$$X(k)=\sum_{n=0}^{N-1} x(n)e^{-\frac{j2\pi kn}{N}}$$

$$\delta(n)=\begin{cases}1 & (n=0)\\ 0 & (n\neq 0)\end{cases}$$

(1) let $x(n)=\delta(n)$:

$$X(k)=\sum_{n=0}^{N-1}\delta(n)e^{\frac{-j2\pi kn}{N}}$$

$$=\delta(0)e^{\frac{-j2\pi k}{N}\times 0}+0+0+\cdots+0$$

$$=1\times e^{0}$$

$$=1$$

(2) when $x(n)=\delta(n-n_0)$

$$X(k)=\sum_{n=0}^{N-1}\delta(n-n_0)e^{\frac{-j2\pi kn}{N}}$$

$$\because \delta(n)=\begin{cases}1 & (n=0)\\ 0 & (n\neq 0)\end{cases}$$

$$\therefore \delta(n-n_0)=\begin{cases}1 & (n=n_0)\\ 0 & (n\neq n_0)\end{cases}$$

let $e^{\frac{-j2\pi k}{N}}=a$

$$\therefore X(k)=\delta(0-n_0)\cdot a^0+\delta(1-n_0)a^1+\cdots+\delta(N-1-n_0)a^{N-1}$$

$$=0+0+\cdots+\delta(n_0-n_0)a^{n_0}+0+\cdots+0$$

$$=a^{n_0}$$

$$=e^{\frac{-j2\pi kn_0}{N}}$$

(3) let $x(n) = a^n$

let $e^{\frac{-j2\pi k}{N}} = b$

$$X(k) = \sum_{n=0}^{N-1} a^n \, b^n$$

$$= \sum_{n=0}^{N-1} (ab)^n$$

by geometric sum: $\sum_{n=0}^{N-1} r^n = \frac{1-r^N}{1-r}$

when $ab = 1$: $X(k) = N$

when $ab \neq 1$: $X(k) = \frac{1-(ab)^N}{1-ab}$

$$\therefore X(k) = \begin{cases} N & \left(ae^{\frac{-j2\pi k}{N}} = 1\right) \\ \frac{1-\left(ae^{\frac{-j2\pi k}{N}}\right)^N}{1-ae^{\frac{-j2\pi k}{N}}} & \left(ae^{\frac{-j2\pi k}{N}} \neq 1\right) \end{cases}$$

(4) when $x(n) = \begin{cases} 1 & (0 \le n \le \frac{N}{2} - 1) \\ 0 & (n < 0 \text{ or } \frac{N}{2} \le n \le N-1) \end{cases}$

let $e^{\frac{-j2\pi k}{N}} = b$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, b^n$$

$$= 1 \times b^0 + b^1 + b^2 + \cdots + b^{\frac{N}{2}-1} + 0 + 0 + 0 \cdots + 0$$

$$= \sum_{n=0}^{\frac{N}{2}-1} b^n$$

by geometric sum: $\sum_{n=0}^{N-1} r^n = \frac{1-r^N}{1-r}$

N is even so $\frac{N}{2} - 1$ is odd

if $b = 1$, $k = 0$

$$X(k) = \frac{N}{2}$$

if $b \ne 1$, $k \ne 0$

$$X(k) = \frac{1 - b^{\frac{N}{2}}}{1-b}$$

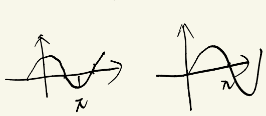
$$= \frac{1 - e^{\frac{-j2\pi k}{N} \times \frac{N}{2}}}{1 - e^{\frac{-j2\pi k}{N}}}$$

$$= \frac{1 - e^{-j\pi k}}{1 - e^{\frac{-j2\pi k}{N}}}$$

by Euler's formula $e^{ix} = \cos(x) + i\sin(x)$

$$e^{-j\pi k} = \cos(\pi k) - j\sin(\pi k)$$

$$= (-1)^k - j \times 0 = (-1)^k$$



$$X(k) = \frac{1 - (-1)^k}{1 - e^{\frac{-j2\pi k}{N}}}$$

when k is even: $(-1)^k = 1$

$$X(k) = 1 - 1 = 0$$

when k is odd: $(-1)^k = -1$

$$\therefore X(k) = \frac{2}{1 - e^{\frac{-j2\pi k}{N}}}$$

To sum up:

$$X(k) = \begin{cases} 0 & (k \text{ is even}) \\ \frac{2}{1 - e^{\frac{-j2\pi k}{N}}} & (k \text{ is odd}) \end{cases}$$

(5) when $x(n)=e^{\frac{j2\pi k_0 n}{N}}$

let $e^{\frac{j2\pi n}{N}}=b$ $\quad x(n)=b^{k_0 n}$

$$X(k)=\sum_{n=0}^{N-1} b^{k_0}\cdot e^{\frac{-j2\pi k n}{N}}$$

$$=\sum_{n=0}^{N-1} b^{k_0}\cdot b^{-k}$$

$$=\sum_{n=0}^{N-1} b^{k_0-k}$$

$$=\sum_{n=0}^{N-1} e^{\frac{j2\pi n(k_0-k)}{N}}$$

let $e^{\frac{j2\pi(k_0-k)}{N}}=a$

$$X(n)=\sum_{n=0}^{N-1} a^n$$

by geometric sum:

$$\sum_{n=0}^{N-1} r^n=\frac{1-r^N}{1-r}$$

when: $a=1$ $\quad X(k)=N$ $\quad k_0=k$

when $a\neq 1$ $\quad X(k)=\frac{1-a^N}{1-a}$

$$=\frac{1-e^{j2\pi(k_0-k)}}{1-a}$$

$\because k_0\neq k$

$\therefore k_0-k\neq 0$

let $k_0-k=m$

$$X(k)=\frac{1-e^{j2\pi m}}{1-a}$$

$$=\frac{1-(e^{-jm\pi})^{-2}}{1-a}$$

by Euler's formular:

$$e^{jx}=\cos(x)+j\sin(x)$$

$\therefore X(k)$

$$=\frac{1-(\cos(\pi m)-j\sin(\pi x))^{-2}}{1-a}$$

$$=\frac{1-(-1)^{-2}}{1-a}$$

$$=0$$

$$\therefore X(k)=\begin{cases} N & (k=k_0)\\ 0 & (k\neq k_0)\end{cases}$$

(b) when $x(n) = \cos\left(\frac{2\pi k_0 n}{N}\right)$

by Euler's formula:

① $e^{ix} = \cos(x) + i\sin(x)$

② $e^{-ix} = \cos(x) - i\sin(x)$

①+②: $\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$

$$x(n) = \cos\left(\frac{2\pi k_0 n}{N}\right)$$

$$= \frac{1}{2}\left(e^{\frac{j2\pi k_0 n}{N}} + e^{\frac{-j2\pi k_0 n}{N}}\right)$$

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{\frac{-j2\pi k n}{N}}$$

$$= \sum_{n=0}^{N-1}\left[\frac{1}{2}\left(e^{\frac{j2\pi k_0 n}{N}} \cdot e^{\frac{-j2\pi k n}{N}}\right) + \frac{1}{2}\left(e^{\frac{-j2\pi k_0 n}{N}} \cdot e^{\frac{-j2\pi k n}{N}}\right)\right]$$

$$= \frac{1}{2}\sum_{n=0}^{N-1}\left(e^{\frac{j2\pi n(k_0-k)}{N}} + e^{\frac{-j2\pi n(k_0+k)}{N}}\right)$$

by Euler's formular

$$e^{jx} = \cos(x) + j\sin(x)$$

let $\frac{2jn(k_0-k)}{N} = a$ $\frac{2jn(k_0+k)}{N} = b$

$$X(k) = \frac{1}{2}\sum_{n=0}^{N-1}\left(e^{j\pi a} + e^{-j\pi \times b}\right)$$

$$= \frac{1}{2}\sum_{n=0}^{N-1}\left(\cos(\pi a) + j\sin(\pi a) + \cos(\pi b) - j\sin(\pi a)\right)$$

$$X(k) = \frac{1}{2}\sum_{n=0}^{N-1}\left((-1)^a + (-1)^b\right)$$

$$= \frac{1}{2}\sum_{n=0}^{N-1}(-1)^a + \frac{1}{2}\sum_{n=0}^{N-1}(-1)^b$$

a and b are even

$\therefore X(k) = \frac{1}{2}N + \frac{1}{2}N = N$