p_A = A.charpoly(lamda)
print("A 의 특성다항식 : ",p_A)
print("특성다항식의 영이 아닌 계수들 : ", p_A.coeffs())

# 행렬 A 의 특성다항식을 먼저 구한 후 직접 행렬다항식을 코드로 작성하는 방법
A2 = np.matmul(A, A)
A3 = np.matmul(A, A2)
char_coeff = p_A.coeffs()
print("p_A(A) = ",char_coeff[0]*A3 + char_coeff[1]*A2 + char_coeff[2]*A +
char_coeff[3]*np.eye(3))
print("\n")

# 행렬 B 의 케일리-해밀턴 정리 만족 여부 확인
print("문제 (b)")
B = sp.Matrix([[0, 1, 0, 1], [1, 0, 1, 0], [0, 1, 0, 1], [1, 0, 1, 0]])
print("B = ", B)
lamda = sp.symbols('lamda')
p_B = B.charpoly(lamda)
print("B 의 특성다항식 : ", p_B)
print("특성다항식의 영이 아닌 계수들 : ", p_B.coeffs())

# 행렬 B 의 특성다항식을 먼저 구한 후 직접 행렬다항식을 코드로 작성하는 방법
B2 = np.matmul(B, B)
B3 = np.matmul(B, B2)
B4 = np.matmul(B, B3)
char_coeff = p_B.coeffs()
print("p_B(B) = \n", char_coeff[0]*B4 + char_coeff[1]*B2 )
print("\n")

# 행렬 C 의 케일리-해밀턴 정리 만족 여부 확인
print("문제 (c)")
C = sp.Matrix([[2, -1, 0, -1], [-1, 2, -1, 0], [0, -1, 2, -1], [-1, 0, -1,
2]])
print("C = ", C)
lamda = sp.symbols('lamda')
p_C = C.charpoly(lamda)
```

```python
print("C 의 특성다항식 : ", p_C)
print("특성다항식의 영이 아닌 계수들 : ", p_C.coeffs())

# 행렬 C 의 특성다항식을 먼저 구한 후 직접 행렬다항식을 코드로 작성하는 방법
C2 = np.matmul(C, C)
C3 = np.matmul(C, C2)
C4 = np.matmul(C, C3)
char_coeff = p_C.coeffs()
print("p_C(C) = ",char_coeff[0]*C4 + char_coeff[1]*C3 +char_coeff[2]*C2 +
char_coeff[3]*C)
```

B. 결과

```
[Running] python -u "c:\Users\man25\Desktop\AI_Math-with-python\work4\work4-5-2.py"
문제 (a)
A =  Matrix([[1, -4, 1], [3, 2, 3], [1, 1, 3]])
A의 특성다항식 :  PurePoly(lamda**3 - 6*lamda**2 + 19*lamda - 28, lamda, domain='ZZ')
특성다항식의 영이 아닌 계수들 :  [1, -6, 19, -28]
p_A(A) =  Matrix([[0, 0, 0], [0, 0, 0], [0, 0, 0]])


문제 (b)
B =  Matrix([[0, 1, 0, 1], [1, 0, 1, 0], [0, 1, 0, 1], [1, 0, 1, 0]])
B의 특성다항식 :  PurePoly(lamda**4 - 4*lamda**2, lamda, domain='ZZ')
특성다항식의 영이 아닌 계수들 :  [1, -4]
p_B(B) =
 [[0 0 0 0]
 [0 0 0 0]
 [0 0 0 0]
 [0 0 0 0]]


문제 (c)
C =  Matrix([[2, -1, 0, -1], [-1, 2, -1, 0], [0, -1, 2, -1], [-1, 0, -1, 2]])
C의 특성다항식 :  PurePoly(lamda**4 - 8*lamda**3 + 20*lamda**2 - 16*lamda, lamda, domain='ZZ')
특성다항식의 영이 아닌 계수들 :  [1, -8, 20, -16]
p_C(C) =  Matrix([[0, 0, 0, 0], [0, 0, 0, 0], [0, 0, 0, 0], [0, 0, 0, 0]])
```

6장 연습문제

9. 다음 행렬 A를 LU분해하라. (6-1)

6.1

a) $A = \begin{bmatrix} 1 & 4 \\ -1 & -1 \end{bmatrix}$

$L = \begin{bmatrix} 1 & 0 \\ L_{21} & 1 \end{bmatrix} \quad U = \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}$

$LU = A \quad \begin{bmatrix} 1 & 0 \\ L_{21} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -1 & -1 \end{bmatrix}$

$U_{11} = 1 \quad U_{11} \cdot L_{21} = -1$

$U_{12} = 4 \quad U_{12} \cdot L_{21} + U_{22} = -1$

$\therefore U_{11} = 1, \ U_{12} = 4, \ L_{21} = -1 \quad U_{22} = 3$

$L = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix}$

6.1

b) $A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix}$

$LU = A$

$L = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \quad U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$

$U_{11} = 1 \qquad U_{11} \cdot L_{12} = 0 \Rightarrow L_{12} = 0 \qquad L_{31} = -1$

$U_{12} = -1 \qquad U_{22} = -2 \qquad 1 + L_{32} \cdot U_{22} = 5 \Rightarrow L_{32} = -2$

$U_{13} = -1 \qquad U_{23} = 2 \qquad L_{31} \cdot U_{13} + L_{32} \cdot U_{23} + U_{33} = 2$

$\Rightarrow 1 + (-4) + U_{33} = 2$

$U_{33} = 5$

$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 2 \\ 0 & 0 & 5 \end{bmatrix}$

6.1

c) $LU = A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 1 & 0 & 1 \\ -2 & 2 & 1 & 1 \end{bmatrix}$

$L = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \quad U = \begin{bmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \end{bmatrix}$

$U_{11} = 2 \qquad L_{21} = 2 \qquad U_{24} = 1 \qquad -U_{13} - 3U_{23} + U_{33} = 1$

$U_{12} = 1 \qquad 2 + U_{22} = 1 \qquad 2L_{31} = -2 \qquad -1 + 6 + U_{33} = 1$

$U_{13} = 1 \qquad U_{22} = -1 \qquad L_{31} = -1 \qquad U_{33} = -4$

$U_{14} = 0 \qquad U_{23} = -2 \qquad L_{32} = -3 \qquad U_{34} = 1$

$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & -4 & 1 \end{bmatrix}$

10. 행렬 A의 LU분해가 존재하지 않음을 보여라. (6-2)

6.2

$$LU = A = \begin{bmatrix} 4 & 2 & 3 \\ 6 & 3 & 6 \\ 5 & 7 & 9 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \quad U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$\begin{cases} U_{11} = 4 \\ U_{12} = 2 \\ U_{13} = 3 \\ L_{21} = 6 \\ L_{21} \cdot U_{12} + U_{22} = 3 \Rightarrow 12 + U_{22} = 3 \Rightarrow U_{22} = -9 \\ L_{21} \cdot U_{13} + U_{23} = 6 \Rightarrow 18 + U_{23} = 6 \Rightarrow U_{23} = -12 \\ L_{31} \cdot U_{31} = 5 \\ L_{31} \cdot U_{12} + L_{32} \cdot U_{22} = 7 \Rightarrow L_{31} \cdot 2 + L_{32} \cdot (-9) = 7 \\ L_{31} \cdot U_{13} + L_{32} \cdot U_{23} + U_{33} = 9 \Rightarrow 3 \cdot L_{31} - 12 L_{32} + U_{33} = 9 \end{cases}$$

U_{33}을 구할 수 없음

11. 행렬 A에 대해 b가 다음과 같을 때, LU분해를 이용하여 Ax = b의 해를 구하라. (6-4)

6.4

$$A = \begin{bmatrix} -1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \quad U = \begin{bmatrix} -1 & -1 & 0 \\ 0 & U_{22} & -1 \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$\begin{cases} -L_{21} = -1 \\ -L_{21} + U_{22} = 2 \\ -L_{31} = 0 \\ -L_{31} + L_{32} \cdot U_{22} = -1 \\ -L_{32} + U_{33} = 2 \end{cases} \Rightarrow \begin{cases} L_{21} = 1 \\ U_{22} = 3 \\ L_{31} = 0 \\ L_{32} = -\frac{1}{3} \\ U_{33} = \frac{1}{3} \end{cases}$$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{bmatrix} \quad U = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & \frac{5}{3} \end{bmatrix}$$

(a)

$$Ly = b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$Ux = y = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -1 & -1 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & \frac{5}{3} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$a = 1,\ b = 0,\ c = 1$

$a + b = 1$

$-\frac{1}{3}b + c = 1$

$-a - b = 1 \quad a = -\frac{6}{5}$

$3b - c = 0 \quad b = \frac{1}{5}$

$\frac{5}{3}c = 1 \quad \therefore c = \frac{3}{5} \quad \therefore x = \begin{bmatrix} -\frac{6}{5} \\ \frac{1}{5} \\ \frac{3}{5} \end{bmatrix}$

(b)

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

$a = 2$

$b = -2$

$\frac{2}{3} + c = -1$

$c = -\frac{5}{3}$

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & \frac{5}{3} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -\frac{5}{3} \end{bmatrix}$$

$-a - b = 2$ $\quad a = -1$

$3b - c = -2$ $\quad b = -1$

$\frac{5}{3}c = -\frac{5}{3}$ $\quad c = -1$ $\quad \therefore x = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$

코드

```
import numpy as np
from scipy.linalg import lu

# 주어진 행렬 A
A = np.array([
    [-1, -1, 0],
    [-1, 2, -1],
    [0, -1, 2]
])

# LU 분해 수행
P, L, U = lu(A)
```

```
print("L 행렬:")
print(L)
print("₩nU 행렬:")
print(U)

# 첫 번째 경우 b 벡터
b1 = np.array([1, 1, 1])

# Ly = b1를 풀어서 y 계산
y1 = np.linalg.solve(L, b1)

# Ux = y1를 풀어서 x 계산
x1 = np.linalg.solve(U, y1)

print("₩n첫 번째 b 벡터에 대한 해 x:")
print(x1)

# 두 번째 경우 b 벡터
b2 = np.array([2, 0, -1])

# Ly = b2를 풀어서 y 계산
y2 = np.linalg.solve(L, b2)

# Ux = y2를 풀어서 x 계산
x2 = np.linalg.solve(U, y2)

print("₩n두 번째 b 벡터에 대한 해 x:")
print(x2)
```

12. 다음과 같은 선형변환 L의 표준행렬을 특잇값 분해하라. (6-8)

A. L(x, y) = (3x, 4y)

6.8

a) $A = \begin{bmatrix}3 & 0\\ 0 & 4\end{bmatrix}$

$\sigma = 4, 3 \quad \therefore \Sigma = \begin{bmatrix}4 & 0\\ 0 & 3\end{bmatrix}$

$\lambda = 16, 9$

$U\Sigma V^T = A$

$U\begin{bmatrix}4 & 0\\ 0 & 3\end{bmatrix}V^T = A$

$U = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \quad \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}4 & 0\\ 0 & 3\end{bmatrix}V^T = \begin{bmatrix}3 & 0\\ 0 & 4\end{bmatrix}$

$V^T = \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$

$\therefore \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}4 & 0\\ 0 & 3\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$

$A^TA = \begin{bmatrix}9 & 0\\ 0 & 16\end{bmatrix}$

$\lambda = 16 \rightarrow x = \begin{bmatrix}0\\ 1\end{bmatrix}$

$\lambda = 9 \rightarrow x = \begin{bmatrix}1\\ 0\end{bmatrix}$

$\therefore U = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$

B. L(x, y) = (x + y, y, x)

b)

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\det \begin{bmatrix} \lambda-2 & -1 \\ -1 & \lambda-2 \end{bmatrix} \quad (\lambda-2)(\lambda-2)-1 = \lambda^2-4\lambda+3 = 0$$

$\lambda = 1$ or $\lambda = 3$

$\sigma = \sqrt{3}, 1$ $\quad \Sigma \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix}$

$$AA^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow \lambda = 2, 1$$

$\lambda=3$ $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow x\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \therefore V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$\lambda=1$ $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow x\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$U\Sigma V^T = U\begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\therefore U\Sigma V^T = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

C. L(x, y, z) = (x + z, y)

c)

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$\lambda = 2$, $\lambda = 1$

$\sigma_1 = \sqrt{2}$, $\sigma_2 = 1$

$$AA^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\therefore \Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

if) $\lambda = 2$

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \quad x = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

if) $\lambda = 1$

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \quad x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore V$ 는 직교 행렬이므로 $V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$

$$\therefore A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

코드

```
import numpy as np

# (a) L(x, y) = (3x, 4y)
L_a = np.array([[3, 0], [0, 4]])
print("(a) 표준행렬:₩n", L_a)

# 특잇값 분해
U_a, S_a, Vt_a = np.linalg.svd(L_a)
print("(a) 특잇값 분해 결과:")
print("U_a:₩n", U_a)
print("S_a:₩n", S_a)
print("Vt_a:₩n", Vt_a)

# (b) L(x, y) = (x + y, y, x)
L_b = np.array([[1, 1], [0, 1], [1, 0]])
print("₩n(b) 표준행렬:₩n", L_b)

# 특잇값 분해
U_b, S_b, Vt_b = np.linalg.svd(L_b)
print("(b) 특잇값 분해 결과:")
print("U_b:₩n", U_b)
print("S_b:₩n", S_b)
print("Vt_b:₩n", Vt_b)

# (c) L(x, y, z) = (x + z, y)
L_c = np.array([[1, 0, 1], [0, 1, 0]])
print("₩n(c) 표준행렬:₩n", L_c)

# 특잇값 분해
U_c, S_c, Vt_c = np.linalg.svd(L_c)
print("(c) 특잇값 분해 결과:")
print("U_c:₩n", U_c)
print("S_c:₩n", S_c)
print("Vt_c:₩n", Vt_c)
```

13. 다음과 같은 행렬 A에 대하여 $A^T A$와 $A A^T$를 계산하고, 계산한 각 행렬을 특잇값 분해하라.

6.9

a) $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

A^TA의 특이값 분해

$A^TA = B$

B의 $\lambda = 9, 4 \Rightarrow \sigma_1 = 3,\ \sigma_2 = 2$

$\Sigma \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

if) $\lambda = 9$ $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\therefore U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

if) $\lambda = 4$ $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\therefore V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore U\Sigma V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

AA^T의 특잇값 분해

$$AA^T = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} = B$$

$$\det\begin{bmatrix} \lambda-1 & -1 & 1 \\ -1 & \lambda-2 & 0 \\ 1 & 0 & \lambda-2 \end{bmatrix} = (\lambda-1)\det\begin{bmatrix} \lambda-2 & 0 \\ 0 & \lambda-2 \end{bmatrix} + \det\begin{bmatrix} -1 & 0 \\ 1 & \lambda-2 \end{bmatrix} + \det\begin{bmatrix} -1 & \lambda-2 \\ 1 & 0 \end{bmatrix}$$

$$= (\lambda-1)(\lambda-2)(\lambda-2) - (\lambda-2) - (\lambda-2)$$

$$= (\lambda-2)\{(\lambda-1)(\lambda-2) - 2\}$$

$$= (\lambda-2)(\lambda^2 - 3\lambda)$$

$$= (\lambda-2)\lambda(\lambda-3)$$

$\therefore \lambda = 3, 2, 0 \Rightarrow B = B^T$ 이므로 $\sigma_1 = 3,\ \sigma_2 = 2,\ \sigma_3 = 0$

$$\Sigma \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

if) $\lambda = 3$

$$\left[\begin{array}{ccc|c} -2 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right] \quad \begin{cases} x_1 = x_2 \\ x_1 = -x_3 \end{cases} \quad span\begin{Bmatrix} x_1 \\ x_1 \\ -x_1 \end{Bmatrix}$$

if) $\lambda = 2$

$$\left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right] \quad \begin{cases} x_2 = x_3 \\ x_1 = 0 \end{cases} \quad span\left\{\begin{bmatrix} 0 \\ x_2 \\ x_2 \end{bmatrix}\right\}$$

if) $\lambda = 0$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \end{array}\right] \quad \begin{cases} -2x_2 + x_2 - x_3 = 0 \\ x_1 = -2x_2 \\ x_1 = 2x_3 \end{cases} \quad \begin{cases} x_2 = -x_3 \\ x_1 = 2x_3 \end{cases} \quad span\begin{bmatrix} 2x_3 \\ -x_3 \\ x_3 \end{bmatrix}$$

$$\therefore AA^T = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} = U\Sigma V^T$$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

b)

6.4

b) $A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & 0 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 3 & 2 \\ 0 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 13 & 0 & 6 \\ 0 & 0 & 0 \\ 6 & 0 & 4 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 0 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 13 & 6 \\ 6 & 4 \end{bmatrix}$$

A^TA의 특잇값

$$\begin{bmatrix} \lambda-13 & 0 & -6 \\ 0 & \lambda & 0 \\ -6 & 0 & \lambda-4 \end{bmatrix} \qquad B = B^T$$

$$(\lambda-13) \det \begin{bmatrix} \lambda & 0 \\ 0 & \lambda-4 \end{bmatrix} - 6 \det \begin{bmatrix} 0 & \lambda \\ -6 & 0 \end{bmatrix}$$

$$(\lambda-13)\lambda(\lambda-4) - 6(6\lambda) = \lambda(\lambda^2-17\lambda+52) - 36\lambda$$

$$= \lambda(\lambda^2-17\lambda+16)$$

$$= \lambda(\lambda-16)(\lambda-1) \quad \therefore \lambda = 16, 1, 0$$

$B=B^T$ 이므로 $\sigma_1 = 16$, $\sigma_2 = 1$, $\sigma_2 = 0$ $\quad \therefore \Sigma \begin{bmatrix} 16 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

AA^T의 특잇값

$$\begin{bmatrix} \lambda-13 & -6 \\ -6 & \lambda-4 \end{bmatrix}$$

$$(\lambda-13)(\lambda-4) - 36 = \lambda^2 - 17\lambda + 52 - 36 = \lambda^2 - 17\lambda + 16$$

$$(\lambda-16)(\lambda-1) = 0$$

$B=B^T$ 이므로 $\sigma_1 = 16$, $\sigma_2 = 1$

$$\therefore \Sigma \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}$$

A^TA 특잇값 분해

if) $\lambda = 16$

$$\left[\begin{array}{ccc|c} -3 & 0 & 6 & 0 \\ 0 & -16 & 0 & 0 \\ 6 & 0 & -12 & 0 \end{array}\right] \quad \begin{array}{c} -3x_1 + 6x_3 = 0 \\ x_2 = 0 \\ x_1 = 2x_3 \end{array} \quad \therefore \text{span}\left\{\begin{bmatrix} 2x_3 \\ 0 \\ x_3 \end{bmatrix}\right\}$$

if) $\lambda = 1$

$$\left[\begin{array}{ccc|c} 12 & 0 & 6 & 0 \\ 0 & -1 & 0 & 0 \\ 6 & 0 & 3 & 0 \end{array}\right] \quad \left\{\begin{array}{c} 12x_1 + 6x_3 = 0 \\ x_2 = 0 \\ x_3 = -2x_1 \end{array}\right. \quad \therefore \text{span}\left\{\begin{matrix} x_1 \\ 0 \\ -2x_1 \end{matrix}\right\}$$

if $\lambda = 0$

$$\left[\begin{array}{ccc|c} 13 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 6 & 0 & 4 & 0 \end{array}\right] \quad \begin{array}{c} 13x_1 + 6x_3 = 0 \\ x_2 \in R \\ 6x_1 + 4x_3 = 0 \end{array} \quad \therefore \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$$

$$\therefore A^TA = U\Sigma V^T = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \end{bmatrix} \begin{bmatrix} 16 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix}$$

AA^T 특잇값 분해

if) $\lambda=16$

$$\left[\begin{array}{cc|c} -3 & 6 & 0 \\ 6 & -12 & 0 \end{array}\right] \quad \begin{array}{l} -3x_1+6x_2=0 \\ x_1=2x_2 \end{array} \qquad \text{span}\left\{\begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix}\right\}$$

if) $\lambda=1$

$$\left[\begin{array}{cc|c} 12 & 6 & 0 \\ 6 & 3 & 0 \end{array}\right] \quad x_2=-2x_1 \qquad \therefore \text{span}\left\{\begin{bmatrix} x_1 \\ -2x_1 \end{bmatrix}\right\}$$

$$U \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{2}{\sqrt5} & \frac{1}{\sqrt5} \\ \frac{2}{\sqrt5} & -\frac{2}{\sqrt5} \end{bmatrix}$$

$$AA^T = U\Sigma V^T = \begin{bmatrix} 13 & 6 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt5} & \frac{1}{\sqrt5} \\ \frac{2}{\sqrt5} & -\frac{2}{\sqrt5} \end{bmatrix}\begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{2}{\sqrt5} & \frac{2}{\sqrt5} \\ \frac{1}{\sqrt5} & \frac{-2}{\sqrt5} \end{bmatrix}$$

코드

```
import numpy as np

# (a) A 행렬
A_a = np.array([
    [1, 0],
    [1, 1],
    [-1, 1]
])

# (a) A^T A와 AA^T 계산
A_a_TA = A_a.T @ A_a
```

```
A_a_AT = A_a @ A_a.T

# (a) 각 행렬의 고유값 계산
eigenvalues_A_a_TA = np.linalg.eigvals(A_a_TA)
eigenvalues_A_a_AT = np.linalg.eigvals(A_a_AT)

# (a) A^T A와 AA^T의 특잇값 분해
U_a_TA, S_a_TA, Vt_a_TA = np.linalg.svd(A_a_TA)
U_a_AT, S_a_AT, Vt_a_AT = np.linalg.svd(A_a_AT)

print("(a) A^T A의 특잇값 분해:")
print("U_a_TA:₩n", U_a_TA)
print("S_a_TA (특잇값):", S_a_TA)
print("Vt_a_TA:₩n", Vt_a_TA)

print("(a) AA^T의 특잇값 분해:")
print("U_a_AT:₩n", U_a_AT)
print("S_a_AT (특잇값):", S_a_AT)
print("Vt_a_AT:₩n", Vt_a_AT)

print("(a) A^T A:₩n", A_a_TA)
print("(a) A^T A의 고유값:", eigenvalues_A_a_TA)
print("(a) AA^T:₩n", A_a_AT)
print("(a) AA^T의 고유값:", eigenvalues_A_a_AT)

# (b) A 행렬
A_b = np.array([
    [3, 0],
    [2, 0]
])

# (b) A^T A와 AA^T 계산
A_b_TA = A_b.T @ A_b
A_b_AT = A_b @ A_b.T
```

```
# (b) 각 행렬의 고유값 계산
eigenvalues_A_b_TA = np.linalg.eigvals(A_b_TA)
eigenvalues_A_b_AT = np.linalg.eigvals(A_b_AT)

# (b) A^T A와 AA^T의 특잇값 분해
U_b_TA, S_b_TA, Vt_b_TA = np.linalg.svd(A_b_TA)
U_b_AT, S_b_AT, Vt_b_AT = np.linalg.svd(A_b_AT)

print("₩n(b) A^T A의 특잇값 분해:")
print("U_b_TA:₩n", U_b_TA)
print("S_b_TA (특잇값):", S_b_TA)
print("Vt_b_TA:₩n", Vt_b_TA)

print("(b) AA^T의 특잇값 분해:")
print("U_b_AT:₩n", U_b_AT)
print("S_b_AT (특잇값):", S_b_AT)
print("Vt_b_AT:₩n", Vt_b_AT)

print("₩n(b) A^T A:₩n", A_b_TA)
print("(b) A^T A의 고유값:", eigenvalues_A_b_TA)
print("(b) AA^T:₩n", A_b_AT)
print("(b) AA^T의 고유값:", eigenvalues_A_b_AT)
```

6장 실습문제

14. LU 분해 함수를 이용하여 다음과 같은 정방행렬 A를 LU분해하라.
    A. 코드

```
import numpy as np

# LU 분해 함수
def LU_decomp(A):
    (n,m) = A.shape
    L = np.zeros((n,n)) # 빈 행렬 L 만들기
    U = np.zeros((n,n)) # 빈 행렬 U 만들기
    # 행렬 L과 U 계산
    for i in range(0, n):
        for j in range(i, n):
            U[i, j] = A[i, j]
            for k in range(0, i):
                U[i, j] = U[i, j] - L[i, k]*U[k, j]
        L[i,i] = 1
        if i < n-1:
            p = i + 1
            for j in range(0,p):
                L[p, j] = A[p, j]
                for k in range(0, j):
                    L[p, j] = L[p, j] - L[p, k]*U[k, j]
                L[p,j] = L[p,j]/U[j,j]
    return L, U

A = np.array([[1, 2, 1], [2, 3, 3], [-3, -10, 2]])

# 행렬 A의 LU 분해
L, U = LU_decomp(A)

print("A = \n", A)
print("\n")
print("L = \n", L)
print("\n")
print("U = \n", U)
print("\n")
```

B. 결과

```
[Running] python -u "c:\Users\man25\Desktop\AI_Math-with-python\work4\work4-6-1.py"
A =
 [[  1   2   1]
 [  2   3   3]
 [ -3 -10   2]]


L =
 [[ 1.  0.  0.]
 [ 2.  1.  0.]
 [-3.  4.  1.]]


U =
 [[ 1.  2.  1.]
 [ 0. -1.  1.]
 [ 0.  0.  1.]]
```

15. [프로그래밍 실습6.1]의 LU분해 함수를 이용하여 A와 b를 입력하여 Ax = B의 해를 구하라

A. 코드

```
import numpy as np

def LU_decomp(A):
    (n,m) = A.shape
    L = np.zeros((n,n)) # 빈 행렬 L 만들기
    U = np.zeros((n,n)) # 빈 행렬 U 만들기
    # 행렬 L과 U 계산
    for i in range(0, n):
        for j in range(i, n):
            U[i, j] = A[i, j]
            for k in range(0, i):
                U[i, j] = U[i, j] - L[i, k]*U[k, j]
        L[i,i] = 1
        if i < n-1:
            p = i + 1
            for j in range(0,p):
                L[p, j] = A[p, j]
                for k in range(0, j):
                    L[p, j] = L[p, j] - L[p, k]*U[k, j]
                    L[p,j] = L[p,j]/U[j,j]
    return L, U

# LU 분해를 이용하여 Ax=b의 해를 구하는 함수
def LU_Solver(A, b):
    L, U = LU_decomp(A)
    n = len(L)
    # Ly=b 계산
    y = np.zeros((n,1))
    for i in range(0,n):
        y[i] = b[i]
        for k in range(0,i):
            y[i] -= y[k]*L[i,k]
    # Ux=y 계산
    x = np.zeros((n,1))
    for i in range(n-1, -1, -1):
        x[i] = y[i]
        if i < n-1:
            for k in range(i+1,n):
                x[i] -= x[k]*U[i,k]
        x[i] = x[i]/float(U[i,i])
    return x

A = np.array([[1, 2, 1], [2, 3, 3], [-3, -10, 2]])
```

```python
b = np.array([[2], [1], [0]])

# LU 분해를 이용하여 Ax=b 의 해 구하기
x = LU_Solver(A,b)
print("x = \n", x)
```

B. 결과

```
[Running] python -u "c:\Users\man25\Desktop\AI_Math-with-python\work4\work4-6-2.py"
x = 
 [[-58.]
 [ 21.]
 [ 18.]]
```

16. 다음 두 행렬을 특잇값 분해하라.

A. 코드

```
import numpy as np

# 행렬 A 의 특잇값 분해
A = np.array([[6, 4], [0,0], [4,0]])
print("A = \n", A)
print("\n")

U, Sig, VT = np.linalg.svd(A) # 특잇값 분해
print("U= \n", U)
print("\n")

m, n = A.shape
Sigma = np.zeros((m, n)) # m×n 행렬 Σ
k = np.size(Sig)
Sigma[:k, :k] = np.diag(Sig) # 특잇값
print("Sigma = \n", Sigma)
print("\n")
print("V^T = \n", VT) # n×n 행렬 V^T
print("\n")

# 행렬 B 의 특잇값 분해
B = np.array([[1, 1, -1], [0,1,1]])
print("B = \n", B)
print("\n")

U, Sig, VT = np.linalg.svd(B) # 특잇값 분해
print("U= \n", U)
print("\n")

m, n = B.shape
Sigma = np.zeros((m, n)) # m×n 행렬 Σ
k = np.size(Sig)
Sigma[:k, :k] = np.diag(Sig) # 특잇값
print("Sigma = \n", Sigma)
print("\n")
print("V^T = \n", VT) # n×n 행렬 V^T
print("\n")
```

B. 결과

```
[Done] exited with code=0 in 0.217 seconds

[Running] python -u "c:\Users\man25\Desktop\AI_Math-with-python\work4\work4-6-3.py"
A =
 [[6 4]
 [0 0]
 [4 0]]


U=
 [[-0.89442719 -0.4472136   0.        ]
 [ 0.          0.          1.        ]
 [-0.4472136   0.89442719  0.        ]]


Sigma =
 [[8. 0.]
 [0. 2.]
 [0. 0.]]


V^T =
 [[-0.89442719 -0.4472136 ]
 [ 0.4472136  -0.89442719]]


B =
 [[ 1  1 -1]
 [ 0  1  1]]


U=
 [[1. 0.]
 [0. 1.]]


Sigma =
 [[1.73205081 0.         0.        ]
 [0.         1.41421356 0.        ]]


V^T =
 [[ 5.77350269e-01  5.77350269e-01 -5.77350269e-01]
 [ 8.75605293e-17  7.07106781e-01  7.07106781e-01]
 [ 8.16496581e-01 -4.08248290e-01  4.08248290e-01]]
```