ECE586 MP1 Markov Chains

Exercise 1

What is the distribution of the number of fair coin tosses before one observes 3 heads in a row? To solve this, consider a 4-state Markov chain with transition probability matrix

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $X_t = 1$ if the previous toss was tails, $X_t = 2$ if the last two tosses were tails then heads, $X_t = 3$ if the last three tosses were tails then heads twice, and $X_t = 4$ is an absorbing state that is reached when the last three tosses are heads.

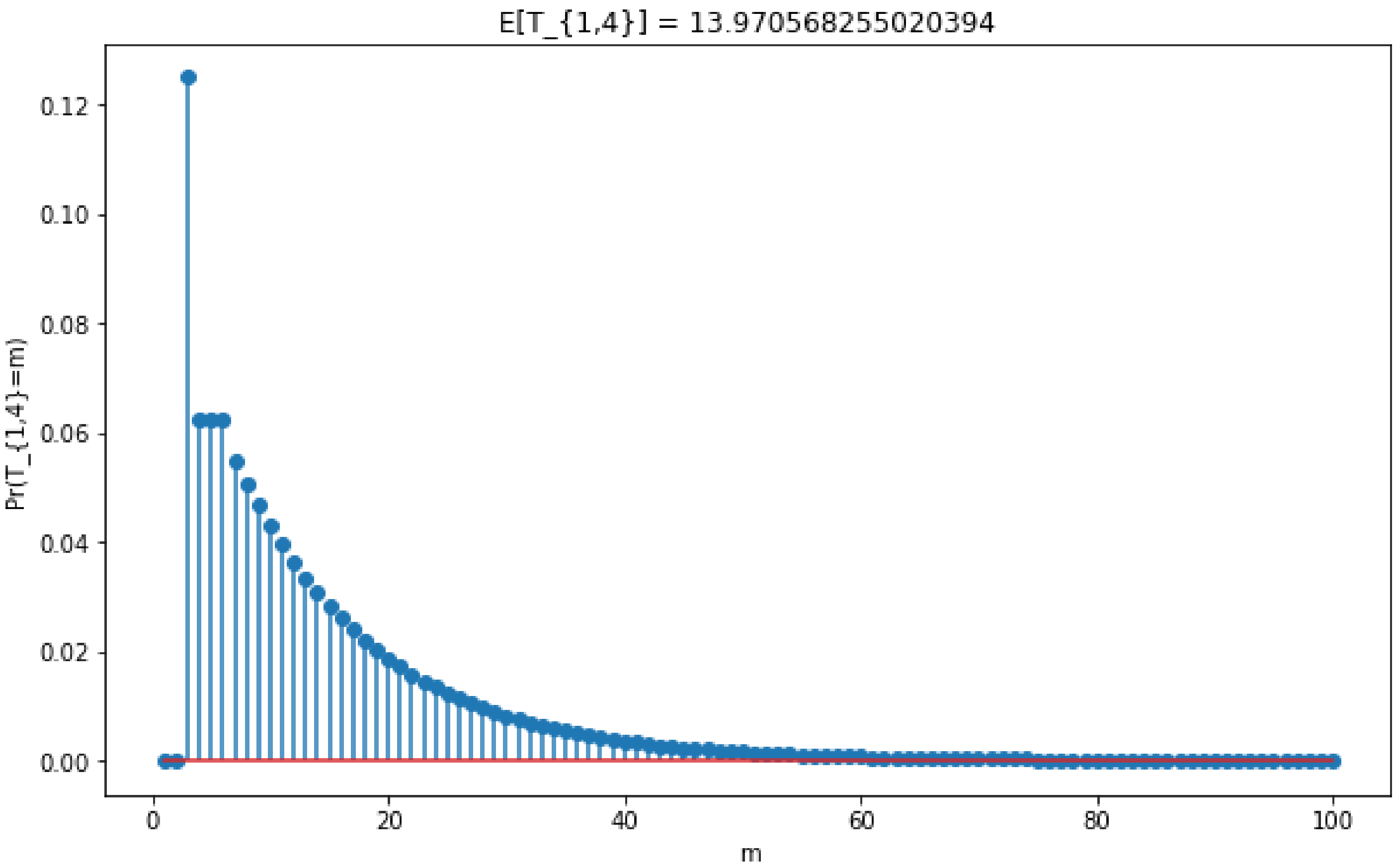
- ** (15 pts) ** Write a computer program (e.g., in Python, Matlab, ...) to compute $\Pr(T_{1,4} = m)$ for $m = 1, 2, \ldots, 100$ and ** (10 pts) ** use this to compute and print an estimate of the expected number of tosses $\mathbb{E}[T_{1,4}]$

In [4]:

```
# See compute_Phi_ET in fsmc_code.py

P = np.array([[0.5, 0.5, 0, 0], [0.5, 0, 0.5, 0], [0.5, 0, 0, 0.5], [0, 0, 0, 1]])
Phi_list, ET = compute_Phi_ET(P, 100)

m = np.arange(1, 101)                               ### steps to be plotted
Pr = Phi_list[m, 0, 3] - Phi_list[(m - 1), 0, 3]   ### \Pr(T_{1,4} = m) for all m
E = ET[0, 3]                                        ### \mathbb{E}[T_{1,4}]

plt.figure(figsize=(10, 6))
plt.stem(m, Pr, use_line_collection=True)
plt.xlabel(r'm')
plt.ylabel(r'Pr(T_{1,4}=m)')
plt.title(r'E[T_{1,4}] = ' + str(E))
```

Out[4]:

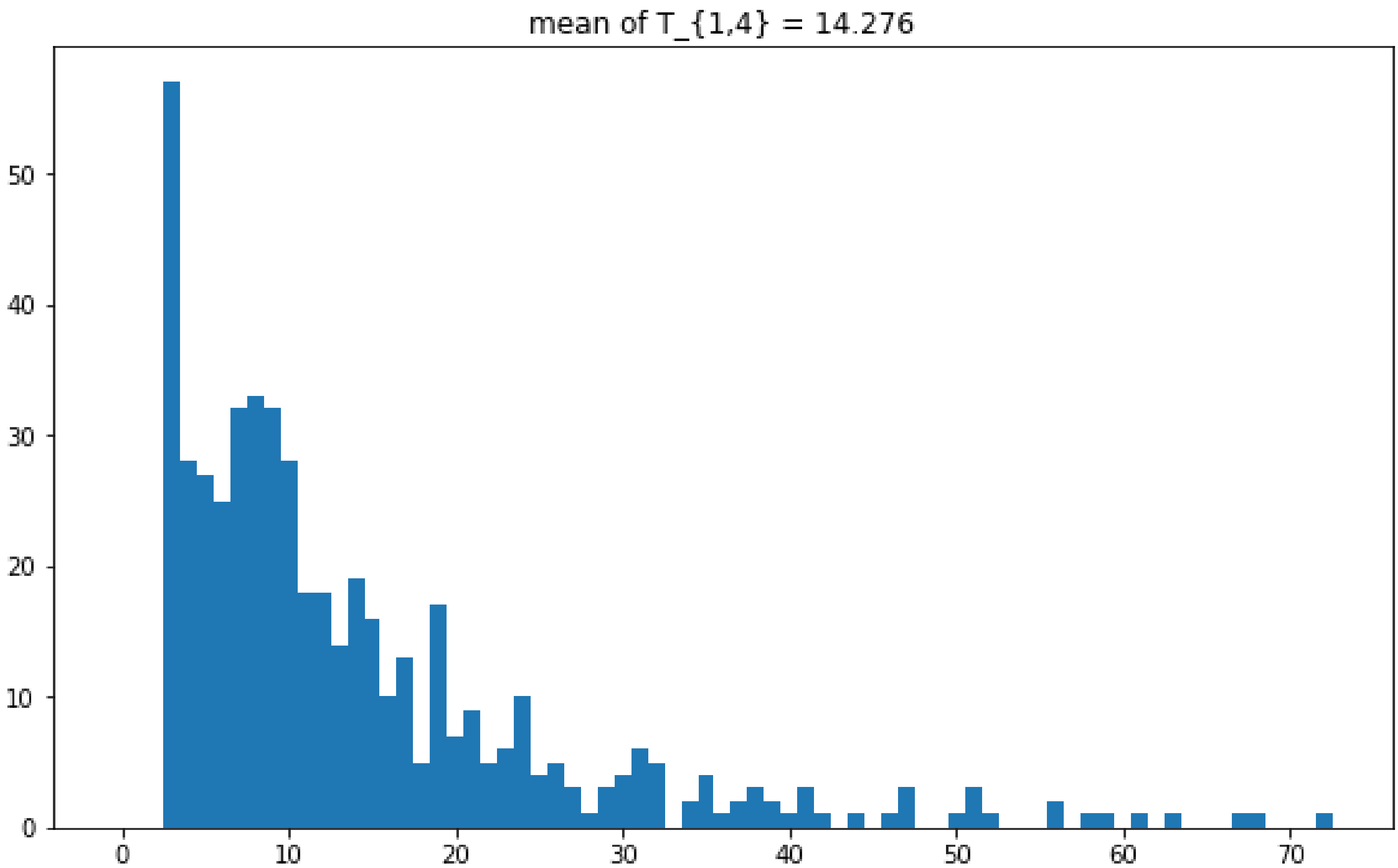
```
Text(0.5, 1.0, 'E[T_{1,4}] = 13.970568255020394')
```



- ** (15 pts) ** Write a computer program that generates 500 realizations from this Markov chain and ** (10 pts) ** uses them to plot a histogram of $T_{1,4}$ and compute/print an estimate of the expected number of tosses $\mathbb{E}[T_{1,4}]$.

In [5]:
```
# implement simulate_hitting_time(P, states, nr) in fsmc_code.py

T = simulate_hitting_time(P, [0, 3], 500)
plt.figure(figsize=(10, 6))
plt.hist(T, bins=np.arange(max(T))-0.5)
plt.title(r'mean of T_{1,4} = ' + str(np.mean(T)))
```

Out[5]:
```
Text(0.5, 1.0, 'mean of T_{1,4} = 14.276')
```



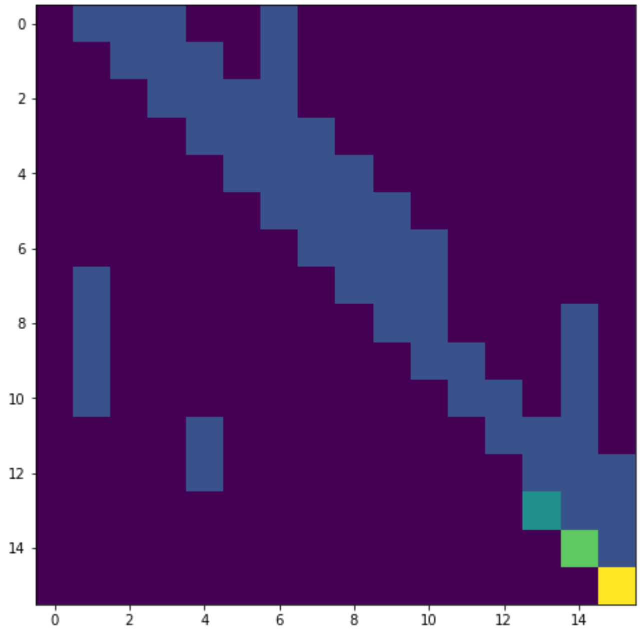
Exercise 2

Consider the miniature chutes and ladders game shown in Figure 1. Assume a player starts on the space labeled 1 and plays by rolling a fair four-sided die and then moves that number of spaces. If a player lands on the bottom of a ladder, then they automatically climb to the top. If a player lands at the top of a slide, then they automatically slide to the bottom. This process can be modeled by a Markov chain with $n = 16$ states where each state is associated with a square where players can start their turn (e.g., players never start at the bottom of a ladder or the top of a slide). To finish the game, players must land exactly on space 20 (moves beyond this are not taken).

- ** (10 pts) ** Compute the transition probability matrix P of the implied Markov chain.

In [6]:
```python
# You can either do this by hand (e.g., look at picture and write down matrix) or by automating the process.

# By hand
# P = np.asarray([[...],[...],[...],...])

###

# Or automated general function for Chutes and Ladders games
def construct_P_matrix(n, dice, chutes, ladders):
    '''
    Arguments:
        n {int} -- size of the state space
        dice {numpy.array} -- probability distribution of the dice outcome
        chutes {list[(int, int)]} -- the list of chutes, in pairs of (start, end)
        ladders {list[(int, int)]} -- the list of ladders, in pairs of (start, end)

    Returns:
        P {numpy.array} -- n x n, transition matrix of the Markov chain
    '''
    # Add code here to build matrix

    #########################################################################################
    P = np.zeros((n, n))
    num_ladders = len(ladders)
    num_chutes  = len(chutes)
    dice_length = len(dice)
    n_expand = n + num_ladders + num_chutes
    P_expand = np.zeros((n_expand, n_expand))
    remove = [0] * (num_ladders + num_chutes)

    for i in range(0, n_expand) :
        if i + dice_length < n_expand :
            P_expand[i, (i+1):(i+1+dice_length)] = dice
        else :
            P_expand[i, i]      = 1 - sum(dice[0:(n_expand-(i+1))])
            P_expand[i, (i+1):n_expand] = dice[0:(n_expand-(i+1))]

    for j in range(0, num_ladders) :
        P_expand[:, (ladders[j][1]-1)] = P_expand[:, (ladders[j][1]-1)] + P_expand[:, (ladders[j][0]-1)]
        remove[j] = ladders[j][0] - 1

    for k in range(0, num_chutes) :
        P_expand[:, (chutes[k][1]-1)] = P_expand[:, (chutes[k][1]-1)] + P_expand[:, (chutes[k][0]-1)]
        remove[num_ladders + k] = chutes[k][0] - 1

    P_remove_row = np.delete(P_expand, remove, 0)
    P_remove_row_column = np.delete(P_remove_row, remove, 1)
    P = P_remove_row_column
    P[n-1, n-1] = 1
    #########################################################################################

    return P

n = 16 ### number of states
dice = np.array([1/4, 1/4, 1/4, 1/4]) ### probability distribution of dice
chutes = [(13, 2), (17, 6)]             ### (sorce, destination) pairs of chutes
ladders = [(4, 8), (14, 19)]            ### (sorce, destination) pairs of ladders
P = construct_P_matrix(n, dice, chutes, ladders)
###

# Plot transition matrix
plt.figure(figsize=(8, 8))
plt.imshow(P)
```

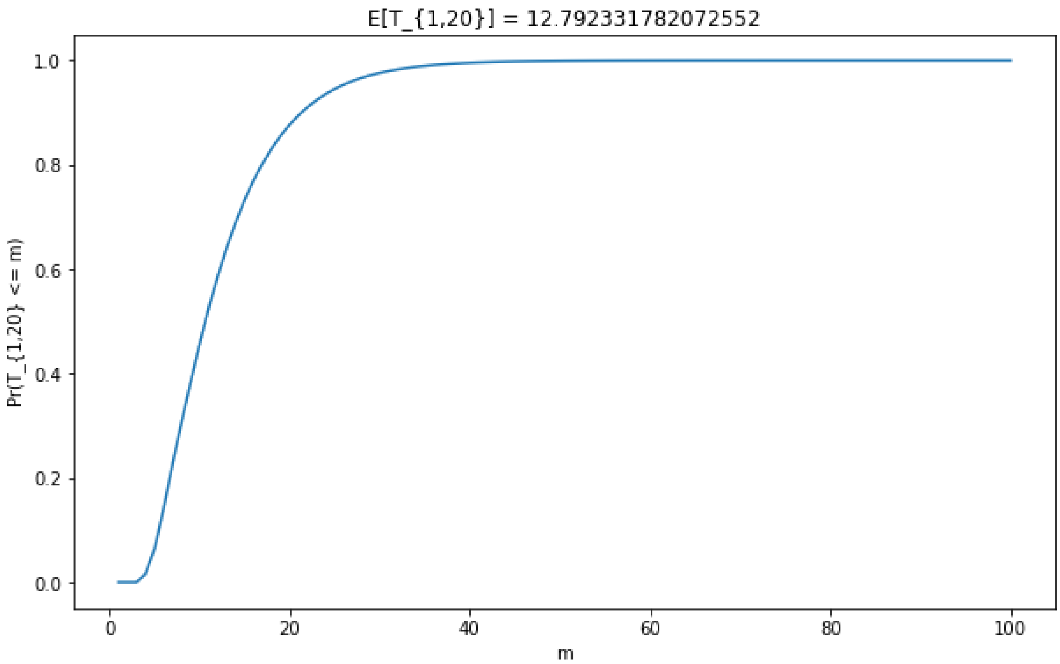
Out[6]: <matplotlib.image.AxesImage at 0x28764411100>



- ** (10 pts) ** For this Markov chain, use function from Exercise 1 to compute and plot the cumulative distribution of the number turns a player takes to finish (i.e., the probability $\Pr(T_{1,20} \leq m)$ where $T_{1,20}$ is the hitting time from state 1 to state 20). Compute and print the mean $\mathbb{E}[T_{1,20}]$.

In [7]:
```
# Use previous functions to complete this exercise
Phi_list, ET = compute_Phi_ET(P, ns=100)

m = np.arange(1, 101)    ### steps to be plotted
Pr = Phi_list[m, 0, 15] ### \Pr(T_{1,20} <= m) for all m
E = ET[0, 15]           ### \mathbb{E}[T_{1,20}]

plt.figure(figsize=(10, 6))
plt.plot(m ,Pr)
plt.xlabel(r'm')
plt.ylabel(r'Pr(T_{1,20} <= m)')
plt.title(r'E[T_{1,20}] = ' + str(E))
```

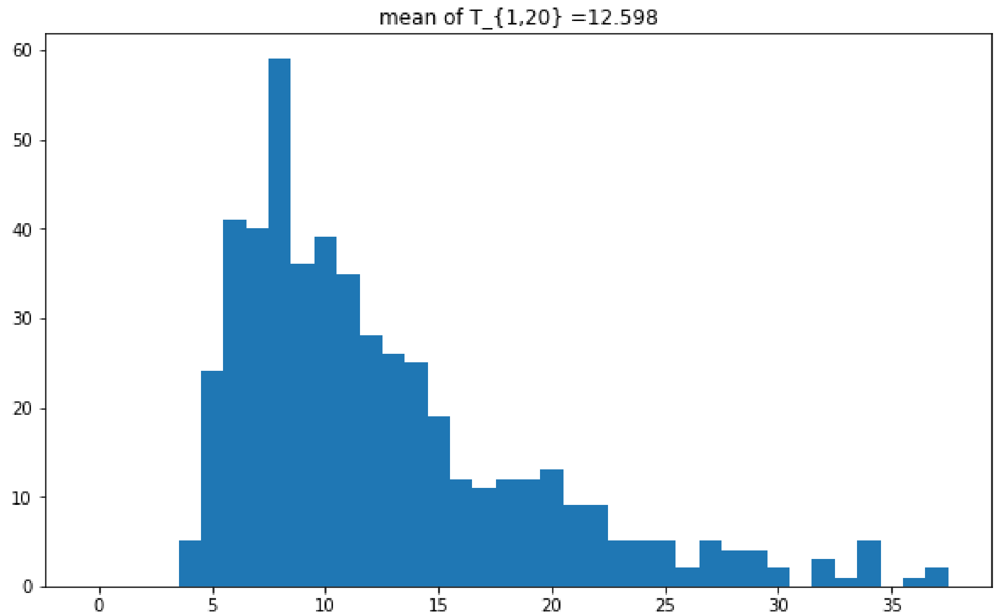
Out[7]:
```
Text(0.5, 1.0, 'E[T_{1,20}] = 12.792331782072552')
```



- ** (10 pts) ** Use function from Exercise 1 to generate 500 realizations from this Markov chain. Then, use them to plot a histogram of $T_{1,20}$ and compute/print an estimate of the expected number of tosses $\mathbb{E}\left[T_{1,20}\right]$.

In [8]:
```
# Use previous funcitons to complete this exercise
T = simulate_hitting_time(P, [0, n-1], 500)
plt.figure(figsize=(10, 6))
plt.hist(T, bins=np.arange(max(T))-0.5)
plt.title(r'mean of T_{1,20} =' + str(np.mean(T)))
```

Out[8]:
```
Text(0.5, 1.0, 'mean of T_{1,20} =12.598')
```



Exercise 3

** (10 pts) ** Write a program to compute the stationary distribution of a Markov chain when it is unique. Consider a game where the gameboard has 8 different spaces arranged in a circle. During each turn, a player rolls two 4-sided dice and moves clockwise by a number of spaces equal to their sum. ** (5 pts) ** Define the transition matrix for this 8-state Markov chain and compute its stationary distribution.

In [11]:
```
# Use previous functions to complete this exercise
### construct the transition matrix
##################################################################
P = np.zeros((8, 8))
###        [1/16, 0, 1/16, 2/16, 3/16, 4/16, 3/16, 2/16]
P[0, :] = [1/16, 0, 1/16, 1/8, 3/16, 1/4, 3/16, 1/8]
for row_state in range(1, 8) :
    P[row_state, :] = np.roll(P[row_state-1, :], 1)
##################################################################
print(P)
stationary_distribution(P)
```

```
[[0.0625 0.     0.0625 0.125  0.1875 0.25   0.1875 0.125 ]
 [0.125  0.0625 0.     0.0625 0.125  0.1875 0.25   0.1875]
 [0.1875 0.125  0.0625 0.     0.0625 0.125  0.1875 0.25  ]
 [0.25   0.1875 0.125  0.0625 0.     0.0625 0.125  0.1875]
 [0.1875 0.25   0.1875 0.125  0.0625 0.     0.0625 0.125 ]
 [0.125  0.1875 0.25   0.1875 0.125  0.0625 0.     0.0625]
 [0.0625 0.125  0.1875 0.25   0.1875 0.125  0.0625 0.    ]
 [0.     0.0625 0.125  0.1875 0.25   0.1875 0.125  0.0625]]
```

Out[11]:
```
array([[0.125],
       [0.125],
       [0.125],
       [0.125],
       [0.125],
       [0.125],
       [0.125],
       [0.125]])
```

Next, suppose that one space is special (e.g., state-1 of the Markov chain) and a player can only leave this space by rolling doubles (i.e., when both dice show the same value). Again, the player moves clockwise by a number of spaces equal to their sum. ** (5 pts) ** Define the transition matrix for this 8-state Markov chain and compute its stationary probability distribution.

In [12]:

```
# Use previous functions to complete this exercise
### construct the transition matrix
####################################################################
P = np.zeros((8, 8))
P[0, :] = [13/16, 0, 1/16, 0, 1/16, 0, 1/16, 0]
###                 [1/16, 0, 1/16, 2/16, 3/16, 4/16, 3/16, 2/16]
P[1, :] = np.roll([1/16, 0, 1/16, 1/8, 3/16, 1/4, 3/16, 1/8], 1)
for row_state in range(2, 8) :
    P[row_state, :] = np.roll(P[row_state-1, :], 1)
####################################################################
print(P)
stationary_distribution(P)
```

```
[[0.8125 0.     0.0625 0.     0.0625 0.     0.0625 0.    ]
 [0.125  0.0625 0.     0.0625 0.125  0.1875 0.25   0.1875]
 [0.1875 0.125  0.0625 0.     0.0625 0.125  0.1875 0.25  ]
 [0.25   0.1875 0.125  0.0625 0.     0.0625 0.125  0.1875]
 [0.1875 0.25   0.1875 0.125  0.0625 0.     0.0625 0.125 ]
 [0.125  0.1875 0.25   0.1875 0.125  0.0625 0.     0.0625]
 [0.0625 0.125  0.1875 0.25   0.1875 0.125  0.0625 0.    ]
 [0.     0.0625 0.125  0.1875 0.25   0.1875 0.125  0.0625]]
```

Out[12]:

```
array([[0.41836864],
       [0.08285234],
       [0.10176963],
       [0.07092795],
       [0.09311176],
       [0.0625555 ],
       [0.09593429],
       [0.07447989]])
```

In [ ]: